Church's thesis and related axioms in Coq's type theory

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- Abstract

"Church's thesis" (CT) as an axiom in constructive logic states that every total function of type $\mathbb{N} \to \mathbb{N}$ is computable, i.e. definable in a model of computation. CT is inconsistent both in classical mathematics and in Brouwer's intuitionism since it contradicts weak Kőnig's lemma and the fan theorem, respectively. Recently, CT was proved consistent for (univalent) constructive type theory.

Since neither weak Kőnig's lemma nor the fan theorem is a consequence of just logical axioms or just choice-like axioms assumed in constructive logic, it seems likely that CT is inconsistent only with a combination of classical logic and choice axioms. We study consequences of CT and its relation to several classes of axioms in Coq's type theory, a constructive type theory with a universe of propositions which proves neither classical logical axioms nor strong choice axioms.

We thereby provide a partial answer to the question as to which axioms may preserve computational intuitions inherent to type theory, and which certainly do not. The paper can also be read as a broad survey of axioms in type theory, with all results mechanised in the Coq proof assistant.

2012 ACM Subject Classification Theory of computation → Constructive mathematics; Type theory

Keywords and phrases Church's thesis, constructive type theory, constructive reverse mathematics, synthetic computability theory, Coq

Supplementary Material https://github.com/uds-psl/churchs-thesis-coq

Acknowledgements I want to thank Gert Smolka, Andrej Dudenhefner, Dominik Kirst, and Dominique Larchey-Wendling for discussions and feedback on drafts of this paper. Special thanks go to the anonymous reviewers for their helpful ideas, constructive comments, and editorial suggestions.

1 Introduction

The intuition that the concept of a constructively defined function and a computable function can be identified is prevalent in intuitionistic logic since the advent of recursion theory and is maybe most natural in constructive type theory, where computation is primitive.

A formalisation of the intuition is the axiom CT ("Church's thesis"), stating that every function is computable, i.e. definable in a model of computation. CT is well-studied as part of Russian constructivism [34] and in the field of constructive reverse mathematics [11,25].

CT allows proving results of recursion theory without extensive references to a model of computation, since one can reason with functions instead. While such synthethic developments of computability theory [1,7,37] can be carried out in principle without assuming any axioms [14], assuming CT allows stronger results: CT essentially provides a universal machine w.r.t. all functions in the logic, allowing to show the non-existence of certain deciding functions – whose existence is logically independent with no axioms present.

It is easy to see that CT is in conflict with traditional classical mathematics, since the law of excluded middle LEM together with a form of the axiom of countable choice $AC_{\mathbb{N},\mathbb{N}}$ allows the definition of non-computable functions [46]. This observation can be sharpened in various ways: To define a non-computable function directly, the weak limited principle of omniscience WLPO and the countable unique choice axiom $AUC_{\mathbb{N},\mathbb{B}}$ suffice. Alternatively, Kleene noticed that there is a decidable tree predicate with infinitely many nodes but no

computable infinite path [28]. If functions and computable functions are identified via CT, a Kleene tree is in conflict with weak Kőnig's lemma WKL and with Brouwer's fan theorem.

It is however well-known that CT is consistent in Heyting arithmetic with Markov's principle MP [27] which given CT states that termination of computation is stable under double negation. Recently, Swan and Uemura [43] proved that CT is consistent in univalent type theory with propositional truncation and MP.

While predicative Martin-Löf type theory as formalisation of Bishop's constructive mathematics proves the full axiom of choice AC, univalent type theory usually only proves the axiom of unique choice AUC. But since $AUC_{\mathbb{N},\mathbb{B}}$ suffices to show that LEM implies $\neg CT$, classical logic is incompatible with CT in both predicative and in univalent type theory.

In the (polymorphic) calculus of (cumulative) inductive constructions, a constructive type theory with a separate, impredicative universe of propositions as implemented by the proof assistant Coq [44], none of AC, AUC, and AUC_{N,B} are provable. This is because large eliminations on existential quantifications are not allowed in general [35], meaning one can not recover a function in general from a proof of $\forall x.\exists y. Rxy$. However, choice axioms as well al LEM can be consistently assumed in Coq's type theory [47]. Furthermore, it seems likely that the consistency proof for CT in [43] can be adapted for Coq's type theory.

This puts Coq's type theory in a special position: Since to disprove CT one needs a (weak) classical logical axiom and a (weak) choice axiom, assuming just classical logical axioms or just choice axioms might be consistent with CT. This paper is intended to serve as a preliminary report towards this consistency question, approximating it by surveying results from intuitionistic logic and constructive reverse mathematics in constructive type theory with a separate universe of propositions, with a special focus on CT and other axioms based on notions from computability theory. Specifically, we discuss these propositional axioms:

- computational enumerability axioms (EA, EPF) and Kleene trees (KT) in Section 5
- extensionality axioms like functional extensionality (Fext), propositional extensionality (Pext), and proof irrelevance (PI) in Section 6
- classical logical axioms like the principle of excluded middle (LEM, WLEM), independence of premises (IP), and limited principles of omniscience (LPO, WLPO, LLPO) in Section 7
- axioms of Russian constructivism like Markov's principle (MP) in Section 8
- choice axioms like the axiom of choice (AC), countable choice (ACC, $AC_{\mathbb{N},\mathbb{N}}$, $AC_{\mathbb{N},\mathbb{B}}$), dependent choice (ADC), and unique choice (AUC, $AUC_{\mathbb{N},\mathbb{B}}$) in Section 9
- axioms on trees like weak Kőnig's lemma (WKL) and the fan theorem (FAN) in Section 10
- axioms regarding continuity and Brouwerian principles (Homeo, Cont, WC-N) in Section 11 The following hyper-linked diagram displays provable implications and incompatible axioms.

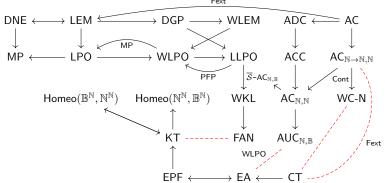


Figure 1 Overview of results. \rightarrow are implications, --- denotes incompatible axioms.

All results in this paper are mechanised in the Coq proof assistant and the proof scripts are accessible at https://github.com/uds-psl/churchs-thesis-coq. The statements in this document are hyperlinked to their Coq proof, indicated by a *p-symbol.

Outline. Section 2 establishes necessary preliminaries regarding Coq's type theory and introduces the notions of (synthetic) decidability, enumerability, and semi-decidability. Section 3 introduces CT formally, together with the related synthetic axioms EA and EPF. Section 4 contains undecidability proofs based on CT. Section 5 introduces decidable binary trees and constructs a Kleene tree. The connection of CT to the classes of axioms as listed above is surveyed in Sections 6 to 11. Section 12 contains concluding remarks.

