Stochastic

coupling

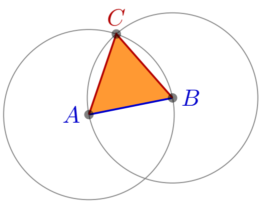
Remark: probability space A probability space is a triple $(E, \mathcal{E}, \mathbb{P})$, that $(E, \mathcal{E})$ is a *measurable space* consisting of:

1. E: a sample space and is a set
2. $\mathcal{E}$: a σ-algebra and is a subsets of E
3. $\mathbb{P}$: a probability measure on $\mathcal{E}$

Typically, E os a Polish space (i.e., complete, separable, and metric) and $\mathcal{E}$ consists of its Borel sets.

Consider two probability measures $\mathbb{P}$ and $\mathbb{P}'$ on the same measueable space $(E, \mathcal{E})$. A product measurable space is defined as: $(E \times E, \mathcal{E} \otimes \mathcal{E})$. $\mathcal{E} \otimes \mathcal{E}$ is the smallest σ-algebra containing $\mathcal{E} \times \mathcal{E}$. If $E = \mathbb{R}$, then $\mathcal{E} = \mathcal{B}(\mathbb{R})$ the Borel σ-algebra.

coupling [fix prob-space]



Given two probability measures $\mathbb{P}$ and $\mathbb{Q}$ on corresponding measurable spaces $(P, \mathcal{P})$ and $(Q, \mathcal{Q})$, the *product measurable space* is defined as $(P \times Q, \mathcal{P} \otimes \mathcal{Q})$,where their the Cartesian product set is:

$$P \times Q := \{(p, q) : p \in P, q \in Q\}$$

where

- (p, q) is a pair of points.
- $P \times Q$ is the set of points, i.e., the *plane* of all coordinates (p, q).
- $\mathcal{P} \otimes \mathcal{Q}$ is the smallest σ-algebra on $P \times Q$, so that all ”regions” in the plane are measurable.

The σ-algebra $\mathcal{P} \otimes \mathcal{Q}$ contains all measurable rectangles $A \times B$ for $A \in \mathcal{P}$ and $B \in \mathcal{Q}$. That is, if a set A and a set B are measurable, then $A \times B$ is measurable in the product space[1].

If $P = \mathbb{R}$ and $Q = \mathbb{R}$, then

$$P \times Q = \mathbb{R}^2, \quad \mathcal{P} \otimes \mathcal{Q} = \mathcal{B}(\mathbb{R}^2),$$

where $\mathcal{B}(\mathbb{R}^2)$ is the Borel σ-algebra on the plane.