Concurrent Dynamic Algebra

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Abstract

We reconstruct Peleg's concurrent dynamic logic in the context of modal Kleene algebras. We explore the algebraic structure of its multirelational semantics and develop an abstract axiomatisation of concurrent dynamic algebras from that basis. In this axiomatisation, sequential composition is not associative. It interacts with concurrent composition through a weak distributivity law. The modal operators of concurrent dynamic algebra are obtained from abstract axioms for domain and antidomain operators; the Kleene star is modelled as a least fixpoint. Algebraic variants of Peleg's axioms are shown to be valid in these algebras and their soundness is proved relative to the multirelational model. Additional results include iteration principles for the Kleene star and a refutation of variants of Segerberg's axiom in the multirelational setting. The most important results have been verified formally with Isabelle/HOL.

1 Introduction

Concurrent dynamic logic (CDL) has been proposed almost three decades ago by Peleg [20] as an extension of propositional dynamic logic (PDL) [9] to study concurrency in "its purest form as the dual notion of nondeterminism". In this setting, a computational process is regarded as a tree with two dual kinds of branchings. According to the first one, the process may choose a transition along one of the possible branches. This is known as angelic, internal or existential choice. According to the second one, it progresses along all possible branches in parallel, which is known as demonic, external or universal choice. This lends itself to a number of interpretations.

One of them associates computations with games processes play against a scheduler or environment as their opponent. A process wins if it can successfully resolve all internal choices and respond to all external choices enforced by the opponent. Another one considers machines which accept inputs by nondeterministically choosing one transition along exixtential branches and executing all transitions in parallel amongst universal ones. In yet another one, universal choices correspond to agents cooperating towards a collective goal while existential choices are made in competition by individual agents. Finally, in shared-variable concurrency, interferences caused by different threads accessing a global variable are observed as nondeterministic assignments by particular threads; hence as external choices imposed by the other threads.

Historically, in fact, CDL has been influenced by work on alternating state machines [3] and Parikh's game logic (GL) [17, 18], which is itself based on PDL. Other aspects of concurrency such as communication or synchronisation, which are at the heart of formalisms such as Petri nets or process algebras, are ignored in its basic axiomatisation.

Standard PDL has a relational semantics. This captures the input/output dependencies of sequential programs. Internal choice is modelled as union, sequential composition as relational composition. External choice, however, cannot be represented by this semantics. It requires relating an individual input to a set of outputs, that is, relations of type $A \times 2^B$ instead of $A \times B$. These are known as multirelations.

In multirelational semantics, external choice still corresponds to union, but sequential composition must be redefined. According to Parikh's definition, a pair (a, A) is in the sequential composition of multirelation R with multirelation S if R relates element a with an intermediate set B and every element of B is related to the set A by S. According to Peleg's more general definition, it suffices that S relates each element $b \in B$ with a set C_b as long as the union of all the sets C_b yields the set A. In addition, a notion of external choice or parallel composition can now be defined. If a pair (a, A) is in a multirelation R and a pair (a, B) in a multirelation S, then the parallel composition of R and S contains the pair $(a, A \cup B)$. Starting from input a, the multirelations R and S therefore produce the collective output $A \cup B$ when executed in parallel. In contrast to Peleg, Parikh also imposes additional conditions on multirelations. In particular, they must be up-closed: $(a, A) \in R$ and $A \subseteq B$ imply $(a, B) \in R$.

In CDL, modal box and diamond operators are associated with the multirelational semantics as they are associated with a relational semantics in PDL. An expression $[\alpha]\varphi$ means that after every terminating execution of program α , property φ holds, whereas $\langle \alpha \rangle \varphi$ means that there is a terminating execution of α after which φ holds. In CDL, as in PDL, boxes and diamonds are related by De Morgan duality: $[\alpha]\varphi$ holds if and only if $\neg \langle \alpha \rangle \neg \varphi$ holds. The axioms of CDL describe how the programming constructs of external choice, sequential and parallel composition, and (sequential) iteration interact with the modalities. CDL can as well be seen as a generalisation of dual-free GL.

Wijesekera and Nerode [14, 27] as well as Goldblatt [8] have generalised CDL to situations where boxes and diamonds are no longer dual. GL has been applied widely in game and social choice theory. A bridge between the two formalisms has recently been built by van Benthem et al. [26] to model simultaneous games as they arise in algorithmic game theory. Peleg has added notions of synchronisation and communication to CDL [19]. Parikh's semantics of upclosed multirelations and its duality to monotone predicate transformers has reappeared in Back and von Wright's refinement calculus [1] and the approach to multirelational semantics of Rewitzky and coworkers [23, 24, 12]. Up-closed multirelations have also been studied more abstractly as a variant of Kleene algebra [7, 16]. Finally, the transitions in alternating automata can be represented as multirelations.

This suggests that CDL and its variants are relevant to games and concurrency; they provide insights in games for concurrency and for concurrency in games. Despite this, beyond the up-closed case, the algebra of multirelations, as a generalisation of Kleene algebras [10] and Tarski's relation algebra (cf. [11]), has never been studied in detail and *concurrent dynamic algebras* as algebraic

companions of CDL remain to be established. This is in contrast to PDL where the corresponding dynamic algebras [22] and test algebras [13, 25, 21] are well studied

An algebraic reconstruction of CDL complements the logical one in important ways. Algebras of multirelations yield abstract yet fine-grained views on the structure of simultaneous games; they might also serve as intermediate semantics for shared-variable concurrency, where interferences have been resolved. The study of dynamic and test algebras shows how modal algebras arise from Kleene and relation algebras in particularly simple and direct ways, and powerful tools from universal algebra and category theory are available for their analysis. Reasoning with modal algebras is essentially first-order equational and therefore highly suitable for mechanisation and automation. In the context of CDL this would make the design of tools for analysing games or concurrent programs particularly simple and flexible.

Our main contribution is an axiomatisation of concurrent dynamic algebras. It is obtained from axiomatisations of the algebra of multirelations which generalise modal Kleene algebras [6, 5, 4]. In more detail, our main results are as follows.

- We investigate the basic algebraic properties of the multirelational semantics of CDL. It turns out that those of sequential composition are rather weak—the operation is, for instance, non-associative—while concurrent composition and union form a commutative idempotent semiring. We also find a new interaction law between sequential and concurrent composition. In addition we investigate special properties of subidentities, which serve as propositions and tests in CDL, and of multirelational domain and antidomain (domain complement) operations.
- We axiomatise variants of semirings (called *proto-dioids* and *proto-trioids*) which capture the basic algebra of multirelations without and with concurrent composition. We expand these structures by axioms for domain and antidomain operations, explore the algebraic laws governing these operations and characterise the subalgebras of domain elements, which serve as state or proposition spaces in this setting. We also prove soundness with respect to the underlying multirelational model.
- We define algebraic diamond and box operators from the domain and antidomain ones as abstract preimage operators and their De Morgan duals and show that algebraic counterparts of the axioms of star-free CDL can be derived in this setting. The diamond axioms of CDL are obtained over a state space which forms a distributive lattice; the additional box axioms are derivable over a boolean algebra.
- We investigate the Kleene star (or reflexive transitive closure operation) in the multirelational model and turn the resulting laws into axioms of proto-Kleene algebras with domain and antidomain as well as proto-bi-Kleene algebras with domain and antidomain. The latter two allow us to derive the full set of CDL axioms; they are therefore informally called concurrent dynamic algebras. Once more we prove soundness with respect to the underlying multirelational model.

• Finally, we study notions of finite iteration for the Kleene star in the multirelational setting and refute the validity of a variant of Segerberg's axiom of PDL.

The complete list of concurrent dynamic algebra axioms can be found in Appendix 1.

Our analysis of the multirelational model and our axiomatisations are minimalistic in the sense that we have tried to elaborate the most general algebraic conditions for deriving the CDL axioms. Many interesting properties of that model have therefore been ignored. Due to the absence of associativity of sequential composition and of left distributivity of sequential composition over union, many proofs seem rather fragile and depend on stronger algebraic properties of special elements. Sequential composition is, for instance, associative if one of the participating multirelations is a domain or antidomain element. This requires a significant generalisation of previous approaches to Kleene algebras with domain and antidomain [5, 4].

Moreover, proofs about multirelations are rather tedious due to the complexity of sequential composition—specifying the family of sets C_b requires second-order quantification. We have therefore formalised and verified the most important proofs with the Isabelle proof assistant [15] (see Appendix 3 for a list). Thus our work is also an exercise in formalised mathematics. The complete code can be found online¹. We also present all manual proofs in order to make this article selfcontained; the less interesting ones have been delegated to Appendix 2.

2 Multirelations

A multirelation R over a set X is a subset of $X \times 2^X$. Inputs $a \in X$ are related by R to outputs $A \subseteq X$; each single input a may be related to many subsets of X. The set of all multirelations over X is denoted M(X).

An intuitive interpretation is the accessibility or reachability in a (directed) graph: (a, A) means that the set A of vertices is reachable from vertex a in the graph. (a, \emptyset) means that no set of vertices is reachable from a, which makes a a terminal node. This is different from (a, A) not being an element of a multirelation for all $A \subseteq X$.

By definition, (a, A) and (a, \emptyset) can be elements of the same multirelation. This can be interpreted as a system, program or player making an "interal", existential or angelic choice to access either A or \emptyset . The elements of A can therefore be seen as "external", universal or demonic choices made by an environment, scheduler or adversary player.

This ability to capture internal and external choices makes multirelations relevant to games and game logics [18], demonic/angelic semantics of programs [1, 12], alternating automata and concurrency [20]. Different applications, however, require different definitions of operations on multirelations. The one used in the concurrent setting by Peleg [20] and Goldblatt [8] is the most general one and we follow it in this article.

Example 1. Let $X = \{a, b, c, d\}$. Then

$$R = \{(a,\emptyset), (a,\{d\}), (b,\{a\}), (b,\{b\}), (b,\{a,b\})(c,\{a\}), (c,\{d\})\}$$

¹http://www.dcs.shef.ac.uk/~georg/isa/cda

is a multirelation over X. Vertex a can alternatively reach no vertex at all—the empty set—or the singleton set $\{d\}$. Vertex b can either reach set $\{a\}$, set $\{b\}$ or their union $\{a,b\}$. Vertex c can either reach set $\{a\}$ or set $\{d\}$, but not their union. Vertex d cannot even reach the empty set; no execution from it is enabled. This is in contrast to the situation (a,\emptyset) , where execution is enabled from a, but no state can be reached.

Peleg defines the following operations of sequential and concurrent composition of multirelations. Let R and S be multirelations over X. The sequential composition of R and S is the multirelation

$$R \cdot S = \{(a, A) \mid \exists B. \ (a, B) \in R \land \exists f. \ (\forall b \in B. \ (b, f(b)) \in S) \land A = \bigcup f(B) \}.$$

The unit of sequential composition is the multirelation

$$1_{\sigma} = \{(a, \{a\}) \mid a \in X\}.$$

The parallel composition of R and S is the multirelation

$$R||S = \{(a, A \cup B) \mid (a, A) \in R \land (a, B) \in S\}.$$

The unit of parallel composition is the multirelation

$$1_{\pi} = \{(a, \emptyset) \mid a \in X\}.$$

The universal multirelation over X is

$$U = \{(a, A) \mid a \in X \land A \subseteq X\}.$$

In the definition of sequential composition, $f(B) = \{f(b) \mid b \in B\}$ is the image of B under f. The intended meaning of $(a,A) \in R \cdot S$ is as follows: the set A is reachable from vertex a by $R \cdot S$ if some intermediate set B is reachable from a by R, and from each vertex $b \in B$ a set A_b is reachable (represented by f(b)) such that $A = \bigcup_{b \in B} A_b = \bigcup f(B)$. Thus, from each vertex $b \in B$, the locally reachable set f(b) contributes to the global reachability of A. We write $G_f(b) = (b, f(b))$ for the graph of f at point b, and $G_f(B) = \{G_f(b) \mid b \in B\}$ for the graph of f on the set B. We can then write

$$(a, A) \in R \cdot S \Leftrightarrow \exists B. \ (a, B) \in R \land \exists f. \ G_f(B) \subseteq S \land A = \bigcup f(B).$$

This definition of sequential composition is subtly different to the one used by Parikh [18] in game logics, which appears also in papers on multirelational semantics and monotone predicate transformers. In addition, Parikh considers up-closed multirelations. This leads not only to much simpler proofs, but also to structural differences. Peleg has argued that up-closure is not desirable for concurrency since it makes all programs—even tests—automatically nondeterministic.

The sequential identity 1_{σ} is defined similarly to the identity relation or identity function. It is given by (the graph of) the embedding $\lambda x.\{x\}$ into singleton sets.

In a parallel composition, $(a, A) \in R || S$ if A is reachable from a by R or S in collaboration, that is, each of R and S must contribute a part of the reachability to A.

The parallel identity 1_{π} is the function λx . \emptyset , which does not reach any set from any vertex. Two interpretations of a pair (a, \emptyset) suggest themselves: it might be the case that nothing is reachable from a due to an error or due to nontermination.

Example 2. Consider the multirelations

$$R = \{(a, \{b, c\})\}, \qquad S = \{(b, \{b\})\}, \qquad T = \{(b, \{b\}), (c, \emptyset)\}.$$

Then $R \cdot S = \emptyset$ because S cannot contribute from c. Moreover, $R \cdot T = \{(a, \{b\})\}$. Finally, $T \cdot S = T$, since, from c, the empty set is the only intermediate set which satisfies the conditions for S and A above.

Example 3. Consider the multirelations

$$R = \{(a, \{a, b\})\}, \qquad S = \{(a, \{b, c\}), (b, \{b\})\}, \qquad T = \{(b, \emptyset)\}.$$
 Then $R||S = \{(a, \{a, b, c\})\}$ and $S||T = \{(b, \{b\})\}.$

3 Basic Laws for Multirelations

The definition of sequential composition is higher-order and some of our proofs of our proofs use higher-order Skolemisation, which is an instance of the Axiom of Choice:

$$(\forall a \in A. \exists b. \ P(a,b)) \Leftrightarrow (\exists f. \forall a \in A. \ P(a,f(a))).$$

First we derive some basic laws of sequential composition.

Lemma 1. Let R, S and T be multirelations.

