Elaborating Inductive Definitions

Pierre-Évariste Dagand   Conor McBride

We present an elaboration of inductive definitions down to a universe of datatypes. The universe of datatypes is an internal presentation of strictly positive families within type theory. By elaborating an inductive definition – a syntactic artifact – to its code – its semantics – we obtain an internalized account of inductives inside the type theory itself: we claim that reasoning about inductive definitions could be carried in the type theory, not in the meta-theory as it is usually the case. Besides, we give a formal specification of that elaboration process. It is therefore amenable to formal reasoning too. We prove the soundness of our translation and hint at its correctness with respect to Coq's `Inductive` definitions. The practical benefits of this approach are numerous. For the type theorist, this is a small step toward bootstrapping, i.e. implementing the inductive fragment in the type theory itself. For the programmer, this means better support for generic programming: we shall present a lightweight `deriving` mechanism, entirely definable by the programmer and therefore not requiring any extension to the type theory.

In a dependent type theory, inductive types come in various shapes and forms. Unsurprisingly, we can define data-types à la ML, following the *sum-of-product* recipe: we offer a choice of constructors and, for each constructor, comes a product of arguments. An example of such definition is the vintage and classic List datatype:

$$\textbf{data } \mathsf{List}\,[A\!:\!\textsc{Set}]\!:\!\textsc{Set} \textbf{ where}$$
$$\mathsf{List}_A \;\ni\; \mathsf{nil}$$
$$\mid\; \mathsf{cons}\,(a\!:\!A)(as\!:\!\mathsf{List}_A)$$

For the working semanticist, this brings fond memory of a golden era: this syntax has a trivial categorical interpretation in term of *signature functor*, here $L_A X = 1 + A \times X$. Without a second thought, we can brush away the syntax, mapping the syntactic representations of sum and product to their categorical counterpart. Handling parameters comes at a minor complexity cost: we merely parameterize the functor itself, for instance with A here.

However, these data-type definition are child's play in a dependently-typed setting: they do not let us use any type dependency in their definition, nor do they let us

define inductive dependent types. To obtain grown-ups' datatypes, we move to inductive families [Dybjer, 1994]: we first introduce the notion of *index* that, unlike parameters, we can *constrain* to a particular value. The archetypal example of an inductive family is the Vec datatype, which can understood as a list indexed by its length:

$$\textbf{data } \mathsf{Vec}\,[A\!:\!\textsc{Set}](n\!:\!\mathsf{Nat})\!:\!\textsc{Set} \textbf{ where}$$
$$\mathsf{Vec}_A \quad (n=0) \quad \ni \ \mathsf{nil}$$
$$\mathsf{Vec}_A\,(n=\mathsf{suc}\,n') \ \ni \ \mathsf{cons}\,(n'\!:\!\mathsf{Nat})(a\!:\!A)(vs\!:\!\mathsf{Vec}_A\,n')$$

In our syntax, we are purposely explicit about constraints on indices. Indeed, these constraints eventually turn into a statement using whatever propositional equality the underlying type theory has to offer.

Alternatively, our model of inductive families [Morris and Altenkirch, 2009] suggests that vectors could be defined by first *matching* over the index n: if n is 0, then the only possible constructor is nil, otherwise, if n is $\mathsf{suc}\,m$ for some m, it must be a cons which tail is of length m. We reflect this definition style – computing over indices – by the following syntax:

$$\textbf{data } \mathsf{Vec}\,[A\!:\!\textsc{Set}](n\!:\!\mathsf{Nat})\!:\!\textsc{Set} \textbf{ where}$$
$$\mathsf{Vec}_A \qquad n \quad \Leftarrow \ \mathsf{Nat\text{-}case}\,n$$
$$\mathsf{Vec}_A \quad 0 \quad \ni \ \mathsf{nil}$$
$$\mathsf{Vec}_A\,(\mathsf{suc}\,m) \ \ni \ \mathsf{cons}\,(a\!:\!A)(vs\!:\!\mathsf{Vec}_A\,m)$$

Note that we are here using the Epigram *by* ($\Leftarrow$) gadget [McBride and McKinna, 2004] to perform the case analysis on n. A pattern-matching notation [Coquand, 1992, Sozeau, 2010] could be used as well. When the patterns are unsurprising, we shall abuse notation and write a standard pattern match.

This definition style lets us maximally use the information provided by the indices to structure datatypes. In the constraint-based definition, we have to store an index n' against which we constrain n. In the computation-based definition, we simply match against the index. Such difference of presentation has been studied by Brady et al. [2003] in the context of various optimizations on inductive types. In our work on ornaments [McBride, 2012, Dagand and McBride, 2012], this definition style is instrumental in structuring our universes of functional ornament and enables us to lift functions across ornamented types.

Now, we ought to make sure that our language of data-type is correct, let alone semantically meaningful. Indeed, if we were to accept the following definition

$$\textbf{data } \mathsf{Bad}\,[A\!:\!\textsc{Set}]\!:\!\textsc{Set} \textbf{ where}$$
$$\mathsf{Bad}\,A \ \ni \ \mathsf{ex}\,(f\!:\!\mathsf{Bad}\,A \to A)$$

we would make many formal developments a lot easier to prove! To ban these bogus definitions, theorem provers such as Agda [Norell, 2007] or Coq [The Coq Development Team] rely on a positivity checker to ensure that all recursive arguments are in a strictly positive position. The positivity checker is therefore part of the trusted computing base of the theorem prover. Besides, by working on the syntactic representation of datatypes,

it is a non negligible piece of software that is a common source of frustration: it either stubbornly prevents perfectly valid definitions – as it sometimes is the case in Coq – or happily accepts obnoxious definitions – as Agda users discover every so often.

For the working semanticist, this is an awakening and a rude one: while reasoning about ML datatypes used to be at a functor away, she now has to cope with equality constraints and computations on indices. Most infuriatingly, we have some elegant models for inductive families but we seem stuck with some clumsy syntactic presentation: quoting Harper and Stone [2000], "the treatment of datatypes is technically complex, but conceptually straightforward". Following Stone and Harper, most authors [Asperti et al., 2012, Luo, 1994, McBride et al., 2004] have no choice but to throw in the towel and proceed over a "..."-filled skeleton of inductive definition. While this does not make these works any less correct, it makes them hard for the author to get right and for the reader to understand.

We attribute these difficulties to the formal gap between the syntax of inductive definitions and their semantics. While inductive families have an interpretation in term of strictly positive functors and their initial algebra, we are unable to leverage this knowledge. Being stuck with a syntactic artifact, the ghost of the de Bruijn criterion haunts our type theories: inductive definitions elude the type checker and must be enforced by a not-so-small positivity checker. Besides, since the syntax of inductive definitions is hardly amenable to formal reasoning, we are left wondering if its intended semantics is indeed always respected. How many inductive skeletons might be hidden in the dark closet of your favorite theorem prover?

An alternative to a purely syntactic approach is to reflect inductive types inside the type theory itself. Following Benke et al. [2003], we extend a Martin-Löf type theory with a universe of inductive families, all that for a minor complexity cost [Chapman et al., 2010]. From within the type theory, we are then able to create and manipulate datatypes but also compute over them. However, from a user perspective, these codes are a no-go: manually coding datatypes, for instance, is too cumbersome. Rather than writing low-level codes, we would like to write a honest-to-goodness inductive definition and get the computer to automatically *elaborate* it to a code in the universe.

In this paper, we elaborate upon (pun intended) the syntax of datatypes introduced in Dagand and McBride [2012]. In our previous work, we presented a conservative extension of an `Inductive`-like syntax, the twist being in our support of computation over indices. While this syntax had been informally motivated, this paper gives a formal specification of its elaboration down to our universe of datatypes. Our contributions are the following:

- In Section 2, we give a crash course in elaboration for dependent types. We will present a bidirectional type checker [Pierce and Turner, 1998] for our type theory. We then extend it to make programming a less cryptic experience. To that purpose, we shall use types as *presentations* of more high-level concepts, such as the notions of finite set or of datatype constructor. While this Section does not contain any new result *per se*, we aim at introducing the reader to a coherent collection of techniques that, put together, form a general framework for type-directed elaboration ;

- In Section 3, we specify the elaboration of inductive types down to a simple universe of inductive types. While the system we present in this Section is restricted to strictly positive types, we take advantage of its simplicity to develop our intuition. The same ideas are at play in the case of inductive families ;

- In Section 4, we specify the elaboration of inductive families down to our universe of inductive families. This system subsumes the one introduced in Section 3 but we should reuse much of the concepts developed in that Section. The novelty of our syntax is to support computation on indices, as made possible by our model of inductive families and its universe presentation ;

- In Section 5, we draw the consequences of our design choice. For the proof-assistant implementer, we show how meta-theoretical results on inductives, such as McBride et al. [2004], can be internalized and formally presented in the type theory. For the programmer, we show how a generic `deriving` mechanism à la Haskell can be implemented from within the type theory.

Scope of this work: This paper aims at *specifying* an elaboration procedure from an inductive definition down to its representation in a universe of inductive types. At the risk of disappointing implementers, we are not describing an implementation. In particular, we shall present the elaboration in a relational style, hence conveniently glancing over the operational details. Our goal is to ease the formal study of inductive definitions, hence the choice of this more abstract style. Nonetheless, this paper is not entirely disconnected from implementation. First, it grew out of our work on the Epigram system [Brady et al.], in which Peter Morris implemented a tactic elaborating an earlier form of inductive definition down to our universe of code. Second, a tutorial implementation of an elaborator for inductive types is currently underway.

1 The Type Theory

For this paper to be self-contained, we shall recall a few definitions from our previous work [Chapman et al., 2010]. We shall not dwell on the meta-theoretical properties of this system: the interested reader should consult Luo [1994] for a study of Martin-Löf type theory and Chapman et al. [2010] for its extension with a universe of datatypes.

We present our core type theory in Figure 1. It is a standard Martin-Löf type theory, with Σ-types and Π-types. We shall write SET_k for the hierarchy of types, implicitly assuming cumulativity of universes. In order to be equality-agnostic, we simply specify our expectations through a judgmental presentation. Our presentation is hopefully not controversial and should adapt easily to regional variations, such as Coq, Agda or an observational type theory [Altenkirch et al., 2007].

A first addition to this core calculus is a universe of enumerations (Fig. 2). The purpose of this universe is to let us define finite collections of labels. Labels are introduced through the UId type and are then used to define finite sets through the EnumU universe. To index a specific element in such a set, we write an EnumT code. To eliminate finite

$$\overline{\vdash \text{VALID}}$$

$$\frac{\Gamma \vdash \text{VALID} \quad \Gamma \vdash S:\text{SET}_k}{\Gamma;x:S \vdash \text{VALID}} \; x \notin \Gamma$$

$$\frac{\Gamma \vdash \text{VALID} \quad \begin{array}{c}\Gamma \vdash S:\text{SET}_k \\ \Gamma \vdash t:S\end{array}}{\Gamma;x \mapsto t:S \vdash \text{VALID}} \; x \notin \Gamma$$

(a) Context validity

$$\frac{\begin{array}{c}\Gamma \vdash S:\text{SET} \quad \Gamma;x:S \vdash t:T \\ \Gamma \vdash s:S\end{array}}{\Gamma \vdash (\lambda x:S.t)\,s \equiv t[s/x]:T[s/x]}$$

$$\frac{\begin{array}{c}\Gamma \vdash s:S \quad \Gamma;x:S \vdash T:\text{SET} \\ \Gamma \vdash t:T[s/x]\end{array}}{\Gamma \vdash \pi_0\,((s,t)_{x.T}) \equiv s:S}$$

$$\frac{\begin{array}{c}\Gamma \vdash s:S \quad \Gamma;x:S \vdash T:\text{SET} \\ \Gamma \vdash t:T[s/x]\end{array}}{\Gamma \vdash \pi_1\,((s,t)_{x.T}) \equiv t:T[s/x]}$$

(b) Judgmental equality

$$\frac{\Gamma;x:S;\Delta \vdash \text{VALID}}{\Gamma;x:S;\Delta \vdash x:S} \qquad \frac{\Gamma \vdash s:S \quad \Gamma \vdash S \equiv T:\text{SET}_k}{\Gamma \vdash s:T}$$

$$\frac{\Gamma \vdash \text{VALID}}{\Gamma \vdash \text{SET}_k:\text{SET}_{k+1}} \qquad \frac{\Gamma \vdash \text{VALID}}{\Gamma \vdash \mathbb{1}:\text{SET}_k} \qquad \frac{\Gamma \vdash \text{VALID}}{\Gamma \vdash *:\mathbb{1}}$$

$$\frac{\Gamma \vdash S:\text{SET}_k \quad \Gamma;x:S \vdash T:\text{SET}_k}{\Gamma \vdash (x:S) \times T:\text{SET}_k}$$

$$\frac{\Gamma \vdash s:S \quad \Gamma;x:S \vdash T:\text{SET}_k \quad \Gamma \vdash t:T[s/x]}{\Gamma \vdash (s,t)_{x.T}:(x:S) \times T}$$

$$\frac{\Gamma \vdash p:(x:S) \times T}{\Gamma \vdash \pi_0\,p:S} \qquad \frac{\Gamma \vdash p:(x:S) \times T}{\Gamma \vdash \pi_1\,p:T[\pi_0\,p/x]}$$

$$\frac{\Gamma \vdash S:\text{SET}_k \quad \Gamma;x:S \vdash T:\text{SET}_k}{\Gamma \vdash (x:S) \to T:\text{SET}_k}$$

$$\frac{\begin{array}{c}\Gamma \vdash S:\text{SET}_k \\ \Gamma;x:S \vdash t:T\end{array}}{\Gamma \vdash \lambda x:S.t:(x:S) \to T} \qquad \frac{\begin{array}{c}\Gamma \vdash f:(x:S) \to T \\ \Gamma \vdash s:S\end{array}}{\Gamma \vdash f\,s:T[s/x]}$$

(c) Typing judgments

Figure 1: Type theory

sets, we form a small Π-type $\pi:(E:\text{EnumU})(P:\text{EnumT } E \to \text{SET}) \to \text{SET}$ that builds a lookup tuple mapping, for all label e in the enumeration, a value of type $P\,e$. To perform the lookup, we define the eliminator:

$$\text{switch}:(E:\text{EnumU})(P:\text{EnumT } E \to \text{SET}) \to \pi\,E\,P \to (x:\text{EnumT } E) \to P\,x$$

Example 1 (Coding $\{'a,'b,'c\}$)**.** We define this set by merely enumerating its labels, in effect building a list of tags:

$$\{'a,'b,'c\} \triangleq \text{consE } 'a\,(\text{consE } 'b\,(\text{consE } 'c\,\text{nilE})):\text{EnumU}$$

Example 2 (Coding $\{'a \mapsto e_a,'b \mapsto e_b,'c \mapsto e_c\}:(x:\text{EnumT }\{'a,'b,'c\}) \to P\,x$)**.** We define this function by a straightforward application of the switch eliminator:

$$\{'a \mapsto e_a,'b \mapsto e_b,'c \mapsto e_c\} \triangleq \text{switch}\,(\text{consE } 'a\,(\text{consE } 'b\,(\text{consE } 'c\,\text{nilE})))\,P\,(e_a,(e_b,(e_c,*)))$$

