Enumerating Error Bounded Polytime Algorithms Through Arithmetical Theories (Extended Version)

Melissa Antonelli

Helsinki Institute for Information Technology, Finland melissa.antonelli@helsinki.fi

Ugo Dal Lago
Bologna University, Italy,
Inria, Université Côte d'Azur, France
ugo.dallago@unibo.it

Davide Davoli
Inria, Université Côte d'Azur, France
davide.davoli@inria.fr

Isabel Oitavem

Nova University Lisbon, Portugal

oitavem@fct.unl.pt

Paolo Pistone
Bologna University, Italy
paolo.pistone2@unibo.it

Abstract

We consider a minimal extension of the language of arithmetic, such that the bounded formulas provably total in a suitably-defined theory à la Buss (expressed in this new language) precisely capture polytime random functions. Then, we provide two new characterizations of the semantic class **BPP** obtained by internalizing the error-bound check within a logical system: the first relies on measure-sensitive quantifiers, while the second is based on standard first-order quantification. This leads us to introduce a family of effectively enumerable subclasses of **BPP**, called **BPP**_T and consisting of languages captured by those probabilistic Turing machines whose underlying error can be proved bounded in T. As a paradigmatic example of this approach, we establish that polynomial identity testing is in **BPP**_T, where $T = I\Delta_0 + Exp$ is a well-studied theory based on bounded induction.

1 Introduction

Since the early days of computer science, numerous and profound interactions with mathematical logic have emerged (think of the seminal works by Turing [56] and Church [9]). Among the sub-fields of computer science that have benefited the most from this dialogue, we should certainly mention the theory of programming languages (e.g. through the Curry-Howard correspondence [18, 39, 55]), the theory of databases (e.g. through Codd's theorem [11]) and computational complexity (e.g. through descriptive complexity [4, 40]). In particular, this last discipline deals with complexity classes [37, 10, 3], the nature of which still remains today, more than fifty years after the introduction of **P** and **NP** [12, 37], somewhat obscure.

The possibility of describing fundamental classes within the language of mathematical logic offered a better understanding of their nature: since the seventies [23, 15], but especially from the eighties and nineties [7, 33, 4, 40, 44], the logical characterization of several crucial classes has made it possible to consider them from a new viewpoint, less dependent on concrete machine models and explicit resource bounds. Characterizing complexity classes by way of a simple enough proof-of-recursion theoretical system also means being able to enumerate the problems belonging to them, and thus to devise sound and complete languages for the class, from which type systems and static analysis methodologies can be derived [38].

Among the various classes of problems considered in computational complexity, those defined on the basis of randomized algorithms [50] have appeared difficult to capture with the tools of logic. These include important and wellstudied classes like **BPP** or **ZPP**. The former, in particular, is often considered as the class of feasible problems, and most complexity theorists conjecture that it actually coincides with P. One might thus expect it to be possible to obtain an enumeration of BPP, along the lines of the many examples known for classes like P, or even PP [19, 20]. However, by simply looking at its definition, **BPP** looks pretty different from **P**. Notably, the former, but not the latter, is an example of what is usually called a *semantic* class: the definition of **BPP** relies on algorithms which are both efficient and not too erratic: once an input is fixed, one of the two possible outputs must clearly prevail over the other; in other words, there is some fixed probability p, bounded away from $\frac{1}{2}$, such that, on any input x, the machine outputs some value $b_x \in \{0,1\}$ with probability at least p. The existence of an effective enumerable family of algorithms deciding all and only the problems in **BPP** is still an open question.

In this paper we make a step towards a logical understanding of semantic complexity classes, and in particular of the logical and proof-theoretic complexity involved in keeping error-bounds under control. Our contributions can be divided in three parts. First, we generalize to the probabilistic setting the path indicated by bounded arithmetic [7, 25], a well-known approach to capture polynomial time algorithms, by extending usual arithmetical languages with a distinguished unary predicate Flip(x), playing the role of a source of randomness. We define a bounded theory $R\Sigma_1^b\text{-NIA}$ as the randomized analogue of Buss' S_2^1 [7] and Ferreira's $\Sigma_1^b\text{-NIA}$ [26], and show that the functions which can be proved total in $R\Sigma_1^b\text{-NIA}$ are precisely the polytime random functions [54], i.e. those functions from strings to distributions of strings which can be computed by polytime probabilistic Turing machines (PTM, for short). Then, we

move towards proper randomized classes by considering ways to keep the probability of error under control from within the logic. We first consider measure quantifiers [49, 47, 2], well-studied second-order quantifiers capable of measuring the extent to which a formula is true; we then show that these quantifiers, when applied to bounded formulas, can be encoded via standard first-order quantification. This way we obtain two characterizations of the problems in **BPP**, yet still semantic in nature: the error-bound check is translated into conditions which are not based on provability in some formal system, but rather on the truth of some formula in the standard model of first-order arithmetic.

While these results, which rely on semantic conditions, do not shed light on the enumeration problem for **BPP** directly, they set the conditions for a proof-theoretic investigation of this class: our last contribution is the introduction of a family of new *syntactic* subclasses of **BPP**, each called **BPP**_T, and consisting of those languages for which the error-bounding condition is not only true, but also *provable* in some (non necessarily bounded) theory T. This reduces the enumeration problem to that of finding a recursively enumerable (r.e., for short) arithmetical theory T such that **BPP** = **BPP**_T. To witness the difficulty of this problem, we show that the error-bounding condition is Π_1^0 -complete and that establishing that **BPP** cannot be enumerated would be at least as hard as refuting the **BPP** = **P** conjecture. At the same time, we show that polynomial identity testing (PIT), one of the few problems in **BPP** currently not known to be in **P** lies in **BPP**_T, where $T = I\Delta_0 + Exp$ is a well-studied [36] sub-theory of PA, thus identifying an interesting and effectively enumerable subclass of **BPP**.

The main technical contributions of this paper can thus be summarized as follows:

- We introduce the arithmetical theory $R\Sigma_1^b$ -NIA and prove that the random functions which are Σ_1^b -representable in it are precisely those which can be computed in polynomial time. The proof of the correspondence goes through the definition of a class of oracle recursive functions, called \mathcal{POR} , which is shown equivalent to the class of probabilistic polytime random functions **RFP**. The overall structure of the proof is described in Section 3, while further details can be found in Appendix A and B.
- We exploit this result to obtain a new syntactic characterization of **PP** and, more interestingly, two semantic characterizations of **BPP**, the first based on measure quantifiers and the second relying on standard, first-order quantification. This is in Section 4.
- Finally, we introduce a family of syntactic subclasses BPP_T ⊆ BPP of provable BPP-problems, relative to a theory T. After showing that the property of being non-erratic is Π₁⁰-complete, we establish that PIT is in BPP_(IΔ0+Exp). We conclude by showing how our approach relates to existing works capturing BPP languages in bounded arithmetic [42]. All this can be found in Sections 5 and 7.

Related Work While a recursion-theoretic characterization of the syntactic class **PP** can be found in [19], most existing characterizations of **BPP** are based on some external, semantic condition [21, 48]. In particular, Eickmeyer and Grohe [22] provide a semantic characterization of **BPP** in a logic with fixed-point operators and a special counting quantifier, associated with a probabilistic

semantics not too different from the quantitative interpretation we present in Section 3. On the other hand, [42] and [41] uses bounded arithmetic to provide characterizations of (both syntactic and semantic) randomized classes, such as **ZPP**, **RP** and **coRP**, and also provides a semantic characterization of **BPP**. An in-depth comparison is thus in order, and can be found in Section 7. Finally, [48] defines a higher-order language for polytime oracle recursive functions based on an adaptation of Bellantoni-Cook's safe recursion.

2 On the Enumeration of Complexity Classes

Before delving into the technical details, it is worth spending a few words on the problem of enumerating complexity classes, and on the reasons why it is more difficult for semantic classes than for syntactic ones.

First of all, it is worth observing that, although the distinction between syntactic and semantic classes appears in many popular textbooks (e.g. in [3, 51), in the literature these notions are not defined in a precise way. Roughly speaking, syntactic classes are those which can be defined via limitations on the amount of resources (i.e. units of either time or space) that the underlying algorithm is allowed to use. Typical examples are the class \mathbf{P} of problems solvable in polynomial time and the class PSPACE of problems solvable in polynomial space. Instead, the definition of a semantic class usually requires, beyond some resource condition, an additional condition, sometimes called a promise, typically expressing that the underlying algorithm returns the correct answer often enough. A typical example here is the class **BPP** considered in this paper (cf. Def. 8), corresponding to problems solvable in polynomial time by probabilistic algorithms with some fixed error bound strictly smaller than $\frac{1}{2}$. Sometimes the distinction between syntactic and semantic classes may be subtle. For instance, as we discuss in Section 4, the class **PP**, whose definition also comprises a promise, is generally considered a syntactic class.

Notice that the sense of the terms "syntactic" and "semantic", as referred to complexity classes, is not clearly related to the sense that these terms have in mathematical logic. To a certain extent, the analysis that we develop in this paper with the tools of bounded arithmetic may help clarify this point. On the one hand, well-known results in bounded arithmetic (cf. [7, 8]) provide a characterization of syntactic classes like **P** in terms of purely proof-theoretic conditions (i.e. provability in some weak fragment of Peano Arithmetic); on the other hand, we establish that, for a semantic class like **BPP**, an arithmetical characterization can be obtained by employing both proof-theoretic and model-theoretic conditions (i.e. truth in the standard model of Peano Arithmetic).

A natural question is whether such genuinely semantic (i.e. model-theoretic) conditions can somehow be eliminated in favor of purely syntactic (i.e. proof-theoretic) ones. In fact, this is a non-trivial problem, since, as proved in Section 5 (cf. Proposition 5.2), the promise underlying **BPP** is expressed by a Π_1^0 -complete arithmetical formula. One should of course recall, however, that the distinction between semantic and syntactic classes refers to how a class is defined and not to the underlying set of problems. It is thus of intensional nature. In other words, even if **P** and **BPP** are defined in a different way, it could well be that someday we discover that $\mathbf{P} = \mathbf{BPP}$: in this case \mathbf{BPP} would become a syntactic class, and, as we show in Section 5 (cf. Proposition 5.1), a purely

proof-theoretic characterization of BPP would be available.

The problem of showing that a complexity class can be enumerated (i.e. that one can devise a recursive enumerations of, say, Turing Machines solving all and only the problems in the class) provides a different, and useful, angle to look at the distinction between syntactic and semantic classes. Ordinary syntactic classes, such as P, PP, and PSPACE, are quite simple to enumerate. While verifying resource bounds for arbitrary programs is very difficult, it is surprisingly easy to define an enumeration of resource bounded algorithms containing at least one algorithm for any problem in one of the aforementioned classes. To clarify what we mean, suppose we want to characterize the class **P**. On the one hand, the class of all algorithms working in polynomial time is recursiontheoretically very hard, actually Σ_2^0 -complete [35]. On the other hand, the class of those programs consisting of a for loop executed a polynomial number of times, whose body itself consists of conditionals and simple enough instructions manipulating string variables, is both trivial to enumerate and big enough to characterize P, at least in an extensional sense: every problem in this class is decided by at least one program in the class and every algorithm in this class works in polytime. Many characterizations of **P** (and of other syntactic classes), as those based on safe-recursion [4, 46], light and soft linear logic [32, 31, 45], and bounded arithmetic [7], can be seen as instances of the just described pattern, where the precise class of polytime programs varies, while the underlying class of *problems* remains unchanged.

But what about semantic classes? Being resource bounded is not sufficient for an algorithm to solve a problem in some semantic class, since there can well be algorithms getting it wrong too often. For instance, it may well be that some probabilistic Turing Machine running in polynomial time does not solve any problem in **BPP**. For this reason, unfortunately, the enumeration strategy sketched above does not seem to be readily applicable to semantic classes. How can we isolate a simple enough subclass of algorithms – which are not only resource bounded, but also not too erratic – at the same time saturating the class?

We think that the results in this paper, concerning proof-theoretic and model-theoretic characterizations of probabilistic complexity classes, may provide new insights on the nature of this problem, without giving a definite answer. Indeed, observe that the existence of a purely proof-theoretic characterization of some complexity class $\mathcal C$ via some recursively enumerable theory T directly leads to providing an enumeration of $\mathcal C$ (by enumerating the theorems of T). In this way, the problem of enumerating a semantic class $\mathcal C$ is directly related to the existence of some strong enough theory T.

In the following sections we do *not* prove **BPP** to be effectively enumerable, which is still out of reach. On the one hand we show that proving the non-enumerability of **BPP** is as hard as proving that $P \neq BPP$. On the other hand, we show that there exist subclasses of **BPP** which are large enough to include interesting problems in **BPP** and still "syntactic enough" to be effectively enumerable via some arithmetical theory.

3 Bounded Arithmetic and Polytime Random Functions

In this section we discuss our first result, namely, the characterization of polytime random functions via bounded arithmetic.

3.1 From Arithmetic to Randomized Computation, Subrecursively

We introduce the two main ingredients on which our characterization of polytime random functions relies: a randomized bounded theory of arithmetic $R\Sigma_1^b$ -NIA, and a Cobham-style function algebra for polytime oracle recursive functions, called \mathcal{POR} .

Recursive Functions and Arithmetical Formulas The study of socalled bounded theories of arithmetic, i.e. subsystems of PA in which only bounded quantifications are admitted, initiated by Parikh and Buss, has led to characterize several complexity classes [52, 14, 7, 8, 25, 43]. At the core of these characterizations lies the well-known fact (dating back to Gödel's [34]) that recursive functions can be represented in PA by means of Σ_1^0 -formulas, i.e. formulas of the form $\exists x_1, \ldots, \exists x_n, A$, where A is a bounded formula. For example, the formula

$$A(x_1, x_2, y) := \exists x_3.x_1 \times x_2 = x_3 \wedge y = succ(x_3)$$

represents the function $f(x_1, x_2) = (x_1 \times x_2) + 1$. Indeed, in PA one can prove that $\forall x_1. \forall x_2. \exists ! y. A(x_1, x_2, y)$, namely that A expresses a functional relation, and check that for all $n_1, n_2, m \in \mathbb{N}$, $A(\overline{n_1}, \overline{n_2}, \overline{m})$ holds (in the standard model \mathscr{N}) precisely when $m = f(n_1, n_2)$. Buss' intuition was then that, by considering theories weaker than PA, it becomes possible to capture functions computable within given resource bounds [7, 8].

In order to extend this approach to classes of random computable functions, we rely on a simple correspondence between first-order predicates over natural numbers and oracles from the Cantor space $\{0,1\}^{\mathbb{N}}$, following [2]. Indeed, suppose the aforementioned recursive function f has now the ability to observe (part of) an infinite sequence of bits. For instance, f might observe the first bit and return $(x_1 \times x_2) + 1$ if this is 0, and return 0 otherwise. Our idea is that we can capture the call by f to the oracle by adding to the standard language of PA a new unary predicate $\mathtt{Flip}(x)$, to be interpreted as a stream of (random) bits. Our function f can then be represented by the following formula:

$$B(x_1,x_2,y) := \left(\operatorname{Flip}(\overline{0}) \wedge \exists x_3.x_1 \times x_2 = x_3 \wedge y = succ(x_3) \right) \vee \left(\neg \operatorname{Flip}(\overline{0}) \wedge y = \overline{0} \right)$$

As in the case above, it is possible to prove that $B(x_1, x_2, y)$ is functional, that is, that $\forall x_1.\forall x_2.\exists !y.B(x_1, x_2, y)$. However, since B now contains the unary predicate symbol $\mathsf{Flip}(x)$, the actual numerical function that B represents depends on the choice of an interpretation for $\mathsf{Flip}(x)$, i.e. on the choice of an oracle for f.

In the rest of this section we develop this idea in detail, establishing a correspondence between polytime random functions and a class of *oracle-recursive*

functions which are provably total in a suitable bounded theory relying on the predicate Flip.

The Language \mathcal{RL} . We let $\mathbb{B} := \{0,1\}$, $\mathbb{S} := \mathbb{B}^*$ indicate the set of finite words from \mathbb{B} , and $\mathbb{O} := \mathbb{B}^{\mathbb{S}}$. We introduce a language for first-order arithmetic incorporating the new predicate symbol $\mathtt{Flip}(x)$ and its interpretation in the standard model. Following [27], we consider a first-order signature for natural numbers in binary notation. Consistently, formulas will be interpreted over \mathbb{S} rather than \mathbb{N} . Working with strings is not essential and all results below could be spelled out in a language for natural numbers. Indeed, bounded theories may be formulated in both ways equivalently, e.g. Ferreira's Σ_1^b -NIA and Buss' S_2^1 [27].

Definition 1. The terms and formulas of \mathcal{RL} are defined by the grammars below:

The function symbol \frown stands for string concatenation, while $t \times u$ indicates the concatenation of t with itself a number of times corresponding to the length of u. The binary predicate \subseteq stands for the initial substring relation. As usual, we let $A \to B := \neg A \vee B$.

We adopt the following abbreviations: ts for $t \sim s$; 1^t for $1 \times t$; $t \leq s$ for $1^t \subseteq 1^s$, i.e. the length of t is smaller than that of s; $t|_r = s$ for $(1^r \subseteq 1^t \land s \subseteq t \land 1^r = 1^s) \lor (1^t \subseteq 1^r \land s = t)$, i.e s is the truncation of t at the length of r. For each string $\sigma \in \mathbb{S}$, we let $\overline{\sigma}$ be the term of \mathcal{RL} representing it (e.g. $\overline{\epsilon} = \epsilon$, $\overline{\sigma0} = \overline{\sigma0}$ and $\overline{\sigma1} = \overline{\sigma1}$).

As for standard bounded arithmetics [7, 24], a defining feature of our theory is the focus on so-called bounded quantifications. In \mathcal{RL} , bounded quantifications are of the forms $\forall x.1^x \subseteq 1^t \to F$ and $\exists x.1^x \subseteq 1^t \wedge F$, abbreviated as $\forall x \leq t.F$ and $\exists x \leq t.F$. Following [24], we adopt subword quantifications as those quantifications of the forms $\forall x.(\exists w \subseteq t.wx \subseteq t) \to F$ and $\exists x.\exists w \subseteq t.wx \subseteq t \wedge F$, abbreviated as $\forall x \subseteq^* t.F$ and $\exists x \subseteq^* t.F$. An \mathcal{RL} -formula F is said to be a Σ_1^b -formula if it is of the form $\exists x_1 \leq t_1.....\exists x_n \leq t_n.G$, where the only quantifications in G are subword ones. The distinction between bounded and subword quantifications is relevant for complexity reasons: if $\sigma \in \mathbb{S}$ is a string of length k, the witness of a subword existentially quantified formula $\exists y.y \subseteq^* \overline{\sigma} \wedge H$ is to be looked for among all possible sub-strings of σ , i.e. within a space of size $\mathcal{O}(k^2)$, while the witness of a bounded formula $\exists y \leq \overline{\sigma}.H$ is to be looked for among all possible strings of length k, i.e. within a space of size $\mathcal{O}(2^k)$.

The Borel Semantics of \mathcal{RL} . We introduce a quantitative semantics for formulas of \mathcal{RL} , inspired by [2]. While the function symbols of \mathcal{RL} , as well as the predicate symbols "=" and " \subseteq ", have a standard interpretation as relations over \mathbb{S} , the idea is that predicate symbol Flip may stand for an arbitrary subset of \mathbb{S} , that is, an arbitrarily chosen $\omega \in \mathbb{O}$. For this reason, we take as the interpretation of a \mathcal{RL} -formula F the set $[\![F]\!] \subseteq \mathbb{O}$ of all possible interpretations of Flip satisfying F. Importantly, such sets $[\![F]\!]$ can be proved

to be *measurable*, a fact that will turn out essential in Section 4. Indeed, the canonical first-order model of \mathcal{RL} over $\mathbb S$ can be extended to a probability space $(\mathbb O,\sigma(\mathscr C),\mu)$ defined in a standard way: here $\sigma(\mathscr C)\subseteq\wp(\mathbb O)$ is the Borel σ -algebra generated by *cylinders* $\mathsf C^b_\sigma=\{\omega\mid\omega(\sigma)=b\},$ with $b\in\{0,1\},$ and μ is the *unique* measure such that $\mu(\mathsf C^b_\sigma)=\frac12$ (see [5]). While the interpretation of terms is standard, the interpretation of formulas is defined below.

Definition 2 (Borel Semantics of \mathcal{RL}). Given a term t, a formula F and an environment $\xi : \mathcal{G} \to \mathbb{S}$, where \mathcal{G} is the set of term variables, the *interpretation* of F under ξ is the measurable set $[\![F]\!]_{\xi} \in \sigma(\mathscr{C})$ inductively defined as follows:

$$\begin{split} & \llbracket t = s \rrbracket_{\xi} := \begin{cases} \mathbb{O} & \text{ if } \llbracket t \rrbracket_{\xi} = \llbracket s \rrbracket_{\xi} \ \llbracket \mathrm{Flip}(t) \rrbracket_{\xi} := \{ \omega \mid \omega(\llbracket t \rrbracket_{\xi}) = 1 \} \ \llbracket \exists x.G \rrbracket_{\xi} := \bigcup_{i \in \mathbb{S}} \llbracket G \rrbracket_{\xi\{x \leftarrow i\}} \\ & \text{ otherwise } & \llbracket \neg G \rrbracket_{\xi} := \mathbb{O} - \llbracket G \rrbracket_{\xi} \end{cases} \\ & \mathbb{I}_{\xi} \subseteq \llbracket s \rrbracket_{\xi} & \mathbb{I}_{\xi} \cap \mathbb{I}_{\xi} := \llbracket G \rrbracket_{\xi} \cap \mathbb{I}_{\xi} \\ & \mathbb{I}_{\xi} \subseteq \mathbb{I}_{\xi} \cap \mathbb{I}_{\xi} := \mathbb{I}_{\xi} \cap \mathbb{I}_{\xi} \cap \mathbb{I}_{\xi} \end{cases} \\ & \mathbb{I}_{\xi} \subseteq \mathbb{I}_{\xi} \cap \mathbb$$

This semantics is well-defined as the sets $[\![\mathtt{Flip}(t)]\!]_{\xi}$, $[\![t=s]\!]_{\xi}$ and $[\![t\subseteq s]\!]_{\xi}$ are measurable and measurability is preserved by all the logical operators.

Observe that an interpretation of the language \mathcal{RL} , in the usual first-order sense, requires some ξ as above as well as an interpretation ω for $\mathsf{Flip}(x)$. One can easily check by induction that, for any formula F and interpretation ξ , $\omega \in \llbracket F \rrbracket_{\xi}$ precisely when F is satisfied in the first-order environment formed by ξ and ω .

The Bounded Theory $R\Sigma_1^b$ -NIA. We now introduce a bounded theory in the language \mathcal{RL} , called $R\Sigma_1^b$ -NIA, which can be seen as a probabilistic counterpart to Ferreira's Σ_1^b -NIA [26]. The theory $R\Sigma_1^b$ -NIA is defined by axioms belonging to two classes:

• Basic axioms (where $b \in \{0, 1\}$):

$$x\epsilon = x \qquad x \times \epsilon = \epsilon \qquad x \subseteq \epsilon \leftrightarrow x = \epsilon \qquad xb = yb \to x = y$$

$$x(yb) = (xy)b \ x \times yb = (x \times y)x \ x \subseteq yb \leftrightarrow x \subseteq y \lor x = yb \ x0 \neq y1 \qquad xb \neq \epsilon$$

• Axiom schema for induction on notation: $B(\epsilon) \wedge \forall x. (B(x) \to B(x0) \wedge B(x1)) \to \forall x. B(x)$, where B is a Σ_1^b -formula in \mathcal{RL} .

The axiom schema for induction on notation adapts the usual induction schema of PA to the binary representation. As standard in bound arithmetic, restriction to Σ_1^b -formulas, is essential to characterize algorithms with *bounded* resources. Indeed, more general instances of this schema would lead to represent functions which are not polytime computable.

An Algebra of Polytime Oracle Recursive Functions We now introduce a Cobham-style function algebra, called \mathcal{POR} , for polytime oracle recursive functions, and show that it is captured by a class of bounded formulas provably representable in the theory $R\Sigma_1^b$ -NIA. This algebra is inspired by Ferreira's PTCA [24, 26]. Yet, a fundamental difference is that the functions we define are of the form $f: \mathbb{S}^k \times \mathbb{O} \to \mathbb{S}$, i.e. they carry an additional argument $\omega: \mathbb{S} \to \mathbb{B}$, to be interpreted as the underlying stream of random bits. Furthermore, our class includes the basic query function, which can be used to observe any bit from ω .

The class \mathcal{POR} is the smallest class of functions from $\mathbb{S}^k \times \mathbb{O}$ to \mathbb{S} , containing the empty function $E(x,\omega) = \epsilon$, the projection functions $P_i^n(x_1,\ldots,x_n,\omega) = x_i$, the word-successor function $S_b(x,\omega) = xb$, the conditional function $C(\epsilon,y,z_0,z_1,\omega) = y$ and $C(xb,y,z_0,z_1,\omega) = z_b$, where $b \in \mathbb{B}$ (corresponding to $b \in \{0,1\}$), the query function $Q(x,\omega) = \omega(x)$, and closed under the following schemata:

- Composition, where f is defined from g, h_1, \ldots, h_k as $f(\vec{x}, \omega) = g(h_1(\vec{x}, \omega), \ldots, h_k(\vec{x}, \omega), \omega)$.
- Bounded recursion on notation, where f is defined from g, h_0, h_1 as

$$f(\vec{x}, \epsilon, \omega) = g(\vec{x}, \omega);$$

$$f(\vec{x}, y0, \omega) = h_0(\vec{x}, y, f(\vec{x}, y, \omega), \omega)|_{t(\vec{x}, y)};$$

$$f(\vec{x}, y1, \omega) = h_1(\vec{x}, y, f(\vec{x}, y, \omega), \omega)|_{t(\vec{x}, y)};$$

with t obtained from $\epsilon, 0, 1, \frown, \times$ by explicit definition, i.e. by applying \frown and \times on constants ϵ , 0, 1, and variables \vec{x} and y.

We now show that functions of \mathcal{POR} are precisely those which are Σ_1^b -representable in $R\Sigma_1^b$ -NIA. To do so, we slightly modify Buss' representability conditions by adding a constraint relating the quantitative semantics of formulas in \mathcal{RL} and the additional functional parameter ω of oracle recursive functions.

Definition 3. A function $f: \mathbb{S}^k \times \mathbb{O} \to \mathbb{S}$ is Σ_1^b -representable in $R\Sigma_1^b$ -NIA if there exists a Σ_1^b -formula $G(\vec{x}, y)$ of \mathcal{RL} such that:

- 1. $R\Sigma_1^b$ -NIA $\vdash \forall \vec{x}. \exists ! y. G(\vec{x}, y),$
- 2. for all $\sigma_1, \ldots, \sigma_k, \tau \in \mathbb{S}$ and $\omega \in \mathbb{O}$, $f(\sigma_1, \ldots, \sigma_k, \omega) = \tau$ iff $\omega \in \llbracket G(\overline{\sigma_1}, \ldots, \overline{\sigma_k}, \overline{\tau}) \rrbracket$.

Condition 1. of Definition 3 does *not* say that the unique value y is obtained as a function of \vec{x} only. Indeed, the truth-value of a formula depends both on the value of its first-order variables and on the value assigned to the random predicate Flip. Hence this condition says that y is uniquely determined as a function both of its first-order inputs and of an oracle from \mathbb{O} , precisely as functions of \mathcal{POR} .

Theorem 3.1. For any $f: \mathbb{S}^k \times \mathbb{O} \to \mathbb{S}$, f is Σ_1^b -representable in $R\Sigma_1^b$ -NIA iff $f \in \mathcal{POR}$.

