Trace models of concurrent valuation algebras

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Abstract This paper introduces Concurrent Valuation Algebras (CVAs), a novel extension of ordered valuation algebras (OVAs). CVAs include two combine operators representing parallel and sequential products, adhering to a weak exchange law. This development offers theoretical and practical benefits for the specification and modelling of concurrent and distributed systems. As a presheaf on a space of domains, CVAs enable localised specifications, supporting modularity, compositionality, and the ability to represent large and complex systems. Furthermore, CVAs align with lattice-based refinement reasoning and are compatible with established methodologies such as Hoare and Rely-Guarantee logics. The flexibility of CVAs is explored through three trace models, illustrating distinct paradigms of concurrent/distributed computing, interrelated by morphisms. The paper also highlights the potential to incorporate a powerful local computation framework from valuation algebras for model checking in concurrent and distributed systems. The foundational results presented have been verified with the proof assistant Isabelle/HOL.

Keywords: Concurrent valuation algebras \cdot Concurrent systems \cdot Distributed systems.

1 Introduction

Valuation algebras are versatile algebraic structures that parameterise information across multiple domains, representing for example subsets of variables or events. These structures have been widely utilised across diverse disciplines such as database theory, logic, probability and statistics, and constraint satisfaction, among others. What sets valuation algebras apart is their robust computational theory, enabling the deployment of highly efficient distributed algorithms for addressing inference problems that involve information combination and querying [16].

In our preceding work [6], we applied ordered valuation algebras to distributed systems, demonstrating their potential as a modular framework for specifying these systems in a refinement paradigm. Moreover, we established a link between sequential consistency—a crucial correctness criterion—and contextuality, an abstract form of information inconsistency which valuation algebras capture.

Paper outline. In Section 2, we introduce ordered valuation algebras (OVAs) based on prior studies [8,1] and extend these to concurrent valuation algebras (CVAs) in Section 3, a structure comprising two OVA structures on a space with combine operators adhering to a weak exchange law. This design takes inspiration from Communicating Sequential Processes (CSP) [11], Concurrent Kleene Algebra (CKA) [12], Concurrent Refinement Algebra (CRA) [9], and duoidal/2-monoidal categories [2]. We also define morphisms between CVAs, and explain their alignment with the refinement reasoning methodologies of Hoare [10] and Rely-Guarantee [13] logics. Section 4 delves into tuple systems and relational OVAs, which underpin our trace models. The subsequent sections, Sections 5 to 7, introduce and contrast various trace models, elaborating on their distinct combine operators and trace characteristics. In Section 8, we reflect on the potential extension of the local computation framework from valuation algebras to CVAs. Section 9 closes our paper, encapsulating our findings and suggesting avenues for future exploration.

The theoretical underpinnings detailed in Sections 2 to 4 have been rigorously formalised using the proof assistant Isabelle/HOL, lending credibility to our study.¹ A separate formalisation of Section 6 is also available.²

2 Ordered valuation algebras

We assume familiarity with foundational ideas in order theory and category theory, including the definitions of a category, a functor, and a natural transformation. For those interested in a more detailed understanding, please refer to [7] for an accessible introduction, or [17] as a thorough reference.

Notation. The category of sets and functions is denoted Set. Posets (partially ordered sets) are identified with their associated (thin) categories, so that the hom-set hom(a, b) is a singleton when $a \leq b$ and empty otherwise, and we write Pos for the category whose objects are posets and whose morphisms are monotone functions. A **topological space** (X,\mathcal{T}) is a set X equipped with a topology \mathcal{T} , which is a family of subsets of X, termed **open sets**, partially ordered by inclusion, and closed under arbitrary unions and finite intersections (in particular, the empty intersection, X, and the empty union, \emptyset , are open). For a category \mathcal{C} , \mathcal{C}^{op} denotes its *opposite category*, that is the category \mathcal{C} with the direction of its arrows reversed. A **presheaf** Φ on a topological space (X,\mathcal{T}) is a functor with domain \mathcal{T}^{op} . We notate the value of a presheaf Φ applied to A by Φ_A instead of $\Phi(A)$, and for $B \subseteq A$ in \mathcal{T} , the **restriction map** $\Phi_A \to \Phi_B$ is denoted $a \mapsto a^{\downarrow B}$. All presheaves considered are valued in Set or Pos, though we adopt the name prealgebra for a poset-valued presheaf, suggesting our intent to develop an OVA structure upon it. A **global element** of a prealgebra Φ is a natural transformation $\epsilon: 1 \Rightarrow \Phi$ from the terminal prealgebra 1, defined $\mathbf{1} := A \mapsto \{\emptyset\}$. For such a global element $\boldsymbol{\epsilon}$, we write $\boldsymbol{\epsilon}_A$ instead of $\boldsymbol{\epsilon}_A(\emptyset)$. The

¹ Available at https://github.com/nasosev/cva .

 $^{^2}$ Available at $\label{eq:attps://github.com/onomatic/icfem23-proofs} \ .$

symbol **P** denotes the *covariant powerset functor* **P**: $\mathcal{S}et \to \mathcal{P}os$, sending a set X to the poset of its subsets $\mathbf{P}(X)$, and sending a function $f: X \to Y$ to its *direct image* $f_* := X \mapsto \{f(x) \mid x \in X\}$. The symbol $\mathbb N$ denotes the set of natural numbers $\{0,1,\ldots\}$, while $\mathbb N_+$ is the set of positive natural numbers $\{1,2,\ldots\}$.

Throughout this paper, we fix a topological space (X, \mathcal{T}) . Here, the open sets symbolise abstract *domains*, representing subsets of system elements like memory locations, resources, or events, as well as their interconnectivity.

Example 1. A network composed of three computer systems a, b, c and three network links d, e, f as pictured in Fig. 1 may be represented by the topological space generated by unions and intersections of the domains $\{d, a, e\}$, $\{e, b, f\}$, $\{f, c, d\}$. More generally, a network defined by a labelled, undirected graph converts to a finite topology where open sets are the upwards-closed sets of the network's face poset, i.e. the poset whose elements are the nodes n and edges e of the network, where $n \le e$ if and only if n is a vertex of e. Alternatively, a set of memory addresses X may be given the discrete topology $\mathcal{T} = \mathbf{P}(X)$.

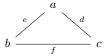


Figure 1. A network of three computers and three links.

A prealgebra $\Phi: \mathcal{T}^{\mathsf{op}} \to \mathcal{P}os$ comprises a family of posets $\{\Phi_A\}_{A \in \mathcal{T}}$ parameterised by the domains of the space \mathcal{T} , and a family of monotone restriction maps $\{a \mapsto a^{\downarrow B}: \Phi_A \to \Phi_B\}_{A,B \in \mathcal{T},B \subseteq A}$ parameterised by the inclusions of the space.

The elements of the posets Φ_A represent abstract units of information pertaining to their domain. Their ordering signifies information refinement: $a \leq b$ means a is *more* deterministic than b, a convention that aligns with the intuitions of program refinement.

The prealgebra's restriction maps $a \mapsto a^{\downarrow B}$ serve to project or query information $a \in A$ onto a subdomain $B \subseteq A$. These mappings facilitate the extraction of specific details from a wider context. Further, restriction maps are transitive and idempotent: for $C \subseteq B \subseteq A$ and $a \in \Phi_A$, we have $(a^{\downarrow B})^{\downarrow C} = a^{\downarrow C}$, and $a^{\downarrow A} = a$.

³ The topology described is the *Alexandrov* topology of the face poset of the network, viewed as a simplicial complex. Another possibility is to take its geometric realisation, but this typically results in an infinite space. These spaces, however, are weakly homotopy equivalent [3].

This family of posets $\{\Phi_A\}_{A\in\mathcal{I}}$ can be unified into a single poset $\int \Phi$, through a canonical process known as the *Grothendieck construction* of Φ (for a detailed explanation within a broader context, refer to [18]).

Definition 1 (covariant Grothendieck construction for a prealgebra). Let $\Phi: \mathcal{T}^{op} \to \mathcal{P}os$ be a prealgebra. The covariant Grothendieck construction of Φ is the poset $(\int \Phi, \preceq)$ whose elements are pairs $(A, a) \in \int \Phi$ where $A \in \mathcal{T}$ and $a \in \Phi_A$, and whose ordering \preceq is defined

$$(A, a) \leq (B, b)$$
 if and only if $B \subseteq A$ and $a^{\downarrow B} \leq_{\Phi_B} b$ (1)

For the projection map $d: \int \Phi \to \mathcal{T}^{op}$, $(A, a) \mapsto A$, call da the domain of a.

Notation. As shorthand, we suppress the domain in the first component of elements (A, a) belonging to $\int \Phi$, writing a instead of (A, a).

Remark 1. In Definition 1, we apply the covariant Grothendieck construction to a contravariant functor, treating it as a covariant functor from its domain's opposite. This choice, though atypical, aligns with a semantic interpretation for refining program specifications, explained in Section 3.1.

Next, the concept of an ordered valuation algebra (OVA) is introduced, which incorporates a prealgebra $\Phi: \mathcal{T}^{op} \to \mathcal{P}os$, a binary operator $\otimes: \int \Phi \times \int \Phi \to \int \Phi$, and a global element $\epsilon: \mathbf{1} \Rightarrow \Phi$, satisfying a number of axioms. Before delving into the formal definition, we illustrate the concept with an example.

Example 2. A familiar instance of an OVA models relational databases. Here, a set X of attributes is fixed (e.g., $X = \{\text{`name'}, \text{`age'}, \text{`height'}\}$). A schema is a subset A of X defining a table's columns, while each row defines a tuple: an assignment of a value to each attribute. A relation on X is a set a of tuples sharing a common schema a, and a relational database is a set of such relations.

To frame this within an OVA, we define a prealgebra $\Phi: \mathbf{P}(X) \to \mathcal{P}os$, mapping a schema $A \in \mathcal{T}$ to the poset Φ_A of all relations with schema A, with ordering given by inclusion. The restriction maps of Φ correspond to querying, by projecting the tuples of a relation a to a sub-schema B; the result is the relation $a^{\downarrow B} = \{t^{\downarrow B} \mid t \in a\}$, where $t^{\downarrow B}$ is the tuple t restricted to the attributes in B.

The operator \otimes is taken to be the *natural join*,

$$\bowtie : \int \mathbf{\Phi} \times \int \mathbf{\Phi} \to \int \mathbf{\Phi}$$

$$a \bowtie b := \left\{ t \in \mathbf{\Phi}_{\mathrm{d}a \cup \mathrm{d}b} \mid t^{\downarrow \mathrm{d}a} \in a \text{ and } t^{\downarrow \mathrm{d}b} \in b \right\}$$
(2)

This operation is associative and monotone (forming an *ordered semigroup*), and the schema of $a \bowtie b$ is $da \cup db$. Moreover, the natural join satisfies the following *combination axiom*:

$$(a \bowtie b)^{\downarrow da} = a \bowtie b^{\downarrow da \cap db}$$
(3)

This identity is fundamental to query optimisation algorithms in relational databases, with its right-hand side referred to as a *semi-join*. Lastly, the global element ϵ assigns to each schema A the universal relation ϵ_A on A, encompassing all possible tuples on A. These universal relations serve as units for the natural join, i.e. for all $a \in \int \Phi$, we have $a \bowtie \epsilon_{da} = a = \epsilon_{da} \bowtie a$.

Please note that our definition of an OVA below deviates from standard ones (e.g. [16,1,8]) in several ways. First, we do not mandate commutativity of the operator \otimes , as a sequential product of programs is noncommutative. This requires a symmetric revision Eq. (7) of the combination axiom. Second, constraints in the classical definition such as the existence of infima in the posets Φ_A are not imposed. Yet, we stipulate that neutral valuations correspond to a global element, which is tantamount to the stability property in [16], though we do not require neutral valuations to combine to neutral valuations—we call an algebra in which this property holds strongly neutral (Definition 4). Last, while Grothendieck constructions have been applied to ordered valuation algebras ([5]), their conventional definition does not involve a Grothendieck ordering.

