RIGID ANALYTIC EISENSTEIN COCYCLES FOR $\mathrm{SL}_n(\mathbb{Z}[1/p])$

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Abstract. We study the Eisenstein class of a torus bundle and explore its application in constructing p-adic rigid analytic classes for SL_n. Using an explicit symbol-based approach to this class, we obtain a new construction of an Eisenstein group cohomology class for SL_n valued on distributions on $\mathbb{Z}_p^n - p\mathbb{Z}_p^n$. By integrating these distributions, we are lead to the desired rigid analytic classes for SL_n. Finally, we explore the relation between the values of these classes at points attached to totally real fields and Gross–Stark units, suggesting they provide a generalization of the modular side of the theory of complex multiplication to totally real fields where p is inert. We confirm this relationship in certain cases when F is Galois over $\mathbb{Q}$ by using the Brumer–Stark conjecture.

Contents

1. Introduction

If F is an imaginary quadratic field, then the theory of modular functions for (subgroups of) $\mathrm{SL}_2(\mathbb{Z})$ acting on the complex upper-half plane $\mathscr{H}$ offers an explicit analytic approach to understanding abelian extensions of F, or in other words a solution to Hilbert's twelfth problem. Namely, there exist certain invertible holomorphic functions

$$ {}_c g_{\alpha,\beta} \in \mathcal{O}_{\mathscr{H}}^{\times}, $$

called modular (or Siegel) units, invariant under arithmetic subgroups of $\mathrm{SL}_2(\mathbb{Z})$, such that the values ${}_c g_{\alpha,\beta}(\tau)$ at imaginary quadratic points lying in F generate such extensions; these values are called *elliptic units* for the field, and have rich internal arithmetic structure.

A naive analogue of this theory for *real* quadratic fields F is not possible: for example, because $\mathscr{H}$ does not contain real quadratic points. However, Darmon and Dasgupta [DD06] proposed a conjectural construction of "elliptic units for real quadratic fields" using a p-adic limiting process involving periods of logarithmic derivatives of modular units along real quadratic geodesics, and conjectured that they enjoyed analogous properties (above all, being algebraic and generating abelian extensions) as classic elliptic units.

Dasgupta suggested an alternate approach to constructing conjectural elements in these extensions, using p-adic integrals against distributions constructed from Shintani's method of calculating L-values of F. Using this, he gave conjectural p-adic analytic formulas for p-units in abelian extensions of general totally real fields F [Das08]. Recently, the remarkable work of Dasgupta and Kakde confirmed this conjecture; more precisely, they proved that their resulting objects are precisely the p-units satisfying the conjectures of Gross-Stark and Brumer-Stark [DK23] [DKSW23]. This furnishes a complete p-adic analytic solution to Hilbert's twelfth problem for totally real fields.

The first steps of a program towards a "modular" framework for understanding these constructions, more closely analogous to the classical theory of modular units, appeared in the work of Darmon, Pozzi, and Vonk [DPV21], who constructed analogs of modular functions which can be evaluated at real quadratic points in the *p-adic* upper half-plane $\mathscr{H}_p$. In fact, their construction produces what they term a rigid analytic cohomology class in

$$H^1(\mathrm{SL}_2(\mathbb{Z}[1/p]), \mathcal{O}_{\mathscr{H}_p}^\times/p^{\mathbb{Z}}),$$

for which a structure theory was developed in [DV21]. The evaluation of these classes involves a specialization and cap product in addition to the usual evaluation of functions. They then expressed the original construction of [DD06] as the value of a rigid class in [DPV], and gave a new proof of the conjecture of [DD06] in this setting.

More broadly, one hopes and expects that the setting of rigid analytic classes/cocycles for p-arithmetic groups will give a conceptual geometric framework for understanding p-adic limiting constructions in arithmetic in general, beyond the case of SL_2. In this paper, for any positive integer n we construct a rigid analytic "Eisenstein" class in

$$H^{n-1}(\mathrm{SL}_n(\mathbb{Z}[1/p]), \mathcal{O}_{\mathscr{X}_p}^\times/p^{\mathbb{Z}}),$$

where $\mathscr{X}_p$ is Drinfeld's p-adic symmetric domain for SL_n, generalizing $\mathscr{H}_p$ in the case $n = 2$. We then study its values at points attached to totally real fields where p is inert; we will show that the norms of these units agree with the norms of Gross-Stark units. We conjecture that the values are in fact Gross–Stark units without taking norms, and provide some theoretical evidence, including a proof of some cases when $F/\mathbb{Q}$ is Galois using the recently-proved Brumer–Stark conjecture. (In forthcoming work, by comparing formulas with the seminal work of [DK23] and [DKSW23] this conjecture, we will prove this conjecture.) Our classes therefore continue the program of establishing the arithmetic significance of rigid analytic classes, showing they give a "p-modular" or "p-automorphic" answer to (a portion of) Hilbert's twelfth problem for totally real fields.

The key ingredients in our construction are the Eisenstein class of a torus bundle of Bergeron, Charollois and García [BCG20] that replaces the role of the modular units in [DD06] and [DPV], which we study from the point of view of singular cohomology and via a symbol-based approach.

This article is a continuation of [RX25], where we produced a class in

$$H^{n-1}(\mathrm{SL}_n(\mathbb{Z}), \Omega^1_{\mathscr{X}_p}),$$

by purely topological methods; this class was the restriction to $\mathrm{SL}_n(\mathbb{Z})$ of the logarithmic derivative of our p-arithmetic class, and consequently produced elements which had analogous comparisons to p-adic logarithms of Gross–Stark units. The contribution of the present article can be viewed as refining this purely topological construction in several ways:

- We provide a means of finding explicit cocycle representatives symbolically.
- We obtain canonical cohomology classes valued in mass-zero measures with *integer* coefficients.
- We are able to construct cocycles for *p-arithmetic* (or even S-arithmetic) groups, and not just arithmetic ones, fully generalizing the original construction of [DPV].

Though we view this article as a refinement of [RX25], we do not prove that the "big" cohomology classes we obtain actually refine the ones in loc. cit.; only that certain families of their specializations coincide after extending scalars (which is enough for our applications). We do believe that the "big" classes do also coincide, but this would require the introduction of additional technical tools; see Remark 3.18.

1.1. **Summary of article and methods.**

1.1.1. *Equivariant and symbolic approaches to Eisenstein cocycles.* Denote by $T = \mathbb{R}^n/\mathbb{Z}^n$ an n-torus. Let $\mathcal{D}_T^i$ be the space of real valued smooth i-currents on T, i.e. the linear dual of (compactly supported) smooth $(n-i)$ forms on T. We then have the distributional de Rham complex

$$\mathcal{D}_T^0 \to \mathcal{D}_T^1 \to \ldots \to \mathcal{D}_T^n.$$

The double complex $C^\bullet(\Gamma, \mathcal{D}_T^\bullet)$ computes Γ-equivariant cohomology of T.

In the previous article [RX25], we considered classes

$${}_c z_{T(\Gamma)} \in H^{n-1}(T(\Gamma) - T(\Gamma)[c], \mathbb{Z}[1/c])$$

in the cohomology of a universal torus *bundle* with Γ-level structure, determined uniquely by their residues along $T[c]$ and their invariance under scalar pushforwards $[a]_*$ for $a \in \mathbb{Z}$ coprime to c. These classes correspond to analogous classes we notate

$${}_c z_\Gamma \in H_\Gamma^{n-1}(T - T[c], \mathbb{Z}[1/c])$$

in *equivariant* cohomology, via a standard geometric-equivariant dictionary, and our primary construction will be to parametrize explicit cocycles representing (refinements of) these classes using special elements in the double complex $C^\bullet(\Gamma, \mathcal{D}_T^\bullet)$.

To accomplish this, we will define a (homologically-indexed) complex of symbols: to sketch a simplified idea, in degree $k \geq 0$, it will generated by sums

$$\sum_P c_P[\ell_1, \ldots, \ell_k]_P$$

where $c_P \in \mathbb{Z}$ are constants, $[\ell_1, \ldots, \ell_k]$ is an oriented c-unimodular set of lines in $\mathbb{Z}^n$, P ranges over points of $(\mathbb{Z}/c)^n$, and the sum must satisfy the condition that the total degree along each $\ell_i[c]$ orbit of a c-torsion point is zero, i.e. for each $1 \leq i \leq k$:

$$\sum_{P \in P_0 + \ell_i[c]} c_P = 0,$$

for every $P_0 \in \frac{1}{c}\mathbb{Z}^n/\mathbb{Z}^n$. The differentials are given by the alternating sum of forgetting each line. In degree zero, with no lines to speak of, we impose only the condition that the coefficients of all points sum to zero.

In practice, we cannot quite prove the above "naive" complex is exact, only that its homology is c-torsion; thus, we will actually use a more technical "Pontryagin/Fourier-dual" variation of the above construction, which we call ${}_c\widetilde{\mathbf{Ber}}(n)$ (and whose precise definition is given in Section 3.2). The theory of matroids allows us to prove that its homology is supported in degree n, whence we construct the fully exact modification ${}_c\mathbf{Ber}(n) := {}_c\widetilde{\mathbf{Ber}}(n)/H_n({}_c\widetilde{\mathbf{Ber}}(n))$. We have a map of complexes

$$\mathbf{Ber}(n)_\bullet \to \mathcal{D}_T^{n-\bullet} \tag{1}$$

induced by the map (written in terms of our naive symbols from above)

$$[\ell_1, \ldots, \ell_k]_P \mapsto (-1)^k (t_P)_* L_* B_1(z_1) B_1(z_2) \ldots B_1(z_k),$$

where L is a matrix whose ith column is given by a generator of the line ℓ_i, viewed as a map $L : \mathbb{R}^k/\mathbb{Z}^k \to \mathbb{R}^n/\mathbb{Z}^n$, and t_P denotes the translation-by-P map.

To simplify the exposition, suppose that $p \equiv 1 \mod c$. Consider

$$T[c] - c^n\{0\} \in {}_c\mathbf{Ber}(n)_0^\Gamma.$$

From there and using the exactness of ${}_c\mathbf{Ber}(n)$, we produce a class in $H^{n-1}(\Gamma, {}_c\mathbf{Ber}(n)_n)$. Denote by ${}_c\mathcal{B}_p(n)$ the image of ${}_c\mathbf{Ber}(n)_n$ by the realization map (1). On the other hand, let ${}_cB_p(n)$ be the space of functions of the form

$$\sum_P (-1)^k c_P (t_P)_* L_* B_1(z_1) B_1(z_2) \ldots B_1(z_k). \tag{2}$$

where the c_P are as above, but now the pushforward L_* is viewed as a pushforward on functions. Using the Q-summation trick of Sczech [Scz93], we show that we have an $\mathrm{SL}_n(\mathbb{Z})$-equivariant isomorphism

$${}_cB_p(n) \xrightarrow{\sim} {}_c\mathcal{B}_p(n).$$

Hence, we obtain a class in $H^{n-1}(\Gamma, {}_cB_p(n))$. Using the values of these functions at p^r torsion sections, yields a class

$$\boldsymbol{\mu}_\Gamma \in H^{n-1}(\Gamma, \mathbb{D}_0(\mathbb{X}, \mathbb{Z})).$$

The formalism described above is not restricted to Γ which act geometrically on the torus: p-arithmetic (or even S-arithmetic, though we do not use this) groups also act on the symbol complexes we consider. Thus, we obtain a refinement:

Theorem 1.1. *There exists a class*

$$\boldsymbol{\mu} \in H^{n-1}(\mathrm{SL}_2(\mathbb{Z}[1/p]), \mathbb{D}_0(\mathbb{Q}_p^n, \mathbb{Z})^{(p)}),$$

where $\mathbb{D}_0(\mathbb{Q}_p^n, \mathbb{Z}[1/p])^{(p)})$ *denotes the space of* p*-invariant distributions such that the mass of* $\mathbb{X}$ *is equal to* 0*, and whose restriction to the subgroup* $\Gamma_1(p^r)$ *fixing the open set*

$$(1/p^r, 0, \dots, 0) + \mathbb{Z}_p^n$$

and then evaluated on this open set yields the corresponding Eisenstein class

$$v_r^* {}_c z_r \in H^{n-1}(\Gamma(p^r), \mathbb{Q})$$

up to torsion. (See Theorem 3.16 for a more precise statement.)

Now, let F be a totally real field of degree n, with totally positive units U_F. Suppose we have a coordinatization $\mathbb{Z}^n \cong \mathfrak{a}$ for an ideal $\mathfrak{a} \subset F$, equivariant for an associated embedding $U_F \hookrightarrow \mathrm{SL}_n(\mathbb{Z})$. As in [RX25, Proposition 6.8], we will show that the restriction of $\boldsymbol{\mu}$ to U_F can be used to recover the Deligne–Ribet partial zeta function attached to F and $\mathfrak{a}$ (Corollary 5.7).

Remark 1.2. There are many prior constructions of closely related "Eisenstein cocycles" in the literature, among which we highlight [CDG15], [BKL18], [GS24], [BCG20], and [BCG23]; see the review in our prior article [RX25, §1.4] for more details on the relationship between these constructions and ours. We will also make use of pioneering work of Sczech [Scz93] on Eisenstein cocycles in the present work. The method of constructing cocycles using symbol complexes was inspired by the work of Sharifi–Venkatesh [SV24] on Eisenstein cocycles in a *motivic* setting. The second author generalized this approach in [Xu24], both in the motivic as well the differential-forms setting; the current article's symbols are closely related to the ones used in that article. In a sequel, we will show the precise relation between our cocycles and those in [Xu24], which turn out to encode essentially the same information.

