

On the derivatives of the integer-valued polynomials

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Abstract

In this paper, we study the derivatives of an integer-valued polynomial of a given degree. Denoting by E_n the set of the integer-valued polynomials with degree $\leq n$, we show that the smallest positive integer c_n satisfying the property: $\forall P \in E_n, c_n P' \in E_n$ is $c_n = \text{lcm}(1, 2, \dots, n)$. As an application, we deduce an easy proof of the well-known inequality $\text{lcm}(1, 2, \dots, n) \geq 2^{n-1}$ ($\forall n \geq 1$). In the second part of the paper, we generalize our result for the derivative of a given order k and then we give two divisibility properties for the obtained numbers $c_{n,k}$ (generalizing the c_n 's). Leaning on this study, we conclude the paper by determining, for a given natural number n , the smallest positive integer λ_n satisfying the property: $\forall P \in E_n, \forall k \in \mathbb{N}: \lambda_n P^{(k)} \in E_n$. In particular, we show that: $\lambda_n = \prod_{p \text{ prime}} p^{\lfloor \frac{n}{p} \rfloor}$ ($\forall n \in \mathbb{N}$).

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1 Introduction and Notation

Throughout this paper, we let \mathbb{N}^* denote the set of positive integers. We let $\lfloor \cdot \rfloor$ denote the integer-part function. For a given prime number p , we let v_p denote the usual p -adic valuation. For a given positive integer n and given positive integers a_1, \dots, a_n , we denote the least common multiple of a_1, \dots, a_n by $\text{lcm}(a_1, \dots, a_n)$ or by one of the two equivalent notations $a_1 \vee \dots \vee a_n$ and $\bigvee_{i=1}^n a_i$, which are sometimes more convenient. For given positive integers n and k , with $1 \leq k \leq n$, and given real numbers x_1, \dots, x_n , we let $x_1 \cdots \widehat{x_k} \cdots x_n$ denote the product $\prod_{\substack{1 \leq i \leq n \\ i \neq k}} x_i$.

We say that a rational number u is a multiple of a non-zero rational number v if the ratio u/v

is an integer. For a given rational number r , we let $\text{den}(r)$ denote the denominator of r ; that is the smallest positive integer d such that $dr \in \mathbb{Z}$. For given $n, k \in \mathbb{N}$, with $n \geq k$, we define

$$F_{n,k} := \sum_{\substack{i_1, \dots, i_k \in \mathbb{N}^* \\ i_1 + \dots + i_k = n}} \frac{1}{i_1 i_2 \dots i_k} \quad \text{and} \quad d_{n,k} := \text{den}(F_{n,k}),$$

with the conventions that $F_{0,0} = 1$ and $F_{n,0} = 0$ for any $n \in \mathbb{N}^*$.

For given $n, k \in \mathbb{N}$, with $n \geq k$, we also define

$$q_{n,k} := \text{lcm}\{i_1 i_2 \dots i_k \mid i_1, \dots, i_k \in \mathbb{N}^*, i_1 + \dots + i_k \leq n\},$$

with the convention that $q_{n,0} = 1$ for any $n \in \mathbb{N}$. Besides, for $n \in \mathbb{N}$, we define

$$\begin{aligned} q_n &:= \text{lcm}\{q_{n,k}; 0 \leq k \leq n\} \\ &= \text{lcm}\{i_1 i_2 \dots i_k \mid k \in \mathbb{N}^*, i_1, \dots, i_k \in \mathbb{N}^*, i_1 + \dots + i_k \leq n\}. \end{aligned}$$

Next, we let $s(n, k)$ ($n, k \in \mathbb{N}$, $n \geq k$) denote the Stirling numbers of the first kind, which are the integer coefficients appearing in the polynomial identity:

$$x(x-1) \dots (x-n+1) = \sum_{k=0}^n s(n, k) x^k$$

(see e.g., [2, Chapter V] or [3, Chapter 6]).

Further, we let I , D and Δ the linear operators on $\mathbb{C}[X]$ which respectively represent the identity, the derivation and the forward difference ($\Delta P(X) = P(X+1) - P(X)$, $\forall P \in \mathbb{C}[X]$). The expression of D in terms of Δ , obtained by using symbolic methods (see e.g., [4, Chapter 1, §6]), is given by:

$$D = \ln(I + \Delta) = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \dots \quad (1.1)$$

Note that this formula will be of crucial importance throughout this paper.

An integer-valued polynomial is a polynomial $P \in \mathbb{C}[X]$ such that $P(\mathbb{Z}) \subset \mathbb{Z}$; that is the value taken by P at every integer is an integer. It is immediate that every polynomial with integer coefficients is an integer-valued polynomial but the converse is false (for example, the polynomial $\frac{X(X+1)}{2}$ is a counterexample to the converse statement). However, an integer-valued polynomial always has rational coefficients (i.e., lies in $\mathbb{Q}[X]$). This can be easily proved by using for example the Lagrange interpolation formula. So the set E of the integer-valued polynomials is a subring of $\mathbb{Q}[X]$. More interestingly, the set E can be also seen as a \mathbb{Z} -module. From this point of view, it is shown (see e.g., [1] or [5]) that E is free with infinite rank and has as a basis the sequence of polynomials:

$$B_n(X) := \frac{X(X-1) \dots (X-n+1)}{n!} = \binom{X}{n}$$

$(n \in \mathbb{N})$, with the convention that $B_0(X) = 1$.

From the definition of the polynomials B_n ($n \in \mathbb{N}$), the following identities are immediate:

$$\Delta^i B_j = \begin{cases} B_{j-i} & \text{if } i \leq j \\ 0 & \text{else} \end{cases} \quad \text{and} \quad B_j(0) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{else} \end{cases} \quad (\forall i, j \in \mathbb{N}).$$

Combining these, we derive that for all $i, j \in \mathbb{N}$, we have

$$(\Delta^i B_j)(0) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases} = \delta_{ij} \quad (1.2)$$

(where δ_{ij} denotes the Kronecker delta). The last formula will be useful later in §3.