Definition 2 (ordered valuation algebra (OVA)). An ordered valuation algebra (OVA) is a triple $(\Phi, \otimes, \varepsilon)$, where Φ is a prealgebra $\Phi : \mathcal{T}^{op} \to \mathcal{P}os$, \otimes is a binary operator $\otimes : \int \Phi \times \int \Phi \to \int \Phi$, called the combine operator, and $\varepsilon : \mathbf{1} \Rightarrow \Phi$ is a global element, called the neutral element, satisfying the below four axioms for all valuations $a, b, c, a', b' \in \int \Phi$:

Ordered semigroup. The combine operator \otimes is associative, and monotone:

$$a \otimes (b \otimes c) = (a \otimes b) \otimes c$$
, and, $a \leq a'$ and $b \leq b' \implies a \otimes b \leq a' \otimes b'$ (4)

Labelling.

$$d(a \otimes b) = da \cup db \tag{5}$$

Neutrality.

$$\mathbf{\epsilon}_{\mathrm{d}a} \otimes a = a = a \otimes \mathbf{\epsilon}_{\mathrm{d}a}$$
(6)

Combination.

$$(a \otimes b)^{\downarrow da} = a \otimes b^{\downarrow da \cap db}, \qquad (a \otimes b)^{\downarrow db} = a^{\downarrow da \cap db} \otimes b$$
 (7)

Remark 2. The functor laws for Φ imply that for all $C \subseteq B \subseteq A$ and $a \in \Phi_A$, we have $(a^{\downarrow B})^{\downarrow C} = a^{\downarrow C}$, and also $a^{\downarrow A} = a$. The requirement that ϵ is a global element says for all $B \subseteq A$ in \mathcal{T} , we have $\epsilon_A^{\downarrow B} = \epsilon_B$.

Remark 3. Monotonicity of \otimes says for $a_1 \in \Phi_{A_1}$, $a_2 \in \Phi_{A_2}$, $b_1 \in \Phi_{B_1}$, $b_2 \in \Phi_{B_2}$, if $a_1 \leq b_1$ and $a_2 \leq b_2$, then $a_1 \otimes a_2 \leq b_1 \otimes b_2$, that is, $(a_1 \otimes a_2)^{\downarrow B_1 \cup B_2} \leq_{\Phi_{B_1 \cup B_2}} b_1 \otimes b_2$. Taking $A_1 = A_2 = B_1 = B_2$, this implies **local monotonicity**, i.e. the combine operator \otimes restricted to each domain A, $\otimes_A : \Phi_A \times \Phi_A \to \Phi_A$, is monotone.

Definition 3 (commutative OVA). We call an OVA $(\Phi, \otimes, \varepsilon)$ commutative if \otimes is commutative.

Definition 4 (strongly neutral). We call an OVA $(\Phi, \otimes, \epsilon)$ strongly neutral if for all inclusions $B \subseteq A$ in \mathcal{T} , we have $\epsilon_B^{\uparrow A} = \epsilon_A$.

Theorem 1. Let $(\Phi, \otimes, \varepsilon)$ be an OVA. Then for each inclusion $B \subseteq A$ in \mathcal{T} , the restriction map $a \mapsto a^{\downarrow B}$ has a right adjoint given by $b \mapsto \varepsilon_A \otimes b$. Moreover, these right adjoints assemble to a functor $\mathcal{T} \to \mathcal{P}$ os. We adopt the notation $b^{\uparrow A} := \varepsilon_A \otimes b$, and call $b^{\uparrow A}$ the extension of $b \in A$.

Proof. We must show that for all $B \subseteq A$ and all $a \in \Phi_A$ and $b \in \Phi_B$,

$$a^{\downarrow B} \leq_{\Phi_B} b \iff a \leq_{\Phi_A} \epsilon_A \otimes b$$

Assume $a^{\downarrow B} \leq_{\Phi_B} b$. Using the fact that $a \leq a^{\downarrow B}$ and monotonicity,

$$a = \mathbf{\epsilon}_A \otimes a \leq_{\mathbf{\Phi}_A} \mathbf{\epsilon}_A \otimes a^{\downarrow B} \leq_{\mathbf{\Phi}_A} \mathbf{\epsilon}_A \otimes b$$

Now assume $a \leq_{\Phi_A} \epsilon_A \otimes b$. Using monotonicity of restriction, the combination axiom, naturality of ϵ , and neutrality,

$$a^{\downarrow B} \leq_{\Phi_B} (\epsilon_A \otimes b)^{\downarrow B} = \epsilon_A^{\downarrow A \cap B} \otimes b = \epsilon_B \otimes b = b$$

So the adjunction holds. That extension is functorial (i.e. for $C \subseteq B \subseteq A$ and $c \in \Phi_C$, both $(c^{\uparrow B})^{\uparrow A} = c^{\uparrow A}$ and $c^{\uparrow C} = c$) is due to the composability and uniqueness of adjoints.

Corollary 1. Let $(\Phi, \otimes, \epsilon)$ be a strongly neutral OVA. Then for all A, B in \mathcal{T} , $\epsilon_A \otimes \epsilon_B = \epsilon_{A \cup B}$. Also, $\int \Phi$ is an ordered monoid.

Proof. We have, $\mathbf{e}_A \otimes \mathbf{e}_B = (\mathbf{e}_{A \cup B} \otimes \mathbf{e}_{A \cup B}) \otimes (\mathbf{e}_A \otimes \mathbf{e}_B) = (\mathbf{e}_{A \cup B} \otimes \mathbf{e}_A) \otimes (\mathbf{e}_{A \cup B} \otimes \mathbf{e}_B) = \mathbf{e}_A^{\uparrow A \cup B} \otimes \mathbf{e}_B^{\uparrow A \cup B} = \mathbf{e}_{A \cup B} \otimes \mathbf{e}_{A \cup B} = \mathbf{e}_{A \cup B}$. If $a \in \int \mathbf{\Phi}$, $a \otimes \mathbf{e}_\emptyset = (a \otimes \mathbf{e}_{da}) \otimes \mathbf{e}_\emptyset = a \otimes (\mathbf{e}_{da} \otimes \mathbf{e}_\emptyset) = a \otimes \mathbf{e}_{da} = a$. Thus, \mathbf{e}_\emptyset is a unit for $\int \mathbf{\Phi}$.

Corollary 2. Let $(\Phi, \otimes, \epsilon)$ be an OVA, and let $B \subseteq A$ in \mathcal{T} , $a \in \Phi_A$, and $b \in \Phi_B$. Then,

- 1. Restriction after extension is the identity map, i.e., $(b^{\uparrow A})^{\downarrow B} = b$.
- 2. Extension after restriction is extensive, i.e., $a \leq_{\Phi_A} (a^{\downarrow B})^{\uparrow A}$.

Proof. For the first claim, by the neutrality and combination axioms and naturality, we have $(b^{\uparrow A})^{\downarrow B} = (\boldsymbol{\epsilon}_A \otimes b)^{\downarrow B} = \boldsymbol{\epsilon}_A^{\downarrow A \cap B} \otimes b = \boldsymbol{\epsilon}_B \otimes b = b$. The second is always true of the composition of a right adjoint after its left adjoint.

Corollary 3. If for each $A \in \mathcal{T}$, Φ_A is a complete lattice, then so is $\int \Phi$.

Proof. See [18], where it is shown in more generality that completeness of the poset $\int \mathbf{\Phi}$ follows from: (i) cocompleteness of the poset \mathcal{T} , (ii) completeness of each poset $\mathbf{\Phi}_A$, and (iii) that the restriction maps $a \mapsto a^{\downarrow B}$ have right adjoints.

Definition 5 (morphism of OVAs). Let $(\Phi, \otimes, \epsilon)$ and $(\Phi', \otimes', \epsilon')$ be OVAs. A lax morphism $f : \Phi \to \Phi'$ is a family of monotone maps $\{f_A : \Phi_A \to \Phi'_A\}_{A \in \mathcal{I}}$ so that the below hold for all $a \in \Phi_A$, $b \in \Phi_B$ and $C \subseteq A$,

Monotonicity.

$$a \leq b \implies f_A(a) \leq f_B(b)$$
 (8)

Lax naturality.

$$f_A(a)^{\downarrow C} \leq f_C(a^{\downarrow C})$$
 (9)

Lax multiplicativity.

$$f_A(a) \otimes' f_B(b) \leq f_{A \cup B}(a \otimes b)$$
 (10)

Lax unitality.

$$\mathbf{\epsilon}_A' \leq f_A(\mathbf{\epsilon}_A)$$
 (11)

Reversing the inequality directions above defines a colax morphism. A morphism that is both lax and colax is termed a strong morphism.

2.1 Extension of local operators

In the following, let Φ be a prealgebra such that for each inclusion $B \subseteq A$ in \mathcal{T} , the restriction map $a \mapsto a^{\downarrow B} : \Phi_A \to \Phi_B$ has a right adjoint $b \mapsto b^{\uparrow A} : \Phi_B \to \Phi_A$.

Definition 6 (extension of a family of local operators). Assume a family of associative binary operators $\{ \odot_A : \Phi_A \times \Phi_A \to \Phi_A \}_{A \in \mathcal{I}}$. We define the extension of $\{ \odot_A \}_{A \in \mathcal{I}}$ to $\int \Phi$ to be the binary operator,

$$\odot: \int \mathbf{\Phi} \times \int \mathbf{\Phi} \to \int \mathbf{\Phi}
a \odot b := a^{\uparrow da \cup db} \odot_{da \cup db} b^{\uparrow da \cup db}$$
(12)

Note that a combine operator of an OVA $(\Phi, \otimes, \epsilon)$ is the extension of the family $\{\otimes_A\}_{A \in \mathcal{I}}$, where \otimes_A is the restriction of \otimes to $\Phi_A \times \Phi_A$, because

$$a \otimes b = (\boldsymbol{\epsilon}_U \otimes \boldsymbol{\epsilon}_U) \otimes (a \otimes b) = (\boldsymbol{\epsilon}_U \otimes a) \otimes (\boldsymbol{\epsilon}_U \otimes b) = a^{\uparrow U} \otimes_U b^{\uparrow U}$$
 (13)

where $U = da \cup db$. Next is a key lemma establishing conditions for the reverse direction, i.e. for when a family of local operators on Φ may give rise to a combine operator.

Lemma 1. [proof in Appendix A.1] Assume $\{ \odot_A : \Phi_A \times \Phi_A \to \Phi_A \}_{A \in \mathcal{I}}$ is a family of local associative operators satisfying:

Local monotonicity. For all $A \in \mathcal{T}$ and $a_1, a'_1, a_2, a'_2 \in \Phi_A$,

$$a_1 \leq_{\Phi_A} a_1'$$
 and $a_2 \leq_{\Phi_A} a_2' \implies a_1 \odot_A a_2 \leq_{\Phi_A} a_1' \odot_A a_2'$ (14)

Extension-commutation. For all $B \subseteq A$ in \mathcal{T} and $b_1, b_2 \in \Phi_B$,

$$(b_1 \odot_B b_2)^{\uparrow A} = b_1^{\uparrow A} \odot_A b_2^{\uparrow A} \tag{15}$$

Then $(\int \Phi, \odot)$ is an ordered semigroup, where \odot is the extension of $\{\odot_A\}_{A \in \mathcal{T}}$.

The next lemma shows that to establish the *weak exchange* axiom for a CVA (Definition 7 below), it suffices to show a local weak exchange law on each domain.

Lemma 2. [proof in Appendix A.2] Let $(\Phi, \|, \text{run})$ and (Φ, \S, skip) be OVAs whose combine operators $\|$ and \S are respectively defined as extensions of $\{\|_A\}_{A \in \mathcal{T}}$ and $\{\S_A\}_{A \in \mathcal{T}}$. Assume that on each $A \in \mathcal{T}$, a weak exchange law holds: for all $a_1, a_2, a_3, a_4 \in \Phi_A$, $(a_1 \|_A a_2) \S_A (a_3 \|_A a_4) \leq_{\Phi_A} (a_1 \S_A a_3) \|_A (a_2 \S_A a_4)$. Then the weak exchange law holds on $\int \Phi$: for all $a, b, c, d \in \int \Phi$, $(a \| b) \S(c \| d) \preceq (a \S_C) \| (b \S_C) \|$.

