#### LINKED TREE-DECOMPOSITIONS INTO FINITE PARTS

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ABSTRACT. We prove that every graph which admits a tree-decomposition into finite parts has a rooted tree-decomposition into finite parts that is linked, tight and componental.

As an application, we obtain that every graph without half-grid minor has a lean tree-decomposition into finite parts, strengthening the corresponding result by Kříž and Thomas for graphs of finitely bounded tree-width. In particular, it follows that every graph without half-grid minor has a tree-decomposition which efficiently distinguishes all ends and critical vertex sets, strengthening results by Carmesin and by Elm and Kurkofka for this graph class.

As a second application of our main result, it follows that every graph which admits a tree-decomposition into finite parts has a tree-decomposition into finite parts that displays all the ends of G and their combined degrees, resolving a question of Halin from 1977. This latter tree-decomposition yields short, unified proofs of the characterisations due to Robertson, Seymour and Thomas of graphs without half-grid minor, and of graphs without binary tree subdivision.

## §1. Introduction

1.1. The main result. Our point of departure is Kříž and Thomas's result on linked tree-decompositions, which forms a cornerstone both in Robertson and Seymour's work [28] on well-quasi-ordering finite graphs, and in Thomas's result [33] that the class of infinite graphs of tree-width  $\langle k \rangle$  is well-quasi-ordered under the minor relation for all  $k \in \mathbb{N}$ .

**Theorem 1.1** (Thomas 1990 [34], Kříž and Thomas 1991 [22]). Every (finite or infinite) graph of tree-width < k has a linked rooted tree-decomposition of width < k.

To make this result precise, recall that a *(rooted) tree-decomposition*  $(T, \mathcal{V})$  of a possibly infinite graph G is given by a (rooted) *decomposition tree* T whose nodes t are assigned *bags*  $V_t \subseteq V(G)$  or *parts*  $G[V_t]$  of the underlying graph G such that  $\mathcal{V} = (V_t)_{t \in T}$  covers G in a way that reflects the separation properties of T: Similarly as the deletion of an edge e = st from T separates it into components  $T_s \ni s$  and  $T_t \ni t$ , the corresponding sets  $A_s^e = \bigcup_{x \in T_s} V_x$  and  $A_t^e = \bigcup_{x \in T_t} V_x$  in the underlying graph G are separated by the *adhesion set*  $V_e = V_s \cap V_t$ . A graph G has

- $\diamond$  tree-width < k if it admits a (rooted) tree-decomposition into parts of size  $\leq k$ ,
- $\diamond$  finitely bounded tree-width if it has tree-width < k for some  $k \in \mathbb{N}$ , and

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<sup>&</sup>lt;sup>1</sup>There is also an unrooted version of this theorem but this is not needed for the well-quasi-ordering applications.

\$\sigma finite tree-width\$ if it admits a (rooted) tree-decomposition into finite parts (of possibly unbounded size).

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Given a tree T rooted at a node r, its tree-order is given by  $x \leq y$  for  $x, y \in V(T) \cup E(T)$  if x lies on the (unique)  $\subseteq$ -minimal path rTy from r to y. For an edge e = st of T with s < t, the  $part\ above\ e$  is  $G \uparrow e = G[A_t^e]$  and the  $part\ strictly\ above\ e$  is  $G \uparrow e = G \uparrow e - V_e$ . A rooted tree-decomposition  $(T, \mathcal{V})$  of a graph G is

- $\diamond$  *linked* if for every two comparable nodes s < t of T there are  $\min\{|V_e|: e \in E(sTt)\}$  pairwise disjoint  $V_s$ – $V_t$  paths in G [33],
- $\diamond$  tight if for every edge e of T some component C of  $G \uparrow e$  satisfies  $N(C) = V_e$ , and
- $\diamond$  componental if for every edge e of T the part  $G \uparrow e$  strictly above e is connected.

Our main result extends Theorem 1.1 to graphs of (possibly unbounded) finite tree-width:

**Theorem 1.** Every graph of finite tree-width admits a rooted tree-decomposition into finite parts that is linked, tight, and componental.

Note that achieving just a subset of the properties in Theorem 1 may be significantly easier. Recall that a normal spanning tree of a graph G is a rooted spanning tree T such that the endvertices of every edge of G are comparable in its tree-order. It is well known that the connected graphs of finite tree-width are precisely the graphs with normal spanning trees (see Theorem 2.2 below). Given a normal spanning tree T with root r, by assigning to each of its nodes t the bag  $V_t := V(rTt)$  we obtain a rooted tree-decomposition  $(T, \mathcal{V}_{NT})$  into finite parts that is componental and linked, the latter albeit for the trivial reason that  $s < t \in T$  implies  $V_s \subseteq V_t$ . However, this tree-decomposition clearly fails to be tight. Following an observation by Diestel [6], one can restore tightness by taking as bags the subsets  $V'_t \subseteq V_t$  that consist only of those vertices in  $V_t$  that send a G-edge to a vertex above t in T. The new tree-decomposition  $(T, \mathcal{V}'_{NT})$  is then tight and componental, but in general no longer linked.

Having all three properties simultaneously is more challenging to achieve. In what follows, we hope to convince the reader of the usefulness of Theorem 1 by demonstrating the surprisingly powerful interplay of the properties of being linked, tight, and componental. We do so by presenting several applications in the following sections.

In [2] we provide examples showing that Theorem 1 appears to lie at the frontier of what is still true. For example, *linked* cannot be strengthened to its 'unrooted' version that requires the *linked* property between every pair of nodes and not just comparable ones [2, Example 2].

1.2. Displaying end structure and excluding infinite minors. As our first application, we show that every tree-decomposition as in Theorem 1 displays the combinatorial and topological end structure of the underlying graph as follows, resolving a question by Halin from 1977 [16, §6].

<sup>&</sup>lt;sup>2</sup>Note that for a graph to have finite tree-width we do not require that the tree-decomposition into finite parts also satisfies that  $\liminf_{e \in E(R)} V_e$  is finite for all rays R in T, as it is sometimes [8,30] done.

(Halin's original work yields proofs of Theorems 1 and 2 for locally finite connected graphs with at most two ends.)

**Theorem 2.** Every graph G of finite tree-width admits a rooted tree-decomposition into finite parts that homeomorphically displays all the ends of G, their dominating vertices, and their combined degrees.

Recall that an end  $\varepsilon$  of a graph G is an equivalence class of rays in G where two rays are equivalent if for every finite set X they have a tail in the same component of G - X. We refer to this component of G - X as  $C_G(X, \varepsilon)$ . Write  $\Omega(G)$  for the set of all ends of G. A vertex v of G dominates an end  $\varepsilon$  of G if it lies in  $C_G(X, \varepsilon)$  for every finite set X of vertices other than v. We denote the set of all vertices of G which dominate an end  $\varepsilon$  by  $Dom(\varepsilon)$ . The degree  $deg(\varepsilon)$  of an end  $\varepsilon$  of G is the supremum over all cardinals  $\kappa$  such that there exists a set of  $\kappa$  pairwise disjoint rays in  $\varepsilon$ , and its combined degree is  $\Delta(\varepsilon) := deg(\varepsilon) + |Dom(\varepsilon)|$ .

In a rooted tree-decomposition  $(T, \mathcal{V})$  of a graph G into finite parts, every end  $\varepsilon$  of G gives rise to a single rooted ray  $R_{\varepsilon}$  in T that starts at the root and then always continues upwards along the unique edge  $e \in T$  with  $C_G(V_t, \varepsilon) \subseteq G \uparrow e$ . This yields a map  $\varphi \colon \Omega(G) \to \Omega(T)$ ,  $\varepsilon \mapsto R_{\varepsilon}$ . A rooted tree-decomposition  $(T, \mathcal{V})$ 

- $\diamond$  displays the ends of G if  $\varphi$  is a bijection [4],
- $\diamond$  displays the ends homeomorphically if  $\varphi$  is a homeomorphism [20],
- $\diamond$  displays all dominating vertices if  $\liminf_{e \in E(R_{\varepsilon})} V_e = \text{Dom}(\varepsilon)^3$  for all  $\varepsilon \in \Omega(G)$ , and
- $\diamond$  displays all combined degrees if  $\liminf_{e \in E(R_{\varepsilon})} |V_e| = \Delta(\varepsilon)$  for all  $\varepsilon \in \Omega(G)$ .

Roughly, every componental, rooted tree-decomposition homeomorphically displays all the ends [20, Lemma 3.1], every tight such tree-decomposition displays all dominating vertices (Lemma 3.2), and every such linked tight tree-decomposition displays all combined degrees (Lemma 3.3).

Diestel's tree-decomposition  $(T, \mathcal{V}'_{NT})$  mentioned above was not linked, but still displays all the ends and their dominating vertices. This observation allowed Diestel [6] to obtain a short proof of Robertson, Seymour and Thomas's characterisation that graphs without subdivided infinite cliques are precisely the graphs that admit a tree-decomposition  $(T, \mathcal{V})$  into finite parts such that for every ray R of T, the set  $\liminf_{e \in E(R)} V_e$  is finite. Extending this idea, we now use Theorem 2 to provide short, unified proofs for two other related results by Robertson, Seymour and Thomas.

Corollary 1.2 (Robertson, Seymour & Thomas 1995 [30, (2.6)]). A graph contains no half-grid minor if and only if it admits a tree-decomposition  $(T, \mathcal{V})$  into finite parts such that for every ray R of T we have  $\liminf_{e \in E(R)} |V_e| < \infty$ .

<sup>&</sup>lt;sup>3</sup>The set-theoretic  $\liminf_{n\in\mathbb{N}}A_n$  consists of all points that are contained in all but finitely many  $A_n$ . For a ray  $R=v_0e_0v_1e_1v_1\ldots$  in T, one gets  $\liminf_{e\in E(R_\varepsilon)}V_e=\bigcup_{n\in\mathbb{N}}\bigcap_{i\geq n}V_{e_i}$ .

*Proof.* We prove here only the hard implication<sup>4</sup>. So assume that G is a graph without half-grid minor. By a result of Halin [17, 24], every connected graph without a subdivided  $K^{\aleph_0}$  has a normal spanning tree. Applying this to every component of G we see that G has finite tree-width. So we may apply Theorem 2.

By Halin's grid theorem [14, 23], every end of infinite degree contains a half-grid minor. And by a routine exercise, every end dominated by infinitely many vertices contains a subdivided  $K^{\aleph_0}$ . So every end of G has finite combined degree. Let  $(T, \mathcal{V})$  be the tree-decomposition into finite parts from Theorem 2. Then  $\liminf_{e \in R} |V_e|$  equals the combined degree of the end of G giving rise to R, and hence is finite, as desired.

Corollary 1.3 (Seymour & Thomas 1993 [31, (1.5)]). A graph contains no subdivision of  $T_2$  if and only if it admits a tree-decomposition  $(T, \mathcal{V})$  into finite parts such that for every ray R in T we have  $\liminf_{e \in E(R)} |V_e| < \infty$  and T contains no subdivision of the binary tree  $T_2$ .

*Proof.* Again, we only prove the hard implication. Assume that G is a graph without a subdivision of  $T_2$ . In particular, G contains no half-grid minor. As above, G has finite tree-width and every end of G has finite combined degree. Let  $(T, \mathcal{V})$  be the tree-decomposition into finite parts from Theorem 2. Then  $\liminf_{e \in E(R)} |V_e|$  equals the combined degree of the end of G giving rise to R, and hence is finite.

We complete the proof by showing that T contains no subdivided binary tree. Recall Jung's characterisation that a graph G contains no end-injective<sup>5</sup> subdivided  $T_2$  if and only if the end space of G is scattered<sup>6</sup> [19, §3]. Hence, if G contains no subdivided binary tree, then its end space is scattered. Since  $(T, \mathcal{V})$  displays the ends of G homeomorphically, we conclude that the end space of T is scattered, so T contains no end-injective subdivided  $T_2$ , by the converse of Jung's result. As all subtrees of trees are end-injective, this yields the desired result.

Note that Theorem 1 shows that one can always require the witnessing tree-decompositions for Corollaries 1.2 and 1.3 to be linked, tight, and componental.

1.3. Lean tree-decompositions. We already mentioned that there exist even stronger versions of the Kříž-Thomas Theorem 1.1. See the articles by Bellenbaum and Diestel [3] and by Erde [12] for a modern treatment of the finite case.

**Theorem 1.4** (Thomas 1990 [34], Kříž and Thomas 1991 [22]). Every (finite or infinite) graph of tree-width < k has a lean tree-decomposition of width < k.

Here, a tree-decomposition of a graph G is *lean* if for every two (not necessarily distinct) nodes  $t_1, t_2 \in T$  and vertex sets  $Z_1 \subseteq V_{t_1}$  and  $Z_2 \subseteq V_{t_2}$  with  $|Z_1| = |Z_2| =: \ell \in \mathbb{N}$ , either G

<sup>&</sup>lt;sup>4</sup> See [30, 31] for details about the 'easy' implications.

 $<sup>{}^{5}</sup>$ A rooted subtree T of a graph G is *end-injective* if distinct rooted rays in T belong to distinct ends of G.

<sup>&</sup>lt;sup>6</sup>The precise definition of 'scattered' is not relevant here, it is only important that 'scattered' is a property of topological spaces and hence preserved under homeomorphisms.

contains  $\ell$  pairwise disjoint  $Z_1$ – $Z_2$  paths or there exists an edge  $e \in t_1Tt_2$  with  $|V_e| < \ell$ . There are two ways in which 'lean' is stronger than 'linked': First, every lean tree-decomposition  $(T, \mathcal{V})$  satisfies that for every two nodes  $s \neq t \in T$  there are min $\{|V_e|: e \in E(sTt)\}$  pairwise disjoint  $V_s$ – $V_t$  paths in G. In a way, this is the unrooted version of the property 'linked' as introduced above which only required this property when s and t are comparable. The other difference between 'linked' and 'lean' is that a bag of a lean tree-decomposition is only large if it is highly connected, which follows by considering the case  $t_1 = t_2$  in the definition of 'lean'.

Now it is natural to ask whether Theorem 1.4 extends to graphs of finite tree-width. Given that the tree-decomposition  $(T, \mathcal{V}_{NT})$  from above is linked but not necessarily lean, the following question appears to be non-trivial:

Does every graph of finite tree-width admit a lean tree-decomposition into finite parts?

Unfortunately, the answer to this question is in the negative. In [2, Example 1] we construct a locally finite and planar graph which does not admit any lean tree-decomposition. In particular, not even graphs without  $K^{\aleph_0}$  minor admit lean tree-decompositions.

Yet, we can extend Theorem 1.4 in the optimal way to graphs without half-grid minor by post-processing the tree-decomposition from Theorem 1 using the finite case of Theorem 1.4.

**Theorem 3.** Every graph G without half-grid minor admits a lean tree-decomposition into finite parts. Moreover, if the tree-width of G is finitely bounded, then the lean tree-decomposition can be chosen to have width  $\operatorname{tw}(G)$ .

The first part of Theorem 3 is a new result, while the 'moreover'-part reobtains the infinite case in Theorem 1.4. We recall that Robertson, Seymour and Thomas's strongest version [30, (12.11)] of Corollary 1.2 yields witnessing tree-decompositions which are additionally linked (even in its unrooted version). An immediate application of our detailed version of Theorem 3 (see Theorem 3' in Section 8) strengthens their result by showing that the witnessing tree-decomposition in Corollary 1.2 may even be chosen to be lean.

1.4. **Displaying all infinities.** A crucial tool for the proof of Theorem 1 are 'critical vertex sets', the second kind of infinities besides ends: A set X of vertices of G is *critical* if there are infinitely many *tight* components of G - X, that is components C of G - X with  $N_G(C) = X$ . Polat [27] already noted that an infinite graph always contains either a ray/end or a critical vertex set.

As an intermediate step in the proof of Theorem 1 we obtain a similar tree-decomposition which 'displays all the infinities' of the underlying graph: A rooted tree-decomposition  $(T, \mathcal{V})$ 

- $\diamond$  displays the critical vertex sets if the map  $t \mapsto V_t$  restricted to the infinite-degree nodes of T whose  $V_t$  is finite is a bijection to the critical vertex sets. In this context, we denote for every critical vertex set X the unique infinite-degree node of T whose bag is X by  $t_X$ .
- $\diamond$  displays the tight components of every critical vertex set cofinitely if it displays the critical vertex sets such that for every critical vertex set X every  $G \uparrow e$  with  $e = t_X t \in T$  with

 $t_X <_T t$  is a tight component of G - X and cofinitely many tight components of G - X are some such  $G \uparrow e$ , then  $(T, \mathcal{V})$ .

♦ displays the infinities if it displays the ends homeomorphically, their combined degree, their dominating vertices, the critical vertex sets and their tight components cofinitely.

A tree-decomposition that cofinitely displays the tight components of every critical vertex set can no longer in general be componental; but we still ensure that it is *cofinally componental*, i.e. along every rooted ray of T there are infinitely many edges e such that  $G \uparrow e$  is connected.

**Theorem 4.** Every graph of finite tree-width admits a linked, tight, cofinally componental, rooted tree-decomposition into finite parts which displays the infinities.

Further, a tree-decomposition  $(T, \mathcal{V})$  efficiently distinguishes all the ends and critical vertex sets if for any pair of ends and/or critical vertex sets one of the adhesion sets of  $(T, \mathcal{V})$  is a size-wise minimal separator for them. Carmesin [4, Theorem 5.12] showed that for every graph there is a nested set of separations which efficiently distinguishes all its ends. Elm and Kurkofka [11, Theorem 1] extended this result by showing that there always exists a nested set of separations which efficiently distinguishes all ends and all critical vertex sets. If one aims not only for a nested set of separations but for a tree-decomposition, then the best current result is that every locally finite graph without half-grid minor admits even a tree-decomposition which distinguishes all its ends efficiently [18, Theorem 1]. In [2, Construction 3.1] we present a planar graph witnessing that this result cannot be extended to graphs without  $K^{\aleph_0}$  minor [2, Lemma 3.2], even if they are locally finite. Yet we extend it from locally finite graphs to arbitrary infinite graphs without half-grid minor:

**Corollary 5.** Every graph G without half-grid minor admits a tree-decomposition of finite adhesion which distinguishes all its ends and critical vertex sets efficiently.

We obtain Corollary 5 as an application of Theorem 3: Our proof of Theorem 3 yields that the obtained tree-decomposition also displays the infinities of G and hence in particular distinguishes them. Since the tree-decomposition is lean, one can conclude that it even does so efficiently, and thus is as desired for Corollary 5.

An equivalent way of stating Corollary 5 is that every graph G without half-grid minor admits a tree-decomposition of finite adhesion which efficiently distinguishes all its 'combinatorially distinguishable infinite tangles'. This fits into a series of results [5, 10, 11, 18] extending the 'tree-of-tangles' theorem from Robertson and Seymour [29, (10.3)] to infinite graphs.

Additionally, the first author [1, Theorem 3] applies Theorem 4 to obtain a 'tangle-tree duality' theorem for infinite graphs<sup>8</sup>, which extends Robertson and Seymour's [29, (4.3) & (5.1)] other fundamental theorem about tangles to infinite graphs.

<sup>&</sup>lt;sup>7</sup>See Section 9 for definitions.

 $<sup>^{8}</sup>$ Every graph with no principal k-tangle not induced by an end admits a tree-decomposition witnessing this.

1.5. **An open problem.** Thomas famously conjectured that the class of countable graphs is well-quasi-ordered under the minor relation [33, (10.3)]. In light of the observation that all known counterexamples to well-quasi-orderings of infinite graphs [9,21,26,32] do not have finite tree-width by the characterisation of finite tree-width graphs (i.e. graphs with normal spanning trees, Theorem 2.2 below) in [25], it might be interesting to also consider the following, even stronger conjecture:

Conjecture 1.5. The graphs of finite tree-width are well-quasi-ordered under the minor relation.

We remark that already the simplest case of the conjecture – namely for the class of infinite graphs where all components are finite – is open.

1.6. How this paper is organised. In Section 2 we give a short introduction into tree-decompositions, ends and critical vertex sets. We show in Section 3 that the tree-decomposition from Theorem 1 already satisfies Theorem 2. Section 4 consists of three parts. First, we give in Section 4.1 a brief overview over the proof of Theorem 1, which includes the statement of Theorems 6 and 7, our two main ingredients to the proof of Theorem 1. In Section 4.2, we state detailed versions of Theorems 1 and 4 which assert further properties of the respective tree-decompositions. In Section 4.3, we then reduce Theorems 1 and 4 to Theorems 6 and 7. We prove Theorem 6 in Section 5. In Section 6, we show some basic behaviour of paths and rays in torsos to prove Theorem 7 in Section 7. Finally, in Section 8, we prove Theorem 3 and Corollary 5. Furthermore, we compare in Appendix A our Theorem 6 to Elm and Kurkofka's [11, Theorem 2].

## §2. Preliminaries

In this section, we gather all concepts needed for the remainder of the paper. We repeat here concepts already defined in the introduction, in the hope that it may be convenient for future reference to have all definitions gathered in one place.

Throughout the paper, graphs may be infinite. The notation and definitions follow [8] unless otherwise specified; in particular,  $\mathbb{N} = \{0, 1, 2, ...\}$ , and we may speak of a vertex  $v \in G$  (rather than  $v \in V(G)$ ), an edge  $e \in G$ , and so on.

In this paper, a tree T often contains a special vertex  $\operatorname{root}(T)$ , its  $\operatorname{root}$ . A rooted tree T has a natural partial order  $\leqslant_T$  on its vertices and edges, which depends on  $r = \operatorname{root}(T)$ : for two  $x,y \in V(T) \cup E(T)$ , we write xTy for the unique  $\subseteq$ -minimal path in T which contains x and y, and we then set  $t \leqslant_T t'$  if  $t \in rTt'$ . In a rooted tree, a  $\operatorname{leaf}$  is any maximal node in the tree-order. The  $\operatorname{down\text{-}closure} [x]$  and  $\operatorname{up\text{-}closure} [x]$  of x in T are  $\{y \in V(T) \mid y \leqslant x\}$  and  $\{y \in V(T) \mid y \geqslant x\}$ , respectively. We write [x] and [x] as shorthand for  $[x] \setminus x$  and  $[x] \setminus x$ .

A rooted tree T in a graph G is *normal* if the endvertices of every T-path in G are  $\leq_{T}$ -comparable.

An *(oriented) separation* of a graph G is a tuple (A, B) of vertex sets A, B of G, its *sides*, such that  $A \cup B = V(G)$  and there are no edges in G joining  $A \setminus B$  and  $B \setminus A$ . Its *separator* is  $A \cap B$ , and its *order* is the size of its separator. Note that if (A, B) is a separation of G, then so is (B, A).

A *star* of separations is a set  $\sigma$  of separations of G such that  $A \subseteq D$  and  $B \subseteq C$  for every two distinct  $(A, B), (C, D) \in \sigma$ . The *interior* of a star  $\sigma$  is  $int(\sigma) := \bigcap_{(A,B) \in \sigma} B$ .

For a star  $\sigma$  of separations of G, we let  $torso(\sigma)$  be the graph arising from  $G[int(\sigma)]$  by making each separator of a separation in  $\sigma$  complete. We call these added edges that lie in  $torso(\sigma)$  but not in G torso edges. We refer to  $torso(\sigma)$  as the torso at  $\sigma$ .

- 2.1. Tree-decompositions. A tree-decomposition of a graph G is a pair  $(T, \mathcal{V})$  that consists of a tree T and a family  $\mathcal{V} = (V_t)_{t \in T}$  of vertex sets of G indexed by the nodes of T and satisfies the following two conditions:
  - (T1)  $G = \bigcup_{t \in T} G[V_t],$
  - (T2) for every vertex  $v \in G$ , the subgraph of T induced by  $\{t \in T \mid v \in V_t\}$  is connected.

The sets  $V_t$  are the *bags* of the tree-decomposition, the induced subgraphs  $G[V_t]$  on the bags are its *parts*, and T is its *decomposition tree*. Whenever a tree-decomposition is introduced as  $(T, \mathcal{V})$  in this paper, we tacitly assume that  $\mathcal{V} = (V_t)_{t \in T}$ . A tree-decomposition  $(T, \mathcal{V})$  is *rooted* if its decomposition tree T is rooted.

If  $(T, \mathcal{V})$  is a tree-decomposition of a graph G, then a tree T' obtained from T by edge-contractions *induces* the tree-decomposition  $(T', \mathcal{V}')$  of G whose bags are  $V'_t = \bigcup_{s \in t} V_s$  for every  $t \in T'$ , where we denote the vertex set of T' as the set of branch sets, that is the  $\subseteq$ -maximal subtrees of T consisting of contracted edges. Whenever  $t \in T'$  is a subtree of T on a single vertex s, we may reference to t by s, as well. If T is a rooted tree, then root(T') is the node of T' containing root(T).

In a tree-decomposition  $(T, \mathcal{V})$  of G every (oriented) edge  $\vec{e} = (t_0, t_1)$  of the decomposition tree T induces a separation of G as follows: Write  $T_0$  for the component of T - e containing  $t_0$  and  $T_1$  for the one containing  $t_1$ . Then  $(\bigcup_{t \in T_0} V_t, \bigcup_{t \in T_1} V_t)$  is a separation of G [8, Lemma 12.3.1]. We say that  $(\bigcup_{t \in T_0} V_t, \bigcup_{t \in T_1} V_t)$  is *induced* by  $\vec{e}$ , or more generally by  $(T, \mathcal{V})$ . Its separator is  $V_e := V_{t_0} \cap V_{t_1}$ , the *adhesion set* of  $(T, \mathcal{V})$  corresponding to e, where e is the undirected edge of T underlying  $\vec{e}$ .

For every node  $t \in T$ , we write  $\sigma_t$  for the set of separations of G induced by the oriented edges  $\vec{e} = (s, t)$  for  $s \in N_T(t)$ . It is easy to see that such  $\sigma_t$  are stars of separations and that their interior is precisely  $V_t$ . We refer to  $torso(\sigma_t)$  as the torso of (T, V) at t.

The *leaf separations* of a tree-decomposition  $(T, \mathcal{V})$  are those separations of G that are induced by the oriented edges (s, t) of T where t is a leaf of T. We refer to this separation also as leaf separation at t.

