Monotonic References for Gradual Typing

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1 Introduction

We describe an alternative approach to handling mutable references (aka. pointers) within a gradually typed language that has different efficiency characteristics than the prior approach of Herman et al. [2010]. In particular, we hope to reduce the costs of reading and writing through references in statically typed regions of code. We would like the costs to be the same as they would in a statically typed language, that is, simply the cost of a load or store instruction (for primitive data types). This reduction in cost is especially important for programmers who would like to use gradual typing to facilitate transitioning from a dynamically-typed prototype of an algorithm to a statically-typed, high-performance implementation. The programmers we have in mind are scientists and engineers who currently prototype in Matlab and then manually translate their algorithms into Fortran.

While our alternative approach succeeds in improving the efficiency of dereference and updates in statically typed code, it does come with some limitations. The approach requires all heap-allocated values to be tagged with their runtime type, which may be an added cost in space (though many languages require such tags for other reasons). In addition, our approach requires all values to be of a uniform size, which is true for many languages (most functional and object-oriented languages) but not true for some (the C family of languages). Finally, our approach is more restrictive than prior ones in that certain usage patterns are not allowed, triggering runtime exceptions, for which we give examples later in this introduction.

The source of inefficiency in the prior approach of Herman et al. [2010], which we refer to as "guarded references", is that two kinds of values have reference type: normal references and guarded references. A guarded reference consists of the underlying reference (a memory address) and two coercions, one to apply when reading and another to apply when writing. A guarded reference is created during runtime when a normal reference is casted from one reference type to another. When a compiler for a language

with guarded references generates code for a dereference or an update, the compiler must emit code to dispatch on the kind of reference. Consider the following function f and two calls to it.

```
\begin{split} & \mathsf{let}\, f = \lambda x : \mathsf{Ref}\, \mathsf{Int.}\, !x \; \mathsf{in} \\ & f(\mathsf{ref}_{\mathsf{Int}}\, 4); \\ & f(\mathsf{ref}_{\star}\, (\mathsf{true}: \mathsf{Bool} \Rightarrow \star) : \mathsf{Ref}\, \star \Rightarrow \mathsf{Ref}\, \mathsf{Int}) \end{split}
```

In the first call to f, a normal reference to an integer flows into the dereference !x whereas in the second call, a guarded reference flows into the dereference !x. The code generated for the dereference in the body of f needs to be general enough to handle both situations.

Our new approach, called "monotonic references", has only one kind of value at reference type, normal references. When a cast is applied to a reference, instead of turning the reference into a guarded reference, we cast the underlying value on the heap, so long as the cast is to an equally or less dynamic type. Otherwise the cast results in a runtime error. Thus, the heap is allowed to change monotonically with respect to the less-dynamic relation. This relation is formally defined in Section 2, but roughly speaking, the fewer occurences of the dynamic type, the less dynamic a type is. Swamy et al. [2014] have independently developed a similar idea, though in their formulation, the monotonicity is with respect to subtyping instead of the less dynamic relation, which reflects a difference in goals compared to this work (security vs. efficiency).

In the above example, a runtime error would be triggered by the cast from Ref \star to Ref Int because it would attempt to cast the Boolean true to Int and fail. On the other hand, the following example in which an integer is passed into f in both calls, would terminate without error.

```
\begin{split} & \mathsf{let}\, f = \lambda x : \mathsf{Ref}\,\mathsf{Int.}\,!x \;\mathsf{in} \\ & f(\mathsf{ref}_{\mathsf{Int}}\,4); \\ & f(\mathsf{ref}_{\star}\,(4:\mathsf{Int} \Rightarrow \star) : \mathsf{Ref}\,\star \Rightarrow \mathsf{Ref}\,\mathsf{Int}) \end{split}
```

In this example, just prior to the second call to f, the cast from Ref \star to Ref Int would cause the heap cell containing $4: \text{Int} \Rightarrow \star$ to be updated with that value cast to Int.

In general, monotonic references maintain the invariant that the type of a value in the heap is at less dynamic than the type of any reference that points to that value. Thus, if a reference has a fully static type, such as Ref Int, the corresponding value on the heap must be an actual integer (and not an injection to \star). If a reference does not have a fully static type, then the corresponding value on the heap might be less dynamic, and a cast needs to be performed during a read or write to mediate between the value's type and the reference's type. However, these two situations can be distinguished during compilation, based on whether the reference expression

in the dereference or update has a fully static type or not. Thus, we can generate efficient code for the first case and less efficient code in the second case.

The behavior of monotonic references is rather different than that of guarded references. We conjecture that monotonic references are more picky than guarded references, that is, they trigger runtime errors strictly more often than guarded references. Here we show two examples of this phenomenon. In this first example, we cast a reference of type Ref \star to both Ref Int and Ref Bool. We dereference at Ref Int then write and subsequently read at Ref Bool.

$$\begin{split} & \mathsf{let} \ x = \mathsf{ref}_{\star} \ (4 : \mathsf{Int} \Rightarrow \star) \ \mathsf{in} \\ & \mathsf{let} \ y = x : \mathsf{Ref} \ \star \Rightarrow \mathsf{Ref} \ \mathsf{Int} \ \mathsf{and} \\ & z = x : \mathsf{Ref} \ \star \Rightarrow \mathsf{Ref} \ \mathsf{Bool} \ \mathsf{in} \\ & ! y; z := \mathsf{true}; ! z \end{split}$$

With guarded references, the above program terminates without error whereas with monotonic references, it halts with an error during the cast from Ref \star to Ref Bool. Of course, if we re-order the sequence of operations so that the write of the Boolean occurs before the read of the integer, then the guarded references approach triggers an error on the read from y.

$$\ldots z := \mathsf{true}; !y; !z$$

It is worth emphasizing that when a cast on a reference causes a value on the heap to be changed, the change is rather permanent. Consider the following example in which a function f takes a reference of type Ref \star and casts it to Ref Int. The caller passes in a reference to an injected integer, which works fine, but then after the call, tries to write an injected Boolean to the reference.

$$\begin{split} & \operatorname{fun} f(y:\operatorname{Ref} \star) = \\ & \operatorname{let} z = (y:\operatorname{Ref} \star \Rightarrow \operatorname{Ref Int})\operatorname{in} !z \\ & \operatorname{let} x = \operatorname{ref}_\star (4:\operatorname{Int} \Rightarrow \star)\operatorname{in} \\ & f x; \\ & x := (\operatorname{true} :\operatorname{Bool} \Rightarrow \star) \end{split}$$

With guarded references, the above program terminates without error, whereas with monotonic references, the write to x triggers an error.

One interesting challenged in defining the dynamic semantics of monotonic references is that references may form cycles and we need to make sure that a cast applied to a reference that is in a cyle does not cause the program to diverge. Consider the following example in which we create a pair whose second element is a reference back to itself.

```
\begin{split} \operatorname{let} r_1 &= \operatorname{ref}_{\operatorname{Int} \times \star} \left( \langle 42, 0 : \operatorname{Int} \Rightarrow \star \rangle \right) \operatorname{in} \\ r_1 &:= \langle 42, r_1 : \operatorname{Ref}(\operatorname{Int} \times \star) \Rightarrow \star \rangle; \\ \operatorname{let} r_2 &= r_1 : \operatorname{Ref}(\operatorname{Int} \times \star) \Rightarrow \operatorname{Ref}(\operatorname{Int} \times \operatorname{Ref} \star) \operatorname{in} \\ \operatorname{fst}(!r_2) \end{split}
```

Once the pair with the cycle is created, we cast the reference to it from type $Ref(Int \times \star)$ to $Ref(Int \times Ref \star)$. The correct result of this program is 42 but in a naive dynamic semantics this program would diverge. Our semantics avoids divergence by checking whether the new type for a heap cell is no less dynamic than the old type; in such cases the heap cell is left unchanged.

The rest of this paper gives a formal definition of the static and dynamic semantics for monotonic references and proves type safety. The formal setting for this work is in an intermediate language that is an extension of the simply-typed lambda calculus with casts, a dynamic type, and mutable references. It is straightforward using standard techniques to compile from a gradually-typed source language to this intermediate language.

2 Types

For the purposes of studying monotonic references, the types of our language consist of integers, Booleans, functions, pairs, references, and the dynamic type.

types
$$A, B, C, D ::=$$
Int | Bool | $A \rightarrow B \mid A \times B \mid$ Ref $A \mid \star$

The less or equally dynamic relation on types is defined by the following equations for its characteristic function. (This relation is also known as naive subtyping.)

```
\begin{array}{lll} A\sqsubseteq \star & = & True \\ \mathit{Int}\sqsubseteq \mathit{Int} & = & True \\ \mathit{Bool}\sqsubseteq \mathit{Bool} & = & True \\ A\times B\sqsubseteq C\times D & = & A\sqsubseteq C\wedge B\sqsubseteq D \\ A\to B\sqsubseteq C\to D & = & A\sqsubseteq C\wedge B\sqsubseteq D \\ (\mathit{Ref}\ A)\sqsubseteq (\mathit{Ref}\ B) & = & A\sqsubseteq B \end{array}
```

In all other cases, the less or equally dynamic function returns False.

Lemma 1. $A \sqsubseteq A$

Lemma 2. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

The meet function on types is defined below. (This corresponds to the meet operator of Siek and Wadler [2010].) Many of the function definitions in this development use monadic notation in which the combination of := and semicolon serve as the notation for the bind operation.

```
\begin{array}{lll} \star \sqcap A & = return \ A \\ A \sqcap \star & = return \ A \\ \textit{Int} \sqcap \textit{Int} & = return \ \textit{Int} \\ \textit{Bool} \sqcap \textit{Bool} & = return \ \textit{Bool} \\ A \times B \sqcap C \times D & = A' := A \sqcap C; \ B' := B \sqcap D; \ return \ A' \times B' \\ A \rightarrow B \sqcap C \rightarrow D & = A' := A \sqcap C; \ B' := B \sqcap D; \ return \ A' \rightarrow B' \\ \textit{Ref} \ A \sqcap \textit{Ref} \ B & = A' := A \sqcap B; \ return \ \textit{Ref} \ A' \end{array}
```

In all other cases, the meet function returns a cast error.

Lemma 3. If
$$A \cap B = C$$
 then $C \subseteq A \wedge C \subseteq B$.

We say that a type is "static" if it does not contain the dynamic type.

$$\begin{array}{lll} static \; \star & = \; False \\ static \; Int & = \; True \\ static \; Bool & = \; True \\ static \; (A \times B) & = \; static \; A \wedge static \; B \\ static \; (A \rightarrow B) & = \; static \; A \wedge static \; B \\ static \; (Ref \; A) & = \; static \; A \end{array}$$

Static types are the least dynamic.

Lemma 4 (Static is Least Dynamic). If static A and $B \sqsubseteq A$ then A = B.

3 Association Lists and Type Environments

We represent environments, type environments, and heap typings as association lists.

```
lookup \ x \ [] = stuck \\ lookup \ x \ ((y, \ v) \cdot bs) = if \ x = y \ then \ return \ v \ else \ lookup \ x \ bs
```

The domain of an association list is the set keys.

$$dom A \equiv map fst A$$

A heap typing is less or equally dynamic as another heap typing if each of its components are.

$$\Sigma' \sqsubseteq \Sigma \equiv dom \ \Sigma = dom \ \Sigma' \land (\forall a \ A. \ lookup \ a \ \Sigma = A \longrightarrow (\exists B. \ lookup \ a \ \Sigma' = B \land B \sqsubseteq A))$$

This ordering relation is transitive.

Lemma 5 (Transitive). If $\Sigma' \sqsubseteq \Sigma''$ and $\Sigma \sqsubseteq \Sigma'$ then $\Sigma \sqsubseteq \Sigma''$.

Typing rules for expressions

$$\frac{lookup \ x \ \Gamma = A}{\Gamma \vdash x : A} \qquad \Gamma \vdash c : typeof \ c$$

$$\frac{typeof \cdot opr \ f = A \to B \qquad \Gamma \vdash e : A}{\Gamma \vdash f(e) : B} \qquad \frac{\Gamma \vdash e : A}{\Gamma \vdash (e1, e2) : A \times B}$$

$$\frac{(x, A) \cdot \Gamma \vdash s : B}{\Gamma \vdash \lambda x : A \cdot s : A \to B} \qquad \frac{\Gamma \vdash e : Ref \ A \qquad static \ A}{\Gamma \vdash !e : A}$$
Typing rules for statements
$$\frac{\Gamma \vdash e : A \qquad (x, A) \cdot \Gamma \vdash s : B}{\Gamma \vdash let \ x = e \ in \ s : B} \qquad \frac{\Gamma \vdash e : A}{\Gamma \vdash return \ e : A}$$

$$\frac{\Gamma \vdash e : A \to B \qquad \Gamma \vdash e' : A \qquad (x, B) \cdot \Gamma \vdash s : C}{\Gamma \vdash let \ x = e(e') \ in \ s : C}$$

$$\frac{\Gamma \vdash e : A \to B \qquad \Gamma \vdash e' : A}{\Gamma \vdash return \ e(e') : B}$$

$$\frac{\Gamma \vdash e : A \qquad (x, Ref \ A) \cdot \Gamma \vdash s : B}{\Gamma \vdash let \ x = ref \ A \ e \ in \ s : B}$$

$$\frac{\Gamma \vdash e : Ref \ A \qquad \Gamma \vdash e' : A \qquad \Gamma \vdash s : B}{\Gamma \vdash e : e : (@A; \ s : B)}$$

$$\frac{\Gamma \vdash e : Ref \ A \qquad \Gamma \vdash e' : A \qquad \Gamma \vdash s : B}{\Gamma \vdash let \ x = e : A \Rightarrow B \ in \ s : C}$$

$$\frac{\Gamma \vdash e : Ref \ A \qquad (x, B) \cdot \Gamma \vdash s : C}{\Gamma \vdash let \ x = e : A \Rightarrow B \ in \ s : C}$$

Figure 1: Typing rules for expressions and statements

4 Type Sytem

Figure 1 defines the typing rules for expressions and statements. The separation of expressions and statements achieves a kind of A-normal form that we found convenient for the purposes of proving type safety. In particular, it avoids the need for evaluation contexts. The expressions include only trivially terminating operations that do not update the heap.

