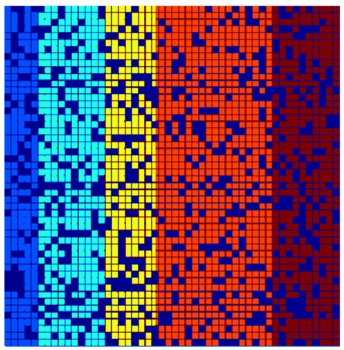
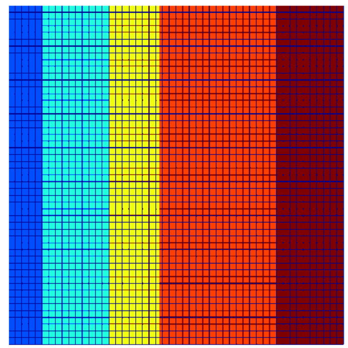
ROBUST PCA

D - observation



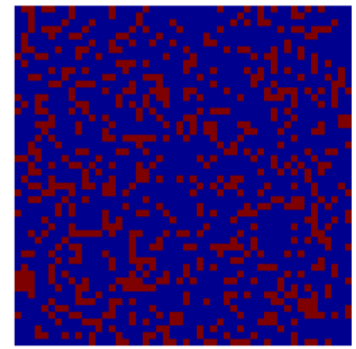
=

A_0 – low-rank



+

E_0 – sparse



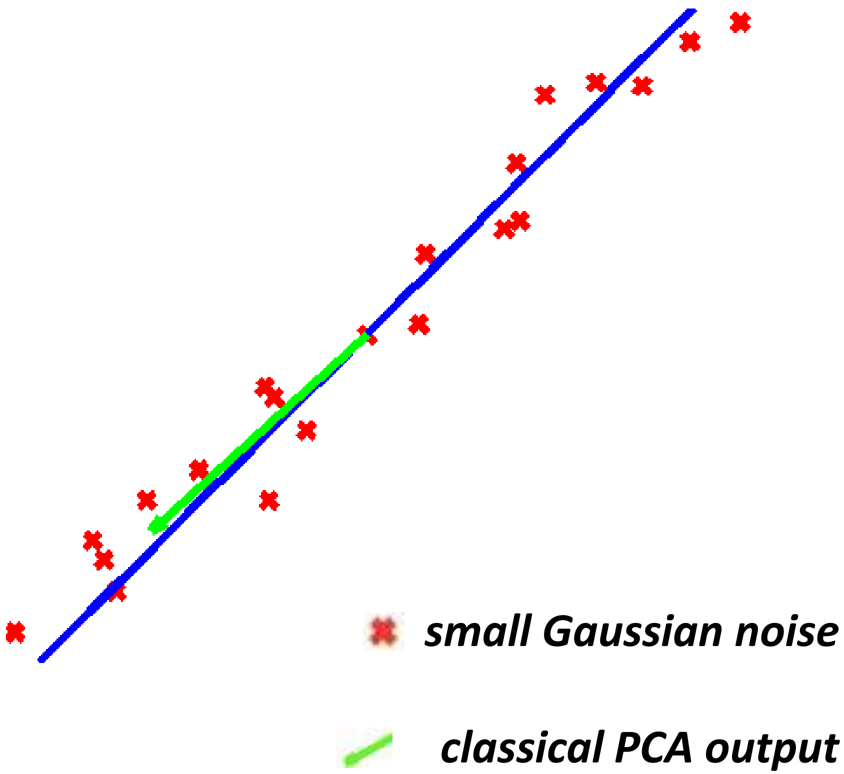
Problem: Given $D = A_0 + E_0$, recover A_0 and E_0