Let  $(T, \mathcal{V})$  be a rooted tree-decomposition. Given an edge  $e = t_0 t_1$  of T with  $t_0 <_T t_1$ , we abbreviate the sides of its induced separation by  $G \downarrow e := G \left[ \bigcup_{t \in T_0} V_t \right]$  and  $G \uparrow e := G \left[ \bigcup_{t \in T_1} V_t \right]$ . Further, we write  $G \downarrow e := G \downarrow e - V_e$  and  $G \uparrow e := G \uparrow e - V_e$ . For a node  $t \in T$ , we set  $G \downarrow t := G \downarrow e$ ,  $G \downarrow t := G \downarrow e$ ,  $G \uparrow t := G \uparrow e$  and  $G \uparrow t := G \uparrow e$  where e = st is the unique edge with  $s <_T t$ . It is easy to see that  $G \uparrow x \supseteq G \uparrow y$ ,  $G \uparrow x \supseteq G \uparrow y$ ,  $G \downarrow x \subseteq G \downarrow y$  and  $G \downarrow x \subseteq G \downarrow y$  for every two nodes or edges  $x, y \in T$  with  $x \leqslant_T y$ , and also  $\bigcap_{e \in R} G \uparrow e = \emptyset$  for every rooted ray R in T.

A rooted tree-decomposition  $(T, \mathcal{V})$  is componental if  $G \uparrow e$  is connected for every edge  $e \in T$ , and it is cofinally componental if  $G \uparrow e$  is connected for cofinally many edges e of every  $\subseteq$ -maximal  $\leq_T$ -chain in T. It is tight if, for every edge  $e \in T$ , there is a component C of  $G \uparrow e$  with  $N_G(C) = V_e$ , and if additionally all components C of  $G \uparrow e$  satisfy  $N_G(C) = V_e$ , then  $(T, \mathcal{V})$  is fully tight. Every tight, componental, rooted tree-decomposition is fully tight. A rooted tree-decomposition  $(T, \mathcal{V})$  is linked if for every two edges  $e \leq_T e'$  of T, there is an edge  $f \in T$  with  $e \leq_T f \leq_T e'$  and a family  $\{P_v \mid v \in V_f\}$  of pairwise disjoint  $V_e - V_{e'}$  paths, or equivalently  $(G \downarrow e) - (G \uparrow e')$  paths, in G such that  $v \in P_v$ . In particular, the size of the family of pairwise disjoint paths equals the size of  $V_f$ . Given a set X of vertices of G, the rooted tree-decomposition  $(T, \mathcal{V})$  of G is X-linked if  $X \subseteq V_{\text{root}(T)}$  and if for every edge  $e \in T$  there exists an edge  $f \in_T e$  and a family  $\{P_v \mid v \in V_f\}$  of pairwise disjoint  $X - V_e$  paths in G such that  $v \in P_v$ .

**Lemma 2.1.** Let  $(T, \mathcal{V})$  be a rooted tree-decomposition of a graph G and  $X \subseteq V(G)$ . Further let  $e, f \in E(T)$ ,  $t \in V(T)$  with  $t \leqslant_T e$  and f is the unique edge in tTe incident with t. If  $V_e = X \subseteq V_t$ ,  $G \uparrow e$  is non-empty and  $G \uparrow f$  is connected, then  $V_s = V_t$  for every node  $s \in tTe$  other than t.

Proof. Suppose for a contradiction that there exists a node  $s \in tTe$  other than t with  $V_s \neq X$ . Let s be a  $\leq_T$ -minimal such node. By (T2),  $X \subseteq V_s$ . Thus, there exists some  $v \in V_s \setminus X$ . Then  $v \in G \uparrow f \setminus G \uparrow e$ , since  $V_e = X$ . Moreover,  $G \uparrow e \subseteq G \uparrow f$ , and  $G \uparrow f$  is connected by assumption. So there is a  $v - G \uparrow e$  path in  $G \uparrow f$ , because  $G \uparrow f$  is non-empty; in particular, this path avoids X, as it avoids  $V_f \supseteq X$ . So it also avoids  $N_G(G \uparrow e) \subseteq X$ , which is a contradiction as  $v \notin G \uparrow e$ .

If all adhesion sets of a tree-decomposition  $(T, \mathcal{V})$  are finite, then  $(T, \mathcal{V})$  has *finite adhesion*. A graph G has tree-width less than k if it admits a tree-decomposition whose parts all have size at most k. If there exists such a minimal  $k \in \mathbb{N}$  we denote it by tw(G), and say it has *finitely bounded tree-width*. We say that a graph has *finite tree-width* if it admits a tree-decomposition into finite parts.

The following result shows that the finite tree-width graphs are essentially the graphs with normal spanning trees; the latter have been heavily investigated and are by now well-understood. In particular, we have Jung's *Normal Spanning Tree Criterion* [19, Satz 6'], that a connected graph G admits a normal spanning tree if its vertex set is a countable union of dispersed sets;

<sup>&</sup>lt;sup>9</sup>These are the ⊂-maximal rooted paths or rays.

here a set of vertices U in G is *dispersed* if for every end  $\varepsilon$  of G there is some finite set  $X \subseteq V(G)$  such that U is disjoint from  $C_G(X, \varepsilon)$  (see the next subsection for a background on ends).

**Theorem 2.2.** A graph has finite tree-width if and only if each of its components admits a normal spanning tree.

Proof. Assume first that each component C of a given graph G admits a normal spanning tree  $T_C$ . This induces a tree-decomposition  $(T_C, \mathcal{V}_C)$  of C into finite parts, given by assigning each vertex  $t \in T_C$  its down-closure  $\lceil t \rceil_{T_C}$  as bag. Consider the tree  $T = \{r\} \sqcup \sqcup_C T_C$  where the new root r is adjacent to each root $(T_C)$ . If we assign to r the empty bag and keep all other bags, then we get a tree-decomposition of G into finite parts as desired.

Conversely, let  $(T, \mathcal{V})$  be a tree-decomposition of a given graph G into finite parts. We may assume that G is connected. Since the parts of  $(T, \mathcal{V})$  are finite, every end  $\varepsilon$  of G gives rise to a rooted ray  $R^{\varepsilon} = v_0, e_0, v_1, e_1, v_2, \ldots$  in T. We claim that the union  $U_i$  of the bags assigned to nodes of the decomposition on a fixed level i is dispersed: Indeed, for every end  $\varepsilon$  of G, the finite adhesion set  $V_{e_i}$  separates  $U_i$  from  $C_G(V_{e_i}, \varepsilon)$ . As  $V(G) = \bigcup_{i \in \mathbb{N}} U_i$ , we are done by Jung's normal spanning tree criterion.

2.2. Ends and tree-decompositions. An *end* of a graph G is an equivalence class of rays in G, where two rays are equivalent if they are joined by infinitely many disjoint paths in G or, equivalently, if for every finite set  $X \subseteq V(G)$  both rays have tails in the same component of G - X. The set of all ends of a graph G is denoted by  $\Omega(G)$ .

For every finite set  $X \subseteq V(G)$  and every end  $\varepsilon$  of G, there is a unique component of G-X which contains a tail of some, or equivalently every, ray in  $\varepsilon$ ; we denote this component by  $C_G(X,\varepsilon)$  and say that  $\varepsilon$  *lives* in  $C_G(X,\varepsilon)$ . We denote by  $\Omega_G(X,\varepsilon)$  all end of G which live in  $C_G(C,\varepsilon)$ . Now the collection of all the  $\Omega_G(X,\varepsilon)$  with finite  $X \subseteq V(G)$  and  $\varepsilon \in \Omega(G)$  form a basis for a topology of  $\Omega(G)$ .

A vertex v of G dominates an end  $\varepsilon$  of G if  $v \in C_G(X, \varepsilon)$  for every finite  $X \subseteq V(G - v)$ . We write  $Dom(\varepsilon)$  for the set of all vertices of G which dominate the end  $\varepsilon$ . For a connected subgraph C of G with finite  $N_G(C)$ , we write Dom(C) for all vertices which dominate some end of G that lives in G. Note that Dom(C) is a subset of  $V(C) \cup N_G(C)$ .

The *degree* of an end  $\varepsilon$  of G is defined as

$$deg(\varepsilon) := \sup\{|\mathcal{R}| \mid \mathcal{R} \text{ is a family of disjoint rays in } \varepsilon\}.$$

Halin [14, Satz 1] showed that this supremum is always attained. Together with the number  $dom(\varepsilon) := |Dom(\varepsilon)|$  of vertices dominating  $\varepsilon$ , this sums up to the *combined degree*  $\Delta_G(\varepsilon) := deg(\varepsilon) + dom(\varepsilon)$  of  $\varepsilon$ . Every end  $\varepsilon$  of a graph with normal spanning tree has countable combined degree: By [8, Lemma 8.2.3]  $deg(\varepsilon)$  is countable, and by [8, Lemma 1.5.5], every vertex in  $Dom(\varepsilon)$  lies on the unique normal ray in  $\varepsilon$ , so  $dom(\varepsilon)$  is countable, too.

For a sequence  $(V_i)_{i\in\mathbb{N}}$  of sets, we set  $\liminf_{i\in\mathbb{N}} V_i := \bigcap_{i\in\mathbb{N}} \bigcup_{j\geqslant i} V_i$ . If the sequence of sets  $V_{e_i}$  is indexed by the edges of a ray  $R = v_0 e_0 v_1 e_1 \dots$  in a graph G, then we also write  $\liminf_{e\in R} V_e$  for  $\liminf_{i\in\mathbb{N}} V_{e_i}$ . If  $(T,\mathcal{V})$  displays the ends of G and additionally has the property that the unique rooted ray R in T which arises from the end  $\varepsilon$  of G satisfies  $\liminf_{e\in R} |V_e| = \Delta_G(\varepsilon)$ , then the tree-decomposition  $(T,\mathcal{V})$  of G displays the combined degrees of every end. If a rooted tree-decomposition  $(T,\mathcal{V})$  of G displays all ends of G and additionally  $\liminf_{e\in R} V_e = \mathrm{Dom}(\varepsilon)$  for every rooted ray R of T and its arising end  $\varepsilon$  of G, then  $(T,\mathcal{V})$  displays the dominating vertices of every end.

Let X and Y be two sets of vertices in a graph G. Then X is *linked to* Y in G if there is a family  $\{R_x \mid x \in X\}$  of pairwise disjoint X-Y paths in G such that  $x \in R_x$ . Let  $\varepsilon$  be an end of G. An  $X-\varepsilon$  ray in G is a ray which starts in X and is contained in  $\varepsilon$ . An  $X-\varepsilon$  path in G is an  $X-\mathrm{Dom}(\varepsilon)$  path in G. A finite set  $S \subseteq V(G)$  is an  $X-\varepsilon$  separator in G if  $C_G(S,\varepsilon)$  does not meet X. The set X is *linked to* an end  $\varepsilon$  of G if there is a family  $\{R_x \mid x \in X\}$  of pairwise disjoint  $X-\varepsilon$  paths and rays such that  $x \in R_x$ . Further, a rooted tree-decomposition  $(T,\mathcal{V})$  is end-linked if for every edge e of T there exists some end  $\varepsilon$  of G which lives in  $G \uparrow e$  and to which  $V_e$  is linked. The following lemma is clear.

**Lemma 2.3.** Every end-linked, rooted tree-decomposition (T, V) is tight.

2.3. Critical vertex sets and tree-decompositions. Given a set X of vertices of a graph G, a component C of G-X is tight at X in G if  $N_G(C)=X$ . By slight abuse of notation, we will refer to such C as tight components of G-X. We write  $\mathcal{C}_X:=\mathcal{C}(G-X)$  for the set of components of G-X and  $\check{\mathcal{C}}_X:=\check{\mathcal{C}}(G-X)\subseteq\mathcal{C}_X$  for the set of all tight components C of G-X. A critical vertex set X of G is a finite set  $X\subseteq V(G)$  such that the set  $\check{\mathcal{C}}_X$  is infinite. We denote by  $\mathrm{crit}(G)$  the set of all critical vertex sets of G. A graph is tough if it has no critical vertex set, or equivalently, if deleting finitely many vertices never leaves infinitely many components.

As two vertices in a critical vertex set can always be joined by a path which avoids any given finite set of other vertices, a greedy argument yields the following.

**Lemma 2.4.** Assume that for a critical vertex set X of a graph G we have two path families  $\mathcal{P}, \mathcal{Q}$  of  $k \in \mathbb{N}$  disjoint Y - X paths and of k disjoint X - Z paths, respectively, for some  $Y, Z \subseteq V(G)$ , such that the paths in  $\mathcal{P} \cup \mathcal{Q}$  are disjoint outside of X. Then there exists a family of k disjoint Y - Z paths in G.

The following theorem was first proved by Polat [27, Theorems 3.3 & 3.8]; we here present a short proof using normal spanning trees.

### **Theorem 2.5.** Every tough, rayless graph is finite.

Proof. Let G be a tough, rayless graph. Since G is rayless, V(G) is trivially dispersed, so every component C of G admits a normal spanning tree  $T_C$  by Jung's normal spanning tree criterion. By normality, for every node  $t \in T_C$  all its successors s are contained in distinct components  $\lfloor s \rfloor$  of  $C - \lceil t \rceil$ . Hence, since G is tough and  $\lceil t \rceil$  is finite,  $T_C$  has no vertices of infinite degree. As every locally finite, rayless tree, such as the  $T_C$ , is finite [8, Proposition 8.2.1], the components C of G are finite as well. Since G is tough, it has only finitely many components. Hence, G is finite.  $\Box$ 

A tree-decomposition  $(T, \mathcal{V})$  of a graph G displays the critical vertex sets if the map  $t \mapsto V_t$  restricted to the infinite-degree nodes of T whose  $V_t$  is finite is a bijection to the critical vertex sets of G. For a critical vertex set X, we denote by  $t_X$  the unique infinite-degree node t with  $V_t = X$ . If a rooted tree-decomposition  $(T, \mathcal{V})$  not only displays the critical vertex sets but also for every critical vertex set X cofinitely many tight components of G - X are  $G \uparrow e$  for some  $e = t_X t \in T$  with  $t_X <_T t$  and every such  $G \uparrow e$  is a tight component of G - X, then  $(T, \mathcal{V})$  displays the tight components of every critical vertex set cofinitely. We remark that if such a  $(T, \mathcal{V})$  is tight, then for every finite proper subset Y of any (possibly infinite) bag  $V_t$  there are at most finitely many edges  $e = ts \in T$  with  $t <_T s$  such that  $V_e = Y$ . A rooted tree-decomposition  $(T, \mathcal{V})$  of G into finite parts displays the infinities of G, if it displays the ends of G homeomorphically, their combined degrees, their dominating vertices, the critical vertex sets and their tight components cofinitely.

- 2.4. Critical vertex sets of torsos. The following lemma, which we will use in the proofs of Theorems 3 and 4, says that the critical vertex sets of a torso are closely related to the critical vertex sets of the underlying graph; in particular, torsos in tough graphs are tough.
- **Lemma 2.6.** Let  $\sigma$  be a star of finite-order separations of a graph G such that for cofinitely many separations  $(A, B) \in \sigma$  the side A contains a tight component of  $G (A \cap B)$ . Then  $crit(torso(\sigma)) \subseteq crit(G)$ . Moreover,
  - (i) If  $X \in \operatorname{crit}(\operatorname{torso}(\sigma))$ , then there are infinitely many tight components of G X which meet  $\operatorname{torso}(\sigma)$ .

(ii) If  $X \subseteq \operatorname{int}(\sigma)$ , then the set  $V(C) \cap \operatorname{int}(\sigma)$  induces a tight component of  $\operatorname{torso}(\sigma) - X$  for cofinitely many  $C \in \check{C}(G - X)$  which meet  $\operatorname{int}(\sigma)$ .

*Proof.* We remark that (i) immediately yields that  $\operatorname{crit}(\operatorname{torso}(\sigma)) \subseteq \operatorname{crit}(G)$ . Let U be the union of the finite separators of those finitely many  $(A, B) \in \sigma$  whose side A does not contain a tight component of  $G - (A \cap B)$ ; in particular, U is finite.

(i): Since U is finite, only finitely many components of  $torso(\sigma) - X$  meet U. For every component C' of  $torso(\sigma) - X$  which avoids U, the subgraph

$$C := C' \cup \bigcup \{G[A] \colon (A, B) \in \sigma, \ V(C') \cap A \neq \emptyset\}$$

is connected by the definition of U. Moreover, since the separators  $A \cap B$  of separations  $(A, B) \in \sigma$  are complete in  $\operatorname{torso}(\sigma)$ , the component C' contains the entire  $(A \cap B) \setminus X$  as soon as it meets  $A \cap B$ . Hence, C is a component of G - X, and by definition it contains no other components of  $\operatorname{torso}(\sigma) - X$  than C'. It thus suffices to show for all infinitely many components C' of  $\operatorname{torso}(\sigma) - X$  that avoid U that the component C of G - X which contains C' is tight. For this, it suffices to prove that whenever there is a torso edge from  $u' \in C'$  to  $v \in X$ , then there also is some edge from C to v in G. By the definition of torso, there is a separation  $(A, B) \in \sigma$  with  $u', v \in A \cap B$ . The side A of (A, B) contains a tight component K of  $G - (A \cap B)$  as C' avoids U; in particular,  $K \subseteq C$  by the definition of C, and K sends an edge to v in G.

(ii): Let  $X \subseteq \operatorname{int}(\sigma)$ . It suffices to show that  $C' := C \cap \operatorname{int}(\sigma)$  is a tight component of  $\operatorname{torso}(\sigma) - X$  for every tight component C of G - X which meets  $\operatorname{int}(\sigma)$  but avoids the finite set U. The definition of torso immediately yields that C' induces a connected subgraph of  $\operatorname{torso}(\sigma) - X$  with  $N_{\operatorname{torso}(\sigma)}(C') \supseteq X$  because  $X \subseteq \operatorname{int}(\sigma)$ . It remains to show that  $N_{\operatorname{torso}(\sigma)}(C') \subseteq X$ . For this it suffices to prove that whenever there is a torso edge from  $u' \in C'$  to  $v \notin C'$  the vertex v is already in X. Let  $(A, B) \in \sigma$  such that  $u', v \in A \cap B$ . As C avoids U, the component C, or equivalently C', only meets separators of separations  $(A, B) \in \sigma$  whose side A contains a tight component of  $G - (A \cap B)$ . Thus, we have  $(A \cap B) \setminus X \subseteq A \setminus X \subseteq C$  as soon as C meets  $A \cap B$ . Since  $v \in (A \cap B)$  but not in C, we thus have  $v \in X$ , as desired.

## §3. Displaying ends

In this short section we show that Theorem 2 follows from Theorem 1, that is, we show that a linked, tight, componental, rooted tree-decomposition into finite parts homeomorphically displays all ends, their dominating vertices and their combined degrees.

The proof is divided into three lemmas. The first follows immediately from [20, Lemma 3.1] applied to the tree-decomposition induced by contracting all edges which violate componental.

**Lemma 3.1.** Let (T, V) be a cofinally componental<sup>10</sup>, rooted tree-decomposition of a graph G which has finite adhesion. Then every rooted ray R of T arises from precisely one end of G. Moreover, if all torsos of (T, V) are rayless, then (T, V) displays all ends of G homeomorphically.

**Lemma 3.2.** Let (T, V) be a tight, componental, rooted tree-decomposition of a graph G which has finite adhesion. Then  $\liminf_{e \in R} V_e = \text{Dom}(\varepsilon)$  for every rooted R of T and the unique end  $\varepsilon$  of G which gives rise to R.

Proof. Since  $(T, \mathcal{V})$  has finite adhesion,  $\liminf_{e \in R} V_e$  contains  $\mathrm{Dom}(\varepsilon)$ . Conversely, let v be a vertex in  $\liminf_{e \in R} V_e$ . Let  $Q \in \varepsilon$  be arbitrary. We aim to construct an infinite v - Q fan in G, which then shows that  $v \in \mathrm{Dom}(\varepsilon)$ . Note that we may recursively find infinitely many pairwise internally disjoint v - Q paths in G if for each finite set  $X \subseteq V(G) \setminus \{v\}$  there is a v - Q path in G avoiding X. As  $(T, \mathcal{V})$  is a tree-decomposition, there is an edge  $e \in R$  for which  $G \uparrow e$  avoids any given finite set  $X \subseteq V(G)$ . Since  $\varepsilon$  gives rise to R, the ray Q has a tail in  $G \uparrow e$ . Thus, we find the desired v - Q path in the connected subgraph  $G \uparrow e$ , as  $(T, \mathcal{V})$  is tight and componental.  $\square$ 

**Lemma 3.3.** Let (T, V) be a linked, rooted tree-decomposition of a graph G which has finite adhesion. Suppose that an end  $\varepsilon$  of G gives rise to a ray R in T which arises from no other end of G and that  $\liminf_{e \in R} V_e = \text{Dom}(\varepsilon)$ . Then  $\liminf_{e \in R} |V_e| = \Delta(\varepsilon)$ .

In particular, if (T, V) displays all ends of G and their dominating vertices, then (T, V) also displays the combined degree of each end of G.

*Proof.* Let  $R = v_0 e_0 v_1 e_1 \dots$  be the unique rooted ray in T which arises from the end  $\varepsilon$ . If  $dom(\varepsilon)$  is infinite, then  $(T, \mathcal{V})$  displays the combined degree of  $\varepsilon$  by assumption. Thus, we may assume  $dom(\varepsilon)$  to be finite. Moving to a tail of R, we may assume that  $Dom(\varepsilon) \subseteq V_e$  for every  $e \in R$ .

From the sequence  $(V_{e_i})_{i\in\mathbb{N}}$  of adhesion sets along R, we extract a subsequence by letting  $i_0 \in \mathbb{N}$  with  $|V_{e_{i_0}}| = \min_{e \in R} |V_e|$  and recursively choosing  $i_{n+1} \in \mathbb{N}$  with  $i_{n+1} > i_n$  and  $|V_{e_{i_{n+1}}}| = \min_{j>i_n} |V_{e_j}|$  for  $n \in \mathbb{N}$ . Write  $S_n := V_{e_{i_n}}$ . Then  $(|S_n|)_{n \in \mathbb{N}}$  is non-decreasing, satisfies  $\liminf_{e \in R} |V_e| = \liminf_{n \in \mathbb{N}} |S_n|$  and  $\liminf_{n \in \mathbb{N}} S_n = \liminf_{e \in R} V_e = \mathrm{Dom}(\varepsilon)$  by definition.

Since  $\liminf_{e\in R} S_n = \operatorname{Dom}(\varepsilon)$ , we may assume that  $S_n \cap S_m \subseteq \operatorname{Dom}(\varepsilon)$  for all  $m \geq n$  by passing to a further subsequence. We also have  $C_G(S_m, \varepsilon) \subseteq C_G(S_n, \varepsilon)$  since  $\varepsilon$  gives rise to R. Moreover,  $\bigcap_{n\in\mathbb{N}} C_G(S_n, \varepsilon) = \emptyset$ , since  $\varepsilon$  gives rise to R and  $(T, \mathcal{V})$  is a tree-decomposition. This implies by [13, Corollary 5.7] that  $\liminf_{e\in R} |V_e| = \liminf_{n\in\mathbb{N}} |S_n| \geq \Delta_G(\varepsilon)$ .

For  $\liminf_{e\in R} |V_e| = \liminf_{n\in \mathbb{N}} |S_n| \leq \Delta_G(\varepsilon)$ , we use that  $(T, \mathcal{V})$  is linked: By the construction of the sequence  $(S_n)_{n\in \mathbb{N}}$ , the linkedness yields  $|S_n|$  disjoint  $S_n - S_{n+1}$  paths in G for every  $n \in \mathbb{N}$ . The resulting rays in G are disjoint and contained in  $\varepsilon$  since  $\varepsilon$  is the unique end of G that gives rise to R, and the resulting trivial paths are by assumption precisely the ones given by  $\mathrm{Dom}(\varepsilon)$ . Moreover, there are  $\liminf_{n\in \mathbb{N}} |S_n|$  many such disjoint rays in  $\varepsilon$  and vertices dominating  $\varepsilon$ , which implies that  $\liminf_{n\in \mathbb{N}} |S_n| \leq \Delta_G(\varepsilon)$ , as desired.

 $<sup>^{10}</sup>$ Our definition of componental agrees with the definition of upwards connected from [20].

Proof of Theorem 2 given Theorem 1. By Lemmas 3.1 to 3.3 the tree-decomposition from Theorem 1 is as desired.  $\Box$ 

### §4. A HIGH-LEVEL PROOF OF THE MAIN RESULT

4.1. A two-step approach to Theorem 1. By Theorem 2.5, every tough, rayless graph is finite. Hence, to arrive at a tree-decomposition into finite parts, we first construct a tree-decomposition  $(T, \mathcal{V})$  whose torsos are tough. Next, for each torso of  $(T, \mathcal{V})$  corresponding to a node  $t \in T$  we construct another tree-decomposition  $(T^t, \mathcal{V}^t)$  with rayless torsos. By refining  $(T, \mathcal{V})$  with all the  $(T^t, \mathcal{V}^t)$ , we obtain a tree-decomposition  $(T', \mathcal{V}')$  into tough and rayless parts, which then must be finite by Theorem 2.5.

But how to guarantee that the resulting tree-decomposition is linked? Lemma 2.4 ensures that if we begin in the above two-step approach with a tree-decomposition  $(T, \mathcal{V})$  whose adhesion sets are all critical vertex sets, and then refine by tree-decompositions  $(T^t, \mathcal{V}^t)$  that are each linked when considered individually, then the arising combined tree-decomposition is again linked.<sup>11</sup>

A little more formally, the first step in our two-step approach is the following theorem:

**Theorem 6.** Every graph of finite tree-width admits a tight, cofinally componental, rooted tree-decomposition whose adhesion sets are critical vertex sets, whose torsos are tough and which displays the critical vertex sets and their tight components cofinitely.

The second step in our two-step approach is formally given by the following theorem:

**Theorem 7.** Every graph of finite tree-width admits a linked, tight, componental, rooted tree-decomposition of finite adhesion whose torsos are rayless.

Our aim in the remainder of this section is to demonstrate formally how Theorem 6 and Theorem 7 can be combined to yield a proof of Theorem 1. The subsequent Sections 5 and 7 are then concerned with proving Theorems 6 and 7 respectively.

4.2. **Detailed versions of the main theorems.** In fact, we will prove a version of Theorem 1 not as stated in the introduction, but with three additional properties that are important for technical reasons, but which we believe are also of interest in their own right.

