§ 8.4　偏导数与全微分

一、 偏导数的定义

设函数$z=f(x,y)$在点(x_0,y_0)的某邻域内有定义

1 $z=f(x,y)$在(x_0,y_0)处对x的偏导数:

$$f'_x(x_0,y_0)=\frac{\partial f(x_0,y_0)}{\partial x}=\frac{\partial z}{\partial x}\Big|_{\substack{x=x_0\\ y=y_0}}=z'_x\Big|_{\substack{x=x_0\\ y=y_0}}$$

$$=\lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

2 $z=f(x,y)$在(x_0,y_0)处对y的偏导数:

$$f'_y(x_0,y_0)=\frac{\partial f(x_0,y_0)}{\partial y}=\frac{\partial z}{\partial y}\Big|_{\substack{x=x_0\\ y=y_0}}=z'_y\Big|_{\substack{x=x_0\\ y=y_0}}$$

$$=\lim_{\Delta y\to 0}\frac{\Delta_y z}{\Delta y}=\lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$$

3 函数$z=f(x,y)$ 对自变量x,y的偏导函数:

$$f'_x(x,y)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x},$$

$$f'_y(x,y)=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

可推广到二元以上函数

二、偏导数的计算

求多元函数对一个自变量的偏导函数，将其他自变量看成常数, 用一元函数的求导法则求.

注1° 偏导数 $\dfrac{\partial f}{\partial x}$ 是一个总体记号, 不能拆分

注2° 偏导数存在 $\not\Rightarrow$ 连续.

连续 $\not\Rightarrow$ 偏导数存在.

三、高阶偏导数

函数的$n-1$阶偏导函数的偏导数称为n阶偏导数.

$$\begin{cases}\text{纯偏导}\\ \text{混合偏导}\end{cases}$$

四、多元函数的全微分

1. 定义 设函数$z=f(x, y)$ 在点(x, y)的某个邻域内有定义, 如果函数 z 的全改变量可以表示为

$$\Delta z=f(x+\Delta x, y+\Delta y)-f(x, y)$$
$$=A\cdot\Delta x+B\cdot\Delta y+o(\rho),\ (\Delta x\to 0, \Delta y\to 0).$$

其中A, B与$\Delta x, \Delta y$ 无关, $o(\rho)$表示一个比ρ 更高阶的无穷小量, 则称函数 $z=f(x, y)$ 点(x, y)处可微, 并称$A\Delta x+B\Delta y$为函数$z=f(x, y)$点(x, y)处的全微分, 记为

$$dz=d f(x, y)=A\cdot\Delta x+B\cdot\Delta y\ .$$

2. 可微的条件

定理1 (必要条件) 如果函数$z=f(x, y)$在点(x, y)处可微，则$z=f(x, y)$点(x, y)的偏导数$f'_x(x, y)$, $f'_y(x, y)$存在，且$z=f(x, y)$点(x, y)的全微分

$$dz = f'_x(x, y)dx + f'_y(x, y)dy .$$

定理2 (充分条件) 设函数$z=f(x, y)$在点(x, y)的某个邻域内有连续的偏导数$f'_x(x, y)$, $f'_y(x, y)$，则函数$z=f(x, y)$点(x, y)处可微，并且

$$dz = df(x, y) = f'_x(x, y)dx + f'_y(x, y)dy.$$

3. 多元函数连续、可导、可微的关系:

偏导数连续 $\longrightarrow$ 函数可微 $\longrightarrow$ 函数可偏导

函数可微 $\longrightarrow$ 函数连续

8. 求下列函数的全微分

(1) $z=\sqrt{\dfrac{x}{y}}$

解
$$\frac{\partial z}{\partial x}=\frac{1}{2\sqrt{\dfrac{x}{y}}}\cdot\frac{1}{y}=\frac{\sqrt{xy}}{2xy},$$

$$\frac{\partial z}{\partial y}=\frac{1}{2\sqrt{\dfrac{x}{y}}}\cdot\left(-\frac{x}{y^2}\right)=-\frac{\sqrt{xy}}{2y^2}.$$

所以

$$\mathrm{d}z=\frac{\sqrt{xy}}{2xy}\mathrm{d}x-\frac{\sqrt{xy}}{2y^2}\mathrm{d}y.$$

8. 求下列函数的全微分

(4) $z = \arctan(xy)$

解
$$\frac{\partial z}{\partial x} = \frac{1}{1+(xy)^2} \cdot y = \frac{y}{1+x^2y^2}$$