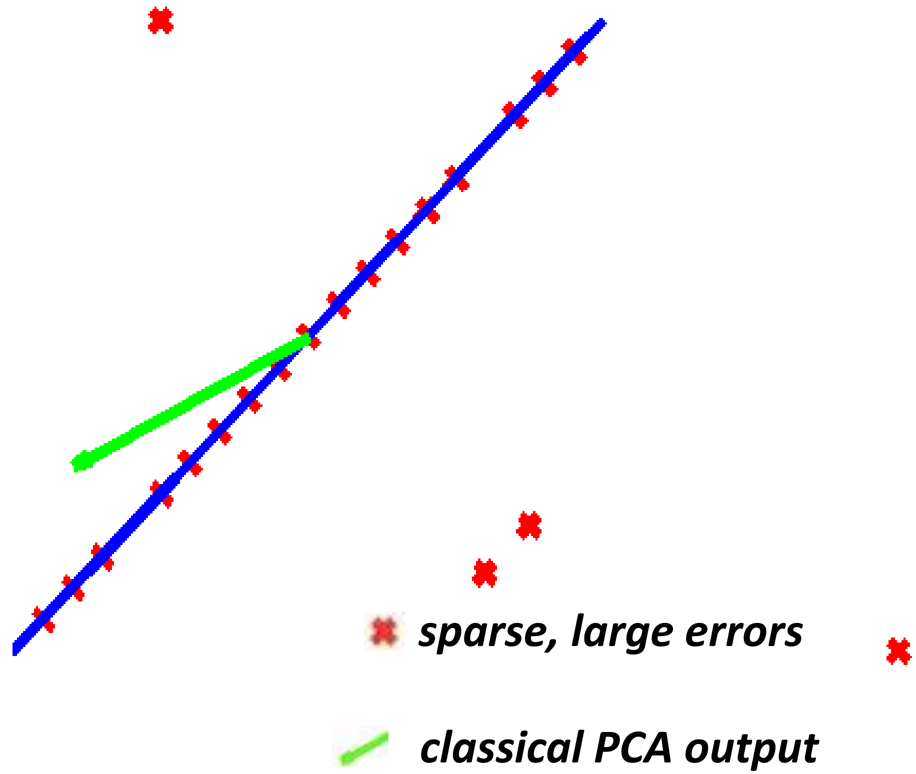
Low-rank component

Sparse component (gross errors)

ROBUST PCA

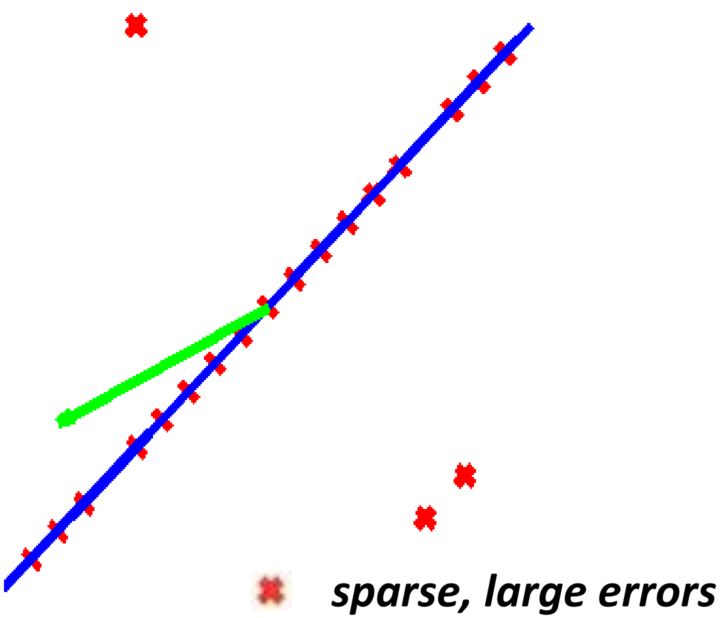
经典PCA局限性





ROBUST PCA

提出 Robust PCA



给出：$D = A_0 + E_0$，恢复 A_0 和 E_0

Low-rank component

Sparse component (gross errors)

$$\min \ \mathrm{rank}(A) + \gamma \|E\|_0 \quad \mathrm{subj} \quad A + E = D.$$