3 Concurrent valuation algebras

We now introduce a concurrent valuation algebra (CVA), structured as two OVAs sharing the same underlying prealgebra, whose combine operators represent parallel and sequential products. These operators are interlinked via a weak exchange law, and their neutral elements are related by a pair of inequalities.

Definition 7 (concurrent valuation algebra (CVA)). A concurrent valuation algebra (CVA) is a structure $(\Phi, \S, \text{skip}, \|, \text{run})$ satisfying the four axioms:

Sequential OVA. $(\Phi, \mathring{9}, \text{skip})$ is an OVA. Parallel OVA. $(\Phi, \parallel, \text{run})$ is a commutative OVA.

Weak exchange. For all $a, b, c, d \in \int \Phi$,

$$(a \parallel b) \circ (c \parallel d) \preceq (a \circ c) \parallel (b \circ d) \tag{16}$$

Neutral laws. For all $A \in \mathcal{T}$,

$$\operatorname{skip}_A \leq \operatorname{skip}_A \| \operatorname{skip}_A, \ and, \ \operatorname{run}_A \ \circ \ \operatorname{run}_A \leq \operatorname{run}_A$$
 (17)

This definition is motivated by the relationship between sequential and parallel products. Sequential product, signifying a temporal juxtaposition, is generally noncommutative. In contrast, parallel product, signifying a spatial juxtaposition, is commutative. These two interlink by the weak exchange law. It states that the sequential composite of two parallel compositions, $a \parallel b$ and $c \parallel d$, results in fewer behaviours than the parallel composite of two sequential compositions, $a \circ c$ and $b \circ d$. Pictorially, this can be represented by a diagram (Fig. 2) where, on the left, a and b must finish together, causing c and d to start simultaneously. On the right, no such constraint is applied.

The neutral element of sequential composition, \mathtt{skip} , acts as a null specification, thus $\mathtt{skip}_A \parallel \mathtt{skip}_A$ must equal \mathtt{skip}_A . Dually, the neutral element of parallel composition, \mathtt{run} , signifies an unconstrained specification, so \mathtt{run}_A ; \mathtt{run}_A equals \mathtt{run}_A . The proof of Proposition 1 below shows that it's enough to assume one direction of these equalities; the other is derivable.

Proposition 1. In a CVA $(\Phi, \S, \text{skip}, \|, \text{run})$, for all $A \in \mathcal{T}$, we have $\text{skip}_A \leq \text{run}_A$, $\text{skip}_A \| \text{skip}_A = \text{skip}_A$, and $\text{run}_A \S \text{run}_A = \text{run}_A$.



Figure 2. Graphical representation of the weak exchange law.

Proof. By neutrality and weak exchange, for each $A \in \mathcal{T}$, $\mathtt{skip}_A = \mathtt{skip}_A$ \S $\mathtt{skip}_A = (\mathtt{run}_A \parallel \mathtt{skip}_A)$ \S $(\mathtt{skip}_A \parallel \mathtt{run}_A) \preceq (\mathtt{run}_A$ \S $\mathtt{skip}_A) \parallel (\mathtt{skip}_A$ \S $\mathtt{run}_A) = \mathtt{run}_A \parallel \mathtt{run}_A = \mathtt{run}_A$. Given the neutral laws, for the remaining two properties, it suffices to show that $\mathtt{skip}_A \parallel \mathtt{skip}_A \preceq \mathtt{skip}_A$ and $\mathtt{run}_A \preceq \mathtt{run}_A$ \S \mathtt{run}_A . By monotonicity of combination, we have $\mathtt{skip}_A \parallel \mathtt{skip}_A \preceq \mathtt{skip}_A \parallel \mathtt{run}_A = \mathtt{skip}_A$. Similarly, $\mathtt{run}_A = \mathtt{skip}_A$ \S \mathtt{run}_A \S \mathtt{run}_A \S \mathtt{run}_A \S \mathtt{run}_A .

Proposition 2. In a CVA $(\Phi, \S, \epsilon, ||, \epsilon)$ in which the neutral elements of parallel and sequential product coincide, for all $a, b \in \int \Phi$, we have $a \S b \leq a || b$.

Proof. Let
$$a \in \Phi_A$$
 and $b \in \Phi_B$. We have, $a \circ b = a^{\uparrow A \cup B} \circ b^{\uparrow A \cup B} = (a \parallel \mathbf{\epsilon}_{A \cup B}) \circ (\mathbf{\epsilon}_{A \cup B} \parallel b) \preceq (a \circ \mathbf{\epsilon}_{A \cup B}) \parallel (\mathbf{\epsilon}_{A \cup B} \circ b) = a^{\uparrow A \cup B} \parallel b^{\uparrow A \cup B} = a \parallel b$.

Definition 8 (morphism of CVAs). Let $(\Phi, \S, \text{skip}, \|, \text{run})$ and $(\Phi', \S', \text{skip}, \|', \text{run}')$ be CVAs. A lax/colax/strong morphism $f : \Phi \to \Phi'$ is a function $f : \int \Phi \to \int \Phi'$ that is both a lax/colax/strong morphism of OVAs $(\Phi, \S, \text{skip}) \to (\Phi', \S', \text{skip})$ and a lax/colax/strong morphism of OVAs $(\Phi, \|, \text{run}) \to (\Phi', \|', \text{run})$.

3.1 Reasoning in a CVA

Refinement. In a CVA Φ , the ordering between elements a and b in $\int \Phi$ is defined as $a \leq b$ if and only if $db \subseteq da$ and $a^{\downarrow db} \leq_{\Phi_{db}} b$. Viewing these elements as system specifications, this ordering is interpreted as refinement: $a \leq b$ means that all behaviour of a within domain db also exists in b, making a on db more deterministic than b. However, the domain da of a may exceed db, as a refined specification may introduce constraints outside the initial domain.

Hoare logic and rely-guarantee reasoning. Hoare triples and Jones quintuples facilitate formal reasoning about program behaviour, leveraging the well-established methodologies of Hoare logic and rely-guarantee reasoning. These constructs may be realised in a CVA by adapting their definitions as framed within Concurrent Kleene Algebras [12].

Let $(\Phi, \S, \text{skip}, \|, \text{run})$ be a CVA, and $p, a, q \in \int \Phi$. We define the **Hoare triple** of a with precondition p and postcondition q as

$$p\{a\} q := p \circ a \leq q \tag{18}$$

⁴ In duoidal categories, morphisms may also be lax with respect to ; and colax with respect to ||, but not the reverse [2].

From this definition, we may derive inference rules⁵ of Hoare logic, such as:

Proposition 3 (concurrency rule). Let $p, p', a, a', q, q' \in \int \Phi$. Then

$$p \{a\} q \text{ and } p' \{a'\} q' \Longrightarrow (p \parallel p') \{a \parallel a'\} (q \parallel q')$$

$$\tag{19}$$

Proof. Assume $p \, \stackrel{\circ}{,} \, a \leq q$ and $p' \, \stackrel{\circ}{,} \, a' \leq q'$. By weak exchange and monotonicity, $(p \parallel p') \, \stackrel{\circ}{,} \, (a \parallel a') \leq (p \, \stackrel{\circ}{,} \, a) \parallel (p' \, \stackrel{\circ}{,} \, a') \leq q \parallel q'$. Thus, $(p \parallel p') \, \{a \parallel a'\} \, (q \parallel q')$.

A **Jones quintuple** with rely r and guarantee g can then be defined as⁶

$$p \ r \{a\} \ g \ q := p \{r \mid | a\} \ q \text{ and } a \leq g$$
 (20)

To employ the standard inference rules of rely-guarantee reasoning, constraints must be placed on the rely variable r and the guarantee variable g. Though this definition serves as a gateway to rely-guarantee reasoning in the context of a CVA, exploration of this aspect is beyond the present study's purview.

4 Tuple systems

In Sections 5 to 7, each CVA examined is based on an underlying OVA of a specific form—they are OVAs of T-relations associated to certain tuple systems T. Tuple systems are presheaves that abstract the characteristic projecting and lifting properties of ordinary tuples. For more on tuple systems and the valuation algebras they induce, please see [14, Section 6.3, p. 169] and [16, Section 7.3.2, p. 286]. A T-relation is a subset of these generalised tuples sharing a common domain. In the trace models to follow, actions, states, traces and valuations themselves are encoded as tuples within tuple systems. The structure of the tuple system T governs how tuples on a larger domain project to a smaller one through the presheaf's restriction maps, as well as how tuples on a smaller domain lift to a larger one via the presheaf's flasque and binary gluing properties.

Definition 9 (tuple system). A tuple system is a presheaf $T: \mathcal{T}^{op} \to \mathcal{S}et$ satisfying the below axioms:

Flasque. For all $B \subseteq A$ in \mathcal{T} , the restriction map $\mathbf{T}_A \to \mathbf{T}_B$ is surjective. Binary gluing. For all $a \in \mathbf{T}_A$ and $b \in \mathbf{T}_B$, if $a^{\downarrow A \cap B} = b^{\downarrow A \cap B}$, then there exists $c \in \mathbf{T}_{A \cup B}$ so that $c^{\downarrow A} = a$ and $c^{\downarrow B} = b$.

Elements of T_A are called **tuples** (on A) or A-tuples.

Theorem 2 (OVAs of T-relations). Let $T: \mathcal{T}^{op} \to \mathcal{S}et$ be a tuple system. Define the prealgebra $\Psi := P \circ T: \mathcal{T}^{op} \to \mathcal{P}os$. Then Ψ , equipped with the relational join as the combine operator, defined

$$\wedge : \int \mathbf{\Psi} \times \int \mathbf{\Psi} \to \int \mathbf{\Psi}$$

$$a \wedge b := \left\{ t \in \mathbf{T}_{\mathrm{d}a \cup \mathrm{d}b} \mid t^{\downarrow \mathrm{d}a} \in a, t^{\downarrow \mathrm{d}b} \in b \right\}$$
(21)

⁵ Other basic rules are verified in the computer formalisation (see Footnote 1).

⁶ The guarantee requirement is stronger than required by Jones, where the guarantee only must hold while the rely does.

is a strongly neutral commutative OVA, that we call the **OVA of T-relations**. Its local orderings $\leq_{\mathbf{T}_A}$ are given by subset inclusion \subseteq , and it has as neutral element $T = A \mapsto \mathbf{T}_A$ for each $A \in \mathcal{T}$. Moreover, $\mathbf{U} \circ \mathbf{\Psi}$ is itself a tuple system, where $\mathbf{U} : \mathcal{P}os \to \mathcal{S}et$ is the forgetful functor that sends a poset to its underlying set, and a monotone map to its underlying function.

Proof. Monotonicity is easily verified. The other details are found in [14, p. 170].

It is worth noting the close resemblance of Eq. (21) with the trace semantics of the CSP parallel operator [11, Section 2.3.3, p. 53].

Proposition 4. Extension is given by the preimage to restriction; i.e. for $a \in \Phi_A$, and $B \in \mathcal{T}$ with $B \subseteq A$, we have $b^{\uparrow A} = \{t \in \mathbf{T}_A \mid t^{\downarrow B} \in b\}$.

Proof. It is a standard proof that direct image is left-adjoint to preimage.

Proposition 5. [proof in Appendix B.1] The relational join of an OVA of relations is the extension of intersection (from Definition 6): for $a, b \in \int \Phi$, $a \wedge b = a^{\uparrow da \cup db} \cap b^{\uparrow da \cup db}$. Moreover, $\int \Phi$ is a complete lattice, and relational join is its meet.

Lemma 3. [proof in Appendix B.2] Let $\Omega : \mathcal{T}^{\mathsf{op}} \to \mathcal{S}et$ be a tuple system, and let $\mathbf{L} : \mathcal{S}et \to \mathcal{S}et$ be the functor that sends a set X to the set of finite lists in X, i.e. $\mathbf{L} := X \mapsto \coprod_{n \in \mathbb{N}} X^n$, and let \mathbf{L}_+ be the functor that sends X to the set of nonempty finite lists in X, i.e. $\mathbf{L}_+ := X \mapsto \coprod_{n \in \mathbb{N}_+} X^n$. Then both $\mathbf{L} \circ \Omega$ and $\mathbf{L}_+ \circ \Omega$ are tuple systems.