For a given $n \in \mathbb{N}$, let E_n denote the set of the integer-valued polynomials with degree $\leq n$. Then, it is clear that E_n is a free submodule of E and has as a basis the polynomials B_0, B_1, \dots, B_n (so E_n is of rank $(n+1)$). Obviously, E is stable by the forward difference operator Δ (i.e., $\forall P \in E : \Delta P \in E$). The stability by Δ also holds for each E_n ($n \in \mathbb{N}$). But remarkably, E is not stable by the operator of derivation D (for example $B_2'(X) = X - \frac{1}{2} \notin E$). This last remark constitutes the starting point of our study. To recover in E (actually in each E_n) something that is close to the stability by derivation, we argue like this: for a given $n \in \mathbb{N}$ and a given $P \in E_n$, we can write P as: $P = a_0 B_0 + a_1 B_1 + \dots + a_n B_n$ ($a_0, \dots, a_n \in \mathbb{Z}$), so we have that $n!P \in \mathbb{Z}[X]$, which implies that $(n!P)' = n!P' \in \mathbb{Z}[X]$. Thus $n!P' \in E_n$. Consequently, the positive integer $n!$ satisfies the following important property:

$$\forall P \in E_n : n!P' \in E_n.$$

This leads us to propose the following problem:

Problem 1: *For a given natural number n , determine the smallest positive integer c_n satisfying the property:*

$$\forall P \in E_n : c_n P' \in E_n.$$

In §2, we show that actually c_n is far enough from $n!$; precisely, we show that: $c_n = \text{lcm}(1, 2, \dots, n)$. Then, we use this result to derive an easy proof of the nontrivial inequality $\text{lcm}(1, 2, \dots, n) \geq 2^{n-1}$ ($\forall n \geq 1$). In §3, we first solve the more general problem:

Problem 2: *For given $n, k \in \mathbb{N}$, determine the smallest positive integer $c_{n,k}$ satisfying the property:*

$$\forall P \in E_n : c_{n,k} P^{(k)} \in E_n.$$

From the definitions of the c_n 's and the $c_{n,k}$'s, it is obvious that $c_{n,1} = c_n$ ($\forall n \in \mathbb{N}$) and that $c_{n,k} = 1$ if $n, k \in \mathbb{N}$ satisfy the condition $k > n$. The last property allows us to restrict our

study of the numbers $c_{n,k}$ to the couples $(n, k) \in \mathbb{N}^2$ such that $n \geq k$. A fundamental result of §3 shows that for every $n, k \in \mathbb{N}$, with $n \geq k$, we have

$$c_{n,k} = \text{lcm}\{d_{m,k}; k \leq m \leq n\}.$$

From this, we deduce that $c_{n,k}$ divides $q_{n,k}$ for any $n, k \in \mathbb{N}$, with $n \geq k$. In the opposite direction, we show that $c_{n,k}$ is a multiple of the rational number $\frac{q_{n,k}}{k!}$ ($\forall n, k \in \mathbb{N}, n \geq k$). Then, as a second part of §3, we solve the following problem:

Problem 3: For a given $n \in \mathbb{N}$, determine the smallest positive integer λ_n satisfying the property:

$$\forall P \in E_n, \forall k \in \mathbb{N}: \lambda_n P^{(k)} \in E_n.$$

As a fundamental result, we show that:

$$\lambda_n = q_n = \prod_{p \text{ prime}} p^{\lfloor \frac{n}{p} \rfloor}$$

(for any $n \in \mathbb{N}$).

In §4, we give some other interesting formulas for the crucial numbers $F_{n,k}$ ($n, k \in \mathbb{N}, n \geq k$); in particular, we express the $F_{n,k}$'s in terms of the Stirling numbers of the first kind. We finally conclude the paper by presenting (in tables) the first values of the numbers $c_{n,k}$, $q_{n,k}$ and λ_n .

2 Results concerning the first derivative of an integer-valued polynomial

In this section, we are going to solve the first problem posed in the introduction. To do so, we need some preparations. For a given $n \in \mathbb{N}$, let:

$$\mathcal{J}_n := \{a \in \mathbb{Z}: \forall P \in E_n, aP' \in E_n\}.$$

Then, it is easy to check that \mathcal{J}_n is an ideal of \mathbb{Z} ; besides, \mathcal{J}_n is non-zero because $n! \in \mathcal{J}_n$ (as explained in the introduction). Since \mathbb{Z} is a principal ring, one deduces that \mathcal{J}_n has the form $\mathcal{J}_n = \alpha_n \mathbb{Z}$ ($\alpha_n \in \mathbb{N}^*$), and α_n is simply the smallest positive integer satisfying the property: $\forall P \in E_n, \alpha_n P' \in E_n$. So α_n is nothing else the constant c_n required in Problem 1. Consequently, we have

$$\mathcal{J}_n = c_n \mathbb{Z}. \tag{2.1}$$

The following theorem solves Problem 1.

Theorem 2.1. For every positive integer n , we have

$$c_n = \text{lcm}(1, 2, \dots, n).$$

Proof. Let $n \in \mathbb{N}^*$ be fixed. For simplicity, we pose $\ell_n := \text{lcm}(1, 2, \dots, n)$. To show that $c_n = \ell_n$, we will show that ℓ_n is a multiple of c_n and then that c_n is a multiple of ℓ_n .

• Let us show that ℓ_n is a multiple of c_n ; that is $\ell_n \in \mathcal{J}_n$ (in view of (2.1)). So, according to the definition of \mathcal{J}_n , this is equivalent to show the property:

$$\forall P \in E_n : \ell_n P' \in E_n. \quad (2.2)$$

Let us show (2.2). So, let $P \in E_n$ and show that $\ell_n P' \in E_n$. By applying the identity of linear operators (1.1) to P , we get

$$P' = \Delta P - \frac{\Delta^2 P}{2} + \frac{\Delta^3 P}{3} - \dots = \sum_{k=1}^n \frac{(-1)^{k-1}}{k} \Delta^k P$$

(because $\Delta^k P = 0$ for $k > n$). Hence

$$\ell_n P' = \sum_{k=1}^n (-1)^{k-1} \frac{\ell_n}{k} (\Delta^k P).$$

Because $\Delta^k P \in E_n$ for any $k \in \mathbb{N}$ (since E_n is stable by Δ) and $\frac{\ell_n}{k} \in \mathbb{Z}$ for any $k \in \{1, 2, \dots, n\}$ (by definition of ℓ_n), the last identity shows that $\ell_n P' \in E_n$, as required.