**Theorem 1'** (Detailed version of Theorem 1). Every graph G of finite tree-width admits a rooted tree-decomposition  $(T, \mathcal{V})$  into finite parts that is linked, tight, and componental. In particular,  $(T, \mathcal{V})$  displays all the ends of G homeomorphically, their combined degrees and their dominating vertices. Moreover, we may assume that

<sup>&</sup>lt;sup>11</sup>This is not completely true: In order to ensure that the arising tree-decomposition is linked we need a 'refinement version' of Theorem 7 (see Theorem 7') that ensures that the adhesion sets of  $(T, \mathcal{V})$  corresponding to edges incident with a given node  $t \in T$  appear as adhesion sets in  $(T^t, \mathcal{V}^t)$ . But this is only a technical issue which does not add much complexity to the proof of Theorem 7.

- (L1) for every  $e \in E(T)$ , the adhesion set  $V_e$  is either linked to a critical vertex set of G that is included in  $G \uparrow e$  or linked to an end of G that lives in  $G \uparrow e$ , and
- (L2) for every  $e <_T e' \in E(T)$  with  $|V_e| \le |V_{e'}|$ , each vertex of  $V_e \cap V_{e'}$  either dominates some end of G that lives in  $G \uparrow e'$ , or is contained in a critical vertex set of G that is included in  $G \uparrow e'$ .
- (L3) the bags of (T, V) are pairwise distinct.

Before we continue, let us quickly comment on these additional properties: Property (L1) is a minimality condition: Since every end of G lives in an end of  $(T, \mathcal{V})$ , there will be infinitely many edges of T whose adhesion set is linked to that end. Moreover, since all parts of  $(T, \mathcal{V})$  are finite, one can easily see that every critical vertex set will appear as some adhesion set of  $(T, \mathcal{V})$  (cf. Lemma 5.1). So (L1) says that we did not decompose G 'more than necessary'. In particular, every bag at a leaf of  $(T, \mathcal{V})$  will be of the form  $X \cup V(C)$  for a critical vertex set X of G and a finite tight component C of G - X.

Let us now turn to property (L2). Recall that Halin [15, Theorem 2] showed that every locally finite connected graph has a linked ray-decomposition into finite parts with disjoint adhesion sets. In light of this, (L2) describes how close we can come to having 'disjoint adhesion sets' in the general case; [2, Example 5.2] shows that even for locally finite graphs, it may be impossible to get a tree-decomposition with disjoint adhesion sets, so the condition 'with  $|V_e| \leq |V_{e'}|$ ' in (L2) is indeed necessary.

Now as already indicated in the introduction, we first show Theorem 4 and then derive Theorem 1 from it. In fact, we derive the detailed version Theorem 1' from the following detailed version of Theorem 4.

**Theorem 4'** (Detailed version of Theorem 4). Every graph G of finite tree-width admits a fully tight, cofinally componental, linked, rooted tree-decomposition into finite parts which displays the infinities of G and which satisfies (L1) and (L2) from Theorem 1'. Moreover,

- (I1) if  $G \uparrow e$  is disconnected for  $e = st \in E(T)$  with  $s <_T t$ , then  $V_s \supseteq V_t \in \operatorname{crit}(G)$  and  $\deg(t) = \infty$ , and
- (I2) if  $V_t = V_e$  for some node  $t \in T$  and the unique edge  $e = st \in T$  with  $s <_T t$ , then  $\deg(t) = \infty$  and  $V_t \in \operatorname{crit}(G)$ .

We remark that whenever a tree-decomposition  $(T, \mathcal{V})$  displays the critical vertex sets and satisfies (I1), then it is automatically cofinally componental, because if two successive comparable edges  $rs, st \in T$  violate componental, then (I1) yields that  $\deg(t), \deg(s) = \infty$  and  $V_s \supseteq V_t$ , which implies that  $V_s \supseteq V_t$ , since  $(T, \mathcal{V})$  displays the critical vertex sets and hence  $V_s \neq V_t$ . Moreover, every tree-decomposition as in Theorem 4' already 'nearly' satisfies (L3), that is, has 'almost' distinct bags.

**Lemma 4.1.** Let (T, V) be a tight, rooted tree-decomposition of a graph G which displays the critical vertex sets of G and satisfies (I1) and (I2). Then the bags of (T, V) are pairwise distinct, unless they are critical vertex sets, which may appear as at most two bags associated with adjacent nodes of T.

Proof. Suppose there are distinct nodes  $t, s \in T$  such that  $V_t = V_s$ . Then tTs contains from at least one of t, s, say from t, its unique down-edge e. Then (T2) ensures that  $V_e = V_t$ , so by (I2) we have  $V_t \in \operatorname{crit}(G)$  and  $\deg(t) = \infty$ . In particular, since  $(T, \mathcal{V})$  displays the critical vertex sets of G, we have  $\deg(s) \neq \infty$  and hence  $s <_T t$ , again by (I2). Now if  $V_{s'} = V_s$  for the unique neighbour of s in sTt, then  $\deg(s') = \infty$  by (I2), so s' = t because  $(T, \mathcal{V})$  displays the critical vertex sets of G. Hence, we may assume that  $V_{s'} \neq V_s$ , so  $V_{s'} \supsetneq V_s$  by (T2). Then (I1) implies that  $G \uparrow f$  is connected. But then Lemma 2.1 implies that  $V_{s'} = V_t = V_s$ , a contradiction.  $\square$ 

Proof of Theorem 1' given Theorem 4'. Let  $(T', \mathcal{V}')$  be the tree-decomposition obtained from Theorem 4'. Let  $(T, \mathcal{V})$  be the tree-decomposition induced by contracting every edge e of T' whose  $G \uparrow e$  is disconnected. It is immediate from the construction that  $(T, \mathcal{V})$  is componental. Since  $(T', \mathcal{V}')$  is fully tight and satisfies (L1) and (L2) from Theorem 1', so does every tree-decomposition induced by edge-contractions from  $(T', \mathcal{V}')$ . We note that whenever an edge  $e = st \in E(T')$  with  $s <_{T'} t$  has been contracted then no other edge f incident with f has been contracted and for f incident with f has been contracted and for f incident with f has been contracted and for the induced tree-decomposition f incident with f has been contracted the induced tree-decomposition f incident with f has been contracted and for the induced tree-decomposition f incident with f has been contracted the induced tree-decomposition f incident with f has been contracted and for the induced tree-decomposition f incident with f has been contracted and for the induced tree-decomposition f incident with f has been contracted and for the induced tree-decomposition f incident with f has been contracted and for the induced tree-decomposition f incident with f has been contracted and for the induced tree-decomposition f incident with f has been contracted and for the induced tree-decomposition f incident with f has been contracted and for the induced tree-decomposition f incident with f has been contracted and for the induced tree-decomposition f incident with f has been contracted and f incident f in

Moreover,  $(T, \mathcal{V})$  satisfies (L3). Indeed, by Lemma 4.1, we only need to check that every critical vertex set appears as at most one bag of  $(T, \mathcal{V})$ . By Lemma 4.1, every  $X \in \text{crit}(G)$  can appear as at most two bags of  $(T', \mathcal{V}')$ , which then need to be adjacent. So assume  $V_s = V_t = X$  with s being the unique down-neighbour of t. Then by (I2),  $\deg(t) = \infty$ . Since  $(T', \mathcal{V}')$  displays the critical vertex sets and their tight components cofinitely, it follows that  $G^{\uparrow}(st)$  is disconnected. Thus, we have contracted the edge st in the construction of  $(T, \mathcal{V})$ . Hence, also every critical vertex set of G appears as at most one bag of  $(T, \mathcal{V})$ , so its bags are paiwise distinct.

In order to prove Theorem 4', we still follow the promised two-step approach, but need the following detailed versions of Theorems 6 and 7.

**Theorem 6'** (Detailed version of Theorem 6). Let G be a graph of finite tree-width. Then G admits a fully tight, cofinally componental, rooted tree-decomposition  $(T, \mathcal{V})$  whose adhesion sets are critical vertex sets, whose torsos are tough and which displays the critical vertex sets and their tight components cofinitely.

Moreover, it satisfies (I1) and (I2) from Theorem 4'.

In order to state the detailed version of Theorem 7, we need one more definition. A separation (A, B) of a graph G is *left-tight* if some components C of  $G[A \setminus B]$  satisfies  $N_G(C) = A \cap B$ . Moreover, a separation (A, B) of a graph G is *left-fully-tight* if all components C of  $G[A \setminus B]$  satisfy  $N_G(C) = A \cap B$ .

**Theorem 7'** (Detailed version of Theorem 7). Let G be a graph, and let  $\sigma$  be a star of left-well-linked left-fully-tight finite-order separations of G such that  $torso(\sigma)$  has finite tree-width. Further, let  $X \subseteq int(\sigma)$  be a finite set of vertices of G. Then G admits a linked, X-linked, fully tight, rooted tree-decomposition (T, V) of finite adhesion such that

- (R1) its torsos at non-leaves are rayless and its leaf separations are precisely  $\{(B, A) \mid (A, B) \in \sigma\}$ ,
- (R2) for all edges e of T, the adhesion set  $V_e$  is either linked to an end living in  $G \uparrow e$  or linked to a set  $A \cap B \subseteq G \uparrow e$  with  $(A, B) \in \sigma$ ,
- (R3) for every  $e <_T e' \in E(T)$  with  $|V_e| \leq |V_{e'}|$ , either each vertex of  $V_e \cap V_{e'}$  dominates some end of G that lives in  $G \uparrow e'$ , or  $V_e \cap V_{e'}$  is contained in  $A \cap B \subseteq G \uparrow e'$  for some  $(A, B) \in \sigma$ ,
- (R4)  $V_s \supseteq V_e \subsetneq V_t$  for all edges  $e = st \in T$  with  $s <_T t$  and  $s \neq r := \text{root}(T)$ . Moreover, if  $X \subsetneq \text{int}(\sigma)$ , G X is connected and  $N_G(G X) = X$ , then  $X \subsetneq V_r$  and also  $V_r \supseteq V_e \subsetneq V_t$  for all edges  $e = rt \in T$ .

Note that (R4) implies that the bags of  $(T, \mathcal{V})$  are pairwise distinct.

4.3. **Proof of the main result.** According to our two-step approach, we prove Theorem 4' by applying Theorem 7' to the torsos of the tree-decomposition given by Theorem 6'. For this, we need to ensure that all torsos again have finite tree-width:

**Lemma 4.2.** Let G be a graph of finite tree-width, and let G' be obtained from G by adding an edge between  $u, v \in V(G)$  whenever there are infinitely many internally disjoint u-v paths in G. Then every tree-decomposition of G of finite adhesion is also a tree-decomposition of G'.

In particular, if G has finite tree-width, then the torso at a star of separations of G whose separators are critical vertex sets in G has finite tree-width.

Proof. Assume that  $(T, \mathcal{V})$  is a tree-decomposition of G of finite adhesion, and consider any two vertices  $u, v \in V(G)$  with  $uv \in E(G') \setminus E(G)$ . If there exists a bag  $V_t$  containing both u and v, then the edge uv in G' cannot violate that  $(T, \mathcal{V})$  is a tree-decomposition of G'. To find such a bag  $V_t$ , recall that there are infinitely many internally disjoint u-v paths in G. In particular, no finite set of vertices other than u and v separates u and v in G. Since  $(T, \mathcal{V})$  has finite adhesion, u and v must be contained in some bag  $V_t$  of  $(T, \mathcal{V})$ , as desired.

For the 'in particular'-part assume that G has finite tree-width and that  $\sigma$  is a star of separations of G whose separators are critical in G. Let  $(T, \mathcal{V})$  be a tree-decomposition of G into finite parts. Then by the first part and because critical vertex sets are infinitely connected,  $(T, \mathcal{V}')$  with  $V'_t := V_t \cap \operatorname{int}(\sigma)$  is a tree-decomposition of the torso at  $\sigma$  in G, as desired.

Proof of Theorem 4' given Theorem 6' and Theorem 7'. Let  $(T^1, \mathcal{V}^1)$  be the rooted tree-decomposition from Theorem 6' whose adhesion sets are critical vertex sets. In particular,  $(T^1, \mathcal{V}^1)$  displays the critical vertex sets of G and their tight components cofinitely. Moreover, its torsos are tough and it satisfies (I1) from Theorem 4'. Let t be a node of  $T^1$ . We describe how we refine the torso at t in  $T^1$  using Theorem 7':

If t is the root of  $T^1$ , then we set  $G^t := G$ ,  $X_t := \emptyset$  and  $\sigma'_t = \sigma_t$ . Else let  $s \in T^1$  be the (unique) predecessor of t and let  $G^t$  be the graph obtained from  $G \uparrow st$  by adding all edges between vertices of  $V^1_{st}$ . Set  $X_t := V^1_{st}$  and  $\sigma'_t := \{(A, B \setminus V(G \downarrow st)) \mid (A_{st}, B_{st}) \neq (A, B) \in \sigma_t\}$ , where  $(A_{st}, B_{st})$  is the separation induced by (s, t).

First, we assume that t is a node of  $T^1$  with  $V_t^1 \in \operatorname{crit}(G)$ ; in particular, all infinite-degree nodes whose corresponding bag is finite are such t, since  $(T^1, \mathcal{V}^1)$  displays the critical vertex sets. Then we set  $(T^t, \mathcal{V}^t)$  to be the tree-decomposition of  $G^t$  whose decomposition tree is a star rooted in its centre t and with bag  $V_t^t := V_t^1$  while its leaf separations are precisely  $((B, A) \mid (A, B) \in \sigma'_t)$ . Note that all adhesion sets of this tree-decomposition  $(T^t, \mathcal{V}^t)$  are  $V_t^1 \in \operatorname{crit}(G)$ . Thus,  $(T^t, \mathcal{V}^t)$  is a rooted tree-decomposition as in the conclusion of Theorem 7'.

Secondly, we now assume that t is a node of  $T^1$  with  $V_t^1 \notin \operatorname{crit}(G)$ . By construction, the torso of  $\sigma_t'$  in  $G^t$  is equal to the torso of  $\sigma_t$  in G; so by Lemma 4.2, this torso has finite treewidth. We claim that  $G^t$ ,  $X_t$  and  $\sigma_t'$  are as required to apply Theorem 7'. For this it suffices that all separations in  $\sigma_t'$  are left-well-linked. Let  $(A,B) \in \sigma_t'$ . Then  $X =: A \cap B$  is some critical vertex set of G, as it is an adhesion set of  $(T^1,\mathcal{V}^1)$ . Since  $(T^1,\mathcal{V}^1)$  displays the critical vertex sets, there is a unique infinite-degree node  $t_X \in T^1$  with  $V_{t_X}^1 = X$ . If infinitely many tight component of G - X are contained in A, then (A,B) is left-well-linked. Thus, it now suffices to show that  $t <_{T^1} t_X$ , as  $(T^1,\mathcal{V}^1)$  cofinitely displays also the tight components of the critical vertex sets. Because  $V_t \notin \operatorname{crit}(G)$ ,  $t \neq t_X$ ; hence, we suppose towards a contradiction that  $t >_{T^1} t_X$ . Now  $G[A \setminus B]$  is in particular non-empty, as  $(T^1,\mathcal{V}^1)$  is fully tight. Also  $G^{\uparrow}f$  is a (tight) component of G - X where f is the unique edge on  $t_X T^1 t$  incident with t, since  $(T^1, \mathcal{V}^1)$  cofinitely displays the tight components of the the critical vertex sets. But then Lemma 2.1 yields  $V_t = X \in \operatorname{crit}(G)$  which contradicts the assumptions on t. All in all, we may now apply Theorem 7' to obtain the rooted tree-decomposition  $(T^t, \mathcal{V}^t)$  of  $G^t$ .

We remark that all these rooted tree-decompositions  $(T^t, \mathcal{V}^t)$  in particular contain  $X_t$  in its root part and have precisely  $((B,A) \mid (A,B) \in \sigma'_t)$  as its leaf separations. To build the desired tree-decomposition of G, we first stick all these tree-decompositions  $(T^t, \mathcal{V}^t)$  together along  $(T^1, \mathcal{V}^1)$ : Formally, the tree  $T^2$  arises from the disjoint union of the trees  $T^t$  by identifying a leaf u of  $T^t$  with the root of  $T^s$  if  $G \uparrow u$  (with respect to  $T^t$ ) is equal to  $G^s - X_s$ ; we say that the edge of  $T^t$  (and hence of  $T^2$ ) incident with the leaf u belongs to  $T^1$  and that it corresponds to  $t^t$ . All edges of  $t^t$  that do not belong to  $t^t$  are said to belong to  $t^t$ . We remark that every edge belongs to precisely one of the  $t^t$  or  $t^t$ . We set the root of  $t^t$  to be root $t^t$  such that  $t^t$  is either a non-leaf of  $t^t$ 

or the unique node of  $T^t$ . We say that s belongs to  $T^t$ . Now we claim that  $(T^2, \mathcal{V}^2)$  has all the desired properties.

Let us first note that the construction of  $T^2$  immediately ensures that

- $\diamond$  if  $e \in T^2$  belongs to  $T^t$  for some  $t \in T^1$ , then  $G \uparrow e = G^t \uparrow e$  and  $V_e^2 = V_e^t$ , and
- $\diamond$  if  $e \in T^2$  belongs to  $T^1$ , then  $G \uparrow e = G \uparrow f$  and  $V_e^2 = V_f^1$  for the edge  $f \in T^1$  which e corresponds to.

Thus,  $(T^2, \mathcal{V}^2)$  is fully tight, since  $(T^1, \mathcal{V}^1)$  and all the  $(T^t, \mathcal{V}^t)$  are fully tight. The tree-decomposition  $(T^2, \mathcal{V}^2)$  satisfies (I1) from Theorem 4', since  $(T^1, \mathcal{V}^1)$  and all the  $(T^t, \mathcal{V}^t)$  satisfy (I1). It also satisfies (I2). Indeed, by Theorem 6',  $(T^1, \mathcal{V}^1)$  satisfies (I2). Moreover, by (R4) from Theorem 7', the  $(T^t, \mathcal{V}^t)$  have the property that  $V_x^t \supseteq V_e^t \subsetneq V_y^t$  for all edges  $e = xy \in T^t$  with  $x <_{T^t} y$  and also  $V_e^1 \subsetneq V_{\text{root}(T^s)}^s$  where  $e = st \in T^1$  with  $s <_{T^1} t$ . We remark that we used here that every node t and its unique edge  $e = st \in T^1$  with  $s <_{T^1} t$  to which we applied Theorem 7' satisfies  $X_t = V_{st}^1 \subsetneq V_t^1 = \text{int}(\sigma) = \text{int}(\sigma'_t)$  by (I2) from Theorem 6' and also  $G^t - X_t$  is connected with  $N_G^t(G^t - X) = X$  by (I1) and since  $(T^1, \mathcal{V}^1)$  is fully tight. It follows that  $V_x^2 \supseteq V_e^2 \subsetneq V_y^2$  for all edges e = xy with  $x <_{T^2} y$  except those where  $V_x^2 = V_x^1 = V_y^1 = V_y^2 \in \text{crit}(G)$ . In particular,  $(T^2, \mathcal{V}^2)$  satisfies (I2).

Let us now show that all parts of  $(T^2, \mathcal{V}^2)$  are finite. By Theorem 2.5, it suffices to show that the torso at every  $s \in T^2$  is rayless and tough. Let t be the node of  $T^1$  to whose  $T^t$  the node s belongs. It is immediate from the construction of  $(T^2, \mathcal{V}^2)$  that the torso  $G_s^2$  at  $s \in T^2$  in  $(T^2, \mathcal{V}^2)$  is equal to the torso  $G_s^t$  at s in  $(T^t, \mathcal{V}^t)$ ; in particular, these torsos are rayless by (R1). Suppose for a contradiction that the torso  $G_s^2$  at  $s \in T^2$  in  $(T^2, \mathcal{V}^2)$  contains a critical vertex set X. By Lemma 2.6 (i), infinitely many tight components of G - X meet the torso  $G_s^2$ ; in particular, they meet the torso  $G_t^1$  at  $t \in T^1$  in  $(T^1, \mathcal{V}^1)$ . Now by Lemma 2.6 (ii), cofinitely many of these tight components of G - X restrict to tight components of the torso  $G_t^1$  at t in  $(T^1, \mathcal{V}^1)$ . Thus, X is a critical vertex set of the tough torso  $G_t^1$  which is a contradiction.

Since the adhesion sets of  $(T^1, \mathcal{V}^1)$  are critical vertex sets of G, (L1) and (L2) follow immediately from (R2) and (R3), respectively.

It remains to show that  $(T^2, \mathcal{V}^2)$  is linked. So let  $e \leqslant_{T^2} e'$  be given. Suppose first that there is a node t of  $T^1$  such that all edges in  $eT^2e'$  belong either to  $T^t$  or correspond to edges of  $T^1$  incident with t. The tree-decomposition  $(T^t, \mathcal{V}^t)$  obtained from Theorem 7' is linked and  $X_t$ -linked, where  $X_t = V_{st}^1$  for the predecessor s of t in  $T^1$ . Thus, there exists an edge  $f \in eT^te'$  with  $e \leqslant_{T^t} f \leqslant_{T^t} e'$  and a family  $\mathcal{P}$  of  $k := |V_f^t|$  disjoint  $V_e^t - V_{e'}^t$  paths in the auxiliary graph  $G^t$  such that  $v \in P_v$ . As  $G^t \subseteq G$ , these paths are also paths in G. Since the adhesion sets of  $T^t$  and the tree-order in  $T^t$  directly transfer to  $T^2$  by construction, this completes the first case.

To conclude the proof that  $(T^2, \mathcal{V}^2)$  is linked, let k be the minimum size of an adhesion set  $V_g$  among all edges  $g \in eT^2e'$ . Further, let  $f_1, \ldots, f_\ell$  be the edges on the path  $eT^2e'$  that belong to  $T^1$  ordered by  $\leq_{T^2}$ . To find k disjoint  $V_e^2 - V_{e'}^2$  paths in G, we apply the above argument

to each subpath  $f_iT^2f_{i+1}$  with  $i \in \{1, ..., n-1\}$ . By the choice of k, we get a family  $\mathcal{P}_i$  of k disjoint  $V_{f_i}^2 - V_{f_{i+1}}^2$  paths in G for every  $i \in \{1, ..., n-1\}$ . As  $(T^2, \mathcal{V}^2)$  is a tree-decomposition, we have that for every  $g_1 \leqslant_{T^2} g_2 \leqslant_{T^2} g_3$ ,  $V_{g_2}$  separates  $V_{g_1}^2$  and  $V_{g_3}^2$ . Therefore,  $P_i \in \mathcal{P}_i$  and  $P_j \in \mathcal{P}_j$  are internally disjoint for  $i \neq j$ . We remark that  $V_{f_{i+1}}^2$  is an adhesion set of  $(T^1, \mathcal{V}^1)$  and thus critical in G. Hence, Lemma 2.4 yields the desired path family.

In particular,  $(T^2, \mathcal{V}^2)$  displays all the ends of G homeomorphically, their dominating vertices and their combined degrees by Lemmas 3.1 to 3.3. It is immediate from the construction that  $(T^2, \mathcal{V}^2)$  still displays the critical vertices and their tight components cofinitely.

#### §5. Tree-decomposition along critical vertex sets

In this section we prove Theorem 6', which we restate here for convenience.

**Theorem 6'** (Detailed version of Theorem 6). Let G be a graph of finite tree-width. Then G admits a fully tight, cofinally componental, rooted tree-decomposition  $(T, \mathcal{V})$  whose adhesion sets are critical vertex sets, whose torsos are tough and which displays the critical vertex sets and their tight components cofinitely.

Moreover,

- (I1) if  $G \uparrow e$  is disconnected for  $e = st \in E(T)$  with  $s <_T t$ , then  $V_s \supseteq V_t \in \operatorname{crit}(G)$  and  $\deg(t) = \infty$ , and
- (I2) if  $V_t = V_e$  for some node  $t \in T$  and the unique edge  $e = st \in T$  with  $s <_T t$ , then  $\deg(t) = \infty$  and  $V_t \in \operatorname{crit}(G)$ .

Recall that these (I1) and (I2) are the same properties as (I1) and (I2) in Theorem 4'.

This tree-decomposition is not difficult to construct: Start from the tree-decomposition  $(T, \mathcal{V}'_{NT})$  described in the introduction. By contracting all edges of the decomposition tree whose corresponding adhesion sets are not critical vertex sets of G, one obtains a tree-decomposition of G that satisfies all properties required for the tree-decomposition in Theorem 6' except that it might not display the critical vertex sets. We then describe how one can turn this tree-decomposition into one that additionally displays all critical vertex sets and their tight components cofinitely.

We first collect two lemmas that describe how the critical vertex sets of a graph interact with a tight, componental, rooted tree-decomposition.

**Lemma 5.1.** Let (T, V) be a rooted tree-decomposition of a graph G of finite adhesion. Then every critical vertex set of G is contained in some bag of (T, V).

Moreover, if (T, V) is tight, componental and its torsos are tough, then for every  $X \in \text{crit}(G)$  cofinitely many tight components of G - X are of the form  $G \uparrow e$  for an edge  $e = t_X t \in T$  with  $t_X <_T t$  where  $t_X$  is the (unique)  $\leq_T$ -minimal node of T whose corresponding bag contains X.

*Proof.* Since critical vertex sets are infinitely connected and  $(T, \mathcal{V})$  has finite adhesion, every critical vertex set is contained in some bag of  $(T, \mathcal{V})$ . Thus, the nodes whose corresponding

bags contain a fixed critical vertex set X form a subtree of T by (T2) which thus has a unique  $\leq_T$ -minimal node  $t_X$ . Let  $e_0 <_T t_X$  be the unique edge of T incident with  $t_X$ . Since  $(T, \mathcal{V})$  is componental, every tight component of G - X which does not meet  $G \downarrow e_0$  is of the form  $G \uparrow e$  for an edge  $e = t_X t \in T$  with  $t_X <_T t$ . By the choice of  $t_X$ , every tight component that meets  $G \downarrow e_0$  also meets  $V_{e_0} \subseteq V_{t_X}$ . Lemma 2.6 (ii) yields that only finitely many tight components of G - X meet  $V_{t_X}$ , as  $(T, \mathcal{V})$  is tight and its torsos are tough. Thus, cofinitely many tight components of G - X are of the desired form.