We recall the definition of our universe of inductive types in Figure 3. This universe captures strictly positive types, a generalization of ML datatypes to dependent types. For pedagogical reason, we choose this simple universe as a first step toward a full-blown universe of inductive families. The idea at work behind this presentation is the following: to define new datatypes, we give their code by *describing* their signature functor in Desc. The interpretation function $\llbracket \_ \rrbracket$ turns such a description into the corresponding endofunctor over SET. Its definition is obvious from the codes: a $'\Sigma$ is interpreted into

Figure 2: Universe of enumerations

$$\frac{\Gamma \vdash \text{VALID}}{\Gamma \vdash \text{UId} : \text{SET}}$$

$$\frac{\Gamma \vdash \text{VALID}}{\Gamma \vdash \text{'}s : \text{UId}} \; s \text{ a valid identifier}$$

(a) Tags

$$\frac{\Gamma \vdash \text{VALID}}{\Gamma \vdash \text{EnumU} : \text{SET}}$$

$$\frac{\Gamma \vdash \text{VALID}}{\Gamma \vdash \text{nilE} : \text{EnumU}}$$

$$\frac{\Gamma \vdash t : \text{UId} \quad \Gamma \vdash E : \text{EnumU}}{\Gamma \vdash \text{consE} \, t \, E : \text{EnumU}}$$

(b) Enumeration

$$\frac{\Gamma \vdash E : \text{EnumU}}{\Gamma \vdash \text{EnumT} \, E : \text{SET}}$$

$$\frac{\Gamma \vdash \text{VALID}}{\Gamma \vdash 0 : \text{EnumT} \, (\text{consE} \, t \, E)}$$

$$\frac{\Gamma \vdash n : \text{EnumT} \, E}{\Gamma \vdash 1\text{+} \, n : \text{EnumT} \, (\text{consE} \, t \, E)}$$

(c) Index

Figure 2: Universe of enumerations

$$\frac{\Gamma \vdash \text{VALID}}{\Gamma \vdash \text{Desc} : \text{SET}_1}$$

$$\frac{\Gamma \vdash \text{VALID}}{\Gamma \vdash \text{'var} : \text{Desc}} \qquad \frac{\Gamma \vdash \text{VALID}}{\Gamma \vdash \text{'1} : \text{Desc}}$$

$$\frac{\Gamma \vdash A : \text{Desc} \quad \Gamma \vdash B : \text{Desc}}{\Gamma \vdash A \, \text{'}\!\times B : \text{Desc}}$$

$$\frac{\Gamma \vdash S : \text{SET} \quad \Gamma \vdash T : S \to \text{Desc}}{\Gamma \vdash \text{'}\Pi \, S \, T : \text{Desc}}$$

$$\frac{\Gamma \vdash S : \text{SET} \quad \Gamma \vdash T : S \to \text{Desc}}{\Gamma \vdash \text{'}\Sigma \, S \, T : \text{Desc}}$$

$$\frac{\Gamma \vdash E : \text{EnumU} \quad \Gamma \vdash T : \text{EnumT} \, E \to \text{Desc}}{\Gamma \vdash \text{'}\sigma \, E \, T : \text{Desc}}$$

(a) Codes

$$[\![(D : \text{Desc})]\!] \, (X : \text{SET}) \; : \; \text{SET}$$
$$[\![\text{'var}]\!] \quad X \mapsto X$$
$$[\![\text{'1}]\!] \quad X \mapsto \mathbb{1}$$
$$[\![A \, \text{'}\!\times B]\!] \; X \mapsto [\![A]\!] \, X \times [\![B]\!] \, X$$
$$[\![\text{'}\Pi \, S \, T]\!] \; X \mapsto (s : S) \to [\![T \, s]\!] \, X$$
$$[\![\text{'}\Sigma \, S \, T]\!] \; X \mapsto (s : S) \times [\![T \, s]\!] \, X$$
$$[\![\text{'}\sigma \, E \, T]\!] \; X \mapsto (e : \text{EnumT} \, E) \times [\![T \, e]\!] \, X$$

$$\frac{\Gamma \vdash D : \text{Desc}}{\Gamma \vdash \mu \, D : \text{SET}} \qquad \frac{\Gamma \vdash xs : [\![D]\!] \, \mu \, D}{\Gamma \vdash \text{in} \, xs : \mu \, D}$$

(b) Fix-point

$$\text{induction} : (D : \text{Desc})(P : \mu D \to \text{SET}) \to ((d : [\![D]\!] \, (\mu D)) \to \text{All} \, D \, (\mu D) \, P \, d \to P(\text{in} \, d)) \to (x : \mu D) \to P \, x$$

(c) Elimination principle

Figure 3: Universe of inductive types

a Σ-type, and so on. The notable exception is 'var, which describes the identity functor. Remark that the resulting functor are strictly positive, by construction. Hence, we can construct their least fix-point using μ and safely provide a generic elimination principle, induction. The All function computes the inductive hypothesis. The interested reader will find their precise definition in Chapman et al. [2010] but the basic intuition we have given here is enough to understand this paper.

Example 3 (Describing natural numbers). Natural numbers can be presented as the least fix-point of the functor $F X = 1 + X$. This is described by:

$$\text{NatD} \; : \; \text{Desc}$$
$$\text{NatD} \mapsto \text{'}\sigma \left\{ \begin{array}{l} \text{'0,} \\ \text{'suc} \end{array} \right\} \left\{ \begin{array}{l} \text{'0} \; \mapsto \text{'1,} \\ \text{'suc} \mapsto \text{'var} \end{array} \right\}$$

$$\text{Nat} \; : \; \text{SET}$$
$$\text{Nat} \mapsto \mu \, \text{NatD}$$

Note that we are using a small 'σ here: a datatype definition always starts with a finite choice of constructors. This corresponds to the *tagged descriptions* of Chapman et al. [2010]: using this structure, we can implement some generic constructions – such as the Zipper or the free monad – and generic theorems – such as in Section 5.1.

6

$$\frac{\Gamma \vdash I : \text{SET}}{\Gamma \vdash \text{IDesc}\, I : \text{SET}}$$

$$\frac{\Gamma \vdash i : I}{\Gamma \vdash \text{'var}\, i : \text{IDesc}\, I}$$

$$\frac{\Gamma \vdash \text{VALID}}{\Gamma \vdash \text{'1} : \text{IDesc}\, I}$$

$$\frac{\Gamma \vdash A : \text{IDesc}\, I \quad \Gamma \vdash B : \text{IDesc}\, I}{\Gamma \vdash A \mathbin{'\!\times} B : \text{IDesc}\, I}$$

$$\frac{\Gamma \vdash S : \text{SET} \quad \Gamma \vdash T : S \to \text{IDesc}\, I}{\Gamma \vdash \text{'}\Pi\, S\, T : \text{IDesc}\, I}$$

$$\frac{\Gamma \vdash S : \text{SET} \quad \Gamma \vdash T : S \to \text{IDesc}\, I}{\Gamma \vdash \text{'}\Sigma\, S\, T : \text{IDesc}\, I}$$

$$\frac{\Gamma \vdash E : \text{EnumU} \quad \Gamma \vdash T : \text{EnumT}\, E \to \text{IDesc}\, I}{\Gamma \vdash \text{'}\sigma\, E\, T : \text{IDesc}\, I}$$

(a) Codes

$$\llbracket (D : \text{IDesc}\, I) \rrbracket \, (X : I \to \text{SET}) \; : \; \text{SET}$$
$$\llbracket \text{'var}\, i \rrbracket \quad X \; \mapsto \; X\, i$$
$$\llbracket \text{'1} \rrbracket \qquad X \; \mapsto \; \mathbb{1}$$
$$\llbracket A \mathbin{'\!\times} B \rrbracket \quad X \; \mapsto \; \llbracket A \rrbracket\, X \times \llbracket B \rrbracket\, X$$
$$\llbracket \text{'}\Pi\, S\, T \rrbracket \quad X \; \mapsto \; (s : S) \to \llbracket T\, s \rrbracket\, X$$
$$\llbracket \text{'}\Sigma\, S\, T \rrbracket \quad X \; \mapsto \; (s : S) \times \llbracket T\, s \rrbracket\, X$$
$$\llbracket \text{'}\sigma\, E\, T \rrbracket \quad X \; \mapsto \; (e : \text{EnumT}\, E) \times \llbracket T\, e \rrbracket\, X$$

$$\frac{\Gamma \vdash f_D : I \to \text{IDesc}\, I}{\Gamma \vdash \mu^i\, f_D : I \to \text{SET}}$$

$$\frac{\Gamma \vdash xs : \llbracket f_D\, i \rrbracket\, (\mu^i\, D)}{\Gamma \vdash \text{in}\, xs : \mu^i\, f_D\, i}$$

(b) Fix-point

$$\text{iinduction} : (R : I \to \text{IDesc}\, I)(P : ((i : I) \times \mu^i\, R\, i) \to \text{SET}) \to$$
$$((i : I)(xs : \llbracket R\, i \rrbracket\, (\mu^i\, R)) \to \text{iAll}\, (R\, i)\, (\mu^i\, R)\, xs\, P \to P\, (i, \text{in}\, xs)) \to$$
$$(i : I)(x : \mu^i\, I\, R\, i) \to P\, (i, x)$$

(c) Elimination principle

Figure 4: Universe of inductive families

Example 4 (Describing binary trees). Categorically, binary trees are modeled by the least fix-point of the functor $T_A X = 1 + A \times X^2$. In our universe, we obtain trees by the following definitions:

$$\text{TreeD}\, (A : \text{SET}) \; : \; \text{Desc}$$
$$\text{TreeD} \quad A \quad \mapsto \text{'}\sigma \left\{ \begin{array}{l} \text{'leaf,} \\ \text{'node} \end{array} \right\} \left\{ \begin{array}{l} \text{'leaf} \mapsto \text{'1,} \\ \text{'node} \mapsto \text{'var} \mathbin{'\!\times} \text{'}\Sigma\, A\, \lambda_{\_}.\, \text{'var} \end{array} \right\}$$
$$\text{Tree}\, (A : \text{SET}) \; : \; \text{SET}$$
$$\text{Tree} \quad A \quad \mapsto \mu\, (\text{TreeD}\, A)$$

Finally, we recall the universe of indexed descriptions (Fig. 4), which captures inductive families. This universe subsumes the previous one by not only allowing parameters but also indices: thanks to this universe, we can describe datatypes such as vector. Just as before, we give a universe of codes, IDesc, that are then interpreted to build the least fix-point. Note that the argument of μ^i is a *function* from indices to codes: this particularity lets us define datatypes by computation over the index. We illustrate this point through two possible implementations of the vector datatype.

Example 5 (Vector, à la Coq). The standard presentation of inductive families is by using equality constraints to force the indexing strategy. In effect, this corresponds to

the following definition of vectors:

$$\mathsf{VecD}\ (A:\textsc{Set})\ (n:\mathsf{Nat})\ :\ \mathsf{IDesc\ Nat}$$

$$\mathsf{VecD}\quad A\qquad n\quad \mapsto\quad {}'\sigma \begin{Bmatrix} \text{'nil,} \\ \text{'cons} \end{Bmatrix}$$
$$\begin{Bmatrix} \text{'nil} \mapsto {}'\Sigma\ (n \equiv 0)\ \lambda\_.\ \text{'1,} \\ \text{'cons} \mapsto {}'\Sigma\ \mathsf{Nat}\ \lambda m.\ {}'\Sigma\ (n \equiv (\mathsf{suc}\ m))\ \lambda\_.\ {}'\Sigma\ A\ \lambda\_.\ \text{'var}\ m \end{Bmatrix}$$

$$\mathsf{Vec}\ (A:\textsc{Set})\ (n:\mathsf{Nat})\ :\ \textsc{Set}$$
$$\mathsf{Vec}\quad A\qquad n\quad \mapsto \mu^i\ (\mathsf{VecD}\ A)\ n$$

Example 6 (Vector, alternatively)**.** Interestingly, in the previous definition, we are defining Vec_D as a function from $(n:\mathsf{Nat})$ to $\mathsf{IDesc\ Nat}$ but we are not making use of our ability to inspect n: we merely constrain it to be what we wish it was. We can make full use of that function space by *first* checking whether n is 0 or $\mathsf{suc}\ m$ and *only then* give a code for the corresponding constructor, respectively nil and cons:

$$\mathsf{VecD}\ (A:\textsc{Set})\ (n:\mathsf{Nat})\ :\ \mathsf{IDesc\ Nat}$$
$$\mathsf{VecD}\quad A\qquad 0\quad \mapsto \text{'1}$$
$$\mathsf{VecD}\quad A\quad (\mathsf{suc}\ m)\ \mapsto {}'\Sigma\ A\ \lambda\_.\ \text{'var}\ m$$

$$\mathsf{Vec}\ (A:\textsc{Set})\ (n:\mathsf{Nat})\ :\ \textsc{Set}$$
$$\mathsf{Vec}\quad A\qquad n\quad \mapsto \mu^i\ (\mathsf{VecD}\ A)\ n$$

2 First Steps in Elaboration

In this Section, we shall make our first steps in elaboration. First, we present a bidirectional type system for the calculus introduced in the previous section. Using the flow of information from type synthesis to type checking, we are then able to declutter our term language. Secondly, we adapt the notion of programming problem [McBride and McKinna, 2004] to our system and we shall see how, with little effort, we can move away from an austere calculus and closer to a proper programming language.