1.1.2. *Rigid analytic cocycles and their values.* Let $\mathscr{X}_p := \mathbb{P}^{n-1}(\mathbb{C}_p) - \cup_\alpha H_\alpha$ be Drinfeld's p-adic symmetric domain, where α runs over all $\mathbb{Q}_p$-rational hyperplanes. It admits an action of $\mathrm{SL}_n(\mathbb{Q})$. The points in $\mathbb{X} = \mathbb{Z}_p^n - p\mathbb{Z}_p^n$, given the equation of a $\mathbb{Q}_p$-rational hyperplanes. This suggests the study of the following function

$$\left(\mathbb{C}_p^n - \cup_\alpha H_\alpha\right) \times \mathbb{X} \to \mathbb{C}_p^\times, \; (\tau, x) \mapsto \tau^t \cdot x.$$

Integrating this function with respect to the variable in $\mathbb{X}$ leads to the following lift.

$$\mathrm{ST} : \mathbb{D}_0(\mathbb{X}^p, \mathbb{Z})^{(p)} \to \mathcal{A}^\times, \; \lambda \mapsto \left(\tau \mapsto \times\!\!\!\!\!\!\int_{\mathbb{X}} \tau^t \cdot x \, d\lambda\right), \tag{3}$$

where we are considering a multiplicative integral in the previous expression. This map is $\mathrm{SL}_n(\mathbb{Z})$-equivariant, and it is $\mathrm{SL}_n(\mathbb{Z}[1/p])$-equivariant after replacing $\mathcal{A}^\times$ by $\mathcal{A}^\times/p^{\mathbb{Z}}$. We can then define the desired rigid analytic classes for SL_n:

Definition 1.3. Let $J_E \in H^{n-1}(\mathrm{SL}_2(\mathbb{Z}[1/p]), \mathcal{A}^\times/p^{\mathbb{Z}})$ be the image of $\boldsymbol{\mu}$ by (3).

We are interested in values of this cocycle: again, let F be a totally real field of degree n, and suppose p is inert in F. Let $\tau \in F^n$ be such that its coordinates give an oriented $\mathbb{Z}$-basis of $\mathfrak{a}^{-1}$, for $\mathfrak{a}$ an ideal of $\mathcal{O}_F$. Since p is inert, it follows that $\tau \in \mathscr{X}_p$. Moreover, τ is a special point in $\mathscr{X}_p$ in the sense that its stabilizer in $\mathrm{SL}_n(\mathbb{Q})$ is isomorphic to the norm 1 elements of

F. In particular, its stabilizer in Γ is a group of rank $n-1$. Following a similar recipe to the case $n=2$, we define the evaluation of $J \in H^{n-1}(\Gamma, \mathcal{A}_{\mathcal{L}})$ at $\tau \in \mathscr{X}_p$, giving $J[\tau] \in \mathbb{C}_p$. From our construction, one readily deduces that $J_E[\tau] \in F_p$, where the evaluation is well-defined by first restricting to $\mathcal{O}_F^\times$ and taking the cap product with a degree-$(n-1)$ fundamental class of this rank-$(n-1)$ (virtually) free abelian group (as in [DPV]).

We conjecture that these values, as in loc. cit., coincide with the Brumer–Stark units constructed by [DK23] in the narrow Hilbert class field H/F; this is a multiplicative refinement of the analogous conjecture of [RX25]. Some evidence is provided by the following multiplicative analogue of [RX25, Theorem 1.6]:

Theorem 1.4. *For $n \geq 2$, $\mathrm{N}_{F_p/\mathbb{Q}_p} J_E[\tau] = \mathrm{N}_{F_p/\mathbb{Q}_p} \log_p(u^{\sigma_{\mathfrak{a}}})$ in $H^\times/p^{\mathbb{Z}}$, where*

$$u \in \mathcal{O}_H[1/p]^\times_- \otimes \mathbb{Q}$$

is the Gross–Stark unit given above and $\sigma_{\mathfrak{a}} \in \mathrm{Gal}\,(H/F)$ is the Frobenius corresponding to $\mathfrak{a}$.

Following the same form of argument as [RX25, §7.2], for certain totally real fields, one can remove the norms from this result and prove the comparison without norms. This is the multiplicative analogue of [RX25, Theorem 7.7]:

Theorem 1.5. *Let F be a cyclic extension of $\mathbb{Q}$ and p an inert prime in F, and let $\tau \in F^n$ be a modulus corresponding to an ideal $\mathfrak{a}$ whose class is fixed by $\mathrm{Gal}(F/\mathbb{Q})$ in the narrow class group. Then we have $J_{\mathrm{Eis}}[\tau] = u_p^{\sigma_{\mathfrak{a}}}$ in $H^\times/p^{\mathbb{Z}}$ $\pmod{\text{roots of unity}}$, where u_p is the Brumer–Stark unit attached to p and $\sigma_{\mathfrak{a}}$ is as previously.*

In forthcoming work, we will explore further the relation between our conjectural formula and the one proven by Dasgupta–Kakde and collaborators [DK23]. We will also treat the more general case where p is not inert, which will necessitate considering finer properties of our cocycles to work at the boundary of the p-adic symmetric space.

1.2. **Acknowledgements.**

2. Cohomological setup

2.1. **Equivariant cohomology via double complexes.** We will take a naive approach to (Borel) equivariant cohomology in this paper. Let G be a discrete group, acting on the left a smooth manifold M, and let $C_M^\bullet$ be a cohomologically-indexed complex computing the usual (singular/de Rham) cohomology $H^\bullet(M)$ of M, either with integer or real coefficients, with suitable functorial properties: in practice, the complexes we consider will at minimum be covariantly functorial for proper maps; i.e. equipped with natural maps

$$f_* : C_Z^{\bullet+d} \to C_M^\bullet$$

for a relative dimension-d proper map $f : Z \to M$, but have varying contravariance properties. However, pullback by an isomorphism ι can always be defined as the pushforward ι_*, so that group actions can equivalently be viewed covariantly/contravariantly.

Then we *define* the equivariant cohomology

$$H_G^\bullet(M)$$

of the G-space M to be the total cohomology of the double complex

$$C^\bullet(G, C^\bullet_M)$$

where G acts by pushforwards on $C^\bullet_M$: in other words, we apply the exact functor $C^\bullet(G, -)$ of taking G-cochains to our cohomology complex.

We will really only use a single "model" for equivariant cohomology in this article, using a chain complex of smooth currents; however, we write down a slightly more flexible framework below for potential future uses: suppose we have two such "models" for equivariant cohomology, coming from functorial complexes $B^\bullet_M$ and $C^\bullet_M$ satisfying the above conditions. Then if there is a natural transformation $B^\bullet_M \to C^\bullet_M$ which is a quasi-isomorphism for each M (maybe after extension of scalars), then this induces an isomorphism between the associated equivariant theories as well (after extension of scalars).

This construction has the following properties/extensions:

- Via the action on the coefficients of the cochains, it has the same covariance/contravariance properties for maps of G-spaces as the original cohomology complex had for maps of spaces.
- There is a Grothendieck spectral sequence with $C^p(\Gamma, C^q_M)$ as its $E_0^{p,q}$ term, and E_2 page
$$E_2^{p,q} = H^p(G, H^q(M))$$
where the action of G on $H^q(M)$ is by pushforward.
- If $j : Z \hookrightarrow M$ is a closed G-submanifold of codimension d, with complementary inclusion $i : M - Z \hookrightarrow M$, then then the homological mapping cone construction
$$\mathcal{C}_Z[-d] \to \mathcal{C}_M \to C(j)$$
is such that the natural composition map
$$\mathcal{C}_{M-Z} \xrightarrow{i_*} \mathcal{C}_M \to C(j)$$
is a quasi-isomorphism, and thus the distinguished triangle above induces the long exact localization sequence for (M, Z) in cohomology. Applying the (exact) G-cochains functor, this induces equally a long exact localization sequence in G-cohomology
$$\ldots \to H^{i-d}_G(Z) \to H^i_G(M) \to H^i_G(M - Z) \to H^{i+1-d}(Z) \to \ldots$$
In cases with good contravariance, there is a direct restriction map $\mathcal{C}_M \xrightarrow{i^*} \mathcal{C}_{M-Z}$ completing the distinguished triangle (which then has a natural quasi-isomorphism to the mapping cone by its universal property), and one may define the localization sequences without reference to the mapping cone. In general, even without such contravariance, we can also use the natural maps $\mathcal{C}_m \to C(j)$ to define contravariant functoriality of (equivariant) cohomology for open immersions.
- If $M_i \to M$ is a G-projective system of immersed submanifolds as i ranges over the objects of some indexing diagram I, with final structure maps $j_i : M_i \to M$, then we

define the ith equivariant cohomology of this pro-system as the cohomology of

$$\varinjlim_i C^\bullet(G, C^\bullet_{M_i}) \text{ or } \varinjlim C^\bullet(G, C^\bullet_{j_i})$$

depending on whether we have contravariance functoriality for open immersions. In either case, we then also have a Grothendieck spectral sequence with second page

$$E_2^{p,q} = H^p(G, \varinjlim H^q(M_\bullet))$$

by exactness of direct limits and of the cochains functor.

All of these similarly are compatible for a pair of "models" which are naturally quasi-isomorphic. See the second author's thesis [Xu23] for more details.

Remark 2.1. The preponderance of pushforward actions is a little unusual for cohomology: this is because in practice, our complexes will all actually be computing *Borel–Moore homology.* For manifolds, reversing the indices identifies Borel–Moore homology canonically with cohomology by Poincaré duality, so this is not a very important distinction for us.

2.2. **An Eisenstein class in equivariant cohomology.** From this section onwards, for convenience we will write V, or $V_{\mathbb{Z}}$, for the defining representation of $\mathrm{GL}_n(\mathbb{Z})$, and V_R for $V_{\mathbb{Z}} \otimes R$ for any $\mathbb{Z}$-module R. We define the torus T, as a $\mathrm{GL}_n(\mathbb{Z})$-space, as the smooth quotient $V_{\mathbb{R}}/V_{\mathbb{Z}}$; then $V_{\mathbb{Z}/c}$ can be naturally be identified with the torsion points $T[c]$, by viewing $\mathbb{Z}/c$ as $\mathbb{Z}[1/c]/\mathbb{Z}$. (The earlier notation $\mathbb{X}$ can then also be identified with $V_{\mathbb{Z}_p} - pV_{\mathbb{Z}_p}$, for example.)

Let $\Gamma \leq \mathrm{GL}_n(\mathbb{Z})$ be any subgroup. We have the localization sequence associated to the closed Γ-fixed subspace $T[c] \subset T$

$$\ldots \to H^{n-1}(T) \to H^{n-1}(T - T[c]) \xrightarrow{\partial} H^0(T[c]) \to H^n(T) \to \ldots$$

Just as in [BCG20, §3], the Γ-equivariant scalar pushforwards $[a]_*$ for all integers a with $(a, c) = 1$ act on this sequence, and in particular act on $H^{n-1}(T)$ only by the eigenvalues $a, a^2, \ldots, a^n$. As explained in [RX25, §2], if c is invertible the coefficients, we can as a result construct a unique lift of

$$[T[c] - c^n\{0\}] \in H^0(T[c])^\Gamma,$$

which we call

$${}_cz_\Gamma \in H^{n-1}_\Gamma(T - T[c], \mathbb{Z}[1/c])$$

characterized by its invariance by all $[a]_*$ for a prime to c. This construction is precisely the analogue of [RX25, §2].

In particular, following the conventions of that previous article, we set $\Gamma_r \subset \mathrm{GL}_n(\mathbb{Z})$ to be the subgroup fixing $(1, 0, \ldots, 0) \in V_{\mathbb{Z}/p^r}$, and $v_r \in T[p^r]$ to be the corresponding point in our torus. Then we have a pullback

$$v_r^* {}_cz_{\Gamma_r} \in H^{n-1}_{\Gamma_r}(*) \cong H^{n-1}(\Gamma_r)$$

which is the same class as $v_r^* {}_cz_r$ of loc. cit., which we showed there is given by taking periods of a weight-2 level-Γ_r holomorphic Eisenstein series.

2.3. **Equivariant distributional de Rham cohomology.** In this section, we introduce the *distributional de Rham complex*, which will be our primary model for equivariant cohomology (with real coefficients).

Write $\mathcal{D}_T^i$ for the real-valued smooth i-currents on T, i.e. the linear dual of the compactly-supported smooth $(n-i)$-forms $\Omega_{T,c}^{n-i}$. The exterior derivative $d : \mathcal{D}_T^i \to \mathcal{D}_T^{i+1}$ is defined as the graded adjoint of the exterior derivative on forms, via

$$(dc)(\omega) := (-1)^{\deg c} c(d\omega).$$

With this differential, the currents form a complex

$$\mathcal{D}_T^0 \to \mathcal{D}_T^1 \to \ldots \to \mathcal{D}_T^n,$$

functorial for flat pullback and finite pushforward, computing the real cohomology of T. There is a quasi-isomorphism from the usual de Rham complex

$$\upsilon : \Omega_T^i \to \mathcal{D}_T^i,\ \omega \mapsto \left(\eta \mapsto \int_T (-1)^{n-i} \eta \wedge \omega\right). \tag{4}$$

Via this map, we can and will implicitly view smooth forms as currents.