• Now, let us show that c_n is a multiple of ℓ_n . By definition of ℓ_n , this is equivalent to show that c_n is a multiple of each of the positive integers $1, 2, \dots, n$. So, let $k \in \{1, 2, \dots, n\}$ be fixed and show that c_n is a multiple of k . Since $k \leq n$, we have $B_k \in E_n$; thus (by definition of c_n): $c_n B'_k \in E_n$. This implies (in particular) that $c_n B'_k(0) \in \mathbb{Z}$. But since

$$\begin{aligned} B'_k(0) &= \lim_{x \rightarrow 0} \frac{B_k(x)}{x} = \lim_{x \rightarrow 0} \frac{(x-1)(x-2) \cdots (x-k+1)}{k!} = \frac{(-1)(-2) \cdots (-k+1)}{k!} \\ &= (-1)^{k-1} \frac{(k-1)!}{k!} = \frac{(-1)^{k-1}}{k}, \end{aligned} \quad (2.3)$$

it follows that $(-1)^{k-1} \frac{c_n}{k} \in \mathbb{Z}$, implying that c_n is a multiple of k , as required.

This completes the proof of the theorem. \square

As an application of Theorem 2.1, we derive a well-known nontrivial lower bound for $\text{lcm}(1, 2, \dots, n)$ ($n \in \mathbb{N}^*$). We have the following:

Corollary 2.2. *For every positive integer n , we have*

$$\text{lcm}(1, 2, \dots, n) \geq 2^{n-1}.$$

To deduce this corollary from Theorem 2.1, we just need the special identity of the following lemma.

Lemma 2.3. *For every positive integer n , we have*

$$\frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{|B'_n(k)|} = 2^{n-1}.$$

Proof. Let $n \in \mathbb{N}^*$ be fixed. From the definition $B_n(X) := \frac{X(X-1)\cdots(X-n+1)}{n!}$, we derive that:

$$B'_n(X) = \sum_{\ell=0}^{n-1} \frac{X \cdots (\widehat{X-\ell}) \cdots (X-n+1)}{n!}.$$

It follows that for any $k \in \{0, 1, \dots, n-1\}$, we have

$$\begin{aligned} B'_n(k) &= \left[\frac{X \cdots (\widehat{X-k}) \cdots (X-n+1)}{n!} \right]_{X=k} \\ &= \frac{k(k-1) \cdots 1 \times (-1)(-2) \cdots (k-n+1)}{n!} \\ &= (-1)^{n-k-1} \frac{k!(n-k-1)!}{n!}, \end{aligned}$$

which gives

$$\frac{1}{|B'_n(k)|} = \frac{n!}{k!(n-k-1)!} = n \binom{n-1}{k}.$$

Thus:

$$\frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{|B'_n(k)|} = \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}$$

(according to the binomial formula). The lemma is proved. \square

Proof of Corollary 2.2. Let $n \in \mathbb{N}^*$ be fixed. Since $B_n \in E_n$, we have $c_n B'_n \in E_n$; that is $c_n B'_n(k) \in \mathbb{Z}$ for any $k \in \mathbb{Z}$. In particular, we have $c_n B'_n(k) \in \mathbb{Z}$ for any $k \in \{0, 1, \dots, n-1\}$. But since $B'_n(k) \neq 0$ for $k \in \{0, 1, \dots, n-1\}$ (see the proof of the preceding lemma), we have precisely $c_n B'_n(k) \in \mathbb{Z}^*$ ($\forall k \in \{0, 1, \dots, n-1\}$), implying that $|c_n B'_n(k)| \geq 1$ ($\forall k \in \{0, 1, \dots, n-1\}$). Using this fact, we get

$$\frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{|c_n B'_n(k)|} \leq \frac{1}{n} \sum_{k=0}^{n-1} 1 = 1.$$

But according to Lemma 2.3, we have

$$\frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{|c_n B'_n(k)|} = \frac{2^{n-1}}{c_n}.$$

Thus $\frac{2^{n-1}}{c_n} \leq 1$, which gives $c_n \geq 2^{n-1}$; that is (according to Theorem 2.1) $\text{lcm}(1, 2, \dots, n) \geq 2^{n-1}$, as required. The corollary is proved. \square

3 Results concerning the higher order derivatives of an integer-valued polynomial

In this section, we are going to solve the second problem posed in the introduction. To do so, we just adapt and generalize the method used in §2. For given $n, k \in \mathbb{N}$, let

$$\mathcal{J}_{n,k} := \{a \in \mathbb{Z} : \forall P \in E_n, aP^{(k)} \in E_n\}.$$

It is easy to check that $\mathcal{I}_{n,k}$ is an ideal of \mathbb{Z} . Besides, for any $P \in E_n$, we have $n!P \in \mathbb{Z}[X]$ (as explained in the introduction), which implies that $(n!P)^{(k)} = n!P^{(k)} \in \mathbb{Z}[X]$ and so $n!P^{(k)} \in E_n$. Hence $n! \in \mathcal{I}_{n,k}$, showing that the ideal $\mathcal{I}_{n,k}$ is non-zero. Since \mathbb{Z} is a principal ring, one deduces that $\mathcal{I}_{n,k}$ has the form $\mathcal{I}_{n,k} = \alpha_{n,k}\mathbb{Z}$ ($\alpha_{n,k} \in \mathbb{N}^*$) and $\alpha_{n,k}$ is simply the smallest positive integer satisfying the property: $\forall P \in E_n, \alpha_{n,k}P^{(k)} \in E_n$. So $\alpha_{n,k}$ is nothing else the constant $c_{n,k}$ required in Problem 2. Thus, we have

$$\mathcal{I}_{n,k} = c_{n,k}\mathbb{Z}. \quad (3.1)$$

The following theorem solves Problem 2.

