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Optimal forecast reconciliation with time series selection

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Abstract

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Keywords: Keyword 1, Keyword 2

1 Introduction

Hierarchical time series and forecast reconciliation.

Single-level approaches, least squares-based reconciliation approaches, geometric intuition, other extensions with constraints.

However... Two issues.

In this paper, our focus will be on...

The remainder of the paper is structured as follows.

2 Preliminaries

2.1 Notation

We denote the set $\{1, \ldots, k\}$ by $[k]$ for any non-negative integer k. A *hierarchical time series* can be considered as an n-dimensional multivariate time series, $\{\boldsymbol{y}_t, t \in [T]\}$, that adheres to known linear constraints. Let $\boldsymbol{y}_t \in \mathbb{R}^n$ be a vector comprising observations of all time series in the hierarchy at time t, and $\boldsymbol{b}_t \in \mathbb{R}^{n_b}$ be a vector comprising observations of all bottom-level time series at time t. The full hierarchy at time t can be written as

$$\boldsymbol{y}_t = S\boldsymbol{b}_t,$$

where S is an $n \times n_b$ *summing matrix* that shows aggregation constraints present in the structure. We can write the summing matrix as $S = \begin{bmatrix} A \\ I_{n_b} \end{bmatrix}$, where A is an $n_a \times n_b$ *aggregation matrix* with $n = n_a + n_b$, and I_{n_b} is an n_b-dimensional identity matrix.
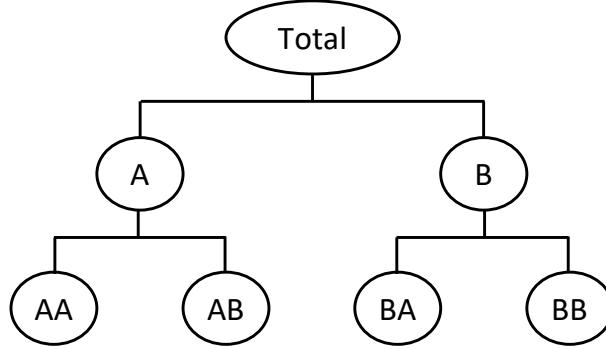

Figure 1: *An example of a two-level hierarchical time series.*

To clarify these notations, consider the example of the hierarchy in Figure 1. For this two-level hierarchy, $n = 7$, $n_b = 4$, $n_a = 3$, $y_t = [y_{\text{Total},t}, y_{A,t}, y_{B,t}, y_{AA,t}, y_{AB,t}, y_{BA,t}, y_{BB,t}]'$, $b_t = [y_{AA,t}, y_{AB,t}, y_{BA,t}, y_{BB,t}]'$, and

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ & I_4 & & \end{bmatrix}.$$
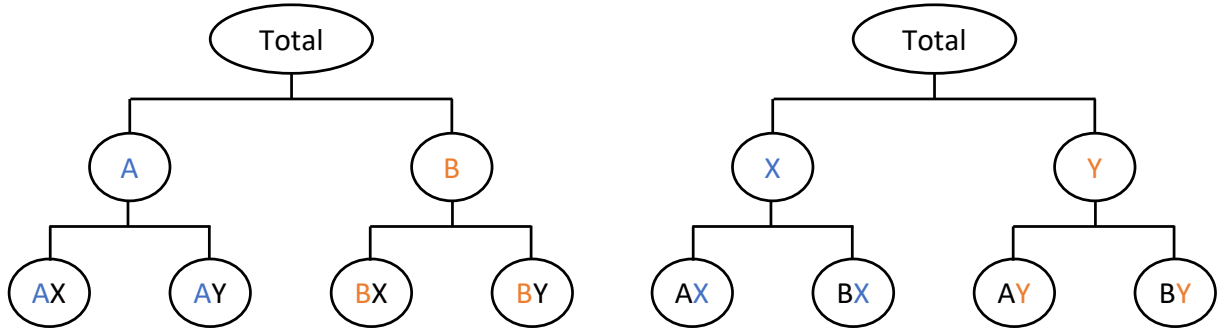

Figure 2: *An example of a two level grouped time series.*

When data structure does not naturally disaggregate in a unique hierarchical manner, we can combine these hierarchical structures to form a *grouped time series*. Thus, grouped time series can also be considered as hierarchical time series with more than one grouping structure. Figure 2 shows an example of a two level grouped time series with two alternative aggregation structures. For this example, $n = 9$, $n_b = 4$, $n_a = 5$, $y_t = [y_{\text{Total},t}, y_{A,t}, y_{B,t}, y_{X,t}, y_{Y,t}, y_{AX,t}, y_{AY,t}, y_{BX,t}, y_{BY,t}]'$, $b_t = [y_{AX,t}, y_{AY,t}, y_{BX,t}, y_{BY,t}]'$, and

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ & I_4 & \end{bmatrix}.$$

2.2 Linear forecast reconciliation

Let $\hat{\boldsymbol{y}}_{T+h|T} \in \mathbb{R}^n$ be a vector of h-step-ahead *base forecasts* for all time series in the hierarchy, given observations up to time T, and stacked in the same order as $\boldsymbol{y}_t$. We can use any method to generate these forecasts, but In general they will not add up especially when we forecast each series independently.