低秩：$\mathrm{rank}(A) = \#\{\sigma_i(A) \neq 0\}.$

稀疏：$\|E\|_0 = \#\{E_{ij} \neq 0\}.$

！ Not always – original problem is NP-hard

ROBUST PCA

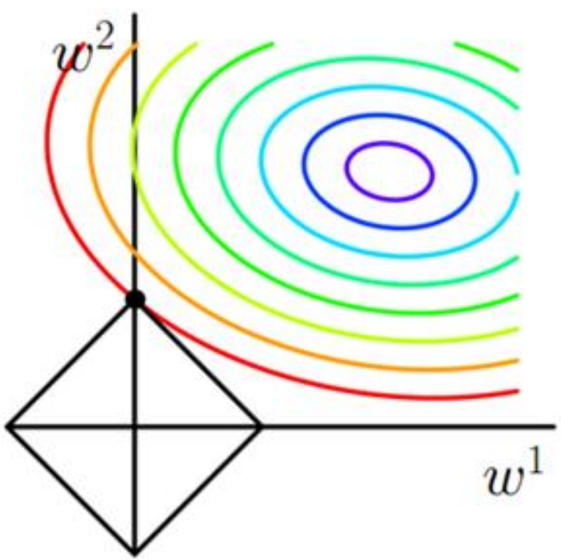
Convex relaxation

0范数(L0)：$\|E\|_0 = \#\{E_{ij} \neq 0\}.$ → 1范数(L1)：$\|E\|_1 = \sum_{ij} |E_{ij}|.$

最优凸近似

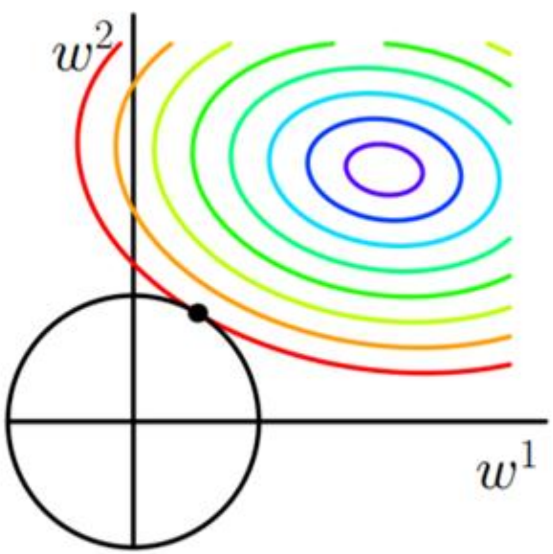
L1范数约束依然稀疏

$$\min_w \frac{1}{n} \| y - Xw \|^2, s.t \| w \|_1 \leq C$$



L2范数约束无稀疏特性

$$\min_w \frac{1}{n} \| y - Xw \|^2, s.t \| w \|_2 \leq C$$



ROBUST PCA

Convex relaxation

$$\min\ \operatorname{rank}(A) + \gamma\|E\|_0 \quad \text{subj} \quad A + E = D.$$