There are two syntactic forms for reference update and dereference, respectively. One pair of forms requires the reference type to be "static" and the other pair includes a type annotation that records the type of the reference. We refer to these later forms as the "dynamic" version of reference update and dereference. The dynamic forms use the type annotation to cast from the runtime type of a value on the heap to the annotated type of the reference. The dynamic dereference is a statement and not an expression because it performs a cast which is a side-effecting operation.

Figure 2 defines the typing rules for run-time structures such as values, environments, stacks, heaps, and states. Most of these typing rules are straightforward and only a few require comment. The typing rule for references allows the type of the reference to be more dynamic than the type in the heap. Our heaps are unusual in that they do not only store values but sometimes also store casted values. We require the values and casted values in a well-typed heap to have the types given by the heap typing. Also, only those addresses in the active address list may contain casted values. The rest must contain (uncasted) values. The typing rule for casted values requires the target of the cast to be at less dynamic than the source, reflecting the invariant that the heap is only allowed to become less dynamic.

Variable lookup always succeeds in well-typed environments.

```
Lemma 6 (Lookup Safety). If \Gamma; \Sigma \vdash \varrho and lookup x \Gamma = A then \exists v. lookup x \varrho = v \land \Sigma \vdash v : A.
```

We can weaken values and environments with respect to the typing environment for term variables.

```
Lemma 7 (Weaken Values).

If \Sigma \vdash v : A and a \notin dom \Sigma then (a, B) \cdot \Sigma \vdash v : A.

Lemma 8 (Weaken Environments).

If \Gamma; \Sigma \vdash \varrho and a \notin dom \Sigma then \Gamma; (a, B) \cdot \Sigma \vdash \varrho.
```

We can strengthen values and environments with respect to the typing of the heap because the typing rule for addresses allows the heap-type to be less dynamic than the static type of the reference.

```
Lemma 9 (Strengthen Values). If \Sigma \vdash v : A and \Sigma' \sqsubseteq \Sigma then \Sigma' \vdash v : A.
```

Typing rules for values

$$\frac{typeof\ c = A}{\Sigma \vdash c : A} \qquad \frac{\Sigma \vdash v : A \qquad \Sigma \vdash v' : B}{\Sigma \vdash \langle v, v' \rangle : A \times B}$$

$$\frac{\Gamma; \Sigma \vdash \varrho \qquad (x, A) \cdot \Gamma \vdash s : B}{\Sigma \vdash \langle \lambda x : A . s, \ \varrho \rangle : A \to B} \qquad \frac{lookup\ a\ \Sigma = A \qquad A \sqsubseteq B}{\Sigma \vdash ref\ a : Ref\ B}$$

$$\frac{\Sigma \vdash v : A}{\Sigma \vdash [v : A \Rightarrow \star] : \star}$$

Typing rules for environments

$$[];\Sigma \vdash [] \qquad \frac{\Sigma \vdash v : A \qquad \Gamma;\Sigma \vdash \varrho}{(x, A) \cdot \Gamma;\Sigma \vdash (x, v) \cdot \varrho}$$

Typing rules for procedure call stacks

$$\begin{array}{cccc} \Sigma \vdash []:A \Rightarrow A \\ \\ \underline{\Gamma; \Sigma \vdash \varrho} & (x,A) \cdot \Gamma \vdash s:B & \Sigma \vdash k:B \Rightarrow C \\ \hline & \Sigma \vdash (x,s,\varrho) \cdot k:A \Rightarrow C \end{array}$$

Typing rules for casted values

$$\frac{\Sigma \vdash v : A}{\Sigma \vdash val \ v : A} \qquad \frac{\Sigma \vdash v : A \qquad B \sqsubseteq A}{\Sigma \vdash v : A \Rightarrow B : B}$$

Well-typed heaps

$$\Sigma \vdash \mu \mid as \equiv (\forall \ a \ A. \ lookup \ a \ \Sigma = A \longrightarrow (\exists \ cv. \ lookup \ a \ \mu = (cv, \ A) \land \Sigma \\ \vdash cv : A \land (a \notin as \longrightarrow (\exists \ v. \ cv = val \ v)))) \land (\forall \ a. \ a \in dom \ \mu \longrightarrow a < |\mu|) \land as \subseteq dom \ \Sigma$$

Well-typed states

$$\frac{\Sigma \vdash \mu \mid as \qquad \Gamma; \Sigma \vdash \varrho \qquad \Gamma \vdash s : A \qquad \Sigma \vdash k : A \Rightarrow B}{\vdash (s, \rho, k, \mu, as) : B}$$

Figure 2: Typing rules for run-time structures.

```
Lemma 10 (Strengthen Environments). If \Gamma; \Sigma \vdash \varrho and \Sigma' \sqsubseteq \Sigma then \Gamma; \Sigma' \vdash \varrho.
```

We can weaken and strengthen stacks as well.

If $\Sigma \vdash k : A \Rightarrow B$ and $\Sigma' \sqsubseteq \Sigma$ then $\Sigma' \vdash k : A \Rightarrow B$.

Lemma 11 (Weaken Stacks). If $\Sigma \vdash k : A \Rightarrow B$ and $a \notin dom \Sigma$ then $(a, T) \cdot \Sigma \vdash k : A \Rightarrow B$. **Lemma 12** (Strengthen Stacks).

Lemma 13 (Strengthen Casted Values). If $\Sigma \vdash cv : A$ and $\Sigma' \sqsubseteq \Sigma$ then $\Sigma' \vdash cv : A$.

One of the defining aspects of monotonic references is that the semantics performs strong updates on the heap. However, we only perform updates that make the types less dynamic. The following lemma shows that we can perform such updates on well-typed heaps and obtain well-typed heaps. We make use of the following auxilliary function in the statement of the lemma.

$$cval\text{-}ads (val \ v) \ a \ ads = ads - \{a\}$$

 $cval\text{-}ads (v:A\Rightarrow B) \ a \ ads = ads \cup \{a\}$

Lemma 14 (Update Heap). If $\Sigma \vdash \mu \mid ads \ and \ lookup \ a \ \Sigma = A \ and \ B \sqsubseteq A$ and $\Sigma \vdash cv : B \ then \ (a, B) \cdot \Sigma \vdash (a, cv, B) \cdot \mu \mid cval\text{-}ads \ cv \ a \ ads.$

The proof of this lemma relies on the above strengthening lemmas.

5 Dynamic Semantics and Type Safety

The following defines the primitive operators.

```
\begin{array}{lll} \delta \ \textit{succ} \ ((\ n)) & = \ \textit{return} \ (\ n+1) \\ \delta \ \textit{prev} \ ((\ n)) & = \ \textit{return} \ (\ n-1) \\ \delta \ \textit{zero?} \ ((\ n)) & = \ \textit{return} \ (\ (n=0)) \\ \delta \ (\textit{fst} \ A \ B) \ (\langle v, v' \rangle) & = \ \textit{return} \ v \\ \delta \ (\textit{snd} \ A \ B) \ (\langle v, v' \rangle) & = \ \textit{return} \ v' \end{array}
```

Lemma 15 (Delta Safety). If typeof-opr $f = A \rightarrow B$ and $\Sigma \vdash v : A$ then $\exists v'. \delta f v = v' \land \Sigma \vdash v' : B$.

The evaluation function uses the following auxilliary function to obtain the address from a reference

$$to\text{-}addr \ (\text{ref } a) = return \ a$$

 $to\text{-}addr \ v = stuck \qquad \text{if } \nexists \ a. \ v = \text{ref } a$

and it uses the below function to extract a value from a potentially-casted value.

$$to\text{-}val\ (val\ v) = return\ v$$

 $to\text{-}val\ (v:A\Rightarrow B) = stuck$

The following is the evaluation function for expressions. The first of the two main accomplishments of monotonic references is that the below equation for dereference is standard (with respect to statically typed languages), that is, it does not need to dispatch based on the kind of reference.

Lemma 16 (Evaluation Safety). If $\Gamma \vdash e : A$ and $\Gamma; \Sigma \vdash \varrho$ and $\Sigma \vdash \mu \mid \emptyset$ then $\exists v. \llbracket e \rrbracket \varrho \ \mu = v \land \Sigma \vdash v : A$.

The case for variables relies on Lookup Safety (Lemma 6) and the case for primitive operators relies on Delta Safety (Lemma 15). The case for dereference makes use of the well-typed heap and that static types are least dynamic (Lemma 4). Expressions may only be safely evaluated when the set of active addresses is empty.

We wrap a cast around a function in the following way. We use integers for variables (but not De Bruijn notation) which makes the following somewhat difficult to read.

wrap
$$v \ A \ B \ C \ D \equiv \langle \lambda 0 : C.(let \ 3=0 : C \Rightarrow A \ in \ (let \ 2=1(3) \ in \ (let \ 4=2 : B \Rightarrow D \ in \ (return \ 4)))), \ [(1,\ v)] \rangle$$

The following auxilliary function creates a casted value that can be stored in the heap.

$$mk$$
- $vcast$ $(val \ v) \ A \ B = v:A \Rightarrow B$
 mk - $vcast$ $(v:A \Rightarrow B) \ C \ D = v:A \Rightarrow D$

The cast function is defined below. We discuss the particulars of this definition in the following paragraph.

```
cast v Int Int \mu as = return (v, \mu, as)
cast \ v \ Bool \ Bool \ \mu \ as = return \ (v, \ \mu, \ as)
cast \ v \star \star \mu \ as = return \ (v, \mu, as)
cast\ v\ (A \to B)\ (C \to D)\ \mu\ as = return\ (wrap\ v\ A\ B\ C\ D,\ \mu,\ as)
cast\ (\langle v1, v2 \rangle)\ (A \times B)\ (C \times D)\ \mu\ as = return\ (\langle v1', v2' \rangle,\ \mu2,\ as2)
     if cast v1 A C \mu as = return (v1', \mu1, as1)
     and cast v2 B D \mu 1 as 1 = return (v2', \mu2, as 2)
cast (ref a) (Ref A) (Ref B) \mu as = return (ref a, \mu, as)
     if lookup a \mu = return (cv, C), B \cap C = return D, C \subseteq D
cast (ref a) (Ref A) (Ref B) \mu as = return (ref a, (a, cv', D)·\mu, a·as)
     if lookup a \mu = return\ (cv,\ C),\ B \sqcap C = return\ D, \neg\ C \sqsubseteq D,
     and cv' = mk\text{-}vcast\ cv\ C\ D
cast\ (ref\ a)\ (Ref\ A)\ (Ref\ B)\ \mu\ as = cast\text{-}error
     if lookup a \mu = return(cv, C), B \sqcap C = cast\text{-}error
cast ([v:A \Rightarrow \star]) \star B \mu \ as = cast \ v \ A \ B \mu \ as
     if B \neq \star, ground A = ground B
cast ([v:A \Rightarrow \star]) \star B \mu \ as = cast\text{-}error
     if B \neq \star, ground A \neq ground B
cast\ ([v:A\Rightarrow\star]) \star B\ \mu\ as = cast\text{-}error
     if B \neq \star, ground A = ground B, cast v A B \mu as = cast-error
cast \ v \ A \star \mu \ as = return \ ([v:A \Rightarrow \star], \ \mu, \ as)
     if A \neq \star
```

In the remaining cases, the result is a cast error.

The case for casting references is the most important to this development and is rather subtle. The main idea is that the value at address a is cast from its current type C to the meet of its current type and the target type of the cast. This cast is accomplished by storing a so-called "casted value" on the heap and returning address a in the list of active addresses (the addresses with pending casts to be performed). One extra wrinkle in the definition of casting a reference is that there may be cycles in the heap. To guard against infinite loops and to improve efficiency, we leave the heap unchanged if the heap type is already less or equally dynamic than the target type of the cast.

The statement of Cast Safety, given below, is rather complex. Given a well-typed value and heap, the result of a cast is either a cast error or a value whose type is the target type and a new heap and active address list. The heap is well-typed in some heap typing that is less dynamic than the typing of the original heap.

Lemma 17 (Cast Safety). If $\Sigma \vdash v : A$ and $\Sigma \vdash \mu \mid ads1$ then $(\exists v' \Sigma' \mu' ads2. cast v A B \mu ads1 = (v', \mu', ads2) \land \Sigma' \vdash v' : B \land \Sigma' \vdash \mu' \mid ads2 \land \Sigma' \sqsubseteq \Sigma) \lor cast v A B \mu ads1 = cast-error.$

The following defines the transitions of the abstract machine. The transitions are defined in terms of a step function but we use the following abbreviation.

$$s \longmapsto s' \equiv step \ s = return \ s'$$

The first four transition rules, listed below, process addresses in the active address list. If the address points to a value, then we can simply remove that address from the active list. If the address points to a casted value, then we need to perform the cast. If successful, the cast produces a new value v', an udpated heap, and a list of addresses that have become active. If the current heap type for a is still the same, then we commit the result of this cast, installing v' at address a. If the heap type has changed, then this cast has been superceded by some other cast, so we do not commit v'. Finally, the cast was unsuccessful, execution halts with a cast error.

$$\frac{lookup\ a\ \mu = return\ (cv,\ A) \qquad cv = val\ v}{(s,\ \varrho,\ k,\ \mu,\ a\cdot as) \longmapsto (s,\ \varrho,\ k,\ \mu,\ as)}$$

$$lookup\ a\ \mu = return\ (cv,\ A)$$

$$cv = v:B \Rightarrow C \qquad cast\ v\ B\ C\ \mu\ (a\cdot as) = return\ (v',\ \mu',\ as')$$

$$lookup\ a\ \mu' = return\ (cv',\ A') \qquad A\ \sqsubseteq A'$$

$$(s,\ \varrho,\ k,\ \mu,\ a\cdot as) \longmapsto (s,\ \varrho,\ k,\ (a,\ val\ v',\ A)\cdot\mu',\ removeAll\ a\ as')$$

$$cv = v:B \Rightarrow C \qquad cast\ v\ B\ C\ \mu\ (a\cdot as) = return\ (v',\ \mu',\ as')$$

$$lookup\ a\ \mu' = return\ (cv',\ A') \qquad \neg\ A\ \sqsubseteq\ A'$$

$$(s,\ \varrho,\ k,\ \mu,\ a\cdot as) \longmapsto (s,\ \varrho,\ k,\ \mu',\ as')$$

$$lookup\ a\ \mu = return\ (cv,\ A)$$

$$cv = v:B \Rightarrow C \qquad cast\ v\ B\ C\ \mu\ (a\cdot as) = cast\text{-}error$$

$$step\ (s,\ \varrho,\ k,\ \mu,\ a\cdot as) = cast\text{-}error$$

The transitions for allocation and "static" reference updates are mostly standard. Note that each value on the heap is paired with its type. The fact that static update is completely standard (with respect to statically-typed languages) is the second of the two main achievements of monotonic references.