2.1 Bidirectional type checking

The idea of bidirectional type checking [Pierce and Turner, 1998] is to capture, in the specification of the type checker, the local flow of typing information. On the one hand, we will *synthesize* types from variables and functions while, on the other hand, we will *check* terms against these synthesized types. By checking terms against their types, we can use types to structure the term language: for example, we remove the need for writing the domain type in an abstraction, or we can use a tuple notation for telescopes of Σ-types. Following [Harper and Stone, 2000], we distinguish two categories of terms. The core type theory defines the *internal* language. The bidirectional approach lets us then extend this core language into a more convenient *external* language. We shall hasten to add that using a bidirectional approach in a dependently-typed framework is not a novel idea: for instance, it is the basis of Matita's refinement system [Asperti et al., 2012]. Also, we gave a similar presentation in some earlier work [Chapman et al., 2010].

$$\boxed{\Gamma \vdash t \overset{Inf}{\leadsto} t' \in T}$$

$$\frac{\Gamma \vdash \mathrm{Set}_k \ni T \overset{Chk}{\leadsto} T' \quad \Gamma \vdash T' \ni t \overset{Chk}{\leadsto} t'}{\Gamma \vdash (t\!:\!T) \overset{Inf}{\leadsto} t' \in T'}$$

$$\frac{\Gamma; x\!:\!S; \Delta \vdash \mathrm{VALID}}{\Gamma; x\!:\!S; \Delta \vdash x \overset{Inf}{\leadsto} x \in S}$$

$$\frac{\begin{array}{c}\Gamma \vdash f \overset{Inf}{\leadsto} f' \in (x\!:\!S) \to T \\ \Gamma \vdash S \ni s \overset{Chk}{\leadsto} s'\end{array}}{\Gamma \vdash f\, s \overset{Inf}{\leadsto} f'\, s' \in T[s'/x]}$$

$$\frac{\Gamma \vdash p \overset{Inf}{\leadsto} p' \in (x\!:\!S) \times T}{\Gamma \vdash \pi_0 p \overset{Inf}{\leadsto} \pi_0 p' \in S}$$

$$\frac{\Gamma \vdash p \overset{Inf}{\leadsto} p' \in (x\!:\!S) \times T}{\Gamma \vdash \pi_1 p \overset{Inf}{\leadsto} \pi_1 p' \in T[\pi_0 p'/x]}$$

(a) Type synthesis

$$\boxed{\Gamma \vdash T \ni t \overset{Chk}{\leadsto} t'}$$

$$\frac{\Gamma \vdash s \overset{Inf}{\leadsto} s' \in S \quad \Gamma \vdash S \equiv T : \mathrm{Set}_k}{\Gamma \vdash T \ni s \overset{Chk}{\leadsto} s'}$$

$$\frac{\Gamma \vdash \mathrm{VALID}}{\Gamma \vdash \mathrm{Set}_{k+1} \ni \mathrm{Set}_k \overset{Chk}{\leadsto} \mathrm{Set}_k}$$

$$\frac{\Gamma \vdash \mathrm{Set}_k \ni S \overset{Chk}{\leadsto} S' \quad \Gamma; x\!:\!S' \vdash \mathrm{Set}_k \ni T \overset{Chk}{\leadsto} T'}{\Gamma \vdash \mathrm{Set}_k \ni (x\!:\!S) \to T \overset{Chk}{\leadsto} (x\!:\!S') \to T'}$$

$$\frac{\Gamma; x\!:\!S \vdash T \ni t \overset{Chk}{\leadsto} t'}{\Gamma \vdash (x\!:\!S) \to T \ni \lambda x.\, t \overset{Chk}{\leadsto} \lambda x\!:\!S.\, t'}$$

$$\frac{\Gamma \vdash \mathrm{Set}_k \ni S \overset{Chk}{\leadsto} S' \quad \Gamma; x\!:\!S' \vdash \mathrm{Set}_k \ni T \overset{Chk}{\leadsto} T'}{\Gamma \vdash \mathrm{Set}_k \ni (x\!:\!S) \times T \overset{Chk}{\leadsto} (x\!:\!S') \times T'}$$

$$\frac{\Gamma \vdash S \ni s \overset{Chk}{\leadsto} s' \quad \Gamma \vdash T[s'/x] \ni t \overset{Chk}{\leadsto} t'}{\Gamma \vdash (x\!:\!S) \times T \ni (s,t) \overset{Chk}{\leadsto} (s',t')_{x.T}}$$

$$\frac{\Gamma \vdash \mathrm{VALID}}{\Gamma \vdash \mathrm{Set}_k \ni \mathbb{1} \overset{Chk}{\leadsto} \mathbb{1}} \qquad \frac{\Gamma \vdash \mathrm{VALID}}{\Gamma \vdash \mathbb{1} \ni * \overset{Chk}{\leadsto} *}$$

(b) Type checking

Figure 5: Bidirectional type checker

Let us present type synthesis (Fig. 5a) first. The judgment $\Gamma \vdash t \overset{Inf}{\leadsto} t' \in T$ states that the external term t elaborates to the internal term t' of type T in the context Γ. By convention, we shall keep inputs to the relation to the left of the $\leadsto$ symbol, while outputs will be on the right. We expect the following soundness property to hold:

Theorem 1 (Soundness of type synthesis). *If $\Gamma \vdash t \overset{Inf}{\leadsto} t' \in T$, then $\Gamma \vdash t'\!:\!T$.*

While type synthesis initiates the flow of typing information, type checking (Fig. 5b) lets us make use of this information to enrich the language of terms. The interpretation of the judgment $\Gamma \vdash T \ni t \overset{Chk}{\leadsto} t'$ is that the external term t is checked against the type T in context Γ and elaborates to an internal term t'. Again, we expect the following soundness property to hold:

Theorem 2 (Soundness of type checking). *If $\Gamma \vdash T \ni t \overset{Chk}{\leadsto} t'$, then $\Gamma \vdash t'\!:\!T$.*

9

2.2 Putting types at work

While the type checker we have specified so far only allows untyped abstractions, we extend it with some convenient features.

Tuples: By design of the interpretation of our universes of enumeration and inductive types, we are often going to build inhabitants of Σ-telescope of the form $(a:A) \times (b:B) \times \ldots \times (z:Z) \times \mathbb{1}$. To reduce the syntactic burden of these nested pairs, we elaborate a LISP-inspired tuple notation:

$$\frac{}{\Gamma \vdash \mathbb{1} \ni () \overset{Chk}{\leadsto} *} \qquad \frac{\Gamma \vdash A \ni x \overset{Chk}{\leadsto} x' \quad \Gamma \vdash B[x'/a] \ni (xs) \overset{Chk}{\leadsto} xs'}{\Gamma \vdash (a:A) \times B \ni (x\ xs) \overset{Chk}{\leadsto} (x', xs')_{a.B}}$$

Finite sets, introduction and elimination: In Section 1, we have used an informal set-like notation for enumerations: we can make that notation formal through elaboration. To do so, we extend the type checker with the following rules:

$$\frac{}{\Gamma \vdash \mathsf{EnumU} \ni \{\} \overset{Chk}{\leadsto} \mathsf{nilE}} \qquad \frac{\Gamma \vdash \mathsf{EnumU} \ni \{ts\} \overset{Chk}{\leadsto} E}{\Gamma \vdash \mathsf{EnumU} \ni \{'a, ts\} \overset{Chk}{\leadsto} \mathsf{consE}\ 'a\ E}$$

$$\frac{\Gamma \vdash \mathsf{EnumU} \ni \{'l_0, \ldots, 'l_k\} \overset{Chk}{\leadsto} E \qquad \Gamma \vdash \pi\ E\ P \ni (e_0 \ldots e_k) \overset{Chk}{\leadsto} es}{\Gamma \vdash (e:\mathsf{EnumT}\ E) \to P\ e \ni \{'l_0 \mapsto e_0, \ldots 'l_k \mapsto e_k\} \overset{Chk}{\leadsto} \mathsf{switch}\ E\ P\ es}$$

Indexing finite sets: Also, rather than indexing into enumerations through the EnumT codes, we would like to be able to write the label and have it elaborate to the corresponding index. This is achieved by the following extension:

$$\frac{}{\Gamma \vdash \mathsf{consE}\ 't\ E \ni 't \overset{Chk}{\leadsto} 0} \qquad \frac{\Gamma \vdash E \ni 't \overset{Chk}{\leadsto} n}{\Gamma \vdash \mathsf{consE}\ 'u\ E \ni 't \overset{Chk}{\leadsto} 1+n}$$

Datatype constructors: Finally, while we do not yet have a proper syntax for *declaring* inductive types, we can already extend our term language with constructors. Upon elaborating an external term $c\ a_0 \ldots a_k$ against the fix-point of a tagged description, we replace this elaboration problem with the one consisting of elaborating the tuple consisting of the constructor label and the arguments:

$$\frac{\Gamma \vdash \mu\ ('\sigma\ E\ T) \ni \mathsf{in}\ ('c\ a_0 \ldots a_k) \overset{Chk}{\leadsto} t}{\Gamma \vdash \mu\ ('\sigma\ E\ T) \ni c\ a_0 \ldots a_k \overset{Chk}{\leadsto} t} \qquad \frac{\Gamma \vdash \mu^i\ ('\sigma\ E\ T)\ i \ni \mathsf{in}\ ('c\ a_0 \ldots a_k) \overset{Chk}{\leadsto} t}{\Gamma \vdash \mu^i\ ('\sigma\ E\ T)\ i \ni c\ a_0 \ldots a_k \overset{Chk}{\leadsto} t}$$

$$\dfrac{\Gamma \vdash \text{VALID}}{\Gamma \vdash \mathbf{f}:\mathsf{label}} \qquad \dfrac{\Gamma \vdash l:\mathsf{label} \quad \Gamma \vdash t:T}{\Gamma \vdash l\, t:\mathsf{label}}$$

$$\dfrac{\Gamma \vdash l:\mathsf{label} \quad \Gamma \vdash T:\text{SET}}{\Gamma \vdash \langle l:T\rangle:\text{SET}} \qquad \dfrac{\Gamma \vdash t:T \quad \Gamma \vdash l:\mathsf{label}}{\Gamma \vdash \mathsf{return}\, t:\langle l:T\rangle} \qquad \dfrac{\Gamma \vdash c:\langle l:T\rangle}{\Gamma \vdash \mathsf{call}\langle l:T\rangle c:T}$$

(a) Programming label

$$\boxed{\Gamma \vdash \mathbf{let}\ f\ \overrightarrow{(x:X)}:T\ \mathbf{where}\ t \overset{L}{\rightsquigarrow} \Delta}$$

$$\dfrac{\begin{array}{c}\Gamma \vdash \text{SET} \ni \overrightarrow{(x:X)} \to T \overset{Chk}{\rightsquigarrow} \overrightarrow{(x:X')} \to T' \\ \Gamma, \overrightarrow{(x:X')} \vdash \langle \mathbf{f}\ \vec{x}:T'\rangle \ni t \overset{P}{\rightsquigarrow} t'\end{array}}{\Gamma \vdash \mathbf{let}\ f\ \overrightarrow{(x:X)}:T\ \mathbf{where}\ t \overset{L}{\rightsquigarrow} \Gamma[f \mapsto \lambda\vec{x}:\vec{X'}.\,\mathsf{call}\langle \mathbf{f}\ \vec{x}:T'\rangle t':\overrightarrow{(x:X')} \to T']}$$

(b) Program declaration

$$\boxed{\Gamma \vdash \langle l:T\rangle \ni t \overset{P}{\rightsquigarrow} t'}$$

$$\dfrac{l \triangleright pt \qquad \Gamma \vdash T \ni e \overset{Chk}{\rightsquigarrow} e'}{\Gamma \vdash \langle l:T\rangle \ni pt \mapsto e \overset{P}{\rightsquigarrow} \mathsf{return}\, e'}$$

$$\dfrac{l \triangleright pt \qquad \begin{array}{c}\Gamma \vdash e \overset{ewm}{\rightsquigarrow} e' \in \overrightarrow{(\overrightarrow{(x_i:X_i)} \to \langle l_i:S_i\rangle)} \to \langle l:T\rangle \\ \Gamma, \overrightarrow{(x_i:X_i)} \vdash \langle l_i:S_i\rangle \ni p_i \overset{P}{\rightsquigarrow} s_i\end{array}}{\Gamma \vdash \langle l:T\rangle \ni pt \Leftarrow e\{\vec{p_i}\} \overset{P}{\rightsquigarrow} e'\,(\lambda\vec{x_i}:\vec{X_i}.\,s_i)}$$

(c) Program definition

Figure 6: Programming labels

2.3 Elaborating Programs

In the previous Section, we have define type-specific languages, relying on the types to guide the elaboration process. In this Section, we go a step further and purposely define finely indexed types for the purpose of elaboration. This idea of using types as a *presentation* of our high-level intentions rather than a *representation* of low-level restrictions originated in McBride and McKinna [2004]. To illustrate our point, we redevelop the elaboration of programs presented in that paper and adapt it to our framework. We shall see how these presentation types guide the elaboration of programs and let us regain a pattern-matching notation without hard-wiring pattern-matching itself.

To guide elaboration of programs, we define *programming label* types (Fig. 6). In essence, a programming label $\langle l:T\rangle$ is a phantom type around the type T. However, the label l is crucial in that it represents the arguments of the function we are implementing. A motivating example of label types is the definition of addition by explicitly using the elimination principle of natural numbers:

$$\begin{array}{lll} (m:\mathsf{Nat}) + (n:\mathsf{Nat}) &:& \langle '+\ m\ n:\mathsf{Nat}\rangle \\ m \quad + \quad n &\mapsto& \mathsf{Nat\text{-}ind}\ (\lambda m'.\,\langle '+\ m'\ n:\mathsf{Nat}\rangle)\ m \\ & & \qquad \{?:\langle '+\ 0\ n:\mathsf{Nat}\rangle\} \\ & & \qquad \{?:(m:\mathsf{Nat}) \to \langle '+\ m\ n:\mathsf{Nat}\rangle \to \langle '+\ (\mathsf{suc}\ m)\ n:\mathsf{Nat}\rangle\} \end{array}$$

11

Thanks to the label types, we can relate every case of the elimination principle with the state of our programming problem.

Using these label types, we can extend our external language to ease programming. First, we introduce a programming problem by the **let** command (Fig. 6b). The elaboration judgment is subject to the following soundness property:

Theorem 3 (Soundness of program elaboration). *If* $\Gamma \vdash$ **let** $f\ \overrightarrow{(x:X)}:T$ **where** $t \overset{L}{\rightsquigarrow} \Delta$, *then* $\Delta \vdash$ VALID.

To elaborate the definition of the program, we restrict our attention to two constructs: returning a value using the *Return* ($\mapsto$) gadget and appealing to an elimination principle using the *By* ($\Leftarrow$) gadget. We shall ignore views and other syntactic conveniences originally introduced by McBride and McKinna [2004]. The elaboration rules are given in Figure 6c. The idea is that they are two rules to realize a programming label: either we return a term, or we decompose the problem further using an eliminator. We have coded the *programming grammar* within this specification. Remark the presence of patterns pt on the left hand side to mimic a pattern-matching notation. Intuitively, the relation $l \triangleright pt$ makes sure that the pattern entered by user corresponds to the label computed by the type-checker. The relation $\Gamma \vdash e \overset{ewm}{\rightsquigarrow} e' \in T$ abstracts away the task of generating a term e' in the internal language from an elimination form e, as described in Goguen et al. [2006], McBride [2002]. We will not recall this construction and simply assume that the generated term is well-typed with respect to T, i.e. $\Gamma \vdash e':T$.

3 Elaborating Datatypes

In this Section, we specify the elaboration of inductive types down to our Desc universe. While this universe only captures strictly positive types, it is a good exercise to understand the general idea governing the elaboration of inductive definitions. Besides, because the syntax is essentially the same, our presentation should be easy to understand for readers familiar with Coq or Agda.

We adopt the standard sum-of-product high-level notation:

$$
\begin{aligned}
&\textbf{data D } \overrightarrow{[p:P]}:\textsc{Set}\ \textbf{where}\\
&\quad \mathsf{D}\ \overrightarrow{p}\ \ni\ \mathsf{c_0}\ \overrightarrow{(a_0:T_0)}\\
&\qquad\quad |\ \dots\\
&\qquad\quad |\ \mathsf{c_k}\ \overrightarrow{(a_k:T_k)}
\end{aligned}
$$

Where the arguments $\vec{p}$ are parameters. A T_i can be recursive, i.e. refers to $\mathsf{D}\ \vec{p}$. Note that it is crucial that the parameters are the same in the definition and the recursive calls.