Notation. Square brackets are used to display the components of a tuple $t \in (\Omega_A)^n$, i.e. we write $t = [t_1, \ldots, t_n]$. Such tuples are referred to as **traces**.

5 Action trace model

Let $\Omega^{\operatorname{act}}: \mathcal{T}^{\operatorname{op}} \to \mathcal{S}et$ be a tuple system whose values $\Omega_A^{\operatorname{act}}$ represent possible actions of a system in the variables A. Some concrete examples: for a semiring $\mathbb S$ of values, $\Omega_A^{\operatorname{act}}$ is the set of matrices $A \times A \to \mathbb S$ (linear actions); the set of pairs $\mathbb S^A \times \mathbb S^A$ (events); the set of relations $\mathbf P(\mathbb S^A \times \mathbb S^A)$ (events with external choice). Let $\mathbf T^{\operatorname{act}} := \mathbf L \circ \Omega^{\operatorname{act}}$, so that for each $A \in \mathcal T$, $\mathbf T_A^{\operatorname{act}}$ is the set of (possibly empty) traces of elements of $\Omega_A^{\operatorname{act}}$. By Lemma 3, $\mathbf T^{\operatorname{act}}$ is a tuple system. Let

$$\begin{split} & \boldsymbol{\Gamma}: \mathcal{T}^{\mathsf{op}} \to \mathcal{P}os \\ & \boldsymbol{\Gamma}:= \mathbf{P} \circ \mathbf{T}^{\mathsf{act}} = A \mapsto \mathbf{P}(\mathbf{L}(\boldsymbol{\Omega}_A^{\mathsf{act}})) \end{split} \tag{22}$$

be the OVA of ${\bf T}^{\sf act}$ -relations. We now develop a CVA structure on ${\bf \Gamma}$ that we call the $action\ trace\ model$.

For each $A \in \mathcal{T}$, define

$$\iota_A := \{[\,]_A\} \tag{23}$$

where $[]_A \in \mathbf{T}_A^{\mathsf{act}}$ is the unique length-0 trace with domain A. As restriction of a trace preserves length, this defines a global element $\iota : \mathbf{1} \Rightarrow \Gamma$.

⁷ This last point follows from the *idempotence* property of \wedge [16, Example 7.7, p. 287].

5.1 Interleaving product

For all $p, q \in \mathbb{N}$, let $\Sigma_{p,q}$ be the set (p,q)-shuffles, i.e. bijections $\{1, \ldots, p+q\} \to \{1, \ldots, p+q\}$ (or permutations) such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. For each $A \in \mathcal{T}$, define an operator on traces,

$$\dot{\mathbf{L}}_{A}: \mathbf{T}_{A}^{\mathsf{act}} \times \mathbf{T}_{A}^{\mathsf{act}} \to \mathbf{\Gamma}_{A}$$
$$[t_{1}, \dots, t_{p}] \, \dot{\mathbf{L}}_{A} \, [t_{p+1}, \dots, t_{p+q}] := \{[t_{\sigma(1)}, \dots, t_{\sigma(p+q)}] \mid \sigma \in \Sigma_{p,q}\}$$
(24)

Then lift each $\dot{\coprod}_A$ to a local operator on valuations,

$$\coprod_{A} : \Gamma_{A} \times \Gamma_{A} \to \Gamma_{A}
a \coprod_{A} a' := \left\{ \int \{ t_{a} \dot{\coprod}_{A} t_{a'} \mid t_{a} \in a, t_{a'} \in a' \} \right\}$$
(25)

It is well-known that \coprod_A is commutative, associative, and has unit ι_A . We then define the *interleaving product* as the extension \coprod of $\{\coprod_A\}_{A\in\mathcal{I}}$ to $\int \Gamma$:

Note that \sqcup is clearly commutative, and has as neutral element ι .

Lemma 4. [proof in Appendix C.1] For all $t, s \in \mathbf{T}_A^{\mathsf{act}}$ and $B \subseteq A$, we have $(t \mathrel{\dot{\coprod}}_A s)^{\downarrow B} = t^{\downarrow B} \mathrel{\dot{\coprod}}_B s^{\downarrow B}$.

Lemma 5. The structure ($\int \Gamma, \sqcup$) is an ordered semigroup.

Proof. By Lemma 1, it suffices to show that the local monotonicity and extension-commutation properties hold. The former follows directly from the definition of \coprod_A . For extension-commutation, let $B\subseteq A$, let $b,b'\in \Gamma_B$, and let $t\in b^{\uparrow A}\coprod_A b'^{\uparrow A}$. By definition of \coprod_A , there exists $r\in b^{\uparrow A}$, $s\in b'^{\uparrow A}$ so that $t\in r \dot \coprod_A s$. By Lemma 4, $t^{\downarrow B}\in (r \dot \coprod_A s)^{\downarrow B}=r^{\downarrow B}\dot \coprod_B s^{\downarrow B}\subseteq b \coprod_B b'$. Thus, $t\in (b \coprod_B b')^{\uparrow A}$. Conversely, let $t'\in (b \coprod_B b')^{\uparrow A}$. Now there is $r\in b$, $s\in b'$ so that $t'^{\downarrow B}\in r \dot \coprod_B s$. We may write $r=[t_1,\ldots,t_p],\ s=[t_{p+1},\ldots,t_{p+q}],\ and\ t'^{\downarrow B}=[t_{\sigma(1)},\ldots,t_{\sigma(p+q)}]$ for a (p,q)-shuffle $\sigma\in \mathcal{L}_{p,q}$. For each $1\leq i\leq p+q$, we then have a lifting $t'_{\sigma(i)}$ of $t_{\sigma(i)}$ so that $t=[t'_{\sigma(1)},\ldots,t'_{\sigma(p+q)}]$. Then $r'=[t'_1,\ldots,t'_p]$ is a lifting of r, $s'=[t'_{p+1},\ldots,t'_{p+q}]$ is a lifting of s, and $t'\in r'\dot \coprod_A s'\in b^{\uparrow A} \coprod_A b'^{\uparrow A}$ is exhibited as a trace associated to the same (p,q)-shuffle σ . The result follows.

Lemma 6. The interleaving product \sqcup satisfies the combination axiom.

Proof. Let $A, B \in \mathcal{T}$, $a \in \Gamma_A$ and $b \in \Gamma_B$. Note that one direction of the combination law follows from monotonicity. It then suffices to show $a \sqcup b^{\downarrow A \cap B} \subseteq (a \sqcup b)^{\downarrow A}$ and $a^{\downarrow A \cap B} \sqcup b \subseteq (a \sqcup b)^{\downarrow B}$. Let $t \in a \sqcup b^{\downarrow A \cap B} = a \sqcup_A (b^{\downarrow A \cap B})^{\uparrow A}$. By definition of \coprod_A , there exists $t_a \in a$ and $t_b \in (b^{\downarrow A \cap B})^{\uparrow A}$ so that $t \in t_a \sqcup_A t_b$. Let $t'_a \in a^{\uparrow A \cup B}$ be a lifting of t_a . Let $s := t_b^{\downarrow A \cap B} \in ((b^{\downarrow A \cap B})^{\uparrow A})^{\downarrow A \cap B} = b^{\downarrow A \cap B}$, where the equality follows by Corollary 2. There then exists $s' \in b$ so that $s'^{\downarrow A \cap B} = s$.

By binary gluing, there exists a common lifting $t_b' \in b^{\uparrow A \cup B}$ of t_b and s'. As in the proof of Lemma 5, it is easily shown that there is $t' \in t_a' \coprod_{A \cup B} t_b' \in a \coprod b$ (associated to the same (p,q)-shuffle as t) so that $t = t'^{\downarrow A} \in (a \coprod b)^{\downarrow A}$. Similarly, $a^{\downarrow A \cap B} \coprod b \subseteq (a \coprod b)^{\downarrow B}$. The result follows.

As strong neutrality easily holds, we have the following.

Proposition 6. The structure (Γ, \sqcup, ι) is a strongly neutral commutative OVA.

5.2 Concatenating product

For each $A \in \mathcal{T}$, define the associative binary operator on traces,

$$\dot{\gamma}_A : \mathbf{T}_A^{\mathsf{act}} \times \mathbf{T}_A^{\mathsf{act}} \to \mathbf{T}_A^{\mathsf{act}}$$
$$[t_1, \dots, t_n] \dot{\gamma}_A [s_1, \dots, s_m] := [t_1, \dots, t_n, s_1, \dots, s_m]$$
(27)

Then lift each $\dot{\neg}_A$ to a local operator on valuations,

We call the extension \smallfrown to $\int \Gamma$ of $\{\smallfrown_A\}_{A\in\mathcal{I}}$ the *concatenating product*.

Proposition 7. [proof in Appendix C.2] The structure (Γ, \neg, ι) is a strongly neutral OVA.

Proposition 8. The structure $(\Gamma, \neg, \iota, \sqcup, \iota)$ is a CVA.

Proof. Both \smallfrown and \sqcup define OVA structures on Γ (Propositions 6 and 7), and the neutral laws $\iota_A \subseteq \iota_A \sqcup \iota_A$ and $\iota_A \smallfrown \iota_A \subseteq \iota_A$ hold trivially. To show the weak exchange law, by Lemma 2, it suffices to show a local exchange law holds on each $A \in \mathcal{T}$. Let $a_1, a_2, a_3, a_4 \in \Gamma_A$ and $t \in (a_1 \sqcup_A a_2) \smallfrown_A (a_3 \sqcup_A a_4)$. By definition of \smallfrown , there is $r \in a_1 \sqcup_A a_2$ and $s \in a_3 \sqcup_A a_4$ so that $t = r \dot{\smallfrown}_A s$. It is clear every action of t coming from a_1 precedes every action of t coming from a_3 , and similarly every action of t coming from a_2 precedes every action of t coming from a_4 . It follows that t is in $(a_1 \smallfrown_A a_3) \sqcup_A (a_2 \smallfrown_A a_4)$. The result follows.

Proposition 9. For all $a, b \in \int \Phi$, we have $a \cap b \leq a \sqcup b$.

Proof. As the units for \neg and \sqcup coincide, this follows from Proposition 2.

6 State trace model

Here we define a CVA whose valuations consist of traces of *states* of an abstract system that progress in lockstep to an implied global clock. For each domain $A \in \mathcal{T}$, denote the hom-functor $\Omega^{\mathsf{state}} := A \mapsto (A \to \mathbb{S})$, i.e., $\Omega_A^{\mathsf{state}}$ is the set of (ordinary) A-tuples in some nonempty set \mathbb{S} of *values*, and the action of Ω^{state} on inclusions in \mathcal{T} is by precomposition. Notably, $\Omega_{\emptyset}^{\mathsf{state}}$ has a unique value \mathbb{S} ,

the *empty state*. By Lemma 3, $\mathbf{T}^{\mathsf{state}} := \mathbf{L}_+ \circ \mathbf{\Omega}^{\mathsf{state}}$ is a tuple system. For traces $t := [t_1, \ldots, t_n] \in \mathbf{T}_A^{\mathsf{state}}$, a component t_i is the system's state at time i. Let

$$\Sigma : \mathcal{T}^{\mathsf{op}} \to \mathcal{P}_{\mathcal{O}\mathcal{S}}$$

$$\Sigma := \mathbf{P} \circ \mathbf{T}^{\mathsf{state}} = A \mapsto \mathbf{P}(\mathbf{L}_{+}(A \to \mathbb{S}))$$
(29)

be the OVA of $\mathbf{T}^{\mathsf{state}}$ -relations. The relational join \wedge on Σ behaves as *synchron-isation*, and we take this as the parallel product for a CVA structure on Σ that we call the *state trace model*.