- 1. $R \cdot 1_{\sigma} = R$ and $1_{\sigma} \cdot R = R$,
- 2. $\emptyset \cdot R = \emptyset$,
- 3. $(R \cdot S) \cdot T \subseteq R \cdot (S \cdot T)$,
- $4. (R \cup S) \cdot T = R \cdot T \cup S \cdot T,$
- 5. $R \cdot S \cup R \cdot T \subseteq R \cdot (S \cup T)$.

See Appendix 2 for proofs. Property (1) confirms that 1_{σ} is indeed an identity of sequential composition. Property (2) shows that \emptyset is a left annihilator. It is, however, not a right annihilator by Lemma 4 below. Similarly, (3) is a weak associativity law which, again by Lemma 4, cannot be strengthened to an identity. In fact, (3) is not needed for the algebraic development in this paper; it is listed for the sake of completeness. Property (5) is a left subdistributivity law for sequential composition, which, again by Lemma 4, cannot be strengthened to an identity. Left subdistributivity and right distributivity imply that sequential composition is left and right isotone:

$$R \subseteq S \Rightarrow T \cdot R \subseteq T \cdot S, \qquad R \subseteq S \Rightarrow R \cdot T \subseteq S \cdot T.$$

Next we verify some basic laws of concurrent composition. These reveal more pleasant algebraic structure.

Lemma 2. Let R, S and T be multirelations.

- 1. (R||S)||T = R||(S||T),
- 2. R||S = S||R,
- 3. $R||1_{\pi} = R$,
- 4. $R \| \emptyset = \emptyset$,
- 5. $R||(S \cup T) = R||S \cup R||T$.

See Appendix 2 for proofs. This show that multirelations under union and parallel composition form a commutative dioid, as introduced in Section 6. It follows that concurrent composition is left and right isotone:

$$R\subseteq S\Rightarrow T\|R\subseteq T\|S, \qquad R\subseteq S\Rightarrow R\|T\subseteq S\|T.$$

Next we establish an important interaction law between sequential and concurrent composition; a right subdistributivity law of sequential over concurrent composition.

Lemma 3. Let R, S and T be multirelations. Then

$$(R||S) \cdot T \subseteq (R \cdot T)||(S \cdot T).$$

See Appendix 2 for a proof. Once more, this general law is not needed for our algebraic development. We use a full right distributivity law that holds in particular cases.

Finally, counterexamples show that the algebraic properties studied so far are sharp.

Lemma 4. There are multirelations R, S and T such that

- 1. $R \cdot \emptyset \neq \emptyset$,
- 2. $R \cdot (S \cdot T) \not\subseteq (R \cdot S) \cdot T$,
- 3. $R \cdot (S \cup T) \not\subseteq R \cdot S \cup R \cdot T$,
- 4. $(R \cdot T) \| (S \cdot T) \not\subseteq (R \| S) \cdot T$.

Proof. 1. Let $R = \{(a, \emptyset)\}$. Then $(a, A) \in R \cdot S \Leftrightarrow \exists f. \ G_f(B) \in S \land A = \bigcup f(\emptyset) \Leftrightarrow A = \emptyset$. Hence, in this particular case, $R \cdot \emptyset = \{(a, \emptyset)\} \neq \emptyset$.

2. Let $R = \{(a, \{a, b\}), (a, \{a\}), (b, \{a\})\}\$ and $S = \{(a, \{a\}), (a, \{b\})\}\$. Then

$$\begin{split} (R \cdot R) \cdot S &= \{(a, \{a\}), (a, \{b\}), (b, \{a\}), (b, \{b\})\} \\ &\subset \{(a, \{a, b\}), ((a, \{a\}), (a, \{b\}), (b, \{a\}), (b, \{b\})\} \\ &= R \cdot (R \cdot S). \end{split}$$

3. Consider $R = \{(a, \{a, b\})\}$, $S = \{(a, \{a\})\}$, and $T = \{(b, \{b\})\}$. It follows that $S \cup T = \{(a, \{a\}), (b, \{b\})\}$ and $R \cdot (S \cup T) = R$, but $R \cdot S = R \cdot T = \emptyset$, whence $R \cdot S \cup R \cdot T = \emptyset$.

4. Let
$$R = \{(a, \{a\})\}$$
 and $S = \{(a, \{a\}), (a, \{b\})\}$. Then
$$(R||R) \cdot T = T \subset \{(a, \{a\}), (a, \{b\}), (a, \{a, b\})\} = (R \cdot T)||(R \cdot T).$$

The following Hasse diagrams are useful for visualising multirelations and finding counterexamples. We depict the multirelations R and S from case (2) in the Hasse diagram of the carrier set in Figure 1. We write ab as shorthand

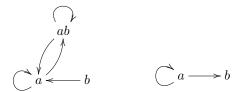


Figure 1: Diagrams for R and S in the proof of Lemma 4(2)

for the set $\{a,b\}$. The arrows $a \to a$, $a \to ab$ and $b \to a$ correspond to the pairs in R. The "virtual" arrows $ab \to ab$ and $ab \to a$ have been added to indicate which states are reachable from the set ab by R. We have omitted the empty set because it is not reachable.

The resulting lifting of the multirelation of type $X \to 2^X$ to a relation $2^X \times 2^X$ allows us to compute powers of R and products such as $R \cdot S$ by using relational composition, that is, by chasing reachability arrows directly in the diagram. It is reminiscent of Rabin and Scott's construction of deterministic finite automata from nondeterministic ones. A systematic study of this lifting will be the subject of another article.

Accordingly, we compute $R \cdot R$, $R \cdot S$, $(R \cdot R) \cdot S$ and $R \cdot (R \cdot S)$ as depicted in Figure 2.

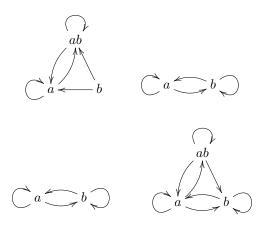


Figure 2: Diagrams for $R \cdot R$, $R \cdot S$, $(R \cdot R) \cdot S$ and $R \cdot (R \cdot S)$ with R and S from the proof of Lemma 4(2)

We even have a counterexample to $R \cdot (R \cdot R) \subseteq (R \cdot R) \cdot R$. Consider the multirelation $R = \{(a, \{c\}), (b, \{a, c\}), (c, \{b\}), (c, \{c\})\}$. Then

$$R \cdot R = \{(a, \{b\}), (a, \{c\}), (b, \{c\}), (b, \{b, c\}), (c, \{b\}), (c, \{c\}), (c, \{a, c\})\}.$$

Therefore

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\begin{split} R\cdot(R\cdot R) = &\{(a,\{b\}),(a,\{c\}),(a,\{a,c\}),\\ &(b,\{b\}),(b,\{c\}),(b,\{a,c\}),(b,\{b,c\}),(b,\{a,b,c\}),\\ &(c,\{b\}),(c,\{c\}),(c,\{a,c\}),(c,\{b,c\})\} \\ \supset &\{(a,\{b\}),(a,\{c\}),(a,\{a,c\}),\\ &(b,\{b\}),(b,\{c\}),(b,\{a,c\}),(b,\{a,b,c\}),\\ &(c,\{b\}),(c,\{c\}),(c,\{b,c\})\} \\ = &(R\cdot R)\cdot R. \end{split}
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This hints at complications in the definition of finite iteration of multirelations, which is considered in Section 13.

4 Stronger Laws for Sequential Subidentities

A multirelation P is a (sequential) subidentity if $P \subseteq 1_{\sigma}$. As mentioned in Section 2, $1_{\sigma} = \lambda x.\{x\}$ embeds X into 2^{X} . Every sequential subidentity is therefore a partial embedding. We usually write P or Q for subidentities. We write $\iota = \lambda x.\{x\}$ for the embedding of X into 2^{X} . One can see $G_{\iota}(a)$ also as a lifting of a point $a \in X$ to a multirelational "point" $(a, \{a\})$ and $G_{\iota}(A)$ as a lifting of a set A to a subidentity.

More generally, this yields an isomorphism between points and multirelational points as well as sets and subidentities.

The next lemma shows that multiplying a multirelation with a subidentity from the left or right amounts to an input or output restriction.

Lemma 5. Let R be a multirelation and P a subidentity.

1.
$$(a, A) \in R \cdot P \Leftrightarrow (a, A) \in R \wedge G_{\iota}(A) \subseteq P$$

2.
$$(a, A) \in P \cdot R \Leftrightarrow G_{\iota}(a) \in P \wedge (a, A) \in R$$
.

See Appendix 2 for proofs. These properties help us to verify that subidentities satisfy equational associativity and interaction laws as well as a left distributivity law.

Lemma 6. Let R, S and T be multirelations.

- 1. $(R \cdot S) \cdot T = R \cdot (S \cdot T)$ if R, S or T is a subidentity.
- 2. $(R||S) \cdot T = (R \cdot T)||(S \cdot T) \text{ if } T \text{ is a subidentity,}$
- 3. $R \cdot (S \cup T) = R \cdot S \cup R \cdot T$ if R is a subidentity.

See Appendix 2 for proofs. Lemma 6 is essential for deriving the axioms of concurrent dynamic algebra.

In addition, it is straightforward to verify that the sequential subidentities form a boolean subalgebra of the algebra of multirelations over X. The empty set is the least element of this algebra and 1_{σ} its greatest element. Join is union and meet coincides with sequential composition, which is equal to parallel composition in this special case. The boolean complement of a subidentity $\bigcup_{a \in A} \{G_{\iota}(a)\}$, for some set $A \subseteq X$, is the subidentity $\bigcup_{b \in X-A} \{G_{\iota}(b)\}$.

Subidentities play an important role in providing the state spaces of modal operators in concurrent dynamic algebras. In our axiomatisation, however, they arise only indirectly through definitions of domain and antidomain elements. In the concrete case of multirelations these are described in the next section.

5 Domain and Antidomain of Multirelations

This section presents the second important step towards concurrent dynamic algebra within the multirelational model: the definitions of domain and antidomain operations and the verification of some of their basic properties. These are then abstracted into algebraic domain and antidomain axioms, which, in turn, allow us to define the modal box and diamond operations of concurrent dynamic algebra.

The domain of a multirelation R is the multirelation

$$d(R) = \{G_{\iota}(a) \mid \exists A. \ (a, A) \in R\}.$$

The antidomain of a multirelation R is the multirelation

$$a(R) = \{G_{\iota}(a)\} \mid \neg \exists A. \ (a, A) \in R\}.$$

Domain and antidomain elements are therefore boolean complements of each

The next lemmas collect some of their basic properties which justify the algebraic axioms in Section 6.

Lemma 7. Let R and S be multirelations.

- $2. \ d(R) \cdot R = R,$
- 3. $d(R \cup S) = d(R) \cup d(S)$,
- $4. \ d(\emptyset) = \emptyset,$

1. $d(R) \subseteq 1_{\sigma}$.

- 5. $d(R \cdot S) = d(R \cdot d(S)),$
- 6. $d(R||S) = d(R) \cap d(S)$,
- 7. $d(R)||d(S) = d(R) \cdot d(S)$.

See Appendix 2 for proofs. Most of these laws are similar to those of relational domain, but properties (6) and (7) are particular to multirelations. Property (1) shows that domain elements are subidentities. According to (2), a multirelation is preserved by multiplying it from the left with its domain element. According to (3) and (4), domain is strict and additive: the domain of the union of two multirelations is the union of their domains and the domain of the empty set is the empty set. The locality property (5) states that it suffices to know the domain of the second multirelation when computing the domain of the sequential composition of two multirelations. By (6), the domain of a parallel composition of two multirelations is the intersection of their domains. Finally, by (7), the parallel composition of two domain elements equals their interesection. More generally, parallel composition of sequential subidentities is meet.

An intuitive explanation of domain is that it yields the set of all states from which a multirelation is enabled. Accordingly, by (3), the union of two multirelations is enabled if one of them is enabled, whereas, by (6), their parallel composition is enabled if both are enabled. It follows immediately from the definition that $d(\{(a,\emptyset)\}) = \{(a,\{a\})\}$. Hence the multirelation $\{(a,\emptyset)\}$ is enabled, but does not yield an output.

The next lemma, proved in Appendix 2, links domain and antidomain. It shows, in particular, that domain and antidomain elements are complemented.

Lemma 8. Let R be a multirelation.

- 1. $a(R) = 1_{\sigma} \cap -d(R)$,
- $2. \ d(R) = a(a(R)),$
- 3. d(a(R)) = a(R).

Many essential properties of antidomain can now be derived by De Morgan duality.

Lemma 9. Let R and S be multirelations.

- 1. $a(R) \cdot R = \emptyset$,
- 2. $a(R \cdot S) = a(R \cdot d(S)),$
- 3. $a(R) \cup d(R) = 1_{\sigma}$,
- 4. $a(R \cup S) = a(R) \cdot a(S)$,
- 5. $a(R||S) = a(R) \cup a(S)$,
- 6. $a(R)||a(S) = a(R) \cdot a(S)$.

See Appendix 15 for proofs. If d(R) describes those states from which mutirelation R is enabled, then a(R) models those where R is not enabled. Property (1) says that antidomain elements are left annihilators: R cannot be executed from states where it is not enabled. Property (2) is a locality property similar to that in Lemma 7(5). Property (3) is a complementation law between domain and antidomain elements. It implies that antidomain elements are sequential subidentities. Properties (4) to (6) are the obvious De Morgan duals of domain properties.

Finally, and crucially for our purposes, domain and antidomain elements support stronger associativity and distributivity properties.

Corollary 1. Let R, S and T be multirelations.

- 1. $(R \cdot S) \cdot T = R \cdot (S \cdot T)$ if R, S or T is a domain or antidomain element,
- 2. $(R||S) \cdot T = (R \cdot T)||(S \cdot T)$, if T is a domain or antidomain element,
- 3. $R \cdot (S \cup T) = R \cdot S \cup R \cdot T$ if R is a domain or antidomain element.

Proof. By Lemma 7(1) and 7(3), domain and antidomain elements are subidentities. The results then follow by Lemma 6.

Domain and antidomain satisfy, of course, additional properties. We have only presented those needed to justify the abstract domain and antidomain axioms in the following section. Further ones can then be derived by simple equational reasoning at the abstract level from those axioms; a considerable simplification.

6 Axioms for Multirelations with Domain and Antidomain

We have now collected sufficiently many facts about multirelations to abstract the domain and antidomain laws from the previous section into algebraic axioms. The approach is inspired by the axiomatisation of domain semirings [5] in the relational setting and the weakening of these axioms to families of near-semirings [4]. In those approaches, however, sequential composition is associative, which considerably simplifies proofs and leads to simpler axiomatisations. Here we can only assume associativity, interaction and left distributivity in the presence of domain and antidomain elements, which holds in the multirelational model according to Corollary 1 and yields just the right assumptions for reconstructing concurrent dynamic logic.

