作业 P_{294} 习题八 (A)

1. 求下列函数的定义域.

(2) $z=\sqrt{1-x^2}+\sqrt{y^2-1}$

(4) $z=\ln(-x-y)$

(6) $u=\sqrt{R^2-x^2-y^2-z^2}+\sqrt{x^2+y^2+z^2-r^2}$ $\quad (R>r)$

1* 求极限 $\lim\limits_{(x,y)\to(0,1)}\dfrac{\sin xy}{x}$

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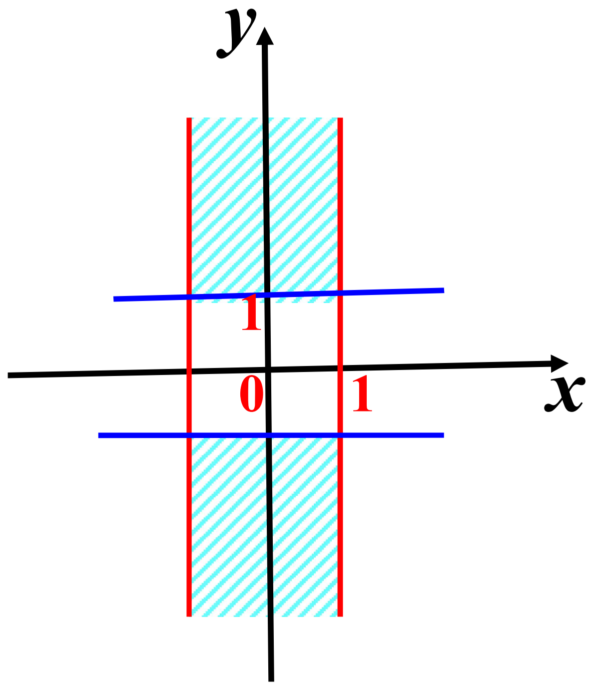


解　由题意知

$$1-x^2\geq 0,$$

且　$$y^2-1\geq 0,$$

所求定义域为

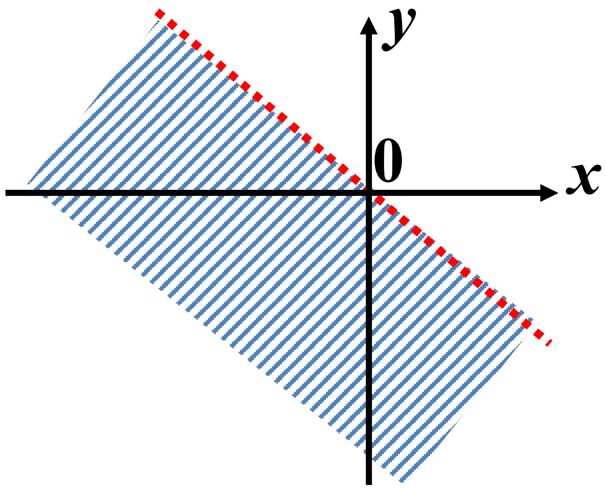
$$D=\{(x,y)\mid |x|\leq 1,\ |y|\geq 1\}.$$

无界闭区域

1. 求下列函数的定义域.

(4) $z = \ln(-x-y)$

解 定义域 $D = \{(x, y) \mid x+y < 0\}$



无界开区域

1. 求下列函数的定义域.

(6) $u=\sqrt{R^2-x^2-y^2-z^2}+\sqrt{x^2+y^2+z^2-r^2}\quad (R>r)$.