We would like to say that the map υ is a natural isomorphism, but this functoriality actually fails, because the integral over T depends on its orientation, and so is reversed in sign by orientation-reversing maps. Consequently, for orientation-reversing maps, the action on currents can fail to give the correct action on cohomology. For example, on S^1, pushforward by the inverse map $[-1]_*$ sends the 1-form $dz \mapsto -dz$, as it does for the associated cohomology class, but if we take the "natural" adjoint action, we have:

$$[-1]_*(\upsilon(dz))(\eta) = (\upsilon(dz))([-1]^*\eta) = \int_{S^1} [-1]^*\eta \wedge dz = \int_{S^1} \eta(-z)\, dz = \int_{S^1} \eta(z)\, dz = (\upsilon(dz))(\eta)$$

for a compactly supported smooth 0-form (i.e. function on S^1) η, so we see that $[-1]_*$ actually fixes the associated current.[1]

This subtlety addressed, we now introduce an important class of currents: associated to closed oriented submanifolds $Z \subset T$ of codimension s, we have a closed *current of integration*

$$\delta_Z \in \mathcal{D}_T^s$$

defined by

$$\delta_Z(\omega) := \int_Z \omega.$$

In line with our discussion above, reversing the orientation of Z turns δ_Z into $-\delta_{-Z}$. Further, a current $\omega \in \mathcal{D}_T^{n-1}$ having residue $\mathcal{C} \in H^0(T[c])$ along the residue map

$$H^{n-1}(T - T[c], \mathbb{R}) \to H^0(T[c])$$

is equivalent to $d\omega = \delta_{\mathcal{C}}$ (where this is interpreted as a suitable linear combination of the currents of integration along points in the support of $\mathcal{C}$); see for example [Xu23, (3.3)] from

[1]Philosophically, this is coming from the distinction between Borel-Moore homology and cohomology: the PoincarÃ© duality identification depends on a choice of orientation.

the author's thesis. In general, [Xu23, §3.2.2] contains more details on the distributional de Rham complex along with proofs (or references to original proofs).

We note that because compactly supported forms can be pushed forward by arbitrary smooth maps and pulled back along proper ones, currents have proper pushforward functoriality and arbitrary pullback functoriality. Thus, the localization sequence for a relative dimension-d pair $j : Z \hookrightarrow X$ arises from the distinguished triangle

$$\mathcal{D}_Z[d] \xrightarrow{j_*} \mathcal{D}_X \to \mathcal{D}_{X-Z}.$$

Note that the pushforward of the 0-current 1 under j_* is simply the current of integration δ_Z. More generally, we can think of the pushforward of a function f on Z (viewed as a 0-current) as a weighted current of integration $f\delta_Z$.

In particular, we observe that an element ω of the double complex $C^\bullet(\Gamma, \mathcal{D}_T^\bullet)$ restricts to a representative of a class in $H_\Gamma^{n-1}(T - T[c])$ with residue

$$[T[c] - c^n\{0\}] \in H^0(T[c])^\Gamma$$

if and only if the total differential of ω is $\delta_{T[c]} - c^n\delta_0 \in C^0(\Gamma, \mathcal{D}_T^n)$. In particular, if we can find such a class which is invariant by $[a]_*$ for $a \in \mathbb{N}^{(c)}$, it will represent the class

$${}_cz_\Gamma \in H_\Gamma^{n-1}(T - T[c], \mathbb{R}).$$

2.4. **An equivariant-geometric dictionary.** We recall the setup of [RX25]: let $\Gamma \leq \mathrm{SL}_n(\mathbb{Z})$, such that it acts freely and discontinuously on the symmetric space X for $\mathrm{GL}_n(\mathbb{R})$. Then we have a torus bundle over the classifying space $B\Gamma := \Gamma\backslash X$

$$T_\Gamma := \Gamma\backslash(X \times T),$$

and a class in the cohomology of the open

$${}_cz_{T_\Gamma} \in H^{n-1}(T_\Gamma - T_\Gamma[c], \mathbb{Z}[1/c])$$

characterized by being trace-fixed for prime-to-c isogenies and having residue $T[c] - c^n\{0\}$, i.e. by a formally identical construction to our class ${}_cz_\Gamma \in H_\Gamma^{n-1}(T - T[c], \mathbb{Z}[1/c])$. We wish to compare this construction with our equivariant class

$${}_cz_\Gamma \in H_\Gamma^{n-1}(T - T[c], \mathbb{Z}[1/c]),$$

after extending scalars to $\mathbb{R}$. Indeed, for any oriented manifold M with Γ-action, we have the chain of quasi-isomorphisms

$$\Omega^\bullet_{(M\times X)/\Gamma} \xrightarrow{\mathcal{I}} C^\bullet(\Gamma, \Omega_M^\bullet) \xrightarrow{\upsilon_*} C^\bullet(\Gamma, \mathcal{D}_M^\bullet)$$

where υ is the quasi-isomorphism of de Rham complexes defined previously, and the $\mathcal{I}$ is defined (and proven to be a weak equivalence) in [BCG23, Appendice A]: here, we make use of the fact that taking geodesic simplices for Γ makes X a smooth model for $E\Gamma$, up to finite order stabilizers (whose contribution to rational cohomology vanishes; e.g., by passing to a finite index torsion-free subgroup and taking norms).

Proposition 2.2. *For any oriented G-manifold M, the above zigzag induces natural isomorphisms*

$$H^i_\Gamma(M,\mathbb{R}) \cong H^i((M\times X)/\Gamma,\mathbb{R}).$$

Since the formal properties characterizing them in cohomology are identical, we can immediately deduce the corollary:

Corollary 2.3. *Under the isomorphism of the preceding proposition, the class ${}_cz_{T_\Gamma}$ is identified with ${}_cz_\Gamma$ after extending scalars to $\mathbb{R}$ (and therefore also after extending scalars to $\mathbb{Q}$, by flatness).*

2.5. **Trace-fixed parts of the distributional de Rham complex.** We now will need the following codification of our previously-used notions of being fixed under various pushforward isogenies $[a]_*$: fix an auxiliary integer $c>1$ which we omit from the notation, and let M be a module for the monoid $\mathbb{N}^\times$, i.e. a module with commuting actions of $[a]$ for $a\in\mathbb{N}^{(c)}$ such that $[a][b]=[ab]$. We write $M^{(0)}$ for the submodule of M on which $[a]=1$ for all $a\in\mathbb{N}^{(c)}$.

In particular, via the pushforward action, spaces of currents on T all have an action of $\mathbb{N}^\times$. The aim of this section is to prove the following proposition:

Proposition 2.4. *The subcomplex of the distributional de Rham complex of T*

$$(\mathcal{D}^0_T)^{(0)}\to(\mathcal{D}^1_T)^{(0)}\to\ldots\to(\mathcal{D}^n_T)^{(0)} \tag{5}$$

is exact except for at the final place, where it has cohomology $\mathbb{R}$, realized by the map

$$(\mathcal{D}^n_T)^{(0)}\to\mathbb{R},\ \omega\mapsto\omega(1_T)$$

where 1_T is the constant function on T.

Let us describe the strategy for the proof: we will find idempotent projection maps

$$\phi_i:\mathcal{D}^i_T\to(\mathcal{D}^i_T)^{(0)} \tag{6}$$

intertwining the differentials and thus giving a map of complexes. We therefore obtain maps on cohomology

$$H^i((\mathcal{D}^\bullet_T)^{(0)})\to H^i(\mathcal{D}^\bullet_T)^{(0)}\xrightarrow{(\phi_i)_*}H^i((\mathcal{D}^\bullet_T)^{(0)})$$

with the first map induced by the natural inclusion of complexes, such that the composition is the identity. We conclude that the first map is an injection. However,

$$H^i(\mathcal{D}^\bullet_T)^{(0)}\cong H^i(T,\mathbb{R})^{(0)}$$

vanishes except when $i=n$ [BCG20, §3], in which case it is spanned by the class of the volume form $dz_1\wedge\ldots\wedge dz_n$ (viewed as a current via the inclusion of smooth forms). Since this volume form is fixed by $[a]_*$ for all $a\in\mathbb{N}$, the result follows.

To construct the projections ϕ_i, we will use Fourier analysis. Under the description $T=V_\mathbb{R}/V_\mathbb{Z}$, write $z_1,z_2,\ldots,z_n$ for the standard coordinates on $V_\mathbb{R}=\mathbb{R}^n$, so that $V_\mathbb{Z}$-periodic functions (forms, etc.) in the z_i yield functions on the torus. Then define the *Fourier coefficients* of a distribution $\omega\in\mathcal{D}^n_T$ by

$$a_{k_1,\ldots,k_n}(\omega):=\omega(\exp(2\pi i(k_1,\ldots,k_n)\cdot(z_1,\ldots,z_n)).$$

for $(k_1, \ldots, k_n) \in V_{\mathbb{Z}}$; this extends the definition of the Fourier coefficients of a function.

We have the following characterization [RT10, §3.1] of smooth distributions on the torus:

Proposition 2.5. *The Fourier transform*

$$\mathcal{F}_T : \omega \mapsto ((k_1, \ldots, k_n) \mapsto a_{k_1,\ldots,k_n}(\omega))$$

defines an isomorphism

$$\mathcal{D}_T^n \to \mathcal{S}'(V_{\mathbb{Z}})$$

where $\mathcal{S}'(V_{\mathbb{Z}})$ is the set of tempered*, or* slowly-increasing*, functions on $V_{\mathbb{Z}}$; i.e, functions with at most polynomial growth at infinity.*

Recall, by contrast, that smooth *functions* map onto rapidly-decreasing functions on $V_{\mathbb{Z}}$, i.e. functions decaying faster than the inverse of any polynomial. Note that the standard definition of the Fourier series of a function is consistent with the definition for its corresponding current.

Example 2.6. For an example of the Fourier transform of a current which is not a smooth function, the Fourier transform of the current of integration at the identity δ_0 is the constant function 1, since every character of the torus takes the value 1 at the identity. Similarly, Fourier transforms of currents of integration on a subtorus embedded as a subgroup will have all 1s along some corresponding linear subspace of $\mathbb{Z}^n$ (corresponding to the subgroup of characters vanishing on that torus) and 0 otherwise.

We can extend this result to currents of arbitrary degree, generalizing how one can take Fourier series of differential forms of arbitrary degree on T by writing them as linear combinations of the basis elements dz_i:

Corollary 2.7. *Define the Fourier transform*

$$\mathcal{F}_T : \omega \mapsto ([(k_1, \ldots, k_n), dz_I] \mapsto \omega(\exp(2\pi i(k_1, \ldots, k_n) \cdot (z_1, \ldots, z_n))\, dz_I)),$$

where $I \subset \{1, 2, \ldots, n\}$ is a subset of cardinality $n - i$, and

$$dz_I := \bigwedge_{i \in I} dz_i \in \Omega_T^{n-i}.$$

Then $\mathcal{F}_T$ yields an isomorphism

$$\mathcal{D}_T^i \to \mathcal{S}'(V_{\mathbb{Z}}) \otimes \left(\bigwedge^{n-i} Lie(V_{\mathbb{R}})^\vee \right)^\vee.$$

For endomorphisms $\gamma : T \to T$ with $\gamma \in M_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q})$, one can check that the natural (non-square bracket) endomorphism γ_* of currents corresponds to the pullback action γ^* on $\mathcal{S}'(\mathbb{Z}^n)$ and the natural action γ_* on the Lie algebra.

One can compute that under the identifications given by the Fourier transform, the ith differential in the distributional de Rham complex is given by the map

$$\mathcal{S}'(V_{\mathbb{Z}}) \otimes \left(\bigwedge^{n-i} \mathrm{Lie}(V_{\mathbb{R}})^\vee \right)^\vee \to \mathcal{S}'(V_{\mathbb{Z}}) \otimes \left(\bigwedge^{n-i-1} \mathrm{Lie}(V_{\mathbb{R}})^\vee \right)^\vee \tag{7}$$

sending

$$\varphi \otimes \partial z_I \mapsto 2\pi i \sum_{i=1}^{n} (-1)^{\mathrm{sgn}(I,i)} (\partial_i \varphi) \otimes \partial z_{I \setminus \{i\}} \tag{8}$$

where here ∂z_I is the dual basis element to dz_I, $dz_{I\setminus\{i\}}$ is to be interpreted as zero if $i \notin I$, $\mathrm{sgn}(I, i)$ is $+1$ if the ordinal place of i inside I (under the usual ordering of integers) is odd and -1 if even, and

$$\partial_i \varphi := ((k_1, \ldots, k_n) \mapsto k_i \varphi(k_1, \ldots, k_n))$$

Another important computation is the fact that the pushforward on i-currents $[a]_*$ sends

$$\varphi \otimes \partial z_I \mapsto a^{n-i}([a]^* \varphi) \otimes \partial z_I. \tag{9}$$

This applies for all $a \in \mathbb{Z}$, not just $a \in \mathbb{N}$.

Proof of Proposition 2.4. We now turn to the construction of ϕ_i on the level of the Fourier transforms. Write $P \subset \mathbb{Z}^n$ for the set of *c-primitive* vectors such that their greatest common denominator is divisible only by primes dividing c; for any vector

$$\underline{k} \in V_{\mathbb{Z}} \setminus \{(0, \ldots, 0)\},$$

write $P(\underline{k})$ for the unique vector in P dividing $\underline{k}$.

We define ϕ_i by sending

$$\varphi \otimes \partial z_I \mapsto \varphi' \otimes \partial z_I$$

where φ' is defined by

$$\underline{k} \mapsto \left(\frac{\underline{k}}{P(\underline{k})} \right)^{i-n} \varphi(P(\underline{k}))$$

for $\underline{k} \neq (0, \ldots, 0)$, and sending $(0, \ldots, 0) \mapsto \phi(0, \ldots, 0)$ if $i = n$ and $(0, \ldots, 0) \mapsto 0$ otherwise. From the formula (9), one immediately sees that the image of ϕ_i is fixed by $[a]_*$ for all $a \in \mathbb{N}^{(c)}$. From (8), it is also a short computation to verify ϕ_i commutes with the exterior derivative. Proposition 2.4 therefore follows. ☐

Remark 2.8. By picking larger and larger c, one can assemble the results of this section into the result that the distributional de Rham complex is exact even if one passes to the subcomplex of elements invariant under *all but finitely many* $[a] \in \mathbb{N}$, which is a nicer formulation since it does not depend on the auxiliary c. For our purposes, taking some fixed c suffices, so we omit this extra argument.