Theorem 3.1. *For every natural numbers n and k , we have*

$$c_{n,k} = \text{lcm} \{d_{m,k} ; k \leq m \leq n\}.$$

In particular, $c_{n,k}$ divides the positive integer $q_{n,k}$.

Proof. Let $n, k \in \mathbb{N}$ be fixed. For simplicity, we pose $\ell_{n,k} := \text{lcm}\{d_{m,k}; k \leq m \leq n\}$. To show that $c_{n,k} = \ell_{n,k}$, we will show that $\ell_{n,k}$ is a multiple of $c_{n,k}$ and then that $c_{n,k}$ is a multiple of $\ell_{n,k}$.

• Let us show that $\ell_{n,k}$ is a multiple of $c_{n,k}$; that is $\ell_{n,k} \in \mathcal{I}_{n,k}$ (in view of (3.1)). So, according to the definition of $\mathcal{I}_{n,k}$, this is equivalent to show the property:

$$\forall P \in E_n : \ell_{n,k}P^{(k)} \in E_n. \quad (3.2)$$

Let us show (3.2). So, let $P \in E_n$ and show that $\ell_{n,k}P^{(k)} \in E_n$. From (1.1), we derive the following identity of linear operators on $\mathbb{C}[X]$:

$$\begin{aligned} D^k &= \left(\sum_{i \in \mathbb{N}^*} \frac{(-1)^{i-1}}{i} \Delta^i \right)^k \\ &= \left(\sum_{i_1 \in \mathbb{N}^*} \frac{(-1)^{i_1-1}}{i_1} \Delta^{i_1} \right) \left(\sum_{i_2 \in \mathbb{N}^*} \frac{(-1)^{i_2-1}}{i_2} \Delta^{i_2} \right) \cdots \left(\sum_{i_k \in \mathbb{N}^*} \frac{(-1)^{i_k-1}}{i_k} \Delta^{i_k} \right) \\ &= \sum_{i_1, \dots, i_k \in \mathbb{N}^*} \frac{(-1)^{i_1+\dots+i_k-k}}{i_1 i_2 \cdots i_k} \Delta^{i_1+\dots+i_k}. \end{aligned}$$

Applying this to P , we obtain (since $\Delta^i P = 0$ for $i > n$) that:

$$\begin{aligned} P^{(k)} &= \sum_{\substack{i_1, \dots, i_k \in \mathbb{N}^* \\ i_1+\dots+i_k \leq n}} \frac{(-1)^{i_1+\dots+i_k-k}}{i_1 i_2 \cdots i_k} \Delta^{i_1+\dots+i_k} P \\ &= \sum_{k \leq m \leq n} (-1)^{m-k} \left(\sum_{\substack{i_1, \dots, i_k \in \mathbb{N}^* \\ i_1+\dots+i_k=m}} \frac{1}{i_1 i_2 \cdots i_k} \right) \Delta^m P \\ &= \sum_{k \leq m \leq n} (-1)^{m-k} F_{m,k} \Delta^m P. \end{aligned} \quad (3.3)$$

Because $\Delta^m P \in E_n$ for any $m \in \mathbb{N}$ (since E_n is stable by Δ) and $\ell_{n,k} F_{m,k} \in \mathbb{Z}$ for any $m \in \{k, k+1, \dots, n\}$ (according to the definition of $\ell_{n,k}$), the last identity shows that $\ell_{n,k} P^{(k)} \in E_n$, as required.

• Now, let us show that $c_{n,k}$ is a multiple of $\ell_{n,k}$. By definition of $\ell_{n,k}$, this is equivalent to show that $c_{n,k}$ is a multiple of each of the positive integers $d_{m,k}$ ($k \leq m \leq n$). So, let $m_0 \in \{k, \dots, n\}$ be fixed and show that $c_{n,k}$ is a multiple of $d_{m_0,k}$. Since $m_0 \leq n$, we have $B_{m_0} \in E_n$; thus (by definition of $c_{n,k}$): $c_{n,k} B_{m_0}^{(k)} \in E_n$. This implies (in particular) that $c_{n,k} B_{m_0}^{(k)}(0) \in \mathbb{Z}$. But, by applying (3.3) for $P = B_{m_0}$ and using (1.2), we get

$$\begin{aligned} B_{m_0}^{(k)}(0) &= \sum_{1 \leq m \leq n} (-1)^{m-k} F_{m,k} (\Delta^m B_{m_0})(0) \\ &= (-1)^{m_0-k} F_{m_0,k}. \end{aligned}$$

Thus $c_{n,k} \cdot (-1)^{m_0-k} F_{m_0,k} \in \mathbb{Z}$, implying that $c_{n,k}$ is a multiple of $\text{den}(F_{m_0,k}) = d_{m_0,k}$, as required. So, the first part of the theorem is proved.

Next, the second part of the theorem immediately follows from its first part and the trivial fact that $d_{m,k}$ divides $\text{lcm}\{i_1 i_2 \cdots i_k \mid i_1, \dots, i_k \in \mathbb{N}^*, i_1 + \dots + i_k = m\}$, which divides $q_{n,k}$ (for every $m, k \in \mathbb{N}$, with $k \leq m \leq n$). This achieves the proof of the theorem. \square

Concerning the divisibility relations between the $c_{n,k}$'s and the $q_{n,k}$'s, we also have the following result:

Theorem 3.2. *For every natural numbers n and k such that $n \geq k$, the positive integer $c_{n,k}$ is a multiple of the rational number $\frac{q_{n,k}}{k!}$.*

Proof. Let $n, k \in \mathbb{N}$ be fixed such that $n \geq k$. Show that the positive integer $c_{n,k}$ is a multiple of the rational number $\frac{q_{n,k}}{k!}$ is equivalent to show that the positive integer $k! c_{n,k}$ is a multiple of the positive integer $q_{n,k}$, which is equivalent (according to the definition of $q_{n,k}$) to show that $k! c_{n,k}$ is a multiple of each of the positive integers having the form $i_1 i_2 \cdots i_k$, where $i_1, \dots, i_k \in \mathbb{N}^*$ and $i_1 + \dots + i_k \leq n$. So, let $i_1, \dots, i_k \in \mathbb{N}^*$ such that $i_1 + \dots + i_k \leq n$ and show that $k! c_{n,k}$ is a multiple of the product $i_1 i_2 \cdots i_k$. To do so, let us consider the integer-valued polynomial

$$P(X) := \binom{X}{i_1} \binom{X}{i_2} \cdots \binom{X}{i_k} = B_{i_1}(X) B_{i_2}(X) \cdots B_{i_k}(X)$$

whose degree is $i_1 + \dots + i_k \leq n$, showing that $P \in E_n$.