When forecasting hierarchical time series, we expect the forecasts to be *coherent* (i.e., aggregation constraints are satisfied). Let $\tilde{\boldsymbol{y}}_{T+h|T} \in \mathbb{R}^n$ denote a vector of h-step-ahead *reconciled forecasts* which are coherent by construction, ψ a mapping that reconciles base forecasts, $\hat{\boldsymbol{y}}_{T+h|T}$. Then we have *forecast reconciliation* $\tilde{\boldsymbol{y}}_{T+h|T} = \psi(\hat{\boldsymbol{y}}_{T+h|T})$, which is essentially a post-processing method. In this paper, we focus on linear forecast reconciliation given by

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}_h \hat{\boldsymbol{y}}_{T+h|T},$$

where

- $\boldsymbol{G}_h$ is an $n_b \times n$ weighting matrix that maps the base forecasts into the bottom level. In other words, it combines all base forecasts to form reconciled forecasts for bottom-level series.
- $\boldsymbol{S}$ is an $n \times n_b$ summing matrix that sums up bottom-level reconciled forecasts to produce coherent forecasts of all levels. It identifies the linear constraints involved in the hierarchy.

2.2.1 Minimum trace reconciliation

Let the h-step-ahead *base forecast errors* be defined as $\hat{\boldsymbol{e}}_{T+h|T} = \boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T}$, and the h-step-ahead *reconciled forecast errors* be defined as $\tilde{\boldsymbol{e}}_{T+h|T} = \boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T}$. Wickramasuriya, Athanasopoulos & Hyndman (2019) formulated a linear reconciliation problem as minimizing the trace (MinT) of the h-step-ahead covariance matrix of the reconciled forecast errors, $\text{Var}(\tilde{\boldsymbol{e}}_{T+h|T})$.

Under the assumption of unbiasedness, the unique solution of the minimization problem is given by

$$G_h = \left(S' W_h^{-1} S \right)^{-1} S' W_h^{-1}, \tag{1}$$

where W_h is the positive definite covariance matrix of the h-step-ahead base forecast errors, $\text{Var}(\hat{e}_{T+h|T})$.

The trace minimization problem can be reformulated as a least squares problem with linear constraints given by

$$\min_{\tilde{y}_{T+h|T}} \quad \frac{1}{2} (\hat{y}_{T+h|T} - \tilde{y}_{T+h|T})' W_h^{-1} (\hat{y}_{T+h|T} - \tilde{y}_{T+h|T})$$

$$\text{s.t.} \quad \tilde{y}_{T+h|T} = S \tilde{b}_{T+h|T},$$

where $\tilde{b}_{T+h|T} \in \mathbb{R}^{n_b}$ is the vector comprising h-step-ahead bottom-level reconciled forecasts, made at time T. Focusing on W_h, the intuitive behind the MinT reconciliation is that the larger the estimated variance of the base forecast errors, the larger the range of adjustments permitted for forecast reconciliation.

It's challenging to estimate W_h, especially for $h > 1$. Assuming that $W_h = k_h W_1$, $\forall h$, where $k_h > 0$, the MinT solution of G does not change with the forecast horizon. Hence, we will drop the subscript h for the ease of exposition. The most popularly used candidate estimators for W in the forecast reconciliation literature are listed as follows.

1. $W_{\text{OLS}} = I$ is the *OLS estimator* proposed by Hyndman et al. (2011), assuming that the base forecast errors are uncorrelated and equivariant.

2. $W_{\text{WLSs}} = \text{diag}(S1)$ is the *WLS estimator applying structural scaling* proposed by Athanasopoulos et al. (2017). This estimator depends only on the aggregation structure of the hierarchy. It assumes that the variance of each bottom-level base forecast error is equivalent and uncorrelated between nodes.

3. $W_{\text{WLSv}} = \text{diag}(\hat{W}_1)$ is the *WLS estimator applying variance scaling* proposed by Hyndman, Lee & Wang (2016), where $\hat{W}_1$ denotes the unbiased covariance estimator based on the in-sample one-step-ahead base forecast errors (i.e., residuals).

4. $W_{\text{MinT}} = \hat{W}_1$ is referred to as the *MinT estimator* based on the sample covariance matrix proposed by Wickramasuriya, Athanasopoulos & Hyndman (2019).

5. $W_{\text{MinTs}} = \lambda \operatorname{diag}(\hat{W}_1) + (1 - \lambda)\hat{W}_1$ is the *MinT shrinkage estimator* suggested by Wickramasuriya, Athanasopoulos & Hyndman ([2019](#)), in which off-diagonal elements of $\hat{W}_1$ are shrunk toward zero.

It's hard to say which estimator for W works better. Pritularga, Svetunkov & Kourentzes ([2021](#)) demonstrated that the performance of forecast reconciliation is affected by two sources of uncertainties, i.e., the base forecast uncertainty and the reconciliation weight uncertainty. Recall that the uncertainty in the MinT solution in Equation [1](#) is introduced by the uncertainty in the reconciliation weighting matrix as the summing matrix is fixed for a certain hierarchy. This indicates that OLS and WLSs estimators for W may lead to less volatile reconcliation performance compared to WLSv, MinT, and MinTs estimators. Panagiotelis et al. ([2021](#)) provided a geometric intuition for reconciliation and showed that, when considering the Euclidean distance loss function, OLS reconciliation yields results that are at least as favorable as the base forecasts, whereas MinT reconciliation performs poorly relative to the base forecasts. However, when considering the mean squared reconciled forecast error, Wickramasuriya ([2021](#)) indicated that MinT reconciliation is better than OLS reconciliation.

2.2.2 Reconciliation with regularization

- their drawbacks

- our contributions

3 Forecast reconciliation with time series selection

For the ease of exposition, we will use the subscript h rather than $T + h|T$.

3.1 Constrained reconciliation with selection

Unbiasedness constraint

3.2 Unconstrained reconciliation with selection

4 Monte Carlo simulations

4.1 Model misspecification in a hierarchy

4.2 Exploring the effect of correlation

5 Applications

5.1 Forecasting Australian domestic tourism

5.2 Forecasting Australian prison population

6 Conclusion

Acknowledgement

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