Proof sketch. (\Leftarrow) The desired Σ_1^b -formula is constructed by induction on the structure of oracle recursive functions. Observe that the formula $\forall \vec{x}.\exists!y.G(\vec{x},y)$ occurring in Condition 1. of Definition 3 is $not \Sigma_1^b$, since it is universally quantified while the existential quantifier is not bounded. Hence, in order to apply the inductive steps (corresponding to functions defined by composition and bounded recursion on notation), we need to adapt Parikh's theorem [52] (which holds for S_2^1 and Σ_1^b -NIA) to $R\Sigma_1^b$ -NIA, to state that if $R\Sigma_1^b$ -NIA $\vdash \forall \vec{x}.\exists y.G(\vec{x},y)$, where $G(\vec{x},y)$ is a Σ_1^b -formula, then we can find a term t such that $R\Sigma_1^b$ -NIA $\vdash \forall \vec{x}.\exists y \leq t.G(\vec{x},y)$. (\Rightarrow) The proof consists in adapting Cook and Urquhart's argument for system IPV $^\omega$ [13], and this goes through a realizability interpretation of the intuitionistic version of $R\Sigma_1^b$ -NIA, called $IR\Sigma_1^b$ -NIA. Further details can be found in the Appendix A.

3.2 Characterizing Polytime Random Functions

Theorem 3.1 shows that it is possible to characterize \mathcal{POR} by means of a system of bounded arithmetic. Yet, this is not enough to deal with classes, like **BPP** or **RP**, which are defined in terms of functions computed by PTMs. Observe that there is a crucial difference in the way in which probabilistic machines and oracle recursive functions access randomness, so our next goal is to fill the gap, by relating these classes of functions.

Let $\mathbb{D}(\mathbb{S})$ indicate the set of distributions over \mathbb{S} , that is, those functions $\lambda: \mathbb{S} \to [0,1]$ such that $\sum_{\sigma \in \mathbb{S}} \lambda(\sigma) = 1$. By a random function we mean a function of the form $f: \mathbb{S}^k \to \mathbb{D}(\mathbb{S})$. Observe that any (polytime) PTM \mathscr{M} computes a random function $f_{\mathscr{M}}$, where, for every $\sigma_1, \ldots, \sigma_k, \tau \in \mathbb{S}$, $f_{\mathscr{M}}(\sigma_1, \ldots, \sigma_k)(\tau)$ coincides with the probability that $\mathscr{M}(\sigma_1\sharp \ldots \sharp \sigma_k) \Downarrow \tau$. However, a random function needs not be computed by a PTM in general. We define the following class of polytime random functions:

Definition 4 (Class **RFP**). The class **RFP** is made of all random functions $f: \mathbb{S}^k \to \mathbb{D}(\mathbb{S})$ such that $f = f_{\mathscr{M}}$, for some PTM \mathscr{M} running in polynomial time.

Functions of **RFP** are closed under monadic composition \diamond , where $(g \diamond f)(\sigma)(\tau) = \sum_{\rho \in \mathbb{S}} g(\rho)(\tau) \cdot f(\sigma)(\rho)$ (one can check $f_{\mathcal{M}'} \diamond f_{\mathcal{M}} = f_{\mathcal{M}' \circ \mathcal{M}}$, where \diamond indicates PTM composition).

Since functions of **RFP** have a different shape from those of \mathcal{POR} , we must adapt the notion of Σ_1^b -representability for them, relying on the fact that any closed \mathcal{RL} -formula F generates a measurable set $\llbracket F \rrbracket \subseteq \mathbb{B}^{\mathbb{N}}$.

Definition 5. A function $f: \mathbb{S}^k \to \mathbb{D}(\mathbb{S})$ is Σ_1^b -representable in $R\Sigma_1^b$ -NIA if there exists a Σ_1^b -formula $G(\vec{x}, y)$ of \mathcal{RL} such that:

- 1. $R\Sigma_1^b$ -NIA $\vdash \forall \vec{x}. \exists ! y. G(\vec{x}, y),$
- 2. for all $\sigma_1, \ldots, \sigma_k, \tau \in \mathbb{S}$, $f(\sigma_1, \ldots, \sigma_k, \tau) = \mu(\llbracket G(\overline{\sigma_1}, \ldots, \overline{\sigma_k}, \overline{\tau}) \rrbracket)$.

Notice that any Σ_1^b -formula $G(\vec{x},y)$ satisfying Condition 1. from Definition 5 actually defines a random function $\langle G \rangle : \mathbb{S} \to \mathbb{D}(\mathbb{S})$ given by $\langle G \rangle (\vec{\sigma})(\tau) = \mu(\llbracket G(\overline{\sigma},\overline{\tau}) \rrbracket)$, where $\langle G \rangle$ is Σ_1^b -represented by G. Moreover, if G represents some $f \in \mathbf{RFP}$, then $f = \langle G \rangle$. In analogy with Theorem 3.1, we can now prove the following result:

Theorem 3.2. For any $f: \mathbb{S}^k \to \mathbb{D}(\mathbb{S})$, f is Σ_1^b -representable in $R\Sigma_1^b$ -NIA iff $f \in \mathbf{RFP}$.

Thanks to Theorem 3.1, the proof of the result above simply consists in showing that POR and RFP can be related as stated below.

Lemma 3.3. For all functions $f: \mathbb{S}^k \times \mathbb{O} \to \mathbb{S}$ in \mathcal{POR} there exists $g: \mathbb{S}^k \to \mathbb{D}(\mathbb{S})$ in **RFP** such that for all $\sigma_1, \ldots, \sigma_k, \tau \in \mathbb{S}$, $\mu(\{\omega \mid f(\vec{\sigma}, \omega) = \tau\}) = g(\sigma_1, \ldots, \sigma_k, \tau)$, and conversely.

Proof sketch. The first step of our proof consists in replacing the class **RFP** by an intermediate class **SFP** corresponding to functions computed by polytime stream Turing machines (STM, for short). These are defined as deterministic TM with one extra read-only tape: at the beginning of the computation the

extra tape is sampled from $\mathbb{B}^{\mathbb{N}}$, and at each computation step the machine reads one new bit from this tape. Then we show that for any function $f: \mathbb{S}^k \to \mathbb{D}(\mathbb{S})$ computed by some polytime PTM there is a function $g: \mathbb{S}^k \times \mathbb{B}^{\mathbb{N}} \to \mathbb{S}$ computed by a polytime STM such that for all $\sigma_1, \ldots, \sigma_k, \tau \in \mathbb{S}$, and $\eta \in \mathbb{B}^{\mathbb{N}}$, $f(\sigma_1, \ldots, \sigma_k, \tau) = \mu(\{\eta \mid g(\sigma_1, \ldots, \sigma_k, \eta) = \tau\})$, and conversely. To conclude, we prove the correspondence between the classes \mathcal{POR} and **SFP**:

- (SFP $\Rightarrow \mathcal{POR}$) The encoding relies on the remark that, given an input $x \in \mathbb{S}$ and an extra-tape $\eta \in \mathbb{B}^{\mathbb{N}}$, an STM \mathscr{S} running in polynomial time can only access a *finite* portion of η , bounded by some polynomial p(|x|). This way the behavior of \mathscr{S} is encoded by a \mathcal{POR} -function h(x,y), where the second string y corresponds to $\eta_{p(|x|)}$, and we can define $f^{\sharp}(x,\omega) = h(x,e(x,\omega))$, where $e : \mathbb{S} \times \mathbb{O} \to \mathbb{S}$ is a function of \mathcal{POR} which mimics the prefix extractor $\eta \mapsto \eta_{p(|x|)}$, in the sense that its outputs have the same distributions of all possible η 's prefixes (yet over \mathbb{O} rather than $\mathbb{B}^{\mathbb{N}}$).
- $(\mathcal{POR} \Rightarrow \mathbf{SFP})$ Here we must consider that these two models not only invoke oracles of different shape, but also that functions of \mathcal{POR} can manipulate such oracles in a much more liberal way than STMs. Notably, the STM accesses oracle bits in a linear way: each bit is used exactly once and cannot be re-invoked. Moreover, at each step of computation the STM queries a new oracle bit, while functions of \mathcal{POR} can access the oracle, so to say, on demand. The argument rests then on a chain of simulations, making use of a class of imperative languages inspired by Winskell's IMP [57], each one taking care of one specific oracle access policy: first nonlinear and on-demand (as for \mathcal{POR}), then linear but still on-demand, and finally linear and not on-demand (as for STMs).

4 Semantic Characterizations of BPP

We now turn our attention to randomized complexity classes. This requires us to consider how random functions (and thus PTMs) may correspond to languages, i.e. subsets of S. The language computed by a random function can naturally be defined via a majority rule:

Definition 6. Let $f: \mathbb{S} \to \mathbb{D}(\mathbb{S})$ be a random function. The language Lang $(f) \subseteq \mathbb{S}$ is defined by $\sigma \in \text{Lang}(f)$ iff $f(\sigma)(\epsilon) > \frac{1}{2}$.

It is instructive to first take a look at the case of the class ${\bf PP},$ recalled below:

Definition 7 (**PP**). Given a language $L \subseteq \mathbb{S}$, $L \in \mathbf{PP}$ iff there is a polynomial time PTM \mathscr{M} such that for any $\sigma \in \mathbb{S}$, $\Pr[\mathscr{M}(\sigma) = \chi_L(\sigma)] > \frac{1}{2}$, where, $\chi_L : \mathbb{S} \to \{0,1\}$ is the characteristic function of L.

At first glance, \mathbf{PP} might be considered a semantic class, since its definition comprises both a resource condition and a promise. However, \mathbf{PP} is generally considered a syntactic class, due to the fact that, when trying to capture the machines solving languages in \mathbf{PP} , the promise condition can actually be eliminated. Indeed, any PTM \mathcal{M} running in polynomial time recognizes some

language in **PP**, namely the language L = Lang(f), where f is the polytime random function computed by \mathcal{M} . Furthermore, the class **PP** can be enumerated (see e.g. [19]).

Using Theorem 3.2, the remarks above readily lead to a proof-theoretic characterization of **PP** via $R\Sigma_1^b$ -NIA.

Proposition 4.1 (Syntactic Characterization of **PP**). For any language $L \subseteq \mathbb{S}$, $L \in \mathbf{PP}$ iff there is a Σ_1^b -formula G(x,y) such that:

- 1. $R\Sigma_1^b$ -NIA $\vdash \forall x.\exists! y.G(x,y)$,
- 2. $L = \text{Lang}(\langle G \rangle)$.

The characterization above provides an enumeration of \mathbf{PP} (by enumerating the pairs made of a formula G and a proof in $R\Sigma_1^b$ -NIA of Condition 1). However, while a majority rule is enough to capture the problems in \mathbf{PP} , the definition of a semantic class like \mathbf{BPP} requires a different condition.

Definition 8 (BPP). Given a language $L \subseteq \mathbb{S}$, $L \in \mathbf{BPP}$ iff there is a polynomial time PTM \mathscr{M} such that for any $\sigma \in \mathbb{S}$, $\Pr[\mathscr{M}(\sigma) = \chi_L(\sigma)] \geq \frac{2}{3}$.

The class **BPP** can be captured by "non-erratic" probabilistic algorithms, i.e. such that, for a fixed input, one possible output is definitely more likely than the others.

Definition 9. A random function $f: \mathbb{S} \to \mathbb{D}(\mathbb{S})$ is non-erratic if for all $\sigma \in \mathbb{S}$, $f(\sigma)(\tau) \geq \frac{2}{3}$ holds for some value $\tau \in \mathbb{S}$.

Lemma 4.2. For any language $L \subseteq \mathbb{S}$, $L \in \mathbf{BPP}$ iff $L = \mathrm{Lang}(f)$, for some non-erratic random function $f \in \mathbf{RFP}$.

Proof. For any non-erratic **RFP**-function f, let \mathscr{M} be the PTM computing $k \diamond f$, where $k(\epsilon) = 1$ and $k(\sigma \neq \epsilon) = 0$; then \mathscr{M} computes $\chi_{\operatorname{Lang}(f)}$ with error $\leq \frac{1}{3}$. Conversely, if $L \in \mathbf{BPP}$, let \mathscr{M} be a PTM accepting L with error $\leq \frac{1}{3}$; then $L = \operatorname{Lang}(h \diamond f_{\mathscr{M}})$, where $h(1) = \epsilon$ and $h(\sigma \neq 1) = 0$.

Lemma 4.2 suggests that, in order to characterize **BPP** in the spirit of Proposition 4.1, a new condition has to be added, corresponding to the fact that G represents a non-erratic random function. In the rest of this section we discuss two approaches to measure error bounds for probabilistic algorithms, leading to two different characterizations of **BPP**: first via measure quantifiers [2], then by purely arithmetical means. While both such methods ultimately consist in showing that the truth of a formula in the standard model of $R\Sigma_1^b$ -NIA, they also suggest a more proof-theoretic approach, that we explore in Section 5.

BPP via Measure Quantifiers. As we have seen, any \mathcal{RL} -formula F is associated with a measurable set $\llbracket F \rrbracket \subseteq \mathbb{O}$. So, a natural idea, already explored in [2], consists in enriching \mathcal{RL} with measure-quantifiers [49, 47], that is, second-order quantifiers of the form $\mathbf{C}^q F$, where $q \in [0,1] \cap \mathbb{Q}$, intuitively expressing that the measure of $\llbracket F \rrbracket$ is greater than (or equal to) q. Then, let $\mathcal{RL}^{\mathsf{MQ}}$ be the extension of \mathcal{RL} with measure-quantified formulas $\mathbf{C}^{t/s} F$, where t,s are terms. The Borel semantics of \mathcal{RL} naturally extends to $\mathcal{RL}^{\mathsf{MQ}}$ letting $\llbracket \mathbf{C}^{t/s} F \rrbracket_{\xi} = \mathbb{O}$

when $|[\![s]\!]_{\xi}| > 0$ and $\mu([\![F]\!]_{\xi}) \ge \frac{|[\![t]\!]_{\xi}|}{|[\![s]\!]_{\xi}|}$ both hold, and $[\![\mathbf{C}^{t/s}F]\!]_{\xi} = \emptyset$ otherwise. To improve readability, for all $n, m \in \mathbb{N}$, we abbreviate $\mathbf{C}^{1^n/1^m}F$ as $\mathbf{C}^{n/m}F$.

Measure quantifiers can now be used to express that the formula representing a random function is non-erratic, as shown below.

Theorem 4.3 (First Semantic Characterization of **BPP**). For any language $L \subseteq \mathbb{S}$, $L \in \mathbf{BPP}$ iff there is a Σ_1^b -formula G(x,y) such that:

- 1. $R\Sigma_1^b$ -NIA $\vdash \forall x \exists ! y. G(x, y)$,
- $2. \models \forall x. \exists y. \mathbf{C}^{2/3} G(x, y),$
- 3. $L = \text{Lang}(\langle G \rangle)$.

Proof. Let $L \in \mathbf{BPP}$ and $g : \mathbb{S} \to \mathbb{D}(\mathbb{S})$ be a function of \mathbf{RFP} computing L with uniform error-bound (which, thanks to Lemma 4.2, we can suppose to be non-erratic). By Theorem 3.2, there is a Σ_1^b -formula G(x,y) such that $g = \langle G \rangle$. So, for all $\sigma \in \mathbb{S}$, $\mu(\llbracket G(\overline{\sigma}, \overline{\tau}) \rrbracket) = g(\sigma)(\tau) \geq \frac{2}{3}$ holds for some $\tau \in \mathbb{S}$, which shows that Condition 2. holds. Conversely, if Conditions 1.-3. hold, then $\langle G \rangle$ computes L with the desired error bound, so $L \in \mathbf{BPP}$.

Arithmetizing Measure Quantifiers. Theorem 4.3 relies on the tight correspondence between arithmetic and probabilistic computation; yet, Condition 2. involves formulas which are not in the language of first-order arithmetic. Lemma 4.4 below shows that measure quantification over bounded formulas of \mathcal{RL} can be expressed arithmetically.

Lemma 4.4 (De-Randomization of Bounded Formulas). For any Σ_1^b -formula $F(\vec{x})$ of \mathcal{RL} , there exists a Flip-free Π_1^0 -formula TwoThirds $[F](\vec{x})$ such that for any $\vec{\sigma} \in \mathbb{S}$, \vDash TwoThirds $[F](\overline{\vec{\sigma}})$ holds iff $\mu(\llbracket F(\overline{\vec{\sigma}}) \rrbracket) \geq \frac{2}{3}$.

Proof. First, observe that for any bounded \mathcal{RL} -formula $F(\vec{x})$, strings $\vec{\sigma}$ and $\omega \in \mathbb{O}$, to check whether $\omega \in \llbracket F(\vec{\sigma}) \rrbracket$ only a finite portions of bits of ω has to be observed. More precisely, we can construct a \mathcal{RL} -term $t_F(\vec{x})$ such that for any $\vec{\sigma} \in \mathbb{S}$ and $\omega, \omega' \in \mathbb{O}$, if ω and ω' agree on all strings shorter than $t_F(\vec{\sigma})$, then $\omega \in \llbracket F(\sigma) \rrbracket$ iff $\omega' \in \llbracket F(\sigma) \rrbracket$. Now, all finitely many relevant bits $\omega(\tau)$, for $|\tau| \leq t_F(\vec{\sigma})$ can be encoded as a unique string w of length $\leq 2^{|t_F(\vec{\sigma})|}$ where the bit w_i corresponds to the value $\omega(\tau)$, where τ is obtained by stripping the right-most bit from the binary representation of i. We obtain in this way a Flip-free formula $F^*(\vec{x}, y)$ such that measuring $\llbracket F(\sigma) \rrbracket$ corresponds to counting the strings y of length $\leq 2^{|t_F(\vec{\sigma})|}$ making $F^*(\vec{x}, y)$ true, i.e. to showing

$$\left| \left\{ \tau \leq 2^{|t_F(\vec{\sigma})|} \mid F^*(\vec{\sigma}, \tau) \right\} \right| \geq \frac{2}{3} \cdot N \tag{*}$$

where $2^{\epsilon} = 1$ and $2^{\sigma b} = 2^{\sigma}2^{\sigma}$ is an exponential function on strings and $N = 2^{\left(2^{|t_F(\sigma)|}\right)}$ is the total amount of the strings to be counted. (*) can be encoded in a standard way yielding a bounded formula $F^{\sharp}(\vec{x})$ in the language of arithmetic extended with the function symbol 2^x . Finally, the function symbol 2^x can be eliminated using a Δ_0^0 -formula $\exp(x,y)$ defining the exponential function (see [29]), yielding a Flip-free Π_1^0 -formula of \mathcal{RL} of the form $\forall z_1, \ldots, \forall z_k, \exp(t_1, z_1) \land \cdots \land \exp(t_k, z_k) \to F^{\sharp}(\vec{x}, z_1, \ldots, z_k)$.

Remark 4.1. It is important to observe at this point that the elimination of Flip via counting takes us beyond the usual machinery of bounded arithmetic, since we employ some operation which is not polytime. This is indeed not surprising, since the counting problems associated with polytime problems (generating the class $\sharp P$) are not even known to belong to the polynomial hierarchy PH (while, by Toda's theorem, we know that PH $\subseteq P^{\sharp P}$).

Theorem 4.3 and Lemma 4.4 together yield a purely arithmetical characterization of **BPP**. Let NotErratic[G] indicate the arithmetical formula $\forall x. \exists y \leq 0.$ TwoThirds[G](x, y).

Theorem 4.5 (Second Semantic Characterization of **BPP**). For any language $L \subseteq \mathbb{S}$, $L \in \mathbf{BPP}$ when there is a Σ_1^b -formula G(x,y) such that:

```
1. R\Sigma_1^b-NIA \vdash \forall x.\exists ! y.G(x, y),
2. \models \mathsf{NotErratic}[G],
3. L = \mathsf{Lang}(\langle G \rangle).
```

5 Provably BPP Problems

The characterization provided by Theorem 4.5 is still semantic in nature, as it provides no way to effectively enumerate **BPP**: the crucial Condition 2 is not checked within a formal system, but over the standard model of \mathcal{RL} . Yet, since the condition is now expressed in purely arithmetical terms, it makes sense to consider *syntactic* variants of Condition 2, where the model-theoretic check is replaced by provability in some sufficiently expressive theory.

We will work in extensions of $R\Sigma_1^b$ -NIA + Exp, where Exp = $\forall x.\exists y.\exp(x,y)$ is the formula expressing the totality of the exponential function (which is used in the de-randomization of Lemma 4.4). This naturally leads to the following definition:

Definition 10 (Class **BPP**_T). Let $T \supseteq R\Sigma_1^b$ -NIA + Exp be a theory in the language \mathcal{RL} . The class **BPP** relative to T, denoted **BPP**_T, contains all languages $L \subseteq \mathbb{S}$ such that for some Σ_1^b -formula G(x, y) the following hold:

```
1. R\Sigma_1^b-NIA \vdash \forall x.\exists ! y.G(x, y),
2. \top \vdash \mathsf{NotErratic}[G],
3. L = \mathsf{Lang}(\langle G \rangle).
```

Whenever T is sound (i.e. $T \vdash F$ implies that F is true in the standard model), it is clear that $\mathbf{BPP}_T \subseteq \mathbf{BPP}$. However, a crucial difference between the syntactic class \mathbf{BPP}_T and the semantic class \mathbf{BPP} is that, when T is recursively enumerable, \mathbf{BPP}_T can be enumerated (by enumerating the proofs of Condition 1. and 2. in T). Hence, the enumerability problem for \mathbf{BPP} translates into the question whether one can find a sound r.e. theory T such that $\mathbf{BPP}_T = \mathbf{BPP}$. Let us first observe that the relevance of this problem is tightly related to the question $\mathbf{BPP} = \mathbf{P}$:

Proposition 5.1. If $\mathbf{BPP} = \mathbf{P}$, then there exists a r.e. theory T such that $\mathbf{BPP} = \mathbf{BPP}_T$.

Proof. If $\mathbf{BPP} = \mathbf{P}$, and $L \in \mathbf{BPP}$, then there is a polytime deterministic TM μ accepting it. μ yields then a PTM μ^* in a trivial way. Since the corresponding formula G of \mathcal{RL} does not contain Flip, NotErratic[G] can be proved in e.g. $R\Sigma_0^b$ -NIA + Exp.

The counter-positive of the result above is even more interesting, as it says that establishing that no r.e. theory T is such that $\mathbf{BPP}_T = \mathbf{BPP}$ is at least as hard as establishing that $\mathbf{BPP} \neq \mathbf{P}$. Yet, without knowing whether $\mathbf{BPP} = \mathbf{P}$, how hard may it be to find a theory T such that $\mathbf{BPP} = \mathbf{BPP}_T$?

Observe that, when G is Σ_1^b , $\operatorname{NotErratic}[G]$ is expressed by a Π_1^0 -formula: as $\operatorname{TwoThirds}[G](\vec{x})$ is of the form $\forall \vec{z}. \land_i \exp(t_i, z_i) \to F^\sharp(\vec{x}, \vec{z})$, the condition is expressed by the Π_1^0 -formula $\forall x. \forall \vec{z}. \land_i \exp(t_i, z_i) \to \exists y \leq 0. F^\sharp(\vec{x}, \vec{z})$. Hence, if we us fix some recursive enumeration $(\mathscr{M}_n)_{n \in \mathbb{N}}$ of polytime PTM as well as a recursive coding $\sharp \mathscr{M}$ of such machines as natural numbers, the fact that \mathscr{M} is non-erratic is expressed by some Π_1^0 -formula $\varphi_{\operatorname{NotErratic}}(\sharp \mathscr{M})$. The Π_1^0 -set $\operatorname{NotErratic} = \{e \mid \varphi_{\operatorname{NotErratic}}(e)\}$ indicates then the sets of codes corresponding to non-erratic machines.

The possibility of finding a theory strong enough to prove *all* positive instances of Condition 2 is then ruled out by the following result.

Proposition 5.2. NotErratic is Π_1^0 -complete.

Proof. We reduce to NotErratic the Π_1^0 -complete problem HALT_{n^2} consisting of codes of TM halting in time at most n^2 (see [30]). With any TM μ associate a polytime PTM μ^* that, on input x, yields TRUE with prob. $\frac{1}{2}$, and otherwise simulates $\mu(x)$ on $|x|^2$ steps, yielding TRUE if the computation of $\mu(x)$ terminated, and FALSE otherwise. Then it is easily seen that $\mu \in \mathsf{HALT}_{n^2}$ iff $\mu^* \in \mathsf{NotErratic}$.

Corollary 5.1. Being (the code of) a PTM solving some BPP-problem is Σ_2^0 -complete.

Proof. As we say, for a PTM, solving some **BPP**-problem is equivalent to being polytime and non-erratic. Being the code of a polytime (P)TM is a Σ_2^0 -complete property [35]. By Proposition 5.2, checking non-erraticity does not increase the logical complexity.

Proposition 5.2 implies that for any consistent theory T one can always find some non-erratic polytime PTM whose non-erraticity is *not provable* in T. Indeed, since NotErratic is Π_1^0 -complete, we can reduce to it the Π_1^0 -set of codes of *consistent* r.e. theories. Hence, if T is some consistent theory such that for any code $e \in \text{NotErratic}$, T proves $\varphi_{\text{NotErratic}}(e)$, then T can prove all Π_1^0 -statement expressing the consistency of some consistent r.e. theory, and thus, in particular, the one expressing its own consistency, contradicting (Rosser's variant of) Gödel's second incompleteness theorem.

Observe that this result suggests that the enumerability problem *might* be very difficult, but it *does not* provide a negative answer to it. Indeed, recall that what we are interested in is not an enumeration of *all* non-erratic polytime PTM, but an enumeration containing *at least one* machine for each problem in **BPP**. In other words, the question remains open whether, for any non-erratic polytime PTM, it is possible to find a machine solving the same problem but whose non-erratic behavior can be proved in some fixed theory T. While we do

not know the answer to this question, we can still show that a relatively weak arithmetical theory is capable of proving the non-erraticity of a machine solving one of the (very few) problems in ${\bf BPP}$ which are currently not known to be in ${\bf PP}$

6 Polynomial Zero Testing is Provably BPP

In this section we establish that PIT is in $\mathbf{BPP}_{(\mathsf{I}\Delta_0 + \mathrm{Exp})}$. We recall that $\mathsf{I}\Delta_0 + \mathrm{Exp}$ is the fragment of Peano Arithmetics with induction restricted to bounded formulas, together with the totality of the exponential function.

Remark 6.1. While $|\Delta_0 + \text{Exp}|$ is a theory in the usual language of PA, here we work in a language for binary strings. Indeed, what we here call $|\Delta_0 + \text{Exp}|$ is actually the corresponding theory Δ_0 -NIA+Exp, formulated for the language \mathcal{RL} without Flip, and defined as Σ_1^b -NIA+Exp with induction extended to all bounded formulas, plus the axiom Exp. Based on [27] Δ_0 -NIA corresponds to Buss' theory S_2 , which, in turn, is known to correspond to $|\Delta_0 + \Omega_1|$, indeed a sub-theory of $|\Delta_0 + \text{Exp}|$.

The PIT problem asks to decide the identity of the polynomial computed by two arithmetical circuits. These are basically DAGs whose nodes can be labeled so as to denote an input, an output, the constants 0,1 or an arithmetic operation. These structures can easily be encoded, e.g. using lists, as terms of \mathcal{RL} .

Definition 11 (cf. [3]). The problem PIT asks to decide whether two arithmetical circuits p, q encoded as lists of nodes describe the same polynomial, i.e. $\mathbb{Z} \models p = q$.