Since the expansion of each polynomial B_i ($i \in \mathbb{N}^*$) in the canonical basis $(1, X, X^2, \dots)$ of $\mathbb{Q}[X]$ begins with

$$\frac{(-1)^{i-1}}{i} X + \dots$$

(because $B_i(0) = 0$ and $B'_i(0) = \frac{(-1)^{i-1}}{i}$, according to (2.3)) then the expansion of the polynomial P in the canonical basis of $\mathbb{Q}[X]$ begins with

$$\frac{(-1)^{i_1-1}}{i_1} \cdot \frac{(-1)^{i_2-1}}{i_2} \cdots \frac{(-1)^{i_k-1}}{i_k} X^k + \dots = \pm \frac{1}{i_1 i_2 \cdots i_k} X^k + \dots$$

It follows from this fact that we have

$$P^{(k)}(0) = \pm \frac{k!}{i_1 i_2 \cdots i_k}.$$

On the other hand, since $c_{n,k} P^{(k)} \in E_n$, we have $c_{n,k} P^{(k)}(0) \in \mathbb{Z}$; that is

$$\pm c_{n,k} \frac{k!}{i_1 i_2 \cdots i_k} \in \mathbb{Z},$$

showing that $k!c_{n,k}$ is a multiple of $i_1 i_2 \cdots i_k$, as required. This completes the proof of the theorem. \square

Remark 3.3. Theorem 2.1 can immediately follow from Theorems 3.1 and 3.2. Indeed, by applying the second part of Theorem 3.1 and Theorem 3.2 for $k = 1$, we obtain that for any $n \in \mathbb{N}$, the positive integer $c_{n,1} = c_n$ is both a divisor and a multiple of the positive integer $q_{n,1} = \text{lcm}(1, 2, \dots, n)$. So, we have $c_n = \text{lcm}(1, 2, \dots, n)$ ($\forall n \in \mathbb{N}$), which is nothing else the result of Theorem 2.1.

Now, we are going to solve Problem 3 and prove the result announced in the introduction. We have the following:

Theorem 3.4. For every natural number n , we have

$$\lambda_n = q_n = \prod_{p \text{ prime}} p^{\lfloor \frac{n}{p} \rfloor}.$$

The proof of Theorem 3.4 needs the two following lemmas.

Lemma 3.5. For every positive integer a and every prime number p , we have

$$v_p(a) \leq \frac{a}{p}.$$

Proof. Let a be a positive integer and p be a prime number. Setting $\alpha := v_p(a)$, we can write $a = p^\alpha b$, where b is a positive integer which is not a multiple of p . For $\alpha = 0$, the inequality of the lemma is trivial. Next, for $\alpha \geq 1$, we have

$$v_p(a) = \alpha \leq 2^{\alpha-1} \leq p^{\alpha-1} \leq p^{\alpha-1} b = \frac{a}{p},$$

as required. This completes the proof of the lemma. \square

Lemma 3.6 (The key lemma). For any positive integer k and any prime number p , we have

$$v_p(F_{kp,k}) = -k.$$

Proof. Let k be a positive integer and p be a prime number. We have by definition:

$$F_{kp,k} := \sum_{\substack{i_1, \dots, i_k \in \mathbb{N}^* \\ i_1 + \dots + i_k = kp}} \frac{1}{i_1 i_2 \cdots i_k}.$$

For any given k -uplet $(i_1, \dots, i_k) \in \mathbb{N}^{*k}$ such that $i_1 + \dots + i_k = kp$, we have

$$\begin{aligned} v_p \left(\frac{1}{i_1 i_2 \dots i_k} \right) &= - \sum_{r=1}^k v_p(i_r) \\ &\geq - \sum_{r=1}^k p^{v_p(i_r)-1} \\ &\geq - \sum_{r=1}^k \frac{i_r}{p} \\ &= - \frac{kp}{p} = -k. \end{aligned}$$

Besides, the last series of inequalities shows that the equality $v_p(\frac{1}{i_1 i_2 \dots i_k}) = -k$ holds if and only if we have for any $r \in \{1, \dots, k\}$:

$$v_p(i_r) = p^{v_p(i_r)-1} \quad \text{and} \quad p^{v_p(i_r)} = i_r.$$

This condition is clearly satisfied if $(i_1, \dots, i_k) = (p, \dots, p)$. Conversely, if the condition in question is satisfied then each i_r ($r = 1, \dots, k$) is a power of p and not equal to 1. This implies in particular that $i_r \geq p$ ($\forall r \in \{1, \dots, k\}$). But since $i_1 + \dots + i_k = kp$, we necessarily have $i_1 = i_2 = \dots = i_k = p$. Consequently, the equality $v_p(\frac{1}{i_1 i_2 \dots i_k}) = -k$ holds if and only if $(i_1, \dots, i_k) = (p, \dots, p)$. It follows (according to the elementary properties of the usual p -adic valuation) that:

$$v_p(F_{kp,k}) = \min_{\substack{i_1, \dots, i_k \in \mathbb{N}^* \\ i_1 + \dots + i_k = kp}} v_p \left(\frac{1}{i_1 i_2 \dots i_k} \right) = -k,$$

as required. The lemma is proved. \square

Proof of Theorem 3.4. Let n be a fixed natural number. From the definition of λ_n , it is clear that λ_n is the smallest positive integer belonging to the ideal of \mathbb{Z} :

$$\begin{aligned} \bigcap_{k \in \mathbb{N}} \mathcal{I}_{n,k} &= \bigcap_{k \in \mathbb{N}} c_{n,k} \mathbb{Z} && \text{(according to (3.1))} \\ &= \text{lcm}\{c_{n,k} ; k \in \mathbb{N}\} \mathbb{Z} \\ &= \text{lcm}\{c_{n,k} ; 0 \leq k \leq n\} \mathbb{Z} && \text{(since } c_{n,k} = 1 \text{ for } k > n). \end{aligned}$$