2 Preliminaries

We work in the polymorphic calculus of cumulative inductive constructions as implemented by the Coq proof assistant [44], which we will refer to as "Coq's type theory". The calculus is a constructive type theory with a cumulative hierarchy of types \mathbb{T}_i (where i is a natural number, but we leave out the index from now on), an impredicative universe of propositions $\mathbb{P} \subseteq \mathbb{T}$, and inductive types in every universe. The inductive types of interest in this paper are

```
\begin{split} n:\mathbb{N} &::= 0 \mid \mathsf{S} \, n \\ o:\mathbb{O} A &::= \mathsf{None} \mid \mathsf{Some} \, a \quad \textit{where} \, a:A \\ A+B &:= \mathsf{inl} \, a \mid \mathsf{inr} \, b \quad \textit{where} \, a:A \, \textit{and} \, b:B \\ A &:= (a,b) \quad \textit{where} \, a:A \, \textit{and} \, b:B \end{split}
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One can easily construct a pairing function $\langle _, _ \rangle : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ and for all $f : \mathbb{N} \to \mathbb{N} \to X$ an inverse construction $\lambda \langle n, m \rangle$. fnm of type $\mathbb{N} \to X$ s.t. $(\lambda \langle n, m \rangle, fnm) \langle n, m \rangle = fnm$.

We write $n =_{\mathbb{B}} m$ for the boolean equality decider on \mathbb{N} , and $\neg_{\mathbb{B}}$ for boolean negation. If $l : \mathbb{L}A$ then $l[n] : \mathbb{O}A$ denotes the *n*-th element of *l*. If n < |l| we can assume l[n] : A.

We write $\forall x: X. Ax$ for both dependent functions and logical universal quantification, $\exists x: X. Ax$ where $A: X \to \mathbb{P}$ for existential quantification and $\Sigma x: X. Ax$ where $A: X \to \mathbb{T}$ for dependent pairs, with elements (x,y). Dependent pairs can be eliminated into arbitrary types, i.e. there is an elimination principle of type $\forall p: (\Sigma x. Ax) \to \mathbb{T}. \ (\forall (x:X)(y:Ax). \ p(x,y)) \to \forall (s:\Sigma x. Ax). \ ps.$ We call such a principle eliminating a proposition into arbitrary types a large elimination principle, following the terminology "large elimination" for Coq's case analysis construct match [35]. Crucially, Coq's type theory proves a large elimination principle for the falsity proposition \bot , i.e. explosion applies to arbitrary types: $\forall A: \mathbb{T}. \bot \to A$. In contrast, existential quantification can only be eliminated for $p: (\exists x. Ax) \to \mathbb{P}$, but the following more specific large elimination principle is provable:

b Lemma 1. There is a guarded minimisation function $\mu_{\mathbb{N}}$ of the following type:

$$\mu_{\mathbb{N}}: \forall f: \mathbb{N} \to \mathbb{B}. \ (\exists n. \ fn = \mathsf{true}) \to \Sigma n. \ fn = \mathsf{true} \land \forall m. \ fm = \mathsf{true} \to m \geq n.$$

There are various implementations of such a minimisation function in Coq's Standard Library. One uses a (recursive) large elimination principle for the accessibility predicate, see e.g. [32, §2.7, §4.1, §4.2] and [6, §14.2.3, §15.4] for a contemporary overview how to implement large eliminations principles. We will not need any other large elimination principle in this paper. A restriction of large elimination in general is necessary for consistency of Coq [8]. As a by-product, the computational universe $\mathbb T$ is separated from the logical universe $\mathbb P$, allowing classical logic in $\mathbb P$ to be assumed while the computational intuitions for $\mathbb T$ remain intact.

¹ The idea was conceived independently by Benjamin Werner and Jean-François Monin in the 1990s.

4 Church's thesis and related axioms in Coq's type theory

| $partA:\mathbb{T}$ | partial values over $A:\mathbb{T}$ | |
|---|------------------------------------|---|
| $\stackrel{!}{=}: part A 	o A 	o \mathbb{P}$ | definedness of values | $x \stackrel{!}{=} a_1 \rightarrow x \stackrel{!}{=} a_2 \rightarrow a_1 = a_2$ |
| $(x:partA)\downarrow\colon\mathbb{P}$ | | $x\downarrow := \exists a. \ x \stackrel{!}{=} a$ |
| $\equiv_{partA}\!:partA	opartA\to\mathbb{P}$ | equivalence | $x \equiv_{part A} y := (\forall a.\ x \stackrel{!}{=} a \leftrightarrow y \stackrel{!}{=} a)$ |
| $ret:A\topartA$ | monadic return | $ret\ a \overset{!}{=} a$ |
| undef:partA | undefined value | $ exists a$. undef $\stackrel{!}{=} a$ |
| $>\!\!\!>\!\!\!=$: part $A 	o (A 	o partB) 	o partB$ | monadic bind | $x \gg f \stackrel{!}{=} b \leftrightarrow (\exists a. \ x \stackrel{!}{=} a \land fa \stackrel{!}{=} b)$ |
| $\mu:(\mathbb{N} 	o \mathbb{B}) 	o part \mathbb{N}$ | unbounded search | $\mu f \stackrel{!}{=} n \leftrightarrow f n = true \land \\ \forall m < n. \ f m = false$ |
| seval : part $A \to \mathbb{N} \to \mathbb{O}A$ | step-indexed evaluation | $x \stackrel{!}{=} a \leftrightarrow \exists n$, seval $xn = Somea$ |

Figure 2 A monad for partial values

2.1 Partial Functions

All definable functions in type theory are total by definition. To model partiality, one often resorts to functional relations $R: A \to B \to \mathbb{P}$ or step-indexed functions $A \to \mathbb{N} \to \mathbb{O}B$, as for instance pioneered by Richman [37] in constructive logic, see e.g. [12] for a comprehensive overview.

For our purpose, we simply assume a type part A for $A : \mathbb{T}$ and a definedness relation $\stackrel{!}{=} : \mathsf{part}\,A \to A \to \mathbb{P}$ and write $A \nrightarrow B$ for $A \to \mathsf{part}\,B$. We assume monadic structure for part (ret and \gg), an undefined value (undef), a minimisation operation (μ) , and a step-indexed evaluator (seval). The operations and their specifications are listed in Figure 2.

2.2 Equivalence relations on functions

Besides intensional equality (=), we will consider other more extensional equivalence relations in this paper. For instance, extensional equality of functions f,g ($\forall x.\ fx=gx$), extensional equivalence of predicates p,q ($\forall x.\ px\leftrightarrow qx$), or range equivalence of functions f,g ($\forall x.\ (\exists y.\ fy=x)\leftrightarrow (\exists y.\ gy=x)$). We will denote all of these equivalence relations with the symbol \equiv and indicate what is meant by an index. For discrete X (e.g. $\mathbb{N},\ \mathbb{O}\mathbb{N},\ \mathbb{L}\mathbb{B},\ \ldots$), \equiv_X denotes equality, $\equiv_{\mathbb{P}}$ denotes logical equivalence, $\equiv_{A\to B}$ denotes an extensional lift of \equiv_B , $\equiv_{A\to \mathbb{P}}$ denotes extensional equivalence, and \equiv_{ran} denotes range equivalence.

Assuming the existence of surjections $A \to (A \to B)$ may or may not be consistent, depending on the particular equivalence relation. We introduce the notion of *surjection w.r.t.* \equiv_B as $\forall b: B$. $\exists a: A.fa \equiv_B b$. We call a function $f: A \to B$ an *injection w.r.t.* \equiv_A and \equiv_B if $\forall a_1 a_2. \ fa_1 \equiv_B fa_2 \to a_1 \equiv_A a_2$ and a *bijection* if it is an injection and surjection.

One formulation of Cantor's theorem is that there is no surjection $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$ w.r.t. =. However, the same proof can be used for the following strengthening of Cantor's theorem:

Fact 2 (Cantor). There is no surjection $\mathbb{N} \to (\mathbb{N} \to \mathbb{N})$ w.r.t. $\equiv_{\mathbb{N} \to \mathbb{N}}$

2.3 Decidability, Semi-decidability, Enumerability, Reducibility

We define decidability, (co-)semi-decidability, and enumerability for predicates $p: X \to \mathbb{P}$:

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\begin{array}{llll} \mathcal{D}p & := \exists f: X \to \mathbb{B}. & \forall x. \ px \leftrightarrow fx = \mathsf{true} & (\text{``p is decidable''}) \\ \mathcal{S}p & := \exists f: X \to \mathbb{N} \to \mathbb{B}. & \forall x. \ px \leftrightarrow \exists n. fxn = \mathsf{true} & (\text{``p is semi-decidable''}) \\ \overline{\mathcal{S}p} & := \exists f: X \to \mathbb{N} \to \mathbb{B}. & \forall x. \ px \leftrightarrow \forall n. fxn = \mathsf{false} & (\text{``p is co-semi-decidable''}) \\ \mathcal{E}p & := \exists f: \mathbb{N} \to \mathbb{O}X. & \forall x. \ px \leftrightarrow \exists n. fn = \mathsf{Some} \ x & (\text{``p is enumerable''}) \\ \end{array}
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Although all notions are defined on unary predicates, we use them on n-ary relations via (implicit) uncurrying. We write \overline{p} for the complement λx . $\neg px$ of p. We call a type X discrete if its equality relation $=_X$ is decidable and enumerable if the predicate λx . \top is enumerable.

Traditionally, propositions P s.t. $P \leftrightarrow (\exists n. fn = \mathsf{true})$ for some f are often called Σ^0_1 or "simply existential", and P s.t. $P \leftrightarrow (\forall n. fn = \mathsf{false})$ are called Π^0_1 or "simply universal". Semi-decidable predicates are pointwise Σ^0_1 , and co-semi-decidable predicates are pointwise Π^0_1 . Note that neither $\overline{\mathcal{S}}p \to \mathcal{S}\overline{p}$ nor the converse is provable, only the following connections:

4 Lemma 3. The following hold:

- 1. Decidable predicates are semi-decidable and co-semi-decidable.
- 2. Semi-decidable predicates on enumerable types are enumerable.
- 3. Enumerable predicates on discrete types are semi-decidable.
- 4. The complement of semi-decidable predicates is co-semi-decidable.

Lemma 4. Decidable predicates are closed under complementation. Decidable, enumerable, and semi-decidable predicates are closed under (pointwise) conjunction and disjunction.

3 Church's thesis in type theory

Church's thesis for total functions (CT) states that every function of type $\mathbb{N} \to \mathbb{N}$ is algorithmic. Thus CT is a relativisation of the function space $\mathbb{N} \to \mathbb{N}$ w.r.t. a given (Turing-complete) model of computation, reminiscent of the axiom V = L in set theory [29].

We first define CT by abstracting away from a concrete model of computation and work with an abstract model of computation, consisting of an abstract computation function Tcxn (with $T: \mathbb{N} \to \mathbb{N} \to \mathbb{N} \to \mathbb{O}$), assigning to a code c (to be interpreted as the code of a partial recursive function in a model of computation), an input number x, and a step index n an output number y if the code terminates in n steps on x with value y. The function Tcx is assumed to be monotonic, i.e. increasing the step index does not change the potential value:

$$Tcxn_1 = \mathsf{Some}\, y \to \forall n_2 \geq n_1. \ Tcxn_2 = \mathsf{Some}\, y.$$

Based on T we define a computability relation between $c: \mathbb{N}$ and $f: \mathbb{N} \to \mathbb{N}$:

$$c \sim f := \forall x. \exists n. \ Texn = \mathsf{Some}\,(fx).$$

Since T is monotonic, \sim is extensional, i.e. $n \sim f_1 \to n \sim f_2 \to \forall x$. $f_1x = f_2x$. We define Church's thesis for total functions relative to an abstract computation function T:

$$\mathsf{CT}_T := \forall f : \mathbb{N} \to \mathbb{N}. \exists n : \mathbb{N}. \ n \sim f$$

Note that CT_T is clearly not consistent for every choice of T. If we write CT without index, we mean T to be the step-indexed evaluation function of a concrete, Turing-complete model of computation. For the mechanisation we could for instance pick the equivalent models of Turing machines [17], λ -calculus [21], μ -recursive functions [30], or register machines [18,31]. It seems likely that the consistency proof of CT in [43] can be adapted to Coq .