We keep the development modular so that it captures also multirelational semirings and Kleene algebras without concurrent composition. We expect that the axioms of Parikh's game logic can be derived from that basis.

A proto-dioid is a structure $(S, +, \cdot, 0, 1)$ such that (S, +, 0) is a semilattice with least element 0 and the following additional axioms hold:

$$\begin{aligned} 1\cdot x &= x, & x\cdot 1 &= x,\\ x\cdot y + x\cdot z &\leq x\cdot (y+z), & (x+y)\cdot z &= x\cdot z + y\cdot z, & 0\cdot x &= 0. \end{aligned}$$

Here, \leq is the semilattice order defined, as usual, by $x \leq y \Leftrightarrow x + y = y$.

We do not include the weak associativity law $(x \cdot y) \cdot z \leq x \cdot (y \cdot z)$, although it is present in multirelations (Lemma 1(3)). It is independent from our axioms.

A *dioid* is a proto-dioid in which multiplication is associative for all elements and the left distributivity law $x \cdot (y+z) = x \cdot y + x \cdot z$ and the right annihilation law $x \cdot 0 = 0$ hold. A dioid is *commutative* if multiplication is commutative: $x \cdot y = y \cdot x$.

A proto-trioid is a structure $(S, +, \cdot, ||, 0, 1_{\sigma}, 1_{\pi})$ such that $(S, +, \cdot, 0, 1_{\sigma})$ is a proto-dioid and $(S, +, ||, 0, 1_{\pi})$ is a commutative dioid.

In every proto-dioid, multiplication is left-isotone, $x \leq y \Rightarrow z \cdot x \leq z \cdot y$.

A domain proto-dioid (dp-dioid) is a proto-dioid expanded by a domain operation which satisfies the domain associativity axiom

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
,

if one of x, y or z is equal to d(w) for some w, and the domain axioms

$$x \le d(x) \cdot x$$
, $d(x \cdot y) = d(x \cdot d(y)), d(x + y) = d(x) + d(y),$
 $d(x) \le 1_{\sigma}, \quad d(0) = 0.$

The first domain axiom is called *left preservation* axiom, the second one *locality* axiom, the third one *additivity* axiom, the fourth one *subidentity* axiom and the fifth one *strictness* axiom.

A domain proto-trioid (dp-triod) is a dp-dioid which is also a proto-trioid and satisfies the domain interaction axiom and the domain concurrency axioms

$$(x||y) \cdot d(z) = (x \cdot d(z))||(y \cdot d(z)),$$
 $d(x||y) = d(x) \cdot d(y),$ $d(x)||d(y) = d(x) \cdot d(y).$

In the presence of antidomain the axioms can be simplified further. An antidomain proto-dioid (ap-dioid) is a proto-dioid expanded by an antidomain operation which satisfies the antidomain associativity axiom

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z,$$

where x, y or z is equal to a(w) for some w, and satisfies the antidomain axioms

$$a(x) \cdot x = 0, \qquad a(x \cdot y) = a(x \cdot a(a(y))), \qquad a(x) + a(a(x)) = 1_{\sigma},$$

$$a(x) \cdot (y + z) = a(x) \cdot y + a(x) \cdot z.$$

The first antidomain axiom is called *left annihilation* axiom, the second one *locality* axiom, the third one *complementation* axiom and the fourth one *antidomain left distributivity* axiom.

An antidomain proto-trioid (ap-trioid) is an ap-dioid which is also a proto-trioid and satisfies the antidomain interaction and antidomain concurrency axioms

$$(x||y) \cdot a(z) = (x \cdot a(z))||(y \cdot a(z)),$$
 $a(x||y) = a(x) + a(y),$ $a(x)||a(y) = a(x) \cdot a(y).$

We have verified irredundancy of all domain and antidomain axioms with Isabelle. The full set of axioms of dp-trioids and ap-trioids (with additional axioms for the Kleene star) is listed in Appendix 1.

We can now relate the multirelational model set up in Sections 3-5 with the abstract algebraic definitions. The theorem is stated only for the smallest axiomatic class; it then holds automatically in all superclasses.

Theorem 1. Let X be a set.

- 1. The structure $(M(X), \cup, \cdot, ||, \emptyset, 1_{\sigma}, 1_{\pi}, d)$ forms a dp-trioid.
- 2. The structure $(M(X), \cup, \cdot, \parallel, \emptyset, 1_{\sigma}, 1_{\pi}, a)$ forms an ap-trioid.

Proof. The union axioms follow from set theory. The remaining proto-dioid axioms of sequential composition have been verified in Lemma 1; the commutative dioid axioms of concurrent composition in Lemma 2; the domain and antidomain axioms in Lemma 7, Lemma 9 and Corollary 1.

We call the structure $(M(X), \cup, \cdot, ||, \emptyset, 1_{\sigma}, 1_{\pi}, d)$ the full multirelational dp-trioid and the structure $(M(X), \cup, \cdot, ||, \emptyset, 1_{\sigma}, 1_{\pi}, a)$ the full multirelational aptrioid over X. Since dp-trioids and ap-trioids are equational classes, they are closed under subalgebras, products and homomorphic images. Hence in particular any subalgebra of a full dp-trioid is a dp-trioid and any subalgebra of a full ap-trioid is an ap-trioid.

7 Modal Operators

Following Desharnais and Struth [5], we define modal box and diamond operators from domain and antidomain. In every dp-dioid we define

$$\langle x \rangle y = d(x \cdot y).$$

This captures the intuition behind the Kripke-style semantics of modal logics. As explained in Section 4, sequential multiplication of a multirelation by a sequential subidentity from the left and right forms an input or output restriction of that multirelation. Therefore, $d(x \cdot y) = d(x \cdot d(y))$ abstractly represents a generalised multirelational preimage of the subidentity d(y) under the element x. In other words, $\langle x \rangle y = \langle x \rangle d(y)$ yields the set of all elements from which, with x, one may reach a set which is a subset of d(y). This can be checked readily in the multirelational model: if $P \subseteq X \times 2^X$ is a sequential subidentity and $R \subseteq X \times 2^X$ a multirelation, then

$$\langle R \rangle P = \{ G_{\iota}(a) \mid \exists B. \ (a, B) \in R \land G_{\iota}(B) \subseteq P \}.$$

This abstractly represents the set of all states $A \subseteq X$ from which R may reach the set B, which is a subset of the set represented by the multirelation P. In particular, if all output sets of the multirelation are singletons, the case of a relational preimage is recovered. The definition of multirelational diamonds thus generalises the relational Kripke semantics in a natural way.

In ap-dioids the situation is similar. Boxes can now be defined by De Morgan duality as well. In accordance with the multirelational model (Lemma 8(2)) we show in Section 11 that $d = a \circ a$. Then

$$\langle x \rangle y = d(x \cdot y) = a(a(x \cdot y)), \qquad [x]y = a(x \cdot a(y)).$$

Intuitively, one might expect that [x]y = [x]d(y) models the set of all states from which, whith x, one must reach sets of elements which are all in d(y). An analysis in the multirelational model, however, shows a subtly different behaviour:

$$[R]P = \{G_{\iota}(a) \mid \neg \exists B. \ (a, B) \in R \cdot a(P)\}$$

$$= \{G_{\iota}(a) \mid \neg \exists B. \ (a, B) \in R \land G_{\iota}(B) \subseteq a(P)\}$$

$$= \{G_{\iota}(a) \mid \neg \exists B. \ (a, B) \in R \land G_{\iota}(B) \cap P = \emptyset\}$$

$$= \{G_{\iota}(a) \mid \forall B. \ (a, B) \in R \Rightarrow G_{\iota}(B) \cap P \neq \emptyset\}.$$

The condition $(a, B) \in R \Rightarrow G_{\iota}(B) \cap P \neq \emptyset$, which is enforced by De Morgan duality, is weaker than what we described above. At least the standard relational case is contained in this definition. Goldblatt [8], following Nerode and Wijesekera [14], has therefore argued for replacing this condition by the more intuitive condition $(a, B) \in R \Rightarrow G_{\iota}(B) \subseteq P$, which breaks De Morgan duality. Here we follow Peleg's De Morgan dual definition and leave the algebraisation of its alternative for future work.

8 The Structure of DP-Trioids

This section presents the basic laws of dp-dioids and dp-trioids. Section 9 shows that algebraic variants of the axioms of concurrent dynamic logic, except the star axiom, can be derived in this setting. The star is then treated in Section 10.

We write d(S) for the image of the carrier set S under the domain operation d and call this set the set of all *domain elements*. We often write p, q, r, \ldots for domain elements.

The following identity is immediate from locality and properties of 1_{σ} .

Lemma 10. In every dp-dioid, operation d is a retraction: $d \circ d = d$.

The next fact is a general property of retractions. Here it gives a syntactic characterisation of domain elements as fixpoints of d (cf. [5]).

Proposition 1. If S is a dp-dioid, then $x \in d(S) \Leftrightarrow d(x) = x$.

This characterisation helps checking closure properties of domain elements. We first prove some auxiliary properties (cf. Appendix 2).

Lemma 11. In every dp-dioid,

- 1. $x < y \Rightarrow d(x) < d(y)$,
- $2. \ d(x) \cdot x = x,$
- 3. $d(x \cdot y) \le d(x)$,
- $4. \ x \le 1_{\sigma} \Rightarrow x \le d(x),$
- 5. $d(d(x) \cdot y) = d(x) \cdot d(y)$.

The domain export law (5) is instrumental in proving further domain laws.

Proposition 2. Let S be a dp-dioid. Then d(S) is a subalgebra of S which forms a bounded distributive lattice.

Proof. First we check that d(S) is closed under the operations, using the fixpoint property d(x) = x from Proposition 1.

- d(0) = 0 is an axiom.
- $d(1_{\sigma}) = 1_{\sigma}$ follows from Lemma 11(1).
- d(d(x) + d(y)) = d(x) + d(y) follows from additivity and idempotency of domain.
- $d(d(x) \cdot d(y)) = d(x) \cdot d(y)$ follows from domain export (Lemma 11(5)) and locality.

Next we verify that the subalgebra forms a distributive lattice with least element 0 and greatest element 1_{σ} .

- It is obvious that 1_{σ} is the greatest and 0 the least element of d(S).
- Associativity of domain elements follows from the dp-dioid axioms.
- $d(x) \cdot d(y) = d(y) \cdot d(x)$. We show that $d(x) \cdot d(y) \le d(y) \cdot d(x)$; the converse direction being symmetric.

$$d(x) \cdot d(y) = d(d(x) \cdot d(y)) \cdot d(x) \cdot d(y) = d(x) \cdot d(y) \cdot d(y) \cdot d(x) \le d(y) \cdot d(x),$$

using Lemma 11(2), domain export, associativity of domain elements and the fact that domain elements are subidentities.

• $d(x) \cdot d(x) = d(x)$ holds since

$$d(x) = d(d(x) \cdot x) = d(x) \cdot d(x)$$

by Lemma 11(2) and domain export.

It follows that $(d(S), \cdot 0, 1)$ is a bounded meet semilattice with meet operation \cdot . It is also clear that (d(S), +, 0, 1) is a bounded join semilattice. Hence it remains to verify the absorption and distributivity laws.

• For $d(x) \cdot (d(x) + d(y)) = d(x)$, we calculate

$$\begin{split} d(x)\cdot (d(y)+d(z)) &= (d(y)+d(x))\cdot d(x)\\ &= d(y)\cdot d(x)+d(x)\cdot d(x)\\ &= d(y)\cdot d(x)+d(x)\\ &= d(x) \end{split}$$

by commutativity and idempotence of meet as well as distributivity.

- $d(x) + d(x) \cdot d(y) = d(x)$. This is the last step of the previous proof.
- $d(x) \cdot (d(y) + d(z)) = d(x) \cdot d(y) + d(x) \cdot d(z)$ is obvious from commutativity of meet and right distributivity.
- The distributivity law $d(x) + d(y) \cdot d(z) = (d(x) + d(y)) \cdot (d(x) + d(z))$ holds by lattice duality.

The next lemma presents additional domain laws; it is proved in Appendix 2.

Lemma 12. In every dp-dioid,

1.
$$x \le d(y) \cdot x \Leftrightarrow d(x) \le d(y)$$
,

2.
$$d(x) \cdot 0 = 0$$
,

3.
$$d(x) = 0 \Leftrightarrow x = 0$$
,

4.
$$d(x) \le d(x+y)$$
.

The least left preservation law (1) is a characteristic property of domain operations. It states that d(x) is the least domain element that satisfies the inequality $x \leq p \cdot x$. Law (2) shows that 0 is a right annihilator in the subalgebra of domain elements.

Next we consider the interaction between domain and the parallel operations.

Lemma 13. In every dp-trioid,

1.
$$d(1_{\pi}) = 1_{\sigma}$$
,

2.
$$d(x||y) = d(x)||d(y),$$

3.
$$d(d(x)||d(y)) = d(x)||d(y),$$

4.
$$d(x)||d(x) = d(x)$$
.

See Appendix 2 for proofs. By (3), the subalgebra of domain elements is also closed with respect to parallel products, which are mapped to meets. Property (4) follows from the fact that parallel products of domain elements, hence of subidentities, are meets.

At the end of this section we characterise domain elements in terms of a weak notion of complementation, following [5]. This further describes the structure of domain elements within the subalgebra of subidentities.

Proposition 3. Let S be a dp-dioid. Then $x \in d(S)$ if $x + y = 1_{\sigma}$ and $y \cdot x = 0$ hold for some $y \in S$.

Proof. Fix x and let p be an element which satisfies $x+p=1_{\sigma}$ and $p\cdot x=0$. We must show that d(x)=x.

- $x \cdot d(x) \le x$ since $d(x) \le 1_{\sigma}$ and $x = (x+p) \cdot x = x \cdot x = x \cdot d(x) \cdot x \le x \cdot d(x)$, whence $x \cdot d(x) = x$.
- $p \cdot d(x) = d(p \cdot d(x)) \cdot p \cdot d(x) = d(p \cdot x) \cdot p \cdot d(x) = d(0) \cdot x \cdot d(x) = 0.$

Therefore
$$d(x) = (x+p) \cdot d(x) = x \cdot d(x) + p \cdot d(x) = x$$
.