The following lemma is kind of a converse of Lemma 5.1:

**Lemma 5.2.** Let (T, V) be a tight, rooted tree-decomposition of a graph G such that for every  $X \in \operatorname{crit}(G)$  there is a node  $t_X \in T$  such that cofinitely many tight components of G - X are of the form  $G \uparrow e$  for an edge  $e \in t_X t \in T$  with  $t_X <_T t$ . Then all torsos of (T, V) are tough.

Proof. Let  $t \in T$  be arbitrary, and suppose for a contradiction that  $\operatorname{torso}(\sigma_t)$  is not tough, that is, there is some  $X \in \operatorname{crit}(\operatorname{torso}(\sigma_t))$ . Since  $(T, \mathcal{V})$  is tight by assumption, we may apply Lemma 2.6 (i) to the star  $\sigma_t$ , which yields that X is also a critical vertex set of G; moreover, infinitely many tight components of G - X meet  $V_t$ . In particular, we have  $t \neq t_X$  by assumption on  $t_X$ . So let  $e \in T$  be the edge incident with  $t_X$  on the unique  $t - t_X$  path in T, and let  $(A, B) \in \sigma_{t_X}$  be the separation induced by the orientation of e towards e0; in particular, e1. Then by the assumption on e2 at most finitely many tight components of e3.

We now use the previous two lemmas to show that every graph of finite tree-width admits a tree-decomposition which satisfies all properties required for the tree-decomposition in Theorem 6' except that it might not display the critical vertex sets.

**Lemma 5.3.** Every graph of finite tree-width admits a tight, componental, rooted tree-decomposition  $(T, \mathcal{V})$  whose adhesion sets are critical vertex sets and whose torsos are tough. Moreover,

(i)  $V_t \setminus V_e$  is non-empty for every node  $t \in T$  and the unique edge  $e = st \in T$  with  $s <_T t$ .

Proof. We may assume that G is connected. Indeed, if G is not connected, we may apply Lemma 5.3 to every component C of G to obtain a tree-decomposition  $(T^C, \mathcal{V}^C)$ . Let T be the disjoint union of all  $T^C$  and a new node r to which we join the root of every  $T^C$ . We assign  $V_r := \emptyset$  and  $V_t := V_t^C$  for the respective component C of G. Now  $(T, \mathcal{V})$  is as desired.

Since the connected graph G has finite tree-width, we may fix a normal spanning tree T of G by Theorem 2.2. Let  $(T, \mathcal{V})$  ne the tree-decomposition into finite parts introduced unter the name tree-decomposition  $(T, \mathcal{V}'_{NT})$  in the introduction; i.e. the tree-decomposition with decomposition tree T whose bags are given by  $V_t := \{t\} \cup N_G(\lfloor t \rfloor) \subseteq \lceil t \rceil$ . By construction, for an edge  $e = st \in T$  with  $s <_T t$ , we have  $G \uparrow e = \lfloor t \rfloor$  and  $V_e = \lceil s \rceil \cap N_G(\lfloor t \rfloor)$ ; in particular, the rooted tree-decomposition  $(T, (V_t)_{t \in T})$  is componental and tight. Moreover, its bags are all finite and hence tough, so

by Lemma 5.1 there exists for every  $X \in \operatorname{crit}(G)$  a node  $t_X \in T$  such that cofinitely many tight components of G - X are of the form  $G \uparrow e$  for an edge  $e = t_X t \in T$  with  $t_X <_T t$ .

Let  $(T', \mathcal{V}')$  be the tree-decomposition induced by  $(T, \mathcal{V})$  given by contracting all edges  $e \in T$  with  $V_e \notin \operatorname{crit}(G)$ . We claim that  $(T', \mathcal{V}')$  is as desired. It is immediate from the construction that  $(T', \mathcal{V}')$  is still tight and componental and that all its adhesion sets are critical vertex sets of G. So we are left to show that all the torsos of  $(T', \mathcal{V}')$  are tough. To this end, for  $X \in \operatorname{crit}(G)$ , let  $t'_X$  be the node of T' whose branch set in T contains  $t_X$ . Since we only contracted edges of T whose adhesion set is not a critical vertex set of G, we still have that cofinitely many tight components of G - X are of the form  $G \uparrow e$  for an edge  $e = t'_X s \in T'$  with  $t'_X <_{T'} s$ . Hence, we may apply Lemma 5.2, which concludes the proof.

To verify that all bags are distinct, let nodes  $t' \neq s' \in T'$  be given, and pick nodes  $t \in t'$  and  $s \in s'$ . In particular,  $t \in V_t \subseteq V'_{t'}$  and  $s \in V_s \subseteq V'_{s'}$  by the definition of  $(T, \mathcal{V})$  and  $(T', \mathcal{V}')$ . If either  $t \notin V'_{s'}$  or  $s \notin V'_{t'}$ , then  $V'_{t'} \neq V'_{s'}$  and we are done; so suppose otherwise. Then, again by construction, there are nodes  $y \in t'$  and  $x \in s'$  such that  $y \geqslant_T s$  and  $x \geqslant_T t$ . But since the branch sets t' and s' are connected in T and disjoint, this contradicts that T is a tree.

Finally,  $(T, \mathcal{V})$  satisfies (i) by construction, as we have  $t \in V_t \setminus V_e$  for every node  $t \in T$  and its unique edge  $e = st \in T$  with  $s <_T t$ . As  $(T', \mathcal{V}')$  was induced by edge-contractions on T, also  $(T', \mathcal{V}')$  satisfies (i).

Next, we describe how one can turn the tree-decomposition from Lemma 5.3 into one that additionally displays all critical vertex sets and their tight components cofinitely.

Construction 5.4. Let  $(T, \mathcal{V})$  be a rooted tree-decomposition of a graph G of finite adhesion. For every  $X \in \operatorname{crit}(G)$ , let  $t_X$  be the  $\leq_T$ -minimal node t with  $X \subseteq V_t$  (which exists by Lemma 5.1). We obtain the tree T' from T by simultaneously adding for each  $X \in \operatorname{crit}(G)$  the node  $t_X'$ , the edge  $t_X t_X'$  and rerouting each edge  $t_X t$  of T with  $V_{t_X t} = X$  to  $t_X' t$  in T'. The tree-decomposition  $(T', \mathcal{V}')$  is given by  $V_t' := V_t$  for all  $t \in T$  and  $V_{t_X'}' := X$  for every  $X \in \operatorname{crit}(G)$ .

To show that the tree-decomposition  $(T', \mathcal{V}')$  from Construction 5.4 displays the critical vertex sets of G we need the following auxiliary lemma.

**Lemma 5.5.** Let  $(T, \mathcal{V})$  be a tight, rooted tree-decomposition of a graph G such that for every  $X \in \operatorname{crit}(G)$  there is a unique node  $t_X \in T$  with  $V_{t_X} = X$  and cofinitely many tight components of G - X are some  $G \uparrow e$  for an edge  $e = t_X t \in T$  with  $t_X <_T t$ . Then  $(T, \mathcal{V})$  displays all critical vertex sets of G.

Proof. We need to show for every infinite-degree node  $t \in T$  with finite  $V_t$  that  $V_t \in \operatorname{crit}(G)$ . So let  $t \in T$  be a node with finite  $V_t$  and  $V_t \notin \operatorname{crit}(G)$ . Since  $V_t$  is finite, some subset  $X \subseteq V_t$  must be the adhesion set corresponding to infinitely many edges e incident with t in T. Since  $(T, \mathcal{V})$  is tight, X is a critical vertex set of G. By assumption on  $(T, \mathcal{V})$  there is a unique node  $t_X \in T$  with  $V_{t_X} = X$ . If  $t_X = t$  we are done so suppose otherwise. Since by assumption only finitely many

tight components of G - X can meet  $G \downarrow f$  for the unique edge  $f = st_X$  with  $s <_T t_X$ , we have that  $t_X <_T t$ . But then, again by the assumption on  $(T, \mathcal{V})$ , at most finitely many edges e = ts with  $t <_T s$  can contain a tight component of G - X contradicting that  $(T, \mathcal{V})$  is tight.

We can now show that the tree-decomposition  $(T', \mathcal{V}')$  from Construction 5.4 is as desired for Theorem 6' if we start with a tree-decomposition  $(T, \mathcal{V})$  that is tight and componental.

**Lemma 5.6.** Let (T, V) be a tight, componental, rooted tree-decomposition of a graph G of finite adhesion whose torsos are all tough. Then the tree-decomposition (T', V') from Construction 5.4 displays all critical vertex sets and their tight components cofinitely. Additionally,

- (i)  $(T', \mathcal{V}')$  is fully tight,
- (ii)  $(T', \mathcal{V}')$  is cofinally componental; moreover, if  $G \uparrow e$  is disconnected for  $e = st \in E(T')$  with  $s <_{T'} t$  then  $V_s \supseteq V_t \in \operatorname{crit}(G)$  and  $\deg(t) = \infty$ ,
- (iii) the torsos of  $(T', \mathcal{V}')$  are tough.

Proof. Let  $(T', \mathcal{V}')$  be the tree-decomposition from Construction 5.4. Then (i), (ii) and (iii) hold for  $(T', \mathcal{V}')$ . Lemma 5.1 applied to  $(T, \mathcal{V})$  ensures that for  $X \in \operatorname{crit}(G)$  cofinitely many tight components of G - X are of the form  $G \uparrow e$  for an edge  $e = t'_X t \in T'$  with  $t'_X <_{T'} t$ . Thus, by Lemma 5.5,  $(T', \mathcal{V}')$  displays all critical vertex sets of G. Moreover, by construction,  $(T', \mathcal{V}')$  displays the tight components of all critical vertex sets cofinitely.

Proof of Theorem 6'. Apply Construction 5.4 to the tree-decomposition  $(T, \mathcal{V})$  from Lemma 5.3. By Lemma 5.6 this tree-decomposition  $(T', \mathcal{V}')$  is as desired; in particular, it satisfies (I1). Moreover, since  $(T, \mathcal{V})$  satisfies (i) from Lemma 5.3,  $(T, \mathcal{V})$  still satisfies it at all  $t \in T'$  which have not been not added in Construction 5.4, i.e.  $(T', \mathcal{V}')$  satisfies (I2).

We remark that a Theorem similar to Theorem 6' was proven by Elm and Kurkofka [11, Theorem 2]. However, their result cannot be used in place our Theorem 6', as their method in general produces only a nested set of separations, and not a tree-decomposition. We refer the reader to the Appendix A for a detailed discussion of the differences between the two results.

# §6. Lifting paths and rays from torso

From Theorem 6' we obtain a tree-decomposition  $(T, \mathcal{V})$  whose torsos are tough. Following our overall proof strategy for Theorem 1 we want to further decompose each of its torsos torso $(\sigma_t)$  with a tree-decomposition  $(T^t, \mathcal{V}^t)$  which is linked and rayless. We have already indicated in the beginning of Section 4.1 that by using the infinite connectivity of critical vertex sets the arising tree-decomposition  $(T', \mathcal{V}')$  is linked, if  $(T^t, \mathcal{V}^t)$  is linked. For this, we need to extend the path families in torso $(\sigma_t)$  which witness the linkedness of  $(T^t, \mathcal{V}^t)$  to path families in G joining up the same sets of vertices. In this section we show that such an extension of (finite) path families is possible, as long as the separations in the star  $\sigma_t$  at t are 'left-well-linked' (Lemma 6.3). Moreover,

we prove similar results, Proposition 6.1 and Lemma 6.2, for extending rays and infinite path families in torsos at stars whose separations are just left-tight.

We recall that for two sets X, Y of vertices of a graph G, an X-Y path meets X precisely in its first vertex and Y precisely in its last vertex. A path through a set C of vertices in G is a path in G whose internal vertices are contained in C.

**Proposition 6.1.** Let  $\sigma$  be a star of left-tight, finite-order separations of a graph G. Let P be a path or ray in  $torso(\sigma)$ . Then there exists a path or ray P' in G, respectively, with  $P' \cap G[int(\sigma)] \subseteq P$  which starts in the same vertex as P and, if P is a path, also ends in the same vertex as P, and such that P' meets V(P) infinitely often if P is ray.

Proof. Fix for every torso edge e = uv of P a separation  $(A_e, B_e) \in \sigma$  such that both  $u, v \in A_e \cap B_e$ . Since each  $(A_e, B_e)$  is left-tight, we may further fix for every torso edge e = uv of P a u-v path  $P_e$  through  $A_e \setminus B_e$ . Note that since all  $(A, B) \in \sigma$  have finite order, we have  $(A_e, B_e) = (A_f, B_f)$  for at most finitely many torso edges f of P. As the strict left sides  $A \setminus B$  of the separations (A, B) in the star  $\sigma$  are pairwise disjoint, every  $P_e$  meets at most finitely many  $P_f$  for the torso edges f of P.

Then the graph H obtained from  $P \cap G$  by adding all the  $P_e$  for torso edges of P is a connected subgraph of G which contains the startvertex and, if P is a path, the endvertex of P. Moreover, H is locally finite: by construction, all vertices in  $P \cap G$  have degree 1 or 2 in H since P is a path or ray, and all vertices in H - V(P) are contained in at most finitely many  $P_e$  by the argument above, and hence also have finite degree in H. Now if P is a path, then  $H \subseteq G$  contains a path P' whose endvertices are the same as P, and if P is a ray, then  $H \subseteq G$  contains a ray P' that starts in the same vertex as P (cf. [8, Proposition 8.2.1]). In particular, if P is a ray, then P' meets V(P) infinitely often since each component of H - V(P) is finite.

Since  $H \cap G[\operatorname{int}(\sigma)] = P \cap G[\operatorname{int}(\sigma)]$  by construction, and because  $P' \subseteq H$ , it follows that  $P' \cap G[\operatorname{int}(\sigma)] \subseteq P$ , so P' is as desired.

We remark that later in this paper we sometimes want to apply Proposition 6.1 to a ray P in the torso at a star  $\sigma$  of finite-order separations of a graph G in which all but one separation (A, B) are left-tight but we are not interested in keeping the startvertex of P. Hence, we may apply Proposition 6.1 instead to the tail of P which avoids the finite set  $A \cap B$  and the star  $\{(C \cap B, D \cap B) \mid (C, D) \in \sigma \setminus \{(A, B)\}\}$  induced by  $\sigma \setminus \{(A, B)\}$  on G[B]. This still yields a ray P' in G with  $P' \cap G[\operatorname{int}(\sigma)] \subseteq P$  which meets V(P) infinitely often.

**Lemma 6.2.** Let  $\sigma$  be a star of left-tight, finite-order separations of a graph G. Let  $X, Y \subseteq \operatorname{int}(\sigma)$  such that there are infinitely many disjoint X-Y paths in  $\operatorname{torso}(\sigma)$ . Then there are infinitely many X-Y paths in G.

*Proof.* Let  $\mathcal{P}$  be an infinite family of disjoint X-Y paths in  $torso(\sigma)$ . We define an infinite family  $\mathcal{P}'$  of disjoint X-Y paths in G recursively. For this, set  $P'_0 := \emptyset$ , let  $n \in \mathbb{N}$ , and assume that we have already constructed a family  $\mathcal{P}'_n$  of n pairwise disjoint X-Y paths in G.

Then  $V(\mathcal{P}'_n)$  is finite, and hence meets the separators of at most finitely many separations in  $\sigma$ . Since all separations in  $\sigma$  have finite order, and because  $\mathcal{P}$  is an infinite family of disjoint paths, there exists a path  $P \in \mathcal{P}$  that avoids both the finite set  $V(\mathcal{P}'_n)$  and the finitely many finite separators of separations in  $\sigma$  that meet  $V(\mathcal{P}'_n)$ . By Proposition 6.1, there exists a path P' in G with the same endvertices as P and with  $P' \cap G[\operatorname{int}(\sigma)] \subseteq P$ . In particular, P' is an X-Y path in G. Moreover, P' is disjoint from the paths in  $\mathcal{P}'_n$  by the choice of P and because  $P' \cap G[\operatorname{int}(\sigma)] \subseteq P$ . Hence, we may define  $\mathcal{P}'_{n+1} := \mathcal{P}'_n \cup \{P'\}$ .

Then 
$$\mathcal{P}' := \bigcup_{n \in \mathbb{N}} \mathcal{P}'_n$$
 is as desired.

Let us call a finite-order separation (A, B) of a graph G left-well-linked if, for every two disjoint sets  $X, Y \subseteq A \cap B$ , there is a family of min $\{|X|, |Y|\}$  disjoint X-Y paths in G through  $A \setminus B$ .

**Lemma 6.3.** Let  $\sigma$  be a star of left-well-linked, finite-order separations of a graph G. Then the following assertions hold:

- (i) For every countable family  $\mathcal{P}$  of disjoint rays in  $torso(\sigma)$  there exists a family  $\mathcal{P}'$  of disjoint rays in G with the same set of startvertices, and such that each ray in  $\mathcal{P}'$  meets  $V(\mathcal{P})$  infinitely often.
- (ii) For every  $k \in \mathbb{N}$  and every family  $\mathcal{P}$  of k disjoint paths in  $torso(\sigma)$  there exists a family  $\mathcal{P}'$  of k disjoint paths in G with the same set of startvertices and the same set of endvertices.

Proof. We prove (i) and (ii) simultaneously; so let  $\mathcal{P}$  be as in (i) or as in (ii). We define the family  $\mathcal{P}'$  recursively, going through the countably many torso edges on  $\mathcal{P}$ . For this, fix an enumeration  $e_1, e_2, \ldots$  of the torso edges in  $\mathcal{P}$  such that if  $e_i \in P \in \mathcal{P}$ , then all torso edges occurring on P before  $e_i$  are enumerated as  $e_j$  for some j < i. Further, for every  $e_i$ , we fix some  $(A_i, B_i) \in \sigma$  such that both endvertices of  $e_i$  are contained in  $A_i \cap B_i$  where we choose  $(A_i, B_i) = (A_j, B_j)$  for some j < i if possible.

We start the recursion with  $\mathcal{P}_0 := \mathcal{P}$ . At step  $i \in \mathbb{N}$ , we assume that we have already constructed a family  $\mathcal{P}_{i-1}$  of disjoint paths/rays in  $\operatorname{torso}(\sigma) \cup \bigcup_{j < i} G[A_j]$  with the same startvertices, and, if  $\mathcal{P}$  is as in (ii), with the same endvertices, as  $\mathcal{P}$ , and without all torso edges whose endvertices are both contained in  $A_j \cap B_j$  for some j < i. We now consider the torso edge  $e_i$ . If  $e_i$  is not contained in any path/ray in  $\mathcal{P}_{i-1}$ , then we set  $\mathcal{P}_i := \mathcal{P}_{i-1}$ . Otherwise, for each  $P \in \mathcal{P}_{i-1}$ , let  $x_P$  be the first vertex on P such that the subsequent edge on P is a torso edge with both endvertices in  $A_i \cap B_i$ , and let  $y_P$  be the respective last vertex on P such that its previous edge on P is a torso edge with both endvertices in  $A_i \cap B_i$ . Now set  $X_i := \{x_P \mid P \in \mathcal{P}_i\}$  and  $Y_i := \{y_P \mid P \in \mathcal{P}_i\}$ . Since the paths/rays in  $\mathcal{P}_{i-1}$  are disjoint,  $X_i$  and  $Y_i$  are disjoint subsets of  $A_i \cap B_i$  with  $k_i := |X_i| = |Y_i|$ .

We can thus use that  $(A_i, B_i)$  is left-well-linked to find a family  $\mathcal{Q}$  of  $k_i$  disjoint  $X_i - Y_i$  paths in  $G[(A_i \setminus B_i) \cup X_i \cup Y_i]$ ; for  $Q \in \mathcal{Q}$ , write  $x_Q$  for its first and  $y_Q$  for its last vertex.

For each  $P \in \mathcal{P}_i$ , we now define  $P^* := Px_Px_QQy_Qy_{P'}P'$  where Q is the unique path in Q with  $x_Q = x_P$  and P' is the unique path/ray in  $\mathcal{P}'$  with  $y_{P'} = y_Q$ . By construction, the set  $\mathcal{P}_i := \{P^* \mid P \in \mathcal{P}_{i-1}\}$  is a family of disjoint paths/rays in  $\operatorname{torso}(\sigma) \cup \bigcup_{j \leqslant i} G[A_j]$  with the same startvertices, and, if  $\mathcal{P}$  is as in (ii), with the same endvertices, as  $\mathcal{P}_{i-1}$  and thus as  $\mathcal{P}$ . Moreover,  $\mathcal{P}_i$  not only avoids all torso edges  $e_j$  with j < i but also all torso edges whose endvertices are both contained in  $A_j \cap B_j$  for some j < i. This completes step i.

To define  $\mathcal{P}'$  from the  $\mathcal{P}_i$ , let X be the set of startvertices of paths/rays in  $\mathcal{P}$  and let Y be the set of endvertices of paths in  $\mathcal{P}$ . By construction, for each vertex  $x \in X$  there is a (unique) path/ray in  $\mathcal{P}_i$  starting in x and we denote it by  $P_i^x$ . We let  $P^x$  be the limit  $\lim \inf_{i \in \mathbb{N}} P_i^x$  of the  $P_i^x$ , and define  $\mathcal{P}' := \{P^x \mid x \in X\}$ . By construction, all the  $P^x$  are disjoint. Note that if  $\mathcal{P}$  is as in (ii), then  $\mathcal{P}' = \mathcal{P}_n$  for some  $n \in \mathbb{N}$ , since the construction yields  $P_i^x = P_j^x$  for all  $i, j \geq |E(\mathcal{P})|$ ; thus  $\mathcal{P}'$  is as desired for (ii). We now prove that if  $\mathcal{P}$  is as in (i), then all  $P^x$  are indeed rays which meet  $V(\mathcal{P})$  infinitely often.

If an initial segment of some  $P_i^x$  is contained in G, then it contains no torso edge and it hence remains untouched by the above construction in all steps  $j \ge i$ ; in other words, this initial segment of  $P_i^x$  in G is also an initial segment of all  $P_j^x$  with  $j \ge i$ . Moreover, if  $P_i^x$  still contains some torso edge and we let  $e_j$  be the first such one occurring along  $P_i^x$ , then the construction at step j implies that the maximal initial segment of  $P_j^x$  in G is strictly longer than the one in  $P_i^x$ . Moreover, it contains a vertex in  $V(\mathcal{P})$ , the respective  $y_{(P_j^x)'}$  in the above construction step, that has not been contained in the maximal initial segment of  $P_i^x$  in G.

Now if  $P_i^x$  contains no torso edge at some step i, then it is a ray in G starting in x, and thus some tail of  $P_i^x$  equals the tail of some ray in  $\mathcal{P}$  by construction; in particular,  $P_i^x$  meets  $V(\mathcal{P})$  infinitely often. Otherwise, the length of the initial segment of  $P_i^x$  in G strictly increases infinitely often, and hence the limit  $P^x$  of the  $P_i^x$  is a ray in G starting in x that meets infinitely many vertices of  $V(\mathcal{P})$ , which is both witnessed by the respective  $y_{(P_j^x)'}$  described above. Thus,  $\mathcal{P}'$  is as desired if  $\mathcal{P}$  is as in (i). This concludes the proof.

# §7. Linked tree-decompositions into rayless parts

In this section we prove Theorem 7', which we restate here for convenience.

**Theorem 7'** (Detailed version of Theorem 7). Let G be a graph, and let  $\sigma$  be a star of left-well-linked, left-fully-tight, finite-order separations of G such that  $torso(\sigma)$  has finite tree-width. Further, let  $X \subseteq int(\sigma)$  be a prescribed finite set of vertices of G.

Then G admits a linked X-linked, fully tight, rooted tree-decomposition  $(T, \mathcal{V})$  of finite adhesion such that

- (R1) its torsos at non-leaves are rayless and its leaf separations are precisely  $\{(B, A) \mid (A, B) \in \sigma\}$ ,
- (R2) for all edges e of T, the adhesion set  $V_e$  is either linked to an end living in  $G \uparrow e$  or linked to a set  $A \cap B \subseteq G \uparrow e$  with  $(A, B) \in \sigma$ ,
- (R3) for every  $e <_T e' \in E(T)$  with  $|V_e| \leq |V_{e'}|$ , each vertex of  $V_e \cap V_{e'}$  either dominates some end of G that lives in  $G \uparrow e'$  or is contained in  $A \cap B \subseteq V(G \uparrow e')$  for some  $(A, B) \in \sigma$ ,
- (R4) for all edges  $e = st \in T$  with  $s <_T t$  and  $s \neq r := root(T)$  we have  $V_s \supseteq V_e \subseteq V_t$ . Moreover, if  $X \subseteq int(\sigma)$ , G - X is connected and  $N_G(G - X) = X$ , then  $X \subseteq V_r$  and also  $V_r \supseteq V_e \subseteq V_t$  for all edges  $e = rt \in T$ , and
- (R5) if G 
  ightharpoonup et al. is disconnected for some edge  $e \in T$ , then e is incident with a leaf of T.

We remark that in Theorem 7' we explicitly allow the case  $\sigma = \emptyset$ , where we go with the convention that the interior of the empty star is V(G).

First we reduce Theorem 7' to the following statement which differs slightly from Theorem 7'. A separation (A, B) of a graph G is *left-connected* if  $G[A \setminus B]$  is connected, and it is *left-end-linked* if  $A \cap B$  is linked to some end  $\varepsilon$  of G with  $C(A \cap B, \varepsilon) \subseteq A$ .

**Theorem 7.1.** Let G be a graph of finite tree-width, let  $\sigma$  be a star of left-end-linked left-connected finite-order separations of G, and let  $X \subseteq \operatorname{int}(\sigma)$  be a finite set of vertices of G. Assume further that for every  $(A, B) \in \sigma$ , the graph  $G[A \cap B]$  is complete and  $A \cap B \subseteq \operatorname{Dom}(G[A \setminus B])$ .

Then G admits a linked, X-linked, tight, componental, rooted tree-decomposition  $(T, \mathcal{V})$  of finite adhesion such that

- (R1') its torsos at non-leaves are rayless, its leaf separations are precisely  $\{(B,A) \mid (A,B) \in \sigma\}$  and no other separation induced by an edge of T is (B,A) for  $(A,B) \in \sigma$ ,
- (R2')  $(T, \mathcal{V})$  is end-linked,
- (R3') for every  $e <_T e' \in E(T)$  with  $|V_e| \leq |V_{e'}|$  each vertex of  $V_e \cap V_{e'}$  dominates some end of G that lives in  $G \uparrow e'$ , and
- (R4') (T, V) satisfies (R4) from Theorem 7'.