Let $\lambda: \mathbf{T}_A^{\mathsf{state}} \to \mathbb{N}_+$ denote the length function, and define

$$\mathbf{\tau} := A \mapsto \left\{ t \in \mathbf{T}_A^{\mathsf{state}} \mid \lambda(t) = 1 \right\} \tag{30}$$

As restriction preserves lengths of traces, this defines a global element $\tau: 1 \Rightarrow \Sigma$.

6.1 Gluing product

Let $A \in \mathcal{T}$. For a trace $t \in \mathbf{T}_A^{\mathsf{state}}$, let t^-, t^+ respectively denote the first and last components of t. Define an associative binary operator $\dot{\smile}_A$ on each $\mathbf{T}_A^{\mathsf{state}}$ by

$$\dot{\mathbf{C}}_A : \mathbf{T}_A^{\mathsf{state}} \times \mathbf{T}_A^{\mathsf{state}} \to \mathbf{T}_A^{\mathsf{state}}
t \dot{\mathbf{C}}_A s := (t_1, \dots, t_{\lambda(t)-1}, s_1, \dots, s_{\lambda(s)})$$
(31)

This then lifts to an associative binary operator on valuations,

We call the extension \vee to $\int \Sigma$ of the family $\{\vee_A\}_{A\in\mathcal{I}}$ the **gluing product**.

Proposition 10. [proof in Appendix D.1] The structure (Σ, \sim, τ) is a strongly neutral OVA.

Proposition 11. The structure $(\Sigma, \smile, \tau, \wedge, \top)$ is a CVA.

Proof. The neutral equalities are clear, and both \wedge and \vee define OVAs on Σ by Theorem 2 and Proposition 10. By Lemma 2, it suffices to show an exchange law holds on each $A \in \mathcal{T}$. Noting that \wedge is the extension of intersection by Proposition 5, let $a_1, a_2, a_3, a_4 \in \Sigma_A$ and let $t \in (a_1 \cap a_2) \vee_A (a_3 \cap a_4)$. By local monotonicity of \vee_A , both $t \in a_1 \vee_A a_3$ and $t \in a_2 \vee_A a_4$. Thus, $t \in (a_1 \vee_A a_3) \cap (a_2 \vee_A a_4)$, and the result follows.

6.2 Strong morphisms between Γ and Σ

There are no interesting strong morphisms between the action trace model Γ and the state trace model Σ . As the neutral elements for parallel and sequential coincide in Γ but not in Σ , there are no strong morphisms $\Gamma \to \Sigma$. On the other hand, a strong morphism $f: \Sigma \to \Gamma$ must map \top_A to ι_A , and by monotonicity this implies that $f_A(a) \subseteq \iota_A$ for all $a \in \Sigma_A$. Whether there are interesting (co)lax morphisms between Γ and Σ is an open question.

7 Relative state trace model

We introduce a variant, Σ^{rel} , of the state trace model from Section 6, that we refer to as the *relative state trace model*. In this model, traces are *stuttering-reduced*, meaning they do not contain duplicate adjacent components. Consequently, only the relative order of the indices in the trace components is significant, indicating independence from a global clock. This may lead to intriguing phenomena like *sequential inconsistency* [6].

An essentially equivalent construction of the underlying relational OVA was already presented in [6] using simplicial sets. Here, we offer a more concise and direct method using *free semigroups with idempotent generators*, previously applied to concurrency theory and quantum computation [4].

Let S be a set. Construct a semigroup $\mathbf{I}(S)$ as the free semigroup on S modulo the relation $x^2 = x$ for all $x \in S$. This is known as the **free semigroup on** S with idempotent generators. For example, if $S := \{0,1\}$, then $\mathbf{I}(S) = \{0,1,01,0,010,101,0101,\ldots\}$, and the semigroup product is concatenation modulo this congruence; e.g., $010 \cdot 01 = 0101$. Given a function $f: S \to S'$, there is a semigroup homomorphism $\mathbf{I}(f): \mathbf{I}(S) \to \mathbf{I}(S')$, defined by $\mathbf{I}(f)(x_1 \cdots x_n) := f(x_1) \cdots f(x_n)$, and moreover, this construction is functorial. Let $\mathbf{U}: \mathcal{S}emi \to \mathcal{S}et$ be the forgetful functor from the category of semigroups to the category of sets, that sends a semigroup to its underlying set, and a semigroup homomorphism to its underlying function. As in Section 6, let Ω^{state} be the contravariant hom-functor $\Omega^{\text{state}} = A \mapsto (A \to \mathbb{S})$ where \mathbb{S} is a fixed set of values. We then define $\mathbf{T}^{\text{rel}} := \mathbf{U} \circ \mathbf{I} \circ \Omega^{\text{state}} : \mathcal{T}^{\text{op}} \to \mathcal{S}et$.

Proposition 12. The presheaf T^{rel} is a tuple system.

Proof (sketch). This is essentially equivalent to [6, Theorem 2]. There, empty traces were included in the tuple system by use of the *augmented* simplicial nerve functor. If the ordinary nerve were used, the same proof goes through, and we would exclude empty traces (problematic here in defining gluing product), yielding a tuple system isomorphic to the one described here with semigroups.

Now let $\Sigma^{\mathsf{rel}} := \mathbf{P} \circ \mathbf{T}^{\mathsf{rel}}$ be the OVA of $\mathbf{T}^{\mathsf{rel}}$ -relations, and denote the relational join \wedge^{rel} and its neutral element $\top^{\mathsf{rel}} = A \mapsto \mathbf{T}_A^{\mathsf{rel}}$. Note that while $\top_{\emptyset} = \mathbf{T}_{\emptyset}^{\mathsf{state}}$ has infinitely many elements $[\heartsuit], [\heartsuit, \heartsuit], \ldots$, the neutral component $\top^{\mathsf{rel}}_{\emptyset} = \mathbf{T}_{\emptyset}^{\mathsf{rel}}$ has only one, namely $[\heartsuit]$. We define a local operator on valuations,

$$\bigcirc_{A}^{\text{rel}} : \Sigma_{A}^{\text{rel}} \times \Sigma_{A}^{\text{rel}} \to \Sigma_{A}^{\text{rel}}
a \smile_{A}^{\text{rel}} a' := \left\{ t_{a} \cdot_{A} t_{a'} \mid t_{a} \in a, t_{a'} \in a', t_{a}^{+} = t_{a'}^{-} \right\}$$
(33)

where \cdot_A is the product⁸ of the semigroup $\mathbf{I}(\Omega_A^{\mathsf{state}})$, and λ and $t \mapsto t^+, t^-$ are defined as in Section 6. We call the extension \vee^{rel} of $\left\{\vee_A^{\mathsf{rel}}\right\}_{A \in \mathcal{I}}$ to $\int \mathbf{\Sigma}^{\mathsf{rel}}$ the **relative gluing product**. Let $\mathbf{\tau}^{\mathsf{rel}} \coloneqq A \mapsto \left\{t \in \mathbf{T}_A^{\mathsf{rel}} \mid \lambda(t) = 1\right\}$. We then have,

⁸ To avoid excessive notation, we apply the semigroup products \cdot_A directly to traces, although their semigroup structure was forgotten by **U**.

Proposition 13. [proof in Appendix E.1] The structure $(\Sigma^{\mathsf{rel}}, {}^{\mathsf{rel}}, \tau^{\mathsf{rel}})$ is an OVA.

Unlike the models Γ and Σ of Sections 5 and 6, we have the following.

Proposition 14. The OVA $(\Sigma^{\text{rel}}, \vee^{\text{rel}}, \tau^{\text{rel}})$ is not strongly neutral.

Proof. We have $\top_{\emptyset}^{\sf rel} = \tau_{\emptyset}^{\sf rel}$ and yet $\top^{\sf rel} \neq \tau^{\sf rel}$. The result follows.

Proposition 15. The structure $(\Sigma^{\text{rel}}, \vee^{\text{rel}}, \tau^{\text{rel}}, \wedge^{\text{rel}}, \top^{\text{rel}})$ is a CVA.

Proof. The neutral laws are immediate, and we are only obliged to show the local weak exchange laws hold by Proposition 5 and Lemma 2. Locally $\wedge^{\mathsf{rel}}_A = \wedge_A = \cap$, and also \vee_A and \vee_A^{rel} have the same effect on traces, i.e. the gluing of two stuttering-reduced traces is already stuttering-reduced, so the proof of Proposition 11 goes through unchanged.

7.1 Colax morphism from Σ to Σ^{rel}

Define the free semigroup functor $\mathbf{F}: \mathcal{S}et \to \mathcal{S}emi$ mapping set $S \in \mathcal{S}et$ to finite lists of its elements, using concatenation as the semigroup product. Please note there is an evident isomorphism $\mathbf{L}_+ \cong \mathbf{U} \circ \mathbf{F}$ that we will apply implicitly. The universal property of the free semigroup leads to a surjective map $q_S: \mathbf{F}(S) \to \mathbf{I}(S)$ for each set S, which acts to eliminate duplicated adjacent elements in a list. This process defines a natural transformation $q: \mathbf{F} \Rightarrow \mathbf{I}$, allowing us to obtain another natural transformation by whiskering on both sides of q.

$$\mathcal{J}^{\text{op}} \xrightarrow{\Omega^{\text{state}}} \mathcal{S}et \xrightarrow{\mathbf{F}} \mathcal{S}emi \xrightarrow{\mathbf{U}} \mathcal{S}et \xrightarrow{\mathbf{P}} \mathcal{P}os \tag{34}$$

We denote this composite $f := (\mathbf{P} \circ \mathbf{U}) \circ q \circ \Omega^{\mathsf{state}} : \Sigma \to \Sigma^{\mathsf{rel}}$

Proposition 16. [proof in Appendix E.2] The map $f: \Sigma \to \Sigma^{rel}$ is a colar morphism of CVAs.

Proposition 16 effectively realises the relative trace model Σ^{rel} as a quotient of the state trace model Σ .

8 Local computation

Valuation algebras provide a foundation for practical computation through a suite of distributed *local computation* algorithms. These algorithms are designed to resolve *inference problems* that arise in the context of valuation algebras. A comprehensive reference to this topic is [16].

⁹ See [17, Remark 1.7.6., p.46].

Definition 10. Let Φ be an OVA. A knowledgebase is a finite subset of valuations $K \subseteq \int \Phi$. Let $\mathcal{A} := \{A_i \in \mathcal{T}\}_{i \in I}$ be a finite family of domains, so that for each $i \in I$, we have $A_i \subseteq \bigcup_{a \in K} \mathrm{d}a$. Then the task of computing $(\bigotimes_{a \in K} a)^{\downarrow A_i}$ for each $i \in I$, is called the inference problem for (K, \mathcal{A}) . In this context, $\bigotimes_{a \in K} a$ is called the joint valuation, and the domains A_i are called queries.

In distributed systems, an inference problem corresponds to determining the local behaviours of a composite system of interacting components. For example, sequential consistency of a specification, as shown in [6], can be framed as an inference problem. The key to local computation is the combination axiom $(a \otimes b)^{\downarrow da} = a \otimes b^{\downarrow da \cap db}$. However, traditional theory falls short in our setting as it presumes a single commutative combine operator. Though the generalised combination axiom of Definition 2 supports local computation for CVAs, further exploration in this area is called for.

9 Conclusion

In this work, we have introduced the concurrent valuation algebra (CVA), a new algebraic structure that expands upon ordered valuation algebras (OVAs) by incorporating parallel and sequential products. This integration places the theory of concurrent and distributed systems within the expansive scope of valuation algebras.

Our CVAs draw inspiration from existing algebraic frameworks in concurrency theory such as Communicating Sequential Processes (CSP) [11], Concurrent Kleene Algebra (CKA) [14], Concurrent Refinement Algebra (CRA) [9], and duoidal/2-monoidal categories [2]. They also facilitate key reasoning methodologies for program specification, like Hoare logic [10], and rely-guarantee reasoning [13].

Within the framework of CVAs, we explored three trace models, each representing distinct computational paradigms, and related them by morphisms.