解 由题意知

$$R^2-x^2-y^2-z^2\geq 0,$$

且 $$x^2+y^2+z^2-r^2\geq 0,$$

所求定义域为

$$D=\{(x,y,z)\mid r^2\leq x^2+y^2+z^2\leq R^2\}.$$

有界闭区域

1* 求极限 $\lim\limits_{(x,y)\to(0,1)} \dfrac{\sin xy}{x}$

解 $\lim\limits_{(x,y)\to(0,1)} \dfrac{\sin xy}{x} = \lim\limits_{(x,y)\to(0,1)} \dfrac{\sin xy}{xy} \cdot y$

$= \lim\limits_{(x,y)\to(0,1)} \dfrac{\sin xy}{xy} \cdot \lim\limits_{y\to 1} y$

$=1$.

§8.4　偏导数与全微分

一、偏增量、全增量的概念

二、偏导数的定义及其计算法

三、高阶偏导数

四、全微分的定义

五、可微的充要条件

六、多元函数连续、可偏导、可微的关系

一、偏增量、全增量

设函数$z=f(x, y)$在点(x_0, y_0)的某邻域内有定义

1. 当x从x_0取得改变量$\Delta x(\Delta x \neq 0)$,而$y=y_0$保持不变时,函数$z$的改变量

$$\Delta_x z=f(x_0+\Delta x, y_0)-f(x_0, y_0)$$

称为函数$z=f(x, y)$对于x的偏改变量或偏增量.

2. 当y从y_0取得改变量$\Delta y(\Delta y \neq 0)$,而$x=x_0$保持不变时,函数$z$的改变量

$$\Delta_y z=f(x_0, y_0+\Delta y)-f(x_0, y_0)$$

称为函数$z=f(x, y)$对于y的偏改变量或偏增量.

3. 当x从x_0取得改变量$\Delta x(\Delta x\neq 0)$, y从y_0取得改变量$\Delta y(\Delta y \neq 0)$时, 函数$z$的改变量

$$\Delta z=f(x_0+\Delta x, y_0+\Delta y)-f(x_0, y_0)$$

称为函数$z=f(x, y)$全改变量或全增量.

二、偏导数的定义及其计算法

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1.定义8.5 设函数$z=f(x,y)$在点(x_0,y_0)的某邻域内有定义

(1) 如果当$\Delta x\to 0$, 极限

$$\lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

存在, 则称此极限值为$z=f(x,y)$在(x_0,y_0)处对x的偏导数.

记为 $f'_x(x_0,y_0),\ \dfrac{\partial f(x_0,y_0)}{\partial x}$ 或 $\dfrac{\partial z}{\partial x}\Big|_{\substack{x=x_0\\y=y_0}},\ z'_x\Big|_{\substack{x=x_0\\y=y_0}}$

即 $$f'_x(x_0,y_0)=\frac{\partial f(x_0,y_0)}{\partial x}=\frac{\partial z}{\partial x}\Big|_{\substack{x=x_0\\y=y_0}}=z'_x\Big|_{\substack{x=x_0\\y=y_0}}$$

$$=\lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

一元函数$z=f(x,y_0)$在点x_0处的导数

(2) 如果当$\Delta y\to 0$, 极限

$$\lim_{\Delta y\to 0}\frac{\Delta_y z}{\Delta y}=\lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$$

存在, 则称此极限值为$z=f(x,y)$在(x_0,y_0)处对y的偏导数.

记为 $f'_y(x_0,y_0), \dfrac{\partial f(x_0,y_0)}{\partial y}$ 或 $\dfrac{\partial z}{\partial y}\Big|_{\substack{x=x_0\\ y=y_0}}, z'_y\Big|_{\substack{x=x_0\\ y=y_0}}$

即 $$f'_y(x_0,y_0)=\frac{\partial f(x_0,y_0)}{\partial y}=\frac{\partial z}{\partial y}\Big|_{\substack{x=x_0\\ y=y_0}}=z'_y\Big|_{\substack{x=x_0\\ y=y_0}}$$

$$=\lim_{\Delta y\to 0}\frac{\Delta_y z}{\Delta y}=\lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$$

一元函数$z=f(x_0,y)$在点y_0处的导数

(3) 如果函数$z=f(x, y)$在区域D在内任意一点(x, y) 处对 x 的偏导数存在，那么这个偏导数就是x、y的函数，它称为$z=f(x, y)$对自变量x的偏导函数.

