Theorem 1

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Theorem 1: Let $\mathcal{X}^h$ be the high component as defined in the main text, then $|\mathcal{X} - \mathcal{X}^h| \leq M * \sqrt{R^2}$, where M is a constant.

Proof:

Following Discrete Fourier Transform (DFT) on the discrete signal,

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N}kn}, \tag{1}$$

where N is the number of terms and k is the frequency index. We introduce $R^2 = 1 - \frac{SSR}{SST}$, as SSR represents the residual sum of squares and SST represents the total sum of squares. SST is the constant and SSR is determined by term N in the Fourier transform.

We can get

$$X^h[k] = \sum_{n=0}^{N-1} X^h[n] \cdot e^{-j\frac{2\pi}{N}kn}, \tag{2}$$

and

$$X^l[k] = \sum_{n=0}^{N-1} X^l[n] \cdot e^{-j\frac{2\pi}{N}kn}, \tag{3}$$

where $X^h[k] = \mathcal{X}^h$, $X^l[k] = \mathcal{X}^l$.

According to Cauchy-Schwarz inequality

$$|\mathcal{X}^l| \leq \sqrt{\sum_{n=0}^{N-1} |X^l[n]|^2} \cdot \sqrt{\sum_{n=0}^{N-1} |e^{-j\frac{2\pi}{N}kn}|^2}, \tag{4}$$

where equality holds when $X^l[n]$ and $e^{-j\frac{2\pi}{N}kn}$ have the same phase. Assuming low component $X^l[n]$ is a truncation error term and has an upper bound represented by a constant M, i.e., $X^l[n] \leq M$ for all n, following equation 4, we have:

$$|\mathcal{X}^l| \leq M \cdot \sqrt{\sum_{n=0}^{N-1} |e^{-j\frac{2\pi}{N}kn}|^2} \tag{5}$$

Notice that $|e^{-j\frac{2\pi}{N}kn}|^2$ is the square of a sine function. This integral can be bounded by a constant less than 1, i.e., $|e^{-j\frac{2\pi}{N}kn}|^2 \leq 1$.

Therefore, we modify equation 5 and obtain:

$$|\mathcal{X}^l| \leq M \cdot \sqrt{\sum_{n=0}^{N-1} 1}, \tag{6}$$

as $\sqrt{\sum_{n=0}^{N-1} 1} \leq 1$, we can obtain:

$$|\mathcal{X}^l| \leq M \cdot \sqrt{N}, \tag{7}$$

then $|\mathcal{X}^l| \leq M \cdot \sqrt{R^2}$ is proved from equation 7.