The "dynamic" dereference and update transitions perform casts to mediate between the reference's type and the type on the heap.

The transition for cast statements is straightforward; all the hard work is all carried out by the auxilliary cast function.

The transition rules for let and for function call and return are standard.

All other states are mapped to stuck.

Lemma 18. If $\vdash s : A$ then final $s \lor (\exists s'. step \ s = s' \land \vdash s' : A) \lor step \ s = cast-error.$

The proof proceeds by cases on the active address list (empty or not) and then by cases on the statement component of the state. The proof is long but straightforward given the above lemmas and the invariants captured in the definition of a well-typed state.

The following function maps values to observables.

```
\begin{array}{lll} observe \ (c) & = \ Con \ c \\ observe \ (\langle v,v'\rangle) & = \ OPair \ (observe \ v') \ (observe \ v') \\ observe \ (\langle \lambda x : T.s, \ \varrho \rangle) & = \ Fun \\ observe \ (\textit{ref } a) & = \ Addr \\ observe \ ([v : T \Rightarrow \star]) & = \ Inj \end{array}
```

Well-typed observables:

```
 \begin{array}{lll} \vdash OPair\ o'\ o'': A \times B &= \ \vdash o': A \wedge \vdash o'': B \\ \vdash Fun: A \to B &= True \\ \vdash Con\ c: T &= typeof\ c = T \\ \vdash OStuck: T &= False \\ \vdash OTimeOut: T &= True \\ \vdash OCastError: T &= True \\ \vdash Addr: Ref\ A &= True \\ \vdash Inj: \star &= True \end{array}
```

Lemma 19. If $\Sigma \vdash v : A$ then \vdash observe v : A.

A final state is one that has finished executing. It is a return statement with an empty procedure call stack and empty active address list.

final (return
$$e, \varrho, [], \mu, []$$
)

The following steps function iterates the step function. We use a counter as a technical device to make this function terminate (which Isabelle requires) even though it otherwise not be guaranteed to terminate.

This language is type safe because, for arbitrary numbers of steps, the result is always well typed.

```
Theorem 1 (Type Safety). If \vdash s : A then \exists r. steps \ n \ s = r \land \vdash r : A.
```

```
theory GTLC-MonoRef-ECDOnANF imports Main begin
```

5.1 Syntax

```
\begin{array}{l} \textbf{datatype} \ ty \\ = IntT \\ \mid BoolT \\ \mid PairT \ ty \ ty \ (\textbf{infixr} \times 201) \\ \mid ArrowT \ ty \ ty \ (\textbf{infixr} \rightarrow 200) \\ \mid RefT \ ty \\ \mid DynT \end{array}
```

datatype const

```
= IntC int
 | BoolC bool
datatype opr
 = Succ
 | Prev
   IsZero
   Fst ty ty
 | Snd ty ty
type-synonym name = nat
datatype stmt
 = \mathit{SLet} \ \mathit{name} \ \mathit{expr} \ \mathit{stmt}
 | SRet expr
   SCall name expr expr stmt
   STailCall\ expr\ expr
   SAlloc\ name\ ty\ expr\ stmt
   SUpdate\ expr\ expr\ stmt
   SDynUpdate\ expr\ expr\ ty\ stmt
   SCast name expr ty ty stmt
   SDynDeref name expr ty stmt
and expr
 = Var name
  Const const
   PrimApp opr expr
   MkPair expr expr
   Lam nat ty stmt
  Deref\ expr
datatype val
 = VConst\ const
 | VPair val val
   Closure nat ty stmt (nat \times val) list
   VRef nat
 | Inject val ty
{\bf datatype} \ {\it casted-val}
 = Val \ val
 | VCast val ty ty
       Result Monad
\mathbf{datatype} 'a result = Result 'a | Stuck | TimeOut | CastError
definition
```

result-bind :: ['a result, 'a => 'b result] => 'b result where

result-bind $m f = (case \ m \ of \ Stuck => Stuck$

```
CastError \Rightarrow CastError
                   Result r => f r)
declare result-bind-def[simp]
syntax -result-bind :: [pttrns,'a result,'b] = > 'c ((-:=-://-) 0)
translations P := E; F == CONST result-bind E (%P. F)
definition return :: 'a \Rightarrow 'a \text{ result } \mathbf{where}
  return \ x \equiv Result \ x
declare return-def[simp]
definition stuck :: 'a result where
 stuck \equiv Stuck
declare stuck-def[simp]
definition cast-error :: 'a result where
  cast-error \equiv CastError
declare cast-error-def[simp]
5.3
       Operational Semantics
type-synonym env = (nat \times val) list
— Heap contains type-tagged possibly-casted values!
type-synonym heap = (nat \times (casted-val \times ty)) \ list
type-synonym \ stack = (name \times stmt \times env) \ list
type-synonym state = stmt \times env \times stack \times heap \times nat \ list
fun lookup :: 'a \Rightarrow ('a \times 'b) \ list \Rightarrow 'b \ result where
  looknil: lookup x [] = stuck []
  lookcons:\ lookup\ x\ ((y,v)\#bs)=(if\ x=y\ then\ return\ v\ else\ lookup\ x\ bs)
fun delta :: opr \Rightarrow val \Rightarrow val result where
  deltas: delta Succ (VConst (IntC n)) = return (VConst (IntC (n + 1)))
  deltap: delta \ Prev \ (VConst \ (IntC \ n)) = return \ (VConst \ (IntC \ (n-1)))
  deltaz: delta \ IsZero \ (VConst \ (IntC \ n)) = return \ (VConst \ (BoolC \ (n = 0))) \mid
  deltafst: delta (Fst \ A \ B) (VPair \ v \ v') = return \ v \ |
  deltasnd: delta (Snd A B) (VPair v v') = return v'
  deltastuck: delta f v = stuck
fun ground :: ty \Rightarrow ty where
  gndi: ground IntT = IntT
  gndb: ground BoolT = BoolT
  gndd: ground DynT = DynT \mid
  gndp: ground (A \times B) = (DynT \times DynT) \mid
  gndf: ground \ (A \to B) = (DynT \to DynT) \ |
 gndr: ground (RefT A) = RefT DynT
fun to-addr :: val \Rightarrow nat result where
  to-addr (VRef a) = return a |
```

```
to-addr v = stuck
fun to-val :: casted-val \Rightarrow val result where
  to\text{-}val (Val v) = return v
  to-val v = <math>stuck
fun eval :: expr \Rightarrow env \Rightarrow heap \Rightarrow val result where
  evalv: eval (Var x) \rho \mu = lookup x \rho
  evalc: eval (Const c) \varrho \mu = return (VConst c)
  evalpa: eval (PrimApp f e) \varrho \mu = (v := eval \ e \ \varrho \ \mu; \ delta \ f \ v) \mid
  evalp: eval (MkPair e1 e2) \varrho \mu = (v1 := eval\ e1\ \varrho\ \mu;\ v2 := eval\ e2\ \varrho\ \mu;
      return (VPair v1 v2))
  evall: eval (Lam x T s) \varrho \mu = return (Closure <math>x T s \varrho) |
  evald: eval (Deref e) \varrho \mu = (v := eval \ e \ \varrho \ \mu; \ a := to - addr \ v;
                                 (cv,A) := lookup \ a \ \mu; \ to\text{-}val \ cv)
fun lesseq-dyn :: ty \Rightarrow ty \Rightarrow bool (infix \sqsubseteq 79) where
  lsda: A \sqsubseteq DynT = True \mid
  lsii: IntT \sqsubseteq IntT = True
  lsbb: BoolT \sqsubseteq BoolT = True
  lspp: (A \times B) \sqsubseteq (C \times D) = (A \sqsubseteq C \land B \sqsubseteq D)
  lsff: (A \to B) \sqsubseteq (C \to D) = (A \sqsubseteq C \land B \sqsubseteq D)
  lsrr: (RefT\ A) \sqsubseteq (RefT\ B) = (A \sqsubseteq B)
  lsnot: A \sqsubseteq B = False
fun meet :: ty \Rightarrow ty \Rightarrow ty result where
  meetda: meet DynT A = return A
  meetad: meet \ A \ DynT = return \ A \ |
  meetii: meet IntT IntT = return IntT \mid
  meetbb: meet BoolT BoolT = return BoolT
  meetpp: meet (A \times B) (C \times D) =
     (A' := meet \ A \ C; \ B' := meet \ B \ D; \ return \ (A' \times B')) \mid
  meetff: meet (A \rightarrow B) (C \rightarrow D) =
     (A' := meet \ A \ C; \ B' := meet \ B \ D; \ return \ (A' \rightarrow B')) \mid
  meetrr: meet (RefT A) (RefT B) =
     (A' := meet \ A \ B; return \ (RefT \ A'))
  meeterr: meet \ A \ B = cast-error
fun mk-vcast :: casted-val \Rightarrow ty \Rightarrow ty \Rightarrow casted-val where
  vcastv: mk-vcast (Val v) A B = VCast v A B
  vcastcv: mk-vcast (VCast v A B) C D = VCast v A D
definition wrap :: val \Rightarrow ty \Rightarrow ty \Rightarrow ty \Rightarrow val where
  wrap \ v \ A \ B \ C \ D \equiv (Closure \ 0 \ C
                (SCast \ 3 \ (Var \ \theta) \ C \ A
                (SCall 2 (Var 1) (Var 3)
                (SCast 4 (Var 2) B D
                    (SRet (Var 4)))))
```

[(1,v)]