Usually, PIT is reduced to another problem: the so-called Polynomial Zero Testing (PZT) problem, which asks to decide whether a polynomial computing a circuit over \mathbb{Z} is zero, i.e. to check whether $\mathbb{Z} \models p = 0$. Indeed, $\mathbb{Z} \models p = q$ if and only if $\mathbb{Z} \models p - q = 0$. Our proof of the fact that the language PZT is in $\mathbf{BPP}_{(\mathsf{I}\Delta_0 + \mathsf{Exp})}$ is structured as follows:

- We identify a Σ_1^b -formula G(x,y) of \mathcal{RL} characterizing the polytime algorithm PZT from [3], and we turn it into a Flip-free formula $G^*(x,y,z)$ as in Lemma 4.4, where the variable z stands for the source of randomness;
- We identify a Flip-free Δ_0^0 -formula H(x,y) which represents the naïve deterministic algorithm for PZT.
- We show that $\mathsf{I}\Delta_0 + \mathsf{Exp}$ proves a statement showing that the formulas G^* and H are equivalent in at least $\frac{2}{3}$ of all (finitely many) relevant values of z. In other words, we establish $\mathsf{I}\Delta_0 + \mathsf{Exp} \vdash \forall x. \forall y. \mathsf{TwoThirds}[G(x,y) \leftrightarrow H(x,y)]$.

From the last step, since the totality of H is provable in $\mathsf{I}\Delta_0 + \mathsf{Exp}$, we can deduce $\mathsf{I}\Delta_0 + \mathsf{Exp} \vdash \forall x. \exists y. \mathsf{TwoThirds}[G](x,y)$, as required in Definition 10.

Each of the aforementioned steps will be described in one of the forthcoming paragraphs, although the details are discussed in the Appendix C.

The Randomized Algorithm. Our algorithm for PZT takes an input x, which encodes a circuit p of size m on the variables v_1, \ldots, v_n , it draws r_1, \ldots, r_n uniformly at random from $\{0, \ldots, 2^{m+3}-1\}$ and k from $\{1, \ldots, 2^{2m}\}$, then it computes the value of $p(r_1, \ldots, r_n)$ mod k, so to ensure that during the evaluation no overflow can take place. This is done linearly many times in |x| (we call this value s), as to ensure that, if the polynomial is not identically zero, the probability to evaluate p on values witnessing this property at least once grows over $\frac{2}{3}$. Finally, if all the evaluations returned 0 as output the input is accepted; otherwise, it is rejected.

The procedure described above is correct only when the size of the input circuit x is greater than some constant ϱ . If this is not the case, our algorithm queries a table T storing all the pairs $(x_i, \chi_{\text{PZT}}(x_i))$ for $|x_i| < \varrho$, to obtain $\chi_{\text{PZT}}(x_i)$. The table T can be pre-computed, having just a constant number of entries. This algorithm, which we call PZT, is inspired by [3] and described in detail in Appendix C.1.

As the input circuit is evaluated modulo some $k \in \mathbb{Z}$, the algorithm works in time polynomial with respect to |x|. Therefore, as a consequence of Theorem 3.1 and Lemma 3.3, there is a Σ_1^b -formula G(x,y) of \mathcal{RL} that represents it. Appendix C.1 also contains a lower-level description of this formula G.

The Underlying Language. We show that there is a predicate H of \mathcal{RL} such that $H(x,\epsilon)$ holds if and only if x is the encoding of a circuit in PZT; otherwise, H(x,0) holds. This predicate realizes the function h described by the following algorithm:

- 1. Take in input x, and check whether it is a polynomial circuit with one output; if it is not, reject it. Otherwise:
- 2. Compute the polynomial term p represented by x, and reduce it to a normal form \overline{p} .
- 3. Check whether all the coefficients of the terms are null. If this is true, output ϵ , otherwise output 1 and terminate.

For reasonable encodings of polynomial circuits and expressions, h is elementary recursive and therefore there is a predicate H which characterizes it, and $\mathsf{I}\Delta_0 + \mathsf{Exp}$ proves the totality of h. Moreover, we have $h = \chi_{\mathrm{PZT}}$, as for every polynomial p with coefficients in \mathbb{Z} , $\mathbb{Z} \models \forall \vec{x}.p(\vec{x}) = 0$ iff all the monomials in the normal form of p have zero as coefficient.

Proving the Error Bound. We now show that the formula G is not-erratic and that it decides $\text{Lang}(\langle G \rangle)$. With the notations G^* and t_G from the the proof of Lemma 4.4, this can be reduced to proving in $|\Delta_0|$ Exp the following two claims:

$$\vdash \forall z. |z| = t_G(x) \land G^*(x, 0, z) \to H(x, 0), \tag{\dagger}$$

$$\vdash \forall x. \left| \left\{ z \leq 2^{t_G(x)} \right| \left| G^*(x, \epsilon, z) \to H(x, \epsilon) \right) \right\} \right| \geq \frac{2}{3} \cdot 2^{|2^{t_G(x)}|}. \tag{\ddagger}$$

(†) states that whenever the randomized algorithm rejects an input, then so does the deterministic one, while (‡), which is reminiscent of (\star) , states that in at

least $\frac{2}{3}$ of all possible cases, if the randomized algorithm accepts the circuit, the deterministic one accepts it too. Jointly, (†) and (‡) imply that the equivalence $G^*(x,y,z) \leftrightarrow H(x,y)$ holds in at least 2/3 of all possible cases.

While Claim (†) is a consequence of the compatibility of the mod k function with addition and multiplication, which are easily proved in $I\Delta_0 + Exp$, the proof of Claim (‡) is more articulated and relies on the Schwartz-Zippel Lemma, providing a lower bound to the probability of evaluating the polynomial on values witnessing that it is not identically zero, and the Prime Number Theorem (whose provability in $I\Delta_0 + Exp$ is known [17]) which bounds the probability to choose a *bad* value for k, i.e. one of those values causing PZT to return the wrong value. Detailed arguments are provided in the Appendix.

Closure under Polytime Reduction Only assessing that a problem belongs to \mathbf{BPP}_T does not tell us anything about other languages of this class; for this reason, we are interested in showing that \mathbf{BPP}_T is closed under polytime reduction. This allows us to start from $PZT \in \mathbf{BPP}_{(I\Delta_0 + \mathrm{Exp})}$ to conclude that all problems which can be reduced to PZT in polynomial time belong to this class, and in particular that $PIT \in \mathbf{BPP}_{(I\Delta_0 + \mathrm{Exp})}$. This is assessed by the following proposition, proved in the Appendix:

Proposition 1. For any theory $T \supseteq R\Sigma_1^b$ -NIA + Exp, language $L \in \mathbf{BPP}_T$ and language $M \subseteq \mathbb{S}$, if there is a polytime reduction from M to L, then $M \in \mathbf{BPP}_T$.

Corollary 1. PIT is in $BPP_{(I\Delta_0+Exp)}$.

7 On Jeřábek's Characterization of BPP

As mentioned in Section 1, a semantic characterization of **BPP** based on bounded arithmetic was already provided by Jeřábek in [42]. This approach relies on checking, against the standard model, the truth of a formula which, rather than expressing that some machine is non-erratic, expresses what can be seen as a second totality condition (beyond the formula expressing the totality of the algorithm). Hence, also within this approach we think it makes sense to investigate which problems can be proved to be in **BPP** within some given theory.

In this section, we relate the two approaches by showing that the problems in $\mathbf{BPP}_{\mathsf{T}}$ are provably definable \mathbf{BPP} problems, in the sense of [42], within some suitable extension of the bounded theory $\mathrm{PV}_1[13]$.

A PTM is represented in this setting by two provably total functions (A, r), where the machine accepts on input x with probability less than p/q when $\Pr_{w < r(x)}(A(x, w)) \le p/q$. Jeřábek focuses on the theory $\Pr_{v \in r(x)}(A(x, w)) \le p/q$. Jeřábek focuses on the theory $\Pr_{v \in r(x)}(A(x, w)) \le p/q$. Jeřábek focuses on the theory $\Pr_{v \in r(x)}(A(x, w)) \le p/q$. PV₁ called the dual weak pigeonhole principle (cf. [42, pp. 962ff.]) for $\Pr_{v \in r(x)}(A(x, w)) \le p/q$. The reason is that this theory is capable of proving approximate counting formulas of the form $\Pr_{v \in r(x)}(A(x, w)) \le p/q$, where " $rectrigodologo of the proving approximate equivalent to "<math>rectrigodologo of the proving that, in order to establish exact counting results, we were forced to use non-polytime operations, cf. Remark 4.1). The representation of <math>\operatorname{\mathbf{BPP}}$ problems hinges on the definition, for any probabilistic algorithm (A, r), of $\operatorname{L}^+_{A,r}(x) := \Pr_{w < r(x)}(\neg A(x, w)) \le 1/3$ and $\operatorname{L}^-_{A,r}(x) := \Pr_{w < r(x)}(A(x, w)) \le 1/3$.

Checking if the algorithm (A, r) solves some problem in **BPP** reduces then to checking the "totality" formula $\vDash \forall x. L_{A,r}^+(x) \lor L_{A,r}^-(x)$.

Now, first observe that, modulo an encoding of strings via numbers, everything which is provable in $R\Sigma_1^b$ -NIA without the predicate Flip can be proved in the theory $S_2^1(PV)$ [13], which extends both PV₁ and Buss' S_2^1 . Moreover, by arguing as in the proof of Lemma 4.2, in our characterization of **BPP** we can w.l.o.g. suppose that the formula G satisfies EpsZero $[G] := \forall x. \forall y. G(x,y) \to y = \epsilon \lor y = 0$. Under this assumption, the de-randomization procedure described in the proof of Lemma 4.4 turns G into a pair (A,r), where $A = G^*$ is Flip-free and $r(x) = t_G(x)$, and the languages $L_{A,r}^+(x)$ and $L_{A,r}^-(x)$ correspond then to the formulas $L_G^+(x) := \text{TwoThirds}[G(-,0)](x)$.

Now, since from $\mathsf{T} \vdash \forall x. \exists y. \mathsf{TwoThirds}[G](x,y)$ and $\mathsf{EpsZero}[G]$ one can deduce $\mathsf{T} \vdash \forall x. L_G^+(x) \lor L_G^-(x)$, we arrive at the following:

Proposition 7.1. Let L be a language with $L = \text{Lang}(\langle G \rangle)$. If $L \in \mathbf{BPP_T}$, then $\forall x. L_G^+(x) \vee L_G^-(x)$ is provable in some recursively enumerable extension of PV_1 . Conversely, if $\mathrm{PV}_1 + \mathrm{dWPHP}(\mathrm{PV}_1) \vdash \forall x. L_G^+(x) \vee L_G^-(x)$, then $L \in \mathbf{BPP}_{R\Sigma_1^b-\mathrm{NIA}+\mathrm{Exp}}$.

The second statement above relies on the fact that approximate counting can be replaced by exact counting in $R\Sigma_1^b$ -NIA+Exp (i.e. " \preceq_0 " can be replaced by " \leq ").

8 Conclusion

The logical characterization of randomized complexity classes, in particular those having a semantic nature, is a great challenge. This paper contributes to the understanding of this problem by showing not only how resource bounded randomized computation can be captured within the language of arithmetic, but also that the latter offers convenient tools to control error bounds, the essential ingredient in the definition of classes like **BPP** and **ZPP**.

We believe that the main contribution of this work is a first example of a sort of reverse computational complexity for probabilistic algorithms. As we discussed in Section 5, while the restriction to bounded theories is crucial in order to capture polytime algorithms via a totality condition, it is not necessary to prove error bounds for probabilistic (even polynomial time) algorithms. In particular, the (difficult) challenge of enumerating **BPP** translates into the challenge of proving $\mathbf{BPP} = \mathbf{BPP}_{\mathsf{T}}$ for some strong enough r.e. theory T. So, it is worth exploring how much can be proved within expressive arithmetical theories. For this reason we focused here on a well-known problem, PIT, which is known to be in \mathbf{BPP} , but not in \mathbf{P} , showing that the whole argument for PIT $\in \mathbf{BPP}$ can be formalized in a fragment of PA, namely $\mathsf{I}\Delta_0 + \mathsf{Exp}$.

Future Work The authors see this work as a starting point for a long-term study on the logical nature of semantic classes. From this point of view, many ides for further work naturally arise.

An exciting direction is the study of the expressiveness of the new syntactic classes $\mathbf{BPP}_{\mathsf{T}}$, that is, an investigation on the kinds of error bounds which can be proved in the arithmetical theories lying *between* standard bounded theories like S_2^1 or PV and PA, but also in theories which are *more expressive* than PA

(like e.g. second-order theories). Surely, classes of the form $\mathbf{BPP}_{\mathsf{T}}$ could be analyzed also as for the existence of complete problems and hierarchy theorems for them, since such results do not hold for \mathbf{BPP} itself [28].

Our approach to **BPP** suggests that extensions to other complexity classes of randomized algorithms like **ZPP**, **RP** and **coRP** could make sense. Notice that this requires to deal not only with beyond error-bounds, but also with either average class complexity or with failure in decision procedures.

Finally, given the tight connections between bounded arithmetics and proof complexity, another natural direction is the study of applications of our work to randomized variations on the theme, for example recent investigations on random resolution refutations [42, 6, 53], i.e. resolution systems where proofs may make errors but are correct most of the time.

References

- [1] M. Antonelli, U. Dal Lago, I. Oitavem, and P. Pistone. Randomized Bounded Arithmetic Technical Report. https://github.com/davidedavoli/RBA, 2022.
- [2] M. Antonelli, U. Dal Lago, and P. Pistone. On Measure Quantifiers in First-Order Arithmetic. In L. De Mol, A. Weiermannn, F. Manea, and D. Fernández-Duque, editors, *Connecting with Computabiliy*, pages 12–24. Springer, 2021.
- [3] S. Arora and B. Barak. Computational Complexity: A Modern Approach. Cambridge University Press, 2009.
- [4] S. Bellantoni and S. Cook. A New Recursion-Theoretic Characterization of the Polytime Functions. *Computational Complexity*, 2:97–110, 1992.
- [5] P. Billingsley. Probability and Measure. Wiley, 1995.
- [6] S. Buss, A.L. Kolodziejczyk, and N. Thapen. Fragments of Approximate Counting. *Journal of Symbolic Logic*, 79(2):496–525, 2014.
- [7] S.R. Buss. Bounded Arithmetic. PhD thesis, Princeton University, 1986.
- [8] S.R. Buss. First-Order Proof Theory of Arithmetic. In S.R. Buss, editor, *Handbook of Proof Theory*. Elsavier, 1998.
- [9] A. Church. An Unsolvable Problem of Elementary Number Theory. American J. of Mathematics, 58(2):345–363, 1992.
- [10] A. Cobham. The Intrinsic Computational Difficulty of Functions. In Y. Bar-Hillel, editor, Logic, Methodology and Philosophy of Science: Proc. of the 1964 International Congress (Studies in Logic and the Foundations of Mathematics), pages 24–30. North-Holland Publishing, 1965.
- [11] E.F. Codd. Relational Completeness of Data Base Sublanguages. In *Data Base Systems. Proc. of 6th Courant Computer Science Symposium, May* 24-25, 1971, New York, N.Y., pages 65–98, 1972.

- [12] S. Cook. The Complexity of Theorem Proving Procedures. In *Proc. Third Annual ACM Symposium on Theory of Computing*, pages 151–158, 1971.
- [13] S. Cook and A. Urquhart. Functional Interpretations of Feasibly Constructive Arithmetic. *Annals of Pure and Applied Logic*, 63(2):103–200, 1993.
- [14] S.A. Cook. Feasibly constructive proofs and the propositional calculus. In ACM Press, editor, Proc. 7th Annual ACM Symposium on Theory of Computing, pages 83–97, 1975.
- [15] S.A. Cook and R.A. Reckhow. Efficiency of Propositional Proof Systems. Journal of Symbolic Logic, 44(1):36–50, 1979.
- [16] M. Coquand, T. andd Hofmann. A New Method for Establishing Conservativity of Classical Systems over their Intuitionistic Version. In Proc. Mathematical Structures in Computer Science, pages 323–333, 1999.
- [17] C. Cornaros and C. Dimitracopoulos. The Prime Number Theorem and Fragments of PA. Archive for Mathematical Logic, 33:265–281, 08 1994.
- [18] H. B. Curry. Functionality in Combinatory Logic*. Proc. of the National Academy of Sciences, 20(11):584–590, 1934.
- [19] U. Dal Lago, R. Kahle, and I. Oitavem. A Recursion-Theoretic Characterization of the probabilistic Class PP. In F. Bonchi and S.J. Puglisi, editors, 46th International Symposium on Mathematical Foundations of Computer Science (MFCS 2021), volume 202 of Leibniz International Proceedings in Informatics (LIPIcs), pages 1–12. Schloss Dagstuhl Leibniz-Zentrum für Informatik, 2021.
- [20] U. Dal Lago, R. Kahle, and I. Oitavem. Implicit Recursion-Theoretic Characterizations of Counting Classes. Archive for Mathematical Logic, May 2022.
- [21] U. Dal Lago and P. Parisen Toldin. A Higher-Order Characterization of Probabilistic Polynomial Time. *Information and Computation*, 241:114–141, 2015.
- [22] K. Eickmeyer and M. Grohe. Randomisation and Derandomisation in Descriptive Complexity Theory. In A. Dawar and H. Veith, editors, Computer Science Logic. Springer Berlin Heidelberg, 2010.
- [23] R. Fagin. Generalized First-Order Spectra and Polynomial-Time Recognizable Sets. In Complexity of Computing: SIAM-AMS Proc., pages 43–73, 1974.
- [24] F. Ferreira. Polynomial-Time Computable Arithmetic and Conservative Extesions. Ph.D. Dissertation, December 1988.
- [25] F. Ferreira. Polynomial-Time Computable Arithmetic. In W. Sieg, editor, Logic and Computation, volume 106 of Contemporary Mathematics, pages 137–156. AMS, 1990.

- [26] Fernando Ferreira. Stockmeyer induction, pages
 161–180. Birkhäuser Boston, Boston, MA, 1990.
 URL: https://doi.org/10.1007/978-1-4612-3466-1_9,
 doi:10.1007/978-1-4612-3466-1_9.
- [27] G. Ferreira and I. Oitavem. An Interpretation of S_2^1 in Σ_1^b -NIA. Portugaliae Mathematica, 63:137–156, 2006.
- [28] L. Fortnow. Comparing notions of full derandomization. In *Proceedings* of the 16th Annual IEEE Conference on Computational Complexity, pages 28–34, Chicago, IL, USA, 2001. IEEE Computer Society.
- [29] H. Gaifman and C. Dimitracopoulos. Fragments of Peano's arithmetic and the MRDP theorem. *Logic and Algorithmic, Monograph. Enseign. Math.*, 30:187–206, 1982.
- [30] D. Gajser. Verifying Time Complexity of Turing Machines. *Informatica*, 40:369–370, 2016, long version available at https://arxiv.org/pdf/1307.3648.pdf.
- [31] J.-Y. Girard. Light Linear Logic. Information and Computation, 2(143):175–204, 1998.
- [32] J.-Y. Girard and Y. Lafont. Advances in Linear Logic. Cambridge University Press, 1995.
- [33] J.-Y. Girard, A. Scedrov, and P. Scott. Bounded Linear Logic: A Modular Approach to Polynomial-Time Computability. *Theoretical Computer Science*, 97(1):1–66, 1992.
- [34] K. Gödel. Über Formal Unentscheidbare Sätze der Principia Mathematica and Verwandter Systeme. Monatshefte für Mathematik und Physik, 38:173– 198, 1931.
- [35] P. Hajek. Arithmetical Hierarchy and Complexity of Computation. *Theoretical Computer Science*, 8(2):227–237, 1979.
- [36] P. Hájek and P. Pudlák. Metamathematics of First-Order Arithmetic, volume 3 of Perspectives in Mathematical Logic. Springer, Berlin-Heidelberg, 1998.
- [37] J. Hartmanis and R.E. Stearns. On the Computational Complexity of Algorithms. *Transactions of the AMS*, 117:285–306, 1965.
- [38] M. Hofmann. Programming Languages Capturing Complexity Classes. SIGACT News, 31(1):31–42, mar 2000.
- [39] H. A. Howard. The Formulae-as-Types Notion of Construction. In To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus, and Formalism. Academic Press, 1980.
- [40] N. Immerman. Descriptive Complexity. Springer, 1999.
- [41] E. Jeřábek. Dual Weak Pigeonhole Principle, Boolean Complexity, and Derandomization. *Annals of Pure and Applied Logic*, 129(1):1–37, 2004.

- [42] E. Jeràbek. Approximate Counting in Bounded Arithmetic. *Journal of Symbolic Logic*, 72(3):959–993, 2007.
- [43] J. Kraijcek, P. Pudlák, and G. Takeuti. Bounded Arithmetic and the Polynomial Hierarchy. *Annals of Pure and Applied Logic*, 52:143–153, 1991.
- [44] J. Krajicek and P. Pudlak. Propositional Proof Systems, the Consistency of First-Order Theories and the Complexity of Computations. *Journal of Symbolic Logic*, 54(3):1063–1079, 1989.
- [45] Y. Lafont. Soft Linear Logic and Polynomial Time. Theoretical Computer Science, 1/2(318):163–180, 2004.
- [46] D. Leivant. Ramified Recurrence and Computational Complexity I: Word Recurrence and Polytime. In P. Clote and J. Remmel, editors, Feasible Mathematics II, pages 320–343. Springer, 1995.
- [47] H. Michalewski and M. Mio. Measure Quantifiers in Monadic Second Order Logic. In *Proc. of Logical Foundations of Computer Science*, pages 267–282, Cham, 2016. Springer.
- [48] J. Mitchell, M. Mitchell, and A. Scedrov. A Linguistic Characterization of Bounded Oracle Computation and Probabilistic Polynomial Time. In Proc. of 39th Annual Symposium on Foundations of Computer science, pages 725–733. IEEE Computer Society, 1998.
- [49] C. Morgenstern. The Measure Quantifier. *Journal of Symbolic Logic*, 44(1):103–108, 1979.
- [50] R. Motwani and P. Raghavan. Randomized Algorithms. Cambridge University Press, Cambridge; NY, 1995.
- [51] C.H. Papadimitriou. Computational Complexity. Pearson Education, 1993.
- [52] R. Parikh. Existence and Feasibility in Arithmetic. Journal of Symbolic Logic, 36:494–508, 1971.
- [53] P. Pudlak and N. Thapen. Random Resolution Refutations. Computational Complexity, 28:185–239, 2019.
- [54] E.S. Santos. Probabilistic Turing Machines and Computability. AMS, 22(3):704-710, 1969.
- [55] M.H. Sorensen and P. Urzyczyn. Lectures on the Curry-Howard Isomorphism. Elsevier, 2006.
- [56] A. Turing. On Computable Numbers, with an Application to the Entscheidungsproblem. Proc. London Mathematical Society, pages 2–42, 230–265, 1936-37.
- [57] G. Winskel. The Formal Semantics of Programming Languages: An Introduction. MIT press, 1993.

A Proofs from Section 3.1

Theorem 3.1. \Leftarrow . As anticipated, in order to apply inductive steps (namely, composition and bounded recursion on notation) we need to adapt Parikh's theorem [52] to $R\Sigma_1^b$ -NIA.¹

Proposition 2 ("Parikh" [52]). Let $F(\vec{x}, y)$ be a bounded \mathcal{RL} -formula such that $R\Sigma_1^b$ -NIA $\vdash \forall \vec{x}. \exists y. F(\vec{x}, y)$. Then, there is a term t such that, $R\Sigma_1^b$ -NIA $\vdash \forall \vec{x}. \exists y \leq t(\vec{x}). F(\vec{x}, y)$.

Proof for Theorem 3.1(\Leftarrow). The proof is by induction on the structure of functions in \mathcal{POR} .

Base Case. Each basic function is Σ_1^b -representable in $R\Sigma_1^b$ -NIA.

• The empty function f = E is Σ_1^b -represented in $R\Sigma_1^b$ -NIA by the formula:

$$F_E(x,y): x = x \wedge y = \epsilon.$$

- 1. Existence is proved considering $y=\epsilon$. For the reflexivity of identity both $R\Sigma_1^b$ -NIA $\vdash x=x$ and $R\Sigma_1^b$ -NIA $\vdash \epsilon=\epsilon$ hold. So, by rules for conjunctions, we obtain $R\Sigma_1^b$ -NIA $\vdash x=x \land \epsilon=\epsilon$, and conclude $R\Sigma_1^b$ -NIA $\vdash \forall x. \exists y. (x=x \land y=\epsilon)$. Uniqueness is proved assuming $R\Sigma_1^b$ -NIA $\vdash x=x \land z=\epsilon$. By rules for conjunction, in particular $R\Sigma_1^b$ -NIA $\vdash z=\epsilon$, and since $R\Sigma_1^b$ -NIA $\vdash y=\epsilon$, by the transitivity of identity, we conclude $R\Sigma_1^b$ -NIA $\vdash y=z$.
- 2. Assume $E(\sigma, \omega^*) = \tau$. If $\tau = \epsilon$, then $\llbracket \overline{\sigma} = \overline{\sigma} \wedge \overline{\tau} = \epsilon \rrbracket = \llbracket \overline{\sigma} = \overline{\sigma} \rrbracket \cap \llbracket \overline{\tau} = \epsilon \rrbracket = \mathbb{O} \cap \mathbb{O} = \mathbb{O}$. So, for any $\omega^*, \omega^* \in \llbracket \overline{\sigma} = \overline{\sigma} \wedge \overline{\tau} = \epsilon \rrbracket$, as clearly $\omega^* \in \mathbb{O}$. If $\tau \neq \epsilon$, then $\llbracket \overline{\sigma} = \overline{\sigma} \wedge \overline{\tau} = \epsilon \rrbracket = \llbracket \overline{\sigma} = \overline{\sigma} \rrbracket \cap \llbracket \overline{\tau} = \epsilon \rrbracket = \mathbb{O} \cap \emptyset = \emptyset$. So, for any $\omega^*, \omega^* \notin \llbracket \overline{\sigma} = \overline{\sigma} \vee \overline{\tau} = \epsilon \rrbracket$, as clearly $\omega^* \notin \emptyset$.
- Functions $f = P_i^n$, $f = S_b$ and f = C are Σ_1^b -represented in $R\Sigma_1^b$ -NIA by respectively the formulas:

$$\begin{split} F_{P_i^n}(x_1,\dots,x_n,y) : \bigwedge_{j \in J} (x_j = x_j) \wedge y &= x_i, \\ F_{S_b}(x,y) : y &= x\mathbf{b}, \\ F_C(x,v,z_0,z_1,y) : (x = \epsilon \wedge y = v) \vee \exists x' \preceq x. (x = x'\mathbf{0} \wedge y = z_0) \\ \vee \exists x' \prec x. (x = x'\mathbf{1} \wedge y = z_1). \end{split}$$

where $1 \leq i \leq n$, $J = \{1, ..., n\} \setminus \{i\}$, and $b \in \{0, 1\}$ corresponding to (resp.) $b \in \{0, 1\}$. Proofs are omitted as straightforward.

• f = Q is Σ_1^b -represented in $R\Sigma_1^b$ -NIA by the formula:

$$F_O(x,y): (\operatorname{Flip}(x) \wedge y = 1) \vee (\neg \operatorname{Flip}(x) \wedge y = 0).$$

Observe that, in this case, the proof crucially relies on the fact that oracle functions invoke *exactly one* oracle:

The theorem is usually presented in the context of Buss' bounded theories, as stating that given a bounded formula F in $\mathcal{L}_{\mathbb{N}}$ such that $S_2^1 \vdash \forall \vec{x}. \exists y.F$, then there is a term $t(\vec{x})$ such that also $S_2^1 \vdash \forall \vec{x}. \exists y \le t(\vec{x}).F(\vec{x},y)$ [7, 8]. Furthermore, due to [27], Buss' syntactic proof can be adapted to Σ_1^b -NIA in a natural way. The same result holds for $R\Sigma_1^b$ -NIA, as not containing specific rules concerning $\text{Flip}(\cdot)$.