由对称性知
$$\frac{\partial z}{\partial y} = \frac{x}{1+x^2y^2}$$

于是有
$$\mathrm{d}z = \frac{y\,\mathrm{d}x + x\,\mathrm{d}y}{1+x^2y^2}$$

8. 求下列函数的全微分

(5) $u=\ln(x^2+y^2+z^2)$

解

$$\frac{\partial u}{\partial x}=\frac{1}{x^2+y^2+z^2}\cdot 2x=\frac{2x}{x^2+y^2+z^2}$$

$$\frac{\partial u}{\partial y}=\frac{2y}{x^2+y^2+z^2}$$

$$\frac{\partial u}{\partial z}=\frac{2z}{x^2+y^2+z^2}$$

$$\mathrm{d}u=\frac{2(x\mathrm{d}x+y\mathrm{d}y+z\,\mathrm{d}z)}{x^2+y^2+z^2}$$

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9 求下列在给定条件下的全微分的值

(2) 函数 $z=e^{xy}$, 当$x=1$, $y=1$, $\Delta x=0.15$, $\Delta y=0.1$时.

解

$$z'_x=e^{xy}\cdot y$$

$$z'_y=e^{xy}\cdot x$$

于是函数的全微分

$$\mathrm{d}z=z'_x\cdot \mathrm{d}x+z'_y\cdot \mathrm{d}y$$

$$=ye^{xy}\mathrm{d}x+xe^{xy}\mathrm{d}y\ .$$

故函数在$x=1$, $y=1$, $\Delta x=0.15$, $\Delta y=0.1$时的全微分:

$$\mathrm{d}z=1\cdot e^{1\cdot 1}\cdot 0.15+1\cdot e^{1\cdot 1}\cdot 0.1$$

$$=0.25\,e\ .$$

§8.5 复合函数的微分法
与隐函数的微分法

一、复合函数的微分法

二、全微分形式的不变性

三、隐函数的微分法

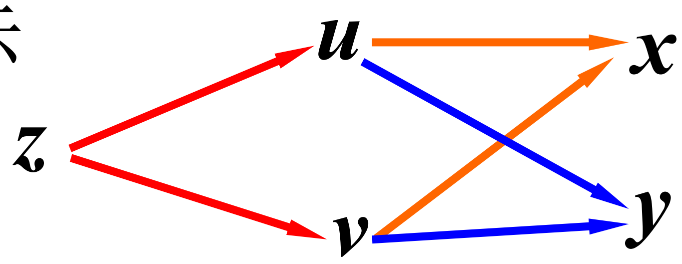
一、复合函数的微分法（链式法则）

定理1 如果函数$u=\varphi(x, y)$及$v=\psi(x, y)$都在点(x, y)具有对x, y的偏导数，且函数$z=f(u, v)$在对应点(u, v)处具有连续偏导数，则复合函数$z=f[\varphi(x, y), \psi(x, y)]$在点$(x, y)$处对$x$及$y$的偏导数都存在，且

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

链式法则如图示



连线相乘
分线相加

证 给x以改变量$\Delta x(\Delta x\neq 0)$, 让y保持不变,

则u, v各得到偏改变量$\Delta_x u$, $\Delta_x v$,

因函数$z=f(u, v)$在对应点(u, v) 处可微, 所以

$$\Delta_x z=\frac{\partial z}{\partial u}\Delta_x u+\frac{\partial z}{\partial v}\Delta_x v+o(\rho)$$

$$\lim_{\Delta x\to 0}\frac{\Delta_x u}{\Delta x}=\frac{\partial u}{\partial x}$$

其中, $\rho=\sqrt{(\Delta_x u)^2+(\Delta_x v)^2}$,

$$\lim_{\Delta x\to 0}\frac{\Delta_x v}{\Delta x}=\frac{\partial v}{\partial x}$$

于是, $\dfrac{\Delta_x z}{\Delta x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\Delta_x u}{\Delta x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\Delta_x v}{\Delta x}+\dfrac{o(\rho)}{\Delta x}$

又函数$u=\varphi(x, y)$及$v=\psi(x, y)$偏导数都存在,

所以, $\Delta x\to 0$, y保持不变时, $\rho\to 0$,

$$\lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x}=\frac{\partial z}{\partial u}\cdot\lim_{\Delta x\to 0}\frac{\Delta_x u}{\Delta x}+\frac{\partial z}{\partial v}\cdot\lim_{\Delta x\to 0}\frac{\Delta_x v}{\Delta x}+\lim_{\Delta x\to 0}\frac{o(\rho)}{\Delta x}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},$$

同理可得, $\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial y}$.