declare wrap-def[simp]

```
fun cast :: val \Rightarrow ty \Rightarrow ty \Rightarrow heap \Rightarrow nat list \Rightarrow (val \times heap \times (nat list)) result
where
  castii: cast v IntT IntT \mu as = return (v,\mu,as)
  castbb: cast v BoolT BoolT \mu as = return (v,\mu,as)
  castdd: cast v DynT DynT \mu as = return (v,\mu,as) |
  castff: cast v (A \rightarrow B) (C \rightarrow D) \mu as = return (wrap <math>v A B C D, \mu, as)
  castpp: cast (VPair v1 v2) (A \times B) (C \times D) \mu as =
     ((v1', \mu 1, as1) := cast \ v1 \ A \ C \ \mu \ as;
      (v2', \mu2, as2) := cast \ v2 \ B \ D \ \mu1 \ as1;
      return (VPair v1'v2', \mu2, as2))
  castrr: cast (VRef a) (RefT A) (RefT B) \mu as =
     ((cv,C) := lookup \ a \ \mu;
      BC := meet B C;
      if C \sqsubseteq BC then return (VRef a, \mu, as)
      else return (VRef a, (a,(mk\text{-}vcast\ cv\ C\ BC,\ BC))\#\mu,\ a\#as))
  castinj: cast (Inject v T1) DynT T2 \mu as =
    (if ground T1 = ground T2 then cast v T1 T2 \mu as
     else cast-error)
  casttd: cast v T DynT \mu as = return (Inject v T, \mu, as)
  casterr: cast \ v \ T1 \ T2 \ \mu \ as = cast-error
fun step :: state \Rightarrow state result where
  step\ (s,\ \varrho,\ k,\ \mu,\ a\#ads) =
        ((cv, A) := lookup \ a \ \mu;
         (case cv of
             Val\ v \Rightarrow return\ (s,\ \varrho,\ k,\ \mu,\ ads)
          \mid VCast \ v \ B \ C \Rightarrow
              (v',\mu',ads2) := cast \ v \ B \ C \ \mu \ (a\#ads);
              (cv',A') := lookup \ a \ \mu';
               if A \sqsubseteq A' then
                   return (s, \varrho, k, (a, (Val\ v', A)) \# \mu', removeAll\ a\ ads2)
               else return (s, \varrho, k, \mu', ads2)))
  step (SLet x e s, \varrho, k, \mu, []) =
    (v := eval \ e \ \rho \ \mu;
     return (s, (x,v)\#\varrho, k, \mu, []))
  step (SRet e, \varrho, (x, s, \varrho')#k, \mu, []) =
      (v := eval \ e \ \varrho \ \mu;
       return (s, (x,v)\#\varrho', k, \mu, [])
  step~(SCall~x~e1~e2~s,~\varrho,~k,~\mu,~[]) =
     (v1 := eval \ e1 \ \varrho \ \mu; \ v2 := eval \ e2 \ \varrho \ \mu;
      case v1 of
        Closure y T s' \varrho' \Rightarrow
          return (s', (y,v2)\#\varrho', (x, s,\varrho)\#k, \mu, [])
      | - \Rightarrow stuck)
  step (STailCall e1 e2, \varrho, k, \mu, []) =
     (v1 := eval \ e1 \ \varrho \ \mu; \ v2 := eval \ e2 \ \varrho \ \mu;
      case v1 of
```

```
Closure y T s' \varrho' \Rightarrow
           return (s', (y,v2)\#\varrho', k, \mu, [])
      | - \Rightarrow stuck) |
  step (SAlloc x A e s, \varrho, k, \mu, []) =
     (v := eval \ e \ \rho \ \mu;
      let a = length \mu in
      return (s, (x, VRef a) \# \varrho, k, (a, (Val v, A)) \# \mu, []))
  step (SUpdate e1 e2 s, \varrho, k, \mu, []) =
      (v1 := eval\ e1\ \varrho\ \mu;\ v2 := eval\ e2\ \varrho\ \mu;\ a := to\text{-}addr\ v1;
        (cv,A) := lookup \ a \ \mu;
        return (s, \varrho, k, (a, (Val\ v2, A)) \# \mu, []))
  step (SDynUpdate e1 e2 A s, \varrho, k, \mu, []) =
      (v1 := eval \ e1 \ \varrho \ \mu; \ v2 := eval \ e2 \ \varrho \ \mu; \ a := to\text{-}addr \ v1;
        (cv,B) := lookup \ a \ \mu;
        return (s, \varrho, k, (a, (VCast \ v2 \ A \ B,B))\#\mu, [a]))
  step (SCast x \in A B s, \varrho, k, \mu, []) =
        (\mathit{v} := \mathit{eval} \ \mathit{e} \ \varrho \ \mu; \ (\mathit{v}',\!\mu',\!\mathit{ads}) := \mathit{cast} \ \mathit{v} \ \mathit{A} \ \mathit{B} \ \mu \ [];
         return (s, (x, v') \# \varrho, k, \mu', ads))
  step (SDynDeref x e A s, \varrho, k, \mu, []) =
         (v := eval \ e \ \varrho \ \mu; \ a := to\text{-}addr \ v; \ (cv,B) := lookup \ a \ \mu;
          v1 := to\text{-}val\ cv;\ (v2,\mu',ads) := cast\ v1\ B\ A\ \mu\ [];
          return (s, (x,v2)\#\varrho, k, \mu', ads))
  step \ s = stuck
{f datatype}\ observable\ =\ OPair\ observable\ observable\ |\ Fun\ |\ Con\ const\ |\ Addr\ |
  | OStuck | OTimeOut | OCastError
\mathbf{primrec}\ \mathit{observe} :: \mathit{val} \Rightarrow \mathit{observable}\ \mathbf{where}
  obsc: observe (VConst c) = Con c
  obsp: observe (VPair v v') = OPair (observe v) (observe v') |
  obsf: observe (Closure x T s \varrho) = Fun
  obsr: observe (VRef a) = Addr
  obsinj: observe (Inject \ v \ T) = Inj
definition final :: state \Rightarrow bool where
  final s \equiv (case \ s \ of \ (SRet \ e, \ \varrho, \ [], \ \mu, \ []) \Rightarrow True \ | \ - \Rightarrow False)
declare final-def[simp]
fun steps :: nat \Rightarrow state \Rightarrow observable where
  stepsz: steps \ 0 \ s = OTimeOut \ |
  stepsret: steps (Suc n) (SRet e, \varrho, [], \mu, []) =
             (case eval e \rho \mu of
                 Stuck \Rightarrow OStuck
             \mid CastError \Rightarrow OCastError
             \mid Result \ v \Rightarrow observe \ v) \mid
  stepsrec: steps (Suc n) s =
          (case step s of
            Stuck \Rightarrow OStuck
```

```
| Result s' \Rightarrow steps n s' |
definition run :: stmt \Rightarrow observable where
  run \ s \equiv steps \ 1000000 \ (s,[],[],[],[])
          Type System
5.4
type-synonym ty-env = (name \times ty) list
primrec typeof :: const \Rightarrow ty where
  typeof (IntC n) = IntT \mid
  typeof (BoolC b) = BoolT
primrec typeof-opr :: opr \Rightarrow ty where
  typeof\text{-}opr\ Succ = (IntT \rightarrow IntT)
  typeof\text{-}opr\ Prev = (IntT \rightarrow IntT)
  typeof\text{-}opr\ IsZero = (IntT \rightarrow BoolT)
  typeof-opr (Fst A B) = ((A \times B) \rightarrow A)
  typeof-opr (Snd\ A\ B) = ((A \times B) \to B)
primrec static :: ty \Rightarrow bool where
  sta-d: static \ DynT = False \mid
  sta-i: static IntT = True |
  sta-b: static BoolT = True
  sta-p: static (A \times B) = (static A \wedge static B) \mid
  sta-f: static (A \rightarrow B) = (static A \land static B)
  sta-r: static (RefT A) = static A
inductive
  wt-expr :: ty-env \Rightarrow expr \Rightarrow ty \Rightarrow bool (- \vdash_e - : - [60,60,60] 59)
  and wt-stmt :: ty-env \Rightarrow stmt \Rightarrow ty \Rightarrow bool (- \vdash_s - : - [60,60,60] 59)
  wt\text{-}var[intro!]: lookup \ x \ \Gamma = Result \ A \Longrightarrow \Gamma \vdash_e Var \ x : A \mid
  wt-const[intro!]: \Gamma \vdash_e Const\ c: typeof\ c
  wt-primapp[intro!]:
  \llbracket typeof\text{-}opr f = A \rightarrow B; \Gamma \vdash_e e : A \rrbracket \Longrightarrow \Gamma \vdash_e PrimApp f e : B \rrbracket
  wt-mkpair[intro!]: [\Gamma \vdash_e e1 : A; \Gamma \vdash_e e2 : B] \Longrightarrow
      \Gamma \vdash_e \mathit{MkPair}\ e1\ e2: A \times B \mid
  wt-lam[intro!]: [(x,A)\#\Gamma \vdash_s s:B] \Longrightarrow
      \Gamma \vdash_e Lam \ x \ A \ s : (A \rightarrow B)
  wt-deref[intro!]: \llbracket \Gamma \vdash_e e : RefT A; static A \rrbracket \Longrightarrow \Gamma \vdash_e Deref e : A \rrbracket
  wt\text{-}let[intro!]: \llbracket \Gamma \vdash_e e : A; (x,A) \# \Gamma \vdash_s s : B \rrbracket \Longrightarrow
     \Gamma \vdash_s SLet \ x \ e \ s : B \mid
  wt\text{-}ret[intro!]: \llbracket \Gamma \vdash_e e : A \rrbracket \Longrightarrow
     \Gamma \vdash_s SRet \ e : A \mid
  wt-call[intro!]: \llbracket \Gamma \vdash_e e : A \to B; \Gamma \vdash_e e' : A; (x,B) \# \Gamma \vdash_s s : C \rrbracket \Longrightarrow
      \Gamma \vdash_s SCall \ x \ e \ e' \ s : C \mid
```

 $CastError \Rightarrow OCastError$

```
wt-tailcall[intro!]: \llbracket \Gamma \vdash_e e : A \to B; \Gamma \vdash_e e' : A \rrbracket \Longrightarrow
       \Gamma \vdash_s STailCall \ e \ e' : B \mid
   wt-alloc[intro!]: \llbracket \Gamma \vdash_e e : A; (x,RefT\ A)\#\Gamma \vdash_s s : B \rrbracket \Longrightarrow
       \Gamma \vdash_s SAlloc \ x \ A \ e \ s : B \mid
   wt-update[intro!]: \llbracket \Gamma \vdash_e e : RefT A; static A; \Gamma \vdash_e e' : A; \Gamma \vdash_s s : B \rrbracket \Longrightarrow
       \Gamma \vdash_s SUpdate\ e\ e'\ s: B \mid
   wt-dynupdate[intro!]: \llbracket \Gamma \vdash_e e : RefT A; \Gamma \vdash_e e' : A; \Gamma \vdash_s s : B \rrbracket \Longrightarrow
       \Gamma \vdash_s SDynUpdate\ e\ e'\ A\ s: B
   wt\text{-}cast[intro!]: \llbracket \Gamma \vdash_e e : A; (x,B) \# \Gamma \vdash_s s : C \rrbracket \Longrightarrow
       \Gamma \vdash_s SCast \ x \ e \ A \ B \ s : C \mid
   wt-dynderef [intro!]: \llbracket \Gamma \vdash_e e : RefT A; (x,A) \# \Gamma \vdash_s s : B \rrbracket \Longrightarrow
       \Gamma \vdash_s SDynDeref \ x \ e \ A \ s : B
inductive wt-val :: ty-env \Rightarrow val \Rightarrow ty \Rightarrow bool (- \vdash v - : - [60,60,60] 59)
   and wt-env :: ty-env \Rightarrow ty-env \Rightarrow env \Rightarrow bool (-; - [60,60,60] 59) where
   wt\text{-}vc[intro!]: typeof \ c = A \Longrightarrow \Sigma \vdash v \ VConst \ c : A \mid
   wt-pair[intro!]: \llbracket \Sigma \vdash v \ v : A; \Sigma \vdash v \ v' : B \rrbracket \implies \Sigma \vdash v \ (VPair \ v \ v') : A \times B \rfloor
   wt\text{-}cl[intro!]: \llbracket \Gamma; \Sigma \vdash \varrho; (x,A) \# \Gamma \vdash_s s : B \rrbracket \Longrightarrow
       \Sigma \vdash v \ (Closure \ x \ A \ s \ \varrho) : A \to B \mid
   wt\text{-ref}[intro!]: \llbracket lookup \ a \ \Sigma = Result \ A; \ A \sqsubseteq B \rrbracket \Longrightarrow \Sigma \vdash v \ (VRef \ a) : RefT \ B \mid
   wt-inject[intro!]: \Sigma \vdash v \ v : A \Longrightarrow \Sigma \vdash v \ (Inject \ v \ A) : DynT \mid
   wt-nil[intro!]: []; \Sigma \vdash [] \mid
   wt\text{-}cons[intro!]: \llbracket \Sigma \vdash v \ v : A; \ \Gamma; \Sigma \vdash \varrho \ \rrbracket \Longrightarrow (x,A)\#\Gamma; \Sigma \vdash (x,v)\#\varrho
inductive wt-stack :: ty-env \Rightarrow stack \Rightarrow ty \Rightarrow ty \Rightarrow bool (- \vdash -: - \Rightarrow -) where
   nil\text{-}stack[intro!]: \Sigma \vdash []: A \Rightarrow A
   cons\text{-}stack[intro!] \colon \llbracket \ \Gamma; \Sigma \vdash \varrho; \ (x,A) \# \Gamma \vdash_s s : B; \ \Sigma \vdash k : B \Rightarrow C \ \rrbracket \Longrightarrow
        \Sigma \vdash (x, s, \varrho) \# k : A \Rightarrow C
definition dom :: ('a \times 'b) \ list \Rightarrow 'a \ set \ where
   dom A \equiv set (map fst A)
inductive wt-casted-val :: ty-env \Rightarrow casted-val \Rightarrow ty \Rightarrow bool (-\vdash cv - : - [60,60,60]
59) where
   wt\text{-}cv\text{-}val[intro!]: \Sigma \vdash v \ v : A \Longrightarrow \Sigma \vdash cv \ Val \ v : A \mid
   wt\text{-}cv[intro!]: \llbracket \Sigma \vdash v \ v : A; B \sqsubseteq A \rrbracket \Longrightarrow \Sigma \vdash cv \ VCast \ v \ A \ B : B
definition wt-heap :: ty-env \Rightarrow heap \Rightarrow nat set \Rightarrow bool where
   wt-heap \Sigma \mu \ as \equiv (\forall \ a \ A. \ lookup \ a \ \Sigma = Result \ A \longrightarrow
                           (\exists cv. lookup \ a \ \mu = Result \ (cv,A) \land \Sigma \vdash cv \ cv : A \land A)
                              (a \notin as \longrightarrow (\exists v. cv = Val v)))
       \land (\forall a. \ a \in dom \ \mu \longrightarrow a < length \ \mu) \land (as \subseteq dom \ \Sigma)
definition lesseq-tyenv :: ty-env \Rightarrow ty-env \Rightarrow bool (infix <math>\sqsubseteq 80) where
  \Sigma' \sqsubseteq \Sigma \equiv (dom \ \Sigma = dom \ \Sigma') \land (\forall \ a \ A. \ lookup \ a \ \Sigma = Result \ A \longrightarrow A)
                       (\exists B. lookup \ a \ \Sigma' = Result \ B \land B \sqsubseteq A))
```

inductive wt-state :: $state \Rightarrow ty \Rightarrow bool$ where

```
wts\text{-}intro[intro!] : \llbracket wt\text{-}heap \ \Sigma \ \mu \ (set \ as); \ \Gamma; \Sigma \vdash \varrho; \ \Gamma \vdash_s \ s : A; \ \Sigma \vdash k : A \Rightarrow B \ \rrbracket \\ \Longrightarrow \\ wt\text{-}state \ (s, \ \varrho, \ k, \ \mu, \ as) \ B \\ \\ \textbf{fun} \ \ wt\text{-}observable :: observable \Rightarrow ty \Rightarrow bool \ \textbf{where} \\ wto\text{-}p : wt\text{-}observable \ (OPair \ o' \ o'') \ (A \times B) = \\ (wt\text{-}observable \ o' \ A \wedge wt\text{-}observable \ o'' \ B) \ | \\ wto\text{-}f : wt\text{-}observable \ Fun \ (A \to B) = True \ | \\ wto\text{-}c : wt\text{-}observable \ (Con \ c) \ T = (typeof \ c = T) \ | \\ wto\text{-}s : wt\text{-}observable \ OTimeOut \ T = True \ | \\ wto\text{-}a : wt\text{-}observable \ OCastError \ T = True \ | \\ wto\text{-}a : wt\text{-}observable \ Addr \ (RefT \ A) = True \ | \\ wto\text{-}inj : wt\text{-}observable \ obs \ T = False \\ \end{cases}
```