1. Existence is proved by cases.² Since our underlying logic is classical, $R\Sigma_1^b\text{-NIA} \vdash \text{Flip}(x) \lor \neg \text{Flip}(x)$ holds. Then, if $R\Sigma_1^b\text{-NIA} \vdash \text{Flip}(x)$, let y = 1. By the reflexivity of identity, $R\Sigma_1^b\text{-NIA} \vdash (\text{Flip}(x) \land 1 = 1) \lor (\neg \text{Flip}(x) \land 0 = 1)$. So, by rules for disjunction, $R\Sigma_1^b\text{-NIA} \vdash (\text{Flip}(x) \land 1 = 1) \lor (\neg \text{Flip}(x) \land 0 = 1)$ and we conclude:

$$R\Sigma_1^b$$
-NIA $\vdash \exists y. ((Flip(x) \land y = 1) \lor (\neg Flip(x) \land y = 0)).$

If $R\Sigma_1^b$ -NIA $\vdash \neg \texttt{Flip}(x)$, let y = 0. By the reflexivity of identity $R\Sigma_1^b$ -NIA $\vdash 0 = 0$ holds. Thus, by the rules for conjunction, $R\Sigma_1^b$ -NIA $\vdash \neg \texttt{Flip}(x) \land 0 = 0$ and for disjunction, we conclude $R\Sigma_1^b$ -NIA $\vdash (\texttt{Flip}(x) \land 0 = 1) \lor (\neg \texttt{Flip}(x) \land 0 = 0)$ and so,

$$R\Sigma_1^b$$
-NIA $\vdash \exists y. ((\texttt{Flip}(x) \land y = 1) \lor (\neg \texttt{Flip}(x) \land y = 0)).$

Uniqueness is established relying on the transitivity of identity.

2. Finally, it is shown that for every $\sigma, \tau \in \mathbb{S}$ and $\omega^* \in \mathbb{O}$, $Q(\sigma, \omega^*) = \tau$ when $\omega^* \in [\![F_Q(\overline{\sigma}, \overline{\tau})]\!]$. Assume $Q(\sigma, \omega^*) = 1$, which is $\omega^*(\sigma) = 1$,

$$\begin{split} \llbracket F_Q(\overline{\sigma}, \overline{\tau}) \rrbracket &= \llbracket \operatorname{Flip}(\overline{\sigma}) \wedge \overline{\tau} = 1 \rrbracket \cup \llbracket \neg \operatorname{Flip}(\overline{\sigma}) \wedge \overline{\tau} = 0 \rrbracket \\ &= (\llbracket \operatorname{Flip}(\overline{\sigma}) \rrbracket \cap \llbracket 1 = 1 \rrbracket) \cup (\llbracket \neg \operatorname{Flip}(\overline{\sigma}) \rrbracket \cap \llbracket 1 = 0 \rrbracket) \\ &= (\llbracket \operatorname{Flip}(\overline{\sigma}) \rrbracket \cap \mathbb{O}) \cup (\llbracket \neg \operatorname{Flip}(\overline{\sigma}) \rrbracket \cap \emptyset) \\ &= \llbracket \operatorname{Flip}(\overline{\sigma}) \rrbracket \\ &= \{ \omega \mid \omega(\sigma) = 1 \}. \end{split}$$

Clearly, $\omega^* \in \llbracket (\mathtt{Flip}(\overline{\sigma}) \wedge \overline{\tau} = 1) \vee (\neg \mathtt{Flip}(\overline{\sigma}) \wedge \overline{\tau} = 0) \rrbracket$. The case $Q(\sigma, \omega^*) = 0$ and the opposite direction are proved in a similar way.

Inductive Case. If f is defined by composition or bounded recursion from Σ_1^b -representable functions, then f is Σ_1^b -representable in $R\Sigma_1^b$ -NIA:

• Composition. Assume that f is defined by composition from functions g, h_1, \ldots, h_k so that $f(\vec{x}, \omega) = g(h_1(\vec{x}, \omega), \ldots, h_k(\vec{x}, \omega), \omega)$ and that g, h_1, \ldots, h_k are represented in $R\Sigma_1^b$ -NIA by the Σ_1^b -formulas $F_g, F_{h_1}, \ldots, F_{h_k}$, respectively. By Proposition 2, there exist suitable terms $t_g, t_{h_1}, \ldots, t_{h_k}$ such that (the existential part of) Condition 1. can be strengthened to $R\Sigma_1^b$ -NIA $\vdash \forall \vec{x}.\exists y \leq t_i.F_i(\vec{x},y)$ for each $i \in \{g,h_1,\ldots,h_k\}$. We conclude that $f(\vec{x},\omega)$ is Σ_1^b -represented in $R\Sigma_1^b$ -NIA by the following formula:

$$F_f(x,y): \exists z_1 \leq t_{h_1}(\vec{x})....\exists z_k \leq t_{h_k}(\vec{x}).(F_{h_1}(\vec{x},z_1) \wedge ... F_{h_k}(\vec{x},z_k) \wedge F_{\sigma}(z_1,...,z_k,y)).$$

Indeed, by IH, $F_g, F_{h_1}, \ldots, F_{h_k}$ are Σ_1^b -formulas. Then, also F_f is in Σ_1^b . Conditions 1.-2. are proved to hold by slightly modifying standard proofs.

• Bounded Recursion. Assume that f is defined by bounded recursion from g, h_0, h_1 , and t, so that:

$$f(\vec{x}, \epsilon, \omega) = g(\vec{x}, \omega)$$

$$f(\vec{x}, yb, \omega) = h_i(\vec{x}, y, f(\vec{x}, y, \omega), \omega)|_{t(\vec{x}, y)},$$

²For the formal proof, see [1].

where $i \in \{0,1\}$ and b=0 when i=0 and b=1 when i=1. Let g,h_0,h_1 be represented in $R\Sigma_1^b$ -NIA by, respectively, the Σ_1^b -formulas F_g,F_{h_0} , and F_{h_1} . Moreover, by Proposition 2, there exist suitable terms t_g,t_{h_0} , and t_{h_1} such that the existential part of condition 1. can be strengthened to its "bounded version". Then, it can be proved that $f(\vec{x},y)$ is Σ_1^b -represented in $R\Sigma_1^b$ -NIA by the formula below:

$$F_{f}(x,y): \exists v \leq t_{g}(\vec{x})t_{f}(\vec{x})(y \times t(\vec{x},y)t(\vec{x},y)\mathbf{1}\mathbf{1}).(F_{lh}(v,\mathbf{1} \times y\mathbf{1})$$

$$\wedge \exists z \leq t_{g}(\vec{x}).(F_{eval}(v,\epsilon,z) \wedge F_{g}(\vec{x},z))$$

$$\wedge \forall u \subset y.\exists z.(\tilde{z} \leq t(\vec{x},y))(F_{eval}(v,\mathbf{1} \times u,z) \wedge F_{eval}(v,\mathbf{1} \times u \times \tilde{z})$$

$$\wedge (u\mathbf{0} \subseteq y \rightarrow \exists z_{0} \leq t_{h_{0}}(\vec{x},u,z).(F_{h_{0}}(\vec{x},u,z,z_{0}) \wedge z_{0}|_{t(\vec{x},u)} = \tilde{z}))$$

$$\wedge (u\mathbf{1} \subseteq y \rightarrow \exists z_{1} \leq t_{h_{1}}(\vec{x},u,z).(F_{h_{1}}(\vec{x},u,z,z_{1}) \wedge z_{1}|_{t(\vec{x},u)} = \tilde{z})))).$$

where F_{lh} and F_{eval} are Σ_1^b -formulas defined as in [24]. Intuitively, $F_{lh}(x,y)$ states that the number of 1s in the encoding of x is yy, while $F_{eval}(x,y,z)$ is a "decoding" formula (strongly resembling Gödel's β -formula), expressing that the "bit" encoded in x as its y-th bit is z. Moreover $x \subset y$ is an abbreviation for $x \subseteq y \land x \neq y$. Then, this formula F_f satisfies all the requirements to Σ_1^b -represent in $R\Sigma_1^b$ -NIA the function f, obtained by bounded recursion form g, h_0 , and h_1 . In particular, Condition 1. concerning existence and uniqueness, have already been proved to hold by Ferreira [24]. Furthermore, F_f expresses that, given the desired encoding sequence v: (i.) the ϵ -th bit of v is (the encoding of) z' such that $F_q(\vec{x}, z')$ holds, where (for IH) F_g is the Σ_1^b -formula representing the function g, and (ii.) given that for each $u \subset y$, z denotes the "bit", encoded in v at position $1 \times u1$, then if $ub \subseteq y$ (that is, if we are considering the initial substring of y the last bit of which correspond to b), then there is a z_b such that $F_{h_b}(\vec{x}, y, z, z_b)$, where F_{h_b} Σ_1^b -represents the function f_{h_b} and the truncation of z_b at $t(\vec{x}, u)$ is precisely \tilde{z} , with b = 0 when b = 0 and b=1 when b=1.

Theorem 3.1.(\Rightarrow). The proof is obtained by adapting that by Cook and Urquhart for IPV^{ω} [13], and is structured as follows:

- 1. We define \mathcal{POR}^{λ} a basic equational theory for a simply typed λ -calculus endowed with primitives corresponding to functions of \mathcal{POR} .
- 2. We introduce a first-order intuitionistic theory $IPOR^{\lambda}$, which extends POR^{λ} with the usual predicate calculus as well as an **NP**-induction schema. It is shown that $IPOR^{\lambda}$ is strong enough to prove all theorems of $IR\Sigma_{1}^{b}$ -NIA.
- 3. We develop a realizability interpretation of $IPOR^{\lambda}$ (inside itself), showing that for any derivation of $\forall x.\exists y.F(x,y)$ (where F is a Σ_0^b -formula) one can extract a λ -term t of POR^{λ} , such that $\forall x.F(x,tx)$ is provable in $IPOR^{\lambda}$. From this we deduce that every function which is Σ_1^b -representable in $IR\Sigma_1^b$ -NIA is in POR.
- 4. We extend this result to classical $R\Sigma_1^b$ -NIA showing that any Σ_1^b -formula provable in $IPOR^{\lambda}$ + Excluded Middle (EM, for short) is already provable in $IPOR^{\lambda}$.

The System \mathcal{POR}^{λ} . We define an equational theory for a simply typed λ -calculus augmented with primitives for functions of \mathcal{POR} . Actually, these do not exactly correspond to the ones of \mathcal{POR} , although the resulting function algebra is proved equivalent.

Definition 12. Types of \mathcal{POR}^{λ} are defined by the grammar below:

$$\sigma := s \mid \sigma \Rightarrow \sigma.$$

Definition 13. Terms of \mathcal{POR}^{λ} are standard, simply typed λ -terms plus the constants:

```
\begin{array}{c} \textbf{0},\textbf{1},\epsilon:s\\ \textbf{0},\mathsf{Trunc}:s\Rightarrow s\Rightarrow s\\ \\ \mathsf{Tail},\mathsf{Flipcoin}:s\Rightarrow s\\ \\ \mathsf{Cond}:s\Rightarrow s\Rightarrow s\Rightarrow s\Rightarrow s\\ \\ \mathsf{Red}:s\Rightarrow (s\Rightarrow s\Rightarrow s)\Rightarrow (s\Rightarrow s\Rightarrow s)\Rightarrow (s\Rightarrow s)\Rightarrow s\Rightarrow s. \end{array}
```

Intuitively, $\mathsf{Tail}(x)$ computes the string obtained by deleting the first digit of x; $\mathsf{Trunc}(x,y)$ computes the string obtained by truncating x at the length of y; $\mathsf{Cond}(x,y,z,w)$ computes the function that yields y when $x=\epsilon,z$ when x=x'0, and w when x=x'1; $\mathsf{Flipcoin}(x)$ indicates a random 0/1 generator; Rec is the operator for bounded recursion on notation.

Notation 1. We abbreviate $x \circ y$ as xy and being T any constant Tail, Trunc, Cond, Flipcoin, Rec of arity n, we indicate $\mathsf{Tu}_1, \ldots, \mathsf{u}_n$ as $\mathsf{T}(\mathsf{u}_1, \ldots, \mathsf{u}_n)$.

We also introduce the following abbreviations for composed functions:

- $B(x) := Cond(x, \epsilon, 0, 1)$ denotes the function computing the last digit of x.
- $\mathsf{BNeg}(x) := \mathsf{Cond}(x, \epsilon, 0, 1)$ denotes the function computing the Boolean negation of $\mathsf{B}(x)$.
- BOr(x, y) := Cond(B(x), B(y), B(y), 1) denotes the function that coerces x and y to Booleans and then performs the OR operation.
- $\mathsf{BAnd}(x,y) := \mathsf{Cond}(\mathsf{B}(x),\epsilon,0,\mathsf{B}(y))$ denotes the function that coerces x and y to Booleans and then performs the AND operation.
- $\mathsf{Eps}(x) := \mathsf{Cond}(x, 1, 0, 0)$ denotes the characteristic function of "x = 0".
- Bool(x) := BAnd(Eps(Tail(x)), BNeg(Eps(x))) denotes the characteristic function of " $x = 0 \lor x = 1$ ".
- $\mathsf{Zero}(x) := \mathsf{Cond}(\mathsf{Bool}(x), \mathsf{0}, \mathsf{Cond}(x, \mathsf{0}, \mathsf{0}, \mathsf{1}), \mathsf{0})$ denotes the characteristic function of predicate " $x = \mathsf{0}$ ".
- Conc(x, y) denotes the concatenation function defined as:

$$\mathsf{Conc}(x,\epsilon) := x \qquad \quad \mathsf{Conc}(x,y\mathsf{b}) := \mathsf{Conc}(x,y)\mathsf{b},$$

with $b \in \{0, 1\}$.

• Eq(x, y) denotes the characteristic function of "x = y" and defined by double recursion by the equations below:

$$\begin{aligned} \mathsf{Eq}(\epsilon,\epsilon) &:= 1 \qquad \mathsf{Eq}(\epsilon,y\mathsf{b}) := 0 \\ \mathsf{Eq}(x\mathsf{b},\epsilon) &= \mathsf{Eq}(x\mathsf{0},y\mathsf{1}) = \mathsf{Eq}(x\mathsf{1},y\mathsf{0}) := 0 \qquad \mathsf{Eq}(x\mathsf{b},y\mathsf{b}) := \mathsf{Eq}(x,y), \\ \text{with } \mathsf{b} &\in \{\mathsf{0},\mathsf{1}\}. \end{aligned}$$

• Times(x,y) denotes the function for self-concatenation, $x,y\mapsto x\times y$ and is defined by the equations below:

$$\mathsf{Times}(x,\epsilon) := \epsilon \qquad \qquad \mathsf{Times}(x,y\mathsf{b}) := \mathsf{Conc}(\mathsf{Times}(x,y),x),$$
 with $\mathsf{b} \in \{\mathsf{0},\mathsf{1}\}.$

• $\mathsf{Sub}(x,y)$ denotes the initial substring function, $x,y\mapsto S(x,y)$, and is defined by bounded recursion as follows:

$$\mathsf{Sub}(x,\epsilon) := \mathsf{Eps}(x) \qquad \qquad \mathsf{Sub}(x,y\mathsf{b}) := \mathsf{BOr}(\mathsf{Sub}(x,y),\mathsf{Eq}(x,y\mathsf{b})),$$
 with $\mathsf{b} \in \{\mathsf{0},\mathsf{1}\}.$

Definition 14. Formulas of \mathcal{POR}^{λ} are equations $\mathsf{t}=\mathsf{u}$, where t and u are terms of type s.

Definition 15 (Theory \mathcal{POR}^{λ}). Axioms of \mathcal{POR}^{λ} are the following ones:

• Defining equations for the constants of \mathcal{POR}^{λ} :

$$\begin{split} \epsilon x &= x\epsilon = x & x(y\mathsf{b}) = (xy)\mathsf{b} \\ & \mathsf{Tail}(\epsilon) = \epsilon & \mathsf{Tail}(x\mathsf{b}) = x \\ \mathsf{Trunc}(x,\epsilon) &= \mathsf{Trunc}(\epsilon,x) = \epsilon & \mathsf{Trunc}(x\mathsf{b},y\mathsf{b}) = \mathsf{Trunc}(x,y)\mathsf{b} \\ & \mathsf{Cond}(\epsilon,y,z,w) = y & \mathsf{Cond}(x0,y,z,w) = z & \mathsf{Cond}(x1,y,z,w) = w \\ & \mathsf{Bool}(\mathsf{Flipcoin}(x)) = 1 \\ & \mathsf{Rec}(x,h_0,h_1,k,\epsilon) = x & \mathsf{Rec}(x,h_0,h_1,k,y\mathsf{b}) = \mathsf{Trunc}(h_by(\mathsf{Rec}(x,h_0,h_1,k,y)),ky), \\ & \mathsf{where} \ \mathsf{b} \in \{0,1\} \ \mathsf{and} \ b \in \{0,1\} \ (\mathsf{correspondingly}). \end{split}$$

• The (β) - and (ν) -axioms:

$$C[(\lambda x.t)u] = C[t\{u/x\}] \tag{\beta}$$

$$C[\lambda x.tx] = C[t]. \tag{ν}$$

where $C[\cdot]$ indicates a context with a unique occurrence of the hole [], so that C[t] denotes the variable capturing replacement of [] by t in C[].

The inference rules of \mathcal{POR}^{λ} are the following ones:

$$t = u \vdash t = u \tag{R1}$$

$$t = u, u = v \vdash t = v \tag{R2}$$

$$\mathsf{t} = \mathsf{u} \vdash \mathsf{v} \{ \mathsf{t}/x \} = \mathsf{v} \{ \mathsf{u}/x \} \tag{R3}$$

$$t = u \vdash t\{v/x\} = u\{v/x\}. \tag{R4}$$

As predictable, $\vdash_{\mathcal{POR}^{\lambda}} \mathsf{t} = \mathsf{u}$ expresses that the equation $\mathsf{t} = \mathsf{u}$ is deducible using instances of the axioms above plus inference rules (R1) – (R4). Similarly, given any set T of equations, $T \vdash_{\mathcal{POR}^{\lambda}} \mathsf{t} = \mathsf{u}$ expresses that the equation $\mathsf{t} = \mathsf{u}$ is deducible using instances of the quoted axioms and rules together with equations from T.

For any string $\sigma \in \mathbb{S}$, let $\overline{\overline{\sigma}} : s$ denote the term of \mathcal{POR}^{λ} corresponding to it, that is:

$$\overline{\overline{\epsilon}} = \epsilon$$
 $\overline{\overline{\sigma}0} = \overline{\overline{\sigma}0$ $\overline{\overline{\sigma}1} = \overline{\overline{\sigma}1.$

For any $\omega \in \mathbb{O}$, let T_{ω} be the set of all equations of the form $\mathsf{Flipcoin}(\overline{\overline{\sigma}}) = \overline{\omega(\overline{\sigma})}$.

Definition 16 (Provable Representability). Let $f: \mathbb{O} \times \mathbb{S}^j \to \mathbb{S}$. A term $\mathsf{t}: s \Rightarrow \ldots \Rightarrow s$ of \mathcal{POR}^{λ} provably represents f when for all strings $\sigma_1, \ldots, \sigma_j, \sigma \in \mathbb{S}$ and $\omega \in \mathbb{O}$,

$$f(\sigma_1, \ldots, \sigma_i, \omega)$$
 iff $T_\omega \vdash_{\mathcal{POR}^\lambda} \mathsf{t}\overline{\overline{\sigma_1}} \ldots \overline{\overline{\sigma_i}} = \overline{\overline{\sigma}}$

Example A.1. The term Flipcoin: $s \Rightarrow s$ provably represents the query function $Q(x,\omega) = \omega(x)$ of \mathcal{POR} , since for any $\sigma \in \mathbb{S}$ and $\omega \in \mathbb{O}$,

$$\textit{Flipcoin}(\overline{\overline{\sigma}}) = \overline{\overline{\omega(\sigma)}} \vdash_{\mathcal{POR}^{\lambda}} \textit{Flipcoin}(\overline{\overline{\sigma}}) = \overline{\overline{Q(\sigma, \omega)}}.$$

We consider some of the terms described above and show them to provably represent the intended functions. Let $Tail(\sigma, \omega)$ indicate the string obtained by chopping the first digit of σ , and $Trunc(\sigma_1, \sigma_2, \omega) = \sigma_1|_{\sigma_2}$.

Lemma A.1. Terms Tail, Trunc and Cond provably represent the functions Tail, Trunc and C, respectively.

Theorem A.2. 1. Any function $f \in POR$ is provably represented by a term $t \in POR^{\lambda}$.

2. For any term $t \in \mathcal{POR}^{\lambda}$, there is a function $f \in \mathcal{POR}$ such that f is provably repesented by t.

Proof Sketch. 1. The proof is by induction on the structure of $f \in \mathcal{POR}$.

Base Case. Each base function is provably represented. Let us consider two examples:

- f = E is provably represented by $\lambda x.\epsilon$. For any string $\sigma \in \mathbb{S}$, $\overline{E(\sigma,\omega)} = \overline{\overline{\epsilon}} = \epsilon$ and $\vdash_{\mathcal{POR}^{\lambda}} \underline{(\lambda x.\epsilon)}\overline{\overline{\sigma}} = \epsilon$ is an instance of (β) -axiom. We conclude, $\vdash_{\mathcal{POR}^{\lambda}} (\lambda x.\epsilon)\overline{\overline{\sigma}} = \overline{E(\sigma,\omega)}$.
- f = Q is provably represented by the term Flipcoin, as observed in Example A.1 above.

Inductive Case. Each function defined by composition or bounded recursion from provably represented functions is provably represented as well. We consider bounded recursion. Let f be defined as:

$$f(\sigma_1, \dots, \sigma_n, \epsilon, \omega) = g(\sigma_1, \dots, \sigma_n, \omega)$$

$$f(\sigma_1, \dots, \sigma_n, \sigma b, \omega) = h_b(\sigma_1, \dots, \sigma_n, \sigma, f(\sigma_1, \dots, \sigma_n, \sigma, \omega), \omega)|_{k(\sigma_1, \dots, \sigma_n, \sigma)}.$$

By IH, g, h_0, h_1 and k are provably represented by the corresponding terms $t_q, t_{h_0}, t_{h_1}, t_k$. So, for any $\sigma_1, \ldots, \sigma_{n+2}, \sigma \in \mathbb{S}$ and $\omega \in \mathbb{O}$:

$$T_{\omega} \vdash_{\mathcal{POR}^{\lambda}} \mathsf{tg}\overline{\overline{\sigma_1}} \dots \overline{\overline{\sigma_n}} = \overline{\overline{g(\sigma_1, \dots, \sigma_n, \omega)}}$$
 (t_g)

$$T_{\omega} \vdash_{\mathcal{POR}^{\lambda}} \mathsf{t}_{h_0} \overline{\overline{\sigma_1}} \dots \overline{\overline{\sigma_{n+2}}} = \overline{\overline{h_0(\sigma_1, \dots, \sigma_{n+2}, \omega)}} \tag{t_{h_0}}$$

$$T_{\omega} \vdash_{\mathcal{POR}^{\lambda}} \mathbf{t}_{h_1} \overline{\overline{\sigma_1}} \dots \overline{\overline{\sigma_{n+2}}} = \overline{h_1(\sigma_1, \dots, \sigma_{n+2}, \omega)}$$
 (\mathbf{t}_{h_1})

$$T_{\omega} \vdash_{\mathcal{POR}^{\lambda}} \mathsf{t}_{k} \overline{\overline{\sigma_{1}}} \dots \overline{\overline{\sigma_{n}}} = \overline{\overline{k}(\sigma_{1}, \dots, \sigma_{n}, \omega)}.$$
 (t_{k})

We prove by induction on σ , that $T_{\omega} \vdash_{\mathcal{POR}^{\lambda}} \mathsf{t}_f \overline{\overline{\sigma_1}} \dots \overline{\sigma_n \sigma} = \overline{f(\sigma_1, \dots, \sigma_n, \omega)}$, where $\mathsf{t}_f = \lambda x_1 \dots \lambda x_n \lambda x$. Rec $(\mathsf{t}_g x_1 \dots x_n, \mathsf{t}_{h_0} x_1 \dots x_n, \mathsf{t}_{h_1} x_1 \dots x_n, \mathsf{t}_{kx_1} \dots x_n, \mathsf{x})$. Then,

- if $\sigma = \epsilon$, then $f(\sigma_1, \dots, \sigma_n, \underline{\sigma}, \omega) = g(\sigma_1, \dots, \sigma_n, \omega)$. Using the $(\underline{\beta})$ -axiom we deduce, $\vdash_{\mathcal{POR}^{\lambda}} \mathsf{t}_f \overline{\sigma_1} \dots \overline{\sigma_n} = \mathsf{Rec}(\mathsf{t}_g \overline{\sigma_1} \dots \overline{\sigma_n}, \mathsf{t}_{h_0} \overline{\sigma_1} \dots \overline{\sigma_n}, \mathsf{t}_{h_1} \overline{\sigma_1} \dots \overline{\sigma_n}, \mathsf{t}_k \overline{\sigma_1} \dots \overline{\sigma_n}, \mathsf{t}_k \overline{\sigma_1} \dots \overline{\sigma_n}, \overline{\sigma})$ and using the axiom $\mathsf{Rec}(\mathsf{t}_g x_1 \overset{\cdot}{\dots} x_n, \mathsf{t}_{h_0} x_1 \dots x_n, \mathsf{t}_{h_1} x_1 \dots x_n, \mathsf{t}_k x_1 \dots x_n, \epsilon = \mathsf{t}_g x_1 \dots x_n$, we obtain $\vdash_{\mathcal{POR}^{\lambda}} \mathsf{t}_f \overline{\sigma_1} \dots \overline{\sigma_n} \overline{\sigma} = \mathsf{t}_g \overline{\sigma_1} \dots \overline{\sigma_n}, \mathsf{by}$ (R2) and (R3). We conclude using (t_g) together with (R2).
- $\sigma = \sigma_m 0$, then $f(\sigma_1, \dots, \sigma_n, \sigma, \omega) = h_0(\sigma_1, \dots, \sigma_n, \sigma_m, f(\sigma_1, \dots, \sigma_n, \sigma, \omega), \omega)|_{k(\sigma_1, \dots, \sigma_n, \sigma_m)}$. By IH, suppose $T_\omega \vdash_{\mathcal{POR}^\lambda} \underline{t_f \overline{\sigma_1}} \dots \overline{\sigma_n \sigma_m} = \overline{f(\sigma_1, \dots, \sigma_n, \sigma', \omega)}$. Thus, using the (β) -axiom $\underline{t_f \sigma_1} \dots \overline{\sigma_n \sigma} = \operatorname{Rec}(\underline{t_g \overline{\sigma}} \dots \overline{\sigma_n}, \underline{t_{h_0} \overline{\sigma_1}} \dots \overline{\sigma_n}, \underline{t_{h_1} \overline{\sigma_1}} \dots \overline{\sigma_n}, \underline{t_k \overline{\sigma_1}} \dots \overline{\sigma_n}, \overline{\sigma}$ the axiom $\operatorname{Rec}(g, h_0, h_1, k, x_0) = \operatorname{Trunc}(\underline{t_{h_0} x(\operatorname{Rec}(g, h_0, h_1, k, 0)), kx)} \text{ and}$ IH we deduce, $\vdash_{\mathcal{POR}^\lambda} \underline{t_f \overline{\sigma_1}} \dots \overline{\sigma_n \sigma} = \operatorname{Trunc}(\underline{t_{h_0} \overline{\sigma_1}} \dots \overline{\sigma_n \sigma_m} \overline{f(\sigma_1, \dots, \sigma_n, \sigma_m, \omega)}, \underline{t_k \overline{\sigma_1}} \dots \overline{\sigma_n})$, by (R2) and (R3). Using $(\underline{t_{h_0}})$ and $(\underline{t_k})$ we conclude using (R3) and (R2): $\vdash_{\mathcal{POR}^\lambda} \underline{t_{\overline{\sigma_1}}} \dots \overline{\sigma_n \sigma} = \overline{h_0(\sigma_1, \dots, \sigma_n, \sigma_m, \overline{\sigma}), \underline{f(\sigma_1, \dots, \sigma, \sigma_m, \omega)}|_{k(\sigma_1, \dots, \sigma_n, \sigma_m)}}$.
- the case $\sigma = \sigma_m 1$ is proved in a similar way.
- 2. It is a consequence of the normalization property for the simply typed λ -calculus: a β -normal term $\mathsf{t}: s \Rightarrow \ldots \Rightarrow s$ cannot contain variables of higher types. By exhaustively inspecting possible normal forms, representability is checked.