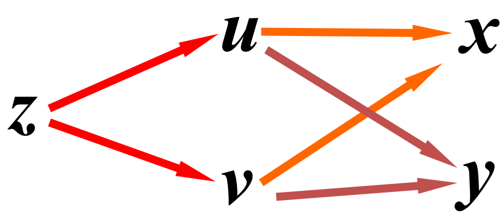
例1* 设 $z=e^{u}\sin v$, 而 $u=xy$, $v=x+y$, 求 $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

解

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=e^{u}\sin v\cdot y+e^{u}\cos v\cdot 1$$

$$=e^{xy}[\,y\sin(x+y)+\cos(x+y)],$$



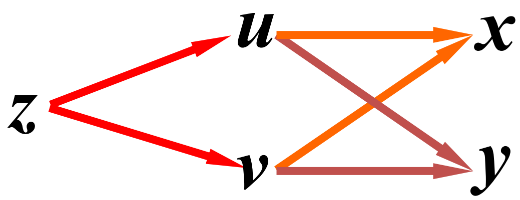
$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$=e^{u}\sin v\cdot x+e^{u}\cos v\cdot 1$$

$$=e^{xy}[x\sin(x+y)+\cos(x+y)].$$

例1 设函数 $z=(3x^2+y^2)^{4x+2y}$ 的偏导数.

解 令 $u=3x^2+y^2$, $v=4x+2y$, 则 $z=u^v$,



$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$=v\cdot u^{v-1}\cdot 6x+u^{v}\cdot\ln u\cdot 4$$

$$=6x(4x+2y)(3x^2+y^2)^{4x+2y-1}$$

$$+4(3x^2+y^2)^{4x+2y}\cdot\ln(3x^2+y^2),$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$=v\cdot u^{v-1}\cdot 2y+u^{v}\cdot\ln u\cdot 2$$

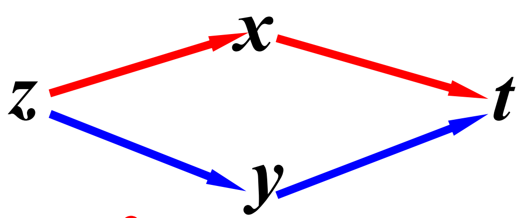
$$=2y(4x+2y)(3x^2+y^2)^{4x+2y-1}$$

$$+2(3x^2+y^2)^{4x+2y}\ln(3x^2+y^2).$$

定理1 的结论可推广到有限个自变量、有限个中间变量的情况.

例2* 设函数 $z=e^{x-2y}$, $x=\sin t$, $y=t^3$, 求 z'_t.

解 $$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial z}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}$$

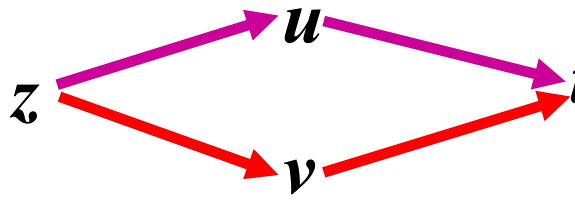


$$=e^{x-2y}\cdot 1\cdot\cos t+e^{x-2y}\cdot(-2)\cdot 3t^2$$

$$=(\cos t-6t^2)e^{\sin t-2t^3}.$$

结论1 如果函数 $u=\varphi(t)$ 及 $v=\psi(t)$ 都在点 t 可导, 函数 $z=f(u,v)$ 在对应点 (u,v) 处具有连续偏导数, 则复合函数 $z=f[\varphi(t),\psi(t)]$ 在点 t 可导, 且

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t}.$$



例3* 设函数 $z=\arctan(xy)$, $y=e^x$, 求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