2.6. **Some equivariant algebra.** Let us now describe the upshot of Proposition 2.4. We have the exact complex

$$(\mathcal{D}^0_T)^{(0)} \to (\mathcal{D}^1_T)^{(0)} \to \ldots \to (\mathcal{D}^n_{T,0})^{(0)}, \tag{10}$$

where $(\mathcal{D}^n_{T,0})^{(0)}$ is defined as the kernel of its natural map to $\mathbb{R}$. since $[a]_*$ acts trivially for all $a \in \mathbb{N}^{(c)}$, this complex has an action by pushforward of not just $\mathrm{GL}_n(\mathbb{Z})$, but even of $\mathrm{GL}_n(\mathbb{Z}_{(c)})$, a necessary extension for our desired p-arithmetic applications. This allows us to construct $(n-1)$-cocycles for this latter group, by the following lemma from homological algebra:

Lemma 2.9. *If a group G acts on an exact complex $C_\bullet$ supported in degrees $[0,n]$, then we have a natural map on cohomology*

$$C_0^G \to H^{n-1}(G, C_n)$$

inhomogenous cocycle representatives of which can be constructed as follows: associated to $e \in C_0^G$, pick a lift c_1 of e to C_1, and consider the 1-cochain

$$\gamma \mapsto (\gamma - 1)c_1 \in C^1(G, C_1).$$

By exactness, this is the boundary of an element $c_2 \in C^1(G, C_2)$; we take the chain coboundary $\partial c_2 \in C^2(G, C_2)$ which again lifts to $c_3 \in C^2(G, C_3)$, etc. The lift $c_n \in C^{n-1}(G, C_n)$ is a cocycle representing the image of e in $H^{n-1}(G, C_n)$.

Proof. See [Xu24, Lemma 2.5]. □

Suppose $\Gamma \subset \mathrm{GL}_n(\mathbb{Z}_{(c)})$ fixes the c-torsion cycle $T[c] - c^n\{0\}$ of degree zero, with associated current of integration $\delta_{T[c]} - c^n\delta_0 \in (\mathcal{D}^n_{T,0})^{(0)}$. We deduce an element $\mathcal{Z}^{(c)}_\Gamma$ as its image under the map

$$\left((\mathcal{D}^n_{T,0})^{(0)}\right)^\Gamma \to H^{n-1}(\Gamma, (\mathcal{D}^0_T)^{(0)})$$

of the above lemma; further, if we can find suitable lifts in the trace-fixed complex, we would be able to find cocycle representatives for $\mathcal{Z}^{(c)}_\Gamma$, and in principle be able to explicitly control their various properties, e.g. rationality/integrality or various relations they satisfy.

As such, this construction will be our primary approach to the construction of Eisenstein classes in the remainder of this article. However, it is important that we can relate this construction to the previously constructed Eisenstein classes $z^{(c)}_\Gamma$; we conclude the section by showing this is indeed the case: for $\Gamma \subset \mathrm{GL}_n(\mathbb{Z})$, the lifting process described in Lemma 2.9 associated to

$$e = \delta_{T[c]} - c^n\delta_0 \in (\mathcal{D}^n_{T,0})^{(0)}$$

produces elements

$$c_i \in C^{i-1}(\Gamma, (\mathcal{D}^{n-i}_T)^{(0)})$$

for $i = 1, 2, \ldots, n$.

Proposition 2.10. *In the setting described above, the sum*

$$c_1 + \ldots + c_n \in C^\bullet(\Gamma, \mathcal{D}^\bullet_T)$$

is a representative for the equivariant Eisenstein class ${}_cz_\Gamma \in H^{n-1}_\Gamma(T - T[c])$.

Proof. Restricted to $T - T[c]$, the element $c_1 + \ldots + c_n$ is an element closed under the total differential, hence representing an equivariant cohomology class. Further, it is a trace-fixed class which is the restriction of a class from $C^\bullet(\Gamma, \mathcal{D}^\bullet_T)$ whose total differential there is $\delta_{T[c]} - c^n\delta_0$, and these two properties characterize the cohomology class ${}_cz_\Gamma$. □

We would like relate the group cohomology class $[c_n]$ to the equivariant cohomology class $z^{(c)}_\Gamma$ in a purely algebraic way. Consider the Hochschild-Serre spectral sequence

$$H^p(G, H^q(X)) \Rightarrow H^{p+q}_G(X)$$

for a group G acting on a space X. Note that if the G-equivariant cohomology of X is defined by the double complex $C^\bullet(G, \mathcal{D}_X^\bullet)$, then the above spectral sequence is precisely the spectral sequence of this double complex.

We apply this to our setting $G = \Gamma$ and $X = T$ as follows: suppose we have some lifts $c_1, c_2, \ldots, c_n$ as in Lemma 2.9 such that each c_i is a closed current upon restriction to some acyclic Γ-subspace $U \subset T$ (or Γ-projective system of subspaces, as the case may be). For this U there then exists a generalized edge map

$$E_U : H_\Gamma^{n-1}(U) \to H^{n-1}(\Gamma, H^0(U))$$

since U has no higher cohomology, e.g. as constructed in [BCG23, Annexe A.3].[2] From that construction, we see that if some equivariant class $c \in H_\Gamma^{n-1}(U)$ is represented by

$$\sum_{i+j=n-1} \omega^{i,j} \in \bigoplus_{i+j=n-1} C^i(\mathcal{D}_U^j)$$

such that each component of $\omega^{\bullet,\bullet}$ is closed under the de Rham differential on U, then the image $E_U(c)$ is represented by $[\omega^{n-1,0}]$ (and in particular independent of choices).

Lemma 2.11. *Suppose, in the context of the discussion following Lemma 2.9, that the lifts c_i for $1 \leq i \leq n$ are all closed under the de Rham differential after restriction to some $U \to T - T[c]$. Then viewed as a cocycle valued in $H^0(U)$, c_n represents the image of the restriction of ${}_cz_\Gamma$ under E_U.*

Proof. By Proposition 2.10, $c_1 + \ldots + c_n$ represents $z_\Gamma^{(c)}$ in $C^\bullet(\Gamma, \mathcal{D}_{T-T[c]}^\bullet)$. Then the result follows by the preceding discussion. $\square$

We will find such suitable lifts $c_1, \ldots, c_n$ using complexes of symbols which map to the distributional de Rham complex, in the context of the equivariant Eisenstein class. The resulting formula for c_n will then give explicit representatives of the Eisenstein classes in group cohomology, by the preceding discussion.

3. Symbol construction of cocycles valued in locally constant functions

3.1. **Bernoulli polynomials and their locally constant stabilizations.** We now know that in principle, one can compute Eisenstein classes by finding suitable lifts inside the trace-fixed distributional de Rham complex. In particular, if we can find lifts providing a cocycle representative c_n of ${}_c\mathcal{Z}_\Gamma$ valued in smooth functions (considered inside the space of 0-distributions) when restricted to some subspace containing Γ-fixed torsion points, we will be able to evaluate at these points to obtain group cocycles. By [Xu23, Lemma 2.2], these cocycles will then represent the classical Eisenstein classes of the form $v_r^* {}_cz_\Gamma$ (for example, when $\Gamma = \Gamma_r = \Gamma_1(p^r)$). However, the problem remains of finding candidates for what these explicit lifts should be: this will be the goal of this section.

[2]There, the edge map is constructed specifically for the double complex of simplicial differential forms, but it is easy to see the construction is formal and applies to any first quadrant cohomologically indexed double complex with vanishing in some right half-plane.

The explicit elements in the trace-fixed complex will be built out of the *periodic weight-*1 *Bernoulli polynomial*

$$B_1(z) := \{z\} - \frac{1}{2}$$

where $\{z\}$ is the fractional part of z, i.e. its unique representative in $[0,1)$ modulo 1, for all $z \notin \mathbb{Z}$. We set the value at all integers n to be $B_1(n) = 0$.

Clearly, this function is not smooth, or even continuous; its graph is in the shape of a sawtooth, with a two-sided jump discontinuity at every integer. However, it has two good properties: it is periodic, meaning it can be considered as a function on the circle, and as a function on the circle it satisfies the distribution property $[a]_* B_1(z) = B_1(z)$ for all $a \in \mathbb{N}$.

Despite not being smooth as a function, $B_1(z)$ can be considered as a 0-current on the circle, in the sense that it yields a well-defined functional

$$\alpha \mapsto \int_{S^1} B_1(z)\alpha$$

on smooth 1-forms α on S^1. Considered as a current, we have that

$$dB_1 = dz - \delta_0,$$

which can be verified by their Fourier series as in Example 2.6, recalling that δ_0 is the current of integration at the identity section.

We want to work with not the Bernoulli polynomial itself, but instead stabilized versions: associated to a degree-zero torsion cycle

$$\mathcal{C} = \sum_{x \in S^1[c]} a_x x \in \mathbb{Z}\{S^1[c]\}^{\deg=0},$$

we define

$$B_1[\mathcal{C}](z) := \sum_{x \in S^1[c]} -a_x B_1(z - x).$$

Considered as a function on the complement of its discontinuity set, $B_1[\mathcal{C}]$ is now locally constant, and it is easy to see that its values are in $\frac{1}{c}\mathbb{Z}$. Considered as a current we find that

$$dB_1[\mathcal{C}] = \delta_{\mathcal{C}} = \sum_{x \in S^1[c]} a_x \delta_x.$$

Further, we have the following distribution property:

Lemma 3.1. *For all $a \in \mathbb{Z}^{(c)}$, we have the distribution property*

$$[a]_* B_1[\mathcal{C}] = B_1[[a]_* \mathcal{C}]$$

where $[a]_*$ is the pushforward of 0-currents. The same holds for the pushforward of functions for $a > 0$, and with reversed signs for $a < 0$.

Proof. It suffices to prove the statement for $a \in \mathbb{N}^{(c)}$ (where we can treat it as the pushforward of functions) and for $a = -1$ separately.

For $a > 0$, we have

$$([a]_* B_1[\mathcal{C}])(z) = \sum_{y \in S^1[a]} \sum_{x \in S^1[c]} -a_x B_1\left(\frac{z}{a} + y - x\right) \tag{11}$$
$$= \sum_{y \in S^1[a]} \sum_{x \in S^1[c]} -a_x B_1\left(\frac{z - ax}{a} + y\right) \tag{12}$$
$$= \sum_{x \in S^1[c]} -a_x([a]_* B_1)(z - ax) \tag{13}$$
$$= \sum_{x \in S^1[c]} -a_x B_1(z - ax) \tag{14}$$
$$= B_1[[a]_* \mathcal{C}]. \tag{15}$$

For $a = -1$, we have for any smooth test 1-form $\eta = f(z)\,dz$ that

$$([-1]_* B_1[\mathcal{C}])(\eta) = B_1[\mathcal{C}](-f(-z)\,dz) \tag{16}$$
$$= -\int_{S^1} \sum_{x \in S^1[c]} -a_x f(-z) B_1(z - x)\,dz \tag{17}$$
$$= -\int_{S^1} f(z) \sum_{x \in S^1[c]} -a_x B_1(-z - x)\,dz \tag{18}$$
$$= \int_{S^1} f(z) \sum_{x \in S^1[c]} -a_x B_1(z + x)\,dz \tag{19}$$
$$= (B_1[[-1]_* \mathcal{C}])(\eta) \tag{20}$$

where the second-to-last step follows from $\{y\} = 1 - \{-y\}$ for all $y \in \mathbb{R}$, together with the constant terms 1 cancelling out since $\sum a_x = 0$.

□

We now pass from S^1 back to the n-torus T, where we need to consider more general combinations of Bernoulli currents. For a concrete example, given an integral linear form $L(z_1, \ldots, z_n)$, we have on T a function-qua-current $B_1[\mathcal{C}](L(z_1, \ldots, z_n))$ with

$$dB_1[\mathcal{C}](L(z_1, \ldots, z_n)) = \delta_{L^{-1}\mathcal{C}},$$

where L is interpreted as a map $T \to S^1$ (i.e. a map $\mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}/\mathbb{Z}$) and $\mathcal{C}$ is a degree-zero c-torsion cycle on S^1. The opposite functoriality will be more relevant for our purposes: if $M : T \to T$ is an invertible (over $\mathbb{Q}$) linear transformation, then we can also take the pushforward

$$M_* B_1[\mathcal{C}_1](z_1) B_1[\mathcal{C}_2](z_2) \ldots B_1[\mathcal{C}_n](z_n),$$

which will have differential

$$M_* \sum_{i=1}^{n} (-1)^i B_1[\mathcal{C}_1](z_1) B_1[\mathcal{C}_2](z_2) \ldots \widehat{B_1[\mathcal{C}_n](z_n)} \ldots B_1[\mathcal{C}_n](z_n)((-1)^i \delta_{z_i = \mathcal{C}_i} + \deg \mathcal{C}_i\, dz_i) \tag{21}$$

where

$$\mathcal{C} = \mathcal{C}_1 \boxtimes \ldots \boxtimes \mathcal{C}_n \in \mathbb{Z}[T[c]] \cong \mathbb{Z}[S^1[c]] \otimes \ldots \otimes \mathbb{Z}[S^1[c]]$$

and we use $\delta_{z_i=\mathcal{C}_i}$ to mean the weighted sum of the currents over the values in the cycle $\mathcal{C}_i$, with weights the associated coefficients.

The currents we will consider for the lifts in (10) will be exactly pushforwards by products of B_1's as above, though we will also consider pushforwards from subtori (whose associated currents will then be supported on that subtorus). The derivative of such a current will be another current of this form, as these derivatives yield currents of integration along subgroup tori translated by c-torsion points, which is also the result of pushforwards from subtori.

In general, we will think of these objects as currents, rather than functions. Note that because the discontinuity locus is of measure zero, the values of these functions on this locus cannot *a priori* be recovered from the corresponding currents. However, it will be important later that when we restrict to a certain subspace of combinations of these currents, these "bad values" on the discontinuity locus will turn out to be recoverable; see Lemma 5.3.