Proof of Theorem 7' given Theorem 7.1. Let H be the graph obtained from  $torso(\sigma)$  by adding, for every separation  $(A, B) \in \sigma$  a disjoint ray  $R_{A,B}$  as well as an edge from every vertex  $v \in A \cap B$  to every vertex of  $R_{A,B}$ . We aim to apply Theorem 7.1 to H with X and the star

$$\sigma' := \{ (V(R_{A,B}) \cup (A \cap B), V(H - R_{A,B})) \mid (A,B) \in \sigma \};$$

but to be able to do so we first have to show that H has finite tree-width.

By assumption in Theorem 7',  $torso(\sigma)$  admits a tree-decomposition  $(T^{\sigma}, \mathcal{V}^{\sigma})$  into finite parts. Any subgraph of H of the form  $H[(A \cap B) \cup V(R_{A,B})]$  clearly also admits such a tree-decomposition  $(T^{A,B}, \mathcal{V}^{A,B})$ . Since  $H[A \cap B]$  is complete for all  $(A,B) \in \sigma$ , it is contained in some part of  $(T^{\sigma}, \mathcal{V}^{\sigma})$  and also it is contained in some part of  $(T^{A,B}, \mathcal{V}^{A,B})$ . We then obtain the desired decomposition tree from the disjoint union of the decomposition trees by adding for each  $(A, B) \in \sigma$  an edge between the nodes corresponding to the respective parts containing  $H[A \cap B]$ . Keeping the parts yields the desired tree-decomposition of H into finite parts.

So by construction and the previous argument, H, X and  $\sigma'$  are as required for Theorem 7.1, which then yields a rooted tree-decomposition  $(T, \mathcal{V}')$  of H. In particular, this  $(T, \mathcal{V}')$  has precisely  $((B,A) \mid (A,B) \in \sigma')$  as its leaf separations. Thus, we obtain a tree-decomposition  $(T,\mathcal{V})$  of G by letting  $V_t := V'_t$  for all non-leaves of T and  $V_t := B$  for all leafs of T whose bag is of the form  $(A \cap B) \cup V(R_{A,B})$ . In particular, its leaf separations are precisely  $\{(B,A) \mid (A,B) \in \sigma\}$ .

We claim that the tree-decomposition  $(T, \mathcal{V})$  of G is as desired. We remark that the adhesion sets corresponding to an edge of the decomposition tree are unchanged. So,  $(T, \mathcal{V})$  still has finite adhesion, as  $(T, \mathcal{V}')$  has finite adhesion. Also,  $(T, \mathcal{V})$  is linked and X-linked: as the  $H[A \cap B]$ are complete, this follows from the (X-)linkedness of  $(T, \mathcal{V}')$  and Lemma 6.3 (ii). Further,  $(T, \mathcal{V})$ is fully tight, since  $(T, \mathcal{V}')$  is fully tight and all separations in  $\sigma$  are left-fully-tight. Also  $(T, \mathcal{V})$ satisfies (R5), since  $(T, \mathcal{V}')$  is componental and the separations in  $\sigma$  are fully tight and no separation of H induced by an edge of T which is not incident with a leaf is (B,A) for some  $(A,B) \in \sigma'$  by (R1'). By construction of  $(T,\mathcal{V})$ , property (R1') of  $(T,\mathcal{V}')$  immediately implies that  $(T, \mathcal{V})$  satisfies (R1). Additionally, for all edges e of T, the adhesion set  $V'_e$  is linked to an end of H by (R2'). If that end corresponds to some ray  $R_{A,B}$ , then  $V_e = V'_e \subseteq \operatorname{int}(\sigma)$  is linked to  $A \cap B \subseteq V(G \uparrow e)$  since  $A \cap B$  separates  $\operatorname{int}(\sigma)$  from  $R_{A,B}$  and because of Lemma 6.3 (ii) as the  $H[A \cap B]$  are complete. Otherwise, Lemma 6.3 (i) and Lemma 6.2 ensure that  $V_e$  is linked to an end of G that lives in  $G \uparrow e$ . Indeed, let  $\mathcal{R}$  be a family of  $|V_e|$  equivalent, disjoint rays in  $torso(\sigma)$  that start in  $V_e$  and that witness that  $V_e$  is end-linked. Then Lemma 6.3 (i) yields a family  $\mathcal{R}'$  of  $|V_e|$  disjoint rays in  $G \uparrow e$  that start in  $V_e$  and that each meet  $V(\mathcal{R})$  infinitely often. In particular, since  $\mathcal{R}$  is finite, for every  $R' \in \mathcal{R}'$  there is  $R \in \mathcal{R}$  such that R' meets V(R)infinitely often. To see that the rays in  $\mathcal{R}'$  are equivalent, let  $R'_0, R'_1 \in \mathcal{R}'$  be given. Since  $R_0$ and  $R_1$  are equivalent in  $torso(\sigma)$ , the infinite sets  $V(R'_0) \cap V(R_0)$  and  $V(R'_1) \cap V(R_1)$  cannot be separated by finitely many vertices. Hence, we may greedily pick infinitely many disjoint  $V(R'_0) \cap V(R_0) - V(R'_1) \cap V(R_1)$  paths in  $torso(\sigma)$ . Since these paths are in fact  $R'_0 - R'_1$  paths in  $torso(\sigma)$ , Lemma 6.2 yields that there are infinitely many disjoint  $R'_0-R'_1$  paths in G, which concludes the proof that  $V_e$  is end-linked in G, and that  $(T, \mathcal{V})$  satisfies (R2).

Also  $(T, \mathcal{V})$  satisfies  $(\mathbf{R3})$  because  $(T, \mathcal{V}')$  satisfies  $(\mathbf{R3}')$ . Finally,  $(T, \mathcal{V})$  satisfies  $(\mathbf{R4})$  since  $(T, \mathcal{V}')$  satisfies  $(\mathbf{R4}')$  too.

Let us briefly sketch the proof of Theorem 7.1. We will construct the tree-decomposition inductively. We start with the trivial tree-decomposition whose decomposition tree is a single vertex whose bag is the whole vertex set of G. In the induction step, we assume that we have already constructed a linked, X-linked, tight, componental, rooted tree-decomposition  $(T^n, \mathcal{V}^n)$  of G of finite adhesion which satisfies  $(\mathbf{R2'})$ ,  $(\mathbf{R3'})$  but it may not satisfy  $(\mathbf{R1'})$ . Instead its torsos at non-leaves are rayless and A is contained in a bag at a leaf of  $T^n$  for every  $(A, B) \in \sigma$ . Then,

for every leaf  $\ell$  of  $T^n$  whose (unique) incident edge e does not induce a separation in  $\sigma$ , we define a set  $Y \subseteq V(G \uparrow e)$  such that the adhesion set  $V_e^n$  is contained in Y. We then replace the bag  $V_\ell^n$  with Y and add for each component C of  $(G \uparrow e) - Y$  a new leaf to  $\ell$  and associate the bag  $V(C) \cup N(C)$  to it. By carefully choosing these sets Y for each such leaf we will ensure that the arising tree-decomposition  $(T^{n+1}, \mathcal{V}^{n+1})$  again satisfies all the properties as assumed for  $(T^n, \mathcal{V}^n)$ . We then show that the pair  $(T, \mathcal{V})$  which arises as the limit for  $n \to \infty$  is indeed a tree-decomposition of G and that it satisfies all the desired properties.

This section is organised as follows. First, we describe in Section 7.1 the Algorithm 7.2 that will construct the bags of the desired tree-decomposition for Theorem 7.1, that is, the sets Y mentioned above. This algorithm is simple to state, but we require some tools to properly analyse it. In Section 7.2 we build a set of tools centred around 'regions', which we then use in Section 7.3 to prove some properties of Algorithm 7.2 that will later on ensure that the resulting tree-decomposition is as desired. Finally, in Section 7.4, we follow the above described approach to construct a tree-decomposition by inductively applying Algorithm 7.2 and then prove with the help of the main result from Section 7.3 that this tree-decomposition is as desired.

7.1. Building the bags of the tree-decomposition. In this section we describe a transfinite recursion, Algorithm 7.2, that will construct the bags of the tree-decomposition for Theorem 7.1. For this, we need the following definitions.

Let G be a graph. A region C of G is a connected subgraph of G. By  $\overline{C}$  we denote the closure  $G[V(C) \cup N_G(C)]$  of C. A k-region for  $k \in \mathbb{N}$  is a region C whose neighbourhood N(C) has size k. A (< k)- or  $(\leq k)$ -region for  $k \in \mathbb{N}$  is a k'-region C for some k' < k or  $k' \leq k$ , respectively. Similarly, an  $(< \aleph_0)$ -region is a k-region for some  $k \in \mathbb{N}$ . Two regions C and D of G touch, if they have a vertex in common or G contains an edge between them. Two regions C and D of a graph G are nested if they do not touch, or  $C \subseteq D$ , or  $D \subseteq C$ . Note that the set of all regions  $G \uparrow e$  given by a rooted componental tree-decomposition of a graph G is nested.

A region C is  $\varepsilon$ -linked for an end  $\varepsilon$  of G if  $\varepsilon$  lives in C and the neighbourhood of C is linked to  $\varepsilon$ . A region C is end-linked if it is  $\varepsilon$ -linked for some end  $\varepsilon$  of G. We emphasise that a region C is  $\varepsilon$ -linked if N(C) is linked to the end  $\varepsilon$  (and not V(C)). To distinguish both cases, we say a region is  $\varepsilon$ -linked and a set of vertices is linked to  $\varepsilon$ .

### **Algorithm 7.2.** (Construction of a bag)

**Input:** a connected graph H; a finite set X of  $k \in \mathbb{N}$  vertices of H; a set  $\mathcal{D}$  of pairwise non-touching end-linked  $(\langle \aleph_0 \rangle)$ -regions D that are disjoint from X and satisfy  $X \cap N_H(D) \subseteq \text{Dom}(D)$ ; a vertex  $x \in V(G) \setminus X$  that lies in no  $D \in \mathcal{D}$ .

**Output:** a transfinite sequence  $C_0, C_1, \ldots, C_i, \ldots$ , indexed by some ordinal  $\langle |H|^+$ , of distinct end-linked ( $\langle \aleph_0 \rangle$ )-regions which are disjoint from X and pairwise nested and which are also nested with all  $D \in \mathcal{D}$ ; a set  $Y := V(H) \setminus (\bigcup_i C_i)$ 

**Recursion:** Iterate the following step:

- Case A: If there is a  $(\langle k \rangle)$ -region  $C_i$  of H that is disjoint from X, is  $\varepsilon$ -linked for some end  $\varepsilon$  which lives in no  $C_j$  for j < i, nested with the  $C_j$  for j < i and with all  $D \in \mathcal{D}$ , then choose a *nicest* such region  $C_i$ . Here, nicest<sup>12</sup> means that
  - (N1)  $C_i$  is such an  $\ell_i$ -region where  $\ell_i \in \mathbb{N}$  is minimum among such regions, and (N2)  $C_i$  is an inclusion-wise maximal such region subject to (N1).
- Case B: If there is no region as in Case A, but there is a  $(<\aleph_0)$ -region  $C_i$  that is disjoint from  $X' := X \cup \{x\}$ , is  $\varepsilon$ -linked for some end  $\varepsilon$  which lives in no  $C_j$  for j < i, nested with the  $C_j$  for j < i and with all  $D \in \mathcal{D}$  such that  $X' \cap N_H(C_i) \subseteq \text{Dom}(C_i)$ , then choose a nicest such region  $C_i$ .
- Case C: If there is no region as in Case A or Case B, then terminate the recursion.

Recall that we will construct the tree-decomposition for Theorem 7.1 recursively, where in each step we replace the leaves of the previous tree-decomposition with stars whose torsos at the centre vertices are rayless. So we can think of the graph H in Algorithm 7.2 as the part  $G \uparrow \ell$  above a leaf  $\ell$  of the decomposition tree  $T^n$  constructed so far and the set X as the adhesion set  $V_e$  that corresponds to the unique edge e incident with that leaf in  $T^n$ . The set Y which Algorithm 7.2 outputs will then be the bag at the centre of the newly added star, and the parts at the new leaves will be the components of H - Y (which will be the  $\supseteq$ -maximal elements of the  $C_i$ , see Theorem 7.9 below) together with their boundaries.

Let us briefly give some intuition on why the set Y is a good candidate for the new bag. Theorem 7.1 requires the torsos of the tree-decomposition to be rayless, so Y should not contain any end of H: for this, Algorithm 7.2 iteratively cuts off all ends of H by choosing  $(<\aleph_0)$ -regions  $C_i$  around them. Additionally, we have to make sure that in the limit of our construction we do end up with a tree-decomposition which in particular must satisfy (T1). The specified vertex x will ensure that we make the appropriate progress when defining the tree-decompositions. In fact, in the final construction of the tree-decomposition we will carefully specify the vertex x in order to ensure that in the end every vertex will lie in some bag of the pair  $(T, \mathcal{V})$  arising as the limit for  $n \to \infty$ . But this is not the only point where we need to be careful. Since the tree-decomposition should be linked, we cannot just choose the regions  $C_i$  arbitrarily; instead, we need to choose those first whose neighbourhood has size less than |X|. This is encoded in Case A, while Case B then will cut off all the remaining ends of G. We will justify these intuitions in Theorem 7.9.

We begin by argueing that Algorithm 7.2 is well-defined: Whenever there exists a region in Case A or Case B, then there obviously exists such a region as in Case A or Case B satisfying (N1), and then Zorn's Lemma and the following lemma (specifically: item (ii)) ensure that there is also such a region satisfying (N2).

<sup>&</sup>lt;sup>12</sup>Note that there might be several nicest regions.

**Lemma 7.3.** Let H be a graph. Let C be a chain of regions of H. Then  $C = \bigcup C$  is a region, and  $C' := \{V(H - C) \cap N_H(C') \mid C' \in C\}$  is a chain with  $N_H(C) = \bigcup C'$ .

Moreover, let  $X \subseteq V(H)$ . Then the following statements hold:

- (i) If all  $C' \in \mathcal{C}$  are disjoint from X, then C is disjoint from X.
- (ii) If, for some  $k \in \mathbb{N}$ , every  $C' \in \mathcal{C}$  is a  $(\leqslant k)$ -region, then C is a  $(\leqslant k)$ -region.
- (iii) If, for every  $C' \in \mathcal{C}$ ,  $X \cap N_H(C') \subseteq \text{Dom}(C')$ , then  $X \cap N_H(C) \subseteq \text{Dom}(C)$ .
- (iv) If  $N_H(C)$  is finite and every  $C' \in C$  is end-linked, then C is end-linked.
- (v) If every  $C' \in \mathcal{C}$  is nested with a region D, then also C is nested with every D.

Proof. Since C is a chain and all the  $C' \in C$  are connected, C is connected. By definition of neighbourhood,  $N_H(C) = V(H - C) \cap \bigcup_{C' \in C} N_H(C')$  and, since C is a chain, the set  $C' = \{V(H - C) \cap N_H(C') \mid C' \in C\}$  forms a chain. This ensures that (ii) and (iii) hold. Next, (i) follows immediately from the definition of C as union of the  $C' \in C$ .

- (iv): If the neighbourhood of C is finite, the fact that C' is a chain with  $\bigcup C' = N_H(C)$  yields that there is a  $C' \in C$  such that  $N_H(C') \supseteq N_H(C)$ . Now, since  $C' \subseteq C$ , the end-linkedness of C' yields the end-linkedness of C.
- (v): Since every  $C' \in \mathcal{C}$  is nested with D, the regions C' and D either do not touch or one is contained in the other. If D is contained in some  $C' \in \mathcal{C}$ , then D is also contained in  $C \supseteq C'$ , as desired. So we may assume that D is contained in no  $C' \in \mathcal{C}$ . Note that whenever  $C'_0 \in \mathcal{C}$  is contained in D or does not touch D, every  $C' \in \mathcal{C}$  with  $C' \subseteq C'_0$  is contained in D or does not touch D, respectively. Thus, either all  $C' \in \mathcal{C}$  are contained in D or they all do not touch D. This yields that their union C either is contained in D or does not touch D, respectively.  $\square$
- 7.2. **Regions.** In this section we collect some statements about regions which we then use in Section 7.3 to analyse the regions  $C_i$  and the set Y which Algorithm 7.2 outputs. Recall that we want Algorithm 7.2 to cut off all ends of H in that every end of H lives in some  $C_i$ . In order to prove that Algorithm 7.2 actually achieves this (at least in the case where every end of H has countable combined degree), we will work in this section towards Lemmas 7.6 and 7.7, which will later on ensure that if at step i of the recursion in Algorithm 7.2 there is still an end of H that does not live in any  $C_j$  for j < i, then there is a region that is a 'candidate for  $C_i$ ' in Case A or Case B. Then Algorithm 7.2 will not terminate as long as there is still some such uncovered end of G that lives in no  $C_i$ .

We say that a region C is a candidate for  $C_i$  if  $C_i$  was chosen in Case A or Case B and C satisfied all properties of the regions in Case A or Case B at step i, respectively, except that C may have not been a nicest such region. We start by showing for that every end of H there is a region  $C_i$  as in in Case A (Lemma 7.4 (i)) or Case B (Lemma 7.4 (ii)) – except that it might not be nested with  $\mathcal{D}$  and with the previously chosen  $C_j$  for j < i.

**Lemma 7.4.** Let H be a graph, let  $X \subseteq V(H)$  be finite, and let  $\varepsilon$  be an end of H with countable combined degree. Then the following statements hold:

- (i) For every  $|\cdot|$ -minimal X- $\varepsilon$  separator S the component of H-S in which  $\varepsilon$  lives is  $\varepsilon$ -linked and disjoint from X.
- (ii) There is an  $\varepsilon$ -linked ( $<\aleph_0$ )-region C disjoint from X which satisfies  $X \cap N_H(C) \subseteq \mathrm{Dom}(\varepsilon)$ .

Proof. (i): Since S separates X and  $\varepsilon$ , the component  $C(S,\varepsilon)$  is disjoint from X. It remains to show that C is  $\varepsilon$ -linked. Because  $\varepsilon$  has countable combined degree, [13, Lemma 5.1] yields an  $\varepsilon$ -defining sequence  $(S_n)_{n\in\mathbb{N}}$ , that is a sequence of finite sets  $S_n\subseteq V(G)$  such that  $C(S_n,\varepsilon)\supseteq C(S_{n+1},\varepsilon)$ ,  $S_n\cap S_{n+1}\subseteq \mathrm{Dom}(\varepsilon)$  and  $\bigcap_{n\in\mathbb{N}}C(S_n,\varepsilon)=\emptyset$ . It suffices to find an  $\varepsilon$ -defining sequence  $(S'_n)_{n\in\mathbb{N}}$  with  $S'_0=S$  such that there are  $|S'_n|$  pairwise disjoint  $S'_n-S'_{n+1}$  paths for every  $n\in\mathbb{N}$ .

So set  $S'_0 := S$ . Assume that we have constructed  $S'_0, \ldots, S'_n$  for some  $n \in \mathbb{N}$ . Since the  $C(S_n, \varepsilon)$  are  $\subseteq$ -decreasing and their intersection is empty, we may choose  $N \in \mathbb{N}$  sufficiently large such that  $S'_n \cap C(S_N, \varepsilon) = \emptyset$ . Then we choose  $S'_{n+1}$  as a  $|\cdot|$ -minimal  $S_{N+1}$ - $\varepsilon$  separator; in particular,  $C(S'_n, \varepsilon) \supseteq C(S_N, \varepsilon) \supseteq C(S_{N+1}, \varepsilon) \supseteq C(S'_{n+1}, \varepsilon)$ , and thus  $S'_n \cap S'_{n+1} \subseteq S_N \cap S_{N+1} \subseteq \mathrm{Dom}(\varepsilon)$ . Note that every  $S'_n$ - $S'_{n+1}$  separator would have been a suitable choice for  $S'_n$  and thus has size at least  $|S'_n|$ . By Menger's Theorem (see for example [8, Proposition 8.4.1]), the desired paths exist. (ii): [13, Lemma 5.1] yields a region C' disjoint from X in which  $\varepsilon$  lives and whose finite

neighbourhood N(C') shares with X only vertices in  $Dom(\varepsilon)$ . Now applying (i) to a  $|\cdot|$ -minimal N(C')- $\varepsilon$  separator yields the desired  $\varepsilon$ -linked region  $C := C(S, \varepsilon) \subseteq C'$ .

In order to get a candidate for  $C_i$  we will use the region from Lemma 7.4 to obtain one which is additionally nested with  $\mathcal{D}$  and with the  $C_j$  for j < i. For this, we first need one auxiliary lemma. A region C of a graph G is well linked if, for every two disjoint finite  $X, Y \subseteq N_G(C)$ , there is a family of  $\min\{|X|, |Y|\}$  pairwise disjoint X-Y paths in G through C (i.e. all internal vertices contained in C).

**Lemma 7.5.** Let H be a graph and let C be an end-linked region of H. Then C is well linked.

Proof. Let  $X,Y \subseteq N(C)$  be disjoint and finite, and suppose for a contradiction that there is no family of  $\min\{|X|,|Y|\}$  pairwise disjoint X-Y paths through C. By Menger's theorem (see for example [8, Proposition 8.4.1]), there is an X-Y separator S of size less than  $\min\{|X|,|Y|\}$  in  $H[V(C) \cup X \cup Y]$ . Since C is end-linked, there is some end  $\varepsilon$  of H which lives in C and a family  $\{R_v \mid v \in N(C)\}$  of pairwise disjoint  $N(C)-\varepsilon$  paths and rays with  $v \in R_v$ . Since  $|S| < \min\{|X|,|Y|\}$  and the  $R_v$  are pairwise disjoint, there is  $x \in X$  and  $y \in Y$  such that  $R_x$  and  $R_y$  both avoid S. Since  $R_x$  and  $R_y$  are  $N(C)-\varepsilon$  paths or rays, there is some  $R_x-R_y$  path P in C which avoids the finite set S. Hence,  $R_x+P+R_y$  is a connected subgraph of  $H[V(C)\cup X\cup Y]$  which meets X and Y but avoids S. This contradicts the fact that S is an X-Y separator in  $H[V(C)\cup X\cup Y]$ .

Let  $\varepsilon$  be an end of H that does not live in any  $C_j$  for j < i or in any  $D \in \mathcal{D}$  and which can be separated from X by fewer than |X| vertices. Then our next lemma yields an  $\varepsilon$ -linked region which is a candidate for the region  $C_i$  in Case A.

**Lemma 7.6.** Let H be a graph and let  $\mathcal{E}$  be a set of pairwise non-touching well-linked  $(\langle \aleph_0 \rangle)$ -regions.

If C is a k-region with  $k \in \mathbb{N}$  which is  $\varepsilon$ -linked for some end  $\varepsilon$  that only lives in  $D \in \mathcal{E}$  which are contained in C, then there exists an  $\varepsilon$ -linked ( $\leqslant k$ )-region C' that is nested with all  $D \in \mathcal{E}$ .

Moreover, the set  $\mathcal{E}'$  of regions  $D \in \mathcal{E}$  which are not nested with C is finite, the region C' is not only nested with all  $D \in \mathcal{E}$  but contains D or does not touch D, and C' can be chosen such that  $C' \subseteq C \cup \bigcup_{D \in \mathcal{E}'} D$  and  $N_H(C') \subseteq N_H(C) \cup \bigcup_{D \in \mathcal{E}'} N_H(D)$ .

*Proof.* We first show that  $\mathcal{E}'$  is finite. For this, it suffices to show that each of the pairwise disjoint regions in  $\mathcal{E}'$  meets the finite set N(C). So let  $D \in \mathcal{E}'$  be given. Because D and C are not nested, they touch, that is, the closure  $\bar{C}$  meets D. Moreover, D is not contained in C, so D - C is non-empty. Thus, since D is connected, there is a  $(D - C) - \bar{C}$  path in D. Its endvertex in  $\bar{C}$  is in N(C), as D - C and C are disjoint. Thus,  $D \in \mathcal{D}$  meets N(C).

Let  $\mathcal{E}'_{<}$  consist of all  $D \in \mathcal{D}'$  with  $|V(D) \cap N(C)| < |V(C) \cap N(D)|$ . Now  $C^* := G[V(C) \cup \bigcup_{D \in \mathcal{E}'_{<}} V(D)]$  has neighbourhood  $(N(C) \setminus \bigcup_{D \in \mathcal{E}'_{<}} V(D)) \cup \bigcup_{D \in \mathcal{E}'_{<}} (N(D) \setminus V(\bar{C}))$ , since the  $D \in \mathcal{E}$  are pairwise non-touching. We claim that  $C^*$  is a  $\varepsilon$ -linked  $(\leqslant k)$ -region.

The subgraph  $C^*$  is connected and thus a region since every  $D \in \mathcal{E}' \supseteq \mathcal{E}'_{<}$  touches C. For  $D \in \mathcal{E}'_{<}$ , the set  $V(D) \cap N(C)$ , which separates  $N(D) \setminus V(\bar{C})$  and  $V(C) \cap N(D)$  in  $\bar{D}$ , must have at least size  $|N(D) \setminus V(\bar{C})|$ , since D is well linked by assumption and  $|V(D) \cap N(C)| < |V(C) \cap N(D)|$  by definition of  $\mathcal{E}'_{<}$ . Hence,  $|N(D) \setminus V(\bar{C})| \le |V(D) \cap N(C)| < |V(C) \cap N(D)|$  for every  $D \in \mathcal{E}'_{<}$  yields that the size of the neigbourhood of  $C^*$  is at most k and a family  $\mathcal{P}_D$  of  $|N(D) \setminus V(\bar{C})|$  pairwise disjoint  $(N(D) \setminus V(\bar{C})) - (V(C) \cap N(D))$  paths through D. We claim that  $C^*$  is  $\varepsilon$ -linked. Indeed, since the  $D \in \mathcal{E}$  are pairwise non-touching, all these paths in  $\mathcal{P}_D$  with  $D \in \mathcal{E}$  and the trivial paths in  $N(C) \setminus (\bigcup_{D \in \mathcal{E}'_{<}} V(D))$  are pairwise disjoint. Hence, we obtain the desired  $N(C^*) - \varepsilon$  paths and rays by extending the collection of all these paths and the trivial paths in  $N(C) \setminus (\bigcup_{D \in \mathcal{E}'_{<}} V(D))$  via the original  $N(C) - \varepsilon$  paths or rays witnessing the  $\varepsilon$ -linkedness of C.