Our translation to code follows the structure of the definition. The first level structure consists of the choice of constructors and is translated to a 'σ code over the finite set of constructors. The second level structure consists of the Σ-telescope of arguments: these are translated to right-nested 'Σ codes. When parsing arguments, we must make sure that the recursive arguments are valid and translate them to the 'var code.

$$\frac{}{\Gamma \vdash \mathbf{D}\ \mathrm{datatel}} \qquad \frac{\Gamma \vdash l\ \mathrm{datatel} \qquad \Gamma \vdash t : T}{\Gamma \vdash l\ t\ \mathrm{datatel}}$$

$$\frac{\Gamma \vdash l\ \mathrm{datatel}}{\Gamma \vdash \langle l \rangle : \mathrm{Set}_1} \qquad \frac{\Gamma \vdash E : \mathsf{EnumU} \qquad \Gamma \vdash T : \mathsf{EnumT}\ E \to \mathsf{Desc}}{\Gamma \vdash \mathsf{return}\ E\ T : \langle l \rangle} \qquad \frac{\Gamma \vdash t : \langle l \rangle}{\Gamma \vdash \mathsf{call}\ \langle l \rangle\ t : \mathsf{Desc}}$$

Figure 7: Description label

3.1 Description labels

To guide the elaboration of inductive definitions, we extend the type theory with *description labels* (Fig. 7). Their role is akin to programming labels: they structure the elaboration task and are used to ensure that recursive arguments are correctly elaborated.

A description label $\langle l \rangle$ is a list starting with the name of the datatype being defined and followed by the parameters of that datatype. It can be thought as a phantom type around Desc. We can introduce such type using return that takes the (finite) set of constructors and their respective code: doing so, we ensure that we are only accepting tagged descriptions. With call $\langle l \rangle$, we eliminate return by joining constructors and their codes in a 'σ code, effectively interpreting the choice of constructors.

3.2 Elaborating inductive types

We shall present our translation in a top-down manner: from a complete definition, we show how the pieces fit together, giving some intuition for the subsequent translations. We then move on to disassemble and interpret each sub-component separately. As we progress, the reader should check that the intuition we gave for the whole is indeed valid. Every elaboration step is backed by a soundness property: proving these properties is inherently bottom-up. Only after having developed all our definition will we be able to prove the soundness of elaboration of inductive definitions. However, the proof is technically unsurprising: we shall briefly sketch it at the end of this Section. To further ease the understanding of our machinery, we shall illustrate every step by elaborating the following definition of binary trees:

> **data** Tree $[A : \mathrm{Set}] : \mathrm{Set}$ **where**
> Tree $A \ni$ leaf
> $\quad\quad$ | node $(l : \mathsf{Tree}\ A)(a : A)(r : \mathsf{Tree}\ A)$

Elaboration of an inductive definition (Fig. 8a): Elaborating an inductive definition extends the input context with the datatype definition. To obtain this definition, we

13

$$\boxed{\Gamma \vdash \textbf{data } D\,\overrightarrow{(p\,{:}\,P)}\,{:}\,\textsc{Set}\ \textbf{where}\ \text{choices} \overset{D}{\leadsto} \Delta}$$

$$\frac{\begin{array}{c}\Gamma \vdash \textsc{Set}_1 \ni \overrightarrow{(p\,{:}\,P)} \to \textsc{Set} \overset{Chk}{\leadsto} \overrightarrow{(p\,{:}\,P')} \to \textsc{Set}\\[2pt] \Gamma, \overrightarrow{p\,{:}\,P'} \vdash \langle \mathbf{D}\ \vec{p}\rangle \ni \text{choices} \overset{Cs}{\leadsto} \text{code}\end{array}}{\Gamma \vdash \textbf{data } D\,\overrightarrow{[p\,{:}\,P]}\,{:}\,\textsc{Set}\ \textbf{where}\ \text{choices} \overset{D}{\leadsto} \Gamma[D \mapsto \lambda\vec{p}.\ \mu\ (\textsf{call}\ \langle \mathbf{D}\ \vec{p}\rangle\ \text{code})\,{:}\,\overrightarrow{(p\,{:}\,P')} \to \textsc{Set}]}$$

<div align="center">(a) Elaboration of definition</div>

$$\boxed{\Gamma \vdash \langle l\rangle \ni \text{choices} \overset{Cs}{\leadsto} \text{code}}$$

$$\frac{\Gamma \vdash \langle l\rangle \ni c_i \overset{C}{\leadsto}[t_i \mapsto \text{code}_i] \qquad \begin{array}{c}\Gamma \vdash \textsf{EnumU} \ni \{t_i\} \overset{Chk}{\leadsto} E\\[2pt] \Gamma \vdash \textsf{EnumT}\ E \to \textsf{Desc} \ni \{t_i \mapsto \text{code}_i\} \overset{Chk}{\leadsto} T\end{array}}{\Gamma \vdash \langle l\rangle \ni c_0 |\ldots| c_n \overset{Cs}{\leadsto} \textsf{return}\ E\ T}$$

<div align="center">(b) Elaboration of choices</div>

$$\boxed{\Gamma \vdash \langle l\rangle \ni c \overset{C}{\leadsto}[t \mapsto \text{code}]}$$

$$\frac{\begin{array}{c}\Gamma \vdash \textsf{UId} \ni {'t} \overset{Chk}{\leadsto} t'\\[2pt] \Gamma \vdash \langle l\rangle \ni \text{args} \overset{A}{\leadsto} \text{code}\end{array}}{\Gamma \vdash \langle l\rangle \ni t\ \text{args} \overset{C}{\leadsto}[t' \mapsto \text{code}]}$$

<div align="center">(c) Elaboration of constructor</div>

$$\boxed{\Gamma \vdash \langle l\rangle \ni \text{args} \overset{A}{\leadsto} \text{code}}$$

$$\frac{\Gamma \vdash \textsc{Set} \ni T \overset{Chk}{\leadsto} T' \qquad \Gamma, x\,{:}\,T' \vdash \langle l\rangle \ni \Delta \overset{A}{\leadsto} \text{code}_\Delta}{\Gamma \vdash \langle l\rangle \ni (x\,{:}\,T)\Delta \overset{A}{\leadsto} {'\Sigma}\ T'\ \lambda x.\,\text{code}_\Delta} \qquad \frac{T \triangleright l \qquad \Gamma \vdash \langle l\rangle \ni \Delta \overset{A}{\leadsto} \text{code}_\Delta}{\Gamma \vdash \langle l\rangle \ni (x\,{:}\,T)\Delta \overset{A}{\leadsto} {'\textsf{var}}\ {'\times}\ \text{code}_\Delta}$$

$$\frac{\begin{array}{c}\Gamma \vdash \textsc{Set} \ni T \overset{Chk}{\leadsto} T' \qquad \Gamma, t\,{:}\,T' \vdash \langle l\rangle \ni \nabla \overset{A}{\leadsto} \text{code}_\nabla\\[2pt] \Gamma \vdash \langle l\rangle \ni \Delta \overset{A}{\leadsto} \text{code}_\Delta\end{array}}{\Gamma \vdash \langle l\rangle \ni (f\,{:}\,(t\,{:}\,T) \to \nabla)\Delta \overset{A}{\leadsto}({'\Pi}\ T'\ \lambda t.\,\text{code}_\nabla)\,{'\times}\,\text{code}_\Delta} \qquad \frac{}{\Gamma \vdash \langle l\rangle \ni \epsilon \overset{A}{\leadsto} {'1}}$$

<div align="center">(d) Elaboration of arguments</div>

$$\boxed{T \triangleright l}$$

$$\frac{}{D \triangleright \mathbf{D}} \qquad \frac{T \triangleright l}{T\ p \triangleright l\ p}$$

<div align="center">(e) Recursion matching</div>

<div align="center">Figure 8: Elaboration of inductive types</div>

first elaborate the parameters and move onto elaborating the choice of constructors, introducing a description label in the process.

Example 7 (Elaborating Tree). Applied to our example, we obtain:

$$\Gamma \vdash \mathbf{data}\; \mathsf{Tree}(A\!:\!\mathrm{SET})\!:\!\mathrm{SET}\;\mathbf{where}[\mathrm{choices}] \overset{D}{\leadsto} \Gamma[\mathsf{Tree} \mapsto \lambda A.\, \mu\,(\mathsf{call}\,\langle \mathbf{Tree}\,A \rangle\,[\mathrm{code}])\!:\!A \rightarrow \mathrm{SET}]$$

where

$$\mathrm{choices} \triangleq \mathsf{leaf} \mid \mathsf{node}\,(l\!:\!\mathsf{Tree}\,A)(a\!:\!A)(r\!:\!\mathsf{Tree}\,A)$$

$$\mathrm{code} \triangleq \mathsf{return}\left\{\begin{array}{l} \text{'leaf,} \\ \text{'node} \end{array}\right\} \left\{\begin{array}{l} \text{'leaf}\;\; \mapsto \text{'1,} \\ \text{'node} \mapsto \text{'var}\,\text{'}{\times}\,\text{'}\Sigma\,A\,\lambda\underline{\;}.\,\text{'var}\,\text{'}{\times}\,\text{'1} \end{array}\right\}$$

Elaboration of constructors (Fig. 8b): To elaborate the choice of constructors, we elaborate each individual constructor, hence obtaining their respective label and code. We then return the finite collection of constructor names and their corresponding codes. This elaboration step is subject to the soundness property:

Lemma 1. *If* $\left\{\begin{array}{l} \Gamma \vdash l\;\mathrm{datatel} \\ \Gamma \vdash \langle l \rangle \ni \mathrm{choices} \overset{Cs}{\leadsto} \mathrm{code} \end{array}\right.$, *then* $\Gamma \vdash \mathrm{code}\!:\!\langle l \rangle$

Example 8 (Elaborating Tree). Applied to our example, we obtain:

$$\Gamma, A\!:\!\mathrm{SET} \vdash \langle \mathbf{Tree}\,A \rangle \ni [\mathrm{choices}] \overset{Cs}{\leadsto} [\mathrm{code}]$$

Where choices and code have been defined above.

Elaboration of constructor (Fig. 8c): The role of this elaboration step is twofold. First, we extract the constructor name and elaborate it into a label. Second, we elaborate the arguments of that constructor, hence obtaining a Desc code. We return the pair of the label and the arguments' code, subject to the following soundness property:

Lemma 2. *If* $\left\{\begin{array}{l} \Gamma \vdash l\;\mathrm{datatel} \\ \Gamma \vdash \langle l \rangle \ni c \overset{C}{\leadsto} [t \mapsto \mathrm{code}] \end{array}\right.$, *then* $\left\{\begin{array}{l} \Gamma \vdash t\!:\!\mathit{UId} \\ \Gamma \vdash \mathrm{code}\!:\!\mathit{Desc} \end{array}\right.$.

Example 9 (Elaborating Tree). Since our datatype definition has two constructors, there are two instances of constructor elaboration:

$$\Gamma, A\!:\!\mathrm{SET} \vdash \langle \mathbf{Tree}\,A \rangle \ni \mathsf{leaf} \overset{C}{\leadsto} [\text{'leaf} \mapsto \text{'1}]$$
$$\Gamma, A\!:\!\mathrm{SET} \vdash \langle \mathbf{Tree}\,A \rangle \ni \mathsf{node}\,(l\!:\!\mathsf{Tree}\,A)(a\!:\!A)(r\!:\!\mathsf{Tree}\,A) \overset{C}{\leadsto} [\text{'node} \mapsto \text{'var}\,\text{'}{\times}\,\text{'}\Sigma\,A\,\lambda\underline{\;}.\,\text{'var}\,\text{'}{\times}\,\text{'1}]$$

Elaboration of arguments (Fig. 8d): The last and perhaps most interesting rule is the elaboration of arguments. Intuitively, the arguments form a telescope of Σ-types, hence our translation to 'Σ and '$\times$ codes. The first two rules are non-deterministic: T could either be a proper type or a recursive call. In the first case, this maps to a standard 'Σ code, while in the second case, we must make sure that the recursive call is valid and, if so, we generate a 'var code. We also support exponentials in the definitions, mapping them to the 'Π code. Once all arguments have been processed, we conclude by generating the '1 code. This translation is subject to the following soundness property:

Lemma 3. *If* $\left\{ \begin{array}{l} \Gamma \vdash l \text{ datatel} \\ \Gamma \vdash \langle l \rangle \ni \text{args} \overset{A}{\rightsquigarrow} \text{code} \end{array} \right.$ *, then* $\Gamma \vdash \text{code} : \textit{Desc}.$

Example 10 (Elaborating Tree). Elaborating the arguments of the leaf constructor is trivial:

$$\overline{\Gamma, A : \textsc{Set} \vdash \langle \mathbf{Tree}\ A \rangle \ni \epsilon \overset{A}{\rightsquigarrow} {'}1}$$

As for the node constructor, we obtain its code through the following sequence of elaborations:

$$\cfrac{\text{Tree}\ A \rhd \mathbf{Tree}\ A \qquad \cfrac{\text{Tree}\ A \rhd \mathbf{Tree}\ A \quad \cfrac{\overline{\Gamma, A : \textsc{Set}, a : A \vdash \langle \mathbf{Tree}\ A \rangle \ni \epsilon \overset{A}{\rightsquigarrow} {'}1}}{\cfrac{\Gamma, A : \textsc{Set}, a : A \vdash \langle \mathbf{Tree}\ A \rangle \ni (r : \text{Tree}\ A) \overset{A}{\rightsquigarrow} {'}\text{var}\ {'}{\times}\ {'}1}{\Gamma, A : \textsc{Set} \vdash \langle \mathbf{Tree}\ A \rangle \ni (a : A)(r : \text{Tree}\ A) \overset{A}{\rightsquigarrow} {'}\Sigma\ A\ \lambda\_.\ {'}\text{var}\ {'}{\times}\ {'}1}}}{\Gamma, A : \textsc{Set} \vdash \langle \mathbf{Tree}\ A \rangle \ni (l : \text{Tree}\ A)(a : A)(r : \text{Tree}\ A) \overset{A}{\rightsquigarrow} {'}\text{var}\ {'}{\times}\ {'}\Sigma\ A\ \lambda\_.\ {'}\text{var}\ {'}{\times}\ {'}1}$$

We can now prove the soundness of the whole translation. We formulate soundness as follow:

Theorem 4 (Soundness of elaboration). *If* $\Gamma \vdash \mathbf{data}\ D\overrightarrow{(p : P)} : \textsc{Set}\ \mathbf{where}\ \text{choices} \overset{D}{\rightsquigarrow} \Delta$, *then* $\Delta \vdash \textsc{valid}.$

Proof. First, we prove Lemma 3 by induction on the list of arguments. We then obtain Lemma 2. By applying this lemma to all constructors, we obtain Lemma 1. From this Lemma, we finally obtain the soundness theorem. $\qquad\square$

While our soundness theorem gives some hint as to the correctness of our specification, we could obtain a stronger result by proving an equivalence between Coq's `Inductive` definitions and the corresponding datatype declaration in our system. This equivalence amounts to proving the equivalence of the associated elimination forms, i.e. `Fix` in Coq and induction in our system. However, since we do not know of any formal description of elimination principles generated from an `Inductive` definition, we shall use the simpler presentation given in Giménez [1995].