We end the section by extending our trace-equivariance result to the more general products of Bernoulli functions:

Proposition 3.2. *For any* $a \in \mathbb{Z}\backslash\{0\}$,

$$[a]_* B_1[\mathcal{C}_1](z_1)\dots B_1[\mathcal{C}_n](z_n) = B_1[[a]_*\mathcal{C}_1](z_1)\dots B_1[[a]_*\mathcal{C}_n](z_n) \in \mathcal{D}_T^0$$

as currents, where $\mathcal{C}_i$, $1 \le i \le n$, *are degree-zero torsion cycles on* S^1. *As functions, this remains true except when* n *is odd and* $a < 0$, *in which case this is wrong by a sign.*

Proof. This is basically is an immediate consequence of the 1-dimensional case, since we have that

$$[a]_* B_1[\mathcal{C}_1](z_1)\dots B_1[\mathcal{C}_n](z_n) = \sum_{a\mathbf{z}'_\bullet=\mathbf{z}} B_1[\mathcal{C}_1](z'_1)\dots B_1[\mathcal{C}_n](z'_n) \tag{22}$$

$$= \prod_{i=1}^{n} \sum_{az'_i=z_i} B_1[\mathcal{C}_i](z'_i) \tag{23}$$

$$= \prod_{i=1}^{n} B_1[[a]_*\mathcal{C}_1](z_i). \tag{24}$$

□

Notice that for any $k, j \in \mathbb{N}$ and "linear" map $L : (S^1)^k \to (S^1)^j$, i.e. a map induced by a linear map $\mathbb{Z}^k \to \mathbb{Z}^j$, $[a]$ commutes with L. Hence similar equivariance results also hold for the pushforward by L of any current of the form $B_1[\mathcal{C}_1](z_1)\dots B_1[\mathcal{C}_n](z_n)$.

3.2. **Matroid symbol complexes.** To understand these elements, we will parameterize them with with a *complex of symbols* defined using linear-algebraic data.

In this subsection we construct this complex; in the next, we will define the $\mathrm{GL}_n(\mathbb{Z}_{(c)})$-equivariant realization map to (10) valued in products of stabilized B_1's and currents of integration as mentioned above.

The idea for us is that any product of Bernoulli polynomials like $B_1(z_1)B_1(z_2)$ (or one of its $\mathrm{GL}_2(\mathbb{Q})$-translates) can be associated to a decomposition of T into S^1-"lines". The non-smooth loci of the B_1s are exactly on codimension-1 embedded subtori (e.g. in this case

$z_1 = 0$ and $z_2 = 0$), which become currents of integration upon differentiating. Thus, the exterior derivative of such a product essentially comes from an alternating sum of restrictions to each of these "hyperplanes" in turn, i.e, an alternating sum of forgetting each of the lines in turn. This suggests that a combinatorially-defined complex capturing the notion of "k-sets of independent lines in T" with its $(k-1)$-simplices could be used to parameterize our products of Bernoulli polynomials.

Unfortunately, the above story is not exactly true, as there are additional terms (coming from forms like dz_1 and dz_2) in addition to the currents of integration along "hyperplane" subtori, because the Bernoulli polynomials are not locally constant. However, the stabilized versions $B_1[\mathcal{C}]$ are constant outside of their discontinuity loci, so the dz_i terms cancel in their (current-wise) derivatives and leave behind only the currents of integration. However, these currents of integration are along not just embedded subtori through the identity, but also their c-torsion translates.

With this rough framework in mind, we turn to the definition of the complexes necessary to capture the *c-stabilized* products of Bernoulli polynomials: recall that a *matroid* is a combinatorial structure abstracting the notion of linear independence:

Definition 3.3. A *matroid* is a set M and a collection $E \subset \mathcal{P}(M)$ of finite subsets of M (i.e. elements of the power set $\mathcal{P}(M)$), called independent sets, satisfying the properties:

- There is at least one independent set.
- If $s \in E$ is independent, every subset of s is as well.
- (*augmentation property*) If s and t are in E and $|t| > |s|$, then there exists an element $m \in t$ such that $s \cup \{m\} \in E$.

The *independence complex* $IC(M)$ of a matroid (M, E) is the simplicial complex with vertex set M such that there is a unique simplex with vertex set s for each $s \in E$. A matroid with maximal independent set of size r has independence complex homotopy equivalent to a wedge of $(r-1)$-spheres [Bjo90]; r is called the *rank* of the matrix. As one sees here, we often leave the collection of independent sets implicit in the notation, since we will rarely consider different matroids on the same base set.

Write $\check{V}_{\mathbb{Z}/c}$ for the Pontryagin dual of $V_{\mathbb{Z}/c}$. It can be identified $\mathrm{GL}_n(\mathbb{Z})$-equivariantly (for the usual right pullback action on both) with the $\mathbb{Z}/c$-linear dual $V^\vee_{\mathbb{Z}/c}$ via the map

$$V^\vee_{\mathbb{Z}/c} \to \check{V}_{\mathbb{Z}/c}, \varphi \mapsto (v \mapsto \exp(2\pi i \varphi(v)/c))$$

We will henceforth implicitly make this identification.

For each $\chi \in \check{V}_{\mathbb{Z}/c} \setminus \{1\}$, we define $(M[\chi], E[\chi])$ to be the matroid of lines in $V_{\mathbb{Z}/c}$ (with the obvious independence condition) such that the corresponding hyperplanes in $V^\vee_{\mathbb{Z}/c}$ do not pass through χ. We will informally say that these lines "avoid" χ for brevity, even though it is technically their corresponding dual hyperplanes which literally avoid the point in the dual space corresponding to χ.

Let $C_\bullet(M[\chi])$ be the reduced simplicial homology complex of $IC(M[\chi])$; by the previously-cited theorem of Bjorn, it is acyclic outisde degree $n-1$.

We define also the free complexes F_V and $F_{V/c}$ to be the reduced homology complexes of the full combinatorial simplicial complexes on the set of lines in V,[3] respectively $V_{\mathbb{Z}/c}$. These are exact by the principle of inclusion-exclusion (or because combinatorial simplicial complexes are contractible).

Define now a complex ${}_c\widetilde{\mathbf{Ber}}(n)_\bullet$ via the $\mathrm{GL}_n(\mathbb{Z}_{(c)})$-equivariant pullback diagram

$$\begin{array}{ccc} {}_c\widetilde{\mathbf{Ber}}(n)[-1]_\bullet & \longrightarrow & \bigoplus_\chi C_\bullet(M[\chi]) \\ \downarrow & & \downarrow \\ \bigoplus_\chi (F_V)_\bullet & \longrightarrow & \bigoplus_\chi (F_{V/c})_\bullet \end{array} \tag{25}$$

Here, the direct sums are over all $\chi \in \check{V}_{\mathbb{Z}/c}\setminus\{1\}$, and the group action simultaneously permutes the χ, via the usual left action $\chi \mapsto \chi\circ g^{-1}$, and acts in the usual way on sets of lines in V. The right vertical and bottom horizontal arrows are the obvious inclusions given by "forgetting" the conditions on the lines with respect to linear independence and avoiding χ, respectively by reduction modulo c.

Proposition 3.4. *The homology of ${}_c\widetilde{\mathbf{Ber}}(n)[-1]$ is concentrated in degree $n-1$; i.e., the homology of ${}_c\widetilde{\mathbf{Ber}}(n)$ is concentrated in degree n.*

Proof. The pullback diagram defining ${}_c\widetilde{\mathbf{Ber}}(n)_\bullet$ leads to a long exact sequence in reduced homology

$$\ldots \to \tilde{H}_i({}_c\widetilde{\mathbf{Ber}}(n)[-1]_\bullet) \to \tilde{H}_i(\bigoplus_\chi C_\bullet(M[\chi])) \oplus \tilde{H}_i(\bigoplus_\chi (F_V)_\bullet) \to \tilde{H}_i(\bigoplus_\chi (F_{V/c})_\bullet) \to \ldots$$

from which the result is immediate from the acyclicity of the other three complexes, outside degree $n-1$ in the case of $\bigoplus C_\bullet(M[\chi])$. □

Our shift was chosen so that the lowest degree is ${}_c\widetilde{\mathbf{Ber}}(n)_0$, given by the free module on $\check{V}_{\mathbb{Z}/c}\setminus\{1\}$. For degrees $d>0$, ${}_c\widetilde{\mathbf{Ber}}(n)_d$ is generated by one copy, for each $\chi \in \check{V}_{\mathbb{Z}/c}\setminus\{1\}$, of the free module on *c-unimodular* sets of lines in V of size d avoiding χ, i.e. sets of lines which remain independent after reduction modulo c.

3.3. **The realization map.** We now define a $\mathrm{GL}_n(\mathbb{Z}_{(c)})$-equivariant realization map, on the "alternate integral structure" we just constructed.

Theorem 3.5. *There exists a $\mathrm{GL}_n(\mathbb{Z}_{(c)})$-equivariant morphism of complexes*

$$\widetilde{\Psi} : {}_c\widetilde{\mathbf{Ber}}(n)_\bullet \to \mathcal{D}_T^{n-\bullet}$$

defined by sending

$$[e_1,\ldots,e_k]_\chi \mapsto (-1)^k \sum_{z_{k+1},\ldots,z_n\in\mathbb{Z}/c} \chi_{k+1}(z_{k+1})\ldots\chi_n(z_n)B_1[\chi_1](z_1)\ldots B_k[\chi_k](z_k)\delta_{z_{k+1}=\ldots=z_n=0}$$

for the standard basis $(e_1,\ldots,e_n)$ and a character

$$\check{V}_{\mathbb{Z}/c} \ni \chi = \chi_1 \boxtimes \ldots \boxtimes \chi_n,$$

[3]That is, the simplicial complex on the prescribed set of vertices such that *every* $(k+1)$-set of vertices forms a unique simplex.

and extended by $\mathrm{GL}_n(\mathbb{Q})$*-equivariance.*

Proof. First note that the definition is independent of the choices of vectors representing the lines ℓ_i, because $B_1(z)$ is invariant by scalar pushforwards prime to c and thus the constituent terms $B_1(z_1)B_1(z_2)\dots B_1(z_k)$ is invariant by diagonal matrices with entries prime to c.

The well-definedness and $\mathrm{GL}_n(\mathbb{Z}_{(c)})$-equivariance of $\widetilde{\Psi}$ is then formal; and commutation with the differential follows by the earlier computation (21). □

We now discuss the issue of trace-invariance. If we consider the multiplicative monoid of prime-to-c integers $(\mathbb{Z}-\{0\},\times)^{(c)}$ to act on both source and target of $\widetilde{\Psi}$ via the corresponding diagonal matrix, we have seen that $\widetilde{\Psi}$ is equivariant for this action. By Proposition 3.2 and the following discussion, it is then immediate that the target of $\widetilde{\Psi}$ is contained in the submodules $(\mathcal{D}_T^{\bullet})^{(f)}$, where

$$(\mathcal{D}_T^{\bullet})^{(0)} \subset (\mathcal{D}_T^{\bullet})^{(f)} \subset \mathcal{D}_T^{\bullet}$$

is defined as the subrepresentation on which $(\mathbb{Z}-\{0\},\times)^{(c)}$ acts via the quotient $(\mathbb{Z}-\{0\},\times)^{(c)} \to (\mathbb{Z}/c)^{\times}$.

The proof of Proposition 2.4 applies with little modification to the trace-finite setting. As such, we obtain the following modification of the realization map on the exact quotient of our symbol complex:

Proposition 3.6. *The map* $\tilde{\Psi}$ *factors through* ${}_c\mathbf{Ber}(n)$*. We call this map*

$$\Psi : {}_c\mathbf{Ber}(n)_{\bullet} \to (\mathcal{D}_T^{n-\bullet})^{(f)}.$$

Proof. This is immediate from Proposition 2.4 and Proposition 3.2. □

Remark 3.7. The realization map Ψ actually also factors through a corresponding *Orlik-Solomon complex* defined from our matroids (see [OT92]), which can be proven using the exactness of (10) in a similar way, though with some additional technicalities. This symbol complex captures more of the relations between products of Bernoulli polynomials (in fact, *all* such relations), but we saw no benefit to proving this for the purposes of the present article. One can observe that these relations are c-stabilized version of the classical "reciprocity laws" for higher Dedekind(-Rademacher) sums.

3.4. **Integral cocycles representing the Eisenstein class.** We now come to the main construction: let $\Gamma \leq \mathrm{GL}_n(\mathbb{Z}_{(c)})$ be a subgroup, and suppose we have a Γ-fixed c-torsion cycle

$$\mathcal{C} \in {}_c\mathbf{Ber}(n)_0 = \mathbb{Z}[\zeta_c]\{T[c]\}^{\deg=0}.$$

In particular, we will take

$$\mathcal{C} = T[c] - c^n\{0\} = -\sum \chi,$$

where the sum is over all nontrivial characters χ of $\check{V}_{\mathbb{Z}/c}$. This cycle is valid for any subgroup of $\mathrm{GL}_n^+(\mathbb{Z}_{(c)})$; however, all of the following would work for more general torsion cycles.

Then Lemma 2.9 affords us a cocycle

$$\theta(n)_{\mathcal{C}} \in C^{n-1}(\Gamma, {}_c\mathbf{Ber}(n)_n)$$

which can be pushed forward via Ψ to a cocycle

$$\Psi_*\theta(n)_{\mathcal{C}} \in C^{n-1}(\Gamma, (\mathcal{D}_T^0)^{(f)})$$

valued in products of c-stabilized weight-1 Bernoulli polynomials; these will be locally constant functions valued in $\mathbb{Z}[\zeta_c]$ outside of specified hyperplanes (corresponding to the discontinuities of the Bernoulli polynomials). To be more specific, the function-qua-0-distribution

$$L_* B_1[\chi_1](z_1) \dots B_n[\chi_n](z_n)$$

on T is locally constant outside of the "hyperplane" (codimension-1) locus

$$H_L := L_* \left(\bigcup_a \left[\bigcup_{i=1}^n \{z_i = a/c\} \right] \right),$$

and its values on each connected component is an element of $\mathbb{Z}[\zeta_c]$.