Convex relaxation

$$\operatorname{rank}(A) = \#\{\sigma_i(A) \neq 0\}.$$

最优 ↓↓ 凸近似

$$\|A\|_* = \textstyle\sum_i \sigma_i(A).$$

$$\|E\|_0 = \#\{E_{ij} \neq 0\}.$$

最优 ↓↓ 凸近似

$$\|E\|_1 = \textstyle\sum_{ij} |E_{ij}|.$$

$$\min \quad \|A\|_* + \gamma\|E\|_1 \quad subj \quad A + E = D$$

NP问题转化为凸函数约束问题

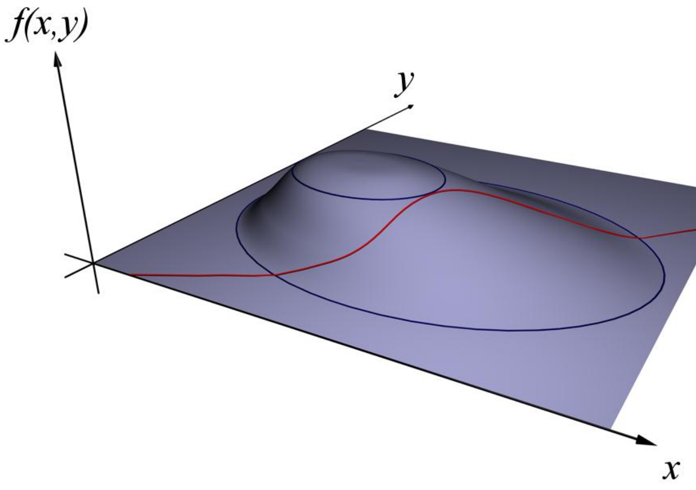
拉格朗日乘子法原理

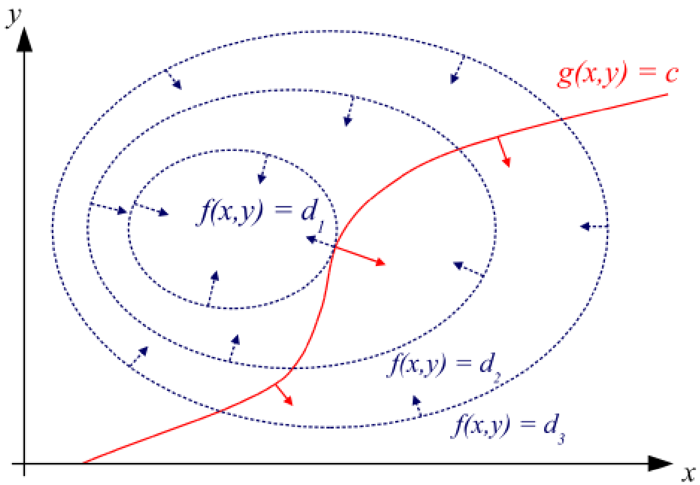
约束问题

$$\max \quad f(x,y)$$
$$s.t. \quad g(x,y)=c$$

极值点位置：

$$\nabla[f(x,y)+\lambda(g(x,y)-c)]=0$$





约束问题

$$\max \quad f(x)$$
$$s.t. \quad g_i(x)=0 \quad i=1,2\ldots,n$$

无约束的目标函数替代原约束问题

极值问题

$$\max \quad L(x,\lambda_i)=f(x)+\sum_{i=1}^{n}\lambda_i g_i(x)$$

ROBUST PCA

增广拉格朗日乘子法

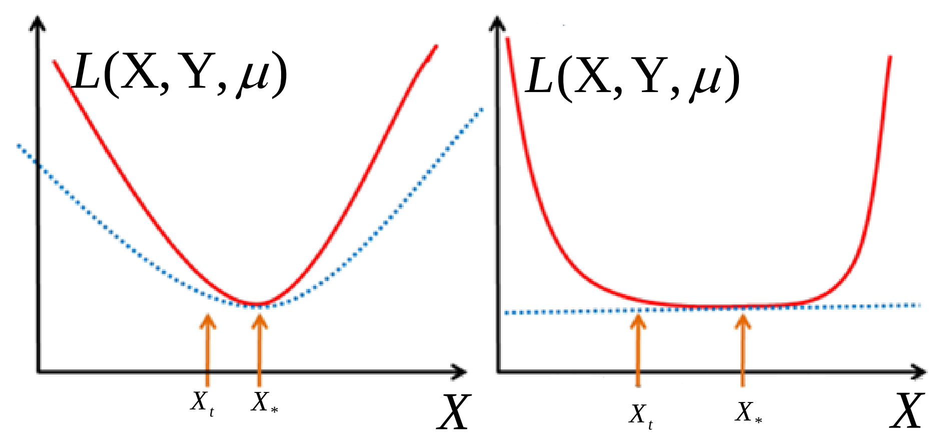
迭代解法

$\rho \geq 1.$
while not converged **do**
  Solve $X_{k+1} = \arg\min_{X} L(X, Y_k, \mu_k)$.
  $Y_{k+1} = Y_k + \mu_k h(X_{k+1})$;
  Update μ_k to μ_{k+1}.
end while
Output: X_k.

$$\min \quad f(x), \quad s.t \quad h(x) = 0$$

$$L(X, Y, \mu) = f(X) + \langle Y, h(X) \rangle + \frac{\mu}{2} \| h(X) \|_2^2$$

强凸



ROBUST PCA

RPCA via the Inexact ALM Method

$$\min \ \operatorname{rank}(A) + \gamma\|E\|_0 \quad \text{subj} \quad A + E = D.$$

$$f(x) = \|A\|_* + \lambda\|E\|_1, \textit{and} \qquad h(x) = D - A - E$$

$$L(A, E, Y, \mu) = \|A\|_* + \lambda\|E\|_1 + \langle Y, D - A - E \rangle + \frac{\mu}{2}\|D - A - E\|_2^{\,2}$$

Input: Observation matrix $D \in \mathbb{R}^{m \times n}$, λ.