The fact that our induction principle is provable in Coq is a known result [Chapman et al., 2010]. The other direction consists in proving that any Coq `Fix` definition can be implemented using our induction principle. To this end, we use Giménez reduction of `Fix` definitions down to elimination rules. To prove this result, we first translate Giménez constructor forms to our universe of code:

$$\lfloor X\vec{N}\rfloor \mapsto \text{'}1$$
$$\lfloor (x\,{:}\,M) \to \mathcal{C}\rfloor \mapsto \text{'}\Sigma\,M\,\lambda x.\,\lfloor\mathcal{C}\rfloor$$
$$\lfloor (x\,{:}\,M \to X\,\vec{N}) \to \mathcal{C}\rfloor \mapsto (\text{'}\Pi\,M\,\lambda\_.\,\text{'var})\,\text{'}{\times}\,\lfloor\mathcal{C}\rfloor$$

We thus get a translation from his recursive types declaration to a code in our universe:

$$\lfloor\mathbf{Ind}(x\,{:}\,A)\langle C_0 \ldots C_n\rangle\rfloor \mapsto \text{'}\sigma\,n\,\{i \mapsto \lfloor C_i\rfloor\}$$

Having done that, it is then a straightforward symbol-pushing exercise to prove that Coq's elimination rules (Section 3.1.1, [Giménez, 1995]) can be reduced to our generic elimination principle. The crux of the matter consists in showing that the minor premises – defined by $\mathcal{E}_1$ in that paper – maps to the induction hypothesis described by $((d\,{:}\,[\![D]\!]\,(\mu D)) \to \mathsf{All}\,D\,(\mu D)\,P\,d \to P(\mathsf{in}\,d))$ in our system.

4 Elaborating inductive families

In this Section, we extend our treatment of inductive definitions to inductive families. To do so, we add support for indices and computation on these indices. The resulting system subsumes the one presented in the previous Section. Hence, we shall reuse many notations used previously: this should not affect reasoning or implementing this elaborator, since only that last system is necessary. We shall however develop our intuition of this more powerful presentation from the simpler translation of inductive types.

The syntax we elaborate is strongly inspired by the syntax used by Agda and Coq. However, to support computation over indices, we support the Epigram-style *by* gadget ($\Leftarrow$). Our language of inductive definition is therefore more complex, following the skeleton:

$$
\begin{array}{ll}
\textbf{data } \mathsf{D}\,\overrightarrow{[p\,{:}\,P]}\,\overrightarrow{(i\,{:}\,I)}\,{:}\,\textsc{Set} \textbf{ where} & \\
\quad \mathsf{D} \quad \vec{p}\,\overrightarrow{(i = t_0)} \;\ni\; c_{0,0}\overrightarrow{(a_{0,0}\,{:}\,T_{0,0})} & \\
\qquad\qquad\qquad\qquad\quad \mid\; \ldots & \\
\quad\vdots & \\
\quad \mathsf{D} \qquad \vec{p}\,\vec{i} \quad \Leftarrow \mathsf{elim}\,i_m & \\
\qquad \mathsf{D} \quad \vec{p}\,\vec{k_0} \;\ni\; \ldots & \\
\quad\vdots &
\end{array}
$$

For anyone wanting to reason about Agda or Coq definitions, it is straightforward to simply discard the elaboration of computation over indices. Thus, with minor adjustement, our formalization of elaboration gives a translation semantics to inductive definitions used in these main-stream theorem provers.

$$\frac{\Gamma \vdash \text{VALID}}{\Gamma \vdash \mathbf{D}\ \text{idatatel}} \qquad \frac{\Gamma \vdash l\ \text{idatatel} \quad \Gamma \vdash p : P}{\Gamma \vdash l\,[p]\ \text{idatatel}} \qquad \frac{\Gamma \vdash l\ \text{idatatel} \quad \Gamma \vdash i : I}{\Gamma \vdash l\,(i)\ \text{idatatel}} \qquad \frac{\Gamma \vdash l\ \text{idatatel} \quad i : I \in \Gamma}{\Gamma \vdash l\,(i = t)\ \text{idatatel}}$$

$$[\![\mathbf{D}]\!]_D \mapsto \mathbb{1} \qquad [\![l\,[p]]\!]_D \mapsto [\![l]\!]_D \qquad [\![l\,(i)]\!]_D \mapsto [\![l]\!]_D \times I \qquad [\![l\,(i = t)]\!]_D \mapsto [\![l]\!]_D \times I$$

$$\frac{\Gamma \vdash l\ \text{datatel}}{\Gamma \vdash \langle l \rangle : \textsc{Set}_1} \qquad \frac{\Gamma \vdash E : \mathsf{EnumU} \quad \Gamma \vdash T : \mathsf{EnumT}\,E \to \mathsf{IDesc}\,[\![l]\!]_D}{\Gamma \vdash \mathsf{return}\,E\,T : \langle l \rangle} \qquad \frac{\Gamma \vdash t : \langle l \rangle}{\Gamma \vdash \mathsf{call}\,\langle l \rangle\,t : \mathsf{IDesc}\,[\![l]\!]_D}$$

Figure 9: Description label (indexed)

4.1 Description labels

In Section 3.1, we have introduced the notion of description labels. While it was enough to deal with parameterized definitions, we have to extend this notion to account for indexing. Besides, indexing can be either unconstrained – some index (i) – or it can be constrained to some particular value – such as $(i = t)$. Besides, a label is not a phantom type around a description but around an $\mathsf{IDesc}\,I$ where I corresponds to the product of all index types the label is taking as arguments.

These requirements lead us to the definition of description labels given in Figure 9. We define label's arguments through a telescope extended with indices and constraints. To compute the index type of the resulting description, we introduce the $[\![\_]\!]_D$ function. Accordingly, we update the introduction and elimination rule of labels to account for the index type computed from the telescope.

As for inductive types, we shall present the elaboration process in a top-down manner. This presentation shares some strong commonality with the simpler elaboration of inductive types: we elaborate choices of constructors (Fig. 10c), followed by individual constructors (Fig. 10d), and finally processing the telescope of arguments (Fig. 11a). However, the presence of indices introduces some new steps too. We support constraint of indices and computation over them through a new top-level rule (Fig. 10b). Besides, we must translate the constraints to actual equalities (Fig. 11c) and pass the correct indices when elaborating a recursive call (Fig. 11d).

4.2 Elaborating inductive families

To give a better intuition of a perhaps intricate system, we should illustrate every inference rule with two examples. Our first example is the definition of vectors relying on

$$\boxed{\Gamma \vdash \textbf{data}\ D\,\overrightarrow{[p:P]}\,\overrightarrow{(i:I)} : \textsc{Set}\ \textbf{where}\ \text{choices} \overset{D}{\leadsto} \Delta}$$

$$\dfrac{\begin{array}{c}\Gamma \vdash \textsc{Set}_1 \ni \overrightarrow{(p:P)}\,\overrightarrow{(i:I)} \to \textsc{Set} \overset{Chk}{\leadsto} \overrightarrow{(p:P')}\,\overrightarrow{(i:I')} \to \textsc{Set} \\ \Gamma;\overrightarrow{p:P'};\overrightarrow{i:I'} \vdash \langle \textbf{D}\ \overrightarrow{[p]}\ \overrightarrow{(i)} \rangle \ni \text{choices} \overset{Patts}{\leadsto} \text{code}\end{array}}{\begin{array}{c}\Gamma \vdash \textbf{data}\ D\,\overrightarrow{[p:P]}\,\overrightarrow{(i:I)} : \textsc{Set}\ \textbf{where}\ \text{choices} \overset{D}{\leadsto} \\ \Gamma[D \mapsto \lambda\vec{p}:\vec{P'}.\,\mu\,(\lambda\vec{i}:\vec{I'}.\,\textsf{call}\,\langle D\ \overrightarrow{[p]}\ \overrightarrow{(i)} \rangle\ \text{code}) : \overrightarrow{(p:P')}\,\overrightarrow{(i:I')} \to \textsc{Set}]\end{array}}$$

(a) Elaboration of definition

$$\boxed{\Gamma \vdash \langle l \rangle \ni \text{patts} \overset{Patts}{\leadsto} \text{code}}$$

$$\dfrac{\begin{array}{cc}\begin{array}{c}l \supseteq pt_i \overset{I}{\leadsto} l_i \\ \Gamma \vdash \langle l_i \rangle \ni cs_i \overset{Cs}{\leadsto} [\{\overrightarrow{c_{i,j}}\} \mapsto \{\overrightarrow{c_{i,j} \mapsto as_{i,j}}\}]\end{array} & \begin{array}{c}l \supseteq pt_{i+1} \overset{I}{\leadsto} l_{i+1} \\ \Gamma \vdash e \overset{ewm}{\leadsto} e' \in \overrightarrow{((\overrightarrow{x_k:X_k}) \to \langle l_k \rangle) \to \langle l_{i+1} \rangle} \\ \Gamma, \overrightarrow{x_k:X_k} \vdash \langle l_k \rangle \ni p_k \overset{Patts}{\leadsto} \text{code}_k\end{array}\end{array}}{\Gamma \vdash \langle l \rangle \ni \begin{array}{l} pt_0 \ni cs_0 \\ \ \vdots \\ pt_i \ni cs_i \\ pt_{i+1} \Leftarrow e\,\{\vec{p_k}\}\end{array} \overset{Patts}{\leadsto} \textsf{return}\,\{\vec{c_{i,j}},\text{'elim}\}\left\{ c_{i,j} \mapsto as_{i,j}, \text{'elim} \mapsto e'\,\overrightarrow{(\lambda\vec{x_k}:\vec{X_k}.\,\textsf{call}\,\langle l\rangle\ \text{code}_k)} \right\}}$$

(b) Elaboration of patterns

$$\boxed{\Gamma \vdash \langle l \rangle \ni \text{choices} \overset{Cs}{\leadsto} [E \mapsto T]}$$

$$\dfrac{\Gamma \vdash \langle l \rangle \ni c_i \overset{C}{\leadsto} [t_i \mapsto \text{code}_i]}{\Gamma \vdash \langle l \rangle \ni c_0 | \ldots | c_n \overset{Cs}{\leadsto} [\{\vec{t_i}\} \mapsto \{\overrightarrow{t_i \mapsto \text{code}_i}\}]}$$

(c) Elaboration of choices

$$\boxed{\Gamma \vdash \langle l \rangle \ni c \overset{C}{\leadsto} [t \mapsto \text{code}]}$$

$$\dfrac{\Gamma \vdash \textsf{UId} \ni \text{'c} \overset{Chk}{\leadsto} c' \qquad \Gamma \vdash \langle l \rangle \ni \text{args} \overset{A}{\leadsto} \text{code}}{\Gamma \vdash \langle l \rangle \ni c\ \text{args} \overset{C}{\leadsto} [c' \mapsto \text{code}]}$$

(d) Elaboration of constructor

Figure 10: Elaboration of inductive families (1)

$$\boxed{\Gamma \vdash \langle l \rangle \ni \mathrm{args} \overset{A}{\rightsquigarrow} \mathrm{code}}$$

$$\frac{\Gamma \vdash \mathrm{SET} \ni T \overset{Chk}{\rightsquigarrow} T' \qquad \Gamma, x{:}T' \vdash \langle l \rangle \ni \Delta \overset{A}{\rightsquigarrow} \mathrm{code}_\Delta}{\Gamma \vdash \langle l \rangle \ni (x{:}T)\,\Delta \overset{A}{\rightsquigarrow} {'\Sigma}\,T\,\lambda x.\,\mathrm{code}_\Delta} \qquad \frac{T \supseteq l \overset{I}{\rightsquigarrow} is \qquad \Gamma \vdash \langle l \rangle \ni \Delta \overset{A}{\rightsquigarrow} \mathrm{code}_\Delta}{\Gamma \vdash \langle l \rangle \ni (x{:}T)\Delta \overset{A}{\rightsquigarrow} ({'\mathsf{var}}\,is)\,{'\times}\,\mathrm{code}_\Delta}$$

$$\frac{\Gamma \vdash \mathrm{SET} \ni T \overset{Chk}{\rightsquigarrow} T' \qquad \begin{array}{c}\Gamma, x{:}T' \vdash \langle l \rangle \ni \nabla \overset{A}{\rightsquigarrow} \mathrm{code}_\nabla \\ \Gamma \vdash \langle l \rangle \ni \Delta \overset{A}{\rightsquigarrow} \mathrm{code}_\Delta\end{array}}{\Gamma \vdash \langle l \rangle \ni (f{:}(x{:}T)\to\nabla)\Delta \overset{A}{\rightsquigarrow} ({'\Pi}\,T'\,\lambda x.\,\mathrm{code}_\nabla)\,{'\times}\,\mathrm{code}_\Delta} \qquad \frac{l \overset{Eq}{\rightsquigarrow} q}{\Gamma \vdash \langle l \rangle \ni \epsilon \overset{A}{\rightsquigarrow} q}$$

(a) Elaboration of arguments

$$\boxed{l \supseteq T \overset{I}{\rightsquigarrow} l_T}$$

$$\frac{}{\mathbf{D} \supseteq D \overset{I}{\rightsquigarrow} \mathbf{D}} \qquad \frac{l \supseteq T \overset{I}{\rightsquigarrow} l_T}{l\,[p] \supseteq T\,p \overset{I}{\rightsquigarrow} l_T\,[p]} \qquad \frac{l \supseteq T \overset{I}{\rightsquigarrow} l_T}{l\,(i) \supseteq T\,i \overset{I}{\rightsquigarrow} l_T\,(i)}$$

$$\frac{l \supseteq T \overset{I}{\rightsquigarrow} l_T}{l\,(i) \supseteq T\,(i{=}t) \overset{I}{\rightsquigarrow} l_T\,(i{=}t)} \qquad \frac{l \supseteq T \overset{I}{\rightsquigarrow} l_T}{l\,(i{=}t) \supseteq T\,(i{=}t) \overset{I}{\rightsquigarrow} l_T\,(i{=}t)}$$

(b) Pattern validation

$$\boxed{l \overset{Eq}{\rightsquigarrow} q}$$

$$\frac{\Gamma \vdash \mathrm{VALID}}{\mathbf{D} \overset{Eq}{\rightsquigarrow} {'1}} \qquad \frac{l \overset{Eq}{\rightsquigarrow} q}{l\,[p] \overset{Eq}{\rightsquigarrow} q} \qquad \frac{l \overset{Eq}{\rightsquigarrow} q}{l\,(i) \overset{Eq}{\rightsquigarrow} q} \qquad \frac{l \overset{Eq}{\rightsquigarrow} q \qquad \begin{array}{c}i{:}I \in \Gamma \\ \Gamma \vdash I \ni t \overset{Chk}{\rightsquigarrow} t'\end{array}}{l\,(i{=}t) \overset{Eq}{\rightsquigarrow} {'\Sigma}\,(i \equiv t')\,\lambda\_.\,q}$$

(c) Elaboration of constraints

$$\boxed{l \ni T \overset{LA}{\rightsquigarrow} is}$$

$$\frac{\Gamma \vdash \mathrm{VALID}}{\mathbf{D} \ni D \overset{LA}{\rightsquigarrow} *} \qquad \frac{l \ni T \overset{LA}{\rightsquigarrow} is}{l\,[p] \ni T\,p \overset{LA}{\rightsquigarrow} is} \qquad \frac{l \ni T \overset{LA}{\rightsquigarrow} is}{l\,(i') \ni T\,i \overset{LA}{\rightsquigarrow} (is,i)} \qquad \frac{l \ni T \overset{LA}{\rightsquigarrow} is}{l\,(i'{=}t) \ni T\,i \overset{LA}{\rightsquigarrow} (is,i)}$$

(d) Extraction of indices

Figure 11: Elaboration of inductive families (2)

constraints to enforce the indexing discipline:

$$\textbf{data } \mathsf{Vec}\,[A\!:\!\mathrm{SET}](n\!:\!\mathsf{Nat})\!:\!\mathrm{SET} \textbf{ where}$$
$$\mathsf{Vec}_A \quad (n\!=\!0) \quad \ni \ \mathsf{nil}$$
$$\mathsf{Vec}_A \ (n\!=\!\mathsf{suc}\ n') \ \ni \ \mathsf{cons}\,(n'\!:\!\mathsf{Nat})(a\!:\!A)(vs\!:\!\mathsf{Vec}_A\,n')$$

While our second example consists of the alternative definition of vector, where we compute over the index to determine which constructor to offer:

$$\textbf{data } \mathsf{Vec}\,[A\!:\!\mathrm{SET}](n\!:\!\mathsf{Nat})\!:\!\mathrm{SET} \textbf{ where}$$
$$\mathsf{Vec}_A \qquad n \quad \Leftarrow \ \mathsf{Nat\text{-}case}\ n$$
$$\mathsf{Vec}_A \quad 0 \quad \ni \ \mathsf{nil}$$
$$\mathsf{Vec}_A \ (\mathsf{suc}\ m) \ \ni \ \mathsf{cons}\,(a\!:\!A)(vs\!:\!\mathsf{Vec}_A\,m)$$

Elaboration of inductive families (Fig. 10a): The elaboration of an inductive definitions sets up the environment to trigger the elaboration of the choices of constructors. To do so, we first elaborate the telescope of parameters and indices types. We can then translate the choices by elaborating against the label type corresponding to the given inductive type.