5.5 Inversion Principles

```
inductive-cases wtv[elim!]: \Gamma \vdash_e Var \ x : A and wti[elim!]: \Gamma \vdash_e Const \ (IntC \ n) : A and wtb[elim!]: \Gamma \vdash_e Const \ (BoolC \ b) : A and wtp[elim!]: \Gamma \vdash_e PrimApp \ f \ e : A and wtpair[elim!]: \Gamma \vdash_e MkPair \ e1 \ e2 : A and wtlam[elim!]: \Gamma \vdash_e Lam \ x \ T \ s : A and wtderef[elim!]: \Gamma \vdash_e (Deref \ e) : A
```

inductive-cases

inductive-cases

```
bool-int[elim!] \colon \Sigma \vdash v \; (VConst \; (BoolC \; b)) \colon IntT \; \text{and} \\ inject-int[elim!] \colon \Sigma \vdash v \; (Inject \; v \; A) \colon IntT \; \text{and} \\ pair-int[elim!] \colon \Sigma \vdash v \; (VPair \; v1 \; v2) \colon IntT \; \text{and} \\ int-any[elim!] \colon \Sigma \vdash v \; (VConst \; (IntC \; n)) \colon A \; \text{and} \\ bool-any[elim!] \colon \Sigma \vdash v \; (VConst \; (BoolC \; b)) \colon A \; \text{and} \\ clos-any[elim!] \colon \Sigma \vdash v \; (VConst \; v1 \; v2) \colon A \; \text{and} \\ pair-any[elim!] \colon \Sigma \vdash v \; (VPair \; v1 \; v2) \colon A \; \text{and} \\ ref-any[elim!] \colon \Sigma \vdash v \; (VRef \; a) \colon A \; \text{and} \\ inject-any[elim!] \colon \Sigma \vdash v \; (Inject \; v \; T) \colon A
```