1: $Y_0 = D/J(D)$; $E_0 = 0$; $\mu_0 > 0$; $\rho > 1$; $k = 0$.
2: **while** not converged **do**
3: // Lines 4-5 solve $A_{k+1} = \arg\min_A L(A, E_k, Y_k, \mu_k)$.
4: $(U, S, V) = \operatorname{svd}(D - E_k + \mu_k^{-1} Y_k)$;
5: $A_{k+1} = U\mathcal{S}_{\mu_k^{-1}}[S]V^T$.
6: // Line 7 solves $E_{k+1} = \arg\min_E L(A_{k+1}, E, Y_k, \mu_k)$.
7: $E_{k+1} = \mathcal{S}_{\lambda\mu_k^{-1}}[D - A_{k+1} + \mu_k^{-1} Y_k]$.
8: $Y_{k+1} = Y_k + \mu_k(D - A_{k+1} - E_{k+1})$.
9: Update μ_k to μ_{k+1}.
10: $k \leftarrow k + 1$.
11: **end while**

Output: (A_k, E_k).

ROBUST PCA

RPCA求解

For a 1000x1000 matrix of rank 50, with 10% (100,000) entries randomly corrupted: $\min \; \|A\|_* + \lambda\|E\|_1 \quad \text{subj} \quad A + E = D.$

Algorithms	Accuracy	Rank	||E||_0	# iterations	time (sec)
IT	5.99e-006	50	101,268	8,550	119,370.3
DUAL	8.65e-006	50	100,024	822	1,855.4
APG	5.85e-006	50	100,347	134	1,468.9
APG_P	5.91e-006	50	100,347	134	82.7
$EALM_P$	2.07e-007	50	100,014	34	37.5
$IALM_P$	3.83e-007	50	99,996	23	11.8

10,000 times speedup!