We call an element y of a dp-dioid a *complement* of an element x whenever $x+y=1_{\sigma}, \ y\cdot x=0$ and $x\cdot y=0$ hold. Thus, if y is a complement of x, then x is a complement of y. We call an element *complemented* if it has a complement. The set of all complemented elements of a dp-dioid S is denoted S.

Corollary 2. Let S be a dp-dioid. Then $B_S \subseteq d(S)$.

Lemma 14. Let S be a dp-dioid. Then B_S is a boolean algebra.

Proof. Since complemented elements are domain elements, they are idempotent and commutative. We use these properties to show that sums and products of complemented elements are complemented. More precisely, if y_1 is a complement of x_1 and y_2 a complement of x_2 , then $y_1 \cdot y_2$ is a complement of $x_1 + x_2$ and $y_1 + y_2$ a complement of $x_1 \cdot x_2$. First,

$$\begin{split} x_1 + x_2 + y_1 \cdot y_2 &= x_1 \cdot (x_2 + y_2) + x_2 \cdot (x_1 + y_1) + y_1 \cdot y_2 \\ &= x_1 \cdot x_2 + x_1 \cdot y_2 + x_2 \cdot x_1 + x_2 \cdot y_1 + y_1 \cdot y_2 \\ &= x_1 \cdot x_2 + x_1 \cdot y_2 + x_2 \cdot y_1 + y_1 \cdot y_2 \\ &= (x_1 + y_1) \cdot (x_2 + y_2) \\ &= 1_{\sigma}. \end{split}$$

Second, $(x_1+x_2)\cdot y_1\cdot y_2 = x_1\cdot y_1\cdot y_2 + x_2\cdot y_1\cdot y_2 = 0$. This proves complementation of sums. The proof of complementation of products is dual, starting from $y_1\cdot y_2$.

These two facts show that B_S is a subalgebra of d(S). It is therefore a bounded distributive sublattice and a boolean algebra, since all elements are complemented and complements in distributive lattices are unique.

The following theorem summarises this investigation of the structure of d(S).

Theorem 2. Let S be a dp-dioid. Then d(S) contains the greatest boolean subalgebra of S bounded by 0 and 1_{σ} .

It is immediately clear that this theorem holds in dp-trioids as well. In the abstract setting, it need not be the case that d(S) contains any Boolean algebra apart from $\{0, 1_{\sigma}\}$. In fact, the sequential subidentities may form a distributive lattice which is not a boolean algebra, for instance a chain.

Example 4. Consider the structure with addition defined by $0 < a < 1_{\pi} < 1_{\sigma}$ and the other operations defined by the following tables.

						0	a	1_{π}	1_{σ}		
0	0	0	0	0	0	0	0	0	0	0	
0	a	a	a	a	a	0	a	a	a	a	
1_{π}	0	a	1_{π}	1_{π}	1_{π}	0	a	1_{π}	1_{σ}	1_{π}	1_{σ}
1_{σ}	0	a	1_{π}	1_{σ}	1_{σ}	0	a	1_{σ}	1_{σ}	1_{π}	1_{π}

It can be checked that this structure forms a dp-trioid (in fact this counterexample was found by Isabelle), but the elements a and 1_{π} are not complemented. For instance, the only element y which satisfies $a+y=1_{\sigma}$ is $y=1_{\sigma}$, but $1_{\sigma} \cdot a=a\neq 0$.

Thus B_s need not be equal to d(S), which justifies Corollary 2. In the multirelational model, however, the set of all sequential subidentities forms a boolean algebra, as mentioned in Section 4. In a multirelational dp-trioid S, therefore, $d(S) = \{P \mid P \subseteq 1_{\sigma}\}.$

9 The Diamond Axioms of Star-Free CDL

We are now equipped for deriving algebraic variants of the diamond axioms of concurrent dynamic logic except the star axioms in dp-trioids. First, note that $\langle x \rangle p = \langle x \rangle d(p)$.

Lemma 15.

- 1. In every dp-dioid, the following CDL-axioms are derivable.
 - (a) $\langle x + y \rangle p = \langle x \rangle p + \langle y \rangle p$.
 - (b) $\langle x \cdot y \rangle p = \langle x \rangle \langle y \rangle p$.
 - (c) $\langle d(p) \rangle q = d(p) \cdot d(q)$.
- 2. In every dp-trioid, the following CDL-axiom is derivable as well.
 - (d) $\langle x || y \rangle p = \langle x \rangle p \cdot \langle y \rangle p$.

Proof. (a) Using right distributivity and additivity of domain, we calculate

$$\begin{aligned} \langle x+y \rangle p &= d((x+y) \cdot p) \\ &= d(x \cdot p + y \cdot p) \\ &= d(x \cdot p) + d(y \cdot p) \\ &= \langle x \rangle p + \langle y \rangle p. \end{aligned}$$

(b) Using domain associativity and locality, we calculate

$$\langle x \cdot y \rangle p = d((x \cdot y) \cdot p)$$

$$= d(x \cdot (y \cdot d(p)))$$

$$= d(x \cdot d(y \cdot d(p)))$$

$$= d(x \cdot \langle y \rangle p)$$

$$= \langle x \rangle \langle y \rangle p.$$

- (c) By domain export, $\langle d(p) \rangle q = d(d(p) \cdot q) = d(p) \cdot d(q)$.
- (d) Using domain interaction and the first domain concurrency axiom, we calculate

$$\langle x || y \rangle p = d((x || y) \cdot d(p))$$

$$= d((x \cdot d(p)) || (y \cdot d(p))$$

$$= d(x \cdot d(p)) \cdot d(y \cdot d(p))$$

$$= \langle x \rangle p \cdot \langle y \rangle p.$$

We can derive additional diamond laws from the domain laws such as $\langle 0 \rangle p = 0$ or $\langle 1_{\sigma} \rangle p = d(p)$. However, we have a counterexample to $\langle 1_P \rangle p = 1_{\sigma}$, which holds in the multirelational model.

Example 5. Consider the structure with addition defined by $0 < 1_{\sigma} < 1_{p}$, concurrent composition defined by meet, and the remaining operations by the conditions $1_{\pi} \cdot 0 = 0$, $1_{\pi} \cdot 1_{\pi} = 1_{\pi}$, d(0) = 0 and $d(1_{\sigma}) = d(1_{\pi}) = 1_{\sigma}$. It can be checked that this defines a dp-trioid, but $\langle 1_{\pi} \rangle 0 = d(1_{\pi} \cdot 0) = d(0) = 0 < 1_{\sigma}$. \square

The following $demodalisation \ law$ is proved in Appendix 2. It is instrumental for deriving the star axioms of CDL.

Lemma 16. In every dp-dioid,

$$\langle x \rangle p < d(q) \Leftrightarrow x \cdot d(p) < d(q) \cdot x.$$

Finally we present two important counterexamples.

Lemma 17. There are multirelations R, P and Q such that the following holds.

- 1. $\langle R \rangle (P \cup Q) \neq \langle R \rangle P \cup \langle R \rangle Q$,
- 2. $\langle R \rangle \emptyset \neq \emptyset$.

Proof. 1. Let $R = \{(a, \{a, b\})\}$, $P = \{(a, \{a\})\}$ and $P = \{(b, \{b\})\}$. Then $\langle R \rangle (P \cup Q) = \{(a, \{a, b\})\} \supset \emptyset = \langle R \rangle P \cup \langle R \rangle Q.$

2. For $R = \{(a, \emptyset)\}$ we have $\langle R \rangle \emptyset = \{(a, \{a\})\} \neq \emptyset$.

The additivity and strictness laws just refuted are defining properties of modal algebras in the sense of Jónsson and Tarski (cf. [2]). Our concurrent dynamic algebra axioms are therefore nonstandard. This situation is analogous to the difference between strict and multiplicative predicate transformers which arise from relational semantics and their isotone counterparts which arise from up-closed multirelations. Predicate transformers are usually obtained from boxes instead of diamonds; the failure of multiplicativity is related to that of additivity by duality.

In the concurrent setting, the above multirelation R models an external choice between a and b from input a. Reflecting this, it is not sufficient that one can observe either one of a and b, but not both after executing R. In contrast to this, $\langle S \rangle (P \cup Q) = \langle S \rangle P \cup \langle S \rangle Q$, for $S = \{(a, \{a\}), (a, \{b\})\}$, models an internal choice.

10 The Star Axioms of CDL

This section derives the star axioms of CDL in expansions of dp-dioids to variants of Kleene algebras. This is not entirely straightforward due to the lack of associativity and left distributivity laws. As before we start at the level of multirelations to derive the appropriate star axioms. We then lift the investigation to the algebraic level.

Let R and S be multirelations. Consider the functions

$$F_{RS} = \lambda X. \ S \cup R \cdot X, \qquad F_R = \lambda X. \ 1_{\sigma} \cup R \cdot X,$$

which generate variants of the Kleene star as their least fixpoints. Existence of these fixpoints is guaranteed by basic fixpoint theory. The universal multirelation U has been introduced in Section 2.

Lemma 18.

- 1. The functions F_{RS} and F_R are isotone.
- 2. $(M(X), \cup, \cap, \emptyset, U)$ forms a complete lattice.
- 3. F_{RS} and F_R have least pre-fixpoints and greatest post-fixpoints which are also least and greatest fixpoints.

See Appendix 2 for proofs.

We write (R^*S) or μF_{RS} for the least fixpoint of F_{RS} and R^* or μF_R for the least fixpoint of F_R . We immediately obtain the fixpoint unfold and induction laws

$$S \cup R \cdot (R^*S) \subseteq (R^*S), \qquad S \cup R \cdot T \subseteq T \Rightarrow (R^*S) \subseteq T$$

for \mathcal{F}_{RS} and the corresponding laws

$$1_{\sigma} \cup R \cdot R^* \subseteq R^*S, \qquad 1_{\sigma} \cup R \cdot T \subseteq T \Rightarrow R^* \subseteq T$$

for F_R . The binary fixpoint (R^*S) is not necessarily equal to $R^* \cdot S$. At least, by definition, $R^* = R^* \cdot 1_{\sigma} = (R^*1_{\sigma})$. The fixpoints μF_R and μF_{RS} can be related by the following well known fixpoint fusion law.

Theorem 3.

- 1. Let f and g be isotone functions and h a continuous function over a complete lattice. If $h \circ g \leq f \circ h$, then $h(\mu g) \leq \mu f$.
- 2. Let f, g and h be isotone functions over a complete lattice. If $f \circ h \leq h \circ g$, then $\mu f \leq h(\mu g)$.

It follows from (1) and (2) that, if f and g are isotone, h is continuous and $h \circ g = f \circ h$, then $\mu f = h(\mu g)$. Applying fixpoint fusion to F_{RS} and F_R yields the following fact.

Corollary 3. Let R, S and T be multirelations. Then

$$R^* \cdot S \subseteq (R^*S), \qquad (R^*S) \cdot T \subseteq (R^*(S \cdot T)).$$

Proof. Let $f = F_{RS}$, $g = F_R$ and $h = H = \lambda X.X \cdot S$.

It is easy to show that H is continuous, that is, $(\bigcup_{i \in I} R_i) \cdot S = \bigcup_{i \in I} (R_i \cdot S)$. The proof is similar to that of Lemma 1(4). Moreover

$$(H \circ F_R)(x) = (1_{\sigma} \cup R \cdot x) \cdot S$$
$$= S \cup (R \cdot x) \cdot S \subseteq S \cup R \cdot (x \cdot S)$$
$$= (F_{RS} \circ H)(x)$$

by weak associativity (Lemma 1(3)), so $R^* \cdot S \subseteq (R^*S)$ by fixpoint fusion. The proof of $(R^*S) \cdot T \subseteq (R^*(S \cdot T))$ follows the same pattern.

Proving the converse direction, $(R^*S) \subseteq R^* \cdot S$, by fixpoint fusion requires associativity in the other direction, which does not hold in our setting (Lemma 4(2), where the counterexample was given for $R \cdot (R \cdot S) \subseteq (R \cdot R) \cdot S$ and extends to the case above). The following counterexample rules out any other proof of this inclusion.

Lemma 19. There are multirelations R and S such that $R^*S \neq R^* \cdot S$.

Proof. Consider R and S from Lemma 4(2) and their diagrams in Figure 3. The

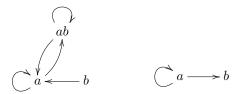


Figure 3: Diagrams for R and S in the proof of Lemma 4(2) (same as Fig. 1)

multirelations $R^* = 1_{\sigma} \cup R \cdot (1_{\sigma} \cup R)$, $R^* \cdot S$ and $R^*S = S \cup R \cdot (S \cup R \cdot (S \cup R)) = R \cdot R \cdot (R \cup S)$ are computed from these diagrams as shown in Figure 4. Clearly, $R^*S \not\subseteq R^* \cdot S$.

At first sight, Lemma 19 seems to invalidate the star-axiom of CDL. However, the identity $R^*S = R^* \cdot S$ is only needed in the modal setting, where S is a subidentity. In this case, as we have seen, stronger algebraic properties for sequential composition are present. We now investigate this restriction.

First, we show that the unfold law for R^*S can be strengthened to an identity.

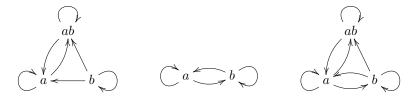


Figure 4: Diagrams for R^* , $R^* \cdot S$ and R^*S for R and S in the proof of Lemma 4(2)

Corollary 4. Let R and S be multirelations. Then $S \cup R \cdot (R^*S) = (R^*S)$.

This holds since every pre-fixpoint of F_{RS} is also a fixpoint.

We now prove the desired fusion of μF_{RS} with μF_R when S is a subidentity.

Proposition 4. Let R be a multirelation and P a subidentity. Then

$$R^*P = R^* \cdot P.$$

Proof. Applying fixpoint fusion as in Corollary 3, but with $H = \lambda X.X \cdot P$, now establishes $H \circ F_R = F_{RS} \circ H$, since we have full associativity for subidentities by Lemma 6(1). This suffices to verify the claim.

We can therefore replace R^*P by $R^* \cdot P$ in the induction law for F_{RS} .

Lemma 20. Let R and S be multirelations and P be a subidentity. Then

$$P \cup R \cdot S \subseteq S \Rightarrow R^* \cdot P \subseteq S.$$

Corollary 4 and Lemma 20 motivate the following algebraic definition. As before we use domain elements instead of sequential subidentities.