inductive-cases

```
clos-int[elim!]: \Sigma \vdash v \ (Closure \ x \ T \ s \ \varrho) : IntT \  and
  inj-int[elim!]: \Sigma \vdash v (Inject \ v \ T) : IntT \  and
  const-fun[elim!]: \Sigma \vdash v (VConst \ c) : (A \to B) and
  const-ref[elim!]: \Sigma \vdash v (VConst \ c) : RefT \ A \ and
  const-any[elim!]: \Sigma \vdash v (VConst c) : A and
  clos-fun[elim!]: \Sigma \vdash v \ (Closure \ x \ T \ s \ \varrho) : (A \to B) and
  pair-times[elim!]: \Sigma \vdash v (VPair v1 v2) : (A \times B) and
  pairval-inv[elim!]: \Sigma \vdash v v : (A \times B) and
  funval-inv[elim!]: \Sigma \vdash v \ v : (A \to B) and
  refval-inv[elim!]: \Sigma \vdash v \ v : (RefT \ A)
inductive-cases
  cv-val[elim!]: \Sigma \vdash cv \ Val \ v : A and
  \mathit{cv\text{-}cv}[\mathit{elim!}] \colon \Sigma \vdash \!\! \mathit{cv} \ \mathit{VCast} \ \mathit{v} \ \mathit{A} \ \mathit{B} \ \colon \mathit{C}
inductive-cases
  wtek[elim!]: \Sigma \vdash []: A \Rightarrow B and
  wtk[elim!]: \Sigma \vdash f \# k : A \Rightarrow B
inductive-cases wts[elim!]: wt-state s A
         Proof of Type Safety
5.6
lemma static-is-most-precise:
  fixes A::ty and B::ty assumes sa: static A and ba: B \subseteq A shows A = B
  using sa ba apply (induct rule: lesseq-dyn.induct)
  apply (case-tac A) apply simp+ done
lemma lookup-dom:
  lookup \ a \ \Sigma = Result \ v \Longrightarrow a \in dom \ \Sigma
  apply (induct \Sigma)
 apply simp apply clarify apply (case-tac a = aa) apply (auto simp: dom-def)
  done
lemma dom-lookup:
  a \in dom \ \Sigma \Longrightarrow (\exists A. \ lookup \ a \ \Sigma = Result \ A)
  apply (induct \Sigma arbitrary: a)
  apply (simp add: dom-def)
  apply clarify apply (case-tac a = aa) apply simp apply (simp add: dom-def)
  done
lemma weaken-value-env:
  (\Sigma \vdash v \ v : A \longrightarrow (\forall \ a \ B. \ a \notin dom \ \Sigma \longrightarrow (a,B) \# \Sigma \vdash v \ v : A))
   \wedge (\Gamma; \Sigma \vdash \varrho \longrightarrow (\forall \ a \ B. \ a \notin dom \ \Sigma \longrightarrow \Gamma; (a,B) \# \Sigma \vdash \varrho))
  apply (induct rule: wt-val-wt-env.induct)
  using lookup-dom apply force+ done
lemma weaken-stack:
  \llbracket \Sigma \vdash k : A \Rightarrow B; a \notin dom \Sigma \rrbracket \Longrightarrow (a,T) \# \Sigma \vdash k : A \Rightarrow B
```

```
apply (induct k arbitrary: A B a \Sigma)
  apply force
  using weaken-value-env apply auto done
lemma delta-safe:
  assumes wtop: typeof-opr f = A \rightarrow B
  and wtv : \Sigma \vdash v \ v : A
  shows \exists v'. delta fv = Result v' \land \Sigma \vdash v v' : B
  using wtop \ wtv \ apply \ (case-tac \ f)
  apply (case-tac v, auto, case-tac const, auto)
  apply (case-tac v, auto, case-tac const, auto)
  apply (case-tac v, auto, case-tac const, auto)
  apply (case-tac c, auto)
  apply (case-tac c, auto)
  done
lemma lookup-safe:
  assumes wtg: \Gamma;\Sigma \vdash \varrho and l: lookup x \Gamma = Result A
 shows \exists v. lookup \ x \ \varrho = Result \ v \land \Sigma \vdash v \ v : A
  using wtq l by (induct \rho) force+
lemma eval-safe:
  assumes wte: \Gamma \vdash_e e : A
  and wtg: \Gamma; \Sigma \vdash \varrho
  and wth: wt-heap \Sigma \mu {}
  shows \exists v. eval \ e \ \varrho \ \mu = Result \ v \land \Sigma \vdash v \ v : A
  using wte wtg wth
  apply (induct e \rho \mu arbitrary: A rule: eval.induct)
  using lookup-safe apply force
  apply (case-tac c) apply force apply force
  using delta-safe apply force
  apply force
  apply force
proof -
  fix e \varrho \mu A
  assume IH: \bigwedge A. \llbracket \Gamma \vdash_e e : A; \Gamma; \Sigma \vdash \varrho; wt\text{-heap } \Sigma \mu \upharpoonright \rrbracket
            \implies \exists v. \ eval \ e \ \varrho \ \mu = Result \ v \land \Sigma \vdash v \ v : A
    and de: \Gamma \vdash_e Deref e: A and wtr: \Gamma; \Sigma \vdash \varrho and wth: wt\text{-}heap \Sigma \mu \{\}
  from de have wte: \Gamma \vdash_e e : (RefT \ A) by auto
  from de have sa: static A by auto
  from IH[of RefT A] wte wtr wth
  have (\exists v. eval \ e \ \varrho \ \mu = Result \ v \land \Sigma \vdash v \ v : RefT \ A) by simp
  from this obtain v where ev: eval e \rho \mu = Result v
    and wtv: \Sigma \vdash v \ v : RefT \ A \ apply \ clarify \ apply \ auto \ done
  from wtv obtain a A' where v: v = VRef a and
    las: lookup a \Sigma = Result A' and aa: A' \sqsubseteq A apply auto
    apply (case-tac c) apply auto done
  from sa aa have aaeq: A = A' using static-is-most-precise by blast
  from las wth aaeq obtain cv \ v' where lam: lookup \ a \ \mu = Result \ (cv, A)
```

```
and wtcv: \Sigma \vdash cv \ cv : A and cv: cv = Val \ v'
    apply (simp only: wt-heap-def) apply blast done
  from wtcv \ cv have wtvp: \Sigma \vdash v \ v': A apply auto done
  from ev lam cv wtvp v
  show (\exists v. eval (Deref e) \rho \mu = Result v \wedge \Sigma \vdash v v : A) by auto
qed
lemma lesseq-refl[simp]: fixes A::ty shows A \sqsubseteq A
  by (induct A) auto
lemma lesseq-env-refl[simp]: fixes \Sigma::ty-env shows \Sigma \sqsubseteq \Sigma
  using lesseq-tyenv-def by simp
lemma lesseq-int-pair[elim!]: A \times B \sqsubseteq IntT \Longrightarrow P by auto
lemma lesseq-int-fun[elim!]: A \to B \sqsubseteq IntT \Longrightarrow P by auto
lemma lesseg-pair-int[elim!]: IntT \subseteq A \times B \Longrightarrow P by auto
lemma lesseq-bool-pair[elim!]: A \times B \sqsubseteq BoolT \Longrightarrow P by auto
lemma lesseq-bool-fun[elim!]: A \to B \sqsubseteq BoolT \Longrightarrow P by auto
lemma lesseq-pair-bool[elim!]: BoolT \sqsubseteq A \times B \Longrightarrow P by auto
lemma lesseq-pair-fun[elim!]: C \to D \sqsubseteq A \times B \Longrightarrow P by auto
lemma lesseq-pair-ref [elim!]: RefT C \sqsubseteq A \times B \Longrightarrow P by auto
lemma lesseq-ref-pair[elim!]: A \times B \sqsubseteq RefT \ C \Longrightarrow P by auto
lemma lesseq-ref-fun[elim!]: A \to B \sqsubseteq RefT \ C \Longrightarrow P by auto
lemma lesseq-pair-dyn[elim!]: DynT \sqsubseteq A \times B \Longrightarrow P by auto
lemma lesseq-fun-pair[elim!]: C \times D \sqsubseteq A \rightarrow B \Longrightarrow P by auto
lemma lesseq-fun-int[elim!]: IntT \sqsubseteq A \rightarrow B \Longrightarrow P by auto
lemma lesseg-fun-bool[elim!]: BoolT \sqsubseteq A \rightarrow B \Longrightarrow P by auto
lemma lesseq-fun-ref[elim!]: RefT C \subseteq A \rightarrow B \Longrightarrow P by auto
lemma lesseq-fun-dyn[elim!]: DynT \sqsubseteq A \rightarrow B \Longrightarrow P by auto
lemma lesseq-ref-inv[elim!]: [ RefT A \sqsubseteq RefT B; A \sqsubseteq B \Longrightarrow P ]] \Longrightarrow P by auto
lemma lesseq-pair-inv[elim!]: [(A1 \times A2) \subseteq (B1 \times B2);
    \llbracket A1 \sqsubseteq B1; A2 \sqsubseteq B2 \rrbracket \Longrightarrow P \rrbracket \Longrightarrow P \text{ by } auto
lemma lesseq-fun-inv[elim!]: [(A1 \rightarrow A2) \subseteq (B1 \rightarrow B2);
    \llbracket A1 \sqsubseteq B1; A2 \sqsubseteq B2 \rrbracket \Longrightarrow P \rrbracket \Longrightarrow P \text{ by } auto
lemma lesseq-dyn-any[intro!]: A \sqsubseteq DynT by auto
lemma less-eq-fun[intro!]: \llbracket A1 \sqsubseteq B1; A2 \sqsubseteq B2 \rrbracket \implies A1 \rightarrow A2 \sqsubseteq B1 \rightarrow B2
lemma less-eq-pair[intro!]: [ A1 \sqsubseteq B1; A2 \sqsubseteq B2 ] \Longrightarrow A1 \times A2 \sqsubseteq B1 \times B2 by
simp
lemma lesseq-prec-trans[rule-format,trans]:
  fixes B::ty
  shows (\forall A \ C. \ A \sqsubseteq B \longrightarrow B \sqsubseteq C \longrightarrow A \sqsubseteq C)
  apply (induct B)
  apply clarify apply (case-tac C) apply force apply force apply force
    apply force apply force apply force
```

```
apply clarify apply (case-tac C) apply force apply force apply force
   apply force apply force apply force
  defer
  defer
 apply clarify apply (case-tac C) apply force apply force apply force
   apply force apply clarify defer apply force
 apply clarify apply (case-tac C) apply force apply force apply force
   apply force apply force apply force
 \mathbf{prefer} \ \mathcal{I}
 apply (case-tac A) apply simp apply simp apply simp apply simp
   apply clarify apply simp apply simp
  apply clarify apply (case-tac C) apply force apply force
   apply clarify apply (case-tac A) apply clarify apply force
   apply clarify defer apply clarify apply clarify apply clarify
   apply clarify apply clarify apply clarify
  apply clarify apply (case-tac C) apply clarify apply clarify apply clarify
   apply clarify apply (case-tac A) apply clarify apply clarify
   apply clarify apply clarify apply blast apply clarify
   apply clarify apply clarify apply clarify
 apply (rule less-eq-pair) apply blast apply blast
 done
\mathbf{lemma}\ \mathit{lesseq\text{-}tyenv\text{-}trans}\colon
  fixes \Sigma::ty\text{-}env and \Sigma'::ty\text{-}env
  assumes s23: \Sigma' \sqsubseteq \Sigma'' and s12: \Sigma \sqsubseteq \Sigma'
 shows \Sigma \sqsubseteq \Sigma''
  using s12 s23 apply (simp add: lesseq-tyenv-def)
 apply clarify
 apply (erule-tac x=a in allE)
 apply (erule-tac x=a in allE)
 apply simp
 apply (erule exE)
 apply (erule conjE)
 apply (erule-tac x=B in allE)
 using lesseq-prec-trans apply blast done
\mathbf{lemma}\ strengthen\text{-}value\text{-}env:
  (\Sigma \vdash v \ v : A \longrightarrow (\forall \ \Sigma'. \ \Sigma' \sqsubseteq \Sigma \longrightarrow \Sigma' \vdash v \ v : A))
  \wedge \ (\Gamma; \Sigma \vdash \varrho \longrightarrow (\forall \ \Sigma'. \ \Sigma' \sqsubseteq \Sigma \longrightarrow \Gamma; \Sigma' \vdash \varrho))
 apply (induct rule: wt-val-wt-env.induct)
 apply force+ defer apply force+
 apply (simp only: lesseq-tyenv-def)
 apply clarify apply (erule-tac x=a in all E) apply (erule-tac x=A in all E)
 apply clarify apply (rule wt-ref) apply simp
 apply (rule lesseq-prec-trans) apply blast apply blast
 done
lemma strengthen-casted-value:
 fixes \Sigma::ty-env
```

```
assumes wtcv: \Sigma \vdash cv \ cv : A and ss: \Sigma' \sqsubseteq \Sigma
 shows \Sigma' \vdash cv \ cv : A
 using wtcv ss apply (induct arbitrary: \Sigma' rule: wt-casted-val.induct)
 using strengthen-value-env apply blast
  using strengthen-value-env apply blast
 done
lemma strengthen-stack:
  fixes \Sigma::ty\text{-}env
 assumes wtk: \Sigma \vdash k: A \Rightarrow B and ss: \Sigma' \sqsubseteq \Sigma shows \Sigma' \vdash k: A \Rightarrow B
  using wtk ss apply (induct arbitrary: \Sigma' rule: wt-stack.induct)
 apply blast
 using strengthen-value-env apply blast done
lemma meet-safe: ((\exists C. meet \ A \ B = Result \ C) \lor meet \ A \ B = CastError)
 apply (induct rule: meet.induct ) by auto
lemma meet-is-meet-aux:
 (\exists \ C. \ meet \ A \ B = Result \ C \land C \sqsubseteq A \land C \sqsubseteq B) \lor meet \ A \ B = CastError
 apply (induct rule: meet.induct) apply auto done
lemma meet-is-meet:
  meet\ A\ B = Result\ C \Longrightarrow C \sqsubseteq A \land C \sqsubseteq B
 using meet-is-meet-aux[of A B] apply auto done
lemma dom-heap:
 assumes as: a \in dom \Sigma and wth: wt-heap \Sigma \mu ads shows a \in dom \mu
proof -
 from as obtain A where las: lookup a \Sigma = Result A
   using dom-lookup[of a \Sigma] by blast
 from las wth obtain cv where
   lam2: lookup \ a \ \mu = Result \ (cv,A)
   using wt-heap-def [of \Sigma \mu \ ads] apply blast done
 from lam2 show ?thesis using lookup-dom[of a \mu] apply simp done
qed
fun cval-ads :: casted-val \Rightarrow nat set \Rightarrow nat set where
  cvads1: cval-ads (Val v) \ a \ ads = ads - \{a\}
  cvads2: cval-ads (VCast \ v \ A \ B) \ a \ ads = ads \cup \{a\}
\mathbf{lemma}\ update\text{-}heap\text{-}val:
  fixes A::ty and B::ty
 assumes wth: wt-heap \Sigma \mu ads and las: lookup a \Sigma = Result A and ab: B \sqsubseteq A
 and wtv: \Sigma \vdash cv \ cv : B
 shows wt-heap ((a,B)\#\Sigma) ((a,(cv,B))\#\mu) (cval-ads cv a ads)
 apply (simp only: wt-heap-def) apply (rule conjI)
 apply clarify defer apply (rule conjI) apply clarify defer
 using wth apply (cases cv) apply (simp add: dom-def wt-heap-def) apply blast
   apply (simp add: dom-def wt-heap-def) apply blast
```

```
proof -
 fix a' A' assume las2: lookup a' ((a,B)\#\Sigma) = Result A'
 let ?S2 = (a,B) \# \Sigma
 from ab las have ss: ?S2 \sqsubseteq \Sigma
   using lookup-dom[of \ a \ \Sigma \ A]
   apply (simp add: lesseq-tyenv-def dom-def) apply auto done
 show \exists cv'. lookup a'((a, cv, B) \# \mu) = Result(cv', A') \land
     (a, B) \# \Sigma \vdash cv \ cv' : A' \land (a' \notin cval\text{-}ads \ cv \ a \ ads \longrightarrow (\exists v. \ cv' = Val \ v))
  proof (cases a' = a)
   assume aa: a' = a
   from aa las2 have abc: A' = B apply simp done
   from wtv ss have wtv2: ?S2 \vdash cv \ cv : B
     using strengthen-casted-value apply blast done
   from aa abc wtv2 show ?thesis apply auto
     apply (case-tac cv) apply auto done
 next
   assume aa: a' \neq a
   from las2 aa have las3: lookup a' \Sigma = Result A' by simp
   from las3 wth obtain cv' v' where
     lam2: lookup \ a' \ \mu = Result \ (cv',A') \ and \ wtcv3: \Sigma \vdash cv \ cv' : A'
     and cval: a' \notin ads \longrightarrow (\exists v. cv' = Val v)
     using wt-heap-def [of \Sigma \mu \ ads] apply blast done
   from wtcv3 ss have wtcv4: ?S2 \vdash cv \ cv' : A'
     using strengthen-casted-value by blast
   from aa wtcv4 lam2 cval show ?thesis
     apply (case-tac cv) apply auto done
 qed
next
 fix a' assume ad: a' \in dom ((a,(cv,B)) \# \mu)
 let ?M2 = ((a,(cv,B))\#\mu)
 show a' < length ?M2
 proof (cases a' = a)
   assume aa: a' = a
   from las have adom: a \in dom \Sigma using lookup-dom[of a \Sigma] by blast
   from wth adom have a \in dom \ \mu \text{ using } dom\text{-}heap \text{ by } blast
   with aa wth show ?thesis
     apply (simp add: wt-heap-def) apply auto done
   assume aa: a' \neq a
   from this ad wth show ?thesis
     apply (simp add: dom-def wt-heap-def)
     apply auto done
 qed
\mathbf{qed}
lemma cast-safe:
 fixes v::val
 assumes wtv: \Sigma \vdash v \ v : A and wth: wt\text{-}heap \ \Sigma \ \mu \ (set \ ads1)
 shows (\exists v' \Sigma' \mu' ads2. cast v \land B \mu ads1 = Result (v', \mu', ads2) \land \Sigma' \vdash v v' : B
```

```
\land wt\text{-}heap \ \Sigma' \ \mu' \ (set \ ads2) \ \land \ \Sigma' \sqsubseteq \Sigma)
      \vee (cast \ v \ A \ B \ \mu \ ads1 = CastError)
 using wtv wth apply (induct arbitrary: \Sigma rule: cast.induct)
 apply force
 apply force
 apply force
 {\bf apply}\ force
 defer
 defer
 apply simp apply (case-tac T1) apply force apply force apply force
  apply force apply force apply force
 apply simp apply (case-tac T1) apply force apply force apply force
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 defer
proof -
 fix a A B \mu ads1 \Sigma
 assume wta: \Sigma \vdash v \ VRef \ a : RefT \ A \ and \ wth: wt-heap \ \Sigma \ \mu \ (set \ ads1)
 from wto obtain C where las: lookup a \Sigma = Result\ C and aa: C \sqsubseteq A by auto
 from wth las obtain cv where lam: lookup a \mu = Result (cv, C)
  and wtcv: \Sigma \vdash cv \ cv : C using wt-heap-def apply force done
 from meet-safe[of B C]
```

```
show (\exists v' \Sigma' \mu' ads2.
            cast\ (VRef\ a)\ (RefT\ A)\ (RefT\ B)\ \mu\ ads1\ =\ Result\ (v',\ \mu',\ ads2)\ \land
            \Sigma' \vdash v \ v' : RefT \ B \ \land
            wt-heap \Sigma' \mu' (set ads2) \wedge \Sigma' \sqsubseteq \Sigma)
       \vee \ cast \ (VRef \ a) \ (RefT \ A) \ (RefT \ B) \ \mu \ ads1 = CastError
 proof
   assume \exists BC. meet BC = Result BC
   from this obtain BC where bc: meet B C = Result BC by blast
   from bc have leq-bbc: BC \sqsubseteq B using meet-is-meet by blast
   show ?thesis
   proof (cases \ C \sqsubseteq BC)
     assume bcc: C \sqsubseteq BC
     from bc bcc lam
    have ca: cast (VRef a) (RefT A) (RefT B) \mu ads1 = Result (VRef a,\mu,ads1)
by simp
     from leq-bbc bcc have leq-bc: C \sqsubseteq B using lesseq-prec-trans by blast
     from las leg-bc have wta2: \Sigma \vdash v \ VRef \ a : RefT \ B \ by \ (rule \ wt-ref)
     from ca wta2 wth show ?thesis by auto
     assume bcc: \neg (C \sqsubseteq BC)
     \mathbf{let}~?VC = \mathit{mk-vcast}~\mathit{cv}~\mathit{C}~\mathit{BC}
     let ?M2 = (a, ?VC, BC) \# \mu
     from bc bcc lam
   have ca: cast (VRef a) (RefT A) (RefT B) \mu ads1 = Result (VRef a,?M2,a#ads1)
by simp
     let ?S2 = (a,BC)\#\Sigma
     from bc have cbc: BC \sqsubseteq C using meet-is-meet by blast
     from las have adom: a \in dom \Sigma using lookup-dom[of a \Sigma] by blast
     from las adom cbc have ss2: ?S2 \sqsubseteq \Sigma using lesseq-tyenv-def[of ?S2 \Sigma]
       apply (simp add: dom-def) apply blast done
     from wtcv ss2 have wtcv2: ?S2 \vdash cv cv : C
       using strengthen-casted-value apply blast done
     from wtcv \ cbc have wtcvbc: \Sigma \vdash cv ?VC : BC
       apply (case-tac cv)
       apply simp apply auto
       using lesseg-prec-trans apply blast done
     from wth las wtcvbc cbc
     have wth2: wt\text{-}heap ?S2 ?M2 (set (a\#ads1))
       using update-heap-val[of \Sigma \mu \text{ set ads1 a } CBC?VC]
       apply (case-tac cv) apply auto done
     have las2: lookup a ?S2 = Result BC by <math>simp
     from las2 leq-bbc have wta2: ?S2 \vdash v VRef \ a : RefT \ B
       using wt-ref[of a ?S2 BC B] apply simp done
     from wta2 ca wth2 ss2 show ?thesis by (auto simp: dom-def)
   qed
  next
   assume meet B C = CastError
   from this lam show ?thesis by simp
  qed
```

```
next
  fix v1 v2 A B C D \mu \Sigma ads1
 let ?P = \lambda \ v \ A \ B \ \mu \ ads1 \ \Sigma.
         (\exists v' \Sigma' \mu' ads2. cast v \land B \mu ads1 = Result (v', \mu', ads2) \land \Sigma' \vdash v v' : B
            \land wt\text{-}heap \ \Sigma' \ \mu' \ (set \ ads2) \ \land \ \Sigma' \sqsubseteq \Sigma)
  assume IH1: \bigwedge \Sigma. \llbracket \Sigma \vdash v \ v1 : A; \ wt\text{-heap} \ \Sigma \ \mu \ (set \ ads1) \rrbracket \Longrightarrow
        (?P \ v1 \ A \ C \ \mu \ ads1 \ \Sigma) \lor (cast \ v1 \ A \ C \ \mu \ ads1 = CastError)
    and IH2: \bigwedge x \ xa \ xb \ \Sigma. \llbracket \ \Sigma \vdash v \ v2 : B; \ wt\text{-}heap \ \Sigma \ xa \ (set \ xb) \rrbracket \Longrightarrow
        (?P \ v2 \ B \ D \ xa \ xb \ \Sigma) \lor (cast \ v2 \ B \ D \ xa \ xb = CastError)
    and wtp: \Sigma \vdash v \ VPair \ v1 \ v2 : A \times B \ and \ wth: \ wt-heap \ \Sigma \ \mu \ (set \ ads1)
  from wtp have wtv1: \Sigma \vdash v v1 : A by auto
  from wtp have wtv2: \Sigma \vdash v v2: B by auto
  from wtv1 wth IH1 have IH1conc:
    ?P v1 A C \mu ads1 \Sigma \vee (cast v1 A C \mu ads1 = CastError) by blast
  from IH1conc
  show (?P (VPair v1 v2) (A \times B) (C \times D) \mu ads1 \Sigma)
        \vee \ cast \ (VPair \ v1 \ v2) \ (A \times B) \ (C \times D) \ \mu \ ads1 = CastError
  proof
    assume ?P v1 A C \mu ads1 \Sigma
    from this obtain v1' \Sigma 1 \mu 1 ads 2 where
      cv1: cast v1 \ A \ C \ \mu \ ads1 = Result \ (v1', \mu1, ads2)
      and wtv1: \Sigma 1 \vdash v v1': C and wth1: wt\text{-}heap \Sigma 1 \mu 1 (set ads2)
      and s1s: \Sigma 1 \sqsubseteq \Sigma by blast
    from wtv2 \ s1s have wtv2p: \Sigma 1 \vdash v \ v2 : B using strengthen-value-env apply
blast done
    from wtv2p wth1 IH2 have IH2conc:
       ?P v2 B D \mu1 ads2 \Sigma1 \vee (cast v2 B D \mu1 ads2 = CastError) apply blast
    from IH2conc
    show ?thesis
    proof
      assume ?P \ v2 \ B \ D \ \mu1 \ ads2 \ \Sigma1
      from this obtain v2' \Sigma 2 \mu 2 ads 3 where
        cv2: cast v2 B D \mu 1 ads2 = Result (<math>v2', \mu 2, ads3)
        and wtv2: \Sigma 2 \vdash v v2': D and wth2: wt\text{-}heap \Sigma 2 \mu 2 (set ads3)
        and s2s1: \Sigma 2 \subseteq \Sigma 1 apply fast done
      let ?V = VPair v1' v2'
      from cv1 cv2
      have cvp: cast (VPair v1 v2) (A \times B) (C \times D) \mu ads1 = Result (?V, \mu2,
ads3) by simp
      from wtv1 s2s1 have wtv1p: \Sigma 2 \vdash v v1': C using strengthen-value-env by
blast
      from wtv1p wtv2 have wtp: \Sigma 2 \vdash v ?V : (C \times D) by blast
      from s1s s2s1 have s2s: \Sigma 2 \sqsubseteq \Sigma using lesseq-tyenv-trans by blast
      from wtp wth2 s2s cvp show ?thesis by blast
      assume cast \ v2 \ B \ D \ \mu1 \ ads2 = CastError
      with cv1 show ?thesis by simp
    qed
```

```
next
   assume cast \ v1 \ A \ C \ \mu \ ads1 = CastError
   from this show ?thesis apply auto done
ged
lemma step-safe:
  assumes wtsA: wt-state s A
 shows final s \vee (\exists s'. step \ s = Result \ s' \wedge wt\text{-state } s' \ A)