We can therefore interpret the values of $\Psi_*\theta_{\mathcal{C}}$ as belonging to

$$H^0(T - H, \frac{1}{c^n}\mathbb{Z}[\zeta_c]) := \varinjlim_{S \subset H} H^0(T - S, \frac{1}{c^n}\mathbb{Z}[\zeta_c])$$

where

$$H := \bigcup_L H_L$$

is the union of all the hyperplanes as L varies over $\mathrm{GL}_n(\mathbb{Z}_{(c)})$, and S varies over finite subarrangements of these hyperplanes. (Here, we use the definition of equivariant cohomology of a pro-system of open immersions we defined previously.)

This perspective leads to the following important comparison result, justifying the relation of all our symbol constructions to Eisenstein classes:

Theorem 3.8. *If Γ is a subgroup of $\mathrm{GL}_n(\mathbb{Z})$, then the cocycle $\Psi_*\theta(n)_{\mathcal{C}}$, interpreted as being a cocycle valued in $H^0(T - H, \mathbb{Z}[\zeta_c])$, is a c-integral representative for the image of the class ${}_c z_\Gamma$ under the edge map*

$$H^{n-1}_\Gamma(T - H, \mathbb{R}) \to H^{n-1}(\Gamma, H^0(T - H, \mathbb{R})).$$

Proof. As in the discussion following Lemma 2.9, take $\ell_1, \dots, \ell_n$ to be the lifts obtained by applying that lemma to the complex ${}_c\widetilde{\mathbf{Ber}}(n)$. Then as in Proposition 2.10, the sum of lifts $\Psi(\ell_1) + \dots + \Psi(\ell_n)$ represents the class $z_\Gamma^{\mathcal{C}}$. Further, each $\Psi(\ell_i)$ is by construction de Rham closed upon restriction to $T - H$ since its locus of non-constancy is only along the hyperplanes H, so the claimed result then follows from Lemma 2.11. □

Example 3.9. Let us spell out how the lifting process yields a cocycle in the simplest interesting case, when $n = 2$ (so $T = (S^1)^2$) and $c = 2$.

We will write (a, b) for $a, b \in \mathbb{Z}/2$ for points in $T[2]$, $\chi_{x,y}$ for the character in $\widehat{T[2]}$ sending $(a, b) \mapsto \exp(\pi i(ax + by))$ for $x, y \in \mathbb{Z}/2$, and $[m : n]$ for the corresponding line in $\mathbb{P}^1(\mathbb{Q})$. We will denote generators of ${}_2\mathbf{Ber}(2)_i$ by

$$(\ell_1, \ell_2; P)$$

where $\ell_1, \ell_2 \in \mathbb{P}^1(\mathbb{Q})$ and $P \in \widehat{T[2]}$ (subject to some avoidance condition).

In this case, the $\mathrm{GL}_2(\mathbb{Z}_{(2)})$-fixed 2-torsion cycle $T[2] - 4\{0\}$ can be written as the sum of characters

$$-\chi_{1,0} - \chi_{0,1} - \chi_{1,1} \in {}_2\mathbf{Ber}(2)_0.$$

There are many possible choices of lift to ${}_2\mathbf{Ber}(2)_1$, but a nice one might be

$$\eta := -([1:0], \chi_{1,0}) - ([1:1], \chi_{0,1}) - ([0:1], \chi_{1,1}) \in {}_2\mathbf{Ber}(2)_0.$$

Notice in particular that, for example, the hyperplane in $\widehat{T[2]} \cong (T[2])^\vee$ corresponding to the line $[1:0]$ is $\mathrm{span}(\chi_{0,1})$, which does not contain the point $\chi_{1,0}$, and similarly for the other pairs.

Then if

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_{(2)})$$

is a matrix, we need then to find the lift to ${}_2\mathbf{Ber}(2)_2$ of

$$\begin{aligned} (\gamma - 1)\eta = ([1:0]; \chi_{1,0}) + ([1:1]; \chi_{0,1}) + ([0:1]; \chi_{1,1}) \\ -([a:c]; \chi_{a,b}) - ([a+b:c+d]; \chi_{c,d}) - ([b:d]; \chi_{a+c,b+d}) \end{aligned} \tag{26}$$

Note by invertibility modulo 2 of γ that the triple $(a,b), (c,d), (a+c, b+d)$ must be a permutation of $(1,0), (0,1), (1,1)$. We can thus pair the lines which have the same corresponding character. If each of these pairs of lines are 2-unimodular, then the lift is easy to find: for example, for

$$\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we get

$$\begin{aligned} (\gamma - 1)\eta = ([1:0]; \chi_{1,0}) + ([1:1]; \chi_{0,1}) + ([0:1]; \chi_{1,1}) \\ -([0:1]; \chi_{0,1}) - ([1:-1]; \chi_{1,0}) - ([1:0]; \chi_{1,1}) \end{aligned} \tag{27}$$

which has lift

$$([1:-1], [1:0]; \chi_{1,0}) + ([0:1], [1:1]; \chi_{0,1}) + ([1:0], [0:1]; \chi_{1,1}).$$

The realization of this cocycle as a locally constant function is then

$$B_1[\chi](z_1) B_1[\chi](z_2) + \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}_* B_1[\chi](z_1) B_1[1](z_2) + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}_* B_1[\chi](z_1) B_1[1](z_2)$$

where here $B_1[\chi](z)$ means the stabilization $B_1(z - 1/2) - B_1(z)$ corresponding to the 2-torsion cycle $(0) - (1/2)$ and $B_1[1](z) = -B_1(z) - B_1(z - 1/2)$ is the stabilization corresponding to $(0) + (1/2)$.

In the case when the pairs of lines resulting from the γ-action are not 2-unimodular, things become more complicated; one must "connect" the resulting pairs of lines by intermediary lines such that adjacent pairs are 2-unimodular, similar to the "connecting sequences" of [SV24] (though with somewhat more complicated conditions than are considered there). This is always possible by a geometry of numbers argument similar to that given in [AR79], but we do not go into the details here, since we have no need to write out general explicit formulas in the present article. Notice that our formalism hides all of this behind the exactness of ${}_2\mathbf{Ber}(2)$, proven by general topological considerations about matroids.

To conclude this section, we will also need a variant of all these constructions "with level structure," for technical reasons relating to the cohomological interpretation of pullbacks. As such, we must reiterate everything preceding, with slight modifications.

Fix any prime p; eventually, we will take $(c,p)=1$, but this is not actually necessary for what follows. Now suppose

$$\Gamma \subset \Gamma_0(p) := \mathrm{Stab}([0:\ldots:0:1] \in \mathbb{P}(V_{\mathbb{Z}/p})) \subset \mathrm{GL}_n(\mathbb{Z}).$$

Then we have a Γ-equivariant pullback diagram of complexes, analogous to and building upon (25):

$$\begin{array}{ccc} {}_c\widetilde{\mathbf{Ber}}^{(p)}(n)[-1]_\bullet & \longrightarrow & {}_c\widetilde{\mathbf{Ber}}(n)[-1]_\bullet \\ \downarrow & & \downarrow \\ \bigoplus_\chi C_\bullet(M^{(K)}) & \longrightarrow & \bigoplus_\chi (F_{V/p})_\bullet \end{array} \tag{28}$$

where $M^{(K)}$ denotes the matroid of sets of independent sets of lines in $V_{\mathbb{Z}/p}$ which $(n-1)$-*avoid the line* $K=[0:\ldots:0:1] \in V_{\mathbb{Z}/p}$: i.e. consisting of sets of lines which have no cardinality $\leq n-1$ subset such that K is in their span.

It is not completely obvious that this forms a matroid:

Lemma 3.10. *The subsets of lines in $V_{\mathbb{Z}/p}$ described above as $M^{(K)}$ do, in fact, satisfy the matroid conditions.*

Proof. The non-obvious condition is the augmentation property: given two sets with $|A|>|B|$ of independent lines which $(n-1)$-avoid K, is there $a \in A$ such that $\{a\}\cup B$ also $(n-1)$-avoids K?

Write $\tilde{A}, \tilde{B}$ for the sets of images of the corresponding lines in $V_{\mathbb{Z}/p}/K$. Then we see that our desired property is implied by the following assertion: if we have two sets of lines in $V_{\mathbb{Z}/p}/K$ with $|\tilde{A}|>|\tilde{B}|$ such that both are $(n-1)$-independent (i.e. every $(n-1)$-set is independent), then there exists $\tilde{a} \in \tilde{A}$ such that $\{\tilde{a}\} \cup \tilde{B}$ is also $(n-1)$-independent.

If $|\tilde{A}|\leq n-1$, this is clear: it is just the usual matroid property for all lines in $V_{\mathbb{Z}/p}/K$. Otherwise, we reduce to the case $|\tilde{A}|=n$. In this case, we might as well assume $|\tilde{B}|=n-1$, since this clearly also implies the property for smaller sets $\tilde{B}$. Then in *some* coordinates on $V_{\mathbb{Z}/p}/K$, $\tilde{B}$ is given by the standard coordinate vectors $\{(1,0\ldots,),\ldots,(0,\ldots,0,1)\}$. Our claim then amounts to showing that there exists some vector in $\tilde{A}$ with no coordinate equal to zero.

Suppose for the sake of contradiction that each of the n vectors in $\tilde{A}$ had one of their $(n-1)$ coordinates equal to zero. By the pigeonhole principle, this implies there are two vectors with zero in the same coordinate; any $(n-1)$-subset containing these two then fails to be independent, which is a contradiction. □

Corollary 3.11. *The complex ${}_c\widetilde{\mathbf{Ber}}^{(p)}(n)$ has reduced homology concentrated in top degree n.*

Proof. As with the diagram (25), this follows immediately from the acyclicity of $\bigoplus_\chi (F_{V/p})_\bullet$ and the fact that the other two complexes have reduced homology concentrated in degree $n-1$ (which is degree n after the shift). $\square$

Then the restriction of Ψ to ${}_c\mathbf{Ber}^{(p)}(n)_n$ consists of elements of the form

$$L_* B_1[\chi_1](z_1) B_1[\chi_2](z_2) \dots B_1[\chi_n](z_n)$$

where L is a matrix whose $(n-1)$-subsets of columns never span K modulo p. As such, its differential is supported on hypersurfaces inside T not containing the (nonzero points of) line of p-torsion corresponding to K, implying that:

Proposition 3.12. *If H_p denotes the $\Gamma_0(p)$-orbit of the hyperplane locus*

$$\bigcup_P t_P \left[\bigcup_{i=1}^n \{z_i = 0\} \right]$$

where P ranges over $T[c]$ (and t_P is translation by P, as previously), then $\Psi({}_c\mathbf{Ber}^{(p)}(n)_n)$ is valued in locally constant functions on $T - H_p$, hence representing classes in

$$H^0(T - H_p, \frac{1}{c^n}\mathbb{Z}[\zeta_c]) := \varinjlim_{S \subset H_p} H^0(T - S, \frac{1}{c^n}\mathbb{Z}[\zeta_c])$$

where as before the limit is over finite subarrangements contained in H_p. In particular, if $(p, c) = 1$, these locally constant functions(-qua-0-distributions) have the property that the nonzero points of the line $K \subset T[p]$ are in the open locus of local constancy.

Proof. By the preceding discussion, the differential of these elements is a sum of currents supported on the hyperplane locus written above. Thus, if one restricts away from this locus, we obtain a closed 0-current, which is isomorphic to the space of locally constant functions by the quasi-isomorphism between the smooth and distributional de Rham complexes. $\square$

Then as with the original construction, we obtain a cocycle

$$\theta(n)_{\mathcal{C}}^{(p)} \in C^{n-1}(\Gamma_0(p), {}_c\mathbf{Ber}^{(p)}(n)_n)$$

which can be pushed forward via Ψ to a cocycle

$$\Psi_* \theta(n)_{\mathcal{C}}^{(p)} \in C^{n-1}(\Gamma, (\mathcal{D}_T^0)^{(f)})$$

representing the image of $z_\Gamma^{(c)}$ in

$$H^{n-1}(\Gamma_0(p), H^0(T - H_p, \mathbb{R})).$$

Note that the restriction of

$$\Psi_* \theta(n)_{\mathcal{C}} \in H^{n-1}(\Gamma, H^0(T - H, \frac{1}{c^n}\mathbb{Z}[\zeta_c]))$$

to $\Gamma_0(p)$ can be identified with

$$\Psi_* \theta^{(p)}(n)_{\mathcal{C}} \in H^{n-1}(\Gamma_0(p), H^0(T - H_p, \frac{1}{c^n}\mathbb{Z}[\zeta_c])).$$

Formally, the restriction of this class to the sublocus $T - H \subset T - H_p$ recovers $\Psi_* \theta(n)_{\mathcal{C}}$.

3.5. **Integral cocycles valued in distributions.** We now wish to obtain an Eisenstein class valued in distributions over all p-power torsion. (In fact, we could consider all prime-to-c torsion, but for our purposes, one prime at a time suffices.)

First, in order for the distribution relations to hold, we need the target of Ψ to be actually fixed under the trace by the isogeny $[p]$, not just have finite orbit under it. Indeed, if o_p is the order of $[p] \in (\mathbb{Z}/c)^\times$, then there exists a projector $e_{[p]} \in \frac{1}{o_p}\mathbb{Z}[(\mathbb{Z}/c)^\times]$ such that the image of $e_{[p]}\Psi$ is trace-fixed. (In particular, when $p \equiv 1 \pmod{c}$, we can take $o_p = 1$ and the image of Ψ is already always $[p]_*$-fixed. Thus for odd p, we may simply always take $c = 2 \Rightarrow o_p = 1$.)