Example 11 (Vector, constrained). The elaboration of constraint-based vectors starts as follow:

$$\frac{A\!:\!\mathrm{SET}, n\!:\!\mathsf{Nat} \vdash \mathbf{Vec}\,[A](n) \ni [\mathrm{choices}_=] \overset{Patts}{\leadsto} [\mathrm{code}_=]}{\vdash \textbf{data } \mathsf{Vec}[A\!:\!\mathrm{SET}](n\!:\!\mathsf{Nat})\!:\!\mathrm{SET} \textbf{ where}[\mathrm{choices}_=] \overset{D}{\leadsto}}$$
$$\Gamma[\mathsf{Vec} \mapsto \lambda A\!:\!\mathrm{SET}.\,\mu\,(\lambda n\!:\!\mathsf{Nat}.\,\mathsf{call}\,\langle\mathbf{Vec}\,[A]\,(n)\rangle\,[\mathrm{code}_=])\!:\!(A\!:\!\mathrm{SET})(n\!:\!\mathsf{Nat}) \to \mathrm{SET}]$$

where

$$\mathrm{choices}_= \triangleq \begin{array}{l} \mathsf{Vec}_A \quad (n\!=\!0) \quad \ni \ \mathsf{nil} \\ \mathsf{Vec}_A \ (n\!=\!\mathsf{suc}\ m) \ \ni \ \mathsf{cons}\,(m\!:\!\mathsf{Nat})(a\!:\!A)(vs\!:\!\mathsf{Vec}_A\,m) \end{array}$$

$$\mathrm{code}_= \triangleq \ \mathsf{return} \left\{ \begin{array}{l} \text{'nil,} \\ \text{'cons} \end{array} \right\}$$
$$\left\{ \begin{array}{l} \text{'nil} \ \mapsto \ \text{'}\Sigma\,(n \equiv 0)\,\lambda\_.\,\text{'}1, \\ \text{'cons} \mapsto \ \text{'}\Sigma\,\mathsf{Nat}\,\lambda m.\,\text{'}\Sigma\,A\,\lambda\_.\,\text{'var}\,(*,m)\,\text{'}{\times}\,\text{'}\Sigma\,(n \equiv \mathsf{suc}\ m)\,\lambda\_.\,\text{'}1 \end{array} \right\}$$

Example 12 (Vector, computed). The same skeleton is used in the alternative definition of vectors, but the choices of constructors – and therefore the resulting code – are different:

$$\mathrm{choices}_\to \triangleq \begin{array}{l} \mathsf{Vec}_A \qquad n \quad \Leftarrow \ \mathsf{Nat\text{-}case}\ n \\ \mathsf{Vec}_A \quad 0 \quad \ni \ \mathsf{nil} \\ \mathsf{Vec}_A \ (\mathsf{suc}\ m) \ \ni \ \mathsf{cons}\,(a\!:\!A)(vs\!:\!\mathsf{Vec}_A\,m) \end{array}$$

$$\mathrm{code}_\to \triangleq \ \mathsf{return}\,\{\text{'elim}\}$$
$$\left\{ \begin{array}{l} \qquad\qquad \mathsf{Nat\text{-}case}\ n\,(\lambda n.\,\langle\mathbf{Vec}\,[A](n)\rangle) \\ \text{'elim} \ \mapsto \quad (\mathsf{return}\,\{\text{'nil}\}\,\{\text{'nil} \ \mapsto \ \text{'}1\}) \\ \qquad\qquad (\lambda m.\,\mathsf{return}\,\{\text{'cons}\}\,\{\text{'cons} \ \mapsto \ \text{'}\Sigma\,A\,\lambda\_.\,\text{'var}\,(*,m)\,\text{'}{\times}\,\text{'}1\}) \end{array} \right\}$$

Elaboration of pattern choices (Fig. 10b): This elaboration process is an extra step that was not necessary with inductive types. With inductive families, we can both constrain the index to some particular value or compute over the index to refine the choice of constructors. Hence, an inductive definition is a list of pattern choices, potentially ending with a computation over the indices. Since the case where no index computation is performed is merely a special case of the rule we give, we save space and do not write it down explicitly.

The elaboration of pattern choices consists in interpreting the datatype patterns of each constructor choices. Then, the resulting labels are used to elaborate these constructors choices. If there is a computation over indices, we rely on elimination with a motive [Goguen et al., 2006, McBride, 2002] to generate a type theoretic term from the elimination principle provided by the user. We then interpret each resulting sub-branch as a pattern choice itself. All in all, this elaboration step satisfies the following invariant:

Lemma 4. *If* $\begin{cases} \Gamma \vdash l \text{ idatatel} \\ \Gamma \vdash \langle l \rangle \ni \text{patts} \overset{Patts}{\leadsto} \text{code} \end{cases}$, *then* $\Gamma \vdash \text{code} : \langle l \rangle$

Note that we rely on the translation from datatype patterns to data telescope (Fig. 11b): since we will be elaborating each individual constructor against these labels, we will generate the valid equality constraints at the end of each telescope of arguments.

Example 13 (Vector, constrained)**.** The elaboration of datatype patterns simply proceeds over the choices of constructors, triggering the elaboration of choices on nil and cons:

$$\mathbf{Vec}\,[A](n) \supseteq \mathsf{Vec}_A\,(n\!=\!0) \overset{I}{\leadsto} \mathbf{Vec}\,[A](n\!=\!0)$$
$$A\!:\!\mathrm{SET}, n\!:\!\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A](n\!=\!0) \rangle \ni \mathsf{nil} \overset{Cs}{\leadsto} [\{\text{'nil}\} \mapsto \{\text{'nil} \mapsto \text{'}\Sigma\,(n \equiv 0)\,\lambda\_.\,\text{'1}\}]$$

$$\mathbf{Vec}\,[A](n) \supseteq \mathsf{Vec}_A\,(n\!=\!\mathsf{suc}\,m) \overset{I}{\leadsto} \mathbf{Vec}\,[A](n\!=\!\mathsf{suc}\,m)$$
$$A\!:\!\mathrm{SET}, n\!:\!\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A](n\!=\!\mathsf{suc}\,m) \rangle \ni \mathsf{cons}\,(m\!:\!\mathsf{Nat})(a\!:\!A)(vs\!:\!\mathsf{Vec}_A\,m)$$
$$\underline{\overset{Cs}{\leadsto} [\{\text{'cons}\} \mapsto \{\text{'cons} \mapsto \text{'}\Sigma\,\mathsf{Nat}\,\lambda m.\,\text{'}\Sigma\,A\,\lambda\_.\,\text{'var}\,(\ast, m)\,\text{'}\!\times\,\text{'}\Sigma\,(n \equiv \mathsf{suc}\,m)\,\lambda\_.\,\text{'1}\}]}$$
$$A\!:\!\mathrm{SET}, n\!:\!\mathsf{Nat} \vdash \mathbf{Vec}\,[A](n) \ni [\text{choices}_=] \overset{Patts}{\leadsto} [\text{code}_=]$$

where $\text{choices}_=$ and $\text{code}_=$ are the same as above.

Example 14 (Vector, computed)**.** Similarly for the alternative definition of vectors, this

triggers the elaboration of a motive, of the nil and cons patterns:

$$\mathbf{Vec}\,[A](n) \supseteq \mathsf{Vec}_A\, n \overset{I}{\leadsto} \mathbf{Vec}\,[A](n)$$
$$A\!:\!\mathrm{SET}, n\!:\!\mathsf{Nat} \vdash \mathsf{Nat\text{-}case}\, n$$
$$\overset{ewm}{\leadsto} \lambda ih_0\ ih_n.\, \mathsf{Nat\text{-}case}\, n\, (\lambda n.\, \langle \mathbf{Vec}\,[A](n)\rangle)\, ih0\ ih_n$$
$$\in \langle \mathbf{Vec}\,[A](0)\rangle \to ((m\!:\!\mathsf{Nat}) \to \langle \mathbf{Vec}\,[A](\mathsf{suc}\, m)\rangle) \to \langle \mathbf{Vec}\,[A](n)\rangle$$
$$A\!:\!\mathrm{SET}, n\!:\!\mathsf{Nat} \vdash \mathbf{Vec}\,[A](0) \ni \mathsf{nil} \overset{Patts}{\leadsto} \mathsf{return}\, \{\text{'nil}\}\, \{\text{'nil} \mapsto\ \text{'}1\}$$
$$A\!:\!\mathrm{SET}, n\!:\!\mathsf{Nat}, m\!:\!\mathsf{Nat} \vdash \mathbf{Vec}\,[A](\mathsf{suc}\, m) \ni \mathsf{cons}\,(a\!:\!A)(vs\!:\!\mathsf{Vec}_A\, m)$$
$$\overset{Patts}{\leadsto} \mathsf{return}\, \{\text{'cons}\}\, \{\text{'cons} \mapsto\ \text{'}\Sigma\, A\, \lambda\_.\, \text{'var}\,(*, m)\, \text{'}\times\, \text{'}1\}$$
$$\rule{10cm}{0.4pt}$$
$$A\!:\!\mathrm{SET}, n\!:\!\mathsf{Nat} \vdash \mathbf{Vec}\,[A](n) \ni [\mathrm{choices}_\to] \overset{Patts}{\leadsto} [\mathrm{code}_\to]$$

with $\mathrm{choices}_\to$ and $\mathrm{code}_\to$ defined above. In turn, this triggers the elaboration of datatype choices for the nil and cons patterns:

$$\frac{A\!:\!\mathrm{SET}, n\!:\!\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A](0)\rangle \ni \mathsf{nil} \overset{Cs}{\leadsto} [\{\text{'nil}\} \mapsto \{\text{'nil} \mapsto\ \text{'}1\}]}{A\!:\!\mathrm{SET}, n\!:\!\mathsf{Nat} \vdash \mathbf{Vec}\,[A](0) \ni \mathsf{nil} \overset{Patts}{\leadsto} \mathsf{return}\, \{\text{'nil}\}\, \{\text{'nil} \mapsto\ \text{'}1\}}$$

$$A\!:\!\mathrm{SET}, n\!:\!\mathsf{Nat}, m\!:\!\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A](\mathsf{suc}\, m)\rangle \ni \mathsf{cons}\,(a\!:\!A)(vs\!:\!\mathsf{Vec}_A\, m)$$
$$\frac{\overset{Cs}{\leadsto} [\{\text{'cons}\} \mapsto \{\text{'cons} \mapsto\ \text{'}\Sigma\, A\, \lambda\_.\, \text{'var}\,(*, m)\, \text{'}\times\, \text{'}1\}]}{\begin{array}{c} A\!:\!\mathrm{SET}, n\!:\!\mathsf{Nat}, m\!:\!\mathsf{Nat} \vdash \mathbf{Vec}\,[A](\mathsf{suc}\, m) \ni \mathsf{cons}\,(a\!:\!A)(vs\!:\!\mathsf{Vec}_A\, m) \\ \overset{Patts}{\leadsto} \mathsf{return}\, \{\text{'cons}\}\, \{\text{'cons} \mapsto\ \text{'}\Sigma\, A\, \lambda\_.\, \text{'var}\,(*, m)\, \text{'}\times\, \text{'}1\} \end{array}}$$

Elaboration of choices (Fig. 10c): The elaboration of the choice of datatypes is the same as for inductive types: we merely collect the tag and code of each individual constructor and return these as enumerations. This step is subject to the following soundness property:

Lemma 5. *If* $\begin{cases} \Gamma \vdash l\ \mathsf{idatatel} \\ \Gamma \vdash \langle l \rangle \ni \mathrm{choices} \overset{Cs}{\leadsto} [E \mapsto T] \end{cases}$, *then* $\begin{cases} \Gamma \vdash E\!:\!\mathit{EnumU} \\ \Gamma \vdash T\!:\!\mathit{EnumT}\, ts \to \mathit{IDesc}\,[\![l]\!]_D \end{cases}$

Example 15 (Vector, constrained). In the particular example of vector, there is only one choice of constructor when $(n = 0)$ – namely nil – and when $(n = \mathsf{suc}\, m)$ – namely cons. Therefore, we obtain the elaboration of choices from the elaboration of the unique constructor, in both cases:

$$\frac{A\!:\!\mathrm{SET}, n\!:\!\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A](n = 0)\rangle \ni \mathsf{nil} \overset{C}{\leadsto} [\text{'nil} \mapsto\ \text{'}\Sigma\, (n \equiv 0)\, \lambda\_.\, \text{'}1]}{A\!:\!\mathrm{SET}, n\!:\!\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A](n = 0)\rangle \ni \mathsf{nil} \overset{Cs}{\leadsto} [\{\text{'nil}\} \mapsto \{\text{'nil} \mapsto\ \text{'}\Sigma\, (n \equiv 0)\, \lambda\_.\, \text{'}1\}]}$$

$$A\!:\!\mathrm{SET}, n\!:\!\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A](n = \mathsf{suc}\, m)\rangle \ni \mathsf{cons}\,(m\!:\!\mathsf{Nat})(a\!:\!A)(vs\!:\!\mathsf{Vec}_A\, m)$$
$$\frac{\overset{C}{\leadsto} [\text{'cons} \mapsto\ \text{'}\Sigma\, \mathsf{Nat}\, \lambda m.\, \text{'}\Sigma\, A\, \lambda\_.\, \text{'var}\,(*, m)\, \text{'}\times\, \text{'}\Sigma\, (n \equiv \mathsf{suc}\, m)\, \lambda\_.\, \text{'}1]}{\begin{array}{c} A\!:\!\mathrm{SET}, n\!:\!\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A](n = \mathsf{suc}\, m)\rangle \ni \mathsf{cons}\,(m\!:\!\mathsf{Nat})(a\!:\!A)(vs\!:\!\mathsf{Vec}_A\, m) \\ \overset{Cs}{\leadsto} [\{\text{'cons}\} \mapsto \{\text{'cons} \mapsto\ \text{'}\Sigma\, \mathsf{Nat}\, \lambda m.\, \text{'}\Sigma\, A\, \lambda\_.\, \text{'var}\,(*, m)\, \text{'}\times\, \text{'}\Sigma\, (n \equiv \mathsf{suc}\, m)\, \lambda\_.\, \text{'}1\}] \end{array}}$$