   \vee step s = CastError
  using wtsA
proof (rule wts)
  \mathbf{fix} \, \Sigma \, \mu \, ads \, \Gamma \, \varrho \, st \, A' \, k
  assume st: s = (st, \varrho, k, \mu, ads) and gr: \Gamma; \Sigma \vdash \varrho and wts: \Gamma \vdash_s st: A'
   and wt-k: \Sigma \vdash k: A' \Rightarrow A and wt-h: wt-heap \Sigma \mu (set ads)
  show ?thesis
  proof (cases ads)
   case (Cons a ads')
   from wt-h Cons have adoms: a \in dom \Sigma apply (simp add: wt-heap-def) done
   from adoms obtain B where las: lookup a \Sigma = Result B
     using dom-lookup[of a \Sigma] by auto
   from las wt-h obtain cv where
     lam: lookup a \mu = Result (cv, B) and wtcv: \Sigma \vdash cv \ cv : B
     and cval: a \notin (set \ ads) \longrightarrow (\exists \ v. \ cv = Val \ v)
     using wt-heap-def [of \Sigma \mu \text{ set ads}] apply blast done
   show ?thesis
   proof (cases cv)
     case (Val\ v)
     have wth2: wt-heap \Sigma \mu (set ads')
       apply (simp only: wt-heap-def)
       apply (rule conjI)
       apply clarify defer
       apply (rule conjI)
       apply clarify defer
       using wt-h Cons apply (simp add: wt-heap-def)
     proof -
       fix a' A' assume las2: lookup a' \Sigma = Result A'
        from las2 wt-h obtain cv' where
         lam2: lookup \ a' \ \mu = Result \ (cv',A') \ {\bf and} \ wtcv2: \Sigma \vdash cv \ cv': A'
         and cval2: a' \notin (set \ ads) \longrightarrow (\exists \ v. \ cv' = \ Val \ v)
         using wt-heap-def [of \Sigma \mu \text{ set ads}] apply blast done
        have ap-ads: (a' \notin set \ ads' \longrightarrow (\exists \ v. \ cv' = \ Val \ v))
        proof
         assume ap\text{-}adsp: a' \notin set \ ads'
         show \exists v. cv' = Val v
         proof (cases a' = a)
           assume aa: a' = a
```

```
from lam2 Val lam aa show ?thesis apply auto done
   next
     assume aa: a' \neq a
     from Cons ap-adsp aa have a' \notin set ads apply auto done
     with cval2 show ?thesis by simp
   qed
  qed
  from lam2 wtcv2 ap-ads
 show \exists cv. lookup a' \mu = Result (cv, A') \land \Sigma \vdash cv cv : A'
   \land (a' \notin set \ ads' \longrightarrow (\exists \ v. \ cv = Val \ v)) by auto
next
 fix a assume a \in dom \mu
 with wt-h show a < length \mu by (simp add: wt-heap-def)
qed
from lam st Cons Val wth2 wt-k qr wts show ?thesis by auto
case (VCast v C B')
from wtcv VCast have wtv: \Sigma \vdash v \ v : C apply blast done
from wtcv\ VCast\ have\ bb:\ B'=B\ by\ blast
from wtcv VCast have cb: B \sqsubseteq C by blast
from wtv wt-h Cons have (\exists v' \Sigma' \mu' ads2.
   cast \ v \ C \ B' \ \mu \ (a\#ads') = Result \ (v', \ \mu', \ ads2) \ \land
  \Sigma' \vdash v \ v' : B' \land wt\text{-}heap \ \Sigma' \ \mu' \ (set \ ads2) \land \Sigma' \sqsubseteq \Sigma) \lor
  cast \ v \ C \ B' \ \mu \ (a\#ads') = CastError
  using cast-safe[of \Sigma \ v \ C \ \mu \ a\#ads' \ B'] by simp
thus ?thesis
proof
 assume (\exists v' \Sigma' \mu' ads2.
  cast \ v \ C B' \ \mu \ (a\#ads') = Result \ (v', \ \mu', \ ads2) \ \land
  \Sigma' \vdash v \ v' : B' \land wt\text{-}heap \ \Sigma' \ \mu' \ (set \ ads2) \land \Sigma' \sqsubseteq \Sigma)
  from this obtain v' \Sigma' \mu' ads2 where
    castv: cast v C B' \mu (a#ads') = Result (v',\mu',ads2) and
   wtvp: \Sigma' \vdash v \ v' : B' and wth2: wt-heap \Sigma' \mu' (set ads2) and
   ss: \Sigma' \sqsubseteq \Sigma apply blast done
  from las ss obtain B2 where las2: lookup a \Sigma' = Result B2
   and bb2: B2 \sqsubseteq B apply (simp add: lesseg-tyenv-def dom-def)
   apply auto apply blast done
  from las2 wth2 obtain cv2 where lam2: lookup a \mu' = Result (cv2,B2)
   apply (simp add: wt-heap-def) apply fast done
  show ?thesis
  proof (cases B \subseteq B2)
   assume b2b: B \sqsubseteq B2
   let ?M2 = (a, (Val\ v', B)) \# \mu'
   let ?S2 = (a,B)\#\Sigma'
   \mathbf{let} ?ads = removeAll a ads2
   let ?S = (st, \varrho, k, ?M2, ?ads)
   from Cons st VCast lam lam2 castv b2b
   have steps: step s = Result ?S by simp
   from wtvp bb have wtvp2: \Sigma' \vdash cv \ Val \ v' : B by blast
```

```
from wth2 las2 b2b
      have wt-heap ?S2 ?M2 (cval-ads (Val v') a (set ads2))
        apply (rule update-heap-val) using wtvp bb apply auto done
      hence wth3: wt-heap ?S2 ?M2 (set ?ads) apply simp done
      from ss\ las2\ b2b have ss2:\ ?S2\ \sqsubseteq\ \Sigma
        apply (simp only: lesseq-tyenv-def)
        apply (frule lookup-dom)
        apply (rule conjI) apply (simp add: dom-def) apply force
        apply auto
        apply (erule-tac x=a in allE)
        apply (erule-tac x=A in allE)
        apply (erule \ impE) apply simp
        apply (erule exE) apply (erule conjE) apply simp
        using lesseq-prec-trans apply blast done
      from qr ss2 have qr2: \Gamma; ?S2 \vdash \rho
        using strengthen-value-env apply blast done
      from wt-k ss2 have wtk2: ?S2 \vdash k : A' \Rightarrow A
        using strengthen-stack apply blast done
      from wth3 qr2 wts wtk2
      have wtS: wt-state ?S A by (rule wts-intro)
      from steps wtS show ?thesis by blast
      assume b2b: \neg B \sqsubseteq B2
      let ?S = (st, \varrho, k, \mu', ads2)
      let ?ads = ads2
      from Cons st VCast lam lam2 castv b2b
      have steps: step s = Result ?S by simp
      have wtS: wt-state ?S A
        apply (rule wts-intro)
        using wth2 apply assumption
        using gr ss strengthen-value-env apply force
        using wts apply assumption
        using wt-k ss strengthen-stack apply force done
      from steps wtS show ?thesis by blast
    qed
   next
    assume cast v \ C \ B' \ \mu \ (a\#ads') = CastError
    with st Cons lam VCast have step s = CastError by simp
    thus ?thesis by simp
   qed
 qed
next
 case Nil
 show ?thesis
 proof (cases st)
   case (SLet \ x \ e \ s2)
   from wts SLet obtain B where wte: \Gamma \vdash_e e : B
```

```
and wts2: (x,B)\#\Gamma \vdash_s s2: A' by blast
     from wte gr wt-h Nil have (\exists v. eval \ e \ \rho \ \mu = Result \ v \land \Sigma \vdash v \ v : B)
        using eval-safe [of \Gamma e B \Sigma \varrho \mu] by simp
     from this obtain v where v: eval e \rho \mu = Result v
       and wtv: \Sigma \vdash v \ v : B \ \mathbf{bv} \ blast
     from v SLet st gr wts2 wt-k wtv wt-h Nil show ?thesis by auto
    next
     case (SRet\ e)
     show ?thesis
     proof (cases k)
        assume k: k = [] from Nil SRet st k have final s by simp
        thus ?thesis by blast
     next
       fix f k' assume k: k = f \# k'
       from wts SRet have wte: \Gamma \vdash_e e : A' by blast
        from wte gr wt-h Nil have (\exists v. eval \ e \ \rho \ \mu = Result \ v \land \Sigma \vdash v \ v : A')
         using eval-safe[of \Gamma e A' \Sigma \varrho \mu] by simp
        from this obtain v where v: eval e \varrho \mu = Result v
         and wtv: \Sigma \vdash v \ v : A' by blast
       from Nil SRet st v wtv k wt-k wt-h show ?thesis by auto
     qed
   next
     case (SCall \ x \ e1 \ e2 \ s2)
     from wts SCall obtain B C where wte1: \Gamma \vdash_e e1 : B \rightarrow C
        and wte2: \Gamma \vdash_e e2: B and wts2: (x,C)\#\Gamma \vdash_s s2: A' by blast
       from wte1 gr wt-h Nil have (\exists v1. eval e1 \varrho \mu = Result v1 \land \Sigma \vdash v v1 :
B \rightarrow C
        using eval-safe[of \Gamma e1 B \rightarrow C \Sigma \rho \mu] by simp
     from this obtain v1 where v1: eval e1 \varrho \mu = Result v1
       and wtv1: \Sigma \vdash v v1: (B \rightarrow C) by blast
     from wte2 gr wt-h Nil have (\exists v2. eval \ e2 \ \varrho \ \mu = Result \ v2 \land \Sigma \vdash v \ v2 : B)
        using eval-safe [of \Gamma e2 B \Sigma \varrho \mu] by simp
     from this obtain v2 where v2: eval\ e2\ \varrho\ \mu=Result\ v2
       and wtv2: \Sigma \vdash v v2 : B by blast
     from SCall st v1 v2 wtv1 gr wtv2 wts2 wt-k wt-h Nil show ?thesis
        by (case-tac v1, auto, case-tac const, auto)
   next
     case (STailCall e1 e2)
     from wts STailCall obtain B where wte1: \Gamma \vdash_e e1 : B \rightarrow A'
       and wte2: \Gamma \vdash_e e2: B by blast
       from wte1 gr wt-h Nil have (\exists v1. eval\ e1\ \varrho\ \mu = Result\ v1\ \land\ \Sigma \vdash v\ v1:
B \rightarrow A'
       using eval-safe [of \Gamma e1 B \rightarrow A' \Sigma \rho \mu] by simp
     from this obtain v1 where v1: eval e1 \varrho \mu = Result v1
       and wtv1: \Sigma \vdash v v1 : (B \rightarrow A') by blast
     from wte2 gr wt-h Nil have (\exists v2. eval\ e2\ \varrho\ \mu = Result\ v2 \land \Sigma \vdash v\ v2:B)
        using eval-safe [of \Gamma e2 B \Sigma \rho \mu] by simp
     from this obtain v2 where v2: eval\ e2\ \varrho\ \mu=Result\ v2
       and wtv2: \Sigma \vdash v v2: B by blast
```

```
from wtv1 show ?thesis
 proof (rule funval-inv)
   fix c assume typeof c = B \rightarrow A' hence False
     apply (case-tac c) apply auto done
   thus ?thesis by simp
 next
   fix \Gamma' \rho' x s' assume clos: v1 = Closure x B s' \rho' and ge: \Gamma'; \Sigma \vdash \rho'
     and wtsp: (x,B)\#\Gamma'\vdash_s s':A'
   show ?thesis apply (rule disjI2) apply (rule disjI1)
   proof -
     from clos v1 v2 Nil have s:
      step (STailCall e1 e2, \varrho, k, \mu,[]) = Result (s', (x,v2)#\varrho', k, \mu,[])
      by simp
     from ge wtsp wt-k wtv2 wt-h Nil
     have wtns: wt-state (s', (x,v2)\#\varrho', k, \mu, ||) A by auto
     from s wtns st STailCall Nil
     show \exists s'. step s = Result s' \land wt-state s' \land A
      by auto
   qed
 qed
next
 case (SAlloc \ x \ B \ e \ s')
 from wts SAlloc have wte: \Gamma \vdash_e e : B by fast
 from wts SAlloc Nil have wts2: (x,RefT\ B)\#\Gamma \vdash_s s':A' by fast
 from wte gr wt-h Nil have (\exists v. eval \ e \ \rho \ \mu = Result \ v \land \Sigma \vdash v \ v : B)
   using eval-safe[of \Gamma e B \Sigma \rho \mu] by simp
 from this obtain v where v: eval e \rho \mu = Result v
   and wtv: \Sigma \vdash v \ v : B \ using \ eval-safe \ by \ blast
 let ?a = length \mu
 let ?S2 = (?a,B)\#\Sigma and ?M2 = (?a,(Val\ v,\ B))\#\mu
 from wt-h dom-lookup[of ?a \Sigma] wt-heap-def[of \Sigma \mu] lookup-dom[of ?a \mu]
 have ads: ?a \notin dom \Sigma apply blast done
 from wtv ads have wtv-2: ?S2 \vdash v v : B
   using weaken-value-env[of \Sigma \ v \ B] apply fast done
 from ads wtv-2 wt-h Nil have wt-h2: wt-heap ?S2 ?M2 {}
   apply (simp only: wt-heap-def) apply (rule conjI)
   apply clarify apply (case-tac a = length \mu)
   apply simp using weaken-value-env apply force
   apply (erule-tac x=a in allE)
   apply (erule-tac x=a in allE)
   apply (erule-tac x=A in allE)
   apply (erule impE) apply (erule impE)
   apply simp apply (erule exE) apply clarify
   using lookup-dom apply force
   apply auto
   using weaken-value-env apply force
   apply (simp add: dom-def) apply auto done
 from ads wt-k have wt-k2: ?S2 \vdash k : A' \Rightarrow A using weaken-stack by force
 from ads gr have gr-2: \Gamma; ?S2 \vdash \varrho
```

```
using weaken-value-env apply auto done
     let ?s2 = (s', (x, VRef ?a) \# \varrho, k, (?a, (Val v, B)) \# \mu, [])
     from Nil v SAlloc st have step-s: step s = Result ?s2
       apply (simp add: Let-def) done
     from qr-2 wts2 wt-k2 wtv-2 wt-h2
     have wts2: wt-state ?s2 A by auto
     from step-s wts2 show ?thesis by fast
     case (SUpdate\ e1\ e2\ s')
     from wts SUpdate obtain B where wte1: \Gamma \vdash_e e1 : RefT B
       and wte2: \Gamma \vdash_e e2: B and wts2: \Gamma \vdash_s s': A'
       and sb: static B by fast
     from wte1 gr wt-h Nil have (\exists v1. eval\ e1\ \varrho\ \mu = Result\ v1\ \land\ \Sigma \vdash v\ v1: RefT
B)
       using eval-safe [of \Gamma e1 RefT B \Sigma \varrho \mu] by simp
     from this obtain v1 where v1: eval e1 \rho \mu = Result v1
       and wtv1: \Sigma \vdash v v1 : RefT \ B apply clarify apply assumption done
     from wte2 gr wt-h Nil have (\exists v2. \ eval \ e2 \ \varrho \ \mu = Result \ v2 \land \Sigma \vdash v \ v2 : B)
       using eval-safe [of \Gamma e2 B \Sigma \varrho \mu] by simp
     from this obtain v2 where v2: eval e2 \rho \mu = Result v2
       and wtv2: \Sigma \vdash v v2: B by fast
     from wtv1 obtain a B' where v1a: v1 = VRef a
       and las: lookup a \Sigma = Result \ B' and bb: B' \subseteq B
       apply auto apply (case-tac c) apply auto done
     from sb bb have bbeq: B = B'
       using static-is-most-precise apply blast done
     from las wt-h Nil obtain v where lam: lookup a \mu = Result (Val v, B')
       and wtv: \Sigma \vdash v \ v : B'
       apply (simp only: wt-heap-def) apply auto
       apply (erule-tac x=a in allE)
       apply (erule-tac x=a in allE)
       apply (erule-tac x=B' in allE)
       apply auto done
     let ?M2 = (a, (Val\ v2, B')) \# \mu
     let ?S2 = (a, B') \# \Sigma
     from las have ss: ?S2 \sqsubseteq \Sigma
       apply (simp add: lesseq-tyenv-def dom-def) apply auto
       using lookup-dom[of\ a\ \Sigma]\ dom-def\ apply\ force\ done
     from Nil wt-h wtv2 las bbeq have wth2: wt-heap ?S2 ?M2 {}
       using update-heap-val[of \Sigma \mu set ads a B' B' Val v2]
       apply simp apply auto done
     have wts2: wt-state (s', \varrho, k, ?M2, []) A
       apply (rule wts-intro)
       apply simp
       using wth2 apply simp
       using gr ss strengthen-value-env apply blast
       using wts2 apply blast
       using wt-k ss strengthen-stack apply blast
       done
```

```
from st Nil v1 v2 SUpdate lam v1a wts2 show ?thesis by simp
   next
     case (SDynUpdate e1 e2 B s')
     from wts SDynUpdate have wte1: \Gamma \vdash_e e1 : RefT B by fast
     from wts SDynUpdate have wte2: \Gamma \vdash_e e2: B by fast
     from wts SDynUpdate have wts2: \Gamma \vdash_s s': A' by fast
    from wte1 gr wt-h Nil have (\exists v1. eval\ e1\ \varrho\ \mu = Result\ v1\ \land\ \Sigma \vdash v\ v1: RefT
B)
       using eval-safe[of \Gamma e1 RefT B \Sigma \varrho \mu] by simp
     from this obtain v1 where v1: eval e1 \varrho \mu = Result v1
       and wtv1: \Sigma \vdash v v1 : RefT \ B apply clarify apply assumption done
     from wte2 gr wt-h Nil have (\exists v2. eval \ e2 \ \varrho \ \mu = Result \ v2 \land \Sigma \vdash v \ v2 : B)
       using eval-safe [of \Gamma e2 B \Sigma \varrho \mu] by simp
     from this obtain v2 where v2: eval e2 \varrho \mu = Result v2
       and wtv2: \Sigma \vdash v v2: B by fast
     from wtv1 obtain a B' where v1a: v1 = VRef a
       and las: lookup a \Sigma = Result \ B' and bb: B' \subseteq B
       apply auto apply (case-tac c) apply auto done
     from las wt-h Nil obtain v where lam: lookup a \mu = Result (Val v, B')
      and wtv : \Sigma \vdash v \ v : B'
      apply (simp only: wt-heap-def) apply auto
      apply (erule-tac x=a in allE)
      apply (erule-tac x=a in allE)
      apply (erule-tac x=B' in allE)
       apply auto done
     from lam Nil v1 v2 SDynUpdate st v1a
     have steps: step s = Result (s', \rho, k, (a, VCast v2 B B', B') \# \mu, [a])
      by simp
     from wtv2 bb have wtcv2: \Sigma \vdash cv VCast v2 B B': B' by blast
     let ?S2 = (a, B') \# \Sigma
     let ?M2 = (a, VCast v2 B B', B') \# \mu
     from wt-h Nil las wtcv2 have wt-h2: wt-heap ?S2 ?M2 {a}
       using update-heap-val[of \Sigma \mu {} a B' B' VCast v2 B B'] by auto
     from las have ss: ?S2 \sqsubseteq \Sigma
      apply (simp add: lesseq-tyenv-def)
      apply (simp add: dom-def)
      apply (frule lookup-dom)
       apply (simp add: dom-def) apply blast done
     have wt-s: wt-state (s', \varrho, k, ?M2, [a]) A
      apply (rule wts-intro)
      using wt-h2 apply simp
      using gr ss strengthen-value-env apply blast
       using wts2 apply simp
      using wt-k ss strengthen-stack apply blast done
     from steps wt-s show ?thesis by simp
   next
     case (SCast \ x \ e \ B \ C \ s')
     from wts SCast have wte: \Gamma \vdash_e e : B by auto
     from wts SCast have wts2: (x,C)\#\Gamma \vdash_s s': A' by auto
```

```
from wte gr wt-h Nil have (\exists v. eval \ e \ \rho \ \mu = Result \ v \land \Sigma \vdash v \ v : B)
      using eval-safe[of \Gamma e B \Sigma \varrho \mu] by simp
    from this obtain v where v: eval e \varrho \mu = Result v
      and wtv: \Sigma \vdash v \ v : B \ \textbf{using} \ eval\text{-safe} \ \textbf{by} \ blast
    from wtv wt-h have (\exists v' \Sigma' \mu' ads2.
      cast \ v \ B \ C \ \mu \ ads = Result \ (v', \mu', ads2) \ \land
      \Sigma' \vdash v \ v' : C \land wt\text{-}heap \ \Sigma' \ \mu' \ (set \ ads2) \land \Sigma' \sqsubseteq \Sigma) \lor
      cast \ v \ B \ C \ \mu \ ads = CastError \ using \ cast-safe \ by \ blast
    thus ?thesis
    proof
      assume (\exists v' \Sigma' \mu' \ ads2.
        cast \ v \ B \ C \ \mu \ ads = Result \ (v', \mu', ads2) \ \land
        \Sigma' \vdash v \ v' : C \land wt\text{-}heap \ \Sigma' \ \mu' \ (set \ ads2) \land \Sigma' \sqsubseteq \Sigma)
      from this obtain v' \Sigma' \mu' ads2 where
        castv: cast v B C \mu ads = Result (v',\mu',ads2) and
        wtvp: \Sigma' \vdash v \ v' : C and wth2: wt-heap \Sigma' \mu' (set ads2) and
        ss: \Sigma' \sqsubseteq \Sigma \text{ by } blast
      let ?R2 = (x,v')\#\varrho
      let ?G2 = (x,C)\#\Gamma
      from qr ss have qr2: \Gamma; \Sigma' \vdash \rho using strengthen-value-env by blast
      from wtvp gr2 have gr3: ?G2;\Sigma' \vdash ?R2 by (rule wt-cons)
      from castv SCast Nil v st
      have steps: step s = Result (s', (x, v') \# \varrho, k, \mu', ads2) by simp
      have wt-s: wt-state (s', (x, v') \# \varrho, k, \mu', ads2) A
        apply (rule wts-intro)
        using wth2 Nil apply simp
        using gr3 apply simp
        using wts2 apply simp
        using wt-k ss strengthen-stack apply blast done
      from steps wt-s show ?thesis by blast
      assume cast\ v\ B\ C\ \mu\ ads = CastError
      from this SCast Nil st v show ?thesis by simp
    qed
next
 case (SDynDeref \ x \ e \ B \ s')
 from wts SDynDeref have wte: \Gamma \vdash_e e : RefT \ B by blast
 from wts SDynDeref have wtsp: (x,B)\#\Gamma \vdash_s s': A' by blast
 from wte gr wt-h Nil have (\exists v. eval \ e \ \rho \ \mu = Result \ v \land \Sigma \vdash v \ v : RefT \ B)
    using eval-safe[of \Gamma e RefT B \Sigma \varrho \mu] by simp
 from this obtain v where v: eval e \varrho \mu = Result v
    and wtv: \Sigma \vdash v \ v : RefT \ B \ using \ eval-safe \ by \ blast
  from wtv obtain a B' where va: v = VRef a
   and las: lookup a \Sigma = Result \ B' and bb: B' \sqsubseteq B apply auto
   apply (case-tac c) apply auto done
  from las wt-h Nil obtain v2 where lam: lookup a \mu = Result (Val v2,B')
    and wtv2: \Sigma \vdash vv2: B' apply (simp\ add:\ wt\text{-}heap\text{-}def) by blast
 from wtv2 wt-h las have (\exists v' \Sigma' \mu' ads2.
    cast \ v2 \ B' \ B \ \mu \ ads = Result \ (v', \mu', ads2) \ \land
```

```
\Sigma' \vdash v \ v' : B \land wt\text{-}heap \ \Sigma' \ \mu' \ (set \ ads2) \land \Sigma' \sqsubseteq \Sigma) \lor
      cast \ v2 \ B' \ B \ \mu \ ads = CastError \ using \ cast-safe \ by \ blast
   thus ?thesis
   proof
      assume (\exists v' \Sigma' \mu' ads2.
        cast \ v2 \ B' \ B \ \mu \ ads = Result \ (v', \mu', ads2) \ \land
        \Sigma' \vdash v \ v' : B \land wt\text{-}heap \ \Sigma' \ \mu' \ (set \ ads2) \land \Sigma' \sqsubseteq \Sigma)
      from this obtain v' \Sigma' \mu' ads2 where
        castv2: cast v2 B' B \mu ads = Result (v', \mu', ads2)
        and wtvp: \Sigma' \vdash v \ v' : B
       and wth2: wt-heap \Sigma' \mu' (set ads2)
       and ss: \Sigma' \sqsubseteq \Sigma by blast
      from st Nil SDynDeref castv2 v va lam
      have steps: step s = Result (s', (x, v') \# \varrho, k, \mu', ads2) by simp
      from gr ss have gr2: \Gamma; \Sigma' \vdash \varrho using strengthen-value-env by blast
      from gr2 wtvp have gr3: (x,B)\#\Gamma;\Sigma'\vdash (x,v')\#\varrho by blast
      have wt-s: wt-state (s', (x, v') \# \varrho, k, \mu', ads2) A
       apply (rule wts-intro)
       using wth2 Nil apply simp
       using qr3 apply blast
        using wtsp apply blast
        using wt-k ss strengthen-stack apply blast done
      from steps wt-s show ?thesis by simp
   next
      assume cast \ v2 \ B' \ B \ \mu \ ads = CastError
      from st Nil SDynDeref this v va lam show ?thesis by simp
    qed
 ged
qed
qed
lemma observe-safe:
  assumes wtv: \Sigma \vdash v \ v : A
  shows wt-observable (observe v) A
  using wtv apply (induct v arbitrary: \Sigma A)
  apply (case-tac const) apply force+ done
```