Since specific properties of the model of computation are not needed, we develop and mechanise all results of this paper parameterised in an arbitrary T. Thus, we could also state all results in terms of a fully synthetic Church's thesis axiom $\Sigma T.\mathsf{CT}_T$.

▶ Fact 5. $CT \rightarrow \Sigma T.CT_T$

Note that the implication is strict: An abstract computation function does not rule out oracles for e.g. the halting problem of Turing machines, whereas CT – with T defined in terms of a standard, Turing-complete model of computation – proves the undecidability of the Turing machine halting problem.

3.1 Bauer's enumerability axiom EA

In proofs of theorems with CT_T as assumption, T can be used as replacement for a universal machine. Bauer [1] develops computability theory synthetically using the axiom "the set of enumerable sets of natural numbers is enumerable", which is equivalent to $\Sigma T.\mathsf{CT}_T$ and thus strictly weaker than CT , but can also be used in place of a universal machine. We introduce Bauer's axiom in our setting as EA' and immediately introduce a strengthening EA s.t. $(\Sigma T.\mathsf{CT}_T) \leftrightarrow \mathsf{EA}$ and $\mathsf{EA} \to \mathsf{EA}'$:

$$\mathsf{EA}' := \Sigma \mathcal{W} : \mathbb{N} \to (\mathbb{N} \to \mathbb{P}). \forall p : \mathbb{N} \to \mathbb{P}. \ \mathcal{E}p \leftrightarrow \exists c. \ \mathcal{W}c \equiv_{\mathbb{N}} p$$

That is, EA' states that there is an enumerator \mathcal{W} of all enumerable predicates, up to extensionality. In contrast, EA poses the existence of an enumerator of all possible enumerators, up to range equivalence:

$$\mathsf{EA} := \Sigma \varphi : \mathbb{N} \to (\mathbb{N} \to \mathbb{ON}). \forall f : \mathbb{N} \to \mathbb{ON}. \exists c. \ \varphi c \equiv_{\mathsf{ran}} f$$

That is, φ is a surjection w.r.t. range equivalence $f \equiv_{\mathsf{ran}} g$, where $\varphi c \equiv_{\mathsf{ran}} f \leftrightarrow \forall x. (\exists n. \varphi cn = \mathsf{Some}\, x) \leftrightarrow (\exists n. fn = \mathsf{Some}\, x)$.

Note the two different roles of natural numbers in the two axioms: If we would consider predicates over a general type X we would have $\mathcal{W}: \mathbb{N} \to (X \to \mathbb{P})$ and $\varphi: \mathbb{N} \to (\mathbb{N} \to \mathbb{O}X)$, i.e. $\mathcal{W}c$ would be an enumerable predicate and φc an enumerator of a predicate $X \to \mathbb{P}$.

We start by proving $\mathsf{CT}_T \to \mathsf{EA}$ by constructing φ from an arbitrary T:

$$\varphi c\langle n,m\rangle :=$$
 if $Tcnm$ is Some x then Sx else 0

\\ \Lemma 6. If CT_T then $\forall f : \mathbb{N} \to \mathbb{O}\mathbb{N}. \exists c. \ \varphi c \equiv_{ran} f$.

Proof. The direction from left to right to establish \equiv_{ran} is based on the fact that if $Tcxn_1 = \mathsf{Some}\,y_1$ and $Tcxn_2 = \mathsf{Some}\,y_2$ then $y_1 = y_2$. The other direction is straightforward.

? Theorem 7. $\forall T. \mathsf{CT}_T \to \mathsf{EA}$

We now prove $\mathsf{EA} \to \mathsf{EA}'$ by constructing \mathcal{W} from $\varphi \colon \mathcal{W} cx := \exists n. \varphi cn = \mathsf{Some}\, x.$

\P Lemma 8. If EA then $\forall p : \mathbb{N} \to \mathbb{P}$. $\mathcal{E}p \leftrightarrow \exists c$. $\mathcal{W}c \equiv_{\mathbb{N} \to \mathbb{P}} p$.

$$\begin{array}{ll} \textbf{Proof.} \ \mathcal{E}p \leftrightarrow \exists f: \mathbb{N} \rightarrow \mathbb{O}\mathbb{N}. \forall x. \ px \leftrightarrow \exists n. \ fn = \mathsf{Some} \ x & (\mathsf{def.} \ \mathcal{E}) \\ \leftrightarrow \exists c. \forall x. \ px \leftrightarrow \exists n. \ \varphi cn = \mathsf{Some} \ x & (\mathsf{EA}) \\ \leftrightarrow \exists c. \ \mathcal{W}c \equiv_{\mathbb{N} \rightarrow \mathbb{P}} p & (\mathsf{def.} \ \equiv_{\mathbb{N} \rightarrow \mathbb{P}}) \end{array}$$

№ Theorem 9. $EA \rightarrow EA'$

3.2 Richman's Enumerability of Partial Functions EPF

Richman [37] introduces a different purely synthetic axiom as replacement for a universal machine and assumes that "partial functions are countable", which is equivalent to EA.

$$\mathsf{EPF} := \Sigma e : \mathbb{N} \to (\mathbb{N} \nrightarrow \mathbb{N}). \forall f : \mathbb{N} \nrightarrow \mathbb{N}. \exists n. \ en \equiv_{\mathbb{N} \nrightarrow \mathbb{N}} f$$

• Theorem 10. EPF \rightarrow EA

Proof. Let e be given. $\varphi c\langle n, m \rangle := \text{seval } (ecn) \ m \text{ is the wanted enumerator.}$

9 Theorem 11. $EA \rightarrow EPF$

Proof. Let φ be given. Then

$$ecx := (\mu (\lambda n. \text{ if } \varphi cn \text{ is Some } \langle x', y' \rangle \text{ then } x =_{\mathbb{B}} x' \text{ else false})) >>\!\!= \lambda n. \text{ if } \varphi cn \text{ is Some } \langle x', y' \rangle \text{ then ret } y' \text{ else undef}$$

is the wanted enumerator.

EPF implies the fully synthetic version of CT:

P Lemma 12. EPF $\rightarrow \Sigma T$. CT $_T$

Proof. Assume $e: \mathbb{N} \to (\mathbb{N} \to \mathbb{N})$ surjective w.r.t. $\equiv_{\mathbb{N} \to \mathbb{N}}$. Define $Tcxn := \mathsf{seval}\ (ecx)\ n$. It is straightforward to prove that T is monotonic and that CT holds.

The axiom EPF can be weakened to cover just boolean functions:

$$\mathsf{EPF}_{\mathbb{B}} := \Sigma e : \mathbb{N} \to (\mathbb{N} \to \mathbb{B}). \forall f : \mathbb{N} \to \mathbb{B}. \exists n. \ en \equiv_{\mathbb{N} \to \mathbb{B}} f$$

9 Lemma 13. $EPF \rightarrow EPF_{\mathbb{B}}$

The reverse direction seems not to be provable.

4 Halting Problems

For this section we assume EA, i.e. $\varphi: \mathbb{N} \to (\mathbb{N} \to \mathbb{O}\mathbb{N})$ s.t. $\forall f: \mathbb{N} \to \mathbb{O}\mathbb{N}. \exists c. \ \varphi c \equiv_{\mathsf{ran}} f$. Recall Lemma 8 stating that $\forall p: \mathbb{N} \to \mathbb{P}$. $\mathcal{E}p \leftrightarrow \exists c. \ \mathcal{W}c \equiv_{\mathbb{N} \to \mathbb{P}} p$.

We define $K_0 n := W n n$ and prove our first negative result:

\rightarrow Lemma 14. $\neg \mathcal{E}\overline{\mathsf{K}_0}$

Proof. Assume $\mathcal{E}(\lambda n. \neg Wnn)$. By specification of W there is c s.t. $\forall n. Wcn \leftrightarrow \neg Wnn$. In particular, $Wcc \leftrightarrow \neg Wcc$, which is contradictory.

a Corollary 15.
$$\neg \mathcal{D}K_0$$
, $\neg \mathcal{D}\overline{K_0}$, $\neg \mathcal{D}W$ and $\neg \mathcal{D}\overline{W}$.

Intuitively, K_0 can be seen as analogous to the self-halting problem: K_0n states that n considered as an enumerator outputs itself in its range (rather than halting on itself).

It is also easy to show that W and thus K_0 are enumerable:

▶ Lemma 16. EW

Proof. Via $f(n,m) := \text{if } \varphi nm \text{ is } \mathsf{Some}\, k \text{ then } \mathsf{Some}\, (n,k) \text{ else } \mathsf{None}.$

♣ Corollary 17. $\mathcal{E}\mathsf{K}_0$

Since Bauer [1] bases his development on EA' , he needs the axiom of countable choice to prove that \mathcal{W} is enumerable, whereas EA allows an axiom-free proof of this fact.

Another well-known traditional result is that a problem is enumerable if and only if it many-one reduces to the halting problem K, which can be proved without reference to EA.

$$p \leq_m q := \exists f : X \to Y . \forall x. \ px \leftrightarrow q(fx)$$
 $\mathsf{K}(f : \mathbb{N} \to \mathbb{B}) := \exists n. \ fn = \mathsf{true}$

\\$ Fact 18. For all $p: X \to \mathbb{P}$, $p \leq_m \mathsf{K} \leftrightarrow \mathcal{S}p$.

- **№ Corollary 19.** SK
- **\rightarrow** Corollary 20. For all $p: \mathbb{N} \to \mathbb{P}$, $p \leq_m \mathsf{K} \leftrightarrow \mathcal{E}p$.

Using the non-enumerability of $\overline{K_0}$ we can now prove our first negative result by reduction:

P Corollary 21. $K_0 \leq_m K$, and thus $\neg \mathcal{E}\overline{K}$, $\neg \mathcal{D}K$, and $\neg \mathcal{D}\overline{K}$.

We can also define $K_{\mathbb{N}} := \lambda f : \mathbb{N} \to \mathbb{N}$. $\exists n. \ fn \neq 0$:

▶ Fact 22. K \leq_m K_N, K_N \leq_m K, $\overline{\mathsf{K}_{\mathbb{N}}} \equiv_{(\mathbb{N} \to \mathbb{N}) \to \mathbb{P}} \lambda f$. $\forall n. \ fn = 0, \ and \ thus \neg \mathcal{D}(\lambda f. \ \forall n. \ fn = 0)$.

5 Kleene Trees

In a lecture in 1953 Kleene [28] gave an example how the axioms of Brouwer's intuitionism fail if all functions are considered computable by constructing an infinite decidable binary tree with no computable infinite path. The existence of such a Kleene tree (KT) is in contradiction to Brouwer's fan theorem, which we will discuss later. We prove that $\mathsf{EPF}_\mathbb{B}$ implies KT.

For this purpose, we call a predicate $\tau: \mathbb{LB} \to \mathbb{P}$ a (decidable) binary tree if

- (a) τ is decidable: $\exists f. \forall u. \tau u \leftrightarrow fu = \mathsf{true}$
- **(b)** τ is non-empty: $\exists u.\tau u$
- (c) τ is prefix-closed: If τu_2 and $u_1 \sqsubseteq u_2$ then τu_1 (where $u_1 \sqsubseteq u_2 := \exists u'. \ u_2 = u_1 + u'$).

We will just speak of trees instead of decidable binary trees in the following.

\rightharpoonup Fact 23. For every tree τ , $\tau[]$ holds.

Furthermore, a decidable binary tree τ ...

- \blacksquare ... is bounded if $\exists n. \forall u. |u| \geq n \rightarrow \neg \tau u$
- \blacksquare ... is well-founded if $\forall f. \exists n. \neg \tau [f0, ..., fn]$
- ... has an *infinite path* if $\exists f. \forall n. \tau [f0, ..., fn]$
- **Fact 24.** A tree is not bounded if and only if it is infinite, defined as $\forall n. \exists u. |u| \geq n \wedge \tau u$.
- Fact 25. Every bounded tree is well-founded and every tree with an infinite path is infinite.

Note that both implications are strict: In our setting we cannot prove boundedness from well-foundedness nor obtain an infinite path from infiniteness, as can be seen from a Kleene tree:

 $\mathsf{KT} := \mathit{There\ exists\ an\ infinite,\ well-founded,\ decidable\ binary\ tree}.$

We follow Bauer [2] to construct a Kleene tree.

Proof. Define $dn := enn \gg \lambda b$. ret $(\neg_{\mathbb{B}}b)$. $\forall f : \mathbb{N} \to \mathbb{B}$. ∃nb. $dn \stackrel{!}{=} b \land fn \neq b$.

We define $\tau_K u := \forall n < |u|. \forall x. \text{ seval } (dn) \ |u| = \text{Some } x \to u[n] = \text{Some } x$. Intuitively, τ_K contains all paths $u = [b_0, b_1, \dots, b_n]$ which might be prefixes of d given n as step index, i.e. where n does not suffice to verify that d is no prefix of d. An infinite path through τ_K would be a totalisation of d.

Proof Theorem 27. $EPF_{\mathbb{B}} \to KT$

Proof. We show that τ_K is a Kleene tree. That τ_K is a decidable tree is immediate. To show that τ_K is infinite let k be given. We define f0 := [] and f(Sn) := fn + [if Dkn is Some x then x else false]. We have |fn| = n. In particular, $|fk| \ge k$ and $\tau_K(fk)$.

For well-foundedness let $f: \mathbb{N} \to \mathbb{B}$ be given. There is n s.t. $dn \stackrel{!}{=} b$ and $fn \neq b$. Thus there is k s.t. seval (dn) $k = \mathsf{Some}\, b$. Now $\neg \tau_K u$ for $u := [f0, \dots, f(n+k)]$.

6 Extensionality Axioms

Coq's type theory is intensional, i.e. $f \equiv_{A \to B} g$ and f = g do not coincide. Extensionality properties can however be consistently assumed as axioms. In this section we briefly discuss the relationship between CT and functional extensionality Fext, propositional extensionality Pext and proof irrelevance PI, defined as follows:

```
\begin{aligned} \mathsf{Fext} &:= \forall AB. \forall fg: A \to B. \ (\forall a.fa = ga) \to f = g \\ \mathsf{Pext} &:= \forall PQ: \mathbb{P}. \ (P \leftrightarrow Q) \to P = Q \\ \mathsf{PI} &:= \forall P: \mathbb{P}. \forall (x_1x_2: P). \ x_1 = x_2 \end{aligned}
```

Pract 28. Pext → PI

Swan and Uemura [43] prove that intensional predicative Martin-Löf type theory remains consistent if CT, the axiom of univalence, and propositional truncation are added. Since functional extensionality and propositional extensionality are a consequence of univalence, and propositions are semantically defined as exactly the irrelevant types, Fext, Pext, and PI hold in this extension of type theory. It seems likely that the consistency result can then be adapted to Coq's type theory, yielding a consistency proof for CT with Fext, Pext, and PI.

It is however crucial to formulate CT using \exists instead of Σ . The formulation as $\mathsf{CT}_{\Sigma} := \forall f. \ \Sigma n. \ n \sim f$ is inconsistent with functional extensionality Fext, as already observed in [46].

```
№ Lemma 29. \mathsf{CT}_\Sigma \to \mathsf{Fext} \to \bot
```

Proof. Since CT_Σ implies EA , it suffices to prove that $\lambda f. \forall n. \ fn=0$ is decidable by Fact 22. Assume $G: \forall f. \ \Sigma c. \ c \sim f$ and let $Ff:=\mathsf{if}\ \pi_1(Gf)=\pi_1(G(\lambda x.0))$ then true else false. If $Ff=\mathsf{true}$, then $\pi_1(Gf)=\pi_1(G(\lambda x.0))$ and by extensionality of \sim , $fn=(\lambda x.0)n=0$. If $\forall n. \ fn=0$, then $f=\lambda x.0$ by Fext, thus $\pi_1(Gf)=\pi_1(G(\lambda x.0))$ and $Ff=\mathsf{true}$.

7 Classical Logical Axioms

In this section we consider consequences of the law of excluded middle LEM. Precisely, besides LEM, we consider the weak law of excluded middle WLEM, the Gödel-Dummett-Principle DGP, and the principle of independence of premises IP, together with their respective restriction of propositions to the satisfiability of boolean functions, resulting in the limited principle of omniscience LPO, the weak limited principle of omniscience WLPO, and the lesser limited principle of omniscience LLPO.