记为 $f'_x(x, y), \dfrac{\partial f(x, y)}{\partial x}$ 或 $\dfrac{\partial z}{\partial x}$、$z'_x$

同理定义函数$z=f(x, y)$对自变量y的偏导数.

记为 $f'_y(x, y), \dfrac{\partial f(x, y)}{\partial y}$ 或 $\dfrac{\partial z}{\partial y}$、$z'_y$

即 $$f'_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x},$$

$$f'_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}.$$

设函数$u=f(x, y, z)$在点(x, y, z)的某邻域内有定义

$$f'_x(x,y,z)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y,z)-f(x,y,z)}{\Delta x},$$

$$f'_y(x,y,z)=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y,z)-f(x,y,z)}{\Delta y},$$

$$f'_z(x,y,z)=\lim_{\Delta z\to 0}\frac{f(x,y,z+\Delta z)-f(x,y,z)}{\Delta z}.$$

偏导数的概念可以推广到二元以上函数

例2 求函数 $f(x,y)=e^{x^2y}$ 的偏导数 $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$.

解
$$\frac{\partial f}{\partial x}=\frac{\partial(e^{x^2y})}{\partial x}=e^{x^2y}\cdot(x^2y)'_x$$
$$=2xye^{x^2y}.$$

$$\frac{\partial f}{\partial y}=\frac{\partial(e^{x^2y})}{\partial y}=e^{x^2y}\cdot(x^2y)'_y$$
$$=x^2e^{x^2y}.$$

二、偏导数的计算

求多元函数对一个自变量的偏导函数, 将其他自变量看成常数, 用一元函数的求导法则求.

例1 求函数$f(x,y)=5x^2y^3$的偏导数$f'_x(x,y)$, $f'_y(x,y)$，并求$f'_x(0,1)$, $f'_x(1,-1)$, $f'_y(1,-2)$, $f'_y(-1,-2)$.

解　因

$$f'_x(x,y)=(5x^2y^3)'_x=5y^3\cdot 2x=10xy^3,$$

$$f'_y(x,y)=(5x^2y^3)'_y=5x^2\cdot 3y^2=15x^2y^2,$$

所以

$$f'_x(0,1)=0,$$

$$f'_x(1,-1)=10\times 1\times(-1)^3=-10,$$

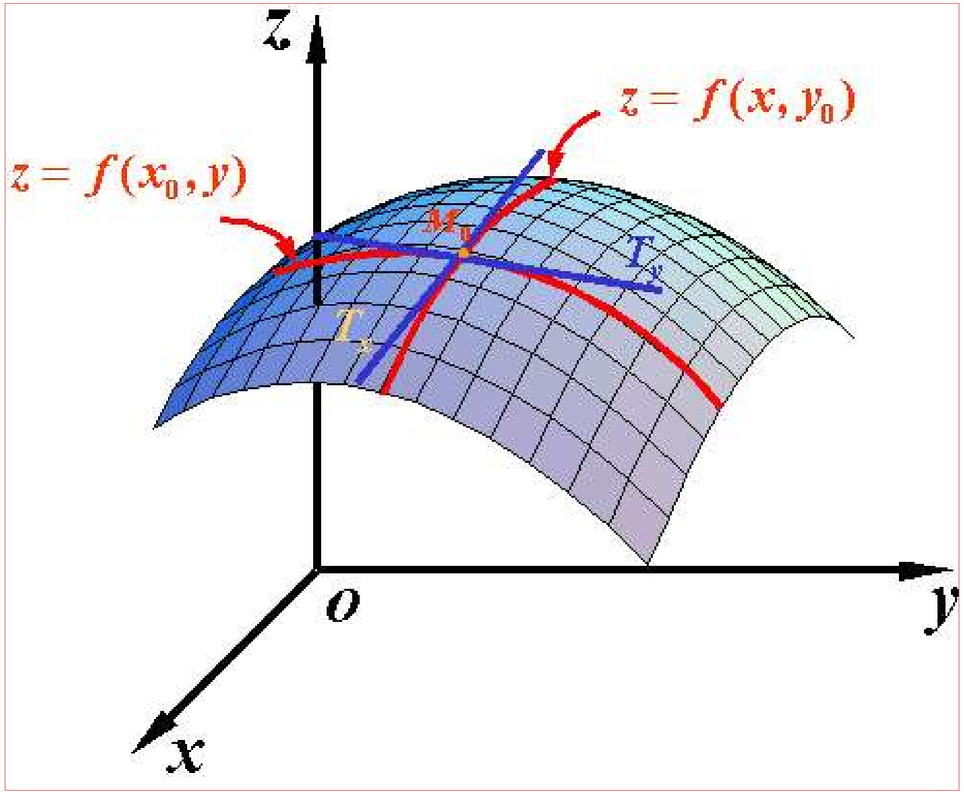
$$f'_y(1,-2)=15\times 1^2\times(-2)^2=60,$$