解
$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\begin{tikzcd} z \arrow[rr] \arrow[dr] & & x \\ & y \arrow[ur] & \end{tikzcd}$$

$$=\frac{1}{1+(xy)^2}\cdot y+\frac{1}{1+(xy)^2}\cdot x\cdot e^x$$

$$=\frac{y+xe^x}{1+(xy)^2}.$$

结论2 如果函数 $z=f(x,y)$,而 $y=\varphi(x)$, 则 $z=f[x,\varphi(x)]$ 对 x 的导数为

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}$$

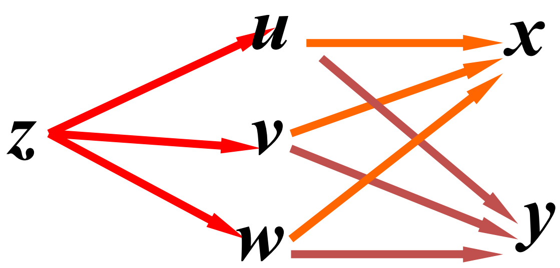
称为**全导数**

$\dfrac{\mathrm{d}z}{\mathrm{d}x}$ 中 z 是作为一个自变量 x 的函数, 对 x 求导;

而 $\dfrac{\partial z}{\partial x}$ 中 z 是作为 x, y 的二元函数, 对 x 求偏导.

例4* 设 $z=\dfrac{1}{\sqrt{u^2+v^2+w^2}}$, $u=x^2+y^2, v=x^2-y^2$,

函数 $w=2xy$, 求 z'_x.



解 $$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x}$$

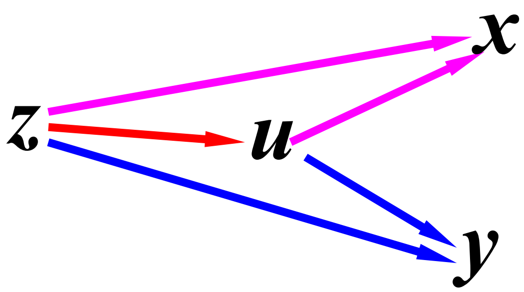
$$=-\frac{1}{2}(u^2+v^2+w^2)^{-\frac{3}{2}}\cdot 2u\cdot 2x-\frac{1}{2}(u^2+v^2+w^2)^{-\frac{3}{2}}\cdot 2v\cdot 2x$$

$$-\frac{1}{2}(u^2+v^2+w^2)^{-\frac{3}{2}}\cdot 2w\cdot 2y$$

$$=\frac{-1}{\sqrt{(u^2+v^2+w^2)^3}}\cdot 2(ux+vx+wy)=\frac{-x}{\sqrt{2}(x^2+y^2)^2}.$$

$$\frac{\partial z}{\partial y}\overset{?}{=}\frac{-1}{\sqrt{(u^2+v^2+w^2)^3}}\cdot 2(uy-vy+wx)=\frac{-y}{\sqrt{2}(x^2+y^2)^2}.$$

例2 设$z=xy+u$，其中$u=\varphi(x,y)$具有二阶连续偏导数，求z'_x，z''_{xx}，z''_{xy}.



解 $z'_x=1\cdot y+1\cdot\varphi'_x(x,y)$

$$z''_{xx}=\varphi''_{xx}(x,y),$$

$$z''_{xy}=1+\varphi''_{xy}(x,y).$$

例3 设 $z=\dfrac{y^2}{2x}+\varphi(xy)$, φ 为可微函数, 求证

$$x^2\frac{\partial z}{\partial x}-xy\frac{\partial z}{\partial y}+\frac{3}{2}y^2=0.$$

证: $\dfrac{\partial z}{\partial x}=-\dfrac{y^2}{2x^2}+\varphi'(xy)\cdot y$

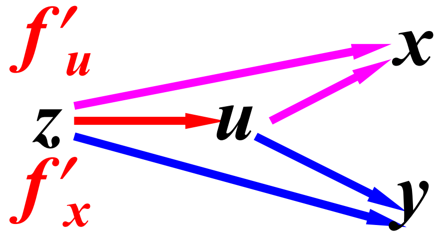
$$\frac{\partial z}{\partial y}=\frac{2y}{2x}+\varphi'(xy)\cdot x=\frac{y}{x}+x\varphi'(xy),$$

$$x^2\frac{\partial z}{\partial x}-xy\frac{\partial z}{\partial y}+\frac{3}{2}y^2=x^2\cdot[-\frac{y^2}{2x^2}+y\varphi'(xy)]$$

$$-xy\cdot[\frac{y}{x}+x\varphi'(xy)]+\frac{3}{2}y^2$$

$$=0.$$

例4 设$z=f(u, x, y)$, $u=xe^{y}$, 其中f具有二阶连续偏导数, 求 z''_{xy}.