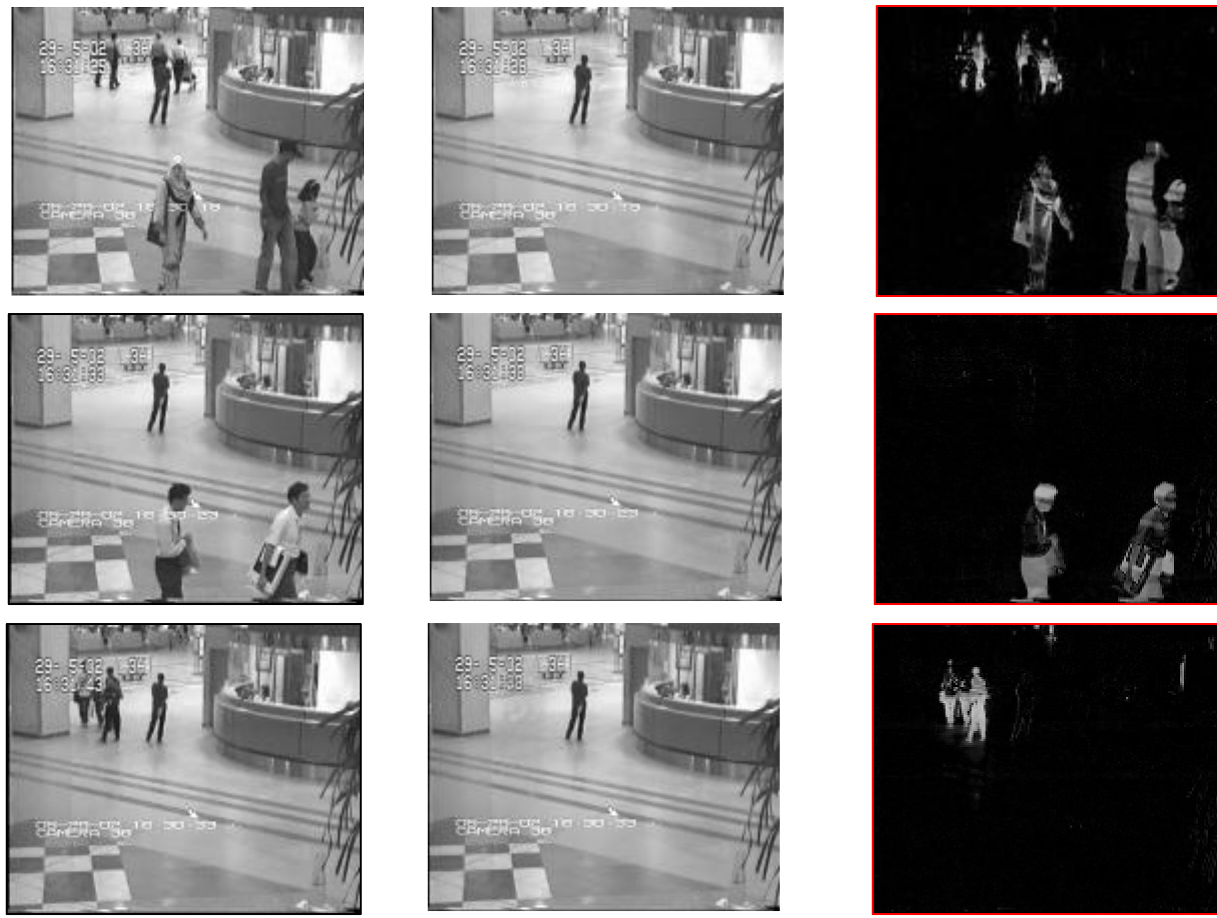
ROBUST PCA

应用

Background Modeling

$$D = A + E$$

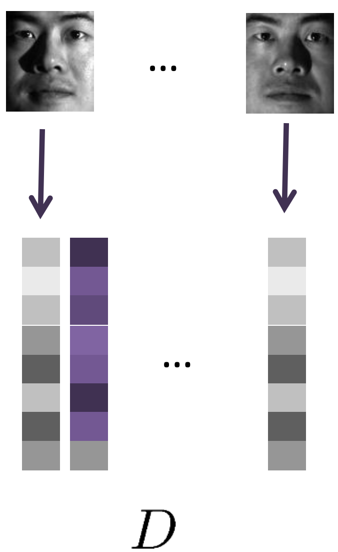
Video D | Low-rank appx. A | Sparse error E