For this lemma, we choose not to use the induction rule for steps because that induction rule is a bit messy, with lots of cases that can be dealt with in a similar fashion. In the following, we just do proof by induction on n.

```
lemma steps-safe:
   assumes wtsA: wt-state s A
   shows \exists r. steps n s = r \land wt-observable r A
   using wtsA
proof (induct \ n \ arbitrary: \ s)
   fix s have steps 0 s = OTimeOut by simp
   thus \exists r. steps 0 s = r \land wt-observable r A by auto
next
   fix n s
```

```
assume IH: \land s. wt-state s A \Longrightarrow
              (\exists r. steps \ n \ s = r \land wt\text{-}observable \ r \ A)
   and wts: wt\text{-}state \ s \ A
  \{ assume final s \}
    from this obtain e \rho \mu where s: s = (SRet e, \rho, [], \mu, [])
     apply (case-tac s) apply (case-tac a) apply auto
     apply (case-tac c) apply auto apply (case-tac e) apply auto done
    from wts \ s obtain \Gamma \ \Sigma where wte: \Gamma \vdash_e e : A and wtg: \Gamma;\Sigma \vdash \varrho
     and wth: wt-heap \Sigma \mu {} by \mathit{auto}
    from wte wtg wth have (\exists v. eval \ e \ \varrho \ \mu = Result \ v \land \Sigma \vdash v \ v : A)
     \vee eval e \varrho \mu = CastError
     using eval-safe [of \Gamma e A \Sigma \varrho \mu] by simp
   hence \exists r. steps (Suc \ n) \ s = r \land wt\text{-}observable \ r \ A
   proof
     assume \exists v. \ eval \ e \ \varrho \ \mu = Result \ v \land \Sigma \vdash v \ v : A
     from this obtain v where ev: eval e \rho \mu = Result v
       and wtv : \Sigma \vdash v \ v : A by blast
     from wtv have wt-observable (observe v) A using observe-safe by blast
     with s ev show \exists r. steps (Suc \ n) \ s = r \land wt\text{-}observable \ r \ A \ by \ auto
     assume eval e \rho \mu = CastError
     with s show ?thesis apply simp done
  } moreover { assume fs: \neg final s
    from fs wts have (\exists s'. step s = Result s' \land wt\text{-state } s' A)
     \vee step s = CastError using step-safe[of s A] by simp
   hence \exists r. steps (Suc n) s = r \land wt\text{-}observable r A
   proof
     assume \exists s'. step s = Result s' \land wt\text{-state } s' A
     from this obtain s' where st: step <math>s = Result s'
       and wtsp: wt-state s' A by blast
     from wtsp IH have ssp:
        \exists r. steps \ n \ s' = r \land wt\text{-}observable \ r \ A \ \mathbf{by} \ blast
     from fs st have ss: steps (Suc n) s = steps n s'
       apply auto apply (case-tac a) apply auto apply (case-tac ab) apply auto
       apply (case-tac b) apply auto done
     from ssp ss show ?thesis by simp
     assume step \ s = CastError
     from fs this show ?thesis apply simp
       apply (case-tac s) apply simp apply (case-tac a) apply auto
       apply (case-tac c) apply auto apply (case-tac e) apply auto done
  } ultimately show \exists r. steps (Suc n) s = r \land wt\text{-}observable r A by blast
qed
theorem type-safety:
 assumes wts: [] \vdash_s s : A
  shows \exists r. run s = r \land wt\text{-}observable \ r \ A
```

```
proof — have wtg: []; [] \vdash [] by fast from wtg wts have 1: wt-state (s, [], [], [], []) A by (auto\ simp:\ wt-heap-def dom-def) let ?n = 1000000 and ?s = (s, [], [], []) from 1 have 2: \exists \ r.\ steps\ ?n\ ?s = r \land wt-observable r A by (rule\ steps-safe) from 2 obtain r where 4:\ steps\ ?n\ ?s = r and 5:\ wt-observable r A by blast from 4 have 6:\ run\ s = r by (simp\ add:\ run-def) from 6\ 5 show \exists\ r.\ run\ s = r \land wt-observable r A by blast qed
```

end

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