Thus, we have

$$\lambda_n = \text{lcm}\{c_{n,k} ; 0 \leq k \leq n\}. \quad (3.4)$$

Using Theorem 3.1, we derive that:

$$\lambda_n = \text{lcm}\{d_{m,k} ; 0 \leq k \leq m \leq n\}. \quad (3.5)$$

Now, from (3.4) and the second part of Theorem 3.1, we immediately derive that λ_n divides the positive integer $\text{lcm}\{q_{n,k} ; 0 \leq k \leq n\} = q_n$. So, to complete the proof of Theorem 3.4,

it remains to prove that λ_n is a multiple of q_n and that $q_n = \prod_{p \text{ prime}} p^{\lfloor \frac{n}{p} \rfloor}$. This is clearly equivalent to prove that for any prime number p , we have

$$v_p(q_n) \leq \left\lfloor \frac{n}{p} \right\rfloor, \quad (I)$$

$$v_p(q_n) \geq \left\lfloor \frac{n}{p} \right\rfloor, \quad (II)$$

$$v_p(\lambda_n) \geq \left\lfloor \frac{n}{p} \right\rfloor. \quad (III)$$

Let p be a prime number and let us begin with proving (I). Since

$$q_n = \text{lcm}\{i_1 i_2 \cdots i_k \mid k \in \mathbb{N}^*, i_1, \dots, i_k \in \mathbb{N}^*, i_1 + \cdots + i_k \leq n\},$$

then we have

$$v_p(q_n) = \max_{\substack{k \in \mathbb{N}^*, i_1, \dots, i_k \in \mathbb{N}^* \\ i_1 + \cdots + i_k \leq n}} v_p(i_1 i_2 \cdots i_k). \quad (3.6)$$

Next, for any $k \in \mathbb{N}^*$ and any $i_1, \dots, i_k \in \mathbb{N}^*$ such that $i_1 + \cdots + i_k \leq n$, we have

$$\begin{aligned} v_p(i_1 i_2 \cdots i_k) &= v_p(i_1) + v_p(i_2) + \cdots + v_p(i_k) \\ &\leq \frac{i_1}{p} + \frac{i_2}{p} + \cdots + \frac{i_k}{p} \quad (\text{according to Lemma 3.5}) \\ &= \frac{i_1 + \cdots + i_k}{p} \leq \frac{n}{p}; \end{aligned}$$

that is (since $v_p(i_1 i_2 \cdots i_k) \in \mathbb{N}$):

$$v_p(i_1 i_2 \cdots i_k) \leq \left\lfloor \frac{n}{p} \right\rfloor.$$

Hence (according to (3.6)):

$$v_p(q_n) \leq \left\lfloor \frac{n}{p} \right\rfloor,$$

as required by (I).

Now, let us prove (II). For $p > n$, the inequality (II) is trivial; so suppose that $p \leq n$ and let $\ell := \lfloor \frac{n}{p} \rfloor \geq 1$ and $i_1 = i_2 = \cdots = i_\ell = p$. Since $\ell \in \mathbb{N}^*$, $i_1, \dots, i_\ell \in \mathbb{N}^*$ and $i_1 + \cdots + i_\ell = \ell p \leq n$, then q_n is a multiple of the product $i_1 i_2 \cdots i_\ell = p^\ell$. Thus

$$v_p(q_n) \geq v_p(p^\ell) = \ell = \left\lfloor \frac{n}{p} \right\rfloor,$$

as required by (II).

Let us finally prove (III). According to (3.5), we have

$$\begin{aligned} v_p(\lambda_n) &= \max_{0 \leq k \leq m \leq n} v_p(d_{m,k}) \\ &\geq v_p\left(d_{p\lfloor \frac{n}{p} \rfloor, \lfloor \frac{n}{p} \rfloor}\right) \\ &\geq -v_p\left(F_{p\lfloor \frac{n}{p} \rfloor, \lfloor \frac{n}{p} \rfloor}\right) \end{aligned}$$

(since $d_{p\lfloor \frac{n}{p} \rfloor, \lfloor \frac{n}{p} \rfloor}$ is the denominator of the rational number $F_{p\lfloor \frac{n}{p} \rfloor, \lfloor \frac{n}{p} \rfloor}$). But, from Lemma 3.6, we have that:

$$v_p \left(F_{p\lfloor \frac{n}{p} \rfloor, \lfloor \frac{n}{p} \rfloor} \right) = - \left\lfloor \frac{n}{p} \right\rfloor.$$

Hence

$$v_p(\lambda_n) \geq \left\lfloor \frac{n}{p} \right\rfloor,$$

as required by (III).

