Theorem 2

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Theorem 2 denotes the estimated expectation and variance following by L_{period} compared to LCM. Under this theorem, we can calculate the information loss.

Theorem 2:

$$0 \le E\left[\mathcal{X}^h(1, L_{max}) - \mathcal{X}^h(1, L_{lcm})\right] \le \sum_{i=1}^{L_n} \int_0^1 A_i \sin(\omega_i t) dt$$

$$\begin{aligned} 0 &\le \mathrm{Var}\left[\mathcal{X}^h(1, L_{max}) - \mathcal{X}^h(1, L_{lcm})\right] \\ &\le \sum_{i=1}^{L_n} \int_0^1 \left[A_i \sin(\omega_i t) - \frac{A_i}{\omega_i}(1-\cos(\omega_i))\right]^2 dt, \end{aligned}$$

where A_i and ω_i are the amplitude and phase of i-th periodic signal and L_n represents the number of terms that cannot divide the largest period L_{max}.

Proof.

To prove the lower bound. Due to sins signal is a periodic signal, we should firstly calculate the probability density function(PDF). Assume the period of the sins function is T. As we know:

$$x(t+T) = A\sin(\omega(t+T)+\phi) = A\sin(\omega t + 2\pi k + \omega T + \phi) = A\sin(\omega t + \phi)$$

Therefore, any two positions in the same period share the same distribution of probability.

We can get the expectations as follows:

$$\begin{aligned} E[x(t)] &= \frac{1}{T}\int_{t_o}^{t_0+T} A\sin(\omega t)\mathrm{d}t = 0 \\ E[x(t)^2] &= \frac{1}{T}\int_{t_o}^{t_0+T} A^2\sin^2(\omega t)\mathrm{d}t = \frac{A^2}{2} \end{aligned} \tag{1}$$

Therefore, the PDF in the period will be:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \sum_{k=-\infty}^{\infty} \exp\left(-\frac{(x - A\sin(\omega t_k))^2}{2\sigma^2}\right) \tag{2}$$

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For the whole time points $(-\infty, \infty)$, the periodic signals can appear in any range. Therefore, the PDF in the whole time range will be:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \sum_{k=-\infty}^{\infty} \exp\left(-\frac{(x - A\sin(\omega t_k))^2}{2\sigma^2}\right)$$

For multiple sins functions on the i_{th} signal, the mixture PDF will be:

$$f(x) = \sum_{i=1}^{N} \frac{p(x,i)}{N}$$

Due to the linear characteristics of the expectations , we get:

$$E[x(t)] = E\left[x_1(t) + x_2(t)\right]$$

If all of the terms N can be divided by Period (L), then there will be no any information loss and the lower bound of expectation is proven as :

$$0 \leq E\left[\mathcal{X}^h(1, L_{max}) - \mathcal{X}^h(1, L_{lcm})\right]$$

For the variance of the lower bound, we can obtain:

$$Var[x(t)] = Var\left[x_1(t) + x_2(t)\right] = Var\left[x_1(t)\right] + Var\left[x_2(t)\right] = \frac{A^2}{2} + \frac{B^2}{2} = \frac{A^2 + B^2}{2}$$

If $L_{max} = L_{lcm}$, we can get $\mathrm{Var}\left[\mathcal{X}^h(1, L_{max}) - \mathcal{X}^h(1, L_{lcm})\right] = 0$, then the lower bound of variance is proved.

To prove the upper bound.

The worst case will be L_n terms which can not be divided by L_{period}, and there will be at most 1 period for every term. Therefore, we only need to calculate the information loss in one period of each periodic term.

To achieve this, we calculate the average value of sins function $A\sin(\omega t)$ in the range $[0, 1]$ as:

$$E = \int_0^1 A\sin(\omega t)dt$$

where A and ω are constant, then we can get

$$E = \frac{A}{\omega}[-\cos(\omega \cdot 1) + \cos(\omega \cdot 0)],$$

Due to $\cos(0) = 1$ and $\cos(\omega)$ are determined by ω, then E :

$$E = \frac{A}{\omega}(1 - \cos(\omega))$$

To calculate the variance of sins function $A\sin(\omega t)$ in the range $[0,1]$, we get:

$$\sigma^2 = \int_0^1 [A\sin(\omega t) - \mu]^2 dt$$

We can also get:

$$\sigma^2 = \int_0^1 \left[A\sin(\omega t) - \frac{A}{\omega}(1-\cos(\omega))\right]^2 dt$$

If all L_n terms lack one period, then we can obtain information loss by multiplying L_n. Then the maximum information loss will be $\sum_{i=1}^{L_n}\int_0^1 A_i\sin(\omega_i t)dt$, so the upper bound is proved.

Similarly, due to the linear characteristics, we can obtain the upper bound of $\sum_{i=1}^{L_n}\int_0^1 \left[A_i\sin(\omega_i t) - \frac{A_i}{\omega_i}(1-\cos(\omega_i))\right]^2 dt$ for the variance.

Then, the upper bound is proved.