A proto-Kleene algebra with domain (dp-Kleene algebra) is a dp-dioid expanded by a star operation which satisfies the (left) star unfold and (left) star induction axioms

$$1_{\sigma} + x \cdot x^* \le x^*, \qquad d(z) + x \cdot y \le y \Rightarrow x^* \cdot d(z) \le y.$$

A proto-bi-Kleene algebra with domain (dp-bi-Kleene algebra) is a dp-Kleene algebra which is also a dp-trioid². In both cases, the unfold law can be strengthened to the identity $1_{\sigma} + x \cdot x^* = x^*$. The full list of dp-bi-Kleene algebra axioms can be found in Appendix 1.

The development so far is summarised in the following soundness result, which links the multirelational layer with the abstract algebraic one.

Theorem 4. $(M(X), \cup, \cdot, ||, \emptyset, 1_{\sigma}, 1_{\pi}, d, *)$ is a dp-bi-Kleene algebra.

Proof. The structure is a dp-trioid as a consequence of Theorem 1. The star axioms hold by Corollary 4 and Lemma 20. \Box

Due to this result we can now continue at the algebraic level. First we derive the modal star unfold axiom of CDL.

²In this article we ignore the star of concurrent composition, which should normally be part of the definition of a bi-Kleene algebra. The reason is that it is not considered in CDL.

Lemma 21. Let K be a dp-Kleene algebra, $x \in K$ and $p \in d(K)$. Then

$$p + \langle x \rangle \langle x^* \rangle p = \langle x^* \rangle p.$$

Proof. $p + \langle x \rangle \langle x^* \rangle p = \langle 1_{\sigma} + x \cdot x^* \rangle p = \langle x^* \rangle p$ by the star unfold axiom and the CDL axioms which have been verified in Lemma 15.

It remains to verify the star induction axiom of CDL. First we show a simulation law.

Lemma 22. Let K be a dp-Kleene algebra, $x \in K$ and $p \in d(K)$. Then

$$x \cdot p \le p \cdot y \Rightarrow x^* \cdot p \le p \cdot y^*$$
.

See Appendix 2 for a proof. The derivation of an algebraic variant of the star unfold axiom of CDL is then trivial.

Proposition 5. Let K be a dp-Kleene algebra, $x \in K$ and $p \in d(K)$. Then

$$\langle x \rangle p \le p \Rightarrow \langle x^* \rangle p \le p.$$

Proof.

$$\langle x \rangle p \le q \Leftrightarrow x \cdot p \le p \cdot x \Rightarrow x^* \cdot p \le p \cdot x^* \Leftrightarrow \langle x^* \rangle p \le p.$$

The first and last step use demodalisation (Lemma 16), the second step uses Lemma 22. \Box

The first main theorem of this article combines these results.

Theorem 5. The CDL axioms are derivable in dp-bi-Kleene algebras.

We therefore call dp-bi-Kleene algebras informally $\it concurrent dynamic algebras.$

Finally, in Appendix 2, we prove a right star unfold law and derive a variant of modal star induction in analogy to the induction axiom of pd-Kleene algebra.

Lemma 23. Let K be a pd-Kleene algebra, $x \in K$ and $p, q \in d(K)$. Then

1.
$$p + \langle x^* \rangle \langle x \rangle p \le \langle x^* \rangle p$$
,

2.
$$p + \langle x \rangle q \le q \Rightarrow \langle x^* \rangle p \le q$$
.

11 The Structure of AP-Trioids

Section 8 shows that the domain elements of a dp-dioid or dp-trioid form a distributive lattice. We now revisit this development for antidomain, where the resulting domain algebras are boolean algebras. We start with a number of auxiliary lemmas. These are needed because the minimality of the axiom set makes it difficult to derive the desirable properties directly.

In the following lemma we abbreviate $d = a \circ a$. This is justified in Proposition 6, which formally verifies that a(a(x)) models the domain of element x.

Lemma 24. In every ap-dioid,

1.
$$a(x) < 1_{\sigma}$$

```
2. a(x) \cdot a(x) = a(x),
```

3.
$$a(x) = 1_{\sigma} \Leftrightarrow x = 0$$
,

4.
$$a(x) \cdot y = 0 \Leftrightarrow a(x) < a(y)$$
,

5.
$$x \le y \Rightarrow a(y) \le a(x)$$
,

6.
$$a(x) \cdot a(y) \cdot d(x+y) = 0$$
,

7.
$$a(x + y) = a(x) \cdot a(y)$$
,

8.
$$a(a(x) \cdot y) = d(x) + a(y)$$
.

See Appendix 2 for proofs. The greatest left annihilation property (4) is a characteristic property of antidomain elements. It states that a(x) is the greatest antidomain element p which satisfy the left annihilation law $p \cdot x = 0$. By (5), the antidomain operation is antitone; by (7) it is multiplicative. Property (8) is an export law for antidomain. These laws are helpful in the following proposition which is proved in Appendix 2.

Proposition 6. Every ap-dioid is a dp-dioid with domain operation $d = a \circ a$.

As in Section 8, we investigate the structure of domain elements.

Proposition 7. Let S be an ap-dioid with $d = a \circ a$. Then d(S) forms a subalgebra which is the greatest boolean algebra in S bounded by 0 and 1.

Proof. First, since every ap-dioid is a dp-dioid, d(S) is a bounded distributive lattice. Second, antidomain elements are closed under the operations because d(a(x)) = a(x): by antidomain locality,

$$d(a(x)) = a(a(a(x))) = a(d(x)) = a(1_{\sigma} \cdot d(x)) = a(1_{\sigma} \cdot x) = a(x).$$

Third, the operation $\lambda x.a(x)$ is complementation in this algebra. One of the complementation properties, $a(d(x))+d(x)=a(x)+d(x)=1_{\sigma}$, is an axiom. The other ones, $a(d(x))\cdot d(x)=a(x)\cdot d(x)=0$ and $d(x)\cdot a(d(y))=d(x)\cdot a(x)=0$, are immediate from antidomain annihilation.

Finally, by Theorem 2, d(S) contains the greatest boolean algebra in S between 0 and 1_{σ} and is therefore equal to the greatest such boolean algebra. \square

We now expand Proposition 6 from the sequential to the concurrent case.

Proposition 8. Every ap-trioid is a dp-trioid.

The proof can be found in Appendix 2.

Finally we investigate the star. A proto-Kleene algebra with antidomain (ap-Kleene algebra) is an ap-dioid expanded by a star operation which satisfies the (left) star unfold and (left) star induction axioms

$$1_{\sigma} + x \cdot x^* \le x^*, \qquad a(z) + x \cdot y \le y \Rightarrow x^* \cdot a(z) \le y.$$

A proto-bi-Kleene algebra with antidomain (ap-bi-Kleene algebra) is an ap-Kleene algebra which is also an ap-trioid. A full list of ap-bi-Kleene algebra axioms can be found in Appendix 1.

The following proposition is immediate from Propositions 6 and 8.

Proposition 9. 1. Every ap-Kleene algebra is a dp-Kleene.

2. Every ap-bi-Kleene algebra is a dp-bi-Kleene algebra.

In combination, these facts establish an analogon to Theorem 4.

Theorem 6. $(M(X), \cup, \cdot, \parallel, \emptyset, 1_{\sigma}, 1_{\pi}, a, *)$ forms an ap-bi-Kleene algebra.

12 The Box Axioms of CDL

The results of the previous section imply that the diamond axioms of concurrent dynamic logic hold in the setting of antidomain algebras. In addition we can now derive algebraic variants of Peleg's De Morgan dual box axioms. Since every ap-bi-Kleene algebra is a dp-Kleene algebra, the diamond axioms of concurrent dynamic algebras hold immediately.

Lemma 25.

- 1. In every ap-dioid, the following CDL-axioms are derivable.
 - (a) $\langle x + y \rangle p = \langle x \rangle p + \langle y \rangle p$.
 - (b) $\langle x \cdot y \rangle p = \langle x \rangle \langle y \rangle p$.
 - (c) $\langle d(p) \rangle q = d(p) \cdot d(q)$.
- 2. In every ap-trioid, the following CDL-axiom is derivable.
 - (d) $\langle x | y \rangle p = \langle x \rangle p \cdot \langle y \rangle p$.
- 3. In every ap-Kleene algebra, the following star axioms are derivable.
 - (e) $1_{\sigma} + \langle x \rangle \langle x^* \rangle p = \langle x^* \rangle p$.
 - (f) $\langle x \rangle p \le p \Rightarrow \langle x^* \rangle p \le p$.

In addition, the following box axioms follow easily from De Morgan duality.

Proposition 10.

- 1. In every ap-dioid, the following CDL-axioms are derivable.
 - (a) $[x+y]p = [x]p \cdot [y]p.$
 - (b) $[x \cdot y]p = [x][y]p$.
 - (c) [d(p)]q = a(p) + d(q).
- 2. In every ap-trioid, the following CDL-axiom is derivable.
 - (d) $[x||y]p = [x]p \cdot [y]p$.
- 3. In every ap-Kleene algebra, the following star axioms are derivable.
 - (e) $1_{\sigma} \cdot [x][x^*]p = [x^*]p$.
 - (f) $p \le [x]p \Rightarrow p \le [x^*]p$.

In sum, these results yield the second main theorem of this article.

Theorem 7. The box and diamond axioms of CDL are derivable in ap-bi-Kleene algebras.

We therefore call ap-bi-Kleene algebras concurrent dynamic algebras as well. In contrast to dp-Kleene algebras, these are based on boolean algebras of domain elements.

Finally we present counterexamples to multiplicativity and co-strictness of boxes.

Lemma 26. There are multirelations R, P and Q such that the following holds.

- 1. $[R](P \cdot Q) \neq [R]P \cdot [R]Q$,
- 2. $[R]1_{\sigma} \neq 1_{\sigma}$.

Proof.

1. Obviously, $\forall p, q$. $\langle x \rangle (p+q) = \langle x \rangle p + \langle x \rangle q$ if and only if $\forall p, q$. $[x](p \cdot q) = [x]p \cdot [x]q$. Hence the counterexample from Lemma 17 applies.

2. Similarly, $\langle x \rangle 0 = 0$ if and only if $[x]1_{\sigma} = 1_{\sigma}$.

The following counterexample is directly related to this lemma. According to Jónsson and Tarski, modal boxes and diamonds are conjugate functions on boolean algebras, that is, they are related by the conjugation law

$$\langle x \rangle p \cdot q = 0 \Leftrightarrow p \cdot [x]q = 0.$$

Conjugate functions are a fortiori additive. By Lemma 17 and 26, this cannot be the case in the multirelational setting, hence the conjugation law cannot hold. This is confirmed directly by the multirelation $R = \{(a, \emptyset)\}$ and the subidentity $P = \{(a, \{a\})\}$ over the set $X = \{a\}$, which satisfy

$$\langle R \rangle P \cdot P = d(R \cdot P) \cdot P = P \supset \emptyset = P \cdot a(R \cdot a(P)) = P \cdot [R]P$$
.

13 The Star and Finite Iteration

It is well known that least fixpoints can be reached by iterating from the least element of a complete lattice up to the first ordinal whenever the function under consideration is not only isotone, but also continuous. Otherwise, if the function is only isotone, transfinite induction beyond the first ordinal is required.

Our counterexample to left distributivity rules out continuity in general, but, in fact, chain continuity or directedness suffices for the star. As in Section 10, we consider

$$F_R = \lambda X.1_{\sigma} \cup R \cdot X.$$

Peleg has provided a counterexample even to chain completeness [20]. We display a proof in Appendix 2 to make this article selfcontained.

Lemma 27 (Peleg). There exists a multirelation R and an ascending chain of multirelations S_i , $i \in \mathbb{N}$, such that $F_R(\bigcup_{i \in \mathbb{N}} S_i) \neq \bigcup_{i \in \mathbb{N}} F_R(S_i)$.

Chain completeness can, however, be obtained if a multirelation R is externally image finite, that is, for all $(a,A) \in R$ the set A has finite cardinality. This notion has been called finitely branched by Peleg. We have chosen a different name to distinguish it from interal image finiteness, which is the case when for each a, the set of all (a,A) has finite cardinality. From a computational point of view, external image finiteness is not a limitation, since infinite sets A correspond to unbounded external nondeterminism or unbounded concurrent composition, which is not implementable.

Lemma 28 (Peleg). If R is externally image finite, then F_R is chain continuous.

See Appendix 2 for a proof.

We define powers of F_R inductively as $F_R^0 = \lambda X.X$ and $F_R^{n+1} = F_R \circ F_R^n$ and can then define iteration to the first limit ordinal as

$$F_R^* = \bigcup_{i \in \mathbb{N}} F^i.$$

General fixpoint theory (Kleene's fixpoint theorem) then implies the following fact.

Proposition 11. If R is externally image finite, then $R^* = F_R^*(\emptyset)$.

We now compare this notion of finite iteration with another one.

$$R^{(0)} = \emptyset, \qquad R^{(n+1)} = 1_{\sigma} \cup R \cdot R^{(n)}, \qquad R^{(*)} = \bigcup_{n \in \mathbb{N}} R^{(n)}.$$

Our next lemma shows that the inductive definition of $R^{(*)}$ captures the iterative function application of F_R^* to \emptyset and hence R^* for external image finiteness. It is proved in Appendix 2.

Lemma 29.

- 1. For all n, $F_R^n(\emptyset) = R^{(n)}$ and therefore $F_R^*(\emptyset) = R^{(*)}$.
- 2. If R is externally image finite, then $R^* = R^{(*)}$.

Finally we show that external image finiteness in Lemma 29(2) is nessesary.

Lemma 30. There exists a multirelation R such that $R^{(*)}$ is not a fixpoint of F_R .

Proof. Consider the multirelation

$$R = \{ (m, \{n \mid n < m\}) \mid m \in \mathbb{N} \cup \{\infty\} \}.$$

It follows that $(0, \emptyset) \in R$ and $R \cdot \emptyset = \{(0, \emptyset)\}.$

Then $(m, \{n \mid n \leq m-2\}) \notin R$ but it is in $R^{(2)}$, and $(m, \{n \mid n \leq m-k\}) \notin R^{(i)}$ for i < k, but it is in $R^{(k)}$; similarly $(m, \emptyset) \in R^{(m)}$ but not in $R^{(l)}$ for all l < m. Consequently, $(\infty, \emptyset) \notin R^{(n)}$ for all $n \in \mathbb{N}$, and therefore $(\infty, \emptyset) \notin R^{(*)}$, but $(\infty, \emptyset) \in F_R(R^{(*)})$.

