Finite Difference Approximations and Tangent Analysis of Functions

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Abstract

This report investigates the use of finite difference methods to approximate derivatives of single-variable and two-variable functions and analyzes the corresponding tangent lines and tangent planes at specific points. Exact analytical derivatives are compared with forward, backward, and central difference methods to evaluate accuracy. Visualizations of functions, tangent lines, and tangent planes are included to illustrate the methods and their errors.

1 Introduction

For functions where analytical derivatives are difficult or impossible to obtain, finite difference methods provide effective numerical approximations. Common finite difference methods include:

- Forward Difference
- Backward Difference
- Central Difference

This work focuses on:

1. A one-dimensional function: $f(x) = e^{-x}\sin(3x)$
2. A two-dimensional function: $f(x, y) = x^2 e^{-x^2-y^2}$

We compute tangent lines and planes at specified points and analyze the accuracy of finite difference approximations compared to exact derivatives.

2 Methods

2.1 One-dimensional function

The first function and its exact derivative are:

$$f(x) = e^{-x}\sin(3x), \quad f'(x) = e^{-x}(3\cos(3x) - \sin(3x))$$

Finite difference approximations are given by:

$$f'_{\text{forward}}(x) = \frac{f(x+h) - f(x)}{h}$$
$$f'_{\text{backward}}(x) = \frac{f(x) - f(x-h)}{h}$$
$$f'_{\text{central}}(x) = \frac{f(x+h) - f(x-h)}{2h}$$

The tangent line at x_0 is:

$$y - f(x_0) = f'(x_0)(x - x_0)$$

The normal vector to the tangent line in 2D is:

$$\vec{n} = [-f'(x_0), 1]$$

2.2 Two-dimensional function

The second function and its partial derivatives are:

$$f(x, y) = x^2 e^{-x^2-y^2}, \quad \frac{\partial f}{\partial x} = 2x(1 - x^2)e^{-x^2-y^2}, \quad \frac{\partial f}{\partial y} = -2x^2 y e^{-x^2-y^2}$$

Finite difference approximations in 2D are:

$$\frac{\partial f}{\partial x} \approx \frac{f(x+h, y) - f(x-h, y)}{2h}, \quad \frac{\partial f}{\partial y} \approx \frac{f(x, y+h) - f(x, y-h)}{2h}$$

(for central difference; forward/backward formulas follow similarly).

The tangent plane at (x_0, y_0) is:

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The normal vector to the tangent plane is:

$$\vec{n} = [f_x(x_0, y_0), f_y(x_0, y_0), -1]$$

3 Results

3.1 One-dimensional function

Table 1: Slope and absolute error of tangent lines at $x_0 = 1.5708$, $h = 0.01$.

Method	Slope	Absolute Error
Exact	0.207880	0
Forward	0.216104	8.22e-03
Backward	0.199475	8.41e-03
Central	0.207790	9.00e-05

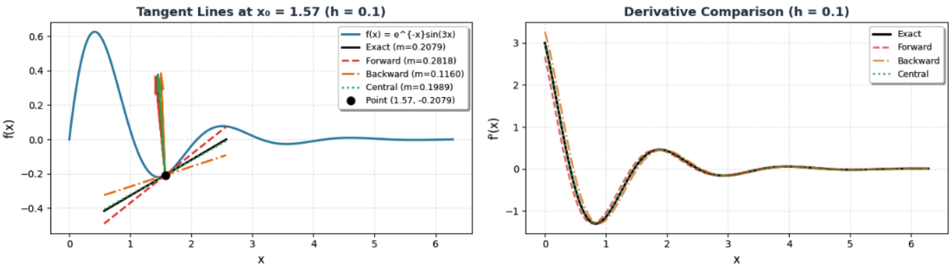

Figure 1: Function $f(x)$ and tangent lines computed by different finite difference methods (h=0.01).

Table 2: Partial derivatives and approximation errors at $(x_0, y_0) = (0.5, 0.5)$, $h = 0.01$.

Method	$\partial f/\partial x$	$\partial f/\partial y$	Error ∂x	Error ∂y
Exact	0.454898	-0.151633	-	-
Forward	0.454062	-0.152378	8.36e-04	7.45e-04
Central	0.454820	-0.151620	7.80e-05	1.30e-05

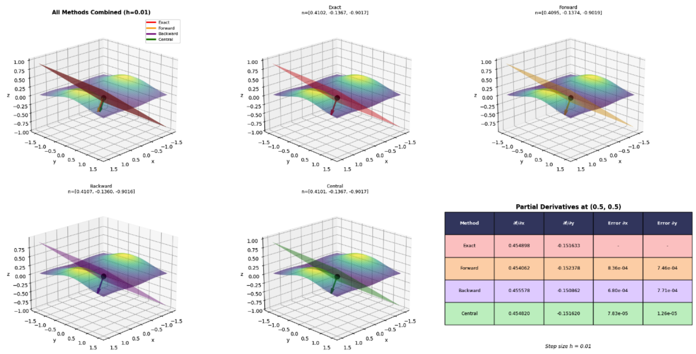

Figure 2: Surface plot of $f(x, y)$ with tangent planes at $(0.5, 0.5)$ for exact and finite difference methods (h=0.01).

3.2 Two-dimensional function

4 Discussion and Conclusion

The results demonstrate that:

- Central difference provides the most accurate numerical derivative approximation in both 1D and 2D cases.
- Normal vectors to tangent lines (1D) and tangent planes (2D) illustrate directional changes and remain nearly identical for central difference, validating its accuracy.
- Finite difference methods are simple to implement, flexible, and effective for approximating derivatives when analytical solutions are unavailable.