Corollary 2. For any function $f: \mathbb{S}^j \times \mathbb{O} \to \mathbb{S}, f \in \mathcal{POR}$ when f is provably represented by some term $\mathsf{t}: s \Rightarrow \ldots \Rightarrow s \in \mathcal{POR}^{\lambda}$.

The Theory $IPOR^{\lambda}$. We introduce a first-order intuitionistic theory $IPOR^{\lambda}$, which extends POR^{λ} with basic predicate calculus and a restricted induction principle. We also define $IR\Sigma_1^b$ -NIA as a variant of $R\Sigma_1^b$ -NIA having the intuitionistic predicate calculus as its logical basis. All theorems of POR^{λ} and $IR\Sigma_1^b$ -NIA are provable in $IPOR^{\lambda}$. In fact, $IPOR^{\lambda}$ can be seen as an extension of POR^{λ} and provides a language to associate derivations in $IR\Sigma_1^b$ -NIA with poly-time computable functions, corresponding to terms of $IPOR^{\lambda}$.

The language of $IPOR^{\lambda}$ extends that of POR^{λ} with (a translation for) all expressions of $R\Sigma_1^b$ -NIA. In particular, the grammar for terms of $IPOR^{\lambda}$ is precisely the same as that of Definition 13, while that for formulas is defined below.

Definition 17. Formulas of $IPOR^{\lambda}$ are defined as follows: i. all equations of POR^{λ} t = u, are formulas of $IPOR^{\lambda}$; ii. for any (possibly open) POR^{λ} -term

 $t, u, t \subseteq u$ and Flip(t) are formulas of $IPOR^{\lambda}$; iii. formulas of $IPOR^{\lambda}$ are closed under $\land, \lor, \rightarrow, \forall, \exists$.

We adopt the standard conventions: $\bot := 0 = 1$ and $\neg F := F \to \bot$. The notions of Σ_0^b and Σ_1^b -formula of $I\mathcal{POR}^{\lambda}$ are precisely as those for $R\Sigma_1^b$ -NIA.

Remark A.1. Any formula of $R\Sigma_1^b$ -NIA can be seen as a formula of $IPOR^{\lambda}$, where each occurrence of 0 is replaced by 0, of 1 by 1, of \frown by \circ (usually omitted), of \times by Times. In the following, we assume that any formula of $R\Sigma_1^b$ -NIA is a formula of $IPOR^{\lambda}$, modulo the substitutions defined above.

Definition 18. The axioms of $IPOR^{\lambda}$ include standard rules of the intuitionistic first-order predicate calculus, usual rules for the equality symbol, plus the following axioms: 1. all axioms of POR^{λ} , 2. $x \subseteq y \leftrightarrow \mathsf{Sub}(x,y) = 1$, 3. $x = \epsilon \lor x = \mathsf{Tail}(x) 0 \lor x = \mathsf{Tail}(x) 1$, 4. $0 = 1 \to x = \epsilon$, 5. $\mathsf{Cond}(x,y,z,w) = w' \leftrightarrow (x = \epsilon \land w' = y) \lor (x = \mathsf{Tail}(x) 0 \land w' = z) \lor (x = \mathsf{Tail}(x) 1 \land w' = w)$, 6. $\mathsf{Flip}(x) \leftrightarrow \mathsf{Flipcoin}(x) = 1$, 7. any formula of the form:

$$(F(\epsilon) \land \forall x. (F(x) \to F(x0)) \land \forall x. (F(x) \to F(x1))) \to \forall y. F(y),$$

where F is of the form $\exists z \leq \mathsf{t.u} = \mathsf{v}$, with t containing only first-order open variables.

Notation 2. We refer to a formula of the form $\exists z \leq \mathsf{t.u} = \mathsf{v}$, with t containing only first-order open variables, as an **NP**-predicate.

Now that $IPOR^{\lambda}$ has been introduced we show that all theorems of both POR^{λ} and the intuitionistic version of $R\Sigma_1^b$ -NIA are derived in it. First, Proposition 3 is established inspecting all rules of POR^{λ} .

Proposition 3. Any theorem of \mathcal{POR}^{λ} is a theorem of $I\mathcal{POR}^{\lambda}$.

Then, we show that every theorem of $IR\Sigma_1^b$ -NIA is derivable in $IPOR^{\lambda}$. To do so, we prove a few properties concerning $IPOR^{\lambda}$. In particular, its recursion schema differs from that of $IR\Sigma_1^b$ -NIA as dealing with formulas of the form $\exists y \leq \mathsf{t.u} = \mathsf{v}$ and not with all the Σ_1^b -ones. The two schemas are related by Proposition 4 proved by induction on the structure of formulas.

Proposition 4. For any Σ_0^b -formula $F(x_1,\ldots,x_n)$ in \mathcal{RL} , there is a term $\mathsf{t}_F(x_1,\ldots,x_n)$ of \mathcal{POR}^λ such that: 1. $\vdash_{I\mathcal{POR}^\lambda} F \leftrightarrow \mathsf{t}_F = 0$, 2. $\vdash_{I\mathcal{POR}^\lambda} \mathsf{t}_F = 0 \lor \mathsf{t}_F = 1$.

This leads us to the following corollary and to Theorem A.3, realting $IPOR^{\lambda}$ and $IR\Sigma_{1}^{b}$ -NIA.

Corollary 3. i. For any Σ_0^b -formula F, $\vdash_{IPOR^{\lambda}} F \lor \neg F$; ii. For any closed Σ_0^b -formula of \mathcal{RL} F and $\omega \in \mathbb{O}$, either $T_{\omega} \vdash_{IPOR^{\lambda}} F$ or $T_{\omega} \vdash_{IPOR^{\lambda}} \neg F$.

Theorem A.3. Any theorem of $IR\Sigma_1^b$ -NIA is a theorem of $IPOR^{\lambda}$.

Proof. First, observe that, as a consequence of Proposition 4, for any $\Sigma^b - 1$ formula $F = \exists x_1 \leq t_1 \dots \exists x_n \leq t_n.G$ in \mathcal{RL} , $\vdash_{I\mathcal{POR}^{\lambda}} F \leftrightarrow \exists x_1 \leq t_1 \dots \exists x_n \leq t_n.t_G = 0$, any instance of the Σ_1^b -recursion schema of $IR\Sigma_1^b$ -NIA is derivable in $I\mathcal{POR}^{\lambda}$ from the **NP**-induction schema. Then, we conclude noticing that basic axioms of $IR\Sigma_1^b$ -NIA are provable in $I\mathcal{POR}^{\lambda}$.

Corollary 4. For any closed Σ_0^b -formula F and $\omega \in \mathbb{O}$, either $T_\omega \vdash_{I\mathcal{POR}^\lambda} F$ or $T_\omega \vdash_{I\mathcal{POR}^\lambda} \neg F$.

Due to Corollary 4 we establish the following Lemma A.4.

Lemma A.4. For any closed Σ_0^b -formula F and $\omega \in \mathbb{O}$, either $T_\omega \vdash_{I\mathcal{POR}^\lambda} F$ iff $\omega \in [\![F]\!]$.

Proof. (\Rightarrow) By induction on the structure of rules for $IP\mathcal{OR}$. (\Leftarrow) For Corollary 4, either $T_{\omega} \vdash_{IP\mathcal{OR}^{\lambda}} F$ or $T_{\omega} \vdash_{IP\mathcal{OR}^{\lambda}} \neg F$. Hence, if $\omega \in \llbracket F \rrbracket$, then it cannot be $T_{\omega} \vdash_{IP\mathcal{OR}^{\lambda}} \neg F$ (by soundness). We conclude $T_{\omega} \vdash_{IP\mathcal{OR}^{\lambda}} F$.

Realizability. We introduce realizability internal to $I\mathcal{POR}^{\lambda}$. As a corollary, we obtain that from any derivation in $IR\Sigma_1^b$ -NIA – actually, in $I\mathcal{POR}^{\lambda}$ – of a formula in the form $\forall x. \exists y. F(x,y)$, one can extract a functional term of \mathcal{POR}^{λ} f: $s \Rightarrow s$, such that $\vdash_{I\mathcal{POR}^{\lambda}} \forall x. F(x, fx)$. This allows us to conclude that if f is Σ_1^b -representable in $IR\Sigma_1^b$ -NIA, then $f \in \mathcal{POR}$.

Notation 3. Let \mathbf{x}, \mathbf{y} denote finite sequences of term variables, (resp.) x_1, \ldots, x_n and y_1, \ldots, y_k and $\mathbf{x}(\mathbf{y})$ be an abbreviation for $y_1(\mathbf{x}), \ldots, y_k(\mathbf{x})$. Let Λ be a shorthand for the empty sequence and $y(\Lambda) := y$.

Definition 19. Formulas $x \otimes F$ are defined by induction as follows:

$$\begin{split} \Lambda & \ \ \mathbb{R} \ F := F \\ \mathbf{x}, \mathbf{y} & \ \ \mathbb{R} \ (G \wedge H) := (\mathbf{x} \ \mathbb{R} \ G) \wedge (\mathbf{y} \ \mathbb{R} \ H) \\ z, \mathbf{x}, \mathbf{y} & \ \ \mathbb{R} \ (G \vee H) := (z = \mathbf{0} \wedge \mathbf{x} \ \mathbb{R} \ G) \vee (z \neq \mathbf{0} \wedge \mathbf{y} \ \mathbb{R} \ H) \\ \mathbf{y} & \ \ \mathbb{R} \ (G \to H) := \forall \mathbf{x}. (\mathbf{x} \ \mathbb{R} \ G \to \mathbf{y}(\mathbf{x}) \ \mathbb{R} \ H) \wedge (G \to H) \\ z, \mathbf{x} & \ \ \mathbb{R} \ \exists y. G := \mathbf{x} \ \mathbb{R} \ G\{z/y\} \\ \mathbf{x} & \ \ \mathbb{R} \ \forall y. G := \forall y. (\mathbf{x}(y) \ \mathbb{R} \ G), \end{split}$$

where no variable in \mathbf{x} is free in F. Given terms $\mathbf{t} = \mathsf{t}_1, \ldots, \mathsf{t}_n$ we let $\mathbf{t} \otimes F := (\mathbf{x} \otimes F)\{\mathbf{t}/\mathbf{x}\}.$

We relate the derivability of these new formulas with that of formulas of $IPOR^{\lambda}$. Proofs below are by induction (resp.) on the structure of $IPOR^{\lambda}$ -formulas and on the height of derivations.

Theorem A.5 (Soundness). If $\vdash_{IPOR^{\lambda}} \mathbf{t} \otimes F$, then $\vdash_{IPOR^{\lambda}} F$.

Notation 4. Given $\Gamma = F_1, \dots, F_n$, let $\mathbf{x} \otimes \Gamma$ be a shorthand for $\mathbf{x}_1 \otimes F_1, \dots, \mathbf{x}_n \otimes F_n$.

Theorem A.6 (Completeness). If $\vdash_{IPOR^{\lambda}} F$, then **t** such that $\vdash_{IPOR^{\lambda}} \mathbf{t} \otimes F$.

Proof. We prove that if $\Gamma \vdash_{I\mathcal{POR}^{\lambda}} F$, there exist terms \mathbf{t} such that $\mathbf{x} \otimes \Gamma \vdash_{I\mathcal{POR}^{\lambda}} \mathbf{tx}_1 \dots, \mathbf{x}_n \otimes F$. The proof is by induction on the derivation of $\Gamma \vdash_{I\mathcal{POR}^{\lambda}} F$. Let us consider just one example:

$$\frac{\Gamma \vdash G}{\Gamma \vdash G \lor H} \lor R_1$$

By IH, there exist terms \mathbf{u} , such that $\mathbf{t} \otimes \Gamma \vdash_{I\mathcal{POR}^{\lambda}} \mathbf{tu} \otimes G$. Since $x, y \otimes G \vee H$ is defined as $(x = 0 \wedge y \otimes G) \vee (x \neq 0 \wedge y \otimes H)$, we can take $\mathbf{t} = 0, \mathbf{u}$.

Corollary 5. Let $\forall x. \exists y. F(x,y)$ be a closed term of $IPOR^{\lambda}$, where F is a Σ_1^b -formula. Then, there is a closed term $t: s \Rightarrow s$ of POR^{λ} such that $\vdash_{IPOR^{\lambda}} \forall x. F(x, tx)$.

Proof. By Theorem A.6, there exist $\mathbf{t} = \mathbf{t}$, w such that $\vdash_{IP\mathcal{OR}^{\lambda}} \mathbf{t} \ \otimes \forall x. \exists y. F(x,y)$. So, $\mathbf{t} \ \otimes \forall x. \exists y. F(x,y) \equiv \forall x. (\mathbf{t}(x) \ \otimes \exists y. F(x,y)) \equiv \forall x. (w(x) \ \otimes F(x,\mathbf{t}x))$. From this, by Theorem A.5, we deduce $\vdash_{IP\mathcal{OR}^{\lambda}} \forall x. F(x,\mathbf{t}x)$.

Now, we have all the ingredients to prove that if a function is Σ_1^b -representable in $IR\Sigma_1^b$ -NIA, then it is in \mathcal{POR} .

Corollary 6. For any function $f: \mathbb{O} \times \mathbb{S} \to \mathbb{S}$, if there is a closed Σ_1^b -formula in \mathcal{RL} F(x,y), such taht:

- 1. $IR\Sigma_1^b$ -NIA $\vdash \forall x.\exists ! y.F(x,y)$
- 2. $[F(\overline{\sigma_1}, \overline{\sigma_2})] = {\omega \mid f(\sigma_1, \omega) = \sigma_2},$

then $f \in \mathcal{POR}$.

Proof. Since $\vdash_{IR\Sigma_1^b\text{-NIA}} \forall x.\exists ! y.F(x,y)$, by Theorem A.3 $\vdash_{I\mathcal{POR}^\lambda} \forall x.\exists ! y.F(x,y)$. Then, from $\vdash_{I\mathcal{POR}^\lambda} \forall x.\exists y.F(x,y)$, we deduce $\vdash_{I\mathcal{POR}^\lambda} \forall x.F(x,gx)$ for some closed term $\mathsf{g} \in \mathcal{POR}^\lambda$, by Corollary 6. Furthermore, by Theorem A.2.2, there is a $g \in \mathcal{POR}$ such that for any $\sigma_1, \sigma_2 \in \mathbb{S}$ and $\omega \in \mathbb{O}$, $g(\sigma_1, \omega) = \sigma_2$, when $T_\omega \vdash_{I\mathcal{POR}^\lambda} \overline{\mathsf{go_1}} = \overline{\sigma_2}$. So, by Proposition 3, for any $\sigma_1, \sigma_2 \in \mathbb{S}$ and $\omega \in \mathbb{O}$ if $g(\sigma_1, \omega) = \sigma_2$, then $T_\omega \vdash_{I\mathcal{POR}^\lambda} \overline{\mathsf{go_1}} = \overline{\sigma_2}$ and so $T_\omega \vdash_{I\mathcal{POR}^\lambda} F(\overline{\sigma_1}, \overline{\sigma_2})$. By Lemma A.4, $T_\omega \vdash_{I\mathcal{POR}^\lambda} F(\overline{\sigma_1}, \overline{\sigma_2})$, when $\omega \in [\![F(\overline{\sigma_1}, \overline{\sigma_2})\!]\!]$, that is $f(\sigma_1, \omega) = \sigma_2$. But then f = g, so since $g \in \mathcal{POR}$ also $f \in \mathcal{POR}$.

 $\forall \mathbf{NP}\text{-}Conservativity of } IP\mathcal{OR}^{\lambda} + EM \text{ over } IP\mathcal{OR}^{\lambda}.$ Corollary 6 is already close to the result we are looking for. The remaining step to conclude our proof is its extension from intuitionistic $IR\Sigma_1^b\text{-}\mathrm{NIA}$ to classical $R\Sigma_1^b\text{-}\mathrm{NIA}$, showing that any function which is $\Sigma_1^b\text{-}\mathrm{representable}$ in $R\Sigma_1^b\text{-}\mathrm{NIA}$ is also in \mathcal{POR} . The proof adapts method by [13]. We start by considering an extension of $I\mathcal{POR}^{\lambda}$ via EM and show that the realizability interpretation extends to it so that for any of its closed theorems $\forall x. \exists y \leq \mathsf{t.} F(x,y)$, being F a $\Sigma_1^b\text{-}\mathrm{formula}$, there is a closed term $\mathsf{t}: s \Rightarrow s$ of \mathcal{POR}^{λ} such that $\vdash_{I\mathcal{POR}^{\lambda}} \forall x. F(x, \mathsf{t}x)$.

Let EM be the excluded-middle schema $F \vee \neg F$, and Markov's principle be defined as follows $\neg \neg (\exists x.F \to (existsx)F)$ where F is a Σ_1^b -formula.

Proposition 5. For any Σ_1^b -formula F, if $\vdash_{IP\mathcal{O}\mathcal{R}^{\lambda}+EM} F$, then $\vdash_{IP\mathcal{O}\mathcal{R}^{\lambda}+(Markov)} F$.

Proof Sketch. The proof is by double negation translation with the following two remarks: 1. for any Σ_0^b -formula F, $\vdash_{I\mathcal{POR}^{\lambda}} \neg \neg F \to F$; 2. using (Markov), the double negation of an instance of the **NP**-induction can be shown equivalent the **NP**-induction schema.

We conclude by that the realizability interpretation defined above extends to $IPOR^{\lambda}+(Markov)$, that is for any closed theorem $\forall x.\exists y \leq \mathsf{t}.F(x,y)$ with F Σ_1^b -formula of $IPOR^{\lambda}+(Markov)$ there is a closed term of POR^{λ} $\mathsf{t}:s\Rightarrow s$ such that $\vdash_{IPOR^{\lambda}} \forall x.F(x,\mathsf{t}x)$.

Let assume given an encoding \sharp : $(s \Rightarrow s) \Rightarrow s$ in $I\mathcal{POR}^{\lambda}$ of first-order unary functions as strings, together with a "decoding" function $\mathsf{app}: s \Rightarrow s \Rightarrow s$ satisfying $\vdash_{I\mathcal{POR}^{\lambda}} \mathsf{app}(\sharp f, x) = \mathsf{f}x$. Moreover, let $x * y := \sharp (\lambda z.\mathsf{BAnd}(\mathsf{app}(x, z), \mathsf{app}(y, z)))$

and $T(x) := \exists y. (\mathsf{B}(\mathsf{app}(x,y)) = 0)$. There is a meet semi-lattice structure on the set of terms of type s defined by $\mathsf{t} \sqsubseteq \mathsf{u}$ when $\vdash_{I\mathcal{POR}^\lambda} T(\mathsf{u}) \to T(\mathsf{t})$ with top element $\underline{1} := \sharp(\lambda x.1)$ and meet given by x * y. Indeed, from $T(x*1) \leftrightarrow T(x)$, $x \sqsubseteq \underline{1}$ follows, Moreover, from $\mathsf{B}(\mathsf{app}(x,\mathsf{u})) = 0$, we obtain $\mathsf{B}(\mathsf{app}(x*y,\mathsf{u})) = \mathsf{BAnd}(\mathsf{app}(x,\mathsf{u}),\mathsf{app}(y,\mathsf{u})) = 0$, whence $T(x) \to T(x*y)$, i.e. $x*y \sqsubseteq x$. One can similarly prove $x*y \sqsubseteq y$. Finally, from $T(x) \to T(y)$ and $T(y) \to T(y)$, we deduce $T(x*y) \to T(y)$, by observing that $\vdash_{I\mathcal{POR}^\lambda} T(x*y) \to T(y)$. Notice that the formula T(x) is not a Σ^b -one, as its existential quantifier is not bounded.

Definition 20. For any $IPOR^{\lambda}$ -formula F and fresh variable x, we define formulas $x \Vdash F$:

$$x \Vdash F := F \vee T(x) \qquad (F \text{ atomic})$$

$$x \Vdash G \wedge H := x \Vdash G \wedge x \Vdash H$$

$$x \Vdash G \vee H := x \Vdash G \vee x \Vdash H$$

$$x \Vdash G \to H := \forall y.(y \Vdash G \to x * y \Vdash H)$$

$$x \Vdash \exists y.G := \exists y.x \Vdash G$$

$$x \Vdash \forall y.G := \forall y.x \Vdash G.$$

Lemma A.7. If F is provable in $IPOR^{\lambda}$ without using **NP**-induction, then $x \Vdash F$ is provable in $IPOR^{\lambda}$.

Proof Sketch. By induction on the structure of formulas of $IPOR^{\lambda}$ as in [16].

Lemma A.8. Let $F = \exists x \leq \mathsf{t}.G$, where F is a Σ_0^b -formula. Then, there is a term $\mathsf{u}_F : s$ with $FV(\mathsf{u}_F) = FV(G)$ such that $\vdash_{IP\mathcal{OR}^{\lambda}} F \leftrightarrow T(\mathsf{u}_F)$.

Proof. Since G(x) is a Σ_0^b -formula, for all terms $\mathbf{u}: s, \vdash_{I\mathcal{POR}^\lambda} G(x) \leftrightarrow \mathbf{u}_{x \preceq \mathsf{t} \wedge G}(x) = \mathbf{0}$, where $\mathsf{t}_{x \preceq \mathsf{t} \wedge G}$ has the free variables of t and G. Let H(x) be a Σ_0^b -formula, it is shown by induction on its structure that for any term $\mathbf{v}: s, \, \mathsf{t}_{H(\mathbf{v})} = \mathsf{t}_H(\mathbf{v})$. Then, $\vdash_{I\mathcal{POR}^\lambda} \vdash F \leftrightarrow \exists x. \mathsf{t}_{x \preceq \mathsf{u} \wedge G}(x) = \mathbf{0} \leftrightarrow \exists x. T(\sharp(\lambda x. \mathsf{t}_{x \preceq \mathsf{t} \wedge G}(x)))$. So, we let $\mathsf{u}_F = \sharp(\lambda x. \mathsf{t}_{x \preceq \mathsf{u} \wedge G}(x))$.

From which we deduce the following three properties: i. $\vdash_{IPOR^{\lambda}} (x \Vdash F) \leftrightarrow (F \lor T(x))$; ii. $\vdash_{IPOR^{\lambda}} (x \Vdash F) \leftrightarrow (F \to T(x))$; iii. $\vdash_{IPOR^{\lambda}} (x \Vdash \neg \neg F) \leftrightarrow (F \lor T(x))$, where F is a Σ_1^b -formula.

Corollary 7 (Markov's Principle). If F is a Σ_1^b -formula, then $\vdash_{IP\mathcal{OR}^{\lambda}} x \Vdash \neg \neg F \to F$.

To define the extension $(I\mathcal{POR}^{\lambda})^*$ of $I\mathcal{POR}^{\lambda}$, we introduce PIND(F) as:

$$(F(\epsilon) \land (\forall x.(F(x) \to F(x0)) \land \forall x.(F(x) \to F(x1)))) \to \forall x.F(x).$$

Observe that if F(x) is a formula of the form $\exists y \leq \mathsf{t.u} = \mathsf{v}$, then $z \Vdash \mathrm{PIND}(F)$ is of the form $\mathrm{PIND}(F(x) \vee T(z))$, which is *not* an instance of the **NP**-induction schema.

Definition 21. Let $(I\mathcal{POR}^{\lambda})^*$ be the theory extending \mathcal{POR}^{λ} with all instances of the induction schema $PIND(F(x) \vee G)$, where F(x) is of the form $\exists y \leq \mathsf{t.u} = \mathsf{v}$, and G is an arbitrary formula with $x \notin FV(G)$.

The following Proposition relates derivability in $IPOR^{\lambda}$ and in $(IPOR^{\lambda})^*$.

Proposition 6. For any Σ_1^b -formula F, if $\vdash_{IP\mathcal{OR}^{\lambda}} F$, then $\vdash_{(IP\mathcal{OR}^{\lambda})^*} x \Vdash F$.

Finally, we extend realizability to $(I\mathcal{POR}^{\lambda})^*$ by constructing a realizer fo $PIND(F(x) \vee G)$.

Lemma A.9. Let $F(x): \exists y \leq \mathsf{t.u} = \mathsf{0}$ and G be any formula not containing free occurrences of x. Then, there exist terms \mathbf{t} such that $\vdash_{I\mathcal{POR}^{\lambda}} \mathbf{t} \ \ \mathbb{R} \ \mathrm{PIND}(F(x) \vee G)$.

So, by Theorem A.5, for any Σ_1^b -formula F and formula G, with $x \notin FV(F)$, $\vdash_{I\mathcal{POR}^{\lambda}} PIND(F(x) \vee G)$.

Corollary 8 (\forall NP-Conservativity). Let F be a Σ_1^b -fromula. If $\vdash_{IP\mathcal{OR}^{\lambda}+EM}$ $\forall x.\exists y \leq \mathsf{t.} F(x,y)$, then $\vdash_{IP\mathcal{OR}^{\lambda}} \forall x.\exists y \leq \mathsf{t.} F(x,y)$.

We conclude the proof establishing the following Proposition 7.

Proposition 7. Let $\forall x. \exists y \leq \mathsf{t}. F(x,y)$ be a closed term of $I\mathcal{POR}^{\lambda} + (\mathrm{Markov})$, where F is a Σ_1^b -formula. Then, there is a closed term of \mathcal{POR}^{λ} $\mathsf{t}: s \Rightarrow s$ such that $\vdash_{I\mathcal{POR}^{\lambda}} \forall x. F(x, \mathsf{t}x)$.

Proof. If $\vdash_{I\mathcal{POR}^{\lambda}+(Markov)} \forall x.\exists y.F(x,y)$, then by Proposition 2, also $\vdash_{I\mathcal{POR}^{\lambda}+(Markov)} \exists y \leq \mathsf{t}.F(x,y)$. Moreover, $\vdash_{(I\mathcal{POR}^{\lambda})^*} z \Vdash \exists y \leq \mathsf{t}.F(x,y)$. Then, let us consider $G = \exists y \leq \mathsf{t}.F(x,y)$. Taking $\mathsf{v} = \mathsf{u}_G$, by Lemma A.8, we deduce $\vdash_{(I\mathcal{POR}^{\lambda})^*} G$ and, thus, by Lemma A.7 and A.9, we conclude that there exist t, u such that $\vdash_{I\mathcal{POR}^{\lambda}} \mathsf{t}, \mathsf{u} \ \mathbb{R} G$, which implies $\vdash_{I\mathcal{POR}^{\lambda}} F(x, \mathsf{t}x)$ and so $\vdash_{I\mathcal{POR}^{\lambda}} \forall x.(F(x), \mathsf{t}x)$.

So, by Proposition 5, if $\vdash_{I\mathcal{POR}^{\lambda}+EM} \forall x.\exists y \leq \mathsf{t}.F(x,y)$, being F a closed Σ_1^b -formula, then there is a closed term of \mathcal{POR} $\mathsf{t}: s \Rightarrow s$ such that $\vdash_{I\mathcal{POR}^{\lambda}} \forall x.F(x,\mathsf{t}x)$. Finally, we conclude the desired Corollary 9.