$$f'_y(-1,-2)=15\times(-1)^2\times(-2)^2=60.$$

3 偏导数的几何意义

设$M_0(x_0, y_0, f(x_0, y_0))$为曲面$z=f(x, y)$上一点，

如图



偏导数$f'_x(x_0, y_0)$就是曲面被平面 $y=y_0$ 所截得的曲线在点M_0处的切线M_0T_x对x轴的斜率

偏导数$f'_y(x_0, y_0)$就是曲面被平面$x=x_0$所截得的曲线在点M_0处的切线M_0T_y对y轴的斜率

例1* 求函数 $u=\sqrt{x^2+y^2+z^2}$ 的偏导数.

解
$$\frac{\partial u}{\partial x}=\frac{\partial(\sqrt{x^2+y^2+z^2})}{\partial x}=\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x$$
$$=\frac{x}{\sqrt{x^2+y^2+z^2}}$$
$$\frac{\partial u}{\partial y}=\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2y=\frac{y}{\sqrt{x^2+y^2+z^2}}$$
$$\frac{\partial u}{\partial z}=\frac{z}{\sqrt{x^2+y^2+z^2}}.$$

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在某函数(或方程)表达式中, 将任意两个自变量互换后, 仍是原来的函数 (或方程), 称函数(或方程)关于自变量对称, 用对称性可简化计算.

例2* 求函数 $z=x^2+3xy+y^2$ 在点(1, 2)处的偏导数.

解 因 $\dfrac{\partial z}{\partial x}=2x+3y$, $\dfrac{\partial z}{\partial y}=3x+2y$,

所以 $\left.\dfrac{\partial z}{\partial x}\right|_{\substack{x=1\\y=2}}=2\times1+3\times2=8,$

$\left.\dfrac{\partial z}{\partial y}\right|_{\substack{x=1\\y=2}}=3\times1+2\times2=7.$

解2

$\left.\dfrac{\partial z}{\partial x}\right|_{\substack{x=1\\y=2}}\overset{?}{=}(x^2+6x+4)'|_{x=1}=(2x+6)|_{x=1}=8$

$\left.\dfrac{\partial z}{\partial y}\right|_{\substack{x=1\\y=2}}\overset{?}{=}(1+3y+y^2)'|_{y=2}=(3+2y)|_{y=2}=7$

例3* 设函数 $z=(1+\frac{x}{y})^{\frac{x}{y}}$，则 $\frac{\partial z}{\partial x}\big|_{(1,1)}=$ <u>2ln2+1</u> .