Example 16 (Vector, computed)**.** The same situation arises in the alternative definition of vector: once we have determined which index we are dealing with, there is a single constructor available. Hence, we move from the elaboration of choices to the elaboration of the unique constructor:

$$\frac{A:\text{SET}, n:\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A](0)\rangle \ni \mathsf{nil} \overset{C}{\rightsquigarrow} [\text{'nil} \mapsto \text{'}1]}{A:\text{SET}, n:\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A](0)\rangle \ni \mathsf{nil} \overset{Cs}{\rightsquigarrow} [\{\text{'nil}\} \mapsto \{\text{'nil} \mapsto \text{'}1\}]}$$

$$\frac{\begin{array}{c}A:\text{SET}, n:\mathsf{Nat}, m:\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A](\mathsf{suc}\,m)\rangle \ni \mathsf{cons}\,(a:A)(vs:\mathsf{Vec}_A\,m)\\ \overset{C}{\rightsquigarrow} [\text{'cons} \mapsto \text{'}\Sigma\,A\,\lambda\_.\,\text{'var}\,(*, m)\,\text{'}\times\,\text{'}1]\end{array}}{\begin{array}{c}A:\text{SET}, n:\mathsf{Nat}, m:\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A](\mathsf{suc}\,m)\rangle \ni \mathsf{cons}\,(a:A)(vs:\mathsf{Vec}_A\,m)\\ \overset{Cs}{\rightsquigarrow} [\{\text{'cons}\} \mapsto \{\text{'cons} \mapsto \text{'}\Sigma\,A\,\lambda\_.\,\text{'var}\,(*, m)\,\text{'}\times\,\text{'}1\}]\end{array}}$$

Elaboration of constructor (Fig. 10d): Again, the elaboration of constructor is not any different from the inductive types case. It is subject to the following invariant:

Lemma 6. *If* $\begin{cases}\Gamma \vdash l \text{ idatatel} \\ \Gamma \vdash \langle l\rangle \ni c \overset{C}{\rightsquigarrow} [t \mapsto \text{code}]\end{cases}$, *then* $\begin{cases}\Gamma \vdash t : \textit{UId} \\ \Gamma \vdash \text{code} : \textit{IDesc}\,\llbracket l\rrbracket_D\end{cases}$

Example 17 (Vector, constrained)**.** Elaboration of the constructors is straightforward, simply switching to the elaboration of the arguments:

$$\frac{A:\text{SET}, n:\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A]\,(n\!=\!0)\rangle \ni \epsilon \overset{A}{\rightsquigarrow} \text{'}\Sigma\,(n \equiv 0)\,\lambda\_.\,\text{'}1}{A:\text{SET}, n:\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A]\,(n\!=\!0)\rangle \ni \mathsf{nil} \overset{C}{\rightsquigarrow} [\text{'nil} \mapsto \text{'}\Sigma\,(n \equiv 0)\,\lambda\_.\,\text{'}1]}$$

$$\frac{\begin{array}{c}A:\text{SET}, n:\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A]\,(n\!=\!\mathsf{suc}\,m)\rangle \ni (m:\mathsf{Nat})(a:A)(vs:\mathsf{Vec}_A\,m)\\ \overset{A}{\rightsquigarrow} \text{'}\Sigma\,\mathsf{Nat}\,\lambda m.\,\text{'}\Sigma\,A\,\lambda\_.\,\text{'var}\,(*, m)\,\text{'}\times\,\text{'}\Sigma\,(n \equiv \mathsf{suc}\,m)\,\lambda\_.\,\text{'}1\end{array}}{\begin{array}{c}A:\text{SET}, n:\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A]\,(n\!=\!\mathsf{suc}\,m)\rangle \ni \mathsf{cons}\,(m:\mathsf{Nat})(a:A)(vs:\mathsf{Vec}_A\,m)\\ \overset{C}{\rightsquigarrow} [\text{'cons} \mapsto \text{'}\Sigma\,\mathsf{Nat}\,\lambda m.\,\text{'}\Sigma\,A\,\lambda\_.\,\text{'var}\,(*, m)\,\text{'}\times\,\text{'}\Sigma\,(n \equiv \mathsf{suc}\,m)\,\lambda\_.\,\text{'}1]\end{array}}$$

Example 18 (Vector, computed)**.** Similarly for the alternative definition, we have:

$$\frac{A:\text{SET}, n:\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A]\,(0)\rangle \ni \epsilon \overset{A}{\rightsquigarrow} \text{'}1}{A:\text{SET}, n:\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A](0)\rangle \ni \mathsf{nil} \overset{C}{\rightsquigarrow} [\text{'nil} \mapsto \text{'}1]}$$

$$\frac{A:\text{SET}, n:\mathsf{Nat}, m:\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A]\,(\mathsf{suc}\,m)\rangle \ni (a:A)(vs:\mathsf{Vec}_A\,m) \overset{A}{\rightsquigarrow} \text{'}\Sigma\,A\,\lambda\_.\,\text{'var}\,(*, m)\,\text{'}\times\,\text{'}1}{\begin{array}{c}A:\text{SET}, n:\mathsf{Nat}, m:\mathsf{Nat} \vdash \langle \mathbf{Vec}\,[A](\mathsf{suc}\,m)\rangle \ni \mathsf{cons}\,(a:A)(vs:\mathsf{Vec}_A\,m)\\ \overset{C}{\rightsquigarrow} [\text{'cons} \mapsto \text{'}\Sigma\,A\,\lambda\_.\,\text{'var}\,(*, m)\,\text{'}\times\,\text{'}1]\end{array}}$$

Elaboration of arguments (Fig. 11a): The elaboration of arguments follows the same principle as for inductive types. However, when elaborating a recursive argument, we must extract the indices for which that recursive step is taken. Besides, after having encoded the list of arguments, we must switch to translating the potential equality constraints, which are dictated by the label type we are elaborating against. This step is subject to the following soundness property:

Lemma 7. *If* $\begin{cases} \Gamma \vdash l \text{ idatatel} \\ \Gamma \vdash \langle l \rangle \ni \text{args} \overset{A}{\leadsto} \text{code} \end{cases}$, *then* $\Gamma \vdash \text{code} : \textsf{IDesc} \llbracket l \rrbracket_D$

Example 19 (Vector, constrained). We then elaborate the arguments by unfolding the telescope, at which point we switch to elaborating the constraints. We do so immediatly in the nil case:

$$\frac{\textbf{Vec}\,[A]\,(n\!=\!0) \overset{Eq}{\leadsto} \text{'}\Sigma\,(n \equiv 0)\,\lambda_{\_}.\,\text{'}1}{A\!:\!\textsc{Set}, n\!:\!\textsf{Nat} \vdash \langle\textbf{Vec}\,[A]\,(n\!=\!0)\rangle \ni \epsilon \overset{A}{\leadsto} \text{'}\Sigma\,(n \equiv 0)\,\lambda_{\_}.\,\text{'}1}$$

While a few steps are necessary to elaborate the arguments in the cons case, including the elaboration of the recursive call:

$$\frac{\dfrac{\dfrac{\textbf{Vec}\,[A]\,(n\!=\!\mathsf{suc}\,m) \overset{Eq}{\leadsto} \text{'}\Sigma\,(n \equiv \mathsf{suc}\,m)\,\lambda_{\_}.\,\text{'}1}{\begin{array}{c}\textbf{Vec}\,[A]\,(n\!=\!\mathsf{suc}\,m) \ni \mathsf{Vec}_A\,m \overset{LA}{\leadsto} (*, m) \\ A\!:\!\textsc{Set}, n\!:\!\textsf{Nat}, m\!:\!\textsf{Nat} \vdash \langle\textbf{Vec}\,[A]\,(n\!=\!\mathsf{suc}\,m)\rangle \ni \epsilon \overset{A}{\leadsto} \text{'}\Sigma\,(n \equiv \mathsf{suc}\,m)\,\lambda_{\_}.\,\text{'}1\end{array}}}{A\!:\!\textsc{Set}, n\!:\!\textsf{Nat}, m\!:\!\textsf{Nat} \vdash \langle\textbf{Vec}\,[A]\,(n\!=\!\mathsf{suc}\,m)\rangle \ni (vs\!:\!\mathsf{Vec}_A\,m) \overset{A}{\leadsto} \text{'var}\,(*, m)\,\text{'}\!\times\text{'}\Sigma\,(n \equiv \mathsf{suc}\,m)\,\lambda_{\_}.\,\text{'}1}}{\dfrac{A\!:\!\textsc{Set}, n\!:\!\textsf{Nat}, m\!:\!\textsf{Nat} \vdash \langle\textbf{Vec}\,[A]\,(n\!=\!\mathsf{suc}\,m)\rangle \ni (a\!:\!A)(vs\!:\!\mathsf{Vec}_A\,m)}{\overset{A}{\leadsto} \text{'}\Sigma\,A\,\lambda_{\_}.\,\text{'var}\,(*, m)\,\text{'}\!\times\text{'}\Sigma\,(n \equiv \mathsf{suc}\,m)\,\lambda_{\_}.\,\text{'}1}}$$
$$\frac{}{\begin{array}{c}A\!:\!\textsc{Set}, n\!:\!\textsf{Nat} \vdash \langle\textbf{Vec}\,[A]\,(n\!=\!\mathsf{suc}\,m)\rangle \ni (m\!:\!\textsf{Nat})(a\!:\!A)(vs\!:\!\mathsf{Vec}_A\,m) \\ \overset{A}{\leadsto} \text{'}\Sigma\,\textsf{Nat}\,\lambda m.\,\text{'}\Sigma\,A\,\lambda_{\_}.\,\text{'}\Sigma\,(n \equiv \mathsf{suc}\,m)\,\lambda_{\_}.\,\text{'}1\end{array}}$$

Example 20 (Vector, computed). In the alternative definition, the process is similar:

$$\frac{\textbf{Vec}\,[A](0) \overset{Eq}{\leadsto} \text{'}1}{A\!:\!\textsc{Set}, n\!:\!\textsf{Nat} \vdash \langle\textbf{Vec}\,[A](0)\rangle \ni \epsilon \overset{A}{\leadsto} \text{'}1}$$

$$\frac{\dfrac{\dfrac{\textbf{Vec}\,[A](\mathsf{suc}\,m) \overset{Eq}{\leadsto} \text{'}1}{\begin{array}{c}\textbf{Vec}\,[A](\mathsf{suc}\,m) \ni \mathsf{Vec}_A\,m \overset{LA}{\leadsto} (*, m) \\ A\!:\!\textsc{Set}, n\!:\!\textsf{Nat}, m\!:\!\textsf{Nat} \vdash \langle\textbf{Vec}\,[A](\mathsf{suc}\,m)\rangle \ni \epsilon \overset{A}{\leadsto} \text{'}1\end{array}}}{A\!:\!\textsc{Set}, n\!:\!\textsf{Nat}, m\!:\!\textsf{Nat} \vdash \langle\textbf{Vec}\,[A](\mathsf{suc}\,m)\rangle \ni (vs\!:\!\mathsf{Vec}_A\,m) \overset{A}{\leadsto} \text{'var}\,(*, m)\,\text{'}\!\times\text{'}1}}{A\!:\!\textsc{Set}, n\!:\!\textsf{Nat}, m\!:\!\textsf{Nat} \vdash \langle\textbf{Vec}\,[A](\mathsf{suc}\,m)\rangle \ni (a\!:\!A)(vs\!:\!\mathsf{Vec}_A\,m) \overset{A}{\leadsto} \text{'}\Sigma\,A\,\lambda_{\_}.\,\text{'var}\,(*, m)\,\text{'}\!\times\text{'}1}$$

Elaboration of constraints (Fig. 11c): In order to generate the equality constraints, we simply traverse the label we are elaborating against. On constraints, we generate the corresponding equality constraint, using whatever propositional equality the system has to offer. On parameters and (unconstrained) indices, we simply go through. This step satisfies the following property:

Lemma 8. *If* $\begin{cases} \Gamma \vdash l \text{ idatatel} \\ l \stackrel{Eq}{\rightsquigarrow} q \end{cases}$ *, then* $\Gamma \vdash q : \mathsf{IDesc}\,[\![l]\!]_D$.

Example 21 (Vector, constrained). We elaborate the constraints in the nil case – constraining n to 0 – and in the cons case – constraining n to $\mathsf{suc}\,m$:

$$\frac{\dfrac{\mathbf{Vec} \stackrel{Eq}{\rightsquigarrow} {}'1}{\mathbf{Vec}\,[A] \stackrel{Eq}{\rightsquigarrow} {}'1}}{\mathbf{Vec}\,[A]\,(n=0) \stackrel{Eq}{\rightsquigarrow} {}'\Sigma\,(n \equiv 0)\,\lambda_{\_}.\,{}'1} \qquad \frac{\dfrac{\mathbf{Vec} \stackrel{Eq}{\rightsquigarrow} {}'1}{\mathbf{Vec}\,[A] \stackrel{Eq}{\rightsquigarrow} {}'1}}{\mathbf{Vec}\,[A]\,(n=\mathsf{suc}\,m) \stackrel{Eq}{\rightsquigarrow} {}'\Sigma\,(n \equiv \mathsf{suc}\,m)\,\lambda_{\_}.\,{}'1}$$

Example 22 (Vector, computed). No equations are generated and, indeed, needed for the alternative definition of vectors:

$$\frac{\dfrac{\mathbf{Vec} \stackrel{Eq}{\rightsquigarrow} {}'1}{\mathbf{Vec}\,[A] \stackrel{Eq}{\rightsquigarrow} {}'1}}{\mathbf{Vec}\,[A](0) \stackrel{Eq}{\rightsquigarrow} {}'1} \qquad \frac{\dfrac{\mathbf{Vec} \stackrel{Eq}{\rightsquigarrow} {}'1}{\mathbf{Vec}\,[A] \stackrel{Eq}{\rightsquigarrow} {}'1}}{\mathbf{Vec}\,[A](\mathsf{suc}\,m) \stackrel{Eq}{\rightsquigarrow} {}'1}$$

Extraction of indices (Fig. 11d): On the recursive calls, we must extract the indices at which the call is made. To do so, we match the type definition with the datatype label. Parameters are ignored while indices are paired together. On the datatype name, we inhabit $\mathbb{1}$. This ensures the following soundness property:

Lemma 9. *If* $\begin{cases} \Gamma \vdash l \text{ idatatel} \\ l \ni T \stackrel{LA}{\rightsquigarrow} is \end{cases}$ *, then* $\Gamma \vdash is : [\![l]\!]_D$.