Corollary 9. Let $R\Sigma_1^b$ -NIA $\vdash \forall x.\exists y \leq t.F(x,y)$, where F is a Σ_1^b -formula with only x,y free. For any $f: \mathbb{S} \times \mathbb{O} \to \mathbb{S}$, if $\forall x.\exists y \leq t.F(x,y)$ represents f so that:

- 1. $R\Sigma_1^b$ -NIA $\vdash \forall x.\exists! y. F(x,y)$
- 2. $[F(\overline{\overline{\sigma_1}}, \overline{\overline{\sigma_2}})] = {\omega \mid f(\sigma_1, \omega) = \sigma_2},$

then $f \in \mathcal{POR}$.

B Proofs from Section 3

B.1 From RFP to POR

The goal of this section is to establish a correspondence between \mathbf{RFP} and \mathcal{POR} . This passes through Proposition 8, which assesses the equivalence between \mathbf{RFP} and the intermediate class \mathbf{SFP} and Proposition 9, which concludes the proof showing the equivalence between \mathcal{POR} and \mathbf{SFP} . To establish rigorously the results mentioned above, we fix some definitions. We start with those of STMs and their configurations:

Definition 22 (Stream Turing Machine). A stream Turing machine is a quadruple $M := \langle \mathcal{Q}, q_0, \Sigma, \delta \rangle$, where:

- Q is a finite set of states ranged over by q_i and similar meta-variables;
- $q_0 \in \mathcal{Q}$ is an initial state;
- Σ is a finite set of characters ranged over by c_i et simila;
- $\delta: \hat{\Sigma} \times \mathcal{Q} \times \hat{\Sigma} \times \mathbb{B} \longrightarrow \hat{\Sigma} \times \mathcal{Q} \times \hat{\Sigma} \times \{L, R\}$ is a transition function describing the new configuration reached by the machine.

L and R are two fixed and distinct symbols, e.g. 0 and 1, $\hat{\Sigma} = \Sigma \cup \{ \circledast \}$ and \circledast represents the *blank character*, such that $\circledast \notin \Sigma$. Without loss of generality, in the following, we will use STMs with $\Sigma = \mathbb{B}$.

Definition 23 (Configuration of STM). The *configuration of an STM* is a quadruple $\langle \sigma, q, \tau, \eta \rangle$, where:

- $\sigma \in \{0, 1, \circledast\}^*$ is the portion of the work tape on the left of the head;
- $q \in \mathcal{Q}$ is the current state of the machine;
- $\tau \in \{0, 1, \circledast\}^*$ is the portion of the work tape on the right of the head;
- $\eta \in \mathbb{B}^{\mathbb{N}}$ is the portion of the oracle tape that has not been read yet.

Thus, we give the definition of the family of reachability relations for STM machines.

Definition 24 (Stream Machine Reachability Functions). Given an STM \mathscr{S} with transition function δ , we denote with \vdash_{δ} its standard step function and we call $\{\triangleright_{\mathscr{S}}^n\}_n$ the smallest family of relations for which:

$$\langle \sigma, q, \tau, \eta \rangle \rhd_M^0 \langle \sigma, q, \tau, \eta \rangle$$

$$\left(\langle \sigma, q, \tau, \eta \rangle \rhd_M^n \langle \sigma', q', \tau', \eta' \rangle \right) \wedge \left(\langle \sigma', q', \tau', \eta' \rangle \vdash_{\delta} \langle \sigma'', q', \tau'', \eta'' \rangle \right) \rightarrow \left(\langle \sigma, q, \tau, \eta \rangle \rhd_M^{n+1} \langle \sigma'', q' \tau'', \eta'' \rangle \right)$$

Definition 25 (STM Computation). Given an STM, $\mathscr{S} = \langle \mathcal{Q}, q_0, \Sigma, \delta \rangle$, $\eta : \mathbb{N} \longrightarrow \mathbb{B}$ and a function $g : \mathbb{N} \longrightarrow \mathbb{B}$, we say that \mathscr{S} computes g, written $f_{\mathscr{S}} = g$ iff for every string $\sigma \in \mathbb{S}$, and oracle tape $\eta \in \mathbb{B}^{\mathbb{N}}$, there are $n \in \mathbb{N}$, $\tau \in \mathbb{S}$, $q' \in \mathcal{Q}$, and a function $\psi : \mathbb{N} \longrightarrow \mathbb{B}$ such that:

$$\langle \epsilon, q_0, \sigma, \eta \rangle \triangleright_{\mathscr{L}}^n \langle \gamma, q', \tau, \psi \rangle,$$

and $\langle \gamma, q', \tau, \psi \rangle$ is a final configuration for $\mathscr S$ with $f_{\mathscr S}(\sigma, \eta)$ being the longest suffix of γ not including \circledast .

Similar notations are employed for all the families of Turing-like machines we define in this paper. However, PTMs are an exception since they compute distributions over $\mathbb S$ instead of functions $\mathbb S \times \mathbb O \longrightarrow \mathbb S$.

Definition 26 (Probabilistic Turing Machines). A Probabilistic Turing Machine (PTM) is a TM with two transition functions δ_0 and δ_1 at each step of the computation, δ_0 is applied with probability $\frac{1}{2}$, otherwise δ_1 is applied. Given a PTM \mathcal{M} , a configuration $\langle \sigma, q, \tau \rangle$, we define its semantics on configurations $\langle \sigma, q, \tau \rangle$ as the following sequence of random variables:

$$\forall \eta \in \mathbb{B}^{\mathbb{N}}.X_{M,0}^{\langle \sigma,q,\tau \rangle} := \eta \mapsto \langle \sigma,q,\tau \rangle$$

$$\forall \eta \in \mathbb{B}^{\mathbb{N}}.X_{M,n+1}^{\langle \sigma,q,\tau \rangle} := \eta \mapsto \begin{cases} \delta_b(X_{M,n}^{\langle \sigma,q,\tau \rangle}(\eta)) & \text{if } \eta(n) = b \land \exists \langle \sigma',q'\tau' \rangle.\delta_b(X_{M,n}^{\langle \sigma,q,\tau \rangle}(\eta)) = \langle \sigma',q',\tau' \rangle \\ X_{M,n}^{\langle \sigma,q,\tau \rangle}(\eta) & \text{if } \eta(n) = b \land \neg \exists \langle \sigma',q'\tau' \rangle.\delta_b(X_{M,n}^{\langle \sigma,q,\tau \rangle}(\eta)) = \langle \sigma',q',\tau' \rangle \end{cases}$$

Intuitively, the variable $X_{\mathcal{M},n}^{\langle\sigma,q,\tau\rangle}$ maps each possible cylinder $\eta:\mathbb{N}\longrightarrow\mathbb{B}$ to the configuration reached by the machine after exactly n transitions where the first transition step has employed $\delta_{\eta(0)}$, the second has employed $\delta_{\eta(1)}$ and so on. We say that a PTM \mathcal{M} computes $Y_{\mathcal{M},\sigma}$ iff $\exists t\in\mathbb{N}.\forall\sigma.X_{\mathcal{M},t}^{\langle\sigma,q_0,\tau\rangle}$ is final. In such case, for every $\eta, Y_{\mathcal{M},\sigma}(\eta)$ is the longest suffix of the leftmost element in $X_{\mathcal{M},t}^{\langle\sigma,q_0,\epsilon\rangle}(\eta)$, which does not contain \circledast .

We start with the proof of the equivalence between the class of the PTM's polytime functions and that of the STMs' polytime ones.

Proposition 8. For any poly-time STM $\mathscr S$ there is a polytime PTM $\mathscr M$ such that for every $\sigma\tau\in\mathbb S$:

$$\mu(\{\omega \in \mathbb{O} \mid N(\sigma\omega) = \tau\}) = \Pr[M(\sigma) = \tau]$$

and viceversa.

Proof. The claim can be restated as follows:

$$\forall \sigma, \tau. \mu(\{\eta \in \mathbb{B}^{\mathbb{N}} \mid N(\sigma, \eta) = \tau\}) = \mu(M(\sigma)^{-1}(\tau))$$
$$\forall \sigma, \tau. \mu(\{\eta \in \mathbb{B}^{\mathbb{N}} \mid N(\sigma, \eta) = \tau\}) = \mu(\{\eta \in \mathbb{B}^{\mathbb{N}} \mid Y_{M, \sigma}(\eta) = \tau\}).$$

Actually, we will show a stronger result: there is bijection $I: \mathrm{STMs} \longrightarrow \mathrm{PTM}$ such that:

$$\forall n \in \mathbb{N}. \{ \eta \in \mathbb{B}^{\mathbb{N}} \mid \langle \sigma, q_0, \tau, \eta \rangle \triangleright_{\delta}^{n} \langle \tau, q, \psi, n \rangle \} = \{ \eta \in \mathbb{B}^{\mathbb{N}} \mid X_{I(N), n}^{\langle \epsilon, q_0, \sigma \rangle}(\eta) = \langle \tau, q, \psi \rangle \}$$

$$\tag{1}$$

which entails:

$$\{\eta \in \mathbb{B}^{\mathbb{N}} \mid N(\sigma, \eta) = \tau\} = \{\eta \in \mathbb{B}^{\mathbb{N}} \mid Y_{I(N), \sigma}(\eta) = \tau\}. \tag{2}$$

For this reason, it suffices to construct I and prove that (1) holds. I splits the function δ of N in such a way that transition is assigned to δ_0 if it matches the character 0 on the oracle-tape, otherwise it is assigned to δ_1 . Observe that I is bijective, indeed, its inverse is a function as well, because it consists in a disjoint union. Claim (1) can be shown by induction on the number of steps required by N to compute its output value, the proof is standard, so we omit it.

Proposition 9. For every $f: \mathbb{S} \times \mathbb{B}^{\mathbb{N}} \longrightarrow Ss$ in **SFP**, there is a function $g: \mathbb{S} \times \mathbb{O} \longrightarrow \mathbb{S}$ in \mathcal{POR} such that:

$$\forall x, y \in \mathbb{S}.\mu(\{\omega \in \mathbb{O} | g(x, \omega) = y\}) = \mu(\{\eta \in \mathbb{B}^{\mathbb{N}} | f(x, \eta) = y\}).$$

To prove the correspondence between the class of polytime STM computable function and \mathcal{POR} , we pass through the class of *finite-stream* TMs. These machines are defined analogously to STMs, but instead of an infinite stream of bits η , they employ a finite sequence of random bits as additional argument.

Lemma B.1. For each $f \in \mathbf{SFP}$ with time-bound $p \in \mathsf{POLY}$, there is a polytime finite-stream TM computable function h such that for any $\eta \in \mathbb{B}^{\mathbb{N}}$ and $x, y \in \mathbb{S}$,

$$f(x,\eta) = h(x,\eta_{p(|x|)}).$$

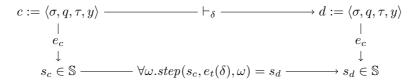


Figure 1: Behavior of the function step.

Proof. Assume that $f \in \mathbf{SFP}$. For this reason there is a polytime STM, $\mathscr{S} = \langle \mathcal{Q}, q_0, \Sigma, \delta \rangle$, such that $f = f_{\mathscr{S}}$. Take the *Finite Stream Turing Machine* (FSTM) \mathscr{S}' which is defined identically to \mathscr{S} . It holds that for any $k \in \mathbb{N}$ and some $\sigma, \tau, y' \in \mathbb{S}$,

$$\langle \epsilon, q_0', x, y \rangle \triangleright_{\mathscr{L}}^k \langle \sigma, q, \tau, y' \rangle \quad \Leftrightarrow \quad \langle \epsilon, q_0', x, y \eta \rangle \triangleright_{\mathscr{L}'}^k \langle \sigma, q, \tau, y' \eta \rangle.$$

Moreover, \mathscr{S}' requires a number of steps which is exactly equal to the number of steps required by \mathscr{S} , and thus the complexity is preserved. We conclude the proof defining $h = f_{\mathscr{S}'}$.

The next step is to show that each polytime FSTM computable function f corresponds to a function $g: \mathbb{S} \times \mathbb{S} \times \mathbb{O} \longrightarrow \mathbb{S}$ of \mathcal{POR} which can be defined without recurring to Q.

Lemma B.2. For any polytime FSTM computable function f and $x \in \mathbb{S}$, there is $g \in \mathcal{POR}$ such that $\forall x, y, \omega. f(x, y) = g(x, y, \omega)$. Moreover, if f is defined without recurring to Q, g can be defined without Q as well.

A formal proof of Lemma B.2 requires too much effort to be done extensively. In this work, we will simply mention the high-level structure of the proof. It relies on the following observations:

- 1. It is possible to encode FSTMs, together with configurations and their transition functions using strings, call these encodings $e_c \in \mathcal{POR}$ and e_t . Moreover, there is a function $step \in \mathcal{POR}$ which satisfies the simulation schema of Figure 1. The proof of this result is done by explicit definition of the functions e_c , e_t and step, proving the correctness of these entities with respect to the simulation schema above.
- 2. For each $f \in \mathcal{POR}$ and $x, y \in \mathbb{S}$, if there is a term t(x) in \mathcal{RL} which bounds the size of $f(x,\omega)$ for each possible input, then the function $m(z,x,\omega) = f^{|z|}(x,\omega)$ is in \mathcal{POR} as well, moreover, if f is defined without recurring to Q, also m can be defined without recurring to Q. This is shown in Lemma B.3.
- 3. Fixed a machine M, if $\sigma \in \mathbb{S}$ is a correct encoding of a configuration of M, for every ω , it holds that $|step(\sigma,\omega)| \leq |\sigma| + c$, for $c \in \mathbb{N}$ fixed once and forall.
- 4. If $c := e_c(\sigma, q, \tau, y, \omega)$ for some omega, then there is a function dectape such that $\forall \omega \in \mathbb{O}.dectape(x, \omega)$ is the longest suffix without occurrences of \circledast of σ .

Lemma B.3. For each $f: \mathbb{S}^{k+1} \times \mathbb{O} \longrightarrow \mathbb{S} \in \mathcal{POR}$, if there is a term $t \in \mathcal{RL}$ such that $\forall x, \vec{z}, \omega. f(x, \vec{z}, \omega)|_t = f(x, \vec{z}, \omega)$ then there is also a function $sa_{f,t}: \mathbb{S}^{k+2} \times \mathbb{O} \longrightarrow \mathbb{S}$ such that:

$$\forall x, n \in \mathbb{S}, \omega \in \mathbb{O}.sa_{f,t}(x, n, \vec{z}, \omega) = \underbrace{f(f(f(x, \vec{z}, \omega), \vec{z}, \omega), \ldots)}_{|n| \text{ times}}.$$

Moreover, if f is defined without recurring to Q, $sa_{f,t}$ can be defined without Q as well.

Proof. Given $f \in \mathcal{POR}$ and $t \in \mathcal{L}_{\mathbb{PW}}$, let $sa_{f,t}$ be defined as follows:

$$sa_{f,t}(x,\epsilon,\vec{z},\omega) := x$$

$$sa_{f,t}(x,yb,\vec{z},\omega) := f(sa_{f,t}(x,y,\omega),\vec{z},\omega)|_t$$

The correctness of sa comes as a direct consequence of its definition by induction on n.

Combining these results, we are able to prove Lemma B.2

Proof of Lemma B.2 (Sketch). As a consequence of points (2) and (3), we obtain that: $k(x, n, y, \omega) = step^{|n|}(x, y, \omega)$ belongs to \mathcal{POR} and can be defined without recurring to Q. As a consequence of (1) we have that:

$$k'(x, y, \omega) := k(e_c(x, q_0, \epsilon, y, \omega), y, \omega)$$

belongs to \mathcal{POR} as well and can be defined without recurring to Q. Finally, as a consequence of (4) and \mathcal{POR} 's closure under composition, there is a function g which returns the longest prefix of the leftmost projection of the output of k'. This function is exactly:

$$g(x, y, \omega) := dectape(k'(x, y, \omega), \omega).$$

As another consequence of Lemma B.2, we show the result we were aiming to: each function $f \in \mathbf{SFP}$ can be simulated by a function in $g \in \mathcal{POR}$, using as an additional input a polynomial prefix of f's oracle.

Corollary 10. For each $f \in \mathbf{SFP}$ and polynomial time-bound $p \in \mathsf{POLY}$, there is a function $g \in \mathcal{POR}$ such that for any $\eta : \mathbb{N} \longrightarrow \mathbb{B}$, $\omega : \mathbb{N} \longrightarrow \mathbb{B}$ and $x \in \mathbb{S}$,

$$f(x,\eta) = g(x,\eta_{p(|x|)},\omega).$$

Now, we need to establish that there is a function $e \in \mathcal{POR}$ which produces strings with the same distribution of the prefixes of the functions in $\mathbb{S}^{\mathbb{N}}$. Intuitively, this function is very simple: it extracts |x|+1 bits from ω and concatenates them in its output. The definition of the function e passes through a bijection $dyad: \mathbb{N} \longrightarrow \mathbb{S}$, called dyadic representation of a natural number. Thus, the function e can simply create the strings $1^0, 1^1, \ldots, 1^k$, and sample the function ω on the coordinates $dy(1^0), dy(1^1), \ldots, dy(1^k)$, concatenating the result in a string.

Definition 27. The function $dyad : \mathbb{N} \longrightarrow \mathbb{S}$ associates each $n \in \mathbb{N}$ to the string obtained stripping the left-most bit from the binary representation of n + 1.

Remark B.1. There is a \mathcal{POR} function $dy : \mathbb{S} \times \mathbb{O} \longrightarrow \mathbb{S}$ such that $\forall \sigma \in \mathbb{S}. \forall \omega \in \mathbb{O}. dy(\sigma, \omega) = dyad(1^{|\sigma|}).$

The construction of this function is not much interesting to the aim of our proof, so we omit it.

Definition 28. Let $e: \mathbb{S} \times \mathbb{O} \longrightarrow \mathbb{S}$ be defined as follows:

$$e(\epsilon, \omega) = \epsilon;$$

 $e(xb, \omega) = e(x, \omega)Q(dy(x, \omega), \omega)|_{xb}.$

Lemma B.4 (Correctness of e). For any $\sigma \in \mathbb{S}$ and $i \in \mathbb{N}$, if $|\sigma| = i + 1$, for any $j \leq i \in \mathbb{N}$ and $\omega \in \mathbb{O}$, (i) $e(\sigma, \omega)(j) = \omega(dy(1^j, \omega))$ and (ii) the length of $e(\sigma, \omega)$ is exactly i + 1.

Proof. Both claims are proved by induction on σ . The latter is trivial, whereas the former requires some more effort:

- ϵ The claim comes from vacuity of the premise $|\sigma| = i + 1$.
- τb It holds that: $e(\tau b, \omega)(j) = e(\tau, \omega)Q(dy(\tau, \omega), \omega) = e(\tau, \omega)\omega(dy(\tau, \omega))$. By (ii), for j = i + 1, the j-th element of $e(\tau b, \omega)$ is exactly $Q(dy(\tau, \omega), \omega)$, which is equal to $\omega(dy(\tau, \omega))$, in turn equal to $\omega(dy(1^j, \omega))$ (by Remark B.1). For smaller values of j, the first claim is a consequence of the definition of e together with IH.

Definition 29. We define \sim_{dy} as the smallest relation in $\mathbb{O} \times \mathbb{B}^{\mathbb{N}}$ such that:

$$\eta \sim_{dy} \omega \leftrightarrow \forall n \in \mathbb{N}. \eta(n) = \omega(dy(1^{n+1}, \omega)).$$

Lemma B.5. It holds that:

- $\forall \eta \in \mathbb{B}^{\mathbb{N}}.\exists !\omega \in \mathbb{O}.\eta \sim_{du} \omega;$
- $\forall \omega \in \mathbb{O}.\exists ! \eta \in \mathbb{B}^{\mathbb{N}}. \eta \sim_{dy} \omega.$

Proof. The proofs of the two claims are very similar. Since dyad is a bijection, applying Remark B.1, we obtain the existence of an ω which is in relation with η . Now suppose that there are ω_1, ω_2 both in relation with η but they are different. Then, there is a $\sigma \in \mathbb{S}$, such that $\omega_1(\sigma) \neq \omega_2(\sigma)$ and, by Remark B.1, $dy(\sigma, \omega_1) = dy(\sigma, \omega_2)$ which entails that $\eta(k) \neq \eta(k)$ for some k, that is a contradiction.

Corollary 11. The relation \sim_{dy} is a bijection.

Proof. Consequence of Lemma B.5.

Lemma B.6.

$$\eta \sim_{dy} \omega \to \forall n \in \mathbb{N}. \eta_n = e(1^{n+1}, \omega).$$

Proof. By contraposition: suppose $\eta_n \neq e(\underline{n}_{\mathbb{N}}, \omega)$. As a consequence of the correctness of e (Lemma B.4), there is an $i \in \mathbb{N}$ such that $\eta(i) \neq \omega(dy(\underline{i}_{\mathbb{N}}, \omega))$, which is a contradiction.

We can finally conclude the proof of Proposition 9.

Proof of Proposition 9. From Corollary 10, we know that there is a function $f' \in \mathcal{POR}$, and a $p \in \mathsf{POLY}$ such that:

$$\forall x, y \in \mathbb{S}. \forall \eta. \forall \omega. y = \eta_{p(x)} \to f(x, \eta) = f'(x, y, \omega). \tag{*}$$

Fixed an $\overline{\eta} \in \{\eta \in \mathbb{B}^{\mathbb{N}} \mid f(x,\eta) = y\}$, its image with respect to \sim_{dy} is in $\{\omega \in \mathbb{O} \mid f'(x,e(p'(s(x,\omega),\omega),\omega),\omega) = y\}$, where s is the \mathcal{POR} -function computing $1^{|x|}$. Indeed, by Lemma B.6, it holds that $\overline{\eta}_{p(x)} = e(p(size(x,\omega),\omega),$ where p' is the \mathcal{POR} -function computing the polynomial p, defined without recurring to Q. By (*), we prove the claim. It also holds that, given a fixed $\overline{\omega} \in \{\omega \in \mathbb{O} \mid f'(x,e(p'(size(x,\omega),\omega),\omega),\omega) = y\}$, its pre-image with respect to \sim_{dy} is in $\{\eta \in \mathbb{B}^{\mathbb{N}} \mid f(x,\eta) = y\}$. The proof is analogous to the one we showed above. Now, since \sim_{dy} is a bijection between the two sets: $\mu(\{\eta \in \mathbb{B}^{\mathbb{N}} \mid f(x,\eta) = y\}) = \mu(\{\omega \in \mathbb{O} \mid f'(x,e(p(size(x,\omega),\omega),\omega),\omega),\omega) = y\})$, which concludes the proof.

B.2 From POR to SFP

We start defining the imperative language $SIFP_{RA}$ and proving its polytime programs equivalent to \mathcal{POR} . To do so, we first introduce the definition of $SIFP_{RA}$ and its big-step semantics.

Definition 30 (Correct programs of $SIFP_{RA}$). The language of $SIFP_{RA}$ programs is $\mathcal{L}(Stm_{RA})$, i.e. the set of strings produced by the non-terminal symbol Stm_{RA} defined by:

$$\begin{split} \operatorname{Id} &::= X_i \mid Y_i \mid S_i \mid R \mid Q \mid Z \mid T \qquad i \in \mathbb{N} \\ \operatorname{Exp} &:= \epsilon \mid \operatorname{Exp.0} \mid \operatorname{Exp.1} \mid \operatorname{Id} \mid \operatorname{Exp} \sqsubseteq \operatorname{Id} \mid \operatorname{Exp} \wedge \operatorname{Id} \mid \neg \operatorname{Exp} \\ \operatorname{Stm}_{\mathbf{RA}} &::= \operatorname{Id} \leftarrow \operatorname{Exp} \mid \operatorname{Stm}_{\mathbf{RA}}; \operatorname{Stm}_{\mathbf{RA}} \mid \operatorname{while}(\operatorname{Exp}) \{\operatorname{Stm}\}_{\mathbf{RA}} \mid \operatorname{Flip}(\operatorname{Exp}) \} \end{split}$$

The *big-step* semantics associated to the language of the **SIFP**_{RA} programs relies on the notion of *Store*, which for us is a function $\Sigma : \mathsf{Id} \to \{0,1\}^*$.

We define the updating of a store Σ with a mapping from $y \in \mathsf{Id}$ to $\tau \in \{\mathsf{0},\mathsf{1}\}^*$ as:

$$\Sigma[y \leftarrow \tau](x) := \begin{cases} \tau & \text{if } x = y \\ \Sigma(x) & \text{otherwise.} \end{cases}$$

Definition 31 (Semantics of **SIFP** expressions). The semantics of an expression $E \in \mathcal{L}(\mathsf{Exp})$ is the smallest relation $\rightharpoonup: \mathcal{L}(\mathsf{Exp}) \times (\mathsf{Id} \longrightarrow \{0,1\}^*) \times \mathbb{O} \times \{0,1\}^*$ closed under the following rules:

$$\frac{\langle e, \Sigma \rangle \rightharpoonup \sigma}{\langle e, \Sigma \rangle \rightharpoonup \sigma \frown 0} \qquad \frac{\langle e, \Sigma \rangle \rightharpoonup \sigma}{\langle e.0, \Sigma \rangle \rightharpoonup \sigma \frown 0} \qquad \frac{\langle e, \Sigma \rangle \rightharpoonup \sigma}{\langle e.1, \Sigma \rangle \rightharpoonup \sigma \frown 1}$$

Definition 32 (big-step Operational Semantics of $\mathbf{SIFP_{RA}}$). The semantics of a program $P \in \mathcal{L}(\mathsf{Stm}_{\mathbf{RA}})$ is the smallest relation $\triangleright \subseteq \mathcal{L}(\mathsf{Stm}_{\mathbf{RA}}) \times (\mathsf{Id} \longrightarrow \{0,1\}^*) \times \mathbb{O} \times (\mathsf{Id} \longrightarrow \{0,1\}^*)$ closed under the following rules:

$$\begin{array}{c|c} \langle e, \Sigma \rangle \rightharpoonup \sigma & \langle s, \Sigma, \omega \rangle \rhd \Sigma' & \langle t, \Sigma', \omega \rangle \rhd \Sigma'' \\ \hline \langle \mathsf{Id} \leftarrow e, \Sigma, \omega \rangle \rhd \Sigma [\mathsf{Id} \leftarrow \sigma] & \langle s; t, \Sigma, \omega \rangle \rhd \Sigma'' \\ \hline \langle e, \Sigma \rangle \rightharpoonup 1 & \langle s, \Sigma, \omega \rangle \rhd \Sigma' & \langle \mathsf{while}(e) \{s\}, \Sigma', \omega \rangle \rhd \Sigma'' \\ \hline & \langle \mathsf{while}(e) \{s\}, \Sigma, \omega \rangle \rhd \Sigma'' \\ \hline & \langle e, \Sigma \rangle \rightharpoonup \sigma & \sigma \neq 1 \\ \hline & \langle \mathsf{while}(e) \{s\}, \Sigma, \omega \rangle \rhd \Sigma \\ \hline & \langle e, \Sigma \rangle \rightharpoonup \sigma & \omega(\sigma) = b \\ \hline & \langle \mathsf{Flip}(e), \Sigma, \omega \rangle \rhd \Sigma [R \leftarrow b] \\ \hline \end{array}$$

This semantics allows us to associate each $\mathbf{SIFP_{RA}}$ program to the function it evaluates:

Definition 33 (Function evaluated by a **SIFP**_{**RA**} program). We say that the function evaluated by a correct **SIFP**_{**RA**} program P is $[\![\cdot]\!]$: $\mathcal{L}(\mathsf{Stm}_{\mathbf{RA}}) \longrightarrow (\mathbb{S}^n \times \mathbb{O} \longrightarrow \mathbb{S})$, defined as below³:

$$\llbracket P \rrbracket := \lambda x_1, \dots, x_n, \omega. \triangleright (\langle P, | [[X_1 \leftarrow x_1], \dots, [X_n \leftarrow x_n], \omega \rangle)(R).$$

Observe that, among all the different registers, the register R is meant to contain the value computed by the program at the end of its execution, similarly the $\{X_i\}_{i\in\mathbb{N}}$ registers are used to store the function's inputs. The correspondence between \mathcal{POR} and $\mathbf{SIFP_{RA}}$ can be stated as follows:

Lemma B.7 (Implementation of \mathcal{POR} in $\mathbf{SIFP_{RA}}$). For every function $f \in \mathcal{POR}$, there is a polytime $\mathbf{SIFP_{RA}}$ program P such that: $\forall x_1, \ldots, x_n . \llbracket P \rrbracket (x_1, \ldots, x_n, \omega) = f(x_1, \ldots, x_n, \omega)$. Moreover, if f can be defined without recurring to Q, then P does not contain any $\mathsf{Flip}(e)$ statement.