解 $\frac{\partial z}{\partial x}\big|_{(1,1)}=[(1+x)^x]'\big|_{x=1}$

$$=[e^{x\ln(1+x)}]'\big|_{x=1}$$

$$=e^{x\ln(1+x)}[\ln(1+x)+\frac{x}{1+x}]\big|_{x=1}$$

$$=2\ln 2+1 .$$

于是，应填 2ln2+1 .

2011数3

三、高阶偏导数

定义 函数的一阶偏导函数的偏导数称为二阶偏导数.

如：$z=f(x,y)$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=z''_{xx}=f''_{xx}(x,y),$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=z''_{yy}=f''_{yy}(x,y)$$

纯偏导

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\partial y}=z''_{xy}=f''_{xy}(x,y),$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y\partial x}=z''_{yx}=f''_{yx}(x,y)$$

混合偏导

类似可定义n阶偏导数,

二阶及二阶以上的偏导数统称为高阶偏导数.

例3 设函数 $z=x^3+y^3-3xy^2$，求

$$\frac{\partial^2 z}{\partial x^2}、\frac{\partial^2 z}{\partial x\partial y}、\frac{\partial^2 z}{\partial y\partial x}、\frac{\partial^2 z}{\partial y^2}、\frac{\partial^3 z}{\partial x^3}.$$

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解

$$\frac{\partial z}{\partial x}=(x^3+y^3-3xy^2)'_x=3x^2-3y^2,$$

$$\frac{\partial z}{\partial y}=(x^3+y^3-3xy^2)'_y=3y^2-6xy,$$

$$\frac{\partial^2 z}{\partial x^2}=(3x^2-3y^2)'_x=6x,\qquad \frac{\partial^3 z}{\partial x^3}=(6x)'_x=6.$$

$$\frac{\partial^2 z}{\partial y^2}=(3y^2-6xy)'_y=6y-6x,$$

$$\frac{\partial^2 z}{\partial x\partial y}=(3x^2-3y^2)'_y=-6y,$$

$$\frac{\partial^2 z}{\partial y\partial x}=(3y^2-6xy)'_x=-6y,$$

结论：高阶混合偏导数函数在连续的条件下与求导次序无关

例4 求函数 $z=x^2ye^y$ 的各二阶偏导数.

解 $\dfrac{\partial z}{\partial x}=(x^2ye^y)'_x=2xye^y$,

$\dfrac{\partial z}{\partial y}=(x^2ye^y)'_y=x^2(1\cdot e^y+y\cdot e^y)=x^2(1+y)e^y$,

$\dfrac{\partial^2 z}{\partial x^2}=(2xye^y)'_x=2ye^y$,

$\dfrac{\partial^2 z}{\partial y^2}=[x^2(1+y)e^y]'_y=x^2[1\cdot e^y+(1+y)\cdot e^y]=x^2(2+y)e^y$,

$\dfrac{\partial^2 z}{\partial y\partial x}=[x^2(1+y)e^y]'_x=2x(1+y)e^y$,

$\dfrac{\partial^2 z}{\partial x\partial y}=2x(1+y)e^y$.

例4* 已知函数 $u=\sqrt{x^2+y^2+z^2}$，证明

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{2}{u}.$$

解 $$\frac{\partial u}{\partial x}=\frac{\partial(\sqrt{x^2+y^2+z^2})}{\partial x}=\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x^2$$

$$=\boxed{\frac{x}{\sqrt{x^2+y^2+z^2}}}$$

$$\frac{\partial^2 u}{\partial x^2}=\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)'_x=\frac{1\cdot\sqrt{x^2+y^2+z^2}-x\cdot\cdot\frac{x}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}$$

$$=\frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}},$$

由对称性知

$$\frac{\partial^2 u}{\partial y^2}=\frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}},\qquad \frac{\partial^2 u}{\partial z^2}=\frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

所以

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{3}{2}}}=\frac{2}{\sqrt{x^2+y^2+z^2}}$$

$$=\frac{2}{u}.$$

有关偏导数的几点说明:

1. 偏导数 $\dfrac{\partial f}{\partial x}$ 是一个总体记号, 不能拆分;

2. 偏导数存在与连续的关系

1° 偏导数存在 $\not\rightarrow$ 连续.