```
\begin{split} \mathsf{LEM} &:= \forall P : \mathbb{P}. \ P \lor \neg P \\ \mathsf{WLEM} &:= \forall P : \mathbb{P}. \ \neg \neg P \lor \neg P \\ \mathsf{DGP} &:= \forall f : \mathbb{N} \to \mathbb{B}. \ (\exists n. \ fn = \mathsf{true}) \lor \neg (\exists n. \ fn = \mathsf{true}) \\ \mathsf{DCP} &:= \forall f : \mathbb{N} \to \mathbb{B}. \ \neg \neg (\exists n. \ fn = \mathsf{true}) \lor \neg (\exists n. \ fn = \mathsf{true}) \\ \mathsf{LLPO} &:= \forall f : \mathbb{N} \to \mathbb{B}. \ ((\exists n. \ fn = \mathsf{true}) \lor \neg (\exists n. \ fn = \mathsf{true})) \\ \lor ((\exists n. \ fn = \mathsf{true}) \to (\exists n. \ fn = \mathsf{true})) \\ \mathsf{V} &:= \forall P : \mathbb{P}. \forall q : \mathbb{N} \to \mathbb{P}. \ (P \to \exists n. qn) \to \exists n. \ P \to qn \end{split}
```

№ Fact 30. LEM \rightarrow DGP, DGP \rightarrow WLEM, LEM \rightarrow IP.

The converses are likely not provable: Diener constructs a topological model where DGP holds but not LEM, and one where WLEM holds but not DGP [11, Proposition 8.5.3]. Pédrot and Tabareau [36] construct a syntactic model where IP holds, but LEM does not.

² We follow Diener [11] in using the abbreviation DGP instead of GDP.

\(\) Fact 31. LPO \rightarrow WLPO and WLPO \rightarrow LLPO.

The converses are likely not provable: Both implications are strict over IZF with dependent choice [23, Theorem 5.1].

LPO is Σ_1^0 -LEM and WLPO is simultaneously Σ_1^0 -WLEM and Π_1^0 -LEM, due to the following:

▶ Fact 32. $(\forall n.fn = \mathsf{false}) \leftrightarrow \neg(\exists n.fn = \mathsf{true})$

Both can also be formulated for predicates:

- **▶ Fact 33.** *The following equivalences hold:*
- 1. LPO $\leftrightarrow \forall X. \forall (p: X \to \mathbb{P}). \ \mathcal{S}p \to \forall x. \ px \lor \neg px$
- **2.** WLPO $\leftrightarrow \forall X. \forall (p: X \to \mathbb{P}). \ \mathcal{S}p \to \forall x. \neg px \lor \neg \neg px$
- 3. WLPO $\leftrightarrow \forall X. \forall (p: X \to \mathbb{P}). \ \overline{\mathcal{S}}p \to \forall x. \ px \lor \neg px$

In our formulation, LLPO is the Gödel-Dummet rule for Σ_1^0 propositions. It can also be formulated as Σ_1^0 or \mathcal{S} De Morgan rule (2, 3 in the following Lemma), \mathcal{S} -DGP (4), or as a double negation elimination principle on $\overline{\mathcal{S}}$ relations into booleans (5):

- ▶ Lemma 34. The following are equivalent:
- 1. LLPO
- 2. $\forall fg: \mathbb{N} \to \mathbb{B}$. $\neg((\exists n.fn = \mathsf{true}) \land (\exists n.gn = \mathsf{true})) \to \neg(\exists n.fn = \mathsf{true}) \lor \neg(\exists n.gn = \mathsf{true})$
- 3. $\forall X. \forall (p \ q: X \to \mathbb{P}). \ \mathcal{S}p \to \mathcal{S}q \to \forall x. \ \neg (px \land qx) \to \neg px \lor \neg qx$
- **4.** $\forall X. \forall (p: X \to \mathbb{P}). \ \mathcal{S}p \to \forall xy. \ (px \to py) \lor (py \to px)$
- **5.** $\forall X. \forall (R: X \to \mathbb{B} \to \mathbb{P}). \ \overline{\mathcal{S}}R \to \forall x. \ \neg\neg(\exists b. \ Rxb) \to \exists b. \ Rxb$
- $\textbf{6.} \ \ \forall f. \ (\forall nm.fn = \mathsf{true} \rightarrow fm = \mathsf{true} \rightarrow n = m) \rightarrow (\forall n.f(2n) = \mathsf{false}) \lor (\forall n.f(2n+1) = \mathsf{false})$

We define the principle of finite possibility as $\mathsf{PFP} := \forall f. \exists g. \ (\forall n. \ fn = \mathsf{false}) \leftrightarrow (\exists n. \ gn = \mathsf{true}).$ PFP unifies WLPO and LLPO :

№ Fact 35. WLPO \leftrightarrow LLPO \land PFP

A principle unifying the classical axioms with their counterparts for Σ_1^0 is Kripke's schema $KS := \forall P : \mathbb{P}.\exists f : \mathbb{N} \to \mathbb{B}. \ P \leftrightarrow \exists n. \ fn = \mathsf{true}$:

- **№ Fact 36.** LEM \rightarrow KS
- **P** Fact 37. Given KS we have LPO \rightarrow LEM, WLPO \rightarrow WLEM, and LLPO \rightarrow DGP.

KS could be strengthened to state that every predicate is semi-decidable (to which KS is equivalent using $AC_{\mathbb{N},\mathbb{N}\to\mathbb{N}}$). The strengthening would be incompatible with CT.

In general, the compatibility of classical logical axioms (without assuming choice principles) with CT seems open. We conjecture that Coq's restriction preventing large elimination principles for non-sub-singleton propositions makes LEM and CT consistent in Coq.

8 Axioms of Russian Constructivism

The Russian school of constructivism morally identifies functions with computable functions, sometimes assuming CT explicitly. Another axiom considered valid is Markov's principle:

$$\mathsf{MP} := \forall f : \mathbb{N} \to \mathbb{B}. \ \neg \neg (\exists n. \ fn = \mathsf{true}) \to \exists n. \ fn = \mathsf{true}$$

Markov's principle is consistent with CT [43] and follows from LPO:

№ Fact 38. LPO ↔ WLPO ∧ MP

Proposition Year Output Output

It seems likely that the converse is not provable: There is a logic where MP holds, but not LPO [24]. As observed by Herbelin [24] and Pedrót and Tabareau [36], $IP \land MP$ yields LPO:

\rightarrow Lemma 40. MP ightarrow IP ightarrow LPO

Proof. Given $f: \mathbb{N} \to \mathbb{B}$ there is $n_0: \mathbb{N}$ s.t. $\forall k. \ fk = \mathsf{true} \to fn_0 = \mathsf{true}$ using MP and IP: By MP, $\neg \neg (\exists k. \ fk = \mathsf{true}) \to \exists n. \ fn = \mathsf{true}$ and by IP, $\exists n. \neg \neg (\exists k. fk = \mathsf{true}) \to fn = \mathsf{true}$, which suffices. Now $fn_0 = \mathsf{true} \leftrightarrow \exists n. \ fn = \mathsf{true}$ and LPO follows.

A nicer factorisation would be to prove $IP \rightarrow WLPO$, but the implication seems unlikely.

> Lemma 41. The following are equivalent:

- 1. MP
- **2.** $\forall X. \forall p: X \to \mathbb{P}. \mathcal{S}p \to \forall x. \neg \neg px \to px$
- **3.** $\forall X. \forall p: X \to \mathbb{P}. \mathcal{S}p \to \mathcal{S}\overline{p} \to \forall x. \ px \lor \neg px$
- **4.** $\forall X. \forall p: X \to \mathbb{P}. \ \mathcal{S}p \to \mathcal{S}\overline{p} \to \mathcal{D}p$
- **5.** $\forall X. \forall (R: X \to \mathbb{B} \to \mathbb{P}). \ \mathcal{S}R \to \forall x. \ \neg \neg (\exists b. \ Rxb) \to \exists b. \ Rxb$

Proof. \blacksquare 1 \rightarrow 2 is immediate.

- 2 → 3: Since S is closed under disjunctions and since $\neg\neg(px \lor \neg px)$ is a tautology.
- **3** \rightarrow **4** is immediate by Lemma 49 with $Rxb := (px \land b = \mathsf{true}) \lor (\neg px \land b = \mathsf{false}).$
- 4 → 1: Let $\neg\neg(\exists n.fn = \text{true})$. Let $p(x : \mathbb{N}) := \exists n.fn = \text{true}$. Now p is semi-decided by $\lambda x.f$, \overline{p} by λxn .false, and $p0 \vee \neg p0$ by 4. One case is easy, the other contradictory.

Note that 4 is often called "Post's theorem". $1 \leftrightarrow 3 \leftrightarrow 4$ is already discussed in [14]. 5 is dual to Lemma 34 (5). Replacing Sp with $\overline{S}p$ in 2 does however not result in an equivalent of LLPO, but turns 2 into an assumption-free fact. While in general $S\overline{p} \leftrightarrow \overline{S}p$ does not hold it seems possible that they can be exchanged in 3 and 4, but we are not aware of a proof.

9 Choice Axioms

We consider the axioms of choice AC, unique choice AUC, dependent choice ADC, and countable choice ACC. $AC_{\mathbb{N},\mathbb{N}}$ and $AC_{\mathbb{N}\to\mathbb{N},\mathbb{N}}$ are often called $AC_{0,0}$ and $AC_{1,0}$ in the literature.