³Instead of the infix notation for \triangleright , we will use its prefixed notation. So, the notation express the store associated to the P, Σ and ω by \triangleright . Moreover, notice that we employed the same function symbol \triangleright to denote two distinct functions: the *big-step* operational semantics of **SIFP**_{RA} programs and the *big-step* operational semantics of **SIFP**_{LA} programs

The proof of this result is quite simple: it relies on the fact that it is possible to associate to each \mathcal{POR} function an equivalent polytime program, and on the observation that it is possible to compose them and to implement bounded recursion on notation in $\mathbf{SIFP_{RA}}$ with a polytime overhead.

Proof Sketch of Lemma B.7. For each function $f \in \mathcal{POR}$ we define a program \mathfrak{L}_f such that $[\![\mathfrak{L}_f]\!](x_1,\ldots,x_n) = f(x_1,\ldots,x_n)$ The correctness of \mathfrak{L}_f is given by the following invariant properties:

- The result of the computation is stored in R.
- The inputs are stored in the registers of the group X.
- The function \mathfrak{L} does not change the values it accesses as input.

We define the function \mathfrak{L}_f as follows: $\mathfrak{L}_E := R \leftarrow \epsilon$; $\mathfrak{L}_{S_0} := R \leftarrow X_0.0$; $\mathfrak{L}_{S_1} := R \leftarrow X_0.1$; $\mathfrak{L}_{P_i^n} := R \leftarrow X_i$; $\mathfrak{L}_C := R \leftarrow X_1 \sqsubseteq X_2$; $\mathfrak{L}_Q := \mathtt{Flip}(X_1)$. The correctness of these base cases is trivial. Moreover, it is simple to see that the only translation containing $\mathtt{Flip}(e)$ for some $e \in \mathcal{L}(\mathsf{Exp})$ is the translation of Q. The encoding of the composition and of the bounded recursion are a little more convoluted: the proof of their correctness requires a conspicuous amount of low-level definitions and technical results, whose presentation is not the aim of this work.

The next step is to sow that every $SIFP_{RA}$ program is equivalent to a $SIFP_{LA}$ program in the sense of Lemma B.8.

Lemma B.8. For each total program $P \in \mathbf{SIFP_{RA}}$ there is a $Q \in \mathbf{SIFP_{LA}}$ such that:

$$\forall x,y.\mu \left(\{\omega \in \mathbb{B}^{\mathbb{S}} | \llbracket P \rrbracket(x,\omega) = y\}\right) = \mu \left(\{\eta \in \mathbb{B}^{\mathbb{N}} | \llbracket Q \rrbracket(x,\eta) = y\}\right).$$

Moreover, if P is polytime Q is polytime, too.

Before delving into the details of the proof of this Lemma, we must define the language $SIFP_{LA}$ together with its standard semantics:

Definition 34 (SIFP_{LA}). The language of the SIFP_{LA} programs is $\mathcal{L}(\mathsf{Stm}_{\mathsf{LA}})$, i.e. the set of strings produced by the non-terminal symbol $\mathsf{Stm}_{\mathsf{LA}}$ described as follows:

$$\mathsf{Stm}_{\mathbf{LA}} ::= \mathsf{Id} \leftarrow \mathsf{Exp} \mid \mathsf{Stm}_{\mathbf{LA}}; \mathsf{Stm}_{\mathbf{LA}} \mid \mathsf{while}(\mathsf{Exp}) \{ \mathsf{Stm} \}_{\mathbf{LA}} \mid \mathsf{RandBit}()$$

Definition 35 (Big Step Operational Semantics of **SIFP_{LA}**). The semantics of a program $P \in \mathcal{L}(\mathsf{Stm}_{\mathbf{LA}})$ is the smallest relation $\triangleright \subseteq (\mathcal{L}(\mathsf{Stm}_{\mathbf{LA}}) \times (\mathsf{Id} \longrightarrow \{0,1\}^*) \times \mathbb{B}^{\mathbb{N}}) \times ((\mathsf{Id} \longrightarrow \{0,1\}^*) \times \mathbb{B}^{\mathbb{N}})$ closed under the following rules:

$$\frac{\langle e, \Sigma \rangle \rightharpoonup \sigma}{\langle \mathsf{Id} \leftarrow e, \Sigma, \eta \rangle \rhd \langle \Sigma [\mathsf{Id} \leftarrow \sigma], \eta \rangle} \\ \frac{\langle s, \Sigma, \eta \rangle \rhd \langle \Sigma', \eta' \rangle \qquad \langle t, \Sigma', \eta \rangle \rhd \langle \Sigma'', \eta'' \rangle}{\langle s; t, \Sigma, \eta \rangle \rhd \langle \Sigma'', \eta'' \rangle} \\ \frac{\langle e, \Sigma \rangle \rightharpoonup 1 \qquad \langle s, \Sigma, \eta \rangle \rhd \langle \Sigma', \eta' \rangle \qquad \langle \mathsf{while}(e) \{s\}, \Sigma', \eta \rangle \rhd \langle \Sigma'', \eta'' \rangle}{\langle \mathsf{while}(e) \{s\}, \Sigma, \eta \rangle \rhd \langle \Sigma'', \eta'' \rangle}$$

$$\frac{\langle e, \Sigma \rangle \rightharpoonup \sigma \qquad \sigma \neq 1}{\langle \mathtt{while}(e)\{s\}, \Sigma, \eta \rangle \rhd \langle \Sigma, \eta \rangle} \qquad \qquad \frac{\langle \mathtt{RandBit}(), \Sigma, \mathtt{b} \eta \rangle \rhd \langle \Sigma[R \leftarrow \mathtt{b}], \eta \rangle}{\langle \mathtt{RandBit}(), \Sigma, \mathtt{b} \eta \rangle \rhd \langle \Sigma[R \leftarrow \mathtt{b}], \eta \rangle}$$

In particular, we prove Lemma B.8 showing that $\mathbf{SIFP_{RA}}$ can be simulated in $\mathbf{SIFP_{LA}}$ with respect to two novel *small-step* semantic relations $(\sim_{\mathbf{LA}}, \sim_{\mathbf{RA}})$ derived splitting the *big-step* semantics into small transitions, one per each :: instruction. Intuitively, the idea behind this novel semantics is to enrich the *big-step* operational semantic with some pieces of information necessary to build a proof that it is possible to each $\mathbf{SIFP_{LA}}$ instruction by meas of a sequence of $\mathbf{SIFP_{RA}}$ instructions preserving the distribution of the values in the register R, i.e. that storing the result. In particular we enrich the configurations of $\mathbf{SIFP_{LA}}$'s and $\mathbf{SIFP_{RA}}$'s *small-step* operational semantics with a list Ψ containing pairs (x,b). In the case of $\mathbf{SIFP_{LA}}$, this list is meant to keep track of the calls to the primitive $\mathbf{Flip}(x)$ and their result b. While, on the side of $\mathbf{SIFP_{RA}}$, the x-th call of the primitive $\mathbf{RandBit}()$ causes the pair (x,b) to be added on top of the list.

This is done to keep track of the accesses to the random tapes done by the simulated and the simulator programs. On the side of $\mathbf{SIFP_{RA}}$ it is possible to show that this table is stored in a specific register. This register plays an important role in the simulation of the $\mathtt{Flip}(e)$ instructions. In particular, this is done as follows:

- At each simulated query Flip(e), the destination program looks up the associative table;
- If it finds the queried coordinate e within a pair (e, b), it returns b, otherwise:
 - It reduces Flip(e) to a call of RandBit() which outputs either b = 0 or b = 1.
 - It records the pair $\langle e, b \rangle$ in the associative table and stores b in R.

Even in this case the construction of the proof is convoluted, but we believe that it is not too much of a problem to see, at least intuitively, that this kind of simulation preserves the distributions of strings computed by the original program. This concludes Lemma B.8.

The next step is to show that $\mathbf{SIFP_{LA}}$ can be reduced to $\mathbf{SFP_{OD}}$: the class corresponding to \mathbf{SFP} defined on a variation of the Stream Machines which are capable to read characters from the oracle tape *on-demand*, i.e. only on those states $q \in \mathcal{Q}_{\natural} \subseteq \mathcal{Q}$. The transition function of the $\mathbf{SFP_{OD}}$ is a function $\delta: \hat{\mathbb{B}} \times \mathcal{Q} \times (\hat{\mathbb{B}} \cup \{\natural\}) \times \mathbb{B} \longrightarrow \hat{\mathbb{B}} \times \mathcal{Q} \times \hat{\mathbb{B}} \times \{L, R\}$ which for all the states in \mathcal{Q}_{\natural} is labeled with the symbol \natural instead of a Boolean value. This peculiarity will be employed in Definition 36 to distinguish those configurations causing the oracle tape to shift to the others.

We do not show the reduction from $\mathbf{SIFP_{LA}}$ to $\mathbf{SFP_{OD}}$ extensively because this kind of reductions are cumbersome and, in literature, it is common to avoid their formal definition on behalf of a more readable presentation. For this reason, we only describe informally but exhaustively how to build the *on-demand* stream machine which corresponds to a generic program $P \in \mathcal{L}(\mathsf{Stm_{LA}})$. The correspondence between $\mathsf{SFP_{OD}}$ is expressed by the following Proposition:

Proposition 10. For every $P \in \mathcal{L}(\mathsf{Stm}_{\mathbf{LA}})$ there is a $M_P \in \mathbf{SFP}$ such that for every $x \in \mathbb{S}$ and $\eta \in \mathbb{B}^{\mathbb{S}}$, $P(x,\eta) = P(x,\eta)$. Moreover, if P is polytime, then M_P is polytime.

Proof. The construction relies on the fact that it is possible to implement a $\mathbf{SIFP_{LA}}$ program by means of a multi-tape on-demand stream machine which uses a tape to store the values of each register, plus an additional tape containing the partial results obtained during the evaluation of the expressions and another tape containing η . We denote the tape used for storing the result coming from the evaluation of the expressions with e.

The machine works thanks to some invariant properties:

- On each tape, the values are stored to the immediate right of the head.
- \bullet The result of the last expression evaluated is stored on the e tape to the immediate right of the head.

The value of a **SIFP** expression can be easily computed using the e tape. We show it by induction on the syntax of the expression:

- Each access to the value stored in a register basically consist in a copy of the content of the corresponding tape to the *e* tape, which is a simple operation, due to the invariant properties properties mentioned above.
- Concatenations (f.0 and f.1) are easily implemented by the addition of a character at the end of the e tape which contains the value of f, as stated by the induction hypothesis on the invariant properties.
- The binary expressions are non-trivial, but since one of the two operands is a register identifier, the machine can directly compare *e* with the tape which corresponding to the identifier, and to replace the content of *e* with the result of the comparison, which in all cases 0 or 1.

All these operations can be implemented without consuming any character on the oracle tape and with linear time with respect to the size of the expression's value. To each statement s_i , we assign a sequence of machine states, $q_{s_i}^I, q_{s_i}^1, q_{s_i}^2, \dots, q_{s_i}^F$.

- Assignments consist in a copy of the value in e to the tape corresponding to the destination register and a deletion of the value on e by replacing its symbols with \circledast characters. This can be implemented without consuming any character on the oracle tape.
- The sequencing operation s;t can be implemented inserting in δ a composed transition from q_s^F to q_t^I , which does not consume the oracle tape.
- A while(){s}tatement $s := \text{while}(f)\{t\}$ requires the evaluation of f and then passing to the evaluation of t, if $f \to 1$, or stepping to the next transition if it exists and $f \not \to 1$. After the evaluation of the body, the machine returns to the initial state of this statement, namely: q_s^I .
- A RandBit() statement is implemented consuming a character on the tape and copying its value on the tape which corresponds to the register *R*.

The following invariant properties hold at the beginning of the execution and are kept true throughout M_P 's execution. In particular, if we assume P to be polytime, after the simulation of each statement, it holds that:

- The length of the non blank portion of the first tapes corresponding to the register is polynomially bounded because their contents are precisely the contents of *P*'s registers, which are polynomially bounded as a consequence of the hypotheses on their polynomial time complexity.
- The head of all the tapes corresponding to the registers point to the leftmost symbol of the string thereby contained.

It is well-known that the reduction of the number of tapes on a polytime Turing Machine comes with a polynomial overhead in time; for this reason, we can conclude that the polytime multi-tape on-demand stream machine we introduced above can be shrinked to a polytime canonical on-demand stream machine. This concludes the proof.

It remains to show that each on-demand stream machine can be reduced to an equivalent STM.

Lemma B.9. For every $\mathscr{S} = \langle \mathcal{Q}, \mathcal{Q}_{\natural}, \Sigma, \delta, q_0 \rangle \in \mathbf{SFP_{OD}}$, the machine $\mathscr{S}' = \langle \mathcal{Q}, \Sigma, H(\delta), q_0 \rangle \in \mathbf{SFP}$ is such that for every $n \in \mathbb{N}$, for every configuration of $\mathscr{S} \langle \sigma, q, \tau, \eta \rangle$ and for every $\sigma', \tau' \in \mathbb{S}, q \in \mathcal{Q}$:

$$\mu\left(\left\{\eta\in\mathbb{B}^{\mathbb{N}}|\exists\eta'.\langle\sigma,q,\tau,\eta\rangle\rhd_{\delta}^{n}\langle\sigma',q',\tau',\eta'\rangle\right\}\right)=\mu\left(\left\{\chi\in\mathbb{B}^{\mathbb{N}}|\exists\chi'.\langle\sigma,q,\tau,\xi\rangle\rhd_{H(\delta)}^{n}\langle\sigma',q',\tau',\chi'\rangle\right\}\right).$$

Even in this case, the proof relies on a reduction. In particular, we show that given an on-demand stream machine $\mathscr S$ it is possible to build a stream machine $\mathscr S'$ which is equivalent to $\mathscr S$. Intuitively, the encoding from an on-demand stream machine $\mathscr S$ to an ordinary stream machine takes the transition function δ of $\mathscr S$ and substitutes each transition not causing the oracle tape to shift — i.e. tagged with \natural — with two distinct transitions, so to match both 0 and 1 on the tape storing ω . This causes the resulting machine to reach the same target state with the same behavior n the work tape, and to shift to the right the head on the oracle tape.

Definition 36 (Encoding from On-Demand to Canonical Stream Machines). We define the encoding from an On-Demand Stream Machine to a Canonical Stream Machine as below:

$$H := \langle \mathbb{Q}, \Sigma, \delta, q_0 \rangle \mapsto \langle \mathbb{Q}, \Sigma, \bigcup \Delta_H(\delta), q_0 \rangle.$$

where Δ_H is defined as follows:

$$\begin{split} &\Delta_H(\langle p, c_r, 0, q, c_w, d \rangle) := \{\langle p, c_r, 0, q, c_w, d \rangle\} \\ &\Delta_H(\langle p, c_r, 1, q, c_w, d \rangle) := \{\langle p, c_r, 1, q, c_w, d \rangle\} \\ &\Delta_H(\langle p, c_r, \natural, q, c_w, d \rangle) := \{\langle p, c_r, 0, q, c_w, d \rangle, \langle p, c_r, 1, q, c_w, d \rangle\}. \end{split}$$

Proof of Lemma B.9. The definition of $\triangleright_{\delta}^{n}$ allows us to rewrite the statement $\exists \eta'. \langle \sigma, q, \tau, \eta \rangle \triangleright_{\delta}^{n} \langle \sigma', q', \tau', \eta' \rangle$ as:

$$\exists \eta', \eta'' \in \mathbb{B}^{\mathbb{N}}. \exists c_1, \dots, c_k.$$

$$\langle \sigma, q, \tau, c_1 c_2 \dots c_k \eta' \rangle \triangleright_{\delta}^{n_1} \langle \sigma_1, q_{i_1}, \tau_1, c_1 c_2 \dots c_k \eta' \rangle \triangleright_{\delta}^{1} \langle \sigma'_1, q'_{i_1}, \tau_1, c_2 \dots c_k \eta' \rangle \wedge$$

$$\langle \sigma'_1, q'_{i_1}, \tau_1, c_2 \dots c_k \eta' \rangle \triangleright_{\delta}^{n_2} \langle \sigma_2, q_{i_2}, \tau_2, c_2 \dots c_k \eta' \rangle \triangleright_{\delta}^{1} \langle \sigma'_2, q'_{i_2}, \tau'_2, c_3 \dots c_k \eta' \rangle \wedge$$

$$\langle \sigma'_2, q'_{i_2}, \tau'_2, c_3 \dots c_k \eta' \rangle \triangleright_{\delta}^{n_3} \dots \triangleright_{\delta}^{n_{k+1}} \langle \sigma', q', \tau', \eta' \rangle.$$

The claim can be rewritten as follows:

Intuitively, this holds because it suffices to take the n_i s as the length of longest sequence of non-shifting transitions of the on-demand stream machine and the correspondence can be proven by induction on the number of steps of each formula in the conjunction. Thus, we can express the sets of the claim as follows:

$$\{\eta \in \mathbb{B}^{\mathbb{N}} | \exists \eta'. \langle \sigma, q, \tau, \eta \rangle \triangleright_{\delta}^{n} \langle \sigma', q', \tau', \eta' \rangle \} = \{\eta \in \mathbb{B}^{\mathbb{N}} | \forall 0 \leq i < k. \eta(i) = c_{i} \rangle \}$$
$$\{\chi \in \mathbb{B}^{\mathbb{N}} | \exists \chi'. \langle \sigma, q, \tau, \xi \rangle \triangleright_{H(\delta)}^{n} \langle \sigma', q', \tau', \chi' \rangle \} = \{\chi \in \mathbb{B}^{\mathbb{N}} | \forall 1 \leq i \leq k. \chi(n_{i} + i) = c_{i} \wedge \chi(0) = c_{1} \rangle \}.$$

The conclusion comes because both these sets are cylinders with the same measure. \Box

Proposition B.10 (From \mathcal{POR} to **SFP**). For any $f: \mathbb{S}^k \times \mathbb{B}^{\mathbb{S}} \to \mathbb{S}$ in \mathcal{POR} there exists a function $f^{\sharp}: \mathbb{S}^k \times \mathbb{B}^{\mathbb{N}} \to \mathbb{S}$ such that for all $n_1, \ldots, n_k, m \in \mathbb{S}$,

$$\mu(\{\eta \in \mathbb{B}^{\mathbb{N}} \mid f(n_1, \dots, n_k, \eta) = m\}) = \mu(\{\omega \in \mathbb{O} \mid f^{\sharp}(n_1, \dots, n_k, \omega) = m\}).$$

Proof. This result is a consequence of Lemma B.7, Lemma B.8, Proposition 10 and Lemma B.9. \Box

C Proofs from Section 5

C.1 The Randomized Algorithm

Let us first describe the randomized algorithm PZT in more detail:

- 1. If the input x is not the output of a circuit, reject it. Otherwise, let n be its arity, d its degree and m its size. Set i to 1.
- 2. Check whether m is smaller than some constant value $\rho \in \mathbb{N}$.
 - If so, walk the table T looking for a pair (x_j, y_j) where $x_j = x$; set $o_i = 0$ if $y_j = 1$, set $o_i = 1$, otherwise.

- Otherwise, proceed as follows. Choose r_1, \ldots, r_n uniformly and independently in $\{0, \ldots, 2^{m+3} 1\} \subseteq \mathbb{N}$. Let k be a random value in $\{1, \ldots, 2^{2m}\} \subseteq \mathbb{N}$. Finally, evaluate the result of x, seen as a circuit, on $r_1, \ldots, r_n \mod k$, with result o_i .
- 3. If i < s, then increase i by 1 and go back to 2.
- 4. If for all $i, 1 \le i \le s, o_i = 0$ output ϵ ; otherwise, output 0.

Let us now discuss the formula G that represents PZT. To do so, we introduce a set of basic predicates, which will be employed for the definition of G:

Circ(x) := x is the encoding of a circuit with a single output

Eval(x, k, y, t) :=When fed with inputs encoded by y, x produces in output t modulo k

NumVar(x, n) := x is the encoding of an arithmetic circuit with n variables

Degree(x, d) :=The degree of the arithmetic circuit encoded by x is d

$$T(x,y) := \bigvee_{\overline{x} \in \bigcup_{i=0}^{\varrho} \{0,1\}^i} (x = \overline{x} \land y = y_{\overline{x}})$$

K(r,z) := z is a uniformly chosen random string in $\{0,1\}^{|r|}$.

All these predicates characterize polytime random functions; for this reason, we can assume without lack of generality that they are Σ_1^b -formulæ of \mathcal{RL} . Using some of these predicates, it is possible to define a formula G_1 which executes one evaluation of the polynomial x:

$$\begin{split} G_1(x,m,n,d,z,y) := \left[m \leq \varrho \wedge T(x,1) \to y = 0 \wedge T(x,0) \to y = 1 \right] \vee \left[(m > \varrho) \wedge \left((y = 0 \wedge \exists z_0, z_1.|z_0| = 2m \wedge |z_1| = n \cdot (m+3) \wedge z_0 \cdot z_1 = z \wedge \exists t \leq z. \left(Eval(x,z_0,z_1,t) \wedge t \neq 0 \right) \right) \vee \left((y = 1 \wedge \exists z_0, z_1.|z_0| = 2m \wedge |z_1| = n \cdot (m+3) \wedge z_0 \cdot z_1 = z \wedge Eval(x,z_0,z_1,0) \right) \right]. \end{split}$$

Remark C.1. The formula G_1 is a Σ_1^b predicate of \mathcal{RL} which characterizes one iteration of the algorithm PZT.

Differently from the original algorithm, the formula G_1 employs an additional parameter z as a source of randomness to determine the values of r_i, \ldots, r_n and k. This way, we are able to isolate the randomization part in a small portion of the formula we are building, i.e. that one where we determine the value of z by means of the predicate Flip.

Thanks to this construction, G_1 is a Σ_1^b formula realizing some Flip-free \mathcal{POR} function g_1 . We can leverage this fact to define another Flip-free function $\iota_{g_1}(x,n,d,y,z,i)$ which iterates the function g_1 i times with the i-th (n(m+3)+2m)-long sub-string of z as source of randomness — i.e. as argument for g's z — and returns ϵ if and only if all these executions of g returned 1, otherwise it returns 0

This proves that there is a Flip-free Σ_1^b formula G^v realizing that function provable under $R\Sigma_1^b$ -NIA — and even S_1^0 . A picture of the proof of the existence of G^v is given in Figure 2. Intuitively, the formula G_v characterizes steps 2-7 of the PZT algorithm.

In order to define G, we will only need to compose G_v with another subformula which characterizes the first step of the PZT algorithm. This is quite

$$\begin{array}{ccc} G_1 & & & & \uparrow \\ & & & \uparrow \\ \text{Realizability} & & \text{Representability} \\ & \downarrow & & \mid \\ g_1 & & & \mathcal{POR} \text{ closure} & \longrightarrow \iota_{g_1} \end{array}$$

Figure 2: Summary of the proof of the existence of G^v

simple because we only need to encode the generation of the values of d, n, m, z and to fix a number of iterations, as we will show in section C.2, 37m is an appropriate choice. Thus, the Σ_1^b formula G can be defined as follows:

$$\begin{split} G(x,y) := \exists m \preceq x. |x| = m \wedge \left[\left((Circ(x) \wedge \exists n \preceq 2^m. \exists d \preceq 2^m. \exists z. |z| = 37m \cdot (n(m+3) + 2m) \wedge \right. \\ \left. NumVar(x,n) \wedge Degree(x,d) \wedge K(1^{37m \cdot (n(m+3) + 2m)}, z) \wedge G^v(x,m,n,d,y,z,37m) \right) \vee \left(\neg Circ(x) \wedge y = 0 \right) \right]. \end{split}$$

C.2 Proving the Error Bound

Within this section, we argument that:

$$\mathsf{I}\Delta_0 + \mathsf{Exp} \vdash \forall x. \forall y. \mathsf{TwoThirds}[G(x,y) \leftrightarrow H(x,y)],$$

which is equivalent to showing that:

$$\mathsf{I}\Delta_0 + \mathsf{Exp} \vdash \forall x. \forall y. \mathsf{Threshold}[\mathsf{NoFlip}[G](x,y,z) \leftrightarrow H(x,y)].$$

In turn, it is possible to obtain a formula equivalent to $\mathsf{NoFlip}[G](x,y,z)$ by removing the quantification over z and the randomization its value by means of K. Doing so, we are able to obtain an equivalent claim:

$$\begin{split} \mathsf{I}\Delta_0 + \mathrm{Exp} \vdash &\forall x. \forall y. \exists m,d,n \preceq 1^{|x|}. |x| = m \land \mathit{NumVar}(x,n) \land \mathit{Degree}(x,d) \\ &\exists^{\frac{2}{3}2^{37m \cdot (n(m+3)+2m)}} z. |z| = 37m \cdot (n(m+3)+2m) \land (G^v(x,n,d,z,y) \leftrightarrow H(x,y)). \end{split}$$

It also is easy to observe $\exists^{\geq h} z. |z| = k \land P(z) \Leftrightarrow |\{z \in \mathbb{B}^k \mid P(z)\}| \geq h$. This allows to replace the threshold quantification with a formula of $|\Delta_0| + \text{Exp}$ measuring the cardinality of some finite set. Therefore, we can reduce out goal to showing Lemma C.1 using only the instruments provided by $|\Delta_0| + \text{Exp}$:

Lemma C.1. For every encoding of a polynomial circuit x, for every $y \in \mathbb{B}$, whereas x has n variables, size m and degree d, it holds that:

$$|\{z \in \mathbb{B}^{37m \cdot (n(m+3)+2m)} \mid (G^v(x,n,d,z,y) \leftrightarrow H(x,y))\}| \geq \frac{2}{3} \cdot 2^{37m \cdot (n(m+3)+2m)}.$$

Since G and H characterize two decisional functions, the claim of Lemma C.1 can be restated in the following way:

$$|\{z \in \mathbb{B}^{37m \cdot (n(m+3)+2m)} \mid (G^v(x,n,d,z,0) \wedge H(x,0)) \vee (G^v(x,n,d,z,\epsilon) \wedge H(x,\epsilon))\}| \geq \frac{2}{3} \cdot 2^{37m \cdot (n(m+3)+2m)} \cdot$$

Thus, it is possible Lemma C.1 as a consequence of (\dagger) and (\ddagger) which, respectively, can be stated as Lemmas C.2 and C.3.

Lemma C.2 (Claim (†)). For every $z \in \mathbb{B}^{37m \cdot (n(m+3)+2m)}$, every encoding of a polynomial circuit x, every $y \in \mathbb{B}$, whereas x has n variables, size m and degree d, it holds that $G^v(x, n, d, z, 0) \to H(x, 0)$.