14 Refutation of Segerberg's Axiom

Segerberg's axiom is the induction axiom of (non-concurrent) propositional dynamic logic (cf. [9]). Goldblatt uses its box version—his box semantics is different from ours—but not the diamond one. This section provides a counterexample to Segerberg's axiom in the multirelational model with box-diamond duality.

In diamond form, Segerberg's axiom is

$$\langle x^* \rangle p \le p + \langle x^* \rangle (\langle x \rangle p - p).$$

In modal Kleene algebra it is equivalent to the star induction axiom. For multirelations, the situation is different.

Proposition 12. There is a multirelation R and a subidentity P such that

$$\langle R^* \rangle P \supset P \cup \langle R^* \rangle (\langle R \rangle P - P).$$

Proof. Let $R = \{(a, \{b, c\}), (b, \{b\}), (b, \{c\}), (c, \{c\})\}$ and $P = \{(c, \{c\})\}$. As previously, we visualise R in the Hasse diagram in Figure 5. The multirelation R^* can be read off as the relational reflexive transitive closure from this diagram by chasing arrows. One can also use the diagram to check that

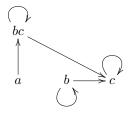


Figure 5: Diagram for R in the proof of Proposition 12

$$R \cdot P = \{(b, \{c\}), (c, \{c\})\},\$$
$$\langle R \rangle P = \{(b, \{b\}), (c, \{c\})\},\$$
$$\langle R \rangle P - P = \{(b, \{b\})\}.$$

One can compute R^* by iterating with $R^{(*)}$ according to Lemma 29(2), since R is externally image finite. Obviously, $R \cdot \emptyset = \emptyset$. Therefore,

$$\begin{split} R^{(1)} &= 1_{\sigma}, \\ R^{(2)} &= 1_{\sigma} \cup R \cdot (1_{\sigma} \cup R) \\ &= \{(a, \{a\}), (a, \{c\}), (a, \{b, c\}), (b, \{b\}), (b, \{c\}), (c, \{c\})\}, \\ R^{(3)} &= 1_{\sigma} \cup R \cdot (1_{\sigma} \cup R \cdot (1_{\sigma} \cup R)) = R^{(2)}, \\ R^{(n)} &= R^{(2)}, \end{split}$$

that is, iteration becomes stationary after four steps. Chain completeness implies that

$$R^* = R^{(*)} = R^{(2)} = \{(a, \{a\}), (a, \{c\}), (a, \{b, c\}), (b, \{b\}), (b, \{c\}), (c, \{c\})\}.$$

On the one hand, his result yields

$$\langle R^* \rangle (\langle R \rangle P - P) = \langle R^* \rangle \{ (b, \{b\}) \} = \{ (b, \{b\}) \}, P \cup \langle R^* \rangle (\langle R \rangle P - P) = \{ (b, \{b\}), (c, \{c\}) \}.$$

On the other hand we obtain

$$R^* \cdot P = \{(a, \{c\}), (b, \{c\}), (c, \{c\})\}, \\ \langle R^* \rangle P = \{(a, \{a\}), (b, \{b\}), (c, \{c\})\}.$$

This confirms that $\langle R^* \rangle P \supset P \cup \langle R^* \rangle (\langle R \rangle P - P)$ and falsifies Segerberg's formula.

Corollary 5. Segerberg's axiom is not derivable in ap-bi-Kleene algebras.

Obviously, this implies that the axiom is not derivable in ap-Kleene algebras. However, at least its converse is derivable.

Lemma 31. In every ap-Kleene algebra,

$$p + \langle x^* \rangle (\langle x \rangle p - p) \le \langle x^* \rangle p.$$

See Appendix 2 for a proof. Hence this fact is derivable in ap-bi-Kleene algebras, too.

Segerberg's axiom is usually presented in box form as $p \cdot [x^*](p \to [x]p) \le [x^*]p$, where $p \to q = a(p) + q$. By De Morgan duality, variants of Proposition 12, Corollary 5 and Lemma 31 hold in the box case. In particular, the box variant of Segerberg's axiom is neither valid in the multirelational model nor derivable in ap-bi-Kleene algebras.

15 Conclusion

We have defined weak variants of Kleene algebras with domain and antidomain which capture essential properties of the algebra of multirelations under union, sequential and concurrent composition and the sequential Kleene star together with multirelational domain and antidomain operations. The relationships between the different algebraic structures defined in this article is summarised in Figure 6. Both dp-bi-Kleene alegebras and ap-bi-Kleene algebras qualify as concurrent dynamic algebras; their axioms are listed in Appendix 1. We have derived algebraic counterparts of Peleg's CDL axioms from these two algebras. We have also proved their soundness with respect to the concrete multirelational model.

The algebra of multirelations is, however, much richer than this article might suggest. First of all, a left interaction law $R \cdot (S||T) \subseteq (R \cdot S)||(R \cdot T)$ complements its dextrous counterpart. Second, domain is characterised by the inclusion $1_{\sigma} \cap R \cdot U \subseteq d(R)$, where U is the universal multirelation defined in Section 2, but an equational definition $d(R) = 1_{\sigma} \cap R \cdot U$ of domain, as in the relational setting, is impossible. Third, sequentiality and concurrency also interact via laws such as $1_{\pi} \cdot R = 1_{\pi}$ and in particular $1_{\pi} \cdot \emptyset = 1_{\pi}$. In fact, whether a multirelation R satisfies $R \cdot \emptyset = \emptyset$, $R \cdot \emptyset \neq \emptyset$, or even $R \cdot \emptyset = R$ depends on whether or not pairs of the form (a, \emptyset) occur in it. This situation is similar to that of languages which

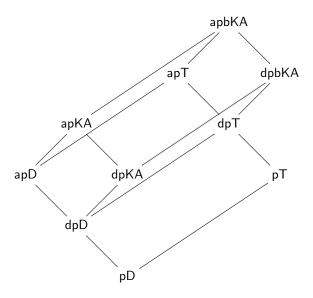


Figure 6: Summary of algebraic subclass relationships. pD stands for the class of proto-dioids, dpD for domain proto-dioids, apD for antidomain proto-dioids, dpKA for dp-Kleene algebras, apKA for ap-Kleene algebras, pT for proto-trioids, dpT for dp-trioids, apT for ap-trioids, dpbKA for dp-bi-Kleene algebras and apbKA for ap-bi-Kleene algebras.

contain finite and infinite words. There one can define the finite part $\operatorname{fin}(L)$ and the infinite part $\operatorname{inf}(L)$ of a language L and prove laws such as $\operatorname{fin}(L) \cdot \emptyset = \emptyset$ and $\operatorname{inf}(L) \cdot \emptyset = \operatorname{inf}(L)$. Here we can consider the multirelations $\tau(R) = R \cap 1_{\pi}$ and $\overline{\tau}(R) = R - 1_{\pi}$, which satisfy $\tau(R) \cdot \emptyset = \tau(R)$ and $\overline{\tau}(R) \cdot \emptyset = \emptyset$, study the sets of these elements, and derive identities for expressions such as $\tau(R \cdot S)$ or $\overline{\tau}(R \cup S)$ in analogy to the language case. Elements (a, \emptyset) can be interpreted as modelling nontermination or program errors; elements $\tau(R)$ can be seen as terminal elements, since $\tau(R) \cdot S = \tau(R)$ holds for any multirelation S. A detailed investigation is the aim of a successor paper.

While up-closed multirelations seem unsuitable for concurrency, another subclass is interesting. Call a multirelation R union-closed if for all a and $X \neq \emptyset$ the condition $X \subseteq \{A \mid (a,A) \in R\}$ implies $(a,\bigcup X) \in R$. If R has only finite internal nondeterminism, that is, for each a there are only finitely many A with $(a,A) \in R$, then R is union closed if and only if $R||R \subseteq R$. It turns out that sequential composition of union-closed multirelations is associative, while, in contrast to the up-closed case, concurrent composition remains nontrivial. In the context of concurrency it seems natural to require that a multirelation can access the union of two separate sets from some state whenever it can acces them individually. Adapting concurrent dynamic algebras to union-closed relations is another promising direction for future work. A further specialisation to Parikh's game logic based on proto-Kleene algebras with domain and antidomain seems another feasible restriction.

In conclusion, the results presented in this article lay the foundation for a

thourough algebraic exploration of Peleg's concurrent dynamic logic with its extensions and variants, Parikh's game logics and monotone predicate transformer semantics. Algebra has been instrumental in taming the tedious syntactic manipulations at the multirelational level in favour of first-order equational reasoning. More succinct descriptions of the algebra of multirelations will be given in successor papers. A unification of related approaches to games and concurrency from this basis seems possible. The integration of more advanced concepts such as communication, synchronisation, knowledge or incentive constraints remains to be explored.

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Appendix 1: Axioms of Concurrent Dynamic Algebras

First we list the complete set of proto-trioid axionms.

$$x + (y + z) = (x + y) + z$$

$$x + y = y + x$$

$$x + 0 = x$$

$$x + x = x$$

$$1_{\sigma} \cdot x = x$$

$$x \cdot 1_{\sigma} = x$$

$$x \cdot y + x \cdot z \le x \cdot (y + z)$$

$$(x + y) \cdot z = x \cdot z + y \cdot z$$

$$0 \cdot x = 0$$

$$x \| (y \| z) = (x \| y) \| z$$

$$x \| y = y \| x$$

$$1_{\pi} \| x = x$$

$$x \| (y + z) = x \cdot y + x \cdot z$$

$$0 \| x = 0$$

Next we list the concurrent dynamics algebra axioms for distributive lattices and boolean algebras. The left-hand column contains the axioms of dp-bi-Kleene algebras, the right-hand column those of ap-bi-Kleene algebras.

$$\begin{array}{lll} d(x) \cdot (y \cdot z) = (d(x) \cdot y) \cdot z & a(x) \cdot (y \cdot z) = (a(x) \cdot y) \cdot z \\ x \cdot (d(y) \cdot z) = (x \cdot d(y)) \cdot z & x \cdot (a(y) \cdot z) = (x \cdot a(y)) \cdot z \\ x \cdot (y \cdot d(z)) = (x \cdot y) \cdot d(z) & x \cdot (y \cdot a(z)) = (x \cdot y) \cdot a(z) \\ x \leq d(x) \cdot x & a(x) \cdot x = 0 \\ d(x \cdot y) = d(x \cdot d(y)) & a(x \cdot y) = a(x \cdot a(a(y)) \\ d(x + y) = d(x) + d(y) & a(x) + a(a(x)) = 1_{\sigma} \\ d(x) \leq 1_{\sigma} & a(x) \cdot (y + z) = a(x) \cdot y + a(x) \cdot z \\ d(0) = 0 & (x || y) \cdot a(z) = (x \cdot a(z)) || (y \cdot a(z)) \\ (x || y) \cdot d(z) = (x \cdot d(z)) || (y \cdot d(z)) & a(x || y) = a(x) + a(y) \\ d(x || y) = d(x) \cdot d(y) & a(x) || a(y) = a(x) \cdot a(y) \\ d(x) || d(y) = d(x) \cdot d(y) & 1_{\sigma} + x \cdot x^* \leq x^* \\ 1_{\sigma} + x \cdot x^* \leq x^* & a(z) + x \cdot y \leq y \Rightarrow x^* \cdot a(z) \leq y \\ d(z) + x \cdot y \leq y \Rightarrow x^* \cdot d(z) \leq y \end{array}$$

To obtain dp-trioids and ap-trioids, the star axioms must be dropped. To obtain proto-algebras, the concurrency axioms must be dropped.

Finally, we show, for ap-bi-Kleene algebras, the definition of domain from antidomain and those for the diamond and box operators.

$$a(a(x)) = d(x)$$
 $\langle x \rangle y = d(x \cdot y)$ $[x]y = a(\langle x \rangle a(y))$

Appendix 2: Proofs

Proof of Lemma 1

- 1. The two facts follow directly from the definition of sequential composition.
- 2. By definition, $(a, A) \notin \emptyset$ for all $a \in X$ and $A \subseteq X$, hence $\emptyset \cdot R = \emptyset$.

3.

$$(a,A) \in (R \cdot S) \cdot T \\ \Leftrightarrow \exists B, C. \ (a,C) \in R \land \exists g. \ G_g(C) \subseteq S \land B = \bigcup g(C) \land \exists f. \ G_f(B) \subseteq T \land A = \bigcup f(B) \\ \Leftrightarrow \exists C. \ (a,C) \in R \land \exists g. \ G_g(C) \subseteq S \land \exists f. \ G_f(\bigcup g(C)) \subseteq T \land A = \bigcup_{c \in C} \bigcup_{x \in g(c)} f(x) \\ \Leftrightarrow \exists C. \ (a,C) \in R \land \exists f,g. \ (\forall c \in C. \ G_g(c) \in S \land G_f(g(c)) \subseteq T) \land A = \bigcup_{c \in C} \bigcup_{x \in g(c)} f(x) \\ \Leftrightarrow \exists C. \ (a,C) \in R \land \exists f. \ \forall c \in C. \exists D. \ (c,D) \in S \land G_f(D) \subseteq T \land A = \bigcup_{c \in C} \bigcup_{d \in D} f(d) \\ \Rightarrow \exists C. \ (a,C) \in R \land \exists h. \ (\forall c \in C. \exists D. \ (c,D) \in S) \\ \land (\forall d \in D. \ (d,h(d,c)) \in T) \land A = \bigcup_{c \in C} \bigcup_{d \in D} h(d,c) \\ \Rightarrow \exists C. \ (a,C) \in R \land \exists f,h. \ \forall c \in C. \exists D. \ (c,D) \in S \\ \land \forall d \in D. \ (d,h(d,c)) \in T \land f(c) = \bigcup_{d \in D} h(d,c) \land A = \bigcup_{c \in C} f(c) \\ \Rightarrow \exists C. \ (a,C) \in R \land \exists f. \ \forall c \in C. \exists D. \ (c,D) \in S \\ \land \exists g. \ G_g(D) \subseteq T \land f(c) = \bigcup_{d \in D} g(d)) \land A = \bigcup_{c \in C} f(c) \\ \Leftrightarrow (a,A) \in R \cdot (S \cdot T).$$

4.

$$(a, A) \in (R \cup S) \cdot T \Leftrightarrow \exists B. \ (a, B) \in R \cup S \land \exists f. \ G_f(B) \subseteq T \land A = \bigcup f(B)$$
$$\Leftrightarrow (\exists B. \ (a, B) \in R \land \exists f. \ G_f(B) \subseteq T \land A = \bigcup f(B))$$
$$\lor (\exists B. \ (a, B) \in S \land \exists f. \ G_f(B) \subseteq T \land A = \bigcup f(B))$$
$$\Leftrightarrow (a, A) \in R \cdot T \lor (a, A) \in S \cdot T$$
$$\Leftrightarrow (a, A) \in R \cdot T \cup R \cdot T.$$

5. We show that $R \cdot S \subseteq R \cdot (S \cup T)$. The claim then follows by symmetry and properties of least upper bounds.

$$(a, A) \in R \cdot S \Leftrightarrow \exists B. \ (a, B) \in R \land \exists f. \ G_f(B) \subseteq S \land A = \bigcup f(B)$$
$$\Rightarrow \exists B. \ (a, B) \in R \land \exists f. \ G_f(B) \subseteq S \cup T \land A = \bigcup f(B)$$
$$\Leftrightarrow (a, A) \in R \cdot (S \cup T).$$

Proof of Lemma 2

1.

$$(a,A) \in (R||S)||T \Leftrightarrow \exists B,C,D.\ A = B \cup C \cup D \land (a,B) \in R \land (a,C) \in S \land (a,D) \in T \Leftrightarrow (a,A) \in R||(S||T).$$

- 2. $(a, A) \in R || S \Leftrightarrow \exists B, C. \ A = B \cup C \land (a, B) \in R \land (a, C) \in S \Leftrightarrow (a, A) \in S || R.$
- 3. Immediate from the definition of parallel composition and 1_{π} .
- 4. Immediate from the definition of parallel composition.