```
\begin{split} \mathsf{AC}_{X,Y} &:= \forall R: X \to Y \to \mathbb{P}. (\forall x. \exists y. Rxy) \to \exists f: X \to Y. \forall x. \ Rx(fx) \\ \mathsf{AUC}_{X,Y} &:= \forall R: X \to Y \to \mathbb{P}. (\forall x. \exists !y. Rxy) \to \exists f: X \to Y. \forall x. \ Rx(fx) \\ \mathsf{ADC}_{X} &:= \forall R: X \to X \to \mathbb{P}. (\forall x. \exists x'. Rxx') \to \forall x_0. \exists f: \mathbb{N} \to X. f0 = x_0 \land \forall n. \ R(fn)(f(n+1))) \\ \mathsf{AC} &:= \forall XY: \mathbb{T}. \ \mathsf{AC}_{X,Y} \quad \mathsf{AUC} &:= \forall XY. \ \mathsf{AUC}_{X,Y} \quad \mathsf{ADC} &:= \forall X: \mathbb{T}. \ \mathsf{ADC}_{X} \quad \mathsf{ACC} &:= \forall X: \mathbb{T}. \ \mathsf{AC}_{\mathbb{N},X} \end{split}
```

▶ Fact 42. $AC_{X,X} \rightarrow ADC_X$, $AC_{X,Y} \rightarrow AUC_{X,Y}$, $ADC \rightarrow ACC$, $ACC \rightarrow AC_{N,N}$, and $AC_{N\rightarrow N,N} \rightarrow AC_{N,N}$.

The following well-known fact is due to Diaconescu [10] and Myhill and Goodman [22]:

№ Fact 43. AC \rightarrow Fext \rightarrow Pext \rightarrow LEM

Given that $AC_{\mathbb{N}\to\mathbb{N},\mathbb{N}}$ turns CT into CT_{Σ} , and that $EA \leftrightarrow \Sigma T.CT_T$ we have:

় Fact 44. $AC_{\mathbb{N}\to\mathbb{N},\mathbb{N}}$ → Fext → EA → \bot

We will later see that $LLPO \wedge AC_{\mathbb{N},\mathbb{N}}$ implies weak Kőnig's lemma, which is incompatible with KT. Already now we can prove that $WLPO \wedge AUC_{\mathbb{N},\mathbb{B}}$ is incompatible with EA:

- **№ Fact 45.** AUC_{N,B} \rightarrow (∀n : N. $pn \lor \neg pn$) $\rightarrow \mathcal{D}p$
- **№ Lemma 46.** WLPO \rightarrow AUC_{N.B} \rightarrow EA \rightarrow $\mathcal{D}\overline{\mathsf{K}}_{0}$

Proof. WLPO implies $\forall n. \neg K_0 n \lor \neg \neg K_0 n$. By $AUC_{\mathbb{N},\mathbb{B}}$ and the last lemma \overline{K}_0 is decidable.

P Corollary 47. WLPO \rightarrow AUC_{N.B} \rightarrow EA \rightarrow \bot

9.1 Provable choice axioms

In contrast to predicative Martin-Löf type theory, Coq's type theory does not prove the axiom of choice, nor the axioms of dependent and countable choice. This is due to the fact that arbitrary large eliminations are not allowed. However, recall that a large elimination principle for the accessibility predicate is provable, resulting in Lemma 1. Using Lemma 1 we can then prove $\mathcal{D}\text{-}\mathsf{AC}_{X,\mathbb{N}}$ for all X, i.e. choice for decidable relations into natural numbers:

\$ Lemma 48.
$$\forall X. \forall R: X \to \mathbb{N} \to \mathbb{P}$$
. $\mathcal{D}R \to (\forall x. \exists n. Rxn) \to \exists f: X \to \mathbb{N}. \forall x. Rx(fx)$.

As a consequence and with no further reference to Lemma 1 we can then prove choice principles for semi-decidable and enumerable relations, i.e. $\mathcal{S}\text{-AC}_{X,\mathbb{N}}$ and $\mathcal{E}\text{-AC}_{\mathbb{N},X}$ for all X:

Lemma 49. The following two choice principles are provable³:

- 1. $\forall X. \forall R: X \to \mathbb{N} \to \mathbb{P}$. $\mathcal{S}R \to (\forall x. \exists n. Rxn) \to \exists f: X \to \mathbb{N}. \forall x. Rx(fx)$
- **2.** $\forall X. \forall R : \mathbb{N} \to X \to \mathbb{P}. \ \mathcal{E}R \to (\forall n. \exists x. \ Rnx) \to \exists f : \mathbb{N} \to X. \forall n. \ Rn(fn)$

Principle 2 can be relaxed to arbitrary discrete types instead of \mathbb{N} , and in particular $S\text{-AC}_{\mathbb{N},\mathbb{B}}$ follows from 1. In Appendix A we discuss consequences of the here mentioned principles with regards to CT for oracles and in the next section $\overline{\mathcal{S}}\text{-AC}_{\mathbb{N},\mathbb{B}}$ will be central.

10 Axioms on Trees

We have already introduced (decidable) binary trees and Kleene trees in Section 5. We now give a broader overview and give formulations of LPO, WLPO, LLPO, and MP in terms of decidable binary trees, following Berger et al. [5].

\Paramodelequiv Fact 50. Let τ be a tree. Then $\tau_u v := \tau(u + v)$ is a tree if and only if τu .

If τu holds we call τ_u a subtree of τ and $\tau_{[b]}$ a direct subtree of τ .

- **▶ Lemma 51.** The following equivalences hold:
- 1. LPO \leftrightarrow every tree is bounded or infinite.
- **2.** WLPO \leftrightarrow every tree is infinite or not infinite.
- **3.** LLPO \leftrightarrow every infinite tree has a direct infinite subtree.
- **4.** MP \leftrightarrow if a tree is not infinite it is bounded.
- **5.** MP \leftrightarrow if a tree has no infinite path it is well-founded.

Recall Fact 25 stating that every bounded tree is well-founded and that every tree with an infinite path is infinite. The respective converse implications are known as Brouwer's fan theorem FAN and weak Kőnig's lemma WKL respectively:

FAN := Every well-founded decidable binary tree is bounded.

WKL := Every infinite decidable binary tree has an infinite path.

№ Fact 52. KT $\rightarrow \neg$ FAN and KT $\rightarrow \neg$ WKL.

Note that FAN is called FAN'_{Δ} in [26] and FAN_{Δ} in [11], and WKL is called WKL_D in [15]. Ishihara [26] shows how to deduce FAN from WKL constructively:

³ A formulation of (1) for disjunctions (equivalently: $R: X \to \mathbb{B} \to \mathbb{P}$) is due to Andrej Dudenhefner and was received in private communication. (2) was anticipated by Larchey-Wendling [30], who formulated it for μ -recursively enumerable instead of synthetically enumerable predicates.

- **▶ Fact 53.** Bounded trees τ have a longest element, i.e. $\exists u. \tau u \land \forall v. \tau v \rightarrow |v| < |u|$.
- **4 Lemma 54.** For every tree τ there is an infinite tree τ' s.t. for any infinite path f of τ' $\forall u. \tau u \to \tau[f0, \ldots, f|u|]$.

Property Theorem 55. WKL \rightarrow FAN

Proof. Let τ be well-founded. By Lemma 54 and WKL, there is f s.t. $\forall a. \ \tau u \to \tau[f0, \ldots, f|u|]$. Since τ is well-founded there is n s.t. $\neg \tau[f0, \ldots, fn]$. Then n is a bound for τ : For u with |u| > n and τu we have $\tau[f0, \ldots, fn, \ldots, f|u|]$. But then $\tau[f0, \ldots, fn]$, contradiction.

9 Corollary 56. $KT \rightarrow \neg WKL$.

Berger and Ishihara [4] show that FAN \leftrightarrow WKL!, a restriction of WKL stating that every infinite decidable binary tree with *at most one* infinite path has an infinite path. Schwichtenberg [40] gives a more direct construction and mechanises the proof in Minlog.

Berger, Ishihara, and Schuster [5] characterise WKL as the combination of the logical principle LLPO and the function existence principle $\overline{\mathcal{S}}$ -AC_{N,B} (called Π_1^0 -ACC $^{\vee}$ in [5]). We observe that WKL can also be characterised as one particular choice or dependent choice principle. The proofs are essentially rearrangements of [5, Theorem 27 and Corollary 5].

▶ Theorem 57. *The following are equivalent:*

- 1. WKL
- 2. LLPO $\wedge \overline{S}$ -AC_{N,B}
- 3. $\forall R: \mathbb{N} \to \mathbb{B} \to \mathbb{P}. \ \overline{\mathcal{S}}R \to (\forall n. \neg \neg \exists b. Rnb) \to \exists f: \mathbb{N} \to \mathbb{B}. \forall n. \ R \ n \ (fn)$
- **4.** $\forall R : \mathbb{LB} \to \mathbb{B} \to \mathbb{P}. \ \overline{\mathcal{S}}R \to (\forall u. \neg \neg \exists b. Rub) \to \exists f : \mathbb{N} \to \mathbb{B}. \forall n. \ R \ [f0, \dots, f(n-1)] \ (fn)$

Proof. For WKL \to LLPO we use the characterisation 3 of LLPO from Lemma 51. Let τ be an infinite tree. By WKL there is an infinite path f. Then $\tau_{[f0]}$ is a direct infinite subtree.

For WKL $\to \overline{\mathcal{S}}$ -AC_{N,B} let R be total and f s.t. $\forall nb.\ Rnb \leftrightarrow \forall m.fnbm = \mathsf{false}$. Define the tree $\tau u := \forall i < |u|. \forall m < |u|.\ fi(u[i])m = \mathsf{false}$. Infinity of τ follows from $\forall n. \exists u. |u| = n \land \forall i < n.Ri(u[i])$, proved by induction on n using totality of R. If g is an infinite path of τ , Rn(gn) follows from $\forall m. \tau[g0, \ldots, g(n+m+1)]$.

 $2 \rightarrow 3$ is immediate using characterisation 3 of LLPO from Lemma 34.

For $3 \to 4$ let $F : \mathbb{N} \to \mathbb{L}\mathbb{B}$ and $G : \mathbb{L}\mathbb{B} \to \mathbb{N}$ invert each other. Let $R : \mathbb{L}\mathbb{B} \to \mathbb{B} \to \mathbb{P}$ and f be the choice function obtained from 3 for $\lambda nb.R(Fn)b$. Then $\lambda n.f(G(gn))$ where g0 := [] and g(Sn) := gn + [f(G(gn))] is a choice function for R as wanted.

For $\mathbf{4} \to \mathbf{1}$ let τ be an infinite tree and let $d_u m := \exists v. |v| = m \wedge \tau_u v$, i.e. $d_u m$ if τ_u has depth at least m and in particular τ_u is infinite iff $\forall m.d_u m$. Define $Rub := \forall m.d_{u+\lfloor b\rfloor} m \vee \neg d_{u+\lfloor \neg Bb\rfloor}$. R is co-semi-decidable (since d is decidable), and $\neg Ru$ true $\wedge \neg Ru$ false is contradictory. Thus $\mathbf{4}$ yields a choice function f which fulfils $\tau[f0,\ldots,fn]$ by induction on n.

11 Continuity: Baire Space, Cantor Space, and Brouwer's Intuitionism

The total function space $\mathbb{N} \to \mathbb{N}$ is often called *Baire space*, whereas $\mathbb{N} \to \mathbb{B}$ is called *Cantor space*. We will from now on write $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{B}^{\mathbb{N}}$ for the spaces.