Example 23 (Vector, constrained). There is only one instance of recursive definition, in the cons case. Its elaboration goes as follow:

$$\frac{\dfrac{\mathbf{Vec} \ni \mathsf{Vec} \stackrel{LA}{\rightsquigarrow} *}{\mathbf{Vec}\,[A] \ni \mathsf{Vec}_A \stackrel{LA}{\rightsquigarrow} *}}{\mathbf{Vec}\,[A]\,(n=\mathsf{suc}\,m) \ni \mathsf{Vec}_A\,m \stackrel{LA}{\rightsquigarrow} (*,m)}$$

Example 24 (Vector, computed). Similarly, the elaboration of the index for the recursive definition is as follow:

$$\frac{\dfrac{\mathbf{Vec} \ni \mathsf{Vec} \stackrel{LA}{\rightsquigarrow} *}{\mathbf{Vec}\,[A] \ni \mathsf{Vec}_A \stackrel{LA}{\rightsquigarrow} *}}{\mathbf{Vec}\,[A](\mathsf{suc}\,m) \ni \mathsf{Vec}_A\,m \stackrel{LA}{\rightsquigarrow} (*,m)}$$

Having stated the soundness properties of each individual elaboration steps, we can now state and prove the soundness of the elaboration of inductive families:

Theorem 5 (Soundness of elaboration)**.** *If* $\Gamma \vdash$ **data** $D\,\overrightarrow{[p:P]}\,\overrightarrow{(i:I)} : \textsc{Set}$ **where** choices $\overset{D}{\leadsto} \Delta$, *then* $\Delta \vdash \textsc{valid}$.

Proof. First, we prove that labels elaborate to a type correct index (Lemma 9). We then prove that the constraints generated by interpreting the label are valid descriptions (Lemma 8). From these lemmas, we can prove the soundness of the elaboration of arguments by induction over the telescope of arguments (Lemma 7). This gives straightforwardly the validity of the elaboration of a constructor (Lemma 6). Using that Lemma over each constructors, we thus obtain the soundness of the elaboration of choices (Lemma 5). Using this result and assuming the soundness of elimination with a motive, we prove Lemma 4. This gives the desired result. $\qquad\square$

5 Reflections on Inductives

Having described our infrastructure to elaborate inductive definitions down to descriptions, we would like to give an overview of the possibilities offered by such a system. Indeed, in a purely syntactic presentation of inductives, we are stuck at the meta-level of the type theory: if we want to provide support to manipulate inductive types, it must be implemented as part of the theorem prover, out of the type theory. Alternatively, a quoting/unquoting mechanism could be provided but guaranteeing the safety of such an extension is likely to be tricky.

Because our type theory reflects inductives in itself, the meta-theory of inductive types is no more than a universe. What used to be meta-theoretical constructions can now be implemented from within the type theory, benefiting from the various amenities offered by a dependently-typed programming language. In this Section, we present two examples of such "reflection on inductives". Our first example, reflecting constructions on constructors [McBride et al., 2004], will appeal to the implementers: we hint at the possibility of implementing key features of the type theory within itself, a baby step toward bootstrapping. Our second example, providing a user defined `deriving` mechanism, should appeal to programmers: we illustrate how programmers could provide generic operations over datatypes and see them automatically integrated in their development. For simplicity and conciseness, we shall define these mechanisms over our universe of inductive types, Desc. Nonetheless, it is straightforward, but more verbose, to extend these constructions to inductive families.

5.1 A few constructions on constructors, internalized

McBride et al. [2004] describe a collection of lemmas that theorem prover's implementer would like to export with every inductive type. In that paper, the authors first show how one can reduce case analysis and course-of-value recursion to standard induction. Then,

27

they describe two lemmas over datatype constructors: *no confusion* – constructors are injective and disjoint – and *acyclicity* – we can automatically disprove equalities of the form $x = t$ where x appears constructor-guarded in t.

However, since this paper works on the syntactic form of datatype definitions, it is rife with "..." definitions. For instance, the authors reduce case analysis to induction with no less than ten ellipsis in the construction. In our system, we generically derive case analysis by a mere definition *within the type theory*:

$$\mathsf{case}(D:\mathsf{Desc})(P:\mu D \to \mathrm{Set})(cases:((d:\llbracket D \rrbracket\,(\mu D)) \to P(\mathsf{in}\,d)))(x:\mu D) \;:\; P\,x$$
$$\mathsf{case}\,D\;P\;cases\;x \;\mapsto\; \mathsf{induction}\,D\;P\;(\lambda d.\,\lambda \_.\,cases\;d)\;x$$

Similarly, the authors specify and prove the no confusion lemma over the skeleton of an inductive definition. In our system, this result is internalized through two definitions. In the following, we will assume that D is a tagged description, i.e. $D = {}'\sigma\,E\,T$ where E is the finite sets of constructor labels. An inhabitant of μD is therefore a pair $\mathsf{in}\,(c, a)$ with c representing the constructor name and a the tuple of arguments. We state the NoConfusion lemma over such code:

$$
\begin{aligned}
&\mathsf{NoConfusion}\;\;(x:\mu D)\quad(y:\mu D)\quad:\mathrm{Set}_1\\
&\mathsf{NoConfusion}\;(\mathsf{in}\,(x, a_x))\,(\mathsf{in}\,(y, a_y))\;|\;\mathsf{decideEq\text{-}EnumT}\;x\;y\\
&\mathsf{NoConfusion}\;(\mathsf{in}\,(x, a_x))\,(\mathsf{in}\,(x, a_y))\;|\quad\quad\mathsf{equal\;refl}\quad\quad\mapsto\;(P:\mathrm{Set}) \to (a_x \equiv a_y \to P) \to P\\
&\mathsf{NoConfusion}\;(\mathsf{in}\,(x, a_x))\,(\mathsf{in}\,(y, a_y))\;|\quad\quad\mathsf{not\text{-}equal}\;q\quad\quad\mapsto\;(P:\mathrm{Set}) \to P
\end{aligned}
$$

and then prove it by deciding the equality of enumeration index to discriminate constructor names:

$$
\begin{aligned}
&\mathsf{noConfusion}\;\;(x:\mu D)\quad(y:\mu D)\;\;(q:x \equiv y) : \mathsf{NoConfusion}\;x\;y\\
&\mathsf{noConfusion}\;(\mathsf{in}\,(x, a_x))\,(\mathsf{in}\,(y, a_y))\quad\;\;q\quad\;|\;\mathsf{decideEq\text{-}EnumT}\;x\;y\\
&\mathsf{noConfusion}\;(\mathsf{in}\,(x, a_x))\,(\mathsf{in}\,(x, a_y))\quad\;\;q\quad\;|\quad\quad\mathsf{equal\;refl}\quad\quad\mapsto\;\lambda P.\,\lambda rec.\,rec\;q\\
&\mathsf{noConfusion}\;(\mathsf{in}\,(x, a_x))\,(\mathsf{in}\,(y, a_y))\quad\;\;q\quad\;|\quad\quad\mathsf{not\text{-}equal}\;neq\quad\mapsto\;\lambda P.\,\mathbb{0}\text{-elim}\,(neq\;q)
\end{aligned}
$$

At this stage, we have proved this lemma generically, for all tagged descriptions. Hence, after having defined a new datatype, a user can directly use this lemma on her definition. For convenience, a subsequent elaboration phase could also specialize such lemma to the particular definition.

5.2 Deriving operations on datatypes

Another possible extension of our system is a generic `deriving` mechanism. In the Haskell language, we can write a definition such as

$$
\begin{aligned}
&\textbf{data}\;\mathsf{Nat}:\mathrm{Set}\;\textbf{where}\\
&\quad\mathsf{Nat}\;\ni\;0\\
&\quad\quad\quad\;|\;\mathsf{suc}\,(n:\mathsf{Nat})\\
&\quad\textbf{deriving}\;\mathsf{Eq}
\end{aligned}
$$

that automatically generates an equality test for the given datatype. Again, since datatypes are a meta-theoretical entity, this deriving mechanism has to be provided

by the implementer and, template programming aside, they cannot be implemented by the programmers themselves.

In our framework, we could extend the elaborator for datatypes with a `deriving` mechanism. However, for such a mechanism to work, we must restrict ourselves to decidable properties: for example, if the user asks to derive equality on a datatype that do not have a decidable equality (e.g. , Brouwer ordinals), the system should fail immediately. To solve this issue, we add one level of indirection: while we cannot decide equality for *any* datatype, we can decide whether the datatype belongs to a sub-universe *for which* equality is decidable. Hence, to introduce a derivable property P in the type theory, the programmer would populate the following record structure:

$$\mathsf{Derivable}\ (P : \mathsf{Desc} \to \mathrm{SET})\ :\ \mathrm{SET}_1$$
$$\mathsf{Derivable}\ P\ \mapsto\ \left\{ \begin{array}{ll} \mathsf{subDesc} & : \mathsf{Desc} \to \mathrm{SET}_1 \\ \mathsf{membership} : (D : \mathsf{Desc}) \to \mathsf{Decidable}\ (\mathsf{subDesc}\ D) \\ \mathsf{derive} & : \mathsf{subDesc}\ D \to P\ D \end{array} \right.$$

For example, in the case of equality, the programmer has first to provide a function $\mathsf{eqDesc} : \mathsf{Desc} \to \mathrm{SET}_1$. One possible (perhaps simplistic, but valid) sub-universe consists only of products, finite sums, recursive call, and unit: it is enough to describe natural numbers and variants thereof. She then implements a procedure $\mathsf{membershipEq} : (D : \mathsf{Desc}) \to \mathsf{Decidable}\ (\mathsf{eqDesc}\ D)$ deciding whether a given Desc code fits into this sub-universe or not. Recall that $\mathsf{Decidable}\ A$ corresponds to $A + \neg\ A$. It should be clear that the membership of a Desc code to our sub-universe of finite products and sums is decidable. Finally, she implements the key operation $\mathsf{deriveEq} : \mathsf{eqDesc}\ D \to (x\ y : \mu\ D) \to \mathsf{Decidable}\ (x \equiv y)$ that decides equality of two objects, assuming that they belong to the sub-universe. This implements the structure $\mathsf{Eq} : \mathsf{Derivable}\ (\lambda D.\ (x\ y : \mu\ D) \to \mathsf{Decidable}\ x \equiv y)$.

While elaborating a datatype, it is then straightforward – and automatic – for us to generate its derivable property, or reject it immediately: we simply compute $\mathsf{membership}$ on the specific code. If we obtain a negative response, we report an error. If we obtain a positive witness, we pass that witness to derive and instantiate the derived property.

For example, since natural numbers fit into the eqDesc universe, the elaboration machinery would automatically generate the following decision procedure

$$\begin{array}{lll} \mathsf{Nat\text{-}eq}\ (x\ y : \mathsf{Nat}) & : & \mathsf{Decidable}\ x \equiv y \\ \mathsf{Nat\text{-}eq}\quad x\ y & \mapsto & \mathsf{deriveEq}\ (\mathsf{witness}\ (\mathsf{membershipEq}\ \mathsf{NatD})\ *) \end{array}$$

without any input from the user but the `deriving` Eq clause. Here, $\mathsf{witness}$ is a library function that extracts the witness from a true decidable property: applied to NatD, the function $\mathsf{membershipEq}$ computes a positive witness that we can simply extract.

6 Conclusion

In this paper, we have striven to give a coherent framework for elaboration in type theory. We have organized our system around two structuring ideas. First, by using the flow of typing information, we obtain a richer and more flexible term language. Second, by using types as presentation of high-level concepts, such as inductive definition, we can effectively guide the elaboration process. This technique is conceptually simple and therefore amenable to formal reasoning. This simplicity together with the soundness proofs should convince the reader of its validity.

From there, we believe that reasoning on inductive definitions can be liberated from the elusive ellipsis: proofs and constructions on inductives ought to happen within the type theory itself. After Harper and Stone [2000], we claim that if the treatment of datatypes is conceptually straightforward then it ought to be technically straightforward and implemented as a generic program in the type theory. For non-straightforward properties, our results should be reusable across calculi – such as the Calculus of Inductive Constructions – and not too rigidly tied to our universe of datatypes. Besides, we were careful to present elaboration as a relation rather than a mere program, making it more amenable to abstract reasoning.

To support our claim that inductives should be bootstrapped, we have presented two possible extensions to the elaboration process. While we did not formalize these examples, our expectations seem reasonable and our experience modeling them in Agda further supports this impression. We have seen how generic theorems on inductive types can be internalized as generic programs: besides the benefit of reducing the trusted computing base, their validity is guaranteed by type checking. Also, we have glimpsed at a generic `deriving` mechanism: with no extension to the type theory, we are able to let the user define sub-universes that support certain operations. These operations could then be automatically specialized to the datatypes that support them, all that without any user intervention.

Future work: A next step would be to adapt our syntax to coinductive types. Similarly, it would be interesting to see if it scales to inductive-recursive definitions. On the implementation side, we are currently implementing these various systems in a toy type theory. This step will require a formalization of our `deriving` mechanism. Also, we will have to generate the specialized induction principles (such as case analysis and course-of-value recursion) as well as the constructions on constructors. Finally, an interesting challenge would be to internalize the elaboration process *itself* in type theory, hence obtaining a correct-by-construction translation.

References

T. Altenkirch, C. McBride, and W. Swierstra. Observational equality, now! In *PLPV*, 2007.

A. Asperti, W. Ricciotti, C. S. Coen, and E. Tassi. A Bi-Directional refinement algorithm for the calculus of (Co)Inductive constructions. 2012. URL `http://arxiv.org/abs/1202.4905`.

M. Benke, P. Dybjer, and P. Jansson. Universes for generic programs and proofs in dependent type theory. *Nordic Journal of Computing*, 2003.

E. Brady, J. Chapman, P.-E. Dagand, A. Gundry, C. McBride, P. Morris, and U. Norell. An Epigram implementation. URL `http://www.e-pig.org/`.

E. Brady, C. McBride, and J. McKinna. Inductive families need not store their indices. In *TYPES*, 2003.

J. Chapman, P.-E. Dagand, C. McBride, and P. Morris. The gentle art of levitation. In *ICFP*, pages 3–14, 2010.

T. Coquand. Pattern matching with dependent types. In *Types for Proofs and Programs*, 1992.

P.-E. Dagand and C. McBride. Transporting functions across ornaments. In *ICFP*, pages 103–114, 2012.

P. Dybjer. Inductive families. *Formal Asp. Comput.*, 6(4):440–465, 1994.

E. Giménez. Codifying guarded definitions with recursive schemes. In *Types for Proofs and Programs*, volume 996, chapter 3, pages 39–59. 1995.

H. Goguen, C. McBride, and J. McKinna. Eliminating dependent pattern matching. In K. Futatsugi, J.-P. Jouannaud, and J. Meseguer, editors, *Algebra, Meaning and Computation*, volume 4060 of *Lecture Notes in Computer Science*, chapter 27, pages 521–540. Springer Berlin / Heidelberg, Berlin, Heidelberg, 2006.

R. Harper and C. Stone. A Type-Theoretic interpretation of standard ML. In *Proof, Language, and Interaction: essays in honour of Robin Milner*, 2000.

Z. Luo. *Computation and Reasoning*. Oxford University Press, 1994.

C. McBride. Elimination with a motive. page 727. 2002.

C. McBride. Ornamental algebras, algebraic ornaments. *Journal of Functional Programming, to appear*, 2012.

C. McBride and J. McKinna. The view from the left. *JFP*, 2004.

C. McBride, H. Goguen, and J. McKinna. A few constructions on constructors. In *Types for Proofs and Programs*, pages 186–200, 2004.

P. Morris and T. Altenkirch. Indexed containers. In *LICS*, 2009.

U. Norell. *Towards a practical programming language based on dependent type theory.* PhD thesis, Chalmers University of Technology, 2007.

B. C. Pierce and D. N. Turner. Local type inference. In *POPL*, 1998.

M. Sozeau. Equations: A dependent pattern-matching compiler. In *ITP*, pages 419–434, 2010.

The Coq Development Team. *The Coq Proof Assistant Reference Manual.*