Proof. This result is a consequence of the compatibility of mod k with respect to addition, multiplication ad inverse in \mathbb{Z} . Precisely: for every $n, m, k \in \mathbb{Z}$ with $k \geq 2$, it holds that:

- 1. $(n \odot m) \mod k = ((n \mod k) \odot (m \mod k)) \mod k \text{ for } 0 \in \{+, \cdot\}.$
- $2. (-n) \mod k = -(n \mod k).$
- 3. $n \mod k \neq 0 \rightarrow n \neq 0$.

These claims are shown as follows:

1. In this case, we will only show the case for +, that of \cdot is analogous. The proof goes by induction on the recursion parameter, e.g. x. The case x=0 is trivial. For the inductive case, let x=ak+b and y=ck+d for b,d < k, the existence and uniqueness of this decomposition can be shown by induction on x. The IH tells that $(x \mod k + y \mod k) \mod k = b + d \mod k$.

$$(x+1 \mod k+y \mod k) \mod k = (x+1 \mod k+d) \mod k$$

= $(b+1 \mod k+d) \mod k$

Now, if b < k - 1, then $(b + 1 \mod k + d) \mod k = b + 1 + d \mod k = ak + b + 1 + ck + d \mod k = x + 1 + y \mod k$. If b = k - 1

$$(x+1 \mod k+y \mod k) \mod k = (d) \mod k$$

= $(b+1+ak+ck+d) \mod k$
= $x+1+y \mod k$

- 2. This proof goes by cases on n. The case where n=0 is trivial. The case n+1 relies on the uniqueness of the decomposition of n=ak+b, which allows us to show that $-((ak+b+1) \mod k) = -((b+1) \mod k)$. If $0 \le b < k-1$, then it is equal to -b-1, otherwise it is equal to 0. On the other hand, $(-ak-b-1) \mod k = (-b-1) \mod k$ and still, if $0 \le b < k-1$, then this is equal to -b-1, otherwise it is equal to 0.
- 3. The counter-nominal is trivial: $n = 0 \rightarrow n \mod k = 0$.

Leveraging points 1 and 2, we show that the evaluation of a polynomial x as performed by the predicate Eval on input \vec{r} is equal to $x(\vec{r}) \mod k$, the assumption $G^v(x, n, d, \vec{r}k, 0)$ allows us to conclude the premise of point 3, thus an application of point 3, allows us to conclude that $\mathbb{Z} \models p(\vec{x}) \neq 0$, by the definition of H, we conclude that H(x, 0) holds.

Lemma C.3 (Claim (‡)). For every encoding of a polynomial circuit x, every $y \in \mathbb{B}$, whereas x has n variables, size m and degree d, it holds that:

$$|\{z \in \mathbb{B}^{37m \cdot (n(m+3)+2m)} \mid G^v(x,n,d,z,\epsilon) \to H(x,\epsilon)\}| \ge \frac{2}{3} \cdot 2^{37m \cdot (n(m+3)+2m)}.$$

C.3 Proof of Lemma C.3

Lemma C.3 is shown finding an upper bound to the probability of error of the algorithm, i.e. to the number of values for z causing $G^v(x, n, d, z, \epsilon) \to H(x, \epsilon)$. Intuitively, there are two possible causes of error within the algorithm underlying the formula G.

- 1. if the outcome of the evaluation of the polynomial on \vec{r} is some value $y \neq 0$ and k divides y, then the algorithm will reject its input even though x belongs to the language.
- 2. if the values \vec{r} are a solution of the *non-identically zero* polynomial, then the algorithm will reject x, even though it belongs to the language.

The bound to the first error is found in section C.3.1, while a bound to the probability of the second error is found in section C.3.2. Lemma C.3 can be shown combining these results.

C.3.1 Estimation of the Error, Case 1

Proposition 11 (Argument in [3]). There is some $\varrho \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ greater than ϱ , every $0 \le y < 2^{(m+3) \cdot 2^m}$:

$$|\{k \in \{1, \dots, 2^{2m}\} |\}| k \text{ is not a prime not dividing } y\}| \le 2^{2m} - \frac{2^{2m}}{16m}$$

This probability depends on the range bounding k and on the number of evaluations taken in exam. The proof of this result relies on the following observations:

- (a) Thanks to the Prime Number Theorem, there is some m' such that, for every $m \ge m'$, the number of primes in $\{1, \ldots, 2^{2m}\}$ is at least $K = \frac{2^{2m}}{4 \cdot 2m}$.
- (b) For sufficiently big m, the number of primes in \mathbb{Z} dividing $0 \leq y < 2^{(m+3)\cdot 2^m}$ is smaller than $L = 3m \cdot 2^m$.
- (c) For every $n \in \mathbb{N}$, it holds that $3(2n+201) \leq 2^{n+96}$.
- (d) For every $m \ge 100$, said L the number of primes dividing y, it holds that $L \le \frac{K}{2} = \frac{2^{2m}}{16m}$. This is shown by induction using (c).

As we anticipated, points (a), (b), (d) hold only for sufficiently big values of m. For this reason, we define ϱ as the greatest of these values.

Point (a) We exploit here the fact, proved in [17], that the PNT is provable in $I\Delta_0 + \text{Exp}$; the formulation of the PNT can be paraphrased as follows:

$$\forall x \in \mathbb{N}. \forall q \in \mathbb{Q}^+. \exists a_x, z_x. z_q \le x \Rightarrow \left| \frac{\pi(x) \log_{a_x}^*(x)}{x} - 1 \right| \le q$$

Where $\pi(x)$ is the number of primes in $\{1, \ldots, x\}$ and $\log_a^*(x)$ is a good approximation of $\log_2(x)$, i.e. $\forall x. \log_2(x) \leq 2 \log_{a_x}^*(x) \leq 4 \log_2(x)$. This can be used to deduce that:

$$z_q \le x \Rightarrow \frac{x}{\log_{a_x}^*(x)} (1-q) \le \pi(x) \le \frac{x}{\log_{a_x}^*(x)} (1+q),$$

Thus, fixing $q = \frac{1}{2}$ we obtain the following claim:

$$z_q \le x \Rightarrow \frac{x}{4 \cdot \log_2(x)} \le \frac{x}{2 \log_{a_x}^*(x)} \le \pi(x)$$

We have shown that:

$$\forall x. \pi(x) = |\{p \in \{1, \dots, x\} \mid p \text{ is prime}\}| \ge \frac{x}{4 \cdot \log_2(x)}.$$

For sake of readability, we name $\Pi(x)$ the set $\{p \in \{1, ..., x\} \mid p \text{ is prime}\}.$

Point(b) To this aim, it suffices to observe that y is the product of q_1, \ldots, q_L primes where $L \leq |y| = \log(2^{(m+3)2^m})$. This is a consequence of the following version of the Fundamental Theorem of Arithmetic.

Theorem C.4. $|\Delta_0 + \text{Exp} \vdash \forall n \geq 2.prime(n) \lor \exists S, s.card(S, s) \land s \leq |n| \land \forall q \in S.prime(q) \land q|n.$

Proof. We observe that the predicates \in , |, prime, card can be easily modeled in the language of arithmetic. Then we go by induction on n to show the claim, observing that the formula $A(n) = prime(n) \lor \exists S, s. card(S, s) \land s \le |n| \land \forall q \in S. prime(q) \land q|n$ is Δ_1^0 . The base case is trivial since 2 is prime and can be divided only by 2, so the cardinality of S is smaller than |2|. For the inductive case suppose that n+1 is prime, in this case the claim is trivial, otherwise, if n+1 is not prime, there are $a,b \in \mathbb{N}$, ab=n+1, thus we can apply the IH on a,b, building S as the union of S_a,S_b .

Thus, $L \leq |y| = (3+m) \cdot 2^m = 3 \cdot 2^m + m \cdot 2^m \leq 3m \cdot 2^m$ — the last step is for $m \geq 1$, and is shown by induction. To sum up:

$$\forall 0 \le y \le 2^{(m+3)2^m}. |\{p \in \mathbb{Z} \mid q \text{ divides } y\}| \le 3m \cdot 2^m.$$

For simplicity's sake, we omit the proof of (c) and (d), which are standard by induction.

Proof of Proposition 11. From (d) we can deduce that $|\{p \in \Pi(2^{2m}) \mid p \text{ does not divide } y\}| > \frac{2^{2m}}{4 \cdot 2m}$. Thus, the claim is a trivial consequence of the following observation, which concludes the proof:

$$|\{1,\ldots,2^{2m}-1\}\setminus \{p\in\Pi(2^{2m})\mid p \text{ does not divide }y\}| \le 2^{2m}-|\{p\in\Pi(2^{2m})\mid p \text{ does not divide }y\}|$$

C.3.2 Estimation of Error, case 2

The goal of this section is to show that for every non-zero n-variate polynomial p of degree d and every set $S \subseteq \mathbb{Z}$, S contains sufficiently many witnesses of the fact that p is a non-zero polynomial.

Remark C.2. We here implicitly assume an encoding of polynomials as finite strings of coefficients, so that a unary polynomial $p(x) = \sum_{i=0}^{i < k} a_i x^i$ translates into (a suitable encoding of) the string $(a_0, a_1, \ldots, a_{k-1})$. Observe that the property " $\forall x.p(x) = q(x)$ " is decidable: once p and q are translated into strings of coefficients it is enough to check equality of these strings. For this reason we can suppose that all statements of the form " $\mathbb{Z} \models \forall x.p(x) = q(x)$ " or " $\mathbb{Z} \models \exists x.p(x) \neq q(x)$ " are encoded via Δ_1^0 -formulas.

Lemma C.5 (Schwartz-Zippel). For every n-variate polynomial p, for every $S \subseteq \mathbb{Z}$, said d the degree of p, it holds that:

$$|\{(x_1,\ldots,x_n)\in S^n\mid \mathbb{Z}\models p(x_1,\ldots,x_n)=0\}|\leq d|S|^{n-1}$$

The proof of this result is by induction on the number of variables and the degree of the polynomial p. It relies on a weak statement of the Fundamental Theorem of Algebra, proved below (Corollary 12), stating that each univariate polynomial has at most d roots in \mathbb{Z} , where d is the degree of the polynomial. We start by defining the notion of non-zero univariate polynomial:

Definition 37. We say that $p \in \mathsf{POLY}$ is a univariate *irreducible* polynomial if and only if it is univariate, and $\mathbb{Z} \models \forall x.p(x) \neq 0$. Moreover, we say that $p \in \mathsf{POLY}$ is a *non-zero* polynomial if and only if $\mathbb{Z} \models \exists x.p(x) \neq 0$.

This notion extends naturally to multivariate polynomials. The first step of the proof is to show that each univariate polynomial in \mathbb{Z} can be expressed as the product of $\prod_{i=0}^{i< k} (x-\overline{x}_i)q(x)$ for k smaller than the degree of d. This result relies itself on the proof of the correctness of the polynomial division algorithm, in particular on the following properties:

Remark C.3.

- 1. Let p be an univariate polynomial in normal form with coefficients in \mathbb{Z} and let \overline{x} be a solution of p, The division algorithm applied on $p, (x \overline{x})$ outputs a polynomial q with no remainder.
- 2. Let p be an univariate polynomial in normal form with coefficients in \mathbb{Z} and let r be a non-zero polynomial in normal form with degree smaller than that of p. If q is obtained with remainder 0 applying the division algorithm to p and q, it holds that rq = p.
- 3. The degree of r = p/q plus the degree of q is equal to the degree of p.

Although these results may seem very simple, for our purpose it is important to ensure that they can be established within $I\Delta_0 + \text{Exp}$. This is true, because the algorithm performing polynomial division is polytime and thus its totality, as well as its correctness, can be proved in $I\Delta_0 + \text{Exp}$.

Now, we want to prove that each uni-variate non-zero polynomial of degree d has at most d roots in \mathbb{Z} . We start by showing that it is always possible to express it as a product of simpler polynomials.

Lemma C.6. For every univariate polynomial p, if p has k distinct roots $\overline{x}_0, \ldots, \overline{x}_{k-1}$, then there is a polynomial q such that deg(q) = deg(p) - k and

$$\mathbb{Z} \models \forall x. p(x) = \prod_{i=0}^{i < k} (x - \overline{x}_i) q(x).$$

Proof. We want to show that for all polynomial p, degree d, natural number k < d and integers x_1, \ldots, x_d , if p is unary, has degree d and $\overline{x}_1, \ldots, \overline{x}_k$ are distinct roots of p, then there is a polynomial q of degree at most deg(p) - k such that $p(x) = \prod_{i=0}^{i < k} (x - \overline{x}_i)q(x)$. Using Remark C.2, this can be encoded as a formula of the form $\forall p. \forall d. \forall s. C(p, d)$, where C(p, d) only contains bounded quantifications, and s is a bound on the values of the roots \overline{x}_i . We will prove $\forall p. \forall d. \forall s. C(p, d)$ by induction on d:

- If d = 0, then the claim also holds trivially.
- If d > 0 and k = 0, then again the claim holds trivially. If k > 0, then polynomial division yields $p(x) \simeq (x \overline{x}_k)p'(x)$, with deg(p'(x)) = d 1 and $\overline{x}_1, \ldots, \overline{x}_{k-1}$ roots of p'(x). Then, by IH, we deduce C(p', d-1), that is, $p'(x) = \prod_{i=0}^{i < k-1} (x \overline{x}_i)q(x)$, and thus $p(x) = \prod_{i=0}^{i < k} (x \overline{x}_i)q(x)$ as desired.

Corollary 12. For every $S \subseteq \mathbb{Z}$ and for every non-zero univariate polynomial, it holds that:

$$|\{x \in S \mid \mathbb{Z} \models p(x) = 0\}| \le deg(p).$$

Proof of Lemma C.5. The proof is by induction on n:

- The claim for n = 1 coincides with Corollary 12.
- In the general case, we take the normal form of p, call that polynomial p'. It holds that

$$\mathbb{Z} \models \forall \vec{x}. p(\vec{x}) = p'(\vec{x}),$$

then we go by induction on d, we can factorize x_1 from each monomial of p and show that

$$\exists k. \mathbb{Z} \models \forall x_1, \dots, x_n. p'(x_1, \dots, x_n) = \sum_{i=0}^k x_1^i \cdot p_i(x_2, \dots, x_n)$$

Where p_k is non-zero. Applying the IH on d to p_k , we obtain that:

$$\{(x_2,\ldots,x_n)\in S^{n-1}\mid \mathbb{Z}\models p_k(x_2,\ldots,x_n)=0\}\le (d-k)|S|^{n-2}$$

Fix some $(y_2, \ldots, y_n) \in S^{n-1}$ and assume that $p_k(y_2, \ldots, y_n) \neq 0$. In this case, the polynomial $\overline{p}_{y_2,\ldots,y_n}(x) := \sum_{i=0}^k x^i \cdot p_i(y_2,\ldots,y_n)$ is not identically zero because it is a normal form and a normal form is identically zero if and only if all its coefficients are different from zero, but the coefficient of x^k is different from 0 for construction. Thus, we can apply the IH on n, showing that:

$$\forall (y_2, \dots, y_n) \in S^{n-1}. |\{x \in S \mid \mathbb{Z} \models \overline{p}_{y_2, \dots, y_n}(x) = 0\}| \le k, \quad (*)$$

We continue by observing that

$$\forall (y_2, \dots, y_n) \in S^{n-1} \{ x \in S \mid \mathbb{Z} \models \overline{p}_{y_2, \dots, y_n}(x) = 0 \} = \{ x \in S \mid \mathbb{Z} \models p(x, y_2, \dots, y_n) = 0 \},\$$

for the definition of \overline{p} . We can conclude that

$$\{(x_1,\dots,x_n)\in S^n\mid \mathbb{Z}\models p(x_1,\dots,x_n)=0\}=\\ \{(x_1,\dots,x_n)\in S^n\mid \mathbb{Z}\models p(x_1,\dots,x_n)=0\land p_k(x_2,\dots,x_n)=0\}\cup\\ \{(x_1,\dots,x_n)\in S^n\mid \mathbb{Z}\models p(x_1,\dots,x_n)=0\land p_k(x_2,\dots,x_n)\neq 0\}=\\ \{(x_1,\dots,x_n)\in S^n\mid \mathbb{Z}\models p(x_1,\dots,x_n)=0\land p_k(x_2,\dots,x_n)=0\}\cup\\ \{(x_1,\dots,x_n)\in S^n\mid \mathbb{Z}\models p(x_1,\dots,x_n)=0\land p_k(x_2,\dots,x_n)\neq 0\land \overline{p_{x_2,\dots,x_n}}(x_1)=0\}\subseteq\\ \{(x_1,\dots,x_n)\in S^n\mid \mathbb{Z}\models p_k(x_2,\dots,x_n)=0\}\cup\{(x_1,\dots,x_n)\in S^n\mid \mathbb{Z}\models \overline{p_{x_2,\dots,x_n}}(x_1)=0\}=\\ S\times\{(x_2,\dots,x_n)\in S^{n-1}\mid \mathbb{Z}\models \overline{p_{x_2,\dots,x_n}}(x_1)=0\}\cup\{(x_1,\dots,x_n)\in S^n\mid \mathbb{Z}\models \overline{p_{x_2,\dots,x_n}}(x_1)=0\}=\\ S\times\{(x_2,\dots,x_n)\in S^{n-1}\mid \mathbb{Z}\models \overline{p_{x_2,\dots,x_n}}(x_1)=0\}\cup\{(x_1,\dots,x_n)\in S^n\mid \mathbb{Z}\models \overline{p_{x_2,\dots,x_n}}(x_1)=0\}$$

This result can be lifted to sizes, obtaining the claim.

Proof of Lemma C.3. We omit the case for $|x| < \varrho$, since the conclusion is trivial. The claim is equivalent to the following one:

$$\left| \left\{ z = \begin{pmatrix} x_{1,1} & \dots & x_{n,1} & k_1 \\ x_{1,2} & \dots & x_{n,2} & k_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,37m} & \dots & x_{n,37m} & k_r \end{pmatrix} \in \left(\{0, 2^{m+3} - 1\}^n \times \{1, 2^{2m}\} \right)^{37m} \mid G^v(p, m, n, d, \epsilon, z, 37m) \wedge H(p, 0) \right\} \right| \leq \frac{1}{3} 2^{37m \cdot (n(m+3) + 2m)}$$

The set on the left is:

$$\left\{ \begin{pmatrix} x_{1,1} & \dots & x_{n,1} & k_1 \\ x_{1,2} & \dots & x_{n,2} & k_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,37m} & \dots & x_{n,37m} & k_r \end{pmatrix} \in \left(\{0, 2^{m+3} - 1\}^n \times \{1, 2^{2m}\} \right)^{37m} \mid \forall 1 \leq j \leq 48m. \\ \mathbb{Z} \models p(\vec{x}_j) = 0 \lor (\mathbb{Z} \models p(\vec{x}_j) \neq 0 \land k_j \text{ does divide } p(\vec{x}_j)) \right\}$$

Thus, it is equal to:

$$\{(x_1, \dots, x_n, k) \in \left(\{0, 2^{m+3} - 1\}^n \times \{1, 2^{2m}\}\right) \mid \mathbb{Z} \models p(\vec{x}) = 0 \lor (\mathbb{Z} \models p(\vec{x}) \neq 0 \land k \text{ does divide } p(\vec{x}))\}^{37m} \subseteq$$

$$(\{(x_1, \dots, x_n, k) \in \{0, 2^{m+3} - 1\}^n \times \{1, 2^{2m}\} \mid \mathbb{Z} \models p(\vec{x}) = 0\} \cup \{(x_1, \dots, x_n, k) \in \{0, 2^{m+3} - 1\}^n \times \{1, 2^{2m}\} \mid \mathbb{Z} \models p(\vec{x}) \neq 0 \land k \text{ does divide } p(\vec{x})\})^{37m}$$

Thus, its size is smaller than:

$$(|\{(x_1,\ldots,x_n,k)\in\{0,2^{m+3}-1\}^n\times\{1,2^{2m}\}\mid \mathbb{Z}\models p(\vec{x})=0\}|+ \\ |\{(x_1,\ldots,x_n,k)\in\{0,2^{m+3}-1\}^n\times\{1,2^{2m}\}\mid \mathbb{Z}\models p(\vec{x})\neq 0 \land k \text{ does divide } p(\vec{x})\}|)^{37m}.$$

Our goal is equivalent to showing the ratio between that value and $2^{37m \cdot (n(m+3)+2m)}$ is smaller than $\frac{1}{3}$. Which, in turn, resolves to:

$$\left(\frac{\left|\{(x_1,\ldots,x_n,k)\in\{0,2^{m+3}-1\}^n\times\{1,2^{2m}\}\mid\mathbb{Z}\models p(\vec{x})=0\}\right|}{2^{n(m+3)+2m}}+\frac{\left|\{(x_1,\ldots,x_n,k)\in\{0,2^{m+3}-1\}^n\times\{1,2^{2m}\}\mid\mathbb{Z}\models p(\vec{x})\neq 0\land k\text{ does divide }p(\vec{x})\}\right|}{2^{n(m+3)+2m}}\right)^{37m}\leq\frac{1}{3}$$

From Lemma C.5 and point (e), it is smaller than:

$$\left(\frac{2^m \cdot 2^{(m+3) \cdot (n-1)} \cdot 2^{2m}}{2^{n(m+3) + 2m}} + \frac{|\{(x_1, \dots, x_n, k) \in \{0, 2^{m+3} - 1\}^n \times \{1, 2^{2m}\} \mid \mathbb{Z} \models p(\vec{x}) \neq 0 \land k \text{ does divide } p(\vec{x})\}|}{2^{n(m+3) + 2m}} \right)^{37m} = \\ \left(\frac{1}{8} - h + \frac{|\{(x_1, \dots, x_n, k) \in \{0, 2^{m+3} - 1\}^n \times \{1, 2^{2m}\} \mid \mathbb{Z} \models p(\vec{x}) \neq 0 \land k \text{ does divide } p(\vec{x})\}|}{2^{n(m+3) + 2m}} \right)^{37m} \leq \frac{1}{8} - \frac{$$

Applying Proposition 11, we can find an upper bound to this value, which is:

$$\left(\frac{1}{8} - h + \left(1 - \frac{1}{8} + h\right) \frac{2^{n(m+3)+2m} - \frac{2^{n(m+3)+2m}}{16m}}{2^{n(m+3)+2m}} d\right)^{37m} =$$

$$\left(\frac{1}{8} - h + \left(\frac{7}{8} + h\right) \frac{2^{n(m+3)+2m} - \frac{2^{n(m+3)+2m}}{16m}}{2^{n(m+3)+2m}}\right)^{37m} =$$

$$\left(\frac{1}{8} - h + \left(\frac{7}{8} + h\right) \left(1 - \frac{1}{16m}\right)\right)^{37m} = \left(\frac{1}{8} - h + \frac{7}{8} - \frac{7}{8 \cdot 16m} + h - \frac{h}{16m}\right)^{37m} =$$

$$\left(1 - \frac{7}{8 \cdot 16m} - \frac{h}{16m}\right)^{37m} = \left(1 - \frac{1}{16m} \left(\frac{7}{8} + h\right)\right)^{37m} \le \left(1 - \frac{1}{\frac{8}{7}16m}\right)^{37m} \le$$

$$\le \left(1 - \frac{1}{\frac{8}{7}16m}\right)^{\frac{8}{7}16m} \cdot \left(1 - \frac{1}{\frac{8}{7}16m}\right)^{\frac{8}{7}16m} \le \frac{1}{4}$$

The very last observation is a consequence that $\forall n \geq 2$. $\left(1 + \frac{1}{n}\right)^n \geq 2$. This, in turn, is done expanding the binomial $\left(1 + \frac{1}{n}\right)^n$ and obtaining:

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^n \frac{n!}{(n-k)!k!} \frac{1}{n^k} = \sum_{k=0}^n \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n^k}\right) = 1 + 1 + \sum_{k=2}^n \frac{1}{k!} \prod_{i=1}^{k-1} \left(1 - \frac{i}{n^k}\right)$$

Where the derivation is justified by:

- The binomial expansion, namely: $\forall a, b, n. (a+b)^n = \sum_{k=0}^n \frac{n!}{k!(n-k)!} a^{n-k} b^k$. The proof is by induction on n. We will omit the details, which can be found in many text-books.
- $\forall k.k! \geq 2^k$. The proof is by induction. The base case is trivial, the inductive one is done as follows: $(k+1)! = k!(k+1) \geq 2^k(k+1)$. Now we go by cases on k: if it is 0, the claim is trivial, otherwise we observe that $(k+1) \geq 2$ and thus $2^k(k+1) \geq 2^{k+1}$.
- $\forall n. \sum_{k=1}^{n} 2^{-k} = 1 2^{-n}$, which is shown by induction on n. The base case is trivial. The inductive one is done as follows: $\sum_{k=1}^{n+1} 2^{-k} = \sum_{k=1}^{n} 2^{-k} + 2^{-n-1} = 1 2^{-n} + 2^{-n-1} = 1 2^{-n-1}$.

These proof are completely syntactic thus, modulo an encoding for rational numbers and for their operations, they can be done in PA without much effort. The proof proceeds observing that

$$\left(1+\frac{1}{n}\right)^n = \left(1+\frac{1}{n}\right)^{-1\cdot -1\cdot n} = \left(\frac{n}{n+1}\right)^{-n} = \left(1-\frac{1}{n}\right)^{-n}.$$

Therefore, $\forall n \geq 4$. $\left(1 - \frac{1}{n}\right)^n \leq \frac{1}{2}$, and so:

$$\left(1 - \frac{1}{\frac{8}{7}16m}\right)^{\frac{8}{7}16m} \cdot \left(1 - \frac{1}{\frac{8}{7}16m}\right)^{\frac{8}{7}16m} \le \frac{1}{4}.$$

As we have shown, even last result can be proved in PA, using an encoding of finite sets and standard arithmetic on \mathbb{N} .

C.4 Closure of BPP_T under polytime reduction

We assume T is any theory containing $R\Sigma_1^b$ -NIA + Exp. The statement of Proposition 1 can be given more precisely as:

Proposition 12. For every language $L \in \mathbf{BPP_T}$ and every language $M \in \mathbb{B}^*$, if there is an polytime function $r_{M,L} : \mathbb{S} \longrightarrow \mathbb{S}$, such that for every string in $\sigma \in \mathbb{S}$, $x \in M \leftrightarrow r(x) \in L$, then M is in $\mathbf{BPP_T}$.

Proof. Assume that $L \in \mathbf{BPP_T}$, and let G_L be the Σ_1^b \mathcal{RL} formula characterizing L as required in Definition 10. Since $r_{M,L}$ is poly-time, there is a Σ_1^b Flip-free formula of \mathcal{RL} R characterizing $r_{M,L}$ as well. This has a consequence that the formula $C(x,y) := \exists w \leq t(x).R(x,w) \land G_L(w,y)$ characterizes the composition of the reduction $r_{M,L}$ and the function χ_L . The function C(x,y) is still a Σ_1^b formula, and characterizes a function deciding M due to the hypothesis on the correctness of the reduction $r_{M,L}$. Form this conclusion and $\operatorname{Lang}(G_L) = L$, we deduce $\operatorname{Lang}(C) = M$, we only need to show point (2) of Definition 10 for C:

$$\begin{split} &\mathsf{T} \vdash \forall x. \exists ! y. \mathsf{TwoThirds}[C] \\ &\mathsf{T} \vdash \forall x. \exists ! y. \mathsf{TwoThirds}[\exists w \leq t(x). R(x,w) \land G_L(w,y)] \\ &\mathsf{T} \vdash \forall x. \exists ! y. \mathsf{Threshold}[\mathsf{NoFlip}[\exists w \leq t(x). R(x,w) \land G_L(w,y)]] \\ &\mathsf{T} \vdash \forall x. \exists w \leq t(x). R(x,w) \land \exists ! y. \mathsf{Threshold}[\mathsf{NoFlip}[G_L(w,y)]] \end{split}$$

This is a consequence of a Σ_1^b Flip-free formula of \mathcal{RL} and the hypothesis on G_L .