Constructively, one cannot prove that $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{B}^{\mathbb{N}}$ are in bijection. However, KT is equivalent to the existence of a continuous bijection $\mathbb{B}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ with a continuous modulus of

⁴ These so called coding functions is easy to construct even formally using e.g. techniques from [14].

continuity, i.e. a modulus function which is continuous (in the point) itself [11]. Furthermore, KT yields a continuous bijection $\mathbb{N}^{\mathbb{N}} \to \mathbb{B}^{\mathbb{N}}$ [3].

We call a function $F:A^{\mathbb{N}}\to B^{\mathbb{N}}$ continuous if $\forall f:A^{\mathbb{N}}.\forall n:\mathbb{N}.\exists L:\mathbb{L}\mathbb{N}.\forall g:A^{\mathbb{N}}$. (map $f:L=\mathbb{N}$ map $g:L)\to Ffn=Fgn$. A function $M:A^{\mathbb{N}}\to\mathbb{N}\to\mathbb{L}\mathbb{N}$ is called the modulus of continuity for F if $\forall n:\mathbb{N}.\forall fg:A^{\mathbb{N}}$. map f $(Mfn)=\mathrm{map}\;g$ $(Mfn)\to Ffn=Fgn$. We define:

 $\mathsf{Homeo}(A^{\mathbb{N}},B^{\mathbb{N}}) := \exists F: A^{\mathbb{N}} \to B^{\mathbb{N}}. \exists M. \ M \ is \ a \ continuous \ modulus \ of \ continuity \ for \ F$

We start by proving that $\mathsf{KT} \leftrightarrow \mathsf{Homeo}(\mathbb{B}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}})$. To do so, we say that u + [b] is a leaf of a Kleene tree τ_K if $\tau_K u$, but $\neg \tau_K (u + [b])$.

\Pi Fact 58. For every τ_K , there is an injective enumeration $\ell: \mathbb{N} \to \mathbb{LB}$ of the leaves of τ_K .

We define $F(f: \mathbb{N} \to \mathbb{N})n := (\ell(f0) + \cdots + \ell(f(n+1)))[n]$. Since leaves cannot be empty, the length of the accessed list is always larger than n and F is well-defined.

- **4 Lemma 59.** F is injective w.r.t. $\equiv_{\mathbb{N}^{\mathbb{N}}}$ and $\equiv_{\mathbb{N}^{\mathbb{N}}}$.
- **4 Lemma 60.** *F* is continuous with continuous modulus of continuity.
- **Lemma 61.** The following hold for a Kleene tree τ_K :
- 1. There is a function $\ell^{-1}: \mathbb{LB} \to \mathbb{N}$ s.t. for all leafs $l, \ell(\ell^{-1}l) = l$.
- **2.** For all l s.t. $\neg \tau_K l$ there exists $l' \sqsubseteq l$ s.t. l' is a leaf of τ_K .
- **3.** There is pref : $(\mathbb{N} \to \mathbb{B}) \to \mathbb{L}\mathbb{B}$ s.t. pref g is a leaf of τ_K and $\exists n$. pref $g = \mathsf{map}\ g\ [0,\ldots,n]$.

We can now define the inverse as $G g n := \ell^{-1}(\mathsf{pref}(\mathsf{nxt}^n g))$ where $\mathsf{nxt} g n := g(n + |\mathsf{pref} g|)$.

- **\rightarrow Lemma 62.** $F(G|g) \equiv_{\mathbb{N} \to \mathbb{B}} g$
- **4 Lemma 63.** *G* is continuous with continuous modulus of continuity.

The following proof is due to Diener [11, Proposition 5.3.2].

4. Lemma 64. Homeo($\mathbb{B}^{\mathbb{N}}, \mathbb{N}^{\mathbb{B}}$) \to KT

Proof. Let F be a bijection with continuous modulus of continuity M. Then $\tau u := \forall 0 < i \le |u| . \exists k < i.k \in M(\lambda n. \text{if } l[n] \text{ is Some } b \text{ then } b \text{ else false}) 0 \text{ is a Kleene tree.}$

7 Theorem 65. KT \leftrightarrow Homeo($\mathbb{B}^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$) and KT \rightarrow Homeo($\mathbb{N}^{\mathbb{N}}$, $\mathbb{B}^{\mathbb{N}}$).

Deiser [9] proves in a classical setting that $\mathsf{Homeo}(\mathbb{N}^{\mathbb{N}}, \mathbb{B}^{\mathbb{N}})$ holds. It would be interesting to see whether the proof can be adapted to a constructive proof $\mathsf{WKL} \to \mathsf{Homeo}(\mathbb{N}^{\mathbb{N}}, \mathbb{B}^{\mathbb{N}})$.

We have already seen that CT is inconsistent with FAN. Besides FAN, in Brouwer's intuitionism the continuity of functionals $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is routinely assumed:

$$\mathsf{Cont} := \forall F : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}. \ \forall f : \mathbb{N} \to A. \\ \exists L : \mathbb{L}\mathbb{N}. \\ \forall g : \mathbb{N} \to A. \ (\mathsf{map} \ f \ L = \mathsf{map} \ g \ L) \to Ff \equiv_{\scriptscriptstyle{B}} Fg$$

Since every computable function is continuous, we believe Cont to be consistent with CT. Combining Cont with $AC_{\mathbb{N}\to\mathbb{N},\mathbb{N}}$ yields *Brouwer's continuity principle*⁵, called WC-N in [46]:

$$\mathsf{WC-N} := \forall R : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N} \to \mathbb{P}. (\forall f. \exists n. Rfn) \to \forall f. \exists Ln. \forall g. \ \mathsf{map} \ f \ L = \mathsf{map} \ g \ L \to Rgn$$

⁵ But note that $Cont \to AC_{\mathbb{N} \to \mathbb{N}, \mathbb{N}} \to \bot$, since the resulting modulus of continuity function allows for the construction of a non-continuous function [13].

Proof Theorem 66. WC-N \rightarrow Cont

WC-N is inconsistent with CT, since the computability relation \sim is not continuous:

? Theorem 67. WC-N \rightarrow CT \rightarrow \bot

Proof. Recall that if two functions have the same code they are extensionally equal. By CT, $\lambda fc.c \sim f$ is a total relation. Using WC-N for this relation and $\lambda x.$ 0 yields a list L and a code c s.t. $\forall g.$ map g $L = [0, \ldots, 0] \rightarrow c \sim g$.

The functions λx . 0 and λx . if $x \in L$ then 0 else 1 both fulfil the hypothesis and thus have the same code – a contradiction since they are not extensionally equal.

12 Conclusion

In this paper we surveyed the known connections of axioms in Coq's type theory, a constructive type theory with a separate, impredicative universe of propositions, with a special focus on Church's thesis CT and formulations of axioms in terms of notions of synthetic computability. Furthermore, all results are mechanised in the Coq proof assistant.

In constructive mathematics, countable choice is often silently assumed, as critised e.g. by Richman [38,39]. In contrast, constructive type theory with a universe of propositions seems to be a suitable base system for matters of constructive (reverse) mathematics sensitive to applications of countable choice. Due to the separate universe of propositions, such a constructive type theory neither proves countable nor dependent choice, allowing equivalences like the one in Theorem 57 to be stated sensitively to choice. We conjecture that Lemma 49 deducing $\mathcal{S}\text{-AC}_{X,\mathbb{N}}$ and $\mathcal{E}\text{-AC}_{\mathbb{N},X}$ directly from $\mathcal{D}\text{-AC}_{X,\mathbb{N}}$ cannot be significantly strengthened. The proof of $\mathcal{D}\text{-AC}_{X,\mathbb{N}}$ in turn crucially relies on a large elimination principle for $\exists n.\ fn=\mathsf{true}$ (Lemma 1). The theory of [5] proves $\mathcal{D}\text{-AC}_{\mathbb{N},\mathbb{B}}$ and thus likely also $\mathcal{S}\text{-AC}_{\mathbb{N},\mathbb{B}}$.

Predicative Martin-Löf type theory proves AC and type theories with propositional truncation and a semantic notion of (homotopy) propositions prove $AUC_{\mathbb{N},\mathbb{B}}$, thus LEM suffices to disprove CT for both these flavours of type theory. Based on the current state of knowledge in the literature it seems likely that $\mathcal{S}\text{-}AC_{\mathbb{N},\mathbb{B}}$ and LEM together do not suffice to disprove CT, which seems to require at least classical logic of the strength of LLPO and a choice axiom for co-semi-decidable predicates. Thus we conjecture that a consistency proof of e.g. LEM \wedge CT might be possible for Coq's type theory.

Another advantage of basing constructive investigations on constructive type theory is that implementations of type theory in proof assistants already exist. For this paper, mechanising the results in Coq was tremendously helpful in keeping track of all details. For example, many of the presented proofs are very sensitive to small changes in formulations, and Coq actually helped in understanding the proofs and getting them right.

Besides consistency, another interesting property of axioms is admissibility. For instance, Pédrot and Tabareau [36] prove MP admissible in constructive type theory. CT seems to be admissible in constructive type theory in the sense that for every defined function $f: \mathbb{N} \to \mathbb{N}$ one can define a program in a model of computation with the same input output behaviour, as witnessed by the certifying extraction for a fragment of Coq to the λ -calculus [16]. An admissibility proof of CT could then serve as a theoretical underpinning of the Coq library of undecidability proofs [19]. However, any formal admissibility proof would have to deal with the intricacies of Coq's type theory. It would be interesting to investigate whether Letouzey's semantic proof for the correctness of type and proof erasure [33] can be connected with the mechanisation of meta-theoretical properties of Coq's type theory [41] in the MetaCoq project [42], yielding a mechanised admissibility proof for CT in Coq's type theory.

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Modesty and Oracles

Using \mathcal{D} -AC_{N,N} from Lemma 48 allows proving a choice axiom w.r.t. models of computation, observed by Larchey-Wendling [30] and called "modesty" by Forster and Smolka [20].

▶ Lemma 68. Let T be an abstract computation function. We have