This research marks a promising pathway to practical applications, particularly through the potent *local computation* framework described in Section 8. Looking ahead, our work will focus on several key areas. We aim to explore a wider range of CVA models, including the trace semantics of CSP, as well as examples founded on different structures, like trees or transition systems, instead of traces. Our study will further involve deepening the understanding of the general theory of CVAs, including the exploration of their categorical structure, and the ways CVAs on different spaces relate via the pull-back and push-forward mechanisms of their underlying presheaves. Of special interest is the examination of potential links between OVAs and the monoidal Grothendieck construction [15].

Acknowledgements. We convey our sincere gratitude to the following for their valuable insights and support: Alexander Evangelou, Brae Webb, Christina Vasilakopoulou, Cliff Jones, Des FitzGerald, Dylan Braithwaite, Brijesh Dongol, Graeme Smith, Igor Dolinka, James East, Jesse Sigal, Joe Moeller, John Baez,

Juerg Kohlas, Kait Lam, Kirsten Winter, Luigi Santocanale, Marc Pouly, Mark Utting, Martti Karvonen, Matt Garcia, Matteo Capucci, Michael Robinson, Mike Shulman, Morgan Rogers, Nick Coughlin, Peter Hoefner, Ralph Sarkis, Reid Barton, Rob Colvin, Scott Heiner, Sori Lee, Ted Goranson, Yannick Chevalier, and the Zulip category theory community. We are thankful for the support of the Australian Government Research Training Program Scholarship of Naso, and funding from the Australian Research Council (ARC) through the Discovery Grant DP190102142. We gratefully acknowledge the use of GitHub Copilot and OpenAI ChatGPT software in refining the readability of this paper, though their contribution did not extend to the semantic substance of the research.

A Proofs of Section 2

A.1 Proof of Lemma 1

Proof. First we show associativity of \odot . Let $a \in \Phi_A$, $b \in \Phi_B$, and $c \in \Phi_C$, and assume extension-commutation holds. We must show that

$$(a \odot b) \odot c = a \odot (b \odot c)$$

Let $U := A \cup B \cup C$. Then

$$\begin{array}{ll} (a\odot b)\odot c\\ = (a\odot b)^{\uparrow U}\odot_{U}c^{\uparrow U} & (\text{definition of }\odot)\\ = (a^{\uparrow A\cup B}\odot_{A\cup B}b^{\uparrow A\cup B})^{\uparrow U}\odot_{U}c^{\uparrow U} & (\text{definition of }\odot)\\ = ((a^{\uparrow A\cup B})^{\uparrow U}\odot_{U}(b^{\uparrow A\cup B})^{\uparrow U})\odot_{U}c^{\uparrow U} & (\text{hypothesis: ext.-comm.})\\ = (a^{\uparrow U}\odot_{U}b^{\uparrow U})\odot_{U}c^{\uparrow U} & (\text{functoriality of extension})\\ = a^{\uparrow U}\odot_{U}(b^{\uparrow U}\odot_{U}c^{\uparrow U}) & (\text{associativity of }\odot_{U})\\ = a^{\uparrow U}\odot_{U}((b^{\uparrow B\cup C})^{\uparrow U}\odot_{U}(c^{\uparrow B\cup C})^{\uparrow U}) & (\text{functoriality of extension})\\ = a^{\uparrow U}\odot_{U}(b^{\uparrow B\cup C}\odot_{B\cup C}c^{\uparrow B\cup C})^{\uparrow U} & (\text{hypothesis: ext.-comm.})\\ = a^{\uparrow U}\odot_{U}(b\odot c)^{\uparrow U} & (\text{definition of }\odot)\\ = a\odot(b\odot c) & (\text{definition of }\odot) \end{array}$$

Thus, \odot is associative.

To see that \odot is monotone, let $a_1, a_2, b_1, b_2 \in \int \Phi$ with $a_1 \leq a_2$ and $b_1 \leq b_2$. Let $U_1 := da_1 \cup db_1$ and $U_2 := da_2 \cup db_2$. Noting that $U_2 \subseteq U_1$, then

$$\begin{split} a_1 \odot b_1 &= a_1^{\uparrow U_1} \odot_{U_1} b_1^{\uparrow U_1} & \text{(definition of } \odot) \\ &\leq_{\Phi_{U_1}} a_2^{\uparrow U_1} \odot_{U_1} b_2^{\uparrow U_1} & \text{(local monotonicity)} \\ &\leq_{\Phi_{U_1}} (a_2^{\uparrow U_2})^{\uparrow U_1} \odot_{U_1} (b_2^{\uparrow U_2})^{\uparrow U_1} & \text{(functoriality of extension)} \\ &\leq_{\Phi_{U_1}} (a_2^{\uparrow U_2} \odot_{U_1} b_2^{\uparrow U_2})^{\uparrow U_1} & \text{(extension-commutation)} \\ &\leq_{\Phi_{U_1}} (a_2 \odot b_2)^{\uparrow U_1} & \text{(definition)} \end{split}$$

By the restriction-extension adjunction, this gives us $(a_1 \odot b_1)^{\downarrow U_2} \leq_{\Phi_{U_2}} a_2 \odot b_2$, i.e. $a_1 \odot b_1 \leq a_2 \odot b_2$. Hence, the extension \odot is monotone.

Thus, $(\int \Phi, \odot)$ is an ordered semigroup.

A.2 Proof of Lemma 2

Proof. First, note that extension commutation holds in any OVA $(\Phi, \otimes, \varepsilon)$, as for all $b_1, b_2 \in \Phi_B$ and $B \subseteq A$,

$$(b_1 \otimes b_2)^{\uparrow A} = \boldsymbol{\epsilon}_A \otimes (b_1 \otimes b_2) = (\boldsymbol{\epsilon}_A \otimes \boldsymbol{\epsilon}_A) \otimes (b_1 \otimes b_2) = (\boldsymbol{\epsilon}_A \otimes b_1) \otimes (\boldsymbol{\epsilon}_A \otimes b_2) = b_1^{\uparrow A} \otimes_A b_2^{\uparrow A}$$

Let $a \in \Phi_A$, $b \in \Phi_B$, $c \in \Phi_C$, $d \in \Phi_D$, and $U = A \cup B \cup C \cup D$. We have,

$$(a \parallel b) \circ (c \parallel d)$$

$$= (a \parallel b)^{\uparrow U} \circ_{U} (c \parallel d)^{\uparrow U}$$
 (definition of \circ)
$$= (a^{\uparrow A \cup B} \parallel_{A \cup B} b^{\uparrow A \cup B})^{\uparrow U} \circ_{U} (c^{\uparrow C \cup D} \parallel_{C \cup D} d^{\uparrow C \cup D})^{\uparrow U}$$
 (definition of \parallel)
$$= ((a^{\uparrow A \cup B})^{\uparrow U} \parallel_{U} (b^{\uparrow A \cup B})^{\uparrow U}) \circ_{U} ((c^{\uparrow C \cup D})^{\uparrow U} \parallel_{U} (d^{\uparrow C \cup D})^{\uparrow U})$$
 (ext.-comm.)
$$= (a^{\uparrow U} \parallel_{U} b^{\uparrow U}) \circ_{U} (c^{\uparrow U} \parallel_{U} d^{\uparrow U})$$
 (functoriality)
$$\leq \Phi_{U} (a^{\uparrow U} \circ_{U} c^{\uparrow U}) \parallel_{U} (b^{\uparrow U} \circ_{U} d^{\uparrow U})$$
 (local exchange)
$$= ((a^{\uparrow A \cup C})^{\uparrow U} \circ_{U} (c^{\uparrow A \cup C})^{\uparrow U}) \parallel_{U} ((b^{\uparrow B \cup D})^{\uparrow U} \circ_{U} (d^{\uparrow B \cup D})^{\uparrow U})$$
 (functoriality)
$$= (a^{\uparrow A \cup C} \circ_{A \cup C} c^{\uparrow A \cup C})^{\uparrow U} \parallel_{U} (a^{\uparrow B \cup D} \circ_{B \cup D} d^{\uparrow B \cup D})^{\uparrow U}$$
 (ext.-comm.)
$$= (a \circ c)^{\uparrow U} \parallel_{U} (b \circ d)^{\uparrow U}$$
 (definition of \circ)
$$= (a \circ c)^{\uparrow U} \parallel_{U} (b \circ d)$$
 (definition of \circ)

The result follows.

B Proofs of Section 4

B.1 Proof of Proposition 5

Proof. The first claim is trivial, and completeness of $\int \Psi$ follows from Corollary 3 as Ψ is actually a presheaf valued in complete lattices. For the third, we must show the universal property of meets: that $a \wedge b$ is the greatest lower bound of a and b. Clearly $a \wedge b$ is a lower bound of a and b. To show it is the greatest, we must show that whenever $(C,c) \preceq (A,a)$ and $(C,c) \preceq (B,b)$, we also have $(C,c) \preceq (A \cup B, a \wedge b)$. So assume (C,c) satisfies the precondition. This means that

$$A \subseteq C$$
, $B \subseteq C$, $c^{\downarrow A} \subseteq a$, $c^{\downarrow B} \subseteq b$

By the universal property of union, we have $A \cup B \subseteq C$. It remains to show that

$$c^{\downarrow A \cup B} \subseteq a \land b = a^{\uparrow A \cup B} \cap b^{\uparrow A \cup B}$$

By Corollary 2, functoriality of restriction, and monotonicity of extension, we have

$$c^{\downarrow A \cup B} \subseteq ((c^{\downarrow A \cup B})^{\downarrow A})^{\uparrow A \cup B} = (c^{\downarrow A})^{\uparrow A \cup B} \subseteq a^{\uparrow A \cup B}$$

and similarly $c^{\downarrow A \cup B} \subseteq b^{\uparrow A \cup B}$. By the universal property of intersection, we have $c \subseteq a^{\uparrow A \cup B} \cap b^{\uparrow A \cup B}$. This shows that $(C,c) \preceq (A \cup B, a \wedge b)$, as required. The result follows.

B.2 Proof of Lemma 3

Proof. We prove just for $\mathbf{L} \circ \Omega$, as the proof for $\mathbf{L}_+ \circ \Omega$ is similar. We have that Ω is a tuple system, and we must show that $\mathbf{T} := \mathbf{L} \circ \Omega$ is a tuple system.

Flasque. Let $B \subseteq A$ and $t \in \mathbf{T}_B$. We must show that there exists $t' \in \mathbf{T}_A$ so that $t'^{\downarrow B} = t$. Write $t = [t_1, \dots, t_n]$. As Ω is a tuple system, each t_i has a lifting $t'_i \in \Omega_A$. Clearly $t' = [t'_1, \dots, t'_n]$ is a lifting of t.

Binary gluing. Let $t_A \in \mathbf{T}_A$, $t_B \in \mathbf{T}_B$ be so that

$$s := t_A^{\downarrow A \cap B} = t_B^{\downarrow A \cap B}$$

The traces $t_A = [t_1^A, \dots, t_n^A]$ and $t_B = [t_1^B, \dots, t_n^B]$ necessarily have the same length and also $(t_i^A)^{\downarrow A \cap B} = (t_i^B)^{\downarrow A \cap B}$ for each i. As Ω has the gluing property, we can find a lifting s_i for each pair t_i^A, t_i^B and clearly the trace $s' = [s_1, \dots, s_n]$ is a common lifting of t_A and t_B .

The result follows.

C Proofs of Section 5

C.1 Proof of Lemma 4

Proof. Let $r' \in (t \, \dot{\sqcup}_A \, s)^{\downarrow B}$. Then there exists $r \in t \, \dot{\sqcup}_A \, s$ so that $r^{\downarrow B} = r'$. Write $t = [r_1, \dots, r_p]$ and $s = [r_{p+1}, \dots, r_{p+q}]$. By definition of $\dot{\sqcup}_A$, there is (p,q)-shuffle $\sigma \in \Sigma_{p,q}$ so that $r = [r_{\sigma(1)}, \dots, r_{\sigma(p+q)}]$. Now $r' = r^{\downarrow B} = [r^{\downarrow B}_{\sigma(1)}, \dots, r^{\downarrow B}_{\sigma(p+q)}]$ is clearly a shuffle of $t^{\downarrow B} = [r^{\downarrow B}_1, \dots, r^{\downarrow B}_p]$ and $s^{\downarrow B} = [r^{\downarrow B}_{p+1}, \dots, r^{\downarrow B}_{p+q}]$, thus $r' \in t^{\downarrow B} \, \dot{\sqcup}_B \, s^{\downarrow B}$.