By definition of  $C^*$  and since the  $D \in \mathcal{E}$  are pairwise non-touching, we have  $C^* \cap \bar{D} = C \cap \bar{D}$ ,  $V(C^*) \cap N(D) = V(C) \cap N(D)$  and  $N(C^*) \cap V(D) = N(C) \cap V(D)$  for every  $D \in \mathcal{E} \setminus \mathcal{E}'_{<}$ . Hence, the regions  $D \in \mathcal{E}$  which are not nested with  $C^*$  are precisely those in  $\mathcal{E}^* := \mathcal{E}' \setminus \mathcal{E}'_{<}$  and every  $D \in \mathcal{E}^*$  satisfies  $|V(D) \cap N(C^*)| = |V(D) \cap N(C)| \geqslant |V(C) \cap N(D)| = |V(C^*) \cap N(D)|$ . Hence, the size of

$$B := \left( N(C^*) \setminus \left( \bigcup_{D \in \mathcal{E}^*} V(D) \right) \right) \cup \bigcup_{D \in \mathcal{E}^*} \left( V(C^*) \cap N(D) \right),$$

which includes the neighbourhood of  $C^* - (\bigcup_{D \in \mathcal{E}^*}) \overline{D}$ , is at most  $|N(C^*)| \leq k$ . Since  $\varepsilon$  lives only in  $D \in \mathcal{E}$  which are contained in  $C \subseteq C^*$  and thus such  $D \notin \mathcal{E}' \supseteq \mathcal{E}^*$ , the end  $\varepsilon$  lives

in  $C^* - (\bigcup_{D \in \mathcal{E}^*} \bar{D})$ . So the  $\varepsilon$ -linkedness of  $C^*$  and  $|B| \leq |N(C^*)|$  yields that B is linked to  $\varepsilon$ . All in all, the component  $C' := C_H(B, \varepsilon)$  is the desired  $\varepsilon$ -linked  $(\leq k)$ -region with  $C' \subseteq (C \cup \bigcup_{D \in \mathcal{E}'} D) - (\bigcup_{D \in \mathcal{E}^*} \bar{D}) \subseteq C \cup \bigcup_{D \in \mathcal{E}'} D$  and  $N(C') = B \subseteq N(C) \cup \bigcup_{D \in \mathcal{E}'} N(D)$ . Note that by definition C' does not touch  $D \in \mathcal{E}^*$ . Since every other  $D \in \mathcal{E} \setminus \mathcal{E}^*$  either does not touch C' or is contained in  $C^*$  and also does not touch any  $D' \in \mathcal{E}^*$ , the region D does not touch C' or is contained in C'.

The next lemma yields for every end  $\varepsilon$  that does not already live in some  $C_j$  for j < i or in some  $D \in \mathcal{D}$  an  $\varepsilon$ -linked region which is a candidate for the region  $C_i$  in Case B.

**Lemma 7.7.** Let H be a graph, let  $Z \subseteq V(H)$  be finite, and let  $\mathcal{E}$  be a set of pairwise non-touching well-linked  $(\langle \aleph_0 \rangle)$ -regions of H.

If  $\varepsilon$  is an end of H of countable combined degree that lives in no  $D \in \mathcal{E}$ , then there exists an  $\varepsilon$ -linked  $(\langle \aleph_0 \rangle)$ -region C that is disjoint from Z, satisfies  $Z \cap N_H(C) \subseteq \text{Dom}(\varepsilon)$  and is nested with all  $D \in \mathcal{E}$ . Moreover, every region in  $\mathcal{E}$  that touches C is contained in C.

Proof. By Lemma 7.4 (ii), there exists an  $\varepsilon$ -linked ( $<\aleph_0$ )-region  $C_0$  of H that is disjoint from Z and satisfies  $Z \cap N(C_0) \subseteq \text{Dom}(\varepsilon)$ . Let  $\mathcal{E}'$  be the set of all regions in  $\mathcal{E}$  that are not nested with  $C_0$ . The 'moreover'-part of Lemma 7.6 ensures that  $\mathcal{E}'$  is finite. Thus, Lemma 7.4 (ii) applied to the finite set  $Z' := Z \cup N(C_0) \cup \bigcup_{D \in \mathcal{E}'} N(D)$  and the end  $\varepsilon$  yields an  $\varepsilon$ -linked region  $C_1$  that is disjoint from Z' and satisfies  $Z \cap N(C_1) \subseteq \text{Dom}(\varepsilon)$ . Since  $\varepsilon$  lives in no  $D \in \mathcal{E} \supseteq \mathcal{E}'$ , the fact that  $C_1$  is disjoint from Z' yields that  $C_1 \subseteq C_0 - (\bigcup_{D \in \mathcal{E}'} \overline{D})$ .

Note that, since  $C_1$  is nested with all regions in D that were not nested with  $C_0$ , every  $D \in \mathcal{E}$  which touches  $C_1$  is contained in  $C_0$ . In particular,  $Z \cap N(D) \subseteq Z \cap N(C_0) \subseteq \text{Dom}(\varepsilon)$  for all such  $D \in \mathcal{D}$  as  $C_0$  avoids Z. Now Lemma 7.6 yields a  $\varepsilon$ -linked  $(\langle \aleph_0 \rangle)$ -region  $C_2$  that is disjoint from Z, is nested with  $\mathcal{E}$  and satisfies  $Z \cap N(C_2) \subseteq Z \cap N(C_0) \subseteq \text{Dom}(\varepsilon)$ . Thus  $C := C_2$  is the desired region.

For the 'moreover'-part we note that  $\varepsilon$  lives in C but in no  $D \in \mathcal{E}$ , and thus every  $D \in \mathcal{E}$  that touches C and that is nested with C is contained in C. Hence, the 'moreover'-part follows since C is nested with all  $D \in \mathcal{E}$ .

7.3. Analysis of Algorithm 7.2. In this section we analyse Algorithm 7.2 and provide in Theorem 7.9 the properties of the regions  $C_i$  and the set Y obtained from Algorithm 7.2 which we need for the proof of Theorem 7.1. First let us take note of some basic properties of Algorithm 7.2 that follow easily from its definition:

**Observation 7.8.** In the setting of Algorithm 7.2, we have that

- (O1) every  $C_i$  with  $|N_H(C_i)| < k$  was chosen in Case A and every  $C_i$  with  $|N_H(C_i)| \ge k$  was chosen in Case B,
- (O2) if  $\ell_i \geqslant k$ , then  $X \cap N_H(C_i) \subseteq \text{Dom}(C_i)$ ,

- (O3) the  $\ell_i$  are increasing,
- (O4) a  $C_i$  is never contained in  $C_j$  with j < i,
- (O5) if  $C_j \subseteq C_i$  with j < i, then  $\ell_j < \ell_i$ ,
- (O6) Algorithm 7.2 terminates at some ordinal  $< |H|^+$ .

*Proof.* (O1): It is immediate from Algorithm 7.2 that every ( $\geq k$ )-region was chosen in Case B. Every ( $\leq k$ )-region as in Case B is already a region as in Case A; thus, every (< k)-region  $C_i$  was chosen in Case A.

- (O2): This is immediate from Algorithm 7.2 and (O1).
- (O3): By (O1), every  $C_i$  with  $|N(C_i)| < k$  was chosen in Case A. Moreover, every  $C_i$  that was chosen in Case A was already a candidate for  $C_j$  in earlier steps j < i where  $C_j$  was chosen due to Case A, but was not the chosen nicest such region (and similar for Case B). Thus (N1) ensures that the  $\ell_i$  are increasing.
- (O4): A  $C_i$  will never be a subgraph of any  $C_j$  with j < i since the end to which  $N(C_i)$  is linked lives in  $C_i$  but not in  $C_j$  by the choice of  $C_i$  in Algorithm 7.2.
- (O5): By (O3) we have  $\ell_j \leq \ell_i$ . So suppose for a contradiction that  $\ell_j = \ell_i$ . By Algorithm 7.2,  $C_i$  was already a candidate for  $C_j$ . Since  $\ell_i = \ell_j$ , the region  $C_i$  satisfied (N1) in step j. But then  $C_j = C_i$  by (N2) of  $C_j$ , which contradicts that the end to which  $N(C_i)$  is linked does not live in  $C_j$  by Algorithm 7.2.
- (O6): It suffices to show that  $C_i$  contains some vertex which is in no other  $C_j$  for j < i. If all  $C_j$  for j < i are disjoint from  $C_i$ , then we are done. So we may assume that there is some  $C_j \subseteq C_i$  with j < i. Consider such a  $\subseteq$ -maximal  $C_{j^*}$ ; it exists, since chains of such regions  $C_j \subseteq C_i$  are finite by (O5). If  $N(C_{j^*}) \subseteq N(C_i)$ , then the connectedness of  $C_i$  and  $C_{j^*}$  yields  $C_i = C_{j^*}$  which contradicts their distinctness. Thus, there exists a vertex  $v \in V(C_i) \cap N(C_{j^*})$ . Since the  $C_j$  for j < i are nested and by the  $\subseteq$ -maximality of  $C_{j^*}$ , the region  $C_{j^*}$  does not touch  $C_j$  or contains  $C_j$  for all  $j \neq j^*$  with  $C_j \subseteq C_i$  and j < i; in particular, in both cases  $V(C_j) \cap N(C_{j^*}) = \emptyset$ . Thus,  $v \in C_i \setminus \bigcup_{j < i} C_j$  as desired.

Using the following properties of Algorithm 7.2, we show in Section 7.4 that the iterative application of Algorithm 7.2 yields the desired tree-decomposition for Theorem 7.1:

**Theorem 7.9.** In the setting of Algorithm 7.2 the following statements hold:

- (A1) Either  $x \in Y$  or  $|X| > |N_H(C)|$  for the component C of H Y containing x.
- (A2) Every  $D \in \mathcal{D}$  is contained in a component of H Y.

Moreover, if every end of H has countable combined degree, then also the following hold:

- (A3) Every end of H lives in some  $C_i$ .
- (A4) Every component of H Y is a  $C_i$ .
- (A5) If  $|N_H(C)| \ge k$  for a component C of H Y, then  $X \cap N_H(C) \subseteq \text{Dom}(C)$ .

- (A6) Set  $H^0 := H$  and  $Y^0 := Y$ . Let  $C'_1 \supseteq C'_2 \supseteq \cdots \supseteq C'_n$  be a sequence of regions of H such that, for every  $m \in \{0, \ldots, n-1\}$ , the region  $C'_{m+1}$  is a component of  $H^m Y^m$  where, for  $m \in \{1, \ldots, n-1\}$ ,  $Y^m$  is given by Algorithm 7.2 applied to  $H^m := \bar{C}'_m$ , the finite set  $X^m := N_H(C'_m)$ , the set  $\mathcal{D}^m := \{D \in \mathcal{D} \mid D \subseteq C'_m\}$  and an arbitrary vertex  $x_m \in C'_m (\bigcup \mathcal{D})$ . Then there are  $\min\{|N_H(C'_m)| \mid m \in \{1, \ldots, n\}\}$  pairwise disjoint  $X N_H(C'_n)$  paths in H.
- (A7) If H X is connected and  $N_H(H X) = X$ , then  $N_H(C) \not\subseteq X$  for all components C of H Y; in particular,  $X \subsetneq Y$ .

Proof. (A1): Assume that  $x \notin Y$ . Let  $\mathcal{C}_x$  be the collection of all those  $C_i$  that contain x. Then  $\mathcal{C}_x$  is non-empty since by the definition of Y at least one  $C_i$  contains x. We claim that  $C := \bigcup \mathcal{C}_x$  is a component of H - Y with |N(C)| < k = |X|. Indeed, it is immediate from Algorithm 7.2 that every  $C_i \in \mathcal{C}_x$  is a region as in Case A, since the regions that where chosen in Case B avoid x; in particular, every  $C_i \in \mathcal{C}_x$  is a (< k)-region. Since all the  $C_i \in \mathcal{C}_x$  meet in x and are nested,  $\mathcal{C}_x$  is a chain (with respect to inclusion). Thus, Lemma 7.3 ensures that C is a region of H with |N(C)| < k = |X|.

To finish the proof of (A1), it thus remains to show that C is a component of H - Y. For this, it suffices to prove that  $N(C) \subseteq Y$  since C is a region and hence connected. As  $C = \bigcup C_x$ , we have  $N(C) \subseteq \bigcup \{N(C_i) \mid C_i \in C_x\}$ . Since all the  $C_i$  are nested, every  $C_j$  which meets the neighbourhood of some  $C_i \in C_x$  already contains  $C_i$ ; in particular, such  $C_j$  are also in  $C_x$  as  $x \in C_i \subseteq C_j$ . Thus N(C) meets no  $C_i$ , and so the definition of Y yields that  $N(C) \subseteq Y$ .

(A2): By the assumptions on  $\mathcal{D}$  and the choice of the  $C_i$ , in every step i of Algorithm 7.2 every  $D \in \mathcal{D}$  is a candidate for the region  $C_i$  in Case B as long as D is  $\varepsilon$ -linked for some end  $\varepsilon$  of H which lives in no  $C_j$  for j < i. Again by the assumptions on  $\mathcal{D}$ , the region  $D \in \mathcal{D}$  is  $\varepsilon$ -linked for some end  $\varepsilon$  of H. So, since Algorithm 7.2 terminates because of Case C by Observation 7.8 (O6), the end  $\varepsilon$  lives is some  $C_i$ . As  $\varepsilon$  lives in both  $C_i$  and D, they touch. Thus,  $C_i \subseteq D$  or  $D \subseteq C_i$  since they are nested by Algorithm 7.2. Hence, we finish the proof of (A2) by showing that if  $C_i \subseteq D$ , then  $C_i = D$ . For this, it suffices to show that in step i of Algorithm 7.2, the region D was a candidate for  $C_i$  and satisfies (N1), because then  $C_i \subseteq D$  together with (N2) yields  $C_i = D$ .

It is immediate from Algorithm 7.2 and the assumptions on  $\mathcal{D}$  that in step i, the region  $D \in \mathcal{D}$  is nested with all  $C_j$  for j < i, disjoint from X and that  $X \cap N(D) \subseteq \text{Dom}(D)$ . Since D is  $\varepsilon$ -linked and  $\varepsilon$  lives in  $C_i \subseteq D$ , there are |N(D)| disjoint  $N(D)-N(C_i)$  paths in  $\overline{D}-C_i$ . These paths can be extended to disjoint  $N(D)-\varepsilon'$  paths and rays where  $\varepsilon'$  is the end so that  $C_i$  is  $\varepsilon'$ -linked. In particular, D is end-linked to the end  $\varepsilon'$  which lives in no  $C_j$  for j < i, and  $\ell_i = |N(C_i)| \ge |N(D)|$ . Thus, D was a candidate for  $C_i$  and satisfies (N1), as desired.

(A3): Suppose towards a contradiction that there is an end  $\varepsilon$  of H that lives in no  $C_i$ . For every  $\ell \in \mathbb{N}$ , let  $\mathcal{C}_{\max}^{<\ell}$  be the set of all  $C_i$  with  $\ell_i < \ell$  that are  $\subseteq$ -maximal among all  $(<\ell)$ -regions  $C_j$ . By Observation 7.8 (O5) every  $C_i$  with  $\ell_i < \ell$  is included in some  $C_j \in \mathcal{C}_{\max}^{<\ell}$ . Moreover, since

all  $C_i$  are nested, the regions in  $\mathcal{C}_{\max}^{<\ell}$  are pairwise non-touching. Hence, by Lemma 7.7 applied to  $Z:=X\cup\{x\}$  and the set  $\mathcal{E}$  of all  $\subseteq$ -maximal elements in  $\mathcal{D}\cup\mathcal{C}_{\max}^{< k}$ , there exists a  $\varepsilon$ -linked  $(<\aleph_0)$ -region C that is disjoint from  $X\cup\{x\}$ , satisfies  $(X\cup\{x\})\cap N(C)\subseteq \mathrm{Dom}(\varepsilon)$  and is nested with all  $\subseteq$ -maximal regions in  $\mathcal{D}\cup\mathcal{C}_{\max}^{< k}$ . Set  $\ell:=|N(C)|$ . Then Lemma 7.6 applied to C and the set  $\mathcal{E}$  of all  $\subseteq$ -maximal regions in  $\mathcal{D}\cup\mathcal{C}_{\max}^{< \ell+1}$  yields a  $\varepsilon$ -linked  $(\leqslant\ell)$ -region C' which is nested with all regions in  $\mathcal{D}\cup\mathcal{C}_{\max}^{< \ell+1}$  and such that  $(X\cup\{x\})\cap N(C')\subseteq \mathrm{Dom}(C')$ , where we have used that  $(X\cup\{x\})\cap N(D)\subseteq \mathrm{Dom}(D)$  for all  $D\in\mathcal{C}_{\max}^{< \ell+1}\setminus\mathcal{C}_{\max}^{< k}$  since any such region D was chosen in Case B. Moreover, by Lemma 7.6, C' is not strictly contained in any region in  $\mathcal{D}\cup\mathcal{C}_{\max}^{< \ell+1}$ . In particular, C' is nested with all  $D\in\mathcal{D}$  and all  $C_i$  with  $\ell_i\leqslant\ell$ . But this contradicts Algorithm 7.2: If there is no  $C_i$  with  $\ell_i>\ell$ , then C' contradicts that Algorithm 7.2 terminated because of Case C by Observation 7.8 (O6). Otherwise, if there exists some  $C_i$  with  $\ell_i>\ell$ , then C' witnesses that the first such  $C_i$  did not satisfy (N1).

(A4): Let C be a component of H-Y. By Zorn's Lemma there exists a  $\subseteq$ -maximal index set I such that  $C_j \subseteq C_i \subseteq C$  for all  $j \leqslant i \in I$ . Then  $\bigcup_{i \in I} C_i = C$ . Indeed, if  $C' := \bigcup_{i \in I} C_i \subsetneq C$ , then  $N(C') \cap C \neq \emptyset$  since C is connected. So by the definition of Y, there is some  $C_j$  such that  $N(C') \cap C_j \neq \emptyset$ . In particular,  $C_j \subseteq C$  since  $C_j$  is connected. But since  $N(C') \subseteq \bigcup_{i \in I} N(C_i)$ , the  $(C_i \mid i \in I)$  are  $\subseteq$ -increasing and all the  $C_i$  are nested, we have  $C_i \subseteq C_j$  for all  $i \in I$ , which contradicts that I is  $\subseteq$ -maximal. By Observation 7.8 (O3), the sequence  $(\ell_i)_{i \in I}$  is strictly increasing. In particular, I is either finite or of the same order type as  $\mathbb{N}$ . In the former case we are done as  $C = C_i$  for  $i = \max(I)$ . So assuming the latter, we now aim towards a contradiction. Enumerate  $I = \{i_n : n \in \mathbb{N}\}$  so that  $i_n < i_m$  for all  $n < m \in \mathbb{N}$ .

Consider the auxiliary graph A that arises from C by contracting  $C_{i_1}$  and, for every  $n \in \mathbb{N}$ , all components of  $C_{i_{n+1}} - C_{i_n}$ . The graph A is obviously infinite and connected. Since the  $N(C_{i_n})$  are finite and the  $C_{i_n}$  are connected, there are at most finitely many components of  $C_{i_{n+1}} - C_{i_n}$  for every  $n \in \mathbb{N}$ . Together with the fact that the  $N(C_{i_n})$  are finite this yields that A is also locally finite. Hence, the minor A of C contains a ray by [8, Proposition 8.2.1]. We lift this ray to a ray R in C by choosing suitable paths in each branch set connecting the endvertices of the incident edges. Then by construction, the end of H that contains R lives in no  $C_i$ , contradicting (A3).

(A5): This follows directly from (A4) and (O2).

(A6): Set  $k' := \min\{|N(C'_m)| \mid m \in \{1, \dots, n\}\}$ . By Menger's Theorem (see for example [8, Proposition 8.4.1]), we either find the desired k' pairwise disjoint  $X-N(C'_n)$  paths or there is an  $X-N(C'_n)$  separator S with |S| < k'. We may assume the latter and let S be a  $|\cdot|$ -minimal such separator. Further, let  $\varepsilon$  be an end such that  $C'_n$  is  $\varepsilon$ -linked, which exists by Algorithm 7.2 and (A4). Since  $C'_n$  is  $\varepsilon$ -linked, the separator S is also a  $|\cdot|$ -minimal  $X-\varepsilon$  separator. Then the component  $\tilde{C}$  of H-S in which  $\varepsilon$  lives is  $\varepsilon$ -linked and disjoint from X by Lemma 7.4 (i). Moreover, we have  $C'_n \subseteq \tilde{C}$  since S avoids  $C'_n$  by its choice as a minimal  $X-N(C'_n)$  separator.

Now let C be a  $\varepsilon$ -linked ( $\leq |S|$ )-region that is disjoint from X, contains  $C'_n$  and is contained in as many of the  $C'_i$  as possible. Note that  $\tilde{C}$  is a candidate for C. Let  $N \geq 0$  be the largest index i

such that  $C \subseteq C_N'$ , and let  $C_i^N$  be the regions obtained from the application of Algorithm 7.2 to  $H^N$  and  $N(C_N')$  (or H and X if N=0). Note that N< n since otherwise we have  $C=C_n'$  and hence  $|N(C_n')|=|N(C)|< k'$ , a contradiction. Let  $\mathcal{C}_{\max}^{\leqslant |S|}$  be the  $\subseteq$ -maximal elements of the  $C_i^N$  with  $\ell_i^N\leqslant |S|$ . Note that by Observation 7.8 (O5) every  $C_i^N$  with  $\ell_i^N<|S|$  is contained in some  $C_j^N\in\mathcal{C}_{\max}^{\leqslant |S|}$ . Since all regions in  $\mathcal{C}_{\max}^{\leqslant |S|}\cup\mathcal{D}$  are end-linked, they are well linked by Lemma 7.5. Moreover, if  $\varepsilon$  lives in  $D\in\mathcal{D}$ , then  $D\subseteq C_n'\subseteq C$  by (A2). To be able to apply Lemma 7.6 to C and the set  $\mathcal{E}$  of  $\subseteq$ -maximal regions in  $\mathcal{C}_{\max}^{\leqslant |S|}\cup\mathcal{D}$  it suffices that  $\varepsilon$  lives in no  $C_j^N$  with  $\ell_j^N\leqslant |S|$ .

So suppose for a contradiction that  $\varepsilon$  lives in some  $C_j^N$  with  $\ell_j^N\leqslant |S|$ . We claim that then  $C''':=C_j^N\cup C'_n$  is an  $(\leqslant |S|)$ -region. Indeed, C'' is a region because  $C_j^N$  and  $C'_n$  are both connected and intersect as  $\varepsilon$  lives in both of them. Moreover,  $N(C''')=(N(C_j^N)\cup N(C'_n))\setminus V(C'')=(N(C_j^N)\setminus V(C'_n))\dot\cup (N(C'_n)\setminus V(\bar C_j^N))$  has size at most  $|N(C_j^N)|=\ell_j^N\leqslant |S|$ : if not, one easily checks by doubling counting that  $Z:=((N(C_j^N)\cup N(C'_n))\cap V(C''))\cup (N(C_j^N)\cap N(C'_n))=(N(C'_n)\cap V(C_j^N))\dot\cup (N(C_j^N)\cap V(C'_n))\dot\cup (N(C_j^N)\cap N(C'_n))$  has size less than  $|N(C'_n)|=(N(C'_n)\cap V(C_j^N))\dot\cup (N(C_j^N)\cap V(C'_n))\dot\cup (N(C_j^N)\cap N(C'_n))$  has size less than  $|N(C'_n)|=(N(C'_n)\cap V(C'_n))\dot\cup (N(C'_n)\cap V(C'_n))\dot\cup (N(C'_n)\cap N(C'_n))$  has size less than  $|N(C'_n)|=(N(C'_n)\cap V(C'_n))$  has size less than  $|N(C'_n)|=(N(C'$ 

So we may apply Lemma 7.6 to C and the set  $\mathcal{E}$  of  $\subseteq$ -maximal separations in  $\mathcal{C}_{\max}^{\leqslant |S|} \cup \mathcal{D}$  to obtain an  $\varepsilon$ -linked ( $\leqslant |S|$ )-region C' which is disjoint from X and which, for every region in  $\mathcal{C}_{\max}^{\leqslant |S|} \cup \mathcal{D}$ , either contains that region or does not touch it. Thus, C' is nested with all  $C_i^N$  with  $\ell_i^N \leqslant |S|$  and with all  $D \in \mathcal{D}$ . This concludes the proof since the existence of C' then contradicts that  $C_I^N$  satisfies (N1) where  $I = \min\{i \mid \ell_i^n > |S|\}$ . Here we used that  $C'_{N+1}$  is some  $C_i^N$  and that  $|N(C'_{N+1})| \geqslant k' > |S|$ , so the minimum is not taken over the empty set.

(A7): Let C be a component of H-Y. By (A4), we have  $C=C_i$  for some suitable i. Suppose first that  $C_i$  that was chosen in Case A. Then  $|N_H(C)| < |X|$ ; since H-X is connected and  $N_H(H-X)=X$ ,  $N_H(C)$  contains a vertex in  $V(H) \setminus X$ , implying  $N_H(C) \not\subseteq X$ . Now suppose that  $C_i$  that was chosen in Case B. Then  $N_H(C)$  separates x and C. Since H-X is connected and  $x \notin X$ , we have  $N_H(C) \not\subseteq X$ . The in-particular part is now also clear.

7.4. **Proof of Theorem 7.1.** Using Theorem 7.9 we now show that the tree-decomposition obtained from iteratively applying Algorithm 7.2 is as desired for Theorem 7.1.

Proof of Theorem 7.1. First, we define the desired tree-decomposition  $(T, \mathcal{V})$  for the case  $X = \operatorname{int}(\sigma)$ . The decomposition tree T is a star whose edges are in a bijective correspondence to  $\sigma$ . We assign to the centre the bag X, and to each leaf  $\ell$  the bag A where  $(A, B) \in \sigma$  corresponds to the edge incident with  $\ell$ . It is immediate to see that this tree-decomposition is as desired. Let us assume from now on that  $X \subseteq \operatorname{int}(\sigma)$ .