```
\forall c. (\forall n. \exists mk. \ Tcnk = \mathsf{Some} \ m) \rightarrow \exists f : \mathbb{N} \rightarrow \mathbb{N}. \forall n. \exists k. \ Tcnk = \mathsf{Some} \ (fn)
```

That is, if c is the code of a function inside the model of computation which is provably total, the total function can be computed outside of the model. This modesty principle simplifies the mechanisation of computability theory in type theory as e.g. in [21]. For instance, it allows to prove that defining decidability as "a total function in the model of computation deciding the predicate" and as "a meta-level function deciding the predicate which is computable in the model of computation" is equivalent.

However, the modesty principle prevents synthetic treatments of computability theory based on oracles. Traditionally, computability theory based on oracles is formulated using a computability function T_p , s.t. for $p:\mathbb{N}\to\mathbb{P}$ there exists a code c_p representing a total function s.t. $\forall n. (\exists k. Tc_p nk = \mathsf{Some}\, 0) \leftrightarrow pn.$

Synthetically, we would now like to assume an abstract computability function for every p as "Church's thesis with oracles". "Church's thesis with oracles" implies CT, and we know that under CT the predicate K_0 is not decidable. However, under the presence of $\mathcal{D}\text{-}\mathsf{AC}_{\mathbb{N},\mathbb{N}}$ we can use T_{K_0} and obtain c_{K_0} which can be turned into a decider $f: \mathbb{N} \to \mathbb{B}$ for K_0 using the choice principle above – a contradiction.

В Cog mechanisation

The Coq mechanisation of the paper comprises 4250 lines of code, with 3300 lines of proofs and 950 lines of statements and definitions, i.e. 77% proofs. The mechanisation is based on

the Coq-std++ library [45], plus around 1500 additional lines of code with custom extensions to Coq's standard library which are shared with the Coq library of undecidability proofs [19].

The 4250 lines of the main development are distributed as follows: The basics of synthetic computability (decidability, semi-decidability, enumerability, many-one reductions) need 1150 lines of code. The mechanisation of Section 3, covering CT, EA, and EPF, comprises 400 lines of code. 120 lines of codes are needed for the undecidability results of Section 4. Section 5 and Section 10, covering trees and in particular Kleene trees, need 1000 lines of code. Section 11 on continuity is mechanised in 800 lines. The rest, i.e. Sections 6 to 9, needs 750 lines of code.

No advanced mechanisation techniques were needed. Discreteness and enumerability proofs for types were eased using type classes to assemble proofs for compound types such as $\mathbb{LB} \times \mathbb{ON}$, as already done in [14]. Defining the notions of $\equiv_{A \to B}$, $\equiv_{A \to \mathbb{P}}$, and so on was made possible by using type classes as well.

The technically most challenging mechanised proofs correspond to Lemmas 59 - 63, i.e. prove $\mathsf{KT} \to \mathsf{Homeo}(\mathbb{B}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}) \wedge \mathsf{Homeo}(\mathbb{N}^{\mathbb{N}}, \mathbb{B}^{\mathbb{N}})$. For these proofs, lots of manipulation of prefixes of lists was needed, and while the functions firstn and dropn are defined in Coq's standard library, the very useful lemmas of Coq-std++ where needed to make the proofs feasible.

In the development of this paper, the Coq proof assistant, while also acting as proof checker, was truly used as an assistant: Lots of proofs were developed and understood directly while working in Coq rather than on paper, allowing to identify for instance the equivalent characterisations of LLPO, MP, and WKL as in Lemma 34 (5), Lemma 41 (5), and Theorem 57 (3,4), which are hard to observe on paper because lots of bookkeeping for side-conditions would have to be done manually then.