Let $\mathbb{D}(V_{\mathbb{Q}_p}, A)$ denote the distributions (i.e. the linear dual of locally constant functions) on $V_{\mathbb{Q}_p}$ valued in an abelian group A, with an action of $\mathrm{GL}_n(\mathbb{Q}_p)$ given by pushforward. We also write $\mathbb{D}(V_{\mathbb{Q}_p}, A)^{(0)}$ for the submodule of distributions μ which are $[p]$-invariant, i.e. distributions μ so that $\mu(U) = \mu(pU)$ for any open set $U \subset V_{\mathbb{Q}_p}$. We view the previously-defined $\mathbb{X}$ as embedded in $V_{\mathbb{Q}_p}$ as the open set

$$\mathbb{X} := \bigcup_{\substack{\mathbf{v} \in \frac{1}{p}V_{\mathbb{Z}_p}/V_{\mathbb{Z}_p} \\ \mathbf{v} \neq 0}} U_{\mathbf{v}},$$

and we write $\mathbb{D}_0(V_{\mathbb{Q}_p}, A)$ for distributions on which $\mu(\mathbb{X}) = 0$, where here $U_{\mathbf{v}} := \mathbf{v} + \mathbb{Z}_p^n$.

Let ${}_c\mathcal{B}_p(n) := e_{[p]}\Psi({}_c\mathbf{Ber}(n)_n) \subset \mathcal{D}_T^0$, and analogously for the P-avoiding submodule ${}_c\mathcal{B}_p^{(p)}(n) := e_{[p]}\Psi({}_c\mathbf{Ber}^{(p)}(n)_n)$.

Write ${}_cB_p(n)$ for the $[p]$-invariant submodule of the *functions* on T generated by *functions* of the form

$$L_*B_1[\chi_1](z_1)\dots B_n[\chi_n](z_n)$$

for a c-unimodular matrix L. These are functions continuous on an open locus which is the complement of codimension-1 subtori.

Lemma 3.13. *Considering the functions in ${}_cB_p(n)$ as 0-currents by viewing them as kernels of integration yields a $\mathrm{GL}_n(\mathbb{Z}[1/p])$-equivariant* isomorphism

$$\nu_B : {}_cB_p(n) \to {}_c\mathcal{B}_p(n)(\det).$$

Proof. The injectivity is the only novel statement here. To obtain it, we define an explicit inverse. The key idea is that given a current $\omega \in {}_c\mathcal{B}_p(n)$, it has a corresponding Fourier series by Proposition 2.5, which as in that proposition we encode as a tempered function

$$\varphi := \mathcal{F}_T(\omega) \in \mathcal{S}'(\mathbb{Z}^n).$$

Viewed as a literal series

$$\sum_{\lambda \in \mathbb{Z}^n} \varphi(\lambda) \exp(2\pi i \lambda \cdot z)) \tag{29}$$

with input variable $z = (z_1, \dots, z_n) \in T$, φ will rarely be absolutely convergent; we thus cannot in general assign an unambiguous value to its evaluation at some given z. It is therefore not obvious how to obtain a function on T.

The workaround is the "Q-summation" method of [Scz93]: one can try to sum series of the form (29) by ordering the terms by increasing value of $|Q(z_1, \dots, z_n)|$, where Q is a nonzero product of linear forms in z with rationally independent coefficients.[4] One denotes the Q-sum by

$$\sum_{\lambda\in\mathbb{Z}^n}\varphi(\lambda)\exp(2\pi i\lambda\cdot z))|_Q. \tag{30}$$

We will also write simply $\varphi|_Q$ for short. If this Q-sum converges, then it is clear that for any $\gamma\in\mathrm{SL}_n(\mathbb{Z})$ we also have

$$\gamma_*(\varphi|_Q)=(\gamma^*\varphi)|_{\gamma\cdot Q} \tag{31}$$

for the action on Q given in loc. cit. (specifics of this action are immaterial to us). Note also that Q-summation is additive (on series and Q for which the corresponding Q series converge), by elementary properties of limits.

From [Scz93, Theorem 2], we have that for the Fourier series φ_{st} corresponding to the *current*

$$B_1(z_1)\dots B_1(z_n),$$

the sum $(\varphi_{st}|_Q)(z)$ is actually *independent* of Q for z in the continuity locus (i.e. with no coordinate zero), and always returns the actual value of the *function* $B_1(z_1)\dots B_1(z_n)$ for these z. If this were true also for the discontinuity locus, we could move this argument around by the $\mathrm{SL}_n(\mathbb{Z})$ action to obtain our equivariant inverse map ${}_c\mathcal{B}_p(n)\to{}_cB_p(n)$. However, the Q-summation in general does have an dependence on Q on the discontinuity loci: in our language, [Scz93, Theorem 2] states that if we have our product of linear forms

$$Q(z)=\prod_{i=1}^m \mathbf{L}_i\cdot z \tag{32}$$

with $\mathbf{L}_i=(L_{i1},\dots,L_{in})$, then the Fourier series for $B_1(z_1)\dots B_1(z_n)$, viewed as an element of $\mathcal{S}'(V_{\mathbb{Z}})$ as in the proof of Proposition 2.4,

$$\varphi:(k_1,\dots,k_n)\mapsto\frac{1}{k_1\dots k_n}$$

has Q-sum

$$\varphi|_Q=\frac{1}{2m}\sum_{i=1}^m\left(\prod_{j=1}^n(B_1(z_j)-[\mathrm{sgn}(L_{ij})/2])+\prod_{j=1}^n(B_1(z_j)+[\mathrm{sgn}(L_{ij})/2])\right)$$

where the terms in the square brackets are included if and only if the corresponding component of the input z_j is zero.

However, our module ${}_c\mathcal{B}_p(n)$ only consists of certain *c-stabilized* Bernoulli currents, for which these Q-ambiguities cancel out in the corresponding series. In particular, denote the product of Bernoulli polynomials associated to a symbol $[\ell_1,\dots,\ell_n]_\chi$ by $B_1[\ell_1,\dots,\ell_n;\chi]$, so that the associated current is precisely $\Psi([\ell_1,\dots,\ell_n]_\chi)$.

By the assumptions on χ, we can write $\chi=\chi_1\otimes\dots\otimes\chi_n$ for some nontrivial characters $\chi_i, 1\le i\le n$ of $\mathbb{Z}/c$, meaning

$$\chi(z_1,\dots,z_n)=\chi_1(z_1)\dots\chi_n(z_n)$$

[4]Note that the rational independence is necessary for the associated ordering on lattice points to be unambiguous, i.e. not have any ties.

for any $(z_1, \dots, z_n) \in T[c]$. Then for any Q in the form (32), Sczech's formula says that the corresponding Fourier series has Q-sum

$$\mathcal{F}_T(\Psi([\ell_1, \dots, \ell_n]_\chi))|_Q$$

equal to

$$\frac{1}{2m}\sum_{i=1}^{m}\left(\prod_{j=1}^{n}\sum_{r\in\mathbb{Z}/c}\chi_j(r)(B_1(z_j-r)-[\operatorname{sgn}(L_{ij})/2])+\prod_{j=1}^{n}\sum_{r\in\mathbb{Z}/c}\chi_j(r)(B_1(z_j-r)+[\operatorname{sgn}(L_{ij})/2])\right) \tag{33}$$

Since each character $\chi_j(r)$ is nontrivial, we find that within each innermost sum, the terms in square brackets cancel to zero (when they appear at all). Hence this expression reduces to simply

$$\prod_{j=1}^{n}\sum_{r\in\mathbb{Z}/c}\chi_j(r)B_1(z_j-r) = B_1[e_1, \dots, e_n; \chi]. \tag{34}$$

i.e. the original Bernoulli function associated to the standard basis $e_1, \dots, e_n$ of $V_{\mathbb{Z}}$. In particular, the Q-sum is independent of Q.

By the equivariance property (31), we can move this argument around by the $\mathrm{GL}_n(\mathbb{Z}[1/p])^+$-action (recalling all of our functions/currents are invariant by $[p]_*$) to apply to *any* generator $B_1[\ell_1, \dots, \ell_n; \chi]$: we conclude that all of the corresponding Fourier series Q-converge to limits independent of Q, equal to the values of the original function, regardless of z lying in any discontinuity strata.

Therefore, this "independent of Q"-summation, for any generator $B_1[\ell_1, \dots, \ell_n; \chi]$ of ${}_cB_p(n)$, sends $\nu_B(B_1[\ell_1, \dots, \ell_n; \chi])$ back to the function $B_1[\ell_1, \dots, \ell_n; \chi]$ on all of T. The additivity of Q-summation ensures this inverse is well-defined, since the zero Fourier series obviously maps to the zero function regardless of Q, so we are done. $\square$

Remark 3.14. The preceding injectivity amounts to saying that no nonzero sum of functions in ${}_cB_p(n)$ can be identically zero outside of its (positive codimension) discontinuity locus, since otherwise it would give the zero current. This really does depend on c-stabilization: it is certainly not true for general functions locally constant on a complement of subtori arrangements. It is even false for general sums of products of Bernoulli polynomials: for instance, the function

$$2(B_1(z_1)B_1(z_2)+B_1(z_2)(B_1(-z_1-z_2))+B_1(-z_1-z_2)B_1(z_1))+B_2(z_1)+B_2(z_2)+B_2(-z_2-z_2)$$

is identically zero on $(S^1)^2$ outside of the identity (where it is equal to 1/6), and therefore yields the zero current. This lack of injectivity is the reason Q-summation is necessary in Sczech's method in the first place.

Note that from an identical proof, the same statement applies also for the restricted map

$$B_p^{(p)}(n) \to \mathcal{B}_p^{(p)}(n).$$

From this lemma, we can deduce the following: write $\rho : \mathbb{Z}[\zeta_c] \to \mathbb{Z}$ for any $\mathbb{Z}$-module map which splits the inclusion; we will also use the same notation for $\frac{1}{c^n}\mathbb{Z}[\zeta_c] \to \frac{1}{c^n}\mathbb{Z}$, or any similar obvious variation.[5] Then we have:

Proposition 3.15. *There is a* $\mathrm{GL}_n^+(\mathbb{Z}[1/p])$*-equivariant morphism*

$$\varphi_p : {}_c\mathcal{B}_p(n) \to \mathbb{D}_0(V_{\mathbb{Q}_p}, \frac{1}{o_p c^n}\mathbb{Z})^{(0)}$$

given by

$$\omega \mapsto \big(U_{\mathbf{v}} \mapsto (\rho)_* x(\mathbf{v})^* \nu_B^{-1}\omega\big)$$

where $\mathbf{v} \in V_{\mathbb{Q}_p}$ *is a vector and* $U_{\mathbf{v}}$ *is the open set* $\mathbf{v} + V_{\mathbb{Z}_p}$*, and extended by the* $[p]$*-invariance of the distribution.*[6] *Here,* $\mathbf{v}$ *defines a* p*-power torsion point* $x(\mathbf{v})$ *in* $T[p^\infty] = V_{\mathbb{Q}_p}/V_{\mathbb{Z}_p}$*, and we write* $x(\mathbf{v})^*\nu_B^{-1}\omega$ *for the pullback of the* function $\nu_B^{-1}\omega \in {}_cB_p(n)$.

Proof. The last thing that needs checking is the condition that $\mu(\mathbb{X}) = 0$, which amounts to the assertion that

$$\sum_{x \in T[p]\setminus\{0\}} x^* f = 0$$

for all $f \in {}_cB_p(n)$. Observe that

$$\sum_{x \in T[p]\setminus\{0\}} x^* = 0^* \circ ([p]_* - 1),$$

so this immediately follows from the fact that ${}_cB_p(n)$ consists of $[p]$-fixed functions. □

3.6. **Cohomological comparisons.** Fix now $(p, c) = 1$. We now come to the importance of defining the variant complex $\mathbf{Ber}^{(p)}(n)$ (and the other corresponding constructions), heretofore unused: it enables us to compare our Bernoulli polynomial-valued cocycles with abstract pullbacks in cohomology.

Let $\Gamma_1(p^r)$ be the level structure fixing a torsion point x_r of order p^r, for any $r \geq 0$. Then for the restricted morphism $\varphi_p : \mathcal{B}_p^{(p)}(n) \to \mathbb{D}_0(V_{\mathbb{Q}_p}, \frac{1}{o_p c^n}\mathbb{Z})^{(0)}$, by Proposition 3.12, we have the commutative diagram of specializations

$$\begin{tikzcd} H^0(T - H_p, \frac{1}{o_p c^n}\mathbb{Z}) \arrow[d, "x_r^*"] & \mathcal{B}_p^{(p)}(n) \arrow[l] \arrow[r, "\varphi_p"] \arrow[d, "\mathrm{ev}_{x_r}"] & \mathbb{D}_0(V_{\mathbb{Q}_p}, \frac{1}{o_p c^n}\mathbb{Z})^{(0)} \arrow[d, "1(U_r)"] \\ \frac{1}{o_p c^n}\mathbb{Z} \arrow[r, equal] & \frac{1}{o_p c^n}\mathbb{Z} \arrow[r, equal] & \frac{1}{o_p c^n}\mathbb{Z} \end{tikzcd} \tag{35}$$

equivariant for $\Gamma_1(p^r)$, where here U_r is the open set of $\mathbb{Z}_p^n$ corresponding to $x_r T[p]$.

We hence deduce:

[5] The necessity of this map is an unfortunate technical necessity, which has no real significance: the specialization we will use for interpolation (and indeed, essentially all specializations of interest) will all be integer/rational-valued, so ρ will act trivially on their extension of scalars. Our need for it is simply an artifact of the fact that our c-stabilization is encoded via a Pontryagin-Fourier dual.

[6] Note that though these open sets do not generate a basis on their own, their $[p]$-translates do, so this uniquely specifies the measure of every open set.