Suppose first that G is connected. By Theorem 2.2, we may fix a normal spanning tree  $T_{NST}$  of G whose root is in X. We denote by  $\mathcal{D}$  the set of subgraphs  $G[A \setminus B]$  of G with  $(A, B) \in \sigma$ . The assumptions on  $\sigma$  ensure that  $\mathcal{D}$  satisfies the assumptions in Algorithm 7.2. Since G admits a normal spanning tree, all its ends have countable combined degree. Hence, G satisfies the assumptions of Theorem 7.9. We now define the desired tree-decomposition recursively as follows. Let  $(T^0, \mathcal{V}^0)$  be the trivial tree-decomposition of G where  $T^0$  is the tree on a single vertex T which is also its root and  $T^0 := T(G)$ . Now let  $T^0 := T(G)$  and suppose that we have already constructed linked,  $T^0 := T(G)$  is the tree-decompositions of  $T^0 := T^0$  of finite adhesion such that for all  $T^0 := T^0$ 

- (i)  $T^m \subseteq T^n$  and  $V_t^m = V_t^n$  for all  $t \in T^m$  that are not at height m, the decomposition tree  $T^m$  has height m, all their leaves are on height m except possibly those whose corresponding leaf separation is a (B,A) with  $(A,B) \in \sigma$ ,
- (ii) the torsos of  $(T^m, \mathcal{V}^m)$  at non-leaves are rayless, and  $(T^m, \mathcal{V}^m)$  satisfies (R2') and (R3').

Note that  $(T^0, \mathcal{V}^0)$  satisfies all these properties immediately, where we remark that we treat the unique node of  $T^0$  as a leaf. Let L be the set of leaves of  $T^n$  whose corresponding leaf separation is not some (B, A) with  $(A, B) \in \sigma$ . Note that, if n = 0, then  $L = \{r\}$ . For every leaf  $\ell \in L$ , we construct a tree-decomposition  $(T^\ell, \mathcal{V}^\ell)$  of  $G[V_\ell^n]$  as follows. Set X' := X if n = 0 and  $\ell$  is the unique node of  $T^0$ , and  $X' := V_f^n$  otherwise where f is the unique edge of  $T^n$  that is incident with  $\ell$ . Further, let x be a  $(\leq_{T_{NST}})$ -minimal vertex in  $V_\ell^n \setminus (X' \cup \bigcup \mathcal{D})$ . Note that, if n = 0, x exists (but might not be unique) as  $X \subseteq \operatorname{int}(\sigma)$ . If n > 0, the vertex x exists and is unique, since  $T_{NST}$  is normal and because  $(T^n, \mathcal{V}^n)$  is componental by construction and so  $G[V_\ell^n] - (X' \cup \bigcup \mathcal{D})$  is connected by the assumption on  $\sigma$ .

Now apply Algorithm 7.2 to  $H := G[V_\ell^n]$  with the finite set X', the vertex x and  $\{D \in \mathcal{D} \mid D \subseteq H\}$  to obtain  $Y \subseteq V(H)$ . Then let  $T^\ell$  be the star with centre  $\ell$  and whose set of leaves is the set  $\mathcal{C}$  of all components of H - Y. Further, set  $V_\ell^\ell := Y$  and  $V_C^\ell := \overline{C}$  for every  $C \in \mathcal{C}$ . Then Theorem 7.9 (A4) yields that for every edge  $e = \ell C$  of  $T^\ell$ , the graph  $H \uparrow e$  is a  $C_i$  in Algorithm 7.2 applied to H. Thus,  $(T^\ell, \mathcal{V}^\ell)$  is componental and end-linked; thus,  $(T^\ell, \mathcal{V}^\ell)$  satisfies (R2'). Moreover, the torso at  $\ell$  is rayless: since  $(T^\ell, \mathcal{V}^\ell)$  is tight, we can obtain from any ray in the torso at  $\ell$  a ray in G that meets  $V_\ell^\ell$  infinitely often by (the comment after) Proposition 6.1. But this contradicts Theorem 7.9 (A3). Furthermore, it satisfies the following (which will ensure (R3') of  $(T^{n+1}, \mathcal{V}^{n+1})$ ) by Theorem 7.9 (A5): if  $|V_e^\ell| \geqslant |V_f^n|$  for an edge  $e \in T^\ell$ , then  $V_\ell^\ell \cap V_f^n \subseteq \text{Dom}(H \uparrow e)$ .

By construction, the  $T^{\ell}$  are pairwise disjoint and only share their centre  $\ell$  with  $T^n$ . We set  $T^{n+1} := T^n \cup \bigcup_{\ell \in L} T^{n+1}_{\ell}$  with  $\operatorname{root}(T^{n+1}) := \operatorname{root}(T^n)(=r)$ , and bags  $V^{n+1}_t := V^n_t$  for every node  $t \in T^n - \ell$  and  $V^{n+1}_t := V^\ell_t$  for every node  $t \in T^{n+1} \setminus T^n$  where  $\ell$  is the unique leaf of  $T^n$  such that  $t \in T^{\ell}$ . In particular, the decomposition tree  $T^{n+1}$  has height n+1, all its leaves are on height n+1, except possibly those whose corresponding leaf separation is (B,A) for some  $(A,B) \in \sigma$ ,

and  $(T^{n+1}, \mathcal{V}^{n+1})$  satisfies all properties that we demanded from the  $(T^m, \mathcal{V}^m)$ : All properties but the (X-)linkedness of  $(T^{n+1}, \mathcal{V}^{n+1})$  follows immediately from the construction and the discussion of the properties of the  $(T^{\ell}, \mathcal{V}^{\ell})$ . Its (X-)linkedness is ensured by Theorem 7.9 (A6).

Let  $(T, \mathcal{V})$  be the limit of  $(T^n, \mathcal{V}^n)$  for  $n \to \infty$ , that is  $T = \bigcup_{n \in \mathbb{N}} T^n$ , root $(T) := r = \operatorname{root}(T^n)$  for all  $n \in \mathbb{N}$  and  $V_t := V_t^N = V_t^n$  for all  $n \ge N$  where N is the minimal number such that  $t \in T^N$  and either t is not a leaf of  $T^N$  or its leaf separation is a (B, A) with  $(A, B) \in \sigma$ .

We first show that  $(T, \mathcal{V})$  is a tree-decomposition of G. For this, by (i), the definition of  $(T, \mathcal{V})$  and since all  $(T^n, \mathcal{V}^n)$  are tree-decompositions of G, it suffices to show that every vertex of G is contained in some bag  $V_t$ . In other words, we need to show that each  $v \in V(G)$  satisfies (\*): there are  $n \in N$  and  $t \in T^n$  such that  $v \in V_t^n$  and t is either a non-leaf of  $T^n$  are its leaf separation is a (B, A) with  $(A, B) \in \sigma$ . Let us first assume that  $v \notin D$  for all  $D \in \mathcal{D}$ . Then (\*) is ensured by considering Theorem 7.9 (A1) along  $\operatorname{root}(T_{NST})T_{NST}v = v_0v_1\dots v_m$ : Suppose for a contradiction that (\*) does not hold for v. Then let i be the minimal index such that  $v_i$  does not satisfy (\*); in particular,  $v_i \in G \uparrow e^n$  for a unique edge  $e^n$  incident with a leaf of  $T^n$  for every  $n \in \mathbb{N}$ . We have  $i \geqslant 1$ , since  $\operatorname{root}(T_{NST}) \in X \subseteq V_r^1$  by assumption on  $T_{NST}$  and construction of  $V_r^1$ . Thus,  $v_{i-1}$  satisfies (\*); let N be a sufficiently large integer given by (\*) of  $v_{i-1}$ . Then  $v_i = \min_{\leqslant S} (V(G \uparrow e^n) \setminus (\bigcup V(\mathcal{D})))$  for every  $n \geqslant N$ . Hence, Theorem 7.9 (A1) ensures that the finite sets  $V_{e^n}^n$  strictly decrease in their size for  $n \geqslant N$ , which is a contradiction.

Second, assume that  $v \in D$  for some  $D \in \mathcal{D}$ . By Theorem 7.9 (A2), we find a path P ending in a leaf  $\ell$  of T or a ray P in T starting in the root r of T such that  $D \subseteq G \uparrow e$  for every  $e \in P$ . If P is a path, then we find  $v \in V_{\ell}$  as desired. In particular, by construction of  $(T, \mathcal{V})$  and since the regions in  $\mathcal{D}$  are nested, the leaf separation at  $\ell$  is  $(V(G) \setminus V(D), V(D) \cup N(D))$ .

But P cannot be a ray. Indeed, applying (\*) to the finite set N(D) yields  $N \in \mathbb{N}$  such that all  $u \in N(D)$  are contained in bags  $V_t^N$  at nodes  $t \in T^N$  which are either not leaves or their corresponding leaf separations are some (B,A) with  $(A,B) \in \sigma$ . Then  $D \subseteq G \uparrow e_N$  for the N-th edge of P by the definition of P, and  $N(D) \subseteq G \downarrow e_N$  by the choice of N. Since  $(T^N, \mathcal{V}^N)$  is componental, this implies that  $V(D) \cup N(D) = V_{\bar{D}}^N$  is a bag at a leaf of  $(T^N, \mathcal{V}^N)$ , and thus, by the construction of  $(T,\mathcal{V})$  also appears as a bag at a leaf of  $(T,\mathcal{V})$ . In particular, the argument above implies that all (B,A) with  $(A,B) \in \sigma$  are leaf separations of  $(T,\mathcal{V})$ . Since  $(T,\mathcal{V})$  has no other leaf separations by construction and (i), it follows that the leaf separations of  $(T,\mathcal{V})$  are precisely  $\{(B,A) \mid (A,B) \in \sigma\}$ .

Now it is immediate from the construction that  $(T, \mathcal{V})$  is linked, X-linked, tight, componental, has finite adhesion and satisfies (R1'), (R2') and (R3') because the  $(T^n, \mathcal{V}^n)$  are linked, X-linked, tight, componental, have finite adhesion and satisfy (ii).

We are thus left to show the 'moreover'-part. For this, recall that  $(T, \mathcal{V})$  is tight and componental, so every  $G[V_{\ell}^n]$  considered in the construction of  $(T, \mathcal{V})$ , except possibly  $G[V_r^0]$ , satisfies the premise of (A7). It follows for all edges  $e = ts \in T$  with  $t \leq_T s$  and  $t \neq r$  that  $V_t \supseteq V_e \subseteq V_s$ . In particular,

if G - X is connected and  $N_G(G - X) = X$ , then also  $G[V_r^0]$  satisfies the premise of (A7), so  $V_r \supseteq V_e \subsetneq t$  for all edges  $e = rt \in T$  and  $X \subsetneq V_r$ .

The proof of the case that G is disconnected is analogous by choosing for each component C of G a normal spanning tree  $T_{NST}^C$  and considering the partial order  $\leq_{NST}$  as the disjoint union of their tree orders.<sup>13</sup>

## §8. Lean tree-decompositions

In this section we prove Theorem 3, the strengthening of Theorem 1 for graphs without half-grid minor in which we replace the linkedness of the tree-decomposition with leanness. We restate it here in its more detailed version:

**Theorem 3'** (Detailed version of Theorem 3). Every graph G without half-grid minor admits a lean, cofinally componental, rooted tree-decomposition into finite parts which displays the infinities. Moreover, if the tree-width of G is finitely bounded, then the tree-decomposition can be chosen to have width tw(G).

The proof of Theorem 3' is structured as follows. We start our construction with the treedecomposition  $(T, \mathcal{V})$  of the graph G from Theorem 4', and we then aim to refine its (finite) parts  $G[V_t]$  via lean tree-decompositions  $(T^t, \mathcal{V}^t)$  given by the finite version of Theorem 3', that is Thomas's Theorem 1.4. If  $V_t$  is a critical vertex set, then we may choose  $(T^t, \mathcal{V}^t)$  as the trivial tree-decomposition into one bag. In order to combine such refinements the tree-decompositions  $(T^t, \mathcal{V}^t)$  along  $(T, \mathcal{V})$ , we apply the finite result not to the parts themselves, but to the torsos. Torsos, however, need not have the same tree-width as G. So our second ingredient to the proof of Theorem 3' provides a sufficient condition on the separations induced by the edges incident with a node  $t \in T$  which allows us to transfer the tree-width bound from G to the torso of  $(T, \mathcal{V})$  at t via [1, Corollary 6.3] (see Lemma 8.1 below). Moreover, it also yields the well-linkedness of the separations on their left side (see Lemma 8.2 below). This property is crucial to the proof of Theorem 3' as it ensures that the tree-decomposition  $(T', \mathcal{V}')$  arising from  $(T, \mathcal{V})$  and the  $(T^t, \mathcal{V}^t)$  by refinement is lean: First, if all separations induced by edges at some node t are left-well-linked, then we can transfer families of disjoint paths from the torso at t to G via Lemma 6.3 (ii), which yields the paths families required for lean between bags that belong to the same  $(T^t, \mathcal{V}^t)$ . Second, whenever the separation induced by an edge  $e = t_0 t_1$  of T is well linked on both sides, this ensures that the learness of the two  $(T^{t_i}, \mathcal{V}^{t_i})$  combines to the leanness of the tree-decomposition resulting from gluing them together along  $V_e$ . This will ensure that we obtain disjoint paths families, as required for lean, also between bags of  $(T', \mathcal{V}')$  that belong to distinct tree-decompositions  $(T^t, \mathcal{V}^t)$  and  $(T^s, \mathcal{V}^s)$ . Our third, and last, ingredient to the proof of Theorem 3', then, is a pre-processing step: We first contract all edges which neither satisfy the well-linked condition on both sides nor are incident with a node whose bag is a critical

<sup>&</sup>lt;sup>13</sup>The only adaption one has to do is to prove (\*) for the roots of every  $T_{NST}^{C}$ .

vertex set. This will ensure that the sufficient condition mentioned above is met by all separations induced by edges of T that are incident with a node of T whose torso we need to refine (recall that we do not need to refine those torsos whose bags are critical vertex sets). Combining these three ingredients then yields a tree-decomposition of G, which we prove to be as desired.

Following [1], we call a finite-order separation (A, B) of a graph G left- $\ell$ -robust for  $\ell \in \mathbb{N}$  if there exist a set  $U \subseteq A$  of size  $\ell$  and a family  $\{P_x \mid x \in A \cap B\}$  of pairwise disjoint paths in G[A] such that  $P_x$  ends in x and for each  $x \in A \cap B$  there are  $\ell$  many  $U-P_x$  paths in  $G[(A \setminus B) \cup \{x\}]$  that do not meet outside  $P_x$ . Analogously, (A, B) is right- $\ell$ -robust for  $\ell \in \mathbb{N}$  if (B, A) is left- $\ell$ -robust. We call (A, B)  $\ell$ -robust if it is both left- and right- $\ell$ -robust.

**Lemma 8.1.** [1, Corollary 6.3] Let G be a graph of tree-width at most  $w \in \mathbb{N}$ , and let  $\sigma$  be a finite star of separations of G of order at most w+1 whose interior is finite. Suppose that all separations in  $\sigma$  are left- $\ell$ -robust for  $\ell = (w+1)^2(w+2) + w + 1$ . Then  $torso(\sigma)$  has tree-width at most w.

**Lemma 8.2.** If a separation of a graph G has order k and is left-(2k + 1)-robust, then it is left-well-linked.

*Proof.* Consider a left-(2k+1)-robust separation (A,B) of G, and let  $X,Y\subseteq A\cap B$  be disjoint. Suppose for a contradiction that there is no family of  $\min\{|X|,|Y|\}$  disjoint X-Y paths through  $A \setminus B$  in G. By Menger's theorem (see for example [8, Proposition 8.4.1]), there then is an X-Y separator S of size less than  $\min\{|X|,|Y|\}$  in  $G[(A \setminus B) \cup X \cup Y]$ .

Now fix a set U and a path family  $\{P_x \mid x \in A \cap B\}$  which witness that (A, B) is left-(2k + 1)-robust, where k is the order of (A, B). Since  $|S| < \min\{|X|, |Y|\}$  and the  $P_x$  are pairwise disjoint, there are  $x \in X$  and  $y \in Y$  such that  $P_x$  and  $P_y$  avoid S. For  $z \in \{x, y\}$ , at least k + 1 of the 2k + 1  $P_z - U$  paths in  $G[(A \setminus B) \cup \{z\}]$  given by the left-(2k + 1)-robustness avoid S. So since U has size 2k + 1, there are such a  $P_x - U$  path  $Q_x$  and such a  $P_y - U$  path  $Q_y$  that both avoid S and end in the same vertex in U. Hence,  $P_x + Q_x + Q_y + P_y$  is a connected subgraph of  $G[(A \setminus B) \cup X \cup Y]$  which meets X and Y but avoids S. This contradicts that S is an X - Y separator in  $G[(A \setminus B) \cup X \cup Y]$ , which completes the proof.

Let us now turn to the third ingredient for our proof of Theorem 3': the pre-processing step in which we contract certain edges of the tree-decomposition from Theorem 4. This ensures that the assumptions of both Lemmas 8.1 and 8.2 are met at all edges incident with nodes whose (finite) torso we later aim to refine using Thomas's Theorem 1.4.

To simplify the wording, let us thus make the following definitions. Given some  $m \in \mathbb{N}_0$ , we call a separation of G of order k left-m-good if it is  $\ell$ -left-robust for  $\ell := \max\{m, 2k+1\}$ . Analogously, (A, B) is right-m-good for  $m \in \mathbb{N}$  if (B, A) is left-m-good. We call (A, B) m-good if it is left-and right-m-good. An left-m-good separation of G is left-good if  $m = (w+1)^2(w+2) + w+1$  for G with  $w = \operatorname{tw}(G) \in \mathbb{N}$  or m = 0 for graphs G whose tree-width is not finitely bounded.

Analogously, we define right-good. We call (A, B) good if it is both left- and right-good. The bounds in the definition of (left-)good are exactly the ones sufficient to apply Lemmas 8.1 and 8.2 in the respective contexts.

Setting out from Theorem 4', our pre-processing step yields the following result:

**Lemma 8.3.** Let G be a graph without half-grid minor, let  $m \in \mathbb{N}$  and let  $(T, \mathcal{V})$  be a fully tight, rooted tree-decomposition into finite parts which displays the infinities and satisfies (II) from Theorem 4'. Then the tree-decomposition  $(T', \mathcal{V}')$  of G induced by contracting every edge st of T with  $\deg(s), \deg(t) < \infty$  whose induced separation is not m-good has finite parts, is cofinally componental, and displays the infinities. Moreover,

(i) if  $deg(t) < \infty$  for  $t \in T'$ , then all separations in  $\sigma'_t$  at t are left-m-good.

We first show three auxiliary lemmas.

**Lemma 8.4.** Let  $\varepsilon$  be a finitely dominated end of a graph G, and  $\ell \in \mathbb{N}$  arbitrary. For every collection  $R_1, \ldots, R_d$  of disjoint rays in  $\varepsilon$  that avoid  $Dom(\varepsilon)$ , there exists a set  $U \subseteq V(R_1)$  of size  $\ell$  such that

- $\diamond$  for every  $i \in \{2, \ldots d\}$ , there are  $\ell$  pairwise disjoint  $U R_i$  paths  $P_1^i, \ldots, P_\ell^i$  in G avoiding  $Dom(\varepsilon)$ , and
- $\diamond$  for every  $v \in \text{Dom}(\varepsilon)$ , there are  $\ell$  independent U-v paths  $P_1^v, \ldots, P_\ell^v$  in G avoiding  $\text{Dom}(\varepsilon) \setminus \{v\}$  that only meet in v.

Proof. Since  $R_1, \ldots, R_d$  belong to the same end and  $\operatorname{Dom}(\varepsilon)$  is finite, there are, for every  $i \in \{2, \ldots, d\}$ , infinitely many pairwise disjoint  $R_1 - R_i$  paths  $Q_1^i, Q_2^i, \ldots$  avoiding  $\operatorname{Dom}(\varepsilon)$ , and we write  $q_k^i$  for the endvertex of  $Q_k^i$  in  $R_1$ . Similarly, for every  $v \in \operatorname{Dom}(\varepsilon)$  there are infinitely many  $v - R_1$  paths  $Q_1^v, Q_2^v, \ldots$  in G avoiding  $\operatorname{Dom}(\varepsilon) \setminus \{v\}$  that only meet in v, and we write  $q_k^v$  for the endvertex of  $Q_k^v$  in  $R_1$ .

We now find the elements  $u_1, \ldots, u_\ell$  of U one by one: Let  $u_1$  be the first vertex of  $R_1$ . Given  $u_1, \ldots, u_{j-1}$  for some  $2 \leq j \leq \ell$ , we then choose  $u_j$  as the first vertex on  $R_1$  such that  $u_{j-1}R_1u_j$  contains as inner vertices some  $q_i^k$  for every  $i \in \{2, \ldots, d\}$  and some  $q_v^k$  for every  $v \in \text{Dom}(\varepsilon)$ . Then  $U := \{u_1, \ldots, u_\ell\}$  is as desired, as witnessed by the  $P_j^i := u_j R_1 q_k^i Q_k^i$  and  $P_j^v := u_j R_1 q_k^v Q_k^v$  for the respective k as given by the choice of  $u_j$ .

**Lemma 8.5.** Let  $m \in \mathbb{N}$ , let  $(T, \mathcal{V})$  be a rooted tree-decomposition of finite adhesion of a graph G without half-grid minor, and let  $\varepsilon$  be an end of G. Assume that  $\varepsilon$  gives rise to a ray  $R = r_0 r_1 \dots$  in T such that  $\liminf_{e \in R} |V_e| = \Delta(\varepsilon)$ . Then cofinitely many edges e of R with  $|V_e| = \Delta(\varepsilon)$  induce m-good separations.

*Proof.* Since G has no half-grid minor, the combined degree  $\Delta(\varepsilon)$  of  $\varepsilon$  is finite. Thus, as  $(T, \mathcal{V})$  has finite adhesion and  $\varepsilon$  gives rise to R, we have  $\liminf_{e \in R} V_e \supseteq \mathrm{Dom}(\varepsilon)$ . In fact,  $\liminf_{e \in R} V_e = \mathrm{Dom}(\varepsilon)$  since  $\liminf_{e \in R} |V_e| = \Delta(\varepsilon)$ , so the  $V_e$  eventually have to meet each ray

in a family of  $\deg(\varepsilon)$  disjoint  $\varepsilon$ -rays avoiding  $\mathrm{Dom}(\varepsilon)$  precisely once. Hence, for some  $N_0 \in \mathbb{N}$  all indices  $i \geqslant N_0$  satisfy  $\mathrm{Dom}(\varepsilon) \subseteq V_{e_i}$  where  $e_i := \{r_i, r_{i+1}\}$ . Let us denote the set of all indices  $i \geqslant N_0$  with  $|V_{e_i}| = \Delta_G(\varepsilon) =: k$  by I. Note that I is infinite since  $\liminf_{e \in R} |V_e| = \Delta(\varepsilon)$ .

Write  $(A_i, B_i)$  for the separation induced by the edge  $\vec{e_i} = (r_i, r_{i+1})$  for all  $i \in \mathbb{N}$ . We now show that cofinitely many of the separations  $(A_i, B_i)$  with  $i \in I$  are m-good, which clearly implies the assertion. So let  $\ell := \max\{2k+1, (w+1)^2(w+2)+w+1\}$  if  $w := \operatorname{tw}(G) \in \mathbb{N}$ , and let  $\ell := 2k+1$  if the tree-width of G is not finitely bounded.

Fix a collection  $R_1, \ldots, R_d$  of  $d := \deg(\varepsilon)$  disjoint rays in  $\varepsilon$  that avoid  $\mathrm{Dom}(\varepsilon)$ , and consider a set  $U \subseteq V(R_1)$  and corresponding paths  $P_1^i, \ldots, P_\ell^i$  and  $P_1^v, \ldots, P_\ell^v$  as given by Lemma 8.4. Let Z be the union of U and all the  $V(P_j^i)$  and  $V(P_j^v)$ . Then Z is finite. So since  $\bigcap_{i \in \mathbb{N}} (B_i \setminus A_i) = \bigcap_{i \in \mathbb{N}} G_i^{\uparrow} e_i = \emptyset$  and  $\bigcap_{i \in \mathbb{N}} (A_i \cap B_i) = \liminf_{i \in \mathbb{N}} V_{e_i} = \mathrm{Dom}(\varepsilon)$ , there exists  $N_1 \in \mathbb{N}$  such that, for all  $i \geq N_1$ , we have  $Z \subseteq (A_i \setminus B_i) \cup \mathrm{Dom}(\varepsilon)$ . Consider  $i \in I$  with  $i \geq N_1$ . We claim that  $(A_i, B_i)$  is left- $\ell$ -robust. Indeed, we have  $|V_{e_i}| = |A_i \cap B_i| = \Delta_G(\varepsilon)$  and  $\mathrm{Dom}(\varepsilon) \subseteq A_i \cap B_i$ . So the set U and the initial segments  $R_j \cap G[A_i]$  together with the trivial paths in  $\mathrm{Dom}(\varepsilon)$  are as required in the definition of left- $\ell$ -robust, as witnessed by the paths  $P_\ell^i, \ldots, P_\ell^i$  and  $P_1^v, \ldots, P_\ell^v$  (and because  $Z \subseteq (A_i \setminus B_i) \cup \mathrm{Dom}(\varepsilon)$ ). Hence,  $(A_i, B_i)$  is left- $\ell$ -robust. To show that  $(A_i, B_i)$  is also right- $\ell$ -robust for  $i \in I$ , we apply Lemma 8.4 in  $G[B_i]$  to the rays  $R_j \cap G[B_i]$  and take the  $P_x$  as suitable finite initial segments of those rays and the trivial paths on  $\mathrm{Dom}(\varepsilon)$ . Altogether, we obtain that all  $(A_i, B_i)$  with  $i \in I$  and  $i \geq N_1$  are  $\ell$ -robust and hence m-good.

**Lemma 8.6.** Let (T, V) be a tight, rooted tree-decomposition of a graph G into finite parts which displays the infinities of G. Assume that F is some set of edges of T which are not incident with any infinite-degree node of T and such that, for every end  $\varepsilon$  of G, the set F avoids cofinitely many edges e of the arising ray  $R_{\varepsilon}$  in T with  $|V_e| = \Delta(\varepsilon)$ . Then the tree-decomposition (T', V') obtained from (T, V) by contracting all edges in F has finite parts and displays the infinities.