5.

$$\begin{split} (a,A) &\in R \| (S \cup T) \\ &\Leftrightarrow \exists B,C. \ A = B \cup C \land (a,B) \in R \land ((a,C) \in S \lor (a,C) \in T) \\ &\Leftrightarrow \exists B,C. \ A = B \cup C \land ((a,B) \in R \land ((a,C) \in S) \lor ((a,B) \in R \land ((a,C) \in T) \\ &\Leftrightarrow (a,A) \in R \| S \cup R \| T. \end{split}$$

PROOF OF LEMMA 3 Since $G_f(A \cup B) \subseteq R \Leftrightarrow G_f(A) \subseteq R \land G_f(B) \subseteq R$, it follows that

$$(a, A) \in (R||S) \cdot T \Leftrightarrow \exists B, C. \ (a, B \cup C) \in R||S \wedge \exists f. \ G_f(B \cup C) \subseteq T \wedge A = \bigcup f(B \cup C)$$
$$\Rightarrow \exists X, Y. \ A = X \cup Y$$
$$\wedge (\exists B. \ (a, B) \in R \wedge \exists f. \ G_f(B) \subseteq T \wedge X = \bigcup f(B))$$
$$\wedge (\exists C. \ (a, C) \in S \wedge \exists f. \ G_f(C) \subseteq T \wedge Y = \bigcup f(C))$$
$$\Leftrightarrow (a, A) \in (R \cdot T) ||(S \cdot T).$$

Proof of Lemma 5

- 1. Suppose $(a,A) \in R \cdot P$. Then there exists a set B such that $(a,B) \in R$ and, for all $b \in B$, $G_{\iota}(b) \in P$, and $A = \bigcup_{b \in B} \{b\} = B$. So $(a,A) \in R$ and $G_{\iota}(A) \subseteq P$.
 - Suppose $(a, A) \in R$ and $G_{\iota}(a) \in P$ for all $a \in A$. Then $(a, A) \in R \cdot P$ by definition of sequential composition with $f = \iota$.
- 2. Suppose $(a, A) \in P \cdot R$. Then $G_{\iota}(a) \in P$ and $(a, A) \in R$ by definition of sequential composition. Suppose that $G_{\iota}(a) \in P$ and $(a, A) \in R$. Then $(a, A) \in P \cdot R$, using $f = \lambda x \cdot A$.

Proof of Lemma 6

1. Let $R \subseteq 1_{\sigma}$. Then

$$(a, A) \in (R \cdot S) \cdot T \Leftrightarrow \exists B. \ G_{\iota}(a) \in R \ \land (a, B) \in S \land \exists f. \ G_{f}(B) \subseteq T \land A = \bigcup f(B)$$
$$\Leftrightarrow G_{\iota}(a) \in R \ \land \exists B.(a, B) \in S \land \exists f. \ G_{f}(B) \subseteq T \land A = \bigcup f(B)$$
$$\Leftrightarrow G_{\iota}(a) \in R \land (a, A) \in S \cdot T$$
$$\Leftrightarrow (a, A) \in R \cdot (S \cdot T).$$

Let $S \subseteq 1_{\sigma}$. Then

$$(a, A) \in (R \cdot S) \cdot T$$

$$\Leftrightarrow \exists B. \ (a, B) \in R \land G_{\iota}(B) \subseteq S \land \exists f. \ G_{f}(B) \in T \land A = \bigcup f(B)$$

$$\Leftrightarrow \exists B. \ (a, B) \in R \land \exists f. \ G_{\iota}(B) \subseteq S \land G_{f}(B) \subseteq T \land A = \bigcup f(B)$$

$$\Leftrightarrow \exists B. \ (a, B) \in R \land \exists f. \ G_{f}(B) \subseteq S \cdot T \land A = \bigcup f(B)$$

$$\Leftrightarrow (a, A) \in R \cdot (S \cdot T).$$

Let $T \subseteq 1_{\sigma}$. Then

$$(a, A) \in (R \cdot S) \cdot T$$

$$\Leftrightarrow (a, A) \in R \cdot S \wedge G_{\iota}(A) \subseteq T$$

$$\Leftrightarrow \exists B. \ (a, B) \in R \wedge \exists f. \ G_{f}(B) \subseteq S \wedge A = \bigcup f(B) \wedge G_{\iota}(A) \subseteq T$$

$$\Leftrightarrow \exists B. \ (a, B) \in R \wedge \exists f. \ G_{f}(B) \subseteq S \wedge G_{\iota}(A) \subseteq T \wedge A = \bigcup f(B)$$

$$\Leftrightarrow \exists B. \ (a, B) \in R \wedge \exists f. \ G_{f}(B) \subseteq S \wedge G_{\iota}(f(b)) \subseteq T \wedge A = \bigcup f(B)$$

$$\Leftrightarrow \exists B. \ (a, B) \in R \wedge \exists f. \ G_{f}(B) \subseteq S \cdot T \wedge A = \bigcup f(B)$$

$$\Leftrightarrow (a, A) \in R \cdot (S \cdot T).$$

2. Let $P \subseteq 1_{\sigma}$.

$$(a,A) \in (R||S) \cdot P$$

$$\Leftrightarrow \exists B,C. \ A = B \cup C \land (a,B) \in R \land (a,C) \in S \land G_{\iota}(A \cup B) \subseteq P$$

$$\Leftrightarrow \exists B,C. \ A = B \cup C \land (a,B) \in R \land (a,C) \in S \land G_{\iota}(A) \subseteq P \land G_{\iota}(B) \subseteq P$$

$$\Leftrightarrow \exists B,C. \ A = B \cup C \land (a,B) \in R \cdot P \land (a,C) \in S \cdot P$$

$$\Leftrightarrow (a,A) \in (R \cdot P) ||(S \cdot P).$$

3. Let again $P \subseteq 1_{\sigma}$.

$$(a, A) \in P \cdot (R \cup S) \Leftrightarrow G_{\iota}(a) \in P \wedge ((a, A) \in R \vee (a, A) \in S)$$
$$\Leftrightarrow (G_{\iota}(a) \in P \wedge (a, A) \in R) \vee (G_{\iota}(a) \in P \wedge (a, A) \in S)$$
$$\Leftrightarrow (a, A) \in P \cdot R \vee (a, A) \in P \cdot S$$
$$\Leftrightarrow (a, A) \in P \cdot R \cup P \cdot S.$$

Proof of Lemma 7

1. Obivous.

2.
$$(a, A) \in d(R) \cdot R \Leftrightarrow G_{\iota}(a) \in d(R) \land (a, A) \in R \Leftrightarrow (a, A) \in R$$
.

3.

$$G_{\iota}(a) \in d(R \cup S) \Leftrightarrow \exists B. \ (a,B) \in R \lor (a,B) \in S$$
$$\Leftrightarrow \exists B. \ (a,B) \in R \lor \exists B.(a,B) \in S$$
$$\Leftrightarrow G_{\iota}(a) \in d(R) \lor G_{\iota}(a) \in d(S)$$
$$\Leftrightarrow G_{\iota}(a) \in d(R) \cup d(S).$$

4. Obvious from the definition of domain.

5.

$$G_{\iota}(a) \in d(R \cdot S) \Leftrightarrow \exists B. \ (a, B) \in R \cdot S$$

$$\Leftrightarrow \exists B, C. \ (a, C) \in R \land \exists f. \ G_{f}(C) \subseteq S \land B = \bigcup f(C)$$

$$\Leftrightarrow \exists C. \ (a, C) \in R \land \exists f. \ G_{f}(C) \subseteq S$$

$$\Leftrightarrow \exists C. \ (a, C) \in R \land G_{\iota}(C) \subseteq d(S)$$

$$\Leftrightarrow \exists C. \ (a, C) \in R \cdot d(S)$$

$$\Leftrightarrow G_{\iota}(a) \in d(R \cdot d(S)).$$

6.

$$G_{\iota}(a) \in d(R||S) \Leftrightarrow \exists B. \ (a,B) \in R||S$$

$$\Leftrightarrow \exists C, D. \ (a,C) \in R \land (a,D) \in S$$

$$\Leftrightarrow \exists C. \ (a,C) \in R \land \exists D. \ (a,D) \in S$$

$$\Leftrightarrow G_{\iota}(a) \in d(R) \land G_{\iota}(a) \in d(S)$$

$$\Leftrightarrow G_{\iota}(a) \in d(R) \cap d(S).$$

7. Obvious.

Proof of Lemma 8

- 1. Obviously, $(a, A) \in a(R)$ iff $A = \{a\}$ and $(a, A) \notin d(R)$, which holds iff $(a, A) \in 1_{\sigma}$ and $(a, A) \notin R$.
- 2. $G_{\iota}(a) \in a(a(R)) \Leftrightarrow \neg \neg \exists A.(a,A) \in R \Leftrightarrow \exists A,(a,A) \in R \Leftrightarrow G_{\iota}(a) \in d(R).$
- 3. $G_{\iota}(a) \in d(a(R)) \Leftrightarrow G_{\iota}(a) \in a(a(a(R))) \Leftrightarrow \neg\neg\neg\exists A. (a,A) \in R \Leftrightarrow G_{\iota}(a) \in a(R).$

Proof of Lemma 9

1. $(a, A) \in a(R) \cdot R \Leftrightarrow G_{\iota}(a) \in a(R) \land (a, A) \in S \Leftrightarrow \neg \exists B \cdot (a, B) \in R \land (a, A) \in R$ which is false.

2.
$$a(R \cdot S) = 1_{\sigma} \cap -d(R \cdot S) = 1_{\sigma} \cap -d(R \cdot d(S)) = a(R \cdot d(S)).$$

3.
$$a(R) \cup d(R) = (1_{\sigma} \cap -d(R)) \cup d(R) = 1_{\sigma} \cap (-d(R) \cup d(R)) = 1_{\sigma} \cap U = 1_{\sigma}$$
.

4.

$$\begin{split} a(R \cup S) &= 1_{\sigma} \cap -d(R \cup S) \\ &= 1_{\sigma} \cap -(d(R) \cup d(S)) \\ &= 1_{\sigma} \cap -d(R) \cap -d(S) \\ &= (1_{\sigma} \cap -d(R) \cap (1_{\sigma} \cap -d(S)) \\ &= a(R) \cap a(S) \\ &= a(R) \cdot a(S). \end{split}$$

5.

$$\begin{split} a(R\|S) &= 1_{\sigma} \cap -d(R\|S) \\ &= 1_{\sigma} \cap -(d(R) \cap d(S)) \\ &= (1_{\sigma} \cap -d(R)) \cup (1_{\sigma} \cap -d(S)) \\ &= a(R) \cup a(S). \end{split}$$

6. $a(R)||a(S) = d(a(R))||d(a(S)) = d(a(R)) \cdot d(a(S)) = a(R) \cdot a(S)$.

Proof of Lemma 11

- 1. Immediate from additivity of domain.
- 2. $x \leq d(x) \cdot x$ is an axiom; $d(x) \cdot x \leq x$ holds since $d(x) \leq 1_{\sigma}$.
- 3. $d(x \cdot y) = d(x \cdot d(y)) \le d(x \cdot 1_{\sigma}) = d(x)$.
- 4. Let $x \leq 1_{\sigma}$. Then $x = d(x) \cdot x \leq d(x) \cdot 1_{\sigma} = d(x)$.

5.

$$\begin{split} d(d(x) \cdot y) &= d(d(d(x) \cdot y)) \cdot d(d(x) \cdot y) \\ &= d(d(x) \cdot y) \cdot d(d(x) \cdot y) \\ &= d(d(x) \cdot d(y)) \cdot d(d(x) \cdot y) \\ &\leq d(d(x)) \cdot d(y) \\ &= d(x) \cdot d(y), \end{split}$$

using (1), (2) and (3). For the converse direction, $d(x) \cdot d(y) \leq 1_{\sigma}$, and therefore $d(x) \cdot d(y) \leq d(d(x) \cdot d(y)) = d(d(x) \cdot y)$ by (4).

Proof of Lemma 12

We consider only the first property. Let $x \leq d(y) \cdot x$. Then

$$d(x) \le d(d(y) \cdot x) = d(y) \cdot d(x) \le d(x).$$

Let
$$d(x) \leq d(y)$$
. Then $x = d(x) \cdot x \leq d(y) \cdot x$. \square

Proof of Lemma 13

- 1. $1_{\sigma} = d(1_{\sigma}) = d(1_{\sigma}||1_{\pi}) = d(1_{\sigma}) \cdot d(1_{\pi}) \leq d(1_{\pi})$. The converse direction is obvious
- 2. $d(d(x)||d(y)) = d(d(x) \cdot d(y)) = d(x) \cdot d(y) = d(x)||d(y)|$ by meet closure.
- 3. $d(x) \| d(x) = d(x) \cdot d(x) = d(x)$.