Conversely, let $r \in t^{\downarrow B}$ $\dot{\coprod}_B s^{\downarrow B}$. Then there exists a (p,q)-shuffle $\sigma \in \Sigma_{p,q}$ so that $t^{\downarrow B} = [r_1, \ldots, r_p]$, $s^{\downarrow B} = [r_{p+1}, \ldots, r_{p+q}]$ and $r = [r_{\sigma(1)}, \ldots, r_{\sigma(p+q)}]$. By definition of restriction, for each $1 \leq i \leq p+q$, there is a lifting r_i' for r_i so that $t' = [r_1', \ldots, r_p']$ is a lifting of t, and $s' = [r_{p+1}', \ldots, r_{p+q}']$ is a lifting of s. Now if $r' := [r_{\sigma(1)}', \ldots, r_{\sigma(p+q)}']$ then clearly $r' \in t \dot{\coprod}_A s$, and thus $r = r'^{\downarrow B} \in (t \dot{\coprod}_A s)^{\downarrow B}$. The result follows.

C.2 Proof of Proposition 7

Lemma 7. For all $t, s \in \mathbf{T}_A^{\mathsf{act}}$ and all $B \subseteq A$, we have $(t \dot{\smallfrown}_A s)^{\downarrow B} = t^{\downarrow B} \dot{\smallfrown}_B s^{\downarrow B}$.

Proof. Let $B \subseteq A$ and $t, s \in \mathbf{T}_A^{\mathsf{act}}$, and write $t = [t_1, \dots, t_n], s = [s_1, \dots, s_m]$. We have

$$\begin{split} (t \,\dot\smallfrown_A \, s)^{\downarrow B} &= ([t_1, \dots, t_n] \,\dot\smallfrown_A \, [s_1, \dots, s_m])^{\downarrow B} \\ &= [t_1, \dots, t_n, s_1, \dots, s_m]^{\downarrow B} \\ &= [t_1^{\downarrow B}, \dots, t_n^{\downarrow B}, s_1^{\downarrow B}, \dots, s_m^{\downarrow B}] \\ &= [t_1^{\downarrow B}, \dots, t_n^{\downarrow B}] \,\dot\smallfrown_B \, [s_1^{\downarrow B}, \dots, s_m^{\downarrow B}] \\ &= t^{\downarrow B} \,\dot\smallfrown_B \, s^{\downarrow B} \end{split}$$

The result follows.

Notation. For $A \in \mathcal{T}$ and a trace $t = [t_1, \dots, t_n] \in \mathbf{T}_A^{\mathsf{state}}$, we write $t_i^j := [t_i, \dots, t_j]$, where $1 \le i \le j \le n$.

Lemma 8. The structure ($\int \Sigma$, \sim) is an ordered semigroup.

Proof. The local operators \smallfrown_A are clearly associative. By Lemma 1, it then suffices to show that the local monotonicity and extension-commutation properties hold. Let $\lambda: \mathbf{T}_A^{\mathsf{act}} \to \mathbb{N}$ denote the length function for each $A \in \mathcal{F}$.

Local monotonicity. Let $A \in \mathcal{T}$ and $a_1, a'_1, a_2, a'_2 \in \Sigma_A^{\mathsf{rel}}$ with $a_1 \subseteq a'_1$ and $a_2 \subseteq a'_2$. Then $a_1 \cap_A a_2 \subseteq a'_1 \cap_A a'_2$ follows from the definition of \cap_A .

Extension-commutation. Let $B \subseteq A$, let $b, b' \in \Sigma_B$, let $t \in (b \curvearrowright_B b')^{\uparrow A}$ and let $t' := t^{\downarrow B}$. Then there is $r \in b$ and $s \in b'$ with $t' = r \stackrel{\cdot}{\smallfrown}_B s$. Let $r' := t_1^{\lambda(r)}$ and $s' := t_{\lambda(r)+1}^{\lambda(t)}$. As the length of a trace is preserved by restriction, $r'^{\downarrow B} = r$ and $s'^{\downarrow B} = s$, so that $r' \in b^{\uparrow A}$ and $s' \in b'^{\uparrow A}$. We then have $t = r' \stackrel{\cdot}{\smallfrown}_A s' \in b^{\uparrow A} \curvearrowright_A b'^{\uparrow A}$.

On the other hand, let $t \in b^{\uparrow A} \smallfrown_A b'^{\uparrow A}$. Then there exists $r \in b^{\uparrow A}$ and $s \in b'^{\uparrow A}$ so that $t = r \dot{\smallfrown}_A s$. Now $r^{\downarrow B} \in b$, $s^{\downarrow B} \in b'$, so that $r^{\downarrow B} \dot{\smallfrown}_B s^{\downarrow B} \in b \smallfrown_B b'$. As $t^{\downarrow B} = r^{\downarrow B} \dot{\smallfrown}_B s^{\downarrow B}$ by Lemma 7, we have that $t \in (b \smallfrown_B b')^{\uparrow A}$.

Lemma 9. The operator \sim satisfies the combination axiom.

Proof. Note that one direction of the combination law follows from monotonicity. It then suffices to show suffices to show $a \cap b^{\downarrow A \cap B} \subseteq (a \cap b)^{\downarrow A}$ and $a^{\downarrow A \cap B} \cap b \subseteq (a \cap b)^{\downarrow B}$. Let $a \in \Gamma_A$, $b \in \Gamma_B$, and $t \in a \cap b^{\downarrow A \cap B}$. By definition of sequential \cap , there exists $t_a \in a$ and $t_b \in (b^{\downarrow A \cap B})^{\uparrow A}$ so that $t = t_a \dot{\cap}_A t_b$. Let $t'_a \in a^{\uparrow A \cup B}$ be a lifting of t_a . Let

$$s := t_b^{\downarrow A \cap B} \in ((b^{\downarrow A \cap B})^{\uparrow A})^{\downarrow A \cap B} = b^{\downarrow A \cap B}$$

where the equality follows by Corollary 2, and let $s' \in b$ be another lifting of s. By binary gluing, there then exists a common lifting $t'_b \in b^{\uparrow A \cup B}$ of t_b and s'. Define $t' := t'_a \stackrel{\cdot}{\smallfrown}_{A \cup B} t'_b \in a \stackrel{\cdot}{\smallfrown} b$. Then by Lemma 7, we have that

$$t = t_a \dot{\smallfrown}_A t_b = t_a^{\prime \downarrow A} \dot{\smallfrown}_A t_b^{\prime \downarrow A} = (t_a^\prime \dot{\smallfrown}_{A \cup B} t_b^\prime)^{\downarrow A} = t^{\prime \downarrow A} \in (a \smallfrown b)^{\downarrow A}$$

Thus, $a \smallfrown b^{\downarrow A \cap B} \subseteq (a \smallfrown b)^{\downarrow A}$. Similarly, $a^{\downarrow A \cap B} \smallfrown b \subseteq (a \smallfrown b)^{\downarrow B}$. The result follows.

Proof (of Proposition 7). We have that \smallfrown is an ordered semigroup by Lemma 8. The labelling axiom is immediate. It is straightforward to see that ι is a neutral element of \smallfrown , and moreover satisfies the strong neutrality condition. Finally, \smallfrown satisfies the combination axiom by Lemma 9. The result follows.

D Proofs of Section 6

D.1 Proof of Proposition 10

Lemma 10. For all $t, s \in \mathbf{T}_A^{\mathsf{state}}$ and all $B \subseteq A$, we have $(t \dot{\smile}_A s)^{\downarrow B} = t^{\downarrow B} \dot{\smile}_B s^{\downarrow B}$ and $(t^{\downarrow B})^+ = (t^+)^{\downarrow B}$.

Proof. Write
$$t = [t_1, \dots, t_n]$$
, $s = [s_1, \dots, s_m]$. We have
$$(t \,\dot{\smile}_A \,s)^{\downarrow B} = ([t_1, \dots, t_n] \,\dot{\smile}_A \,[s_1, \dots, s_m])^{\downarrow B}$$

$$= [t_1, \dots, t_{n-1}, s_1, \dots, s_m]^{\downarrow B}$$

$$= [t_1^{\downarrow B}, \dots, t_{n-1}^{\downarrow B}, s_1^{\downarrow B}, \dots, s_m^{\downarrow B}]$$

$$= [t_1^{\downarrow B}, \dots, t_n^{\downarrow B}] \,\dot{\smile}_B \,[s_1^{\downarrow B}, \dots, s_m^{\downarrow B}]$$

$$= t^{\downarrow B} \,\dot{\smile}_B \,s^{\downarrow B}$$

The second claim is immediate.

Lemma 11. The structure (Σ, \smile) is an ordered semigroup.

Proof. By Lemma 1, it suffices to show that the local monotonicity and extension-commutation properties hold.

Local monotonicity. Let $A \in \mathcal{T}$ and $a_1, a'_1, a_2, a'_2 \in \Sigma_A$ with $a_1 \subseteq a'_1$ and $a_2 \subseteq a'_2$. Then $a_1 \sim_A a_2 \subseteq a'_1 \sim_A a'_2$ follows from the definition of \sim_A . **Extension-commutation.** Let $B \subseteq A$, let $b, b' \in \Sigma_B$, let $t \in (b \sim_B b')^{\uparrow A}$ and let $t' := t^{\downarrow B}$. Then there is $r \in b$ and $s \in b'$ with $t' = r \circ_B s$ and $r^+ = s^-$. Using the notation of Appendix C.2, let $r' := t_1^{\lambda(r)}$ and $s' := t_{\lambda(r)}^{\lambda(t)}$. As the length of a trace is preserved by restriction, $r'^{\downarrow B} = r$ and $s'^{\downarrow B} = s$, so that $r' \in b$, $s' \in b'$, and also $r'^+ = s'^-$. It follows that $t = r' \circ_A s' \in b^{\uparrow A} \circ_A b'^{\uparrow A}$. On the other hand, let $t \in b^{\uparrow A} \circ_A b'^{\uparrow A}$. Then there exists $r \in b^{\uparrow A}$ and $s \in b'^{\uparrow A}$ so that $t = r \circ_A s$ and $r^+ = s^-$. Now $r^{\downarrow B} \in b$, $s^{\downarrow B} \in b'$, and $(r^{\downarrow B})^+ = (s^{\downarrow B})^-$, so that $r^{\downarrow B} \circ_B s^{\downarrow B} \in b \circ_B b'$. As $t^{\downarrow B} = r^{\downarrow B} \circ_B s^{\downarrow B}$ by Lemma 10, we have that $t \in (b \circ_B b')^{\uparrow A}$.

Lemma 12. The operator \sim satisfies the combination axiom.

Proof. Note that one direction of the combination law follows from monotonicity. It then suffices to show $a \smile b^{\downarrow A \cap B} \subseteq (a \smile b)^{\downarrow A}$ and $a^{\downarrow A \cap B} \smile b \subseteq (a \smile b)^{\downarrow B}$.