*Proof.* We first note that  $(T', \mathcal{V}')$  displays the infinities: Since  $(T, \mathcal{V})$  displays the ends homeomorphically, their combined degrees and their dominating vertices, so does  $(T', \mathcal{V}')$ , since by the assumptions on F every end  $\varepsilon$  of G still gives rise to a ray in T', and this ray still contains infinitely many edges e with  $|V_e| = \Delta(\varepsilon)$ . By assumption, F contains no edges incident with nodes of infinite degree; thus,  $(T', \mathcal{V}')$  displays the critical vertex sets and their tight components cofinitely, as  $(T, \mathcal{V})$  does so.

To prove that all bags of  $(T', \mathcal{V}')$  are finite, it suffices to show by Theorem 2.5 that its torsos are tough and rayless. We first show that the torsos are tough. By construction of  $(T', \mathcal{V}')$ , we did not contract edges of T which are incident with some node  $t \in T$  whose corresponding bag  $V_t$  is critical. In particular, for every  $X \in \text{crit}(G)$ , the unique node  $t_X \in T$  with  $V_t = X$  is also in T' with  $V'_{t_X} = X$ , and cofinitely many tight components of G - X are some  $G \uparrow e$  for an edge  $e = t_X t \in T'$  with  $t_X <_{T'} t$ , since  $(T, \mathcal{V})$  displays all critical vertex sets and their tight components

cofinitely. Thus, Lemma 5.2 ensures that all torsos of the tight rooted tree-decomposition  $(T', \mathcal{V}')$  are tough.

It remains to show that the torsos are rayless. Note that the torso of every node  $t \in T'$  with  $V'_t \in \operatorname{crit}(G)$  is rayless, as it is finite. Thus, it remains to consider  $t \in T'$  with  $V'_t \notin \operatorname{crit}(G)$ . So suppose that there is such a node t whose torso contains a ray R'. Since  $(T', \mathcal{V}')$  is tight, we may apply (the comment after) Proposition 6.1 to R' and the star  $\sigma'_t$  at t to obtain a ray of G that meets the part  $V'_t$  infinitely often, which contradicts the fact that  $(T', \mathcal{V}')$  displays the ends of G.

Proof of Lemma 8.3. Let F be the set of edges in T that we contracted. By Lemma 8.5, the set F satisfies the premise of Lemma 8.6, and hence  $(T', \mathcal{V}')$  has finite parts and displays the infinities.

Further,  $(T', \mathcal{V}')$  is cofinally componental: Let R' be a rooted ray in T'. It suffices to show that  $G \uparrow e$  is disconnected for at most finitely many consecutive edges e of T'. Let  $e = rs, f = st \in R'$  with  $r <_{T'} s <_{T'} t$  be two successive edges such that  $G \uparrow e$  and  $G \uparrow f$  are disconnected. Since  $(T', \mathcal{V}')$  is obtained from  $(T, \mathcal{V})$  by edge-contractions, e and f are also edges of T. It follows from (I1) from Theorem 4' of  $(T, \mathcal{V})$  together with the construction of  $(T', \mathcal{V}')$  from  $(T, \mathcal{V})$  that s and t were already nodes of T and  $V_t, V_s \in \mathrm{crit}(G)$  as well as  $\deg(t), \deg(s) = \infty$  and  $V_s \supseteq V_t$ . Since  $(T, \mathcal{V})$  displays the critical vertex sets of G, this implies that  $V_s \neq V_t$ , and thus  $V_s \supseteq V_t$ . Hence, this can only happen finitely many times consecutively, as critical vertex sets are finite.

It remains to show (i). For this, let  $t \in T'$  be a node with finite degree, and denote with  $T_t$  the subtree  $T_t$  of T whose contraction yields t. Suppose for a contradiction that some separation  $(A, B) \in \sigma'_t$  is not left-m-good. We remark that as  $(T', \mathcal{V}')$  is obtained from  $(T, \mathcal{V})$  by edge-contractions, the separations induced by the edges of T with precisely one endvertex in  $T_t$  are the same as the separations induced by the edges incident with t in T', i.e. the ones in  $\sigma'_t$ ; so let e = t's be such an edge with  $t' \in T_t$  which induces (A, B). The construction of T' yields that either (A, B) is m-good or one of t', s has infinite degree in T. If  $T_t$  is a singleton, then  $\deg(t') < \infty$  by assumption on t. If  $T_t$  contains at least one edge, then all nodes of  $T_t$ , in particular t', have finite degree, as the edges in  $T_t$  have been contracted. In both cases,  $\deg(s) = \infty$  and thus  $V_s =: X \in \operatorname{crit}(G)$ . Since  $(T, \mathcal{V})$  displays the critical vertex sets and their tight components cofinitely and because  $\deg(s) = \infty$ , cofinitely, and thus infinitely, many of the tight components of G - X are contained in G[A]. Since  $X = V_s$  and thus  $X \supseteq A \cap B$ , this shows that (A, B) is left-m-good as witnessed by the trivial paths in  $A \cap B$  and a set U consisting of m vertices that lie in pairwise distinct tight components of G - X contained in G[A].

With the three ingredients at hand, we are ready to prove the main result of this section.

Proof of Theorem 3'. Let G be a graph without half-grid minor; in particular, G has no  $K^{\aleph_0}$  minor. So G has finite tree-width, as it has a normal spanning tree by [17]. Let  $(T, \mathcal{V})$  be the (rooted)

tree-decomposition of G from Theorem 4', and let  $(T', \mathcal{V}')$  be the rooted tree-decomposition obtained from  $(T, \mathcal{V})$  by applying Lemma 8.3 with  $m = (w+1)^2(w+2) + w + 1$  if  $w := \operatorname{tw}(G) \in \mathbb{N}$  for finitely bounded tree-width G or with m = 0 for other G. Then Lemma 8.3 yields that all the bags of  $(T', \mathcal{V}')$  are finite and that, moreover, for every node  $t \in T'$  either  $V'_t \in \operatorname{crit}(G)$  or each separation in the star  $\sigma'_t$  at t is left-good. If  $V'_t$  is critical in G, then  $|V'_t| \leq \operatorname{tw}(G) + 1$  since critical vertex sets are infinitely connected and thus every tree-decomposition of G, in particular those witnessing  $\operatorname{tw}(G)$ , contains  $V'_t$  in one of its bags. If  $V'_t$  is not critical in G, then all separations in  $\sigma'_t$  are left-good, and we can apply Lemma 8.1 to find that the torso of  $(T', \mathcal{V}')$  at t has again at most the tree-width of G. Altogether, every torso of  $(T', \mathcal{V}')$  is finite and has tree-width at most  $\operatorname{tw}(G)$ .

We may thus apply Thomas's result [34, Theorem 5] (cf. Theorem 1.1 for finite graphs) to the torsos of  $(T', \mathcal{V}')$  and obtain, for every node  $t \in T'$ , an (unrooted) lean tree-decomposition  $(T^t, \mathcal{V}^t)$  of torso $(\sigma_t)$  where  $T^t$  is a finite tree<sup>14</sup> and whose parts have size at most the tree-width of G; in particular, all bags in  $\mathcal{V}^t$  are finite, as the torso at t is finite. Furthermore, we may assume that for  $t \in T'$  with  $V'_t \in \text{crit}(G)$ , the tree-decomposition  $(T^t, \mathcal{V}^t)$  is the trivial tree-decomposition consisting of a single node-tree, since the torso of  $(T', \mathcal{V}')$  at t is complete.

Now for every edge  $e = st \in T'$ , the adhesion set  $V'_e$  induces a complete subgraph in both  $\operatorname{torso}(\sigma'_s)$  and  $\operatorname{torso}(\sigma'_t)$ . Thus, we may fix  $u \in T^s$  and  $w \in T^t$  with  $V'_e \subseteq V^s_u, V^t_w$  for every edge  $e = st \in T'$ . We now build a tree  $\tilde{T}$  from the disjoint union of the  $T^t$  by joining the corresponding u and w for every  $e \in E(T')$ ; we say that these new edges uv of  $\tilde{T}$  belong to T' and correspond to the respective  $st \in T'$ . Keeping the respective parts from the  $(T^t, \mathcal{V}^t)$ , we obtain a tree-decomposition  $(\tilde{T}, \tilde{\mathcal{V}})$  of G which has finite parts of size at most the tree-width of G. We remark that the separation induced by an edge  $uv \in \tilde{T}$  which belongs to T' is the same as the separation induced by the corresponding edge in T'.

We claim that  $(\tilde{T}, \tilde{\mathcal{V}})$  is as desired. For this, let us first note that  $(\tilde{T}, \tilde{\mathcal{V}})$  is cofinally componental since  $(T', \mathcal{V}')$  is cofinally componental by Lemma 8.3 and because the  $T^t$  are finite. Moreover, since  $(T', \mathcal{V}')$  displays the infinities by Lemma 8.3, it follows that  $(\tilde{T}, \tilde{\mathcal{V}})$  also does so, as the  $T^t$  are finite and consist of a single node, if  $V'_t \in \text{crit}(G)$  and  $\deg_{T'}(t) = \infty$ .

It remains to show that  $(\tilde{T}, \tilde{\mathcal{V}})$  is lean. For this, fix nodes  $t_1, t_2 \in \tilde{T}$  and sets  $Z_1 \subseteq \tilde{V}_{t_1}$  and  $Z_2 \subseteq \tilde{V}_{t_2}$  with  $|Z_1| = |Z_2| =: k$ . We prove the claim by induction on the number of edges on  $t_1 \tilde{T} t_2$  that belong to T'.

If there is no such edge, then there exists a node  $t \in T'$  with  $t_1, t_2 \in T^t$ . Since  $(T^t, \mathcal{V}^t)$  is a lean tree-decomposition of  $torso(\sigma'_t)$ , either there exists an edge  $e \in t_1T^tt_2$  with  $|V_e^t| < k$  or there are k disjoint  $Z_1 - Z_2$  paths in  $torso(\sigma'_t)$ . In the first case, the construction of  $(\tilde{T}, \tilde{V})$  yields  $t_1\tilde{T}t_2 = t_1T^tt_2$  and  $\tilde{V}_e = V_e^t$ , so e is as desired. In the second case, we distinguish between the case whether  $V'_t$  is critical or not. If  $V'_t$  is critical, then  $t_1 = t_2$  and  $\tilde{V}_{t_1} = V'_t$  as  $T^t$  has a

 $<sup>^{14}</sup>$ We remark that this is not explicitly stated in [34, Theorem 5] but follows directly from its proof.

single node by construction. We then use the infinitely many tight components of  $G - V'_t$  to find the desired disjoint  $Z_1 - Z_2$  paths. And if  $V'_t$  is not critical, then all separations in  $\sigma'_t$  are left-good by (i) from Lemma 8.3 and because  $(T', \mathcal{V}')$  displays the critical vertex sets of G. Hence, they are left-well-linked by Lemma 8.2. This allows us to apply Lemma 6.3 (ii) to lift the k pairwise disjoint  $Z_1 - Z_2$  paths in  $torso(\sigma'_t)$  to G.

Now suppose that there is an edge  $f = s_1 s_2$  on  $t_1 \tilde{T} t_2$  which belongs to T'. We may assume by renaming that  $s_1$  appears before  $s_2$  on  $t_1 \tilde{T} t_2$ . If  $|\tilde{V}_f| < k$ , then f is the desired edge e; so suppose otherwise. For  $i \in \{1,2\}$ , the induction hypothesis then yields either an edge  $e \in t_i \tilde{T} s_i$  with  $|\tilde{V}_e| < k$  or a family  $\mathcal{P}_i$  of k pairwise disjoint  $Z_i - \tilde{V}_f$  paths in G. The first case already yields the desired edge e; so we may assume that the second case holds for both  $i \in \{1,2\}$ .

If  $\tilde{V}_{s_1} \in \operatorname{crit}(G)$ , Lemma 2.4 yields the desired family of k disjoint  $Z_1 - Z_2$  paths, as  $\tilde{V}_f \subseteq \tilde{V}_{s_1}$ . The same argument applies if  $\tilde{V}_{s_2} \in \operatorname{crit}(G)$ .

Suppose now that  $\tilde{V}_{s_1}, \tilde{V}_{s_2} \notin \operatorname{crit}(G)$ . Since there are no k disjoint  $Z_1 - Z_2$  paths in G, Menger's theorem (see for example [8, Proposition 8.4.1]) yields a separation (C, D) of G of order less than k with  $Z_1 \subseteq C$  and  $Z_2 \subseteq D$ . We now use (C, D) to find a  $Z_1 - \tilde{V}_f$  or a  $Z_2 - \tilde{V}_f$ -separator of size less than k, which gives the desired contradiction.

Let (A,B) be the separation of G induced by  $\vec{f}=(s_1,s_2)$ , and let  $X:=(A\cap B)\cap (C\smallsetminus D)$  and  $Y:=(A\cap B)\cap (D\smallsetminus C)$ . By symmetry on the assumptions up to this point, we may assume  $|Y|\leqslant |X|$ . Since  $\tilde{V}_{s_1}\notin \mathrm{crit}(G)$ , (A,B) is left-good by the construction of  $(\tilde{T},\tilde{V})$  and (i) from Lemma 8.3. Hence, (A,B) is left-well-linked by Lemma 8.2. Thus, there exist |Y| pairwise disjoint X-Y paths through  $A\smallsetminus B$  in G. All these paths meet  $(C\cap D)\cap (A\smallsetminus B)$ , so  $|(C\cap D)\cap (A\smallsetminus B)|\geqslant |Y|$ . But then

$$|(A \cap C) \cap (B \cup D)| = |(A \cap B) \cap (C \setminus D)| + |(A \cap B) \cap (C \cap D)| + |(C \cap D) \cap (A \setminus B)|$$
  
$$\geqslant |X| + |(A \cap B) \cap (C \cap D)| + |Y| = |A \cap B|,$$

which in turn yields by double counting that

$$|(A \cup C) \cap (B \cap D)| = |A \cap B| + |C \cap D| - |(A \cap C) \cap (B \cup D)| \leqslant |C \cap D| < k.$$

But  $(A \cup C) \cap (B \cap D)$  is a  $Z_2 - \tilde{V}_f$ -separator (or a  $Z_1 - \tilde{V}_f$  separator in the symmetric case), a contradiction.

## §9. Tree-decompositions distinguishing infinite tangles

An *infinite tangle* of a graph G is a set  $\tau$  of (oriented) finite-order separations of G such that

- $\diamond$   $\tau$  contains precisely one orientation of every finite-order separation of G, and
- $\diamond$  there are no three (not necessarily distinct) separations  $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \tau$  with  $G[A_1] \cup G[A_2] \cup G[A_3] = G$ .

A separation of G distinguishes two infinite tangles if they orient it differently. It distinguishes them efficiently if they are not distinguished by any separation of smaller order. A tree-decomposition  $(T, \mathcal{V})$  of G distinguishes some set of infinite tangles of G efficiently if every two tangles in this set are distinguished efficiently by a separation that is induced by an edge of  $(T, \mathcal{V})$ .

By [7, Theorem 3], for every infinite tangle  $\tau$  of a graph G there is either an end  $\varepsilon$  of G such that a finite-order separation (A, B) lies in  $\tau$  if and only if B contains a tail of every, or equivalently some, ray in  $\varepsilon$ , or there is a critical vertex set X of G such that  $(V(C) \cup X, V(G - C)) \in \tau$  for all  $C \in \mathcal{C}(G - X)$ . If the first case holds for  $\tau$ , then we say that  $\tau$  is *induced* by  $\varepsilon$ .

Following [11], we say that two infinite tangles  $\tau, \tau'$  of G are combinatorially distinguishable if at least one of them is induced by an end or there exists a finite set  $X \subseteq V(G)$  such that  $(V(C) \cup X, V(G-C)) \in \tau$  for all  $C \in \mathcal{C}_X$  and such that  $(V(G-C), V(C) \cup X) \in \tau'$  for a component  $C \in \mathcal{C}_X$ .

With the above characterization of infinite tangles the following observation is immediate:

**Observation 9.1.** Every tree-decomposition of a graph G that displays its infinities distinguishes all combinatorially distinguishable infinite tangles of G.

We emphasise though that such a tree-decomposition need not distinguish those infinite tangles efficiently. In fact, we remark that the graph in [2, Construction 3.1] has finite tree-width and thus by Theorem 4 a tree-decomposition that displays its infinities, but no tree-decomposition of that graph efficiently distinguishes all its infinite tangles that are induced by ends [2, Lemma 3.2].

However, Elm and Kurkofka [11, Theorem 1] showed that every graph G has a nested set of separations that efficiently distinguishes all the combinatorially distinguishable infinite tangles of G. They also showed that this is best possible in the following sense: every graph G that has at least two combinatorially indistinguishable infinite tangles has no nested set of separations that efficiently distinguishes all infinite tangles of G [11, Corollary 3.4]. In the special case where G has no half-grid minor we obtain the following strengthening of their result:

Corollary 5' (Tangle version of Corollary 5). Every graph G without half-grid minor has a tree-decomposition that efficiently distinguishes all the combinatorially distinguishable infinite tangles of G.

Proof. We show that the tree-decomposition  $(T, \mathcal{V})$  from Theorem 3' is as desired. For this, let any pair  $\tau_1, \tau_2$  of combinatorially distinguishable infinite tangle be given. Then by Observation 9.1, there is an edge  $e \in E(T)$  such that the separation  $\{A_e, B_e\}$  induced by e distinguishes  $\tau_1$  and  $\tau_2$  (though not necessarily efficiently). For i = 1, 2, if there is a critical vertex set  $X_i$  of G such that  $(V(C) \cup X_i, V(G - C)) \in \tau_i$  for all  $C \in \mathcal{C}(G - X_i)$ , then there is unique infinite-degree node  $t_i \in T$  with  $V_{t_i} = X_i$ , and we then let  $f_i$  be the unique edge of the  $t_i Te$  path incident with  $t_i$ . Otherwise,  $\tau_i$  is induced by an end  $\varepsilon_i$  of G, and we then let  $f_i$  be any edge on the unique ray R of T starting in e and arising from  $\varepsilon_i$  such that  $|V_f| \geqslant |V_{f_i}|$  for all edges  $f >_T f_i$  on R.

Now using the fact that  $(T, \mathcal{V})$  is lean, we find an edge e' and a family  $\{P_x \mid x \in V_{e'}\}$  of pairwise disjoint  $V_{f_1}$ – $V_{f_2}$  paths in G. Since for i = 1, 2 either  $V_{f_i}$  is linked to  $\varepsilon_i$  or  $V_{f_i} = X_i$ , these paths  $P_x$  can be extended to paths or rays witnessing that  $\tau_1$  and  $\tau_2$  cannot be distinguished by a separation of order less than  $V_{e'}$ , which shows that  $(T, \mathcal{V})$  distinguishes  $\tau_1$  and  $\tau_2$  efficiently.  $\square$ 

Proof of Corollary 5. Apply Corollary 5'.

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## §APPENDIX A. A CLOSER LOOK AT THEOREM 6'

A theorem which is similar to Theorem 6' was proven by Elm and Kurkofka [11, Theorem 2]:

**Theorem A.1.** Every connected graph G has a nested<sup>15</sup> set of separations whose separators are precisely the critical vertex sets of G and all whose torsos<sup>16</sup> are tough.

In the remainder of this section we compare Theorem 6' and Theorem A.1. For this, let us first note that the set of separations induced by a tree-decomposition is always nested, so for graphs of finite tree-width, our Theorem 6' immediately implies Theorem A.1 from Elm and Kurkofka. However, not every nested set of separations is induced by a tree-decomposition. In fact, in

 $<sup>^{15}</sup>$ A set N of separations is *nested* if for every two separations  $\{A,B\}, \{C,D\}$  one of  $A \subseteq C \& B \supseteq D, A \subseteq D \& B \supseteq C, B \subseteq C \& A \supseteq D$  and  $B \subseteq D \& A \supseteq C$  holds.

<sup>&</sup>lt;sup>16</sup>We refer the reader to [11, Section 2.6] for a definition of *torsos* in this context. We remark that if the nested set is the set of induced separations of some tree-decomposition, then these torso are precisely the torsos at nodes of the decomposition tree.

[11, Example 5.12] Elm and Kurkofka describe a graph which does not admit a tree-decomposition whose induced nested of separation is of their form <sup>17</sup>. We remark that this graph does not have finite tree-width.

But also for graphs of finite tree-width it is not only not obvious that their result does not yield a tree-decomposition but also not true: We present in the following Example A.2 a graph G of finite tree-width such that no tree-decomposition of G induces their constructed nested set of separations. For this let us recall the nested set that Elm and Kurkofka constructed in more detail. Their nested set [11, Proof of Theorem 2, i.e. Theorem 5.10 & Theorem 5.11] is of the form

$$\{\{V(G) \setminus \bigcup \mathcal{K}(X), X \cup \bigcup \mathcal{K}(X)\} \mid X \in \operatorname{crit}(G)\} \cup \{\{V(G) \setminus C, X \cup C\} \mid X \in \operatorname{crit}(G), C \in \mathcal{K}(X)\}$$

$$\tag{1}$$

for some choice of  $\mathcal{K}(X) \subseteq \check{\mathcal{C}}(G-X)$  containing all but at most one element of  $\check{\mathcal{C}}(G-X)$ .

The nested set induced by the tree-decomposition  $(T', \mathcal{V}')$  from Theorem 6' is also of the form as in (1) but for some choice of  $\mathcal{K}(X) \subseteq \mathcal{C}(G-X)$  consisting of cofinitely many (instead of one) elements of  $\check{\mathcal{C}}(G-X)$ . The following example shows that there are graphs of finite tree-width that have no tree-decomposition inducing a nested set as in (1) for some choice of  $\mathcal{K}(X)$  containing all but at most one element of  $\mathcal{C}(G-X)$ . In particular, we cannot strengthen the definition of 'displaying the tight components of all critical vertex sets cofinitely' by dropping the 'cofinitely'.

**Example A.2.** There is a graph G of finite tree-width such that no tree-decomposition of G induces a set of separations of the form as in (1) for a choice of  $\mathcal{K}(X) \subseteq \check{\mathcal{C}}(G-X)$  containing all but at most one element of  $\check{\mathcal{C}}(G-X)$ .

Proof. Let  $R = dr_0r_1,...$  be a ray, and let H be the graph obtained from R by first adding the edges  $dr_i$  for all  $i \in 2\mathbb{N}$  and the edges  $r_ir_{i+2}$  and  $r_ir_{i+3}$  for all  $i \in 2\mathbb{N} + 1$ , and then adding vertices  $u_{ij}$  for  $i \in 2\mathbb{N} \setminus \{0\}$  and  $j \in \mathbb{N}$  and joining them to R with edges  $r_iu_{ij}$ . To obtain G, we further add the edges

$$\{u_{ij}r_{i-1}, u_{ij}r_k \mid i \in 2\mathbb{N} \setminus \{0\}, j \in \mathbb{N}, k \in 2\mathbb{N} \cap \{0, 1, \dots, i-2\}\}$$

(see Figure 1). Note that G has a normal spanning tree as depicted in blue in Figure 1.

We claim that G is as desired, i.e. that no tree-decomposition of G induces a set of separations of the form as in (1) for a choice of  $\mathcal{K}(X) \subseteq \check{\mathcal{C}}(G-X)$  containing all but at most one element of  $\check{\mathcal{C}}(G-X)$ . For this, let us first observe that all critical vertex sets of G are of the form

$$X_i := \{r_i \mid j \in 2\mathbb{N}, j \leqslant i\} \cup \{r_{i-1}\},\$$

 $<sup>^{17}</sup>$ Moreover, one can show that their example does not even admit a tree-decomposition whose adhesion sets are the critical vertex sets of G and whose torsos are tough.

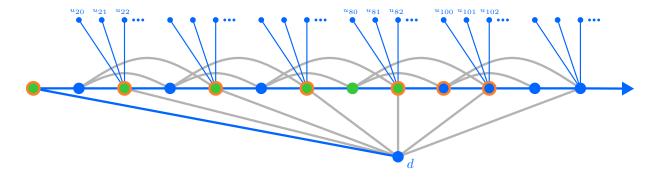


FIGURE 1. Depicted is the graph H. The green (orange) vertices are precisely the neighbours in G of  $u_{8j}$  ( $u_{10j}$ ) for all  $j \in \mathbb{N}$ . The blue edges induce a normal spanning tree of G with root d.

and the components of  $G-X_i$  either consist of a single vertex  $u_{ij}$  or are of the form

$$C_i := G[\{r_j \mid j \in 2\mathbb{N} + 1, j \leqslant i - 2\} \cup \{u_{jk} \mid j < i, k \in \mathbb{N}\}\}] \text{ or } C_i' := G[\{r_i \mid j \geqslant i\} \cup \{u_{jk} \mid j > i, k \in \mathbb{N}\} \cup \{d\}].$$

In particular, for all  $X_i$  all components of  $G - X_i$  are tight (cf. Figure 2). Hence, the unique

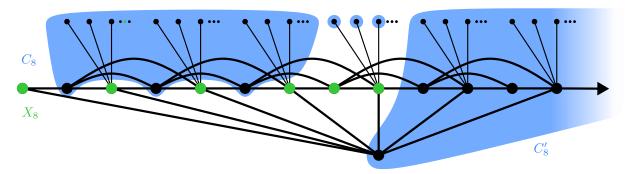


FIGURE 2. Indicated in green is the critical vertex set  $X_8$  of G. The components of  $G - X_8$  are indicated in blue.

nested set that is of the form as in (1) for a choice of  $\mathcal{K}(X_i) \subseteq \check{\mathcal{C}}(G - X_i)$  containing all but at most one element of  $\check{\mathcal{C}}(G - X_i)$  is the set

$$N:=\{\{V(C)\cup X_i,V(G-C)\}\mid X_i\in \mathrm{crit}(G),C\in \mathcal{C}(G-X_i)\}.$$

But N is not induced by any tree-decomposition of G. Indeed, suppose for a contradiction that N is induced by a tree-decomposition  $(T, \mathcal{V})$  of G, and consider the separations  $\{V(G - C_i'), V(C_i') \cup X_i\}$ . Since  $C_i' \supseteq C_j'$  for all  $i \leqslant j$ , it is easy to see that these separations have to be induced by edges  $e_i$  of T that all lie on a common ray in T. In particular, there is some  $j \in 2\mathbb{N}$  such that the  $e_i$  all lie on a common rooted ray in T and that  $G \uparrow e_i = V(C_i')$ . But since  $d \in C_i'$  for all  $i \in 2\mathbb{N} \setminus \{0\}$ ,

one can conclude that d does not lie in any bag of  $(T, \mathcal{V})$ , which contradicts property (T1) of tree-decompositions.

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