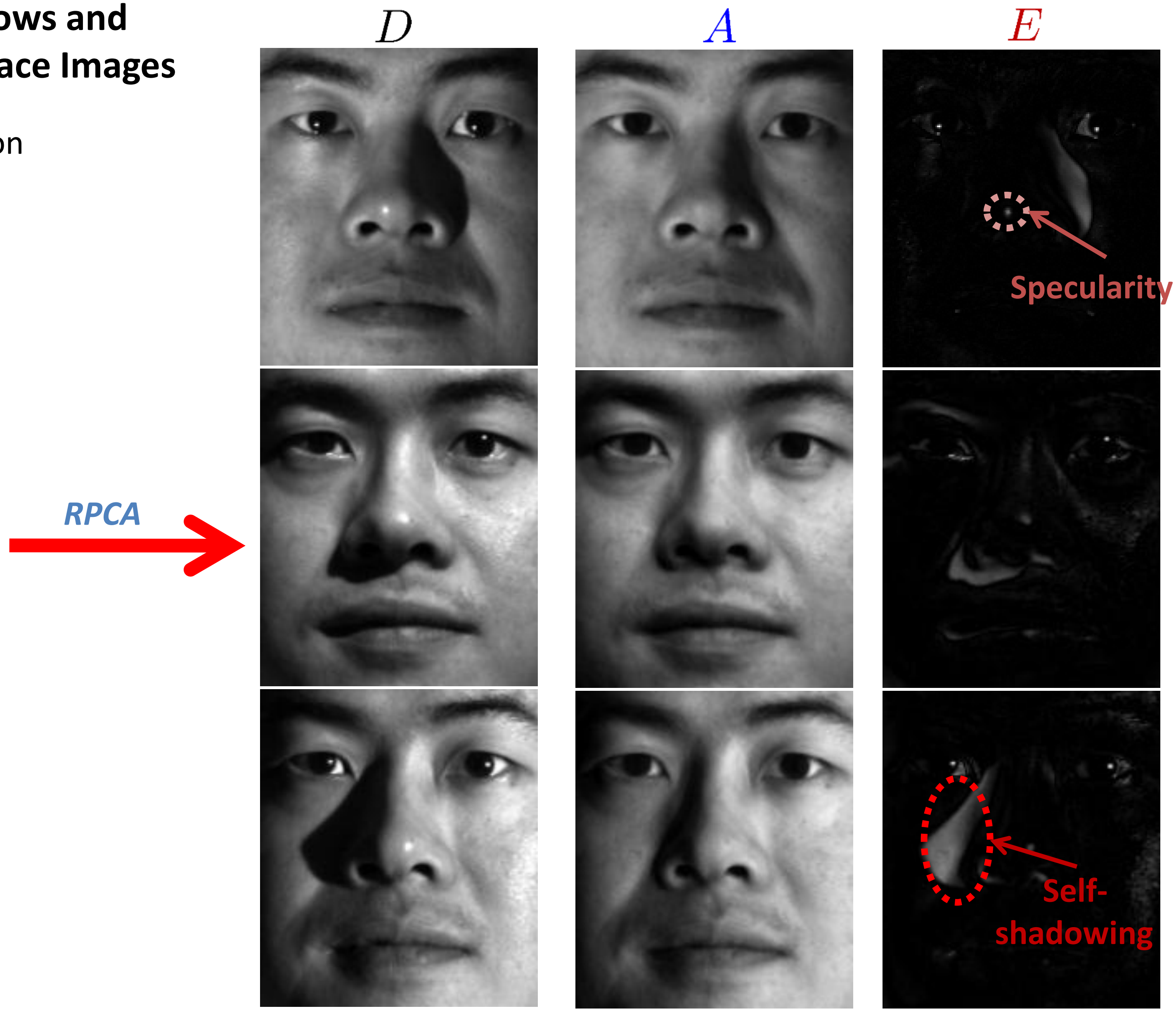


ROBUST PCA

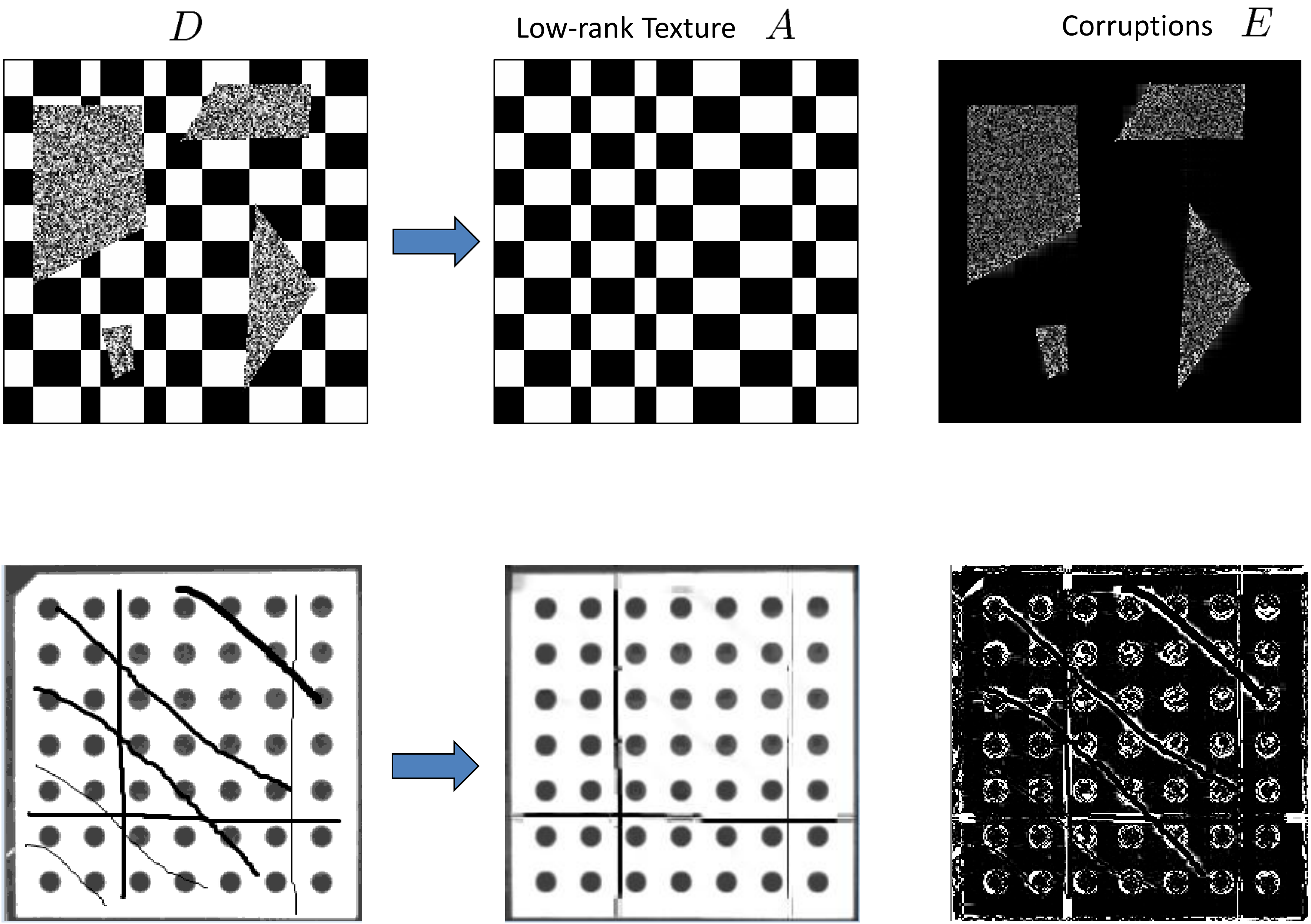
Removing Shadows and Specularities from Face Images

58 images of one person under varying lighting:



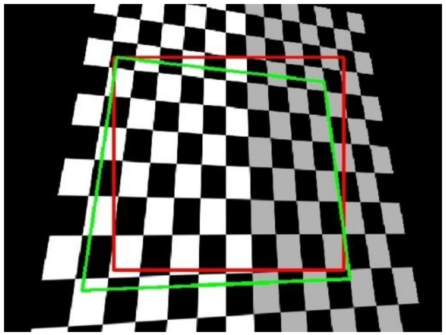


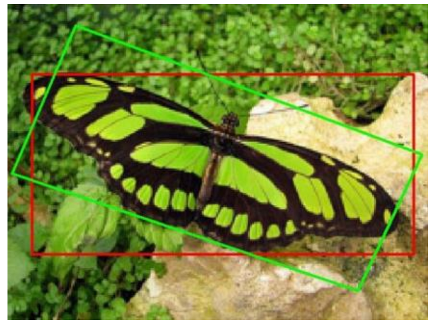
ROBUST PCA



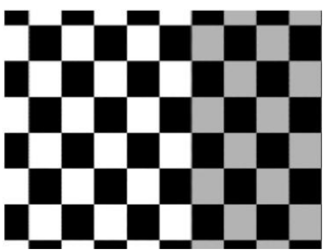
ROBUST PCA

Input (red window D)





Output (rectified green window A)



ROBUST PCA

参考：

Prof. Yi Ma	http://perception.csl.illinois.edu/matrix-rank/sample_code.html
github	https://github.com/andrewssobral/lrslibrary
Optimization	http://www.stat.purdue.edu/~vishy/introml/notes/Optimization.pdf
Lagrange multiplier	https://en.wikipedia.org/wiki/Lagrange_multiplier
RPCA	Robust Principal Component Analysis?
IALM	The Augmented Lagrange Multiplier Method for Exact Recovery of Corrupted Low-Rank Matrices