This completes the proof of Theorem 3.4. □

4 Some other formulas for the numbers $F_{n,k}$ and tables of the $c_{n,k}$'s, the $q_{n,k}$'s and the λ_n 's

The following proposition gives some other useful formulas for the numbers $F_{n,k}$ ($n, k \in \mathbb{N}$, $n \geq k$):

Proposition 4.1. *For every $n, k \in \mathbb{N}$, with $n \geq k$, we have*

$$F_{n,k} = (-1)^{n+k} \frac{k!}{n!} s(n, k) = \frac{k!}{n!} |s(n, k)|, \quad (4.1)$$

$$F_{n,k} = \left| \binom{X}{n}^{(k)}(0) \right|. \quad (4.2)$$

If in addition $k \geq 2$, then we have

$$F_{n,k} = \frac{k!}{n} \sum_{1 \leq i_1 < i_2 < \dots < i_{k-1} \leq n-1} \frac{1}{i_1 i_2 \dots i_{k-1}}. \quad (4.3)$$

Proof. Let us prove Formula (4.1). To do so, consider the following formal power series generating function in two indeterminates:

$$S(X, Y) := \sum_{\substack{n, k \in \mathbb{N} \\ n \geq k}} F_{n,k} X^n \frac{Y^k}{k!}.$$

Using the definition of the $F_{n,k}$'s, we have

$$\begin{aligned} S(X, Y) &= \sum_{\substack{n, k \in \mathbb{N} \\ n \geq k}} \left(\sum_{\substack{i_1, \dots, i_k \in \mathbb{N}^* \\ i_1 + \dots + i_k = n}} \frac{1}{i_1 i_2 \dots i_k} \right) X^n \frac{Y^k}{k!} \\ &= \sum_{k \in \mathbb{N}} \left(\sum_{i_1, \dots, i_k \in \mathbb{N}^*} \frac{X^{i_1 + \dots + i_k}}{i_1 i_2 \dots i_k} \right) \frac{Y^k}{k!} \\ &= \sum_{k \in \mathbb{N}} \left(\sum_{i=1}^{+\infty} \frac{X^i}{i} \right)^k \frac{Y^k}{k!} \\ &= \sum_{k \in \mathbb{N}} (-\log(1 - X))^k \frac{Y^k}{k!} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k \in \mathbb{N}} \frac{(-Y \log(1-X))^k}{k!} \\
&= e^{-Y \log(1-X)} \\
&= (1-X)^{-Y} \\
&= \sum_{n=0}^{+\infty} \binom{-Y}{n} (-X)^n \quad (\text{according to the generalized binomial theorem}) \\
&= \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\sum_{k=0}^n s(n, k) (-Y)^k \right) (-X)^n \\
&= \sum_{\substack{n, k \in \mathbb{N} \\ n \geq k}} \frac{(-1)^{n+k}}{n!} s(n, k) X^n Y^k.
\end{aligned}$$

By identifying the coefficients in the first and last formal power series of the above series of equalities, we derive that for all $n, k \in \mathbb{N}$, with $n \geq k$, we have

$$F_{n,k} = (-1)^{n+k} \frac{k!}{n!} s(n, k),$$

as required by (4.1). The second equality of (4.1) simply follows from the positivity of the $F_{n,k}$'s.

Now, let us prove Formula (4.2). For any given $n, k \in \mathbb{N}$, with $n \geq k$, we have

$$\binom{X}{n} = \frac{1}{n!} X(X-1) \cdots (X-n+1) = \frac{1}{n!} \sum_{i=0}^n s(n, i) X^i.$$

Thus

$$\binom{X}{n}^{(k)} = \frac{1}{n!} \sum_{k \leq i \leq n} s(n, i) i(i-1) \cdots (i-k+1) X^{i-k},$$

which gives

$$\binom{X}{n}^{(k)}(0) = \frac{k!}{n!} s(n, k)$$

and concludes (according to (4.1)) that

$$\left| \binom{X}{n}^{(k)}(0) \right| = \frac{k!}{n!} |s(n, k)| = F_{n,k},$$

as required by (4.2).

Let us finally prove (4.3). For any given $n, k \in \mathbb{N}$, with $n \geq k \geq 2$, the sum

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_{k-1} \leq n-1} \frac{1}{i_1 i_2 \cdots i_{k-1}}$$

is nothing else the coefficient of X^{n-k} in the polynomial $(X + \frac{1}{1})(X + \frac{1}{2}) \cdots (X + \frac{1}{n-1})$ of $\mathbb{Q}[X]$, which is also the coefficient of $\frac{1}{X^{n-k}} = X^{k-n}$ in the rational fraction $(\frac{1}{X} + \frac{1}{1})(\frac{1}{X} + \frac{1}{2}) \cdots (\frac{1}{X} + \frac{1}{n-1})$

of $\mathbb{Q}[\frac{1}{X}]$, when expanded in the basis $(1, \frac{1}{X}, \frac{1}{X^2}, \dots)$. But since we have

$$\begin{aligned} \left(\frac{1}{X} + \frac{1}{1}\right) \left(\frac{1}{X} + \frac{1}{2}\right) \cdots \left(\frac{1}{X} + \frac{1}{n-1}\right) &= \frac{1}{(n-1)!X^n} X(X+1) \cdots (X+n-1) \\ &= \frac{1}{(n-1)!X^n} \sum_{i=0}^n |s(n, i)| X^i \\ &= \sum_{i=0}^n \frac{|s(n, i)|}{(n-1)!} X^{i-n}, \end{aligned}$$

then the coefficient of X^{k-n} in this expression is also equal to $\frac{|s(n, k)|}{(n-1)!}$. Thus, we have

$$\frac{|s(n, k)|}{(n-1)!} = \sum_{1 \leq i_1 < i_2 < \cdots < i_{k-1} \leq n-1} \frac{1}{i_1 i_2 \cdots i_{k-1}}.$$

Then, Formula (4.3) immediately follows by using the second equality of Formula (4.1).

The proof of the proposition is complete. \square

Tables

Now, we explain how we can easily calculate the numbers $F_{n,k}$, $d_{n,k}$, $c_{n,k}$, $q_{n,k}$ and λ_n ($n, k \in \mathbb{N}$, $n \geq k$). Starting from the well-known recurrent relation for Stirling numbers of the first kind:

$$s(n+1, k) = s(n, k-1) - n s(n, k) \quad (\forall n, k \in \mathbb{N}^*, \text{ with } n \geq k)$$

and using Formula (4.1) of Proposition 4.1, we immediately show that the rational numbers $F_{n,k}$ satisfy the recurrent relation:

$$F_{n+1,k} = \frac{k}{n+1} F_{n,k-1} + \frac{n}{n+1} F_{n,k} \quad (\forall n, k \in \mathbb{N}^*, \text{ with } n \geq k). \quad (4.4)$$