例5* 设函数 $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$

求 $f'_x(0,0),\ f'_y(0,0)$.

解

$$f'_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0,$$

同理 $f'_y(0,0)=0.$

但函数在该点处并不连续.

例6* 设 $z=\sqrt{x^2+y^2}$，求 $z'_x(0,0)$，$z'_y(0,0)$.

解 $$\lim_{\Delta x\to 0}\frac{\Delta z_x}{\Delta x}=\lim_{\Delta x\to 0}\frac{z(\Delta x,0)-z(0,0)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sqrt{(\Delta x)^2+0^2}-0}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{|\Delta x|}{\Delta x}\quad \text{不存在,}$$

于是 $z'_x(0,0)$不存在，

同理 $z'_y(0,0)$ 不存在.

但函数 $z=\sqrt{x^2+y^2}$ 在$(0,0)$处连续.

2° 连续 $\not\rightarrow$ 偏导数存在.

四、小结

1. 偏导数的定义 设函数$z=f(x,y)$在点(x_0,y_0)的某邻域内有定义

1° $z=f(x,y)$在(x_0,y_0)处对x的偏导数:

$$f'_x(x_0,y_0)=\frac{\partial f(x_0,y_0)}{\partial x}=\frac{\partial z}{\partial x}\Big|_{\substack{x=x_0\\y=y_0}}=z'_x\Big|_{\substack{x=x_0\\y=y_0}}$$

$$=\lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

2° $z=f(x,y)$在(x_0,y_0)处对y的偏导数:

$$f'_y(x_0,y_0)=\frac{\partial f(x_0,y_0)}{\partial y}=\frac{\partial z}{\partial y}\Big|_{\substack{x=x_0\\y=y_0}}=z'_y\Big|_{\substack{x=x_0\\y=y_0}}$$

$$=\lim_{\Delta y\to 0}\frac{\Delta_y z}{\Delta y}=\lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$$

3° 函数$z=f(x, y)$ 对自变量x, y的偏导函数:

$$f_x'(x,y)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x},$$

$$f_y'(x,y)=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

可推广到二元以上函数

2. 偏导数的计算

求多元函数对一个自变量的偏导函数, 将其他自变量看成常数, 用一元函数的求导法则求.

3. 高阶偏导数

函数的$n-1$阶偏导函数的偏导数称为n阶偏导数.

$$\begin{cases}\text{纯偏导}\\\text{混合偏导}\end{cases}$$

注1° 偏导数 $\dfrac{\partial f}{\partial x}$ 是一个总体记号, 不能拆分

注2° 偏导数存在 $\not\Rightarrow$ 连续.

连续 $\not\Rightarrow$ 偏导数存在.

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4. 求下列函数的偏导数

(1) $z = e^{xy} + yx^2$ (10) $z = \arctan\dfrac{x+y}{x-y}$

5. 计算下列函数在给定点的偏导数

(1) $z = e^{x^2+y^2}$，求 $z'_x\big|_{\substack{x=1\\y=0}}, z'_y\big|_{\substack{x=0\\y=1}}$.

(3) $z = (1+xy)^y$，求 $z'_x\big|_{\substack{x=1\\y=1}}, z'_y\big|_{\substack{x=1\\y=1}}$.

6. 求下列函数的偏导数

(4) $z = e^{xyz}$，求 $z = \dfrac{\partial^3 u}{\partial x \partial y \partial z}$

7. 求下列函数的偏导数

(1) 设 $z = \ln(\sqrt[n]{x} + \sqrt[n]{y})$，且 $n \geq 2$，求证 $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \dfrac{1}{n}$.