解 $z'_x=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial x}=f'_u\cdot e^{y}+f'_x$,

$$z''_{xy}=(e^{y}f'_u+f'_x)'_y=e^{y}\cdot f'_u+e^{y}\cdot(f'_u)'_y+(f'_x)'_y$$

$$=e^{y}f'_u+e^{y}\cdot(f''_{uu}\cdot u'_y+f''_{uy})+f''_{xu}\cdot u'_y+f''_{xy}$$

$$=e^{y}f'_u+xe^{2y}f''_{uu}+e^{y}f''_{uy}+xe^{y}f''_{xu}+f''_{xy}$$

对抽象复合函数求高阶偏导数时, 需注意: 导函数仍是复合函数. 故对导函数再求偏导数时,仍需用复合函数求导的方法.

$z''_{xx}=(e^{y}f'_u+f'_x)'_x$?

$$=e^{y}\cdot(f''_{uu}\cdot u'_x+f''_{ux})+f''_{xu}\cdot u'_x+f''_{xx}$$

$$=e^{2y}f''_{uu}+2e^{y}f''_{ux}+f''_{xx}$$

对复合函数求高阶偏导数时, 常用简记法:

$$f'_1 = f'_u,\quad f'_2 = f'_v,\quad f''_{12} = f''_{uv},\quad \cdots$$

下标1表示对第一个变量求偏导数, 下标2表示对第二个变量求偏导数.

例5* 设函数 $z = f(xy^2, x^2y)$ 具有二阶连续偏导数,
求 z'_x 和 z''_{xy}.

解 $z'_x = f'_1 \cdot y^2 + f'_2 \cdot 2xy$

$$= y^2 f'_1 + 2xyf'_2 .$$

$$z''_{xy} = (y^2 f'_1 + 2xy f'_2)'_y$$

$$= 2y \cdot f'_1 + y^2 \cdot (f''_{11} \cdot 2xy + f''_{12} \cdot x^2) + 2x \cdot f'_2$$

$$+ 2xy \cdot (f''_{21} \cdot 2xy + f''_{22} \cdot x^2)$$

$$= 2yf'_1 + 2xf'_2 + 2xy^3 f''_{11} + 5x^2y^2 f''_{12} + 2x^3y f''_{22} .$$

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13. 求下列函数的导数或偏导数

(1) 设 $z=u^2\ln v$, 而 $u=\dfrac{x}{y}$ ，$v=3x-2y$, 求 $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

(2) $z=\dfrac{x^2-y}{x+y}$， 而 $y=2x-3$, 求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

14. 计算下列函数的偏导数

(2) $z=f(xy, x^2+y^2)$, 其中f具有二阶连续偏导数, 求 z''_{xx}, z''_{xy}.

15. 证明下列各题

(2) $z=f[e^{xy}, \cos(xy)]$, 其中f是可微函数, 求证:

$$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=0.$$

例6* 设函数 $g(r)$ 有二阶导数, 且满足

$$f(x,y)=g(r),\qquad r=\sqrt{x^2+y^2}$$

求证: $f''_{xx}+f''_{yy}=g''(r)+\dfrac{1}{r}g'(r)$. $(x,y)\neq(0,0)$.

证 $f'_x=g'(r)\cdot r'_x=\dfrac{x}{\sqrt{x^2+y^2}}\cdot g'(r)$,

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$$f''_{xx}=\frac{1\cdot\sqrt{x^2+y^2}-x\cdot\dfrac{x}{\sqrt{x^2+y^2}}}{x^2+y^2}\cdot g'(r)+\frac{x}{\sqrt{x^2+y^2}}\cdot g''(r)\cdot r'_x$$

$$=\frac{y^2}{(x^2+y^2)^{3/2}}\cdot g'(r)+\frac{x^2}{x^2+y^2}\cdot g''(r),$$

由对称性知 $f''_{yy}=\dfrac{x^2}{(x^2+y^2)^{3/2}}\cdot g'(r)+\dfrac{y^2}{x^2+y^2}\cdot g''(r)$,

因此

$$f''_{xx}+f''_{yy}=\frac{1}{\sqrt{x^2+y^2}}\cdot g'(r)+g''(r)=g''(r)+\frac{1}{r}\cdot g'(r).$$

$(x,y)\neq(0,0)$.