This last relation allows us to easily generate, via some programming language, the numbers $F_{n,k}$. Then, to generate the positive integers $d_{n,k}$, we simply use their definition: $d_{n,k} := \text{den}(F_{n,k})$ and to generate the positive integers $c_{n,k}$, we use the Formula of Theorem 3.1: $c_{n,k} = \text{lcm}\{d_{m,k}; k \leq m \leq n\}$. Because $c_{n,k} = 0$ for $n < k$, it is practical to arrange the $c_{n,k}$'s (for $0 \leq k \leq n$) in a triangular array in which each $c_{n,k}$ is the entry in the n^{th} row and k^{th} column. The calculations (using Maple software) give the following triangle of the $c_{n,k}$'s ($n \geq k$) up to $n = 10$:

1											
1	1										
1	2	1									
1	6	1	1								
1	12	12	2	1							
1	60	12	4	1	1						
1	60	180	8	6	2	1					
1	420	180	120	6	6	1	1				
1	840	5040	240	240	6	4	2	1			
1	2520	5040	15120	240	144	4	12	1	1		
1	2520	25200	30240	15120	288	240	24	3	2	1	

Table 1: The triangle of the $c_{n,k}$'s for $0 \leq k \leq n \leq 10$

Remark 4.2. Another way to generate the $F_{n,k}$'s consists to use the following recurrent relation:

$$F_{n,k} = \frac{k}{n} \sum_{m=k-1}^{n-1} F_{m,k-1} \quad (\forall n, k \in \mathbb{N}^*, \text{ with } n \geq k),$$

which is easily derived by induction from (4.4).

Next, to generate the positive integers $q_{n,k}$ ($n, k \in \mathbb{N}, n \geq k$), we can use the recurrent relation given by the following proposition:

Proposition 4.3. For every $n, k \in \mathbb{N}^*$, with $n \geq k$, we have

$$q_{n,k} = \text{lcm}\{(n-m+1)q_{m-1,k-1} ; k \leq m \leq n\}.$$

Proof. Let $n, k \in \mathbb{N}^*$ be fixed such that $n \geq k$. We have by definition:

$$q_{n,k} := \bigvee_{\substack{i_1, \dots, i_k \in \mathbb{N}^* \\ i_1 + \dots + i_k \leq n}} (i_1 \cdots i_k) = \bigvee_{\substack{i_1, \dots, i_{k-1}, i \in \mathbb{N}^* \\ i_1 + \dots + i_{k-1} + i \leq n}} (i_1 \cdots i_{k-1} i).$$

Since for any $i_1, \dots, i_{k-1}, i \in \mathbb{N}^*$, the inequality $i_1 + \dots + i_{k-1} + i \leq n$ implies $i \leq n - (i_1 + \dots + i_{k-1}) \leq n - k + 1$, then we derive that:

$$\begin{aligned}
q_{n,k} &= \bigvee_{i=1}^{n-k+1} \left(\bigvee_{\substack{i_1, \dots, i_{k-1} \in \mathbb{N}^* \\ i_1 + \dots + i_{k-1} \leq n-i}} (i_1 \cdots i_{k-1} i) \right) \\
&= \bigvee_{i=1}^{n-k+1} \left(i \left(\bigvee_{\substack{i_1, \dots, i_{k-1} \in \mathbb{N}^* \\ i_1 + \dots + i_{k-1} \leq n-i}} i_1 \cdots i_{k-1} \right) \right) \\
&= \bigvee_{i=1}^{n-k+1} i q_{n-i, k-1} \\
&= \bigvee_{m=k}^n (n-m+1) q_{m-1, k-1} \quad (\text{by setting } m = n-i+1),
\end{aligned}$$

as required. The proposition is proved. \square

Similarly as for the $c_{n,k}$'s, it is also practical to arrange the $q_{n,k}$'s ($0 \leq k \leq n$) in a triangular array in which each $q_{n,k}$ is the entry in the n^{th} row and k^{th} column. Leaning on Formula of Proposition 4.3 and using Maple software, we get the following triangle of the $q_{n,k}$'s up to $n = 10$:

1										
1	1									
1	2	1								
1	6	2	1							
1	12	12	2	1						
1	60	12	12	2	1					
1	60	360	24	12	2	1				
1	420	360	360	24	12	2	1			
1	840	5040	720	720	24	12	2	1		
1	2520	5040	15120	720	720	24	12	2	1	
1	2520	25200	30240	30240	1440	720	24	12	2	1

Table 2: The triangle of the $q_{n,k}$'s for $0 \leq k \leq n \leq 10$

To generate finally the positive integers λ_n ($n \in \mathbb{N}$), we have the choice to use one of the three formulas: $\lambda_n = \text{lcm}\{c_{n,k}; 0 \leq k \leq n\}$ (according to (3.4)); $\lambda_n = q_n := \text{lcm}\{q_{n,k}; 0 \leq k \leq n\}$ (according to Theorem 3.4) or $\lambda_n = \prod_{p \text{ prime}} p^{\lfloor \frac{n}{p} \rfloor}$ (according to Theorem 3.4). The calculations give the following first terms of the sequence $(\lambda_n)_{n \in \mathbb{N}}$:

1 , 1 , 2 , 6 , 12 , 60 , 360 , 2520 , 5040 , 15120 , 151200 , ...

References

- [1] P.-J. CAHEN & J.-L. CHABERT. Integer-valued polynomials, *Mathematical Surveys and Monographs*, Providence, RI: American Mathematical Society, **48**, (1997).
- [2] L. COMTET. Advanced Combinatorics, The Art of Finite and Infinite Expansions (revised and enlarged edition), *D. Reidel Publishing Company*, 1974.
- [3] R.L. GRAHAM, D.E. KNUTH & O. PATASHNIK. Concrete Mathematics: A Foundation for Computer Science, *Addison-Wesley*, 1994.
- [4] CH. JORDAN. Calculus of finite differences, Chelsea Publishing Co., New York, third edition, 1965.
- [5] G. PÓLYA. Über ganzwertige ganze Funktionen, *Palermo Rend.* (in German), **40** (1915), p. 1-16.