Proof of Lemma 16

Suppose $\langle x \rangle p \leq d(q)$, that is, $d(x \cdot p) \leq d(q)$. Then, by Lemma 11(2),

$$x \cdot d(p) = d(x \cdot p) \cdot x \cdot d(p) \le d(q) \cdot x \cdot d(p) \le d(q) \cdot x.$$

For the converse implication, suppose $x \cdot d(p) \leq d(q) \cdot x$. Then $(x \cdot d(p)) \cdot d(p) \leq (d(q) \cdot x) \cdot d(p)$ and therefore $x \cdot d(p) \leq d(q) \cdot (x \cdot d(p))$ by domain associativity and idempotency. Hence

$$\langle x \rangle p = d(x \cdot p) \le d(d(q) \cdot x \cdot d(p)) = d(q) \cdot d(x \cdot d(p)) \le d(q) = q$$

by isotonicity of domain, domain export and properties of meet. \Box

Proof of Lemma 18

- 1. The functions λX . $R \cdot X$ and λX . $S \cup X$ are isotone for all R and S, hence so are their compositions.
- 2. Every ring of sets forms a complete lattice.
- 3. This follows from (1) and (2) by standard fixpoint theory (Knaster-Tarski Theorem).

Proof of Lemma 22

Suppose $x \cdot p \leq p \cdot y$. For $x^*p \leq p \cdot y^*$, it suffices to show that $p + x \cdot (p \cdot y^*) \leq p \cdot y^*$ by star induction. First, $p \leq p \cdot y^*$ by left isotonicity of multiplication and star unfold. Moreover, by the assumption and domain associativity

$$x \cdot (p \cdot y^*) = (x \cdot p) \cdot y^* \le (p \cdot y) \cdot y^* = p \cdot (y \cdot y^*) \le p \cdot y^*.$$

Proof of Lemma 23

- 1. Obviously, $p \leq \langle x^* \rangle p$ by the left unfold law. For $\langle x^* \rangle \langle x \rangle p \leq \langle x^* \rangle p$ it suffices, by star induction, to show that $\langle x \rangle p \leq \langle x^* \rangle p$ and $\langle x \rangle \langle x^* \rangle p \leq \langle x^* \rangle p$. The first inequality follows from $\langle 1_{\sigma} \rangle p \leq \langle x^* \rangle p$ and $\langle x \rangle p = \langle x \rangle \langle 1_{\sigma} \rangle p$ by isotonicity. The second one holds by left star unfold.
- 2. Let $p \leq q$ and $\langle x \rangle q \leq q$. Hence $\langle x^* \rangle q \leq q$ by Proposition 5 and the claim follows by domain isotoniticy.

Proof of Lemma 24 Note that d is an appbreviation of $a \circ a$.

- 1. Obvious from the third antidomain axiom.
- 2. $a(x) = (a(x) + d(x)) \cdot a(x) = a(x) \cdot a(x) + a(a(x)) \cdot a(x) = a(x) \cdot a(x) + 0 = a(x) \cdot a(x)$.
- 3. This holds since $a(1_{\sigma}) = 0$ and $1_{\sigma} \cdot x = 0$ implies x = 0.
- 4. Let $a(x) \le a(y)$. Then $a(x) \cdot y \le a(y) \cdot y = 0$. For the converse direction,

$$a(x) \cdot y = 0 \Leftrightarrow a(a(x) \cdot y) = 1_{\sigma} \Leftrightarrow a(a(x) \cdot d(y)) = 1_{\sigma} \Leftrightarrow a(x) \cdot d(y) = 0.$$

and therefore

$$a(x) = a(x) \cdot (d(y) + a(y)) = a(x) \cdot d(y) + a(x) \cdot a(y) = a(x) \cdot a(y) \le a(y).$$

- 5. $a(y) \cdot x \le a(y) \cdot y = 0$, so $a(y) \le a(x)$ by (3).
- 6.

$$a(x) \cdot a(y) \cdot (x+y) = a(x) \cdot a(y) \cdot x + a(x) \cdot a(y) \cdot y \le a(x) \cdot x + a(y) \cdot y = 0.$$

Moreover, by (4),

$$\begin{aligned} a(x) \cdot a(y) \cdot (x+y) &= 0 \Leftrightarrow a(a(x) \cdot a(y) \cdot (x+y)) = 1_{\sigma} \\ &\Leftrightarrow a(a(x) \cdot a(y) \cdot d(x+y)) = 1_{\sigma} \\ &\Leftrightarrow a(x) \cdot a(y) \cdot d(x+y) = 0. \end{aligned}$$

7. $a(x+y) \le a(x)$ and $a(x+y) \le a(y)$ by (5), so

$$a(x+y) = a(x+y) \cdot a(x+y) < a(x) \cdot a(y).$$

For the converse direction, by (6),

$$a(x) \cdot a(y) = a(x) \cdot a(y) \cdot a(x+y) + a(x) \cdot a(y) \cdot d(x+y)$$
$$= a(x) \cdot a(y) \cdot a(x+y) \le a(x+y).$$

8. First $a(y) \leq a(a(x) \cdot y)$ and $d(x) \leq a(a(x) \cdot y)$ by antitonicity, so

$$d(x) + a(y) \le a(a(x) \cdot y)$$

by properties of least upper bounds.

For the converse direction, we have $a(a(x) \cdot y) \cdot a(x) \cdot d(y) = 0$. Therefore,

$$\begin{split} a(a(x) \cdot y) &= a(a(x) \cdot y) \cdot d(y) + a(a(x) \cdot y) \cdot a(y) \\ &\leq a(a(x) \cdot y) \cdot d(y) + a(y) \\ &= a(a(x) \cdot y) \cdot a(x) \cdot d(y) + a(a(x) \cdot y) \cdot d(x) \cdot d(y) + a(y) \\ &= a(a(x) \cdot y) \cdot d(x) \cdot d(y) + a(y) \\ &\leq d(x) + a(y). \end{split}$$

Proof of Proposition 6

We verify the domain axioms in the setting of ap-dioids.

• The associativity laws

$$\begin{aligned} d(x) \cdot (y \cdot z) &= (d(x) \cdot y) \cdot z, & x \cdot (d(y) \cdot z) \\ &= (x \cdot d(y)) \cdot z, & x \cdot (y \cdot d(z)) \\ &= (x \cdot y) \cdot d(z) \end{aligned}$$

are immediate from antidomain associativity.

- $d(x) \leq 1_{\sigma}$ is immediate from the complementation axiom.
- $d(x) \cdot x = x$ holds because $x = (d(x) + a(x)) \cdot x = d(x) \cdot x + 0$ by the complementation and left annihilation axiom.
- $d(x \cdot y) = d(x \cdot d(y))$ is immediate from antidomain locality.
- d(0) = 0 holds because $a(0) = 1_{\sigma}$ and $a(1_{\sigma}) = 0$.
- d(x + y) = d(x) + d(y) holds because, by antidomain multiplicativity and export,

$$d(x+y) = a(a(x+y)) = a(a(x) \cdot a(y)) = d(x) + a(a(y)) = d(x) + d(y).$$

PROOF OF PROPOSITION 8

Every ap-dioid is a dp-dioid by Proposition 6. We verify the remaining axioms for parallel composition.

- The domain interaction axiom $(x \cdot d(z)) \| (y \cdot d(z)) = (x \| y) \cdot d(z)$ follows immediately from the antidomain interaction axiom.
- $d(x||y) = d(x) \cdot d(y)$ holds because

$$d(x||y) = a(a(x||y)) = a(a(x) + a(y)) = a(a(x)) \cdot a(a(y)) = d(x) \cdot d(y),$$

using the De Morgan law for a and the first antidomain concurrency axiom.

• $d(x)||d(y) = d(x) \cdot d(y)$ is immediate from the second antidomain concurrency axiom.

Proof of Lemma 27

Let $R = \{(n, \mathbb{N}) \mid n \in \mathbb{N}\}$ and $S_i = \{(n, \{m\}) \mid n \in \mathbb{N} \land 0 \leq m \leq i\}$. Thus clearly $S_i \subset S_j$ whenever i < j. Moreover,

$$(n,A) \in R \cdot \bigcup_{i \in \mathbb{N}} S_i \Leftrightarrow (n,\mathbb{N}) \in R \wedge \exists f. \ G_f(\mathbb{N}) \subseteq \bigcup_{i \in \mathbb{N}} S_i \wedge A = \bigcup_{n \in \mathbb{N}} f(n)$$
$$\Leftrightarrow (n,\mathbb{N}) \in R \wedge \exists m \in \mathbb{N}. \ (\forall n \in \mathbb{N}. \ (n,\{m\}) \in \bigcup_{i \in \mathbb{N}} S_i) \wedge A = \bigcup_{n \in \mathbb{N}} \{n\}$$
$$\Leftrightarrow (n,\mathbb{N}) \in R \wedge \exists m \in \mathbb{N}. \ (\forall n \in \mathbb{N}. \ (n,\{m\}) \in \bigcup_{i \in \mathbb{N}} S_i) \wedge A = \mathbb{N}$$

and therefore $(n, \mathbb{N}) \in F_R(\bigcup_{i \in \mathbb{N}} R_i)$ for all (n, \mathbb{N}) . However,

$$(n,A) \in R \cdot S_i \Leftrightarrow (n,\mathbb{N}) \in R \land \exists m \le i. (\forall n \in \mathbb{N}. (n,\{m\})) \in R_i \land A = \bigcup_{0 \le k \le i} \{k\},$$

hence no $F_R(R_i)$ contains (n, \mathbb{N}) for any n and therefore also not the union $\bigcup_{i \in \mathbb{N}} F_R(R_i)$. \square

Proof of Lemma 28

Suppose a family $\{S_i \mid i \in \mathbb{N}\}$ such that $S_i \subset S_j$ whenever i < j. We must show that $F_R(\bigcup_{i \in \mathbb{N}} S_i) \subseteq \bigcup_{i \in \mathbb{N}} F_R(S_i)$. So suppose $(a, A) \in F_R(\bigcup_{i \in \mathbb{N}} S_i)$. If $(a, A) \in \mathbb{I}_{\sigma}$, then $(a, A) \in \bigcup_{i \in \mathbb{N}} F_R(S_i)$.

 $(a,A) \in 1_{\sigma}$, then $(a,A) \in \bigcup_{i \in \mathbb{N}} F_R(S_i)$. Otherwise, if $(a,A) \in R \cdot \bigcup_{i \in \mathbb{N}} R_i$, then there is a finite set $B = \{b_1, \dots b_k\}$ and there are sets A_1, \dots, A_k such that $(a,B) \in R$, all $(b_i,A_i) \in \bigcup_{i \in \mathbb{N}} R_i$ and $A = \bigcup_{i \in \mathbb{N}} A_i$. Hence for all $1 \le i \le k$ there exists a l_i such that $(b_i,A_i) \in S_{l_i}$. Because of the ascending chain condition there exists a maximal S_m such that all $(b_i,A_i) \in S_m$. Then $(a,A) \in F_R(S_m)$ and finally also $(a,A) \in \bigcup_{i \in \mathbb{B}} F_R(S_i)$.

Proof of Lemma 29

1. In the base case, $F_R^0(\emptyset) = \emptyset = R^{(0)}$. In the induction step,

$$F_R^{(n+1)}(\emptyset) = F_R(F_R^n(\emptyset)) = 1_{\sigma} \cup R \cdot R^{(n)} = R^{(n+1)}.$$

Finally,
$$F_R^*(\emptyset) = \bigcup_{n \in \mathbb{N}} F_R^n(\emptyset) = \bigcup_{n \in \mathbb{N}} R^{(n)} = R^{(*)}$$
.

2. Immediate from (1).

PROOF OF LEMMA 31 $p + \langle x^* \rangle (\langle x \rangle p - p) \le p + \langle x^* \rangle \langle x \rangle p \le \langle x^* \rangle p$, by Lemma 23(1). \square

Appendix 3: Proof Automation with Isabelle/HOL

Some of the proofs at the multirelational level in this article are technically tedious, in partiular those using second-order Skolemisation. Reasoning algebraically about domain and antidomain in the absence of associativity of sequential composition is intricate for different reasons. We have therefore formalised the mathematical structures used in this article and verified many of our proofs with the interactive proof assistant Isabelle/HOL [15]. In particular, the complete technical development in this article from multirelations to star-free concurrent dynamic algebras and the complete algebraic layer have been formally verified. Finally, Isabelle's built-in counterexample generators Quickcheck and Nitpick have helped in finding some counterexamples.

We now list in detail the facts which have and have not been formally verified.

Section 3 We have verified Lemma 1, except for part (3), which is not needed for our results, the isotonicity properties of sequential composition, Lemma 2, isotonicity of concurrent composition and Lemma 3. Isabelle also provided the counterexamples in Lemma 4.

- **Section 4** All statements (Lemma 5 and 6) have been verified. The subalgebra of subidentities has not been formalised.
- Section 5 All statements, Lemma 7 to Corollary 1, have been verified.
- Section 6 We have verified irredundancy of the domain and antidomain axiom sets of domain and antidomain proto-dioids and proto-trioids. We have not explicitly formalised Theorem 1, but all facts needed in the proof have been verified.
- Section 8 Lemma 10 has been verified, but not Proposition 1, which is a well known consequence. Lemma 11 and the individual equational proof steps for Proposition 2 have been verified; the precise statement of Proposition 2 has not been formalised. Lemma 12 and Lemma 13 have been verified. The remaining facts in this section (Proposition 3 to Theorem 2) have not been verified.
- **Section 9** All proofs and counterexamples, Lemma 15 to 17, have been verified.
- Section 10 Lemma 18 to Theorem 4 have not been verified; formalising the underlying concepts seems excessive relative to the moderate difficulty of proofs. Lemma 21 to Lemma 23 have been verified. Theorem 5, which combines these results, as not been formalised as such.
- Section 11 Lemma 24 has been verified. All the equational proof steps for Proposition 6, Proposition 7 and Proposition 9 have been verified, but the individual statements have not been formalised. Proposition 7 has not been verified. Theorem 4 has not been verified, because the star in the multirelational model has not been formalised.
- **Section 12** Lemma 25 and Proposition 10 have been verified. Theorem 7 has not been formalised as, but individual proof steps have been verified. Lemma 26 has not been verified because it holds by duality between box and diamonds.
- Section 13 No results have been verified.
- Section 14 No results have been verified.

As mentioned in the Introduction, the complete Isabelle development with all proofs listed above can be found online.