Let $a \in \Sigma_A$, $b \in \Sigma_B$, and $t \in a \smile b^{\downarrow A \cap B}$. By definition of \smile , there exists $t_a \in a$ and $t_b \in (b^{\downarrow A \cap B})^{\uparrow A}$ so that $t = t_a \smile_A t_b$ and $t_a^+ = t_b^-$. Let $t_a' \in a^{\uparrow A \cup B}$ be a lifting of t_a . Let

$$s \coloneqq t_b^{\downarrow A \cap B} \in ((b^{\downarrow A \cap B})^{\uparrow A})^{\downarrow A \cap B} = b^{\downarrow A \cap B}$$

where the equality follows by Corollary 2, and let $s' \in b$ be another lifting of s. By binary gluing, there then exists a common lifting $t'_b \in b^{\uparrow A \cup B}$ of t_b and s'. We can assume that $(t_a)^+ = (t'_b)^-$; if not, simply replace t'_a by t''_a where t''_a is t'_a with its final component replaced by the first component of t'_b . Note that t''_a is then still a lifting of t_a ; for this we must only check its last component restricts onto the last component of t_a . By Lemma 10, we have $(t''_a)^{\downarrow B} = (t'_b)^{\downarrow B} = t_b^- = t_a^+$. Define $t' := t'_a \dot{\circ}_{A \cup B} t'_b \in a \smile b$. Again by Lemma 10, we have that

$$t = t_a \dot{\circ}_A t_b = t_a^{\prime \downarrow A} \dot{\circ}_A t_b^{\prime \downarrow A} = (t_a^{\prime} \dot{\circ}_{A \cup B} t_b^{\prime})^{\downarrow A} = t^{\prime \downarrow A} \in (a \lor b)^{\downarrow A}$$

Thus, $a \smile b^{\downarrow A \cap B} \subseteq (a \smile b)^{\downarrow A}$. Similarly, $a^{\downarrow A \cap B} \smile b \subseteq (a \smile b)^{\downarrow B}$. The result follows.

Lemma 13. Gluing product has as neutral element τ , and τ has the strong neutrality property.

Proof. Let $a \in \Sigma_A$. Then

$$\begin{split} a &\sim \mathbf{\tau}_A = a^{\uparrow A} \smile_A \mathbf{\tau}_A^{\uparrow A} \\ &= a \smile_A \mathbf{\tau}_A \\ &= \left\{ t_a \mathrel{\dot{\smile}}_A t_\mathbf{\tau} \mid t_a \in a, t_\mathbf{\tau} \in \mathbf{\tau}_A, t_a^+ = t_\mathbf{\tau}^- \right\} \\ &= \left\{ t_a \mathrel{\dot{\smile}}_A t_\mathbf{\tau} \mid t_a \in a, t_\mathbf{\tau} \in \mathbf{T}_A^{\mathsf{state}}, \lambda(t_\mathbf{\tau}) = 1, t_a^+ = t_\mathbf{\tau}^- \right\} \\ &= \left\{ t_a \mathrel{\dot{\smile}}_A [x] \mid t_a \in a, x \in \mathbf{\Omega}_A^{\mathsf{state}}, t_a^+ = x \right\} \\ &= \left\{ t_a \mid t_a \in a \right\} \\ &= a \end{split}$$

and similarly $\tau_A \smile a = a$. Thus, τ is a neutral element for gluing product. For $A, B \in \mathcal{T}$, with $B \subseteq A$, we have

$$\begin{split} & \boldsymbol{\tau}_B^{\uparrow A} = \boldsymbol{\tau}_A \smile \boldsymbol{\tau}_B \\ & = \boldsymbol{\tau}_A \smile_A \boldsymbol{\tau}_B^{\uparrow A} \\ & = \left\{ t_1 \mathrel{\dot{\smile}}_A t_2 \mid t_1 \in \boldsymbol{\tau}_A, t_2 \in \boldsymbol{\tau}_B^{\uparrow A}, t_1^+ = t_2^- \right\} \\ & = \left\{ t_1 \mathrel{\dot{\smile}}_A t_2 \mid t_1 \in \boldsymbol{\Sigma}_A, t_2 \in \boldsymbol{\Sigma}_A, t_1^+ = t_2^-, \lambda(t_1) = 1 = \lambda(t_2^{\downarrow B}) \right\} \\ & = \left\{ [x] \mathrel{\dot{\smile}}_A [y] \mid x \in \boldsymbol{\Omega}_A^{\mathrm{state}}, y \in \boldsymbol{\Omega}_A^{\mathrm{state}}, x = y \right\} \\ & = \left\{ [x] \mid x \in \boldsymbol{\Omega}_A^{\mathrm{state}} \right\} \\ & = \boldsymbol{\tau}_A \end{split}$$

Above, we used the fact that restriction of a trace does not change its length. Thus, τ has the strong neutrality property.

Proof (proof of Proposition 10). The ordered semigroup axiom was verified in Lemma 11. The labelling axiom is immediate from definitions. We have shown τ is a neutral element for \sim that has the strongly neutral property in Lemma 13. The combination axiom is proved in Lemma 12. The result follows.

E Proofs of Section 7

E.1 Proof of Proposition 13

Lemma 14. For all $t, s \in \mathbf{T}_A^{\mathsf{rel}}$ and all $B \subseteq A$, we have $(t \cdot_A s)^{\downarrow B} = t^{\downarrow B} \cdot_A s^{\downarrow B}$, and $(t^{\downarrow B})^+ = (t^+)^{\downarrow B}$.

Proof. The first claim is due to restriction being a semigroup homomorphism (recall Footnote 8). The second is clear.

Lemma 15. The structure ($\int \Sigma^{\text{rel}}$, \smile^{rel}) is an ordered semigroup.

Proof. By Lemma 1, it suffices to show that the local monotonicity and extension-commutation properties hold.

Local monotonicity. Let $A \in \mathcal{T}$ and $a_1, a'_1, a_2, a'_2 \in \Sigma^{\mathsf{rel}}_A$ with $a_1 \subseteq a'_1$ and $a_2 \subseteq a'_2$. Now $a_1 {}^{\mathsf{rel}}_A a_2 \subseteq a'_1 {}^{\mathsf{rel}}_A a'_2$ follows from the definition of ${}^{\mathsf{rel}}_A$. **Extension-commutation.** Let $B \subseteq A$, let $b, b' \in \Sigma^{\mathsf{rel}}_B$, let $t \in (b {}^{\mathsf{rel}}_B b')^{\uparrow A}$ and let $t' := t^{\downarrow B}$. Then there is $r \in b$ and $s \in b'$ with $t' = r \cdot_B s$ and $r^+ = s^-$. Using the notation of Appendix C.2, let $n \in \mathbb{N}_+$ be so that $(t_1^n)^{\downarrow B} = r$ and $(t_n^{\lambda(t)})^{\downarrow B} = s$ (note that n may not be unique, and this is the only point of difference with the proof of Lemma 11). Let $r' := t_1^n$ and $s' := t_n^{\lambda(t)}$. Then $r' \in b^{\uparrow A}$, $s' \in b'^{\uparrow A}$, and $r'^+ = s'^-$. It follows that $t = r' \cdot_A s' \in b^{\uparrow A} {}^{\mathsf{rel}}_A b'^{\uparrow A}$. On the other hand, let $t \in b^{\uparrow A} {}^{\mathsf{rel}}_A b'^{\uparrow A}$. Then there exists $r \in b^{\uparrow A}$ and $s \in b'^{\uparrow A}$ so that $t = r \cdot_A s$ and $r^+ = s^-$. Now $r^{\downarrow B} \in b$ and $s^{\downarrow B} \in b'$, and by Lemma 14, $(r^{\downarrow B})^+ = (s^{\downarrow B})^-$, and so $t^{\downarrow B} = r^{\downarrow B} \cdot_B s^{\downarrow B} \in b {}^{\mathsf{rel}}_B b'$. Thus, $t \in (b {}^{\mathsf{rel}}_B b')^{\uparrow A}$.

Proof (of Proposition 13). The ordered semigroup axiom was shown to hold in Lemma 15. The labelling axiom is immediate. Proofs for the neutrality and combination axioms go through exactly as in the proof of Proposition 10. The result follows.

E.2 Proof of Proposition 16

Lemma 16. For all $B \subseteq A$ in \mathcal{T} and $b \in \Sigma_B$, we have

$$f_A(b^{\uparrow A}) \subseteq f_B(b)^{\uparrow A}$$

Proof. We have

$$f_B(b) \subseteq f_B(b)$$
 (reflexivity)
 $\implies f_B((b^{\uparrow A})^{\downarrow B}) \subseteq f_B(b)$ (Corollary 2)
 $\implies f_A(b^{\uparrow A})^{\downarrow B} \subseteq f_B(b)$ (naturality of f)
 $\implies f_A(b^{\uparrow A}) \subseteq f_B(b)^{\uparrow A}$ (adjunction)

The result follows.

Lemma 17. Let $t, s \in \mathbf{T}_A^{\mathsf{state}}$ so that $t^+ = s^-$. Then $q_A(t \,\dot{\sim}_A \, s) = q_A(t) \,\cdot_A \, q_A(s)$ and $q_A(t)^+ = q_A(s)^-$.

Proof. The action of q is to eliminate duplicate adjacent components, so the first claim is immediate by observing that we cannot have $t_{\lambda(t)-1} = s^-$ or $t^+ = s_2$ ($t^+ = t_{\lambda(t)} = s^-$). For the second, note that q cannot change the first or last components of a trace.

Proof (of Proposition 16). Let $a \in \Sigma_A^{\mathsf{rel}}$ and $b \in \Sigma_B^{\mathsf{rel}}$ for some $A, B \in \mathcal{T}$.

Colax naturality. Naturality of q is clear, and this directly implies (strict) naturality of f.

Monotonicity. For monotonicity, first note that each f_A is monotone as $f_A = \mathbf{P}(\mathbf{U}(q_{\mathbf{\Omega}^{\text{state}}}))$ and \mathbf{P} is a functor valued in posets. Now if $a \leq b$ then $B \subseteq A$ and $a^{\downarrow B} \subseteq b$. Note we have d(f(a)) = A and d(f(b)) = B. Then by naturality and local monotonicity, $f_A(a)^{\downarrow B} = f_B(a^{\downarrow B}) \subseteq f_B(b)$, thus by definition $f(a) \leq f(b)$.

Colax unitality. Clearly, we have in fact $f(\top) = \top^{\text{rel}}$ and $f(\tau) = \tau^{\text{rel}}$. Colax multiplicativity. First we show colaxity with respect to \wedge^{rel} . We have,

$$f(a \wedge b) = f_{A \cup B}(a^{\uparrow A \cup B} \cap b^{\uparrow A \cup B}) \qquad \text{(definition)}$$

$$\subseteq f_{A \cup B}(a^{\uparrow A \cup B}) \cap f_{A \cup B}(b^{\uparrow A \cup B}) \qquad \text{(property of image)}$$

$$\subseteq f_A(a)^{\uparrow A \cup B} \cap f_B(b)^{\uparrow A \cup B} \qquad \text{(Lemma 16, monotonicity of } \cap)$$

$$= f(a) \wedge^{\mathsf{rel}} f(b) \qquad \text{(definition)}$$

Finally, we show colaxity with respect to \smile^{rel} , i.e. $f(a \smile b) \preceq f(a) \smile^{\mathsf{rel}} f(b)$. Let

$$t \in f(a \smile b) = f_{A \cup B}(a^{\uparrow A \cup B} \smile_{A \cup B} b^{\uparrow A \cup B})$$

Then there is $r \in a^{\uparrow A \cup B}$ and $s \in b^{\uparrow A \cup B}$ with $r^+ = s^-$ so that $t = q_{A \cup B}(r \,\dot{\sim}_{A \cup B}\, s)$. By Lemma 17, $t = q_{A \cup B}(r) \,\dot{\sim}_{A \cup B}\, q_{A \cup B}(s)$, and $q_{A \cup B}(r)^+ = q_{A \cup B}(s)^-$. Note $q_{A \cup B}(r) \in f_{A \cup B}(a^{\uparrow A \cup B})$ and $q_{A \cup B}(s) \in f_{A \cup B}(b^{\uparrow A \cup B})$, so t is a trace in $f_{A \cup B}(a^{\uparrow A \cup B}) \,\dot{\sim}_{A \cup B}^{el}\, f_{A \cup B}(b^{\uparrow A \cup B})$. By Lemma 16,

$$f_{A \cup B}(a^{\uparrow A \cup B}) \smile^{\mathsf{rel}}_{A \cup B} f_{A \cup B}(b^{\uparrow A \cup B}) \subseteq f_{A}(a)^{\uparrow A \cup B} \smile^{\mathsf{rel}}_{A \cup B} f_{B}(b)^{\uparrow A \cup B}$$
$$= f(a) \smile^{\mathsf{rel}} f(b)$$

Thus $f(a \smile b) \preceq f(a) \smile^{\mathsf{rel}} f(b)$.

The result follows.

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