练习题

1. 设 $z=\arctan\dfrac{y}{x}$, 求 $\dfrac{\partial z}{\partial x}\Big|_{\substack{x=1\\y=1}},\dfrac{\partial z}{\partial y}\Big|_{\substack{x=1\\y=0}}$.

2. 求函数 $u=x\ln(x+y)$ 的二阶偏导数.

3. 设函数 $z=x^y\,(x>0, x\neq 1)$, 求证

$$\frac{x}{y}\frac{\partial z}{\partial x}+\frac{1}{\ln x}\frac{\partial z}{\partial y}=2z.$$

练习题1. 设 $z=\arctan\dfrac{y}{x}$，求 $\dfrac{\partial z}{\partial x}\Big|_{\substack{x=1\\y=1}}, \dfrac{\partial z}{\partial y}\Big|_{\substack{x=1\\y=0}}$.

解

$$\frac{\partial z}{\partial x}=\frac{1}{1+(\frac{y}{x})^2}\cdot\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2},$$

$$\frac{\partial z}{\partial y}=\frac{1}{1+(\frac{y}{x})^2}\cdot\frac{1}{x}=\frac{x}{x^2+y^2},$$

因此 $$\frac{\partial z}{\partial x}\Big|_{\substack{x=1\\y=1}}=-\frac{y}{x^2+y^2}\Big|_{\substack{x=1\\y=1}}=-\frac{1}{2}.$$

$$\frac{\partial z}{\partial y}\Big|_{\substack{x=1\\y=0}}=\frac{x}{x^2+y^2}\Big|_{\substack{x=1\\y=0}}=1.$$

练习题1. 设 $z=\arctan\dfrac{y}{x}$，求 $\left.\dfrac{\partial z}{\partial x}\right|_{\substack{x=1\\y=1}},\left.\dfrac{\partial z}{\partial y}\right|_{\substack{x=1\\y=0}}$.

解2 $$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=1\\y=1}}=\left.\left(\arctan\frac{1}{x}\right)'\right|_{x=1}=\left.\frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\left(-\frac{1}{x^2}\right)\right|_{x=1}$$

$$=-\frac{1}{2}.$$

$$\left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\y=0}}=(\arctan y)'\big|_{y=0}=\left.\frac{1}{1+y^2}\right|_{y=0}=1.$$

练习题2 求函数 $u = x\ln(x+y)$ 的二阶偏导数.

解

$$\frac{\partial u}{\partial x} = 1\cdot\ln(x+y) + x\cdot\frac{1}{x+y}\cdot 1 = \ln(x+y) + \frac{x}{x+y},$$

$$\frac{\partial u}{\partial y} = x\cdot\frac{1}{x+y}\cdot 1 = \frac{x}{x+y},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{x+y} + \frac{1\cdot(x+y) - x\cdot 1}{(x+y)^2} = \frac{x+2y}{(x+y)^2};$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{x}{(x+y)^2};$$

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{1}{x+y} - \frac{x}{(x+y)^2} = \frac{y}{(x+y)^2};$$

$$\frac{\partial^2 u}{\partial y\partial x} = \frac{y}{(x+y)^2}.$$

练习题3 设函数 $z=x^y\ (x>0, x\neq 1)$，求证

$$\frac{x}{y}\frac{\partial z}{\partial x}+\frac{1}{\ln x}\frac{\partial z}{\partial y}=2z.$$

证 $$\frac{\partial z}{\partial x}=(x^y)'_x=yx^{y-1},$$

$$\frac{\partial z}{\partial y}=(x^y)'_y=x^y\cdot\ln x,$$

于是有

$$\frac{x}{y}\frac{\partial z}{\partial x}+\frac{1}{\ln x}\frac{\partial z}{\partial y}=\frac{x}{y}\cdot yx^{y-1}+\frac{1}{\ln x}\cdot x^y\ln x$$

$$=x^y+x^y=2z.$$

原结论成立.