Theorem 3.16. *If we restrict our distribution-valued cocycle*

$$\boldsymbol{\mu} := (\varphi_p)_*\Psi_*\theta(n)_{\mathcal{C}} \in H^{n-1}(\mathrm{GL}_n^+(\mathbb{Z}[1/p]), \mathbb{D}_0(V_{\mathbb{Q}_p}, \frac{1}{o_p c^n}\mathbb{Z})^{(0)})$$

to $\Gamma_1(p^r)$ *and evaluate at the open set* U_r*, then* with $\mathbb{R}$-coefficients*, this coincides with the sum of pullbacks of the Eisenstein class*

$$(x_r)^*{}_c z_{\Gamma_1(p^r)} \in H^{n-1}(\Gamma_1(p^r), \mathbb{Z}[1/c]) \otimes \mathbb{R}$$

Proof. This essentially follows formally from the fact that pullback by x_r commutes with restriction and the Hochschild-Serre edge map, since we know from Proposition 2.10, Lemma 2.9, Lemma 2.11 that $\Psi_*\theta(n)_{\mathcal{C}}$ represents the image of ${}_c z_\Gamma$ under said edge map after extending scalars to $\mathbb{R}$, at any level Γ.

The only non-formal thing which needs to be checked is that applying the rationalizing projector ρ does not change the specialization. Indeed, the cohomological construction of the Eisenstein class tells us that in fact

$$x_r^* {}_c z_{\Gamma_1(p^r)} \in H^{n-1}(\Gamma_0(p), \mathbb{R})$$

is the extension of scalars of a class with $\mathbb{Z}[1/c]$ coefficients; thus, it is fixed by ρ. Hence if

$$1(U_r)_*\Psi_*\theta(n)_{\mathcal{C}} \in H^{n-1}(\Gamma_0(p), \frac{1}{o_p c^n}\mathbb{Z}[\zeta_c])$$

agrees with it after extending coefficients to $\mathbb{C}$, then after applying ρ to both integral cocycles and extending coefficients to $\mathbb{R}$, the same continues to hold. $\square$

By the equivariant-geometric comparison from Section 2.4, the classes ${}_c z_{\Gamma_1(p^r)}$ coincide with the geometric classes ${}_c z_r$ of [RX25]. (There, the torsion point we call x_r is notated as a torsion section v_r.) Thus, the above proposition is enough to write finite-level specializations of $\Psi_*\theta(n)_{\mathcal{C}}$ in terms of explicit weight-2 Eisenstein series (cf. [RX25, §2]).

Analogous to [RX25], we also have an independence-of-c result for our distribution-valued cocycles (which we here decorate $\boldsymbol{\mu}$ with, but generally omit from the notation):

Proposition 3.17. *If* c *and* d *are prime to* p *and each other, and the* d*-torsion cycle* $\mathcal{D}$ *is defined analogously to* $\mathcal{C}$*, we have*

$$([c]_*^{-1} - c^n)\boldsymbol{\mu}_{\mathcal{D}} = ([d]_*^{-1} - d^n)\boldsymbol{\mu}_{\mathcal{C}}$$

where $[a]_*$ *means pushforward of distributions along the multiplication-by-*a *map (which is invertible if* $(a,p) = 1$*).*

Proof. By working in the cd-stabilized complex ${}_{cd}\mathbf{Ber}(n)_\bullet$, this follows formally from the fact that

$$([c]_* - c^n)\mathcal{D} = ([d]_* - d^n)\mathcal{C} = ([c]_* - c^n)([d]_* - d^n)\{0\}$$

and the easily checkable $[c]^*$ and $[d]^*$-equivariance of Ψ and φ_p. Note that $[c]^*$ and $[d]^*$ on functions induce $[c]_*^{-1}$ and $[d]_*^{-1}$ on distributions over their torsion specializations. $\square$

Remark 3.18. This section obtains the same finite-level specializations for our cohomology class as the classes μ_0 considered in [RX25]. However, this does not show they are equal as *distribution*-vauled cohomology classes, and in fact we do not do so in this article. For such a comparison, even up to torsion, we would need a model of equivariant cohomology allowing us to identify $\boldsymbol{\mu}$ as the image under edge maps of *integral* versions of ${}_cz_\Gamma$, which our (real-coefficients) distributional de Rham complex does not do. We believe this may be achievable using *locally finite cubical chains* as a model for $\mathbb{Z}$-coefficients equivariant cohomology (or equivariant Borel–More homology), but since we are able to recover all the applications of [RX25] for $\boldsymbol{\mu}$ below without such a comparison, we found that it did not merit the extra technicalities involved.

4. Drinfeld's p-adic symmetric domain and rigid cocycles

Let X_p be Drinfeld's p-adic symmetric domain, namely $X_p = \mathbb{P}^{n-1}(\mathbb{C}_p) - \cup_\alpha H_\alpha$, where the union runs over all $\mathbb{Q}_p$-rational hyperplanes in $\mathbb{P}^{n-1}(\mathbb{C}_p)$. In this section, we introduce the multiplicative Schneider–Teitelbaum lift ST, which is an $\mathrm{SL}_n(\mathbb{Z}[1/p])$-equivariant map from $\mathbb{D}_0(\mathbb{X}, \mathbb{Z})$ to the space $\mathcal{A}^\times$ of invertible functions on X_p, modulo $p^{\mathbb{Z}}$. Then, we use the distribution valued cocycles of the previous sections to construct cocycles for $\mathrm{SL}_n(\mathbb{Z}[1/p])$ valued in $\mathcal{A}^\times/p^{\mathbb{Z}}$. Finally, we define the evaluation of these cocycles at totally real fields of degree n where p is inert, and study properties of these values.

4.1. **Schneider–Teitelbaum lift and rigid cocycles.** The group $\mathrm{SL}_n(\mathbb{Q}_p)$ acts on $\mathbb{P}^{n-1}(\mathbb{C}_p)$ by matrix multiplication and the $\mathbb{Q}_p$-rational hyperplanes are preserved by this action. From there, we define a left action of $\mathrm{SL}_n(\mathbb{Q}_p)$ on the space of functions on X_p given as follows. If $g \in \mathrm{SL}_n(\mathbb{Q}_p)$, f is a function on X_p, and $\tau \in X_p$

$$(g \cdot f)(\tau) := f(g^t \tau),$$

Let $\log_p : \mathbb{C}_p^\times \to \mathbb{C}_p$ be the p-adic logarithm satisfying $\log_p(p) = 0$.

Definition 4.1. For a distribution $\lambda \in \mathbb{D}_0(\mathbb{X}, \mathbb{Z})$, define by ST the function on X_p given by

$$\mathrm{ST}(\lambda)([\tau]) := \times\!\!\!\!\!\!\int_{\mathbb{X}} \tau^t \cdot x \, d\lambda(x).$$

for $\tau \in X_p$. We will use the notation ST for the map

$$\mathrm{ST} : \mathbb{D}_0(\mathbb{X}, \mathbb{Z}) \to \mathcal{A}^\times/p^{\mathbb{Z}}$$

whose target is quotiented by $p^{\mathbb{Z}}$, due to its nicer equivariance properties (see below).

Remark 4.2. Consider the same notation as above. Since λ has total mass zero, the integral defining $\mathcal{L}\mathrm{ST}(\lambda)([\tau])$ is independent of the choice of representative of $[\tau] \in \mathbb{P}^{n-1}(\mathbb{C}_p)$.

Proposition 4.3. *The morphism* ST *is* $\mathrm{SL}_n(\mathbb{Z}[1/p])$*-equivariant.*

Proof. Our proof is essentially identical to [DPV, §1.4]. Observe that if f is an integrable function on $\mathbb{X}$ and $\gamma \in \mathrm{SL}_n(\mathbb{Z}[1/p])$, we have the equality

$$\times\!\!\!\!\!\!\int_{\mathbb{X}} (\gamma \cdot f) d\mu = \times\!\!\!\!\!\!\int_{\gamma^{-1}\mathbb{X}} f d(\gamma^{-1}\mu). \tag{36}$$

Furthermore, for any γ, and any p-invariant measure μ, we have

$$\times\!\!\!\!\!\!\int_{\gamma\mathbb{X}} \tau^t \cdot x \, d\mu(x) \equiv \times\!\!\!\!\!\!\int_{\mathbb{X}} \tau^t \cdot x \, d\mu(x) \pmod{p^{\mathbb{Z}}} \tag{37}$$

because there exist $a, b \in \mathbb{Z}$ for which

$$p^a\mathbb{X} \subseteq \gamma\mathbb{X} \subseteq p^b\mathbb{X},$$

so one can always find a decomposition into open sets

$$\mathbb{X} = U_1 \sqcup \ldots \sqcup U_t$$

for which

$$\gamma\mathbb{X} = p^{r_1}U_1 \sqcup \ldots \sqcup p^{r_t}U_t$$

for some integers $r_1, \ldots, r_t$, and then we have

$$\times\!\!\!\!\!\!\int_{p^{r_i}U_i} \tau^t \cdot x \, d\mu(x) \equiv \times\!\!\!\!\!\!\int_{U_i} \tau^t \cdot x \, d\mu(x) \pmod{p^{\mathbb{Z}}}$$

for each $1 \leq i \leq t$.

Now, we proceed to verify the desired equivariance. Let $\gamma \in \mathrm{SL}_n(\mathbb{Z}[1/p])$, $\mu \in \mathbb{D}_0(\mathbb{X}, \mathbb{Z}_p)$ and $\tau \in X_p$. Then,

$$\gamma \cdot (\mathrm{ST}(\mu))([\tau]) = \times\!\!\!\!\!\!\int_{\mathbb{X}} \tau^t \gamma x \, d\mu(x) = \times\!\!\!\!\!\!\int_{\gamma\mathbb{X}} \tau^t x \, d(\gamma \cdot \mu)(x) \equiv \mathrm{ST}(\gamma \cdot \mu)([\tau]) \pmod{p^{\mathbb{Z}}}$$

where in the second to last equality we used (36) and in the last we used (37). $\square$

We therefore obtain a $\mathcal{A}^\times/p^{\mathbb{Z}}$-valued cohomology class by pushing forward our previously-defined distribution-valued class:

Definition 4.4. Define ${}_cJ_E = \mathrm{ST}_*\boldsymbol{\mu} \in \frac{1}{o_p(c)c^n} \cdot H^{n-1}(\mathrm{SL}_n(\mathbb{Z}), \mathcal{A}^\times/p^{\mathbb{Z}})$, where $o_p(c)$ denotes the multiplicative order of p modulo c.

We will generally work with a fixed c and omit the pre-superscript c as implicit. However, we first have the following corollary of Proposition 3.17:

Corollary 4.5. *We have, for any c and d prime to p and each other, that*

$$(1 - d^n)_c J_E = (1 - c^n)_d J_E.$$

In particular, $(1 - c^n)^{-1}{}_c J_E$ is independent of c.

Proof. This follows immediately from Proposition 3.17 and the fact that ST factors through $[c]_*^{-1} - 1$, as $[c]_*^{-1}$ preserves $\mathcal{X}$ and the measures involved have mass zero on $\mathbb{X}$. $\square$

4.2. **Evaluation of cocycles at "real multiplication" points.** Let F be a totally real field of degree n where p is inert. Let $\mathfrak{a}$ be an integral ideal of F of norm coprime to pc, and fix $\{\tau_1, \ldots, \tau_n\}$ an oriented $\mathbb{Z}$-basis of $\mathfrak{a}^{-1}$. Let τ be the column vector of size $n \times 1$ whose ith entry is equal to τ_i. It yields to an embedding

$$\iota_F : \mathbb{Q}^n \xrightarrow{\sim} F, \; x \mapsto \tau^t \cdot x.$$

By considering the action of multiplication by $F^\times$ on F, which is $\mathbb{Q}$-linear, we obtain an embedding

$$F \hookrightarrow M_n(\mathbb{Q}),\ \alpha \mapsto A_\alpha$$

determined by $\tau^t A_\alpha = \alpha\tau^t$, for every $\alpha \in F$. For $\alpha \in F$ and $x \in \mathbb{Q}^n$, we have

$$\alpha(\tau^t \cdot x) = \tau \cdot (A_\alpha x). \tag{38}$$

Lemma 4.6. *The element $[\tau] \in \mathbb{P}^{n-1}(\mathbb{C}_p)$ belongs to X_p.*

Proof. The coordinates of τ form a $\mathbb{Q}$-basis of F. Since p is inert in F, the coordinates of τ also form a $\mathbb{Q}_p$-basis of the completion of F at p. This implies that they are independent over $\mathbb{Q}_p$, i.e. τ does not belong to any $\mathbb{Q}_p$-rational hyperplane in $\mathbb{P}^{n-1}(\mathbb{C}_p)$. □

Note that $F^\times \subset \mathrm{GL}_n(\mathbb{Q})$ fixes $\tau \in X_p$. Let U_F be the group of totally positive units in $\mathcal{O}_F^\times$; it is a free group of rank $n-1$, which we view as embedded in $\mathrm{SL}_n(\mathbb{Z})$ via the coordinatization given by $[\tau]$ (so that it is the stabilizer of this point in X_p). We have a morphism in cohomology induced by restriction and then evaluation at $[\tau]$

$$H^{n-1}(\mathrm{SL}_n(\mathbb{Z}), \mathcal{A}^\times) \xrightarrow{\mathrm{ev}_{[\tau]}} H^{n-1}(U_F, \mathbb{C}_p^\times).$$

Denote by $c_{U_F} \in H_{n-1}(U_F, \mathbb{Z})$ the fundamental class whose orientation corresponds to our embedding into $\mathrm{SL}_n(\mathbb{Z})$ (see [BCG20, §12.4]).

Definition 4.7. Consider the same notation as above, and let $J \in H^{n-1}(\mathrm{SL}_n(\mathbb{Z}), \mathcal{A}^\times)$. Define the evaluation of J at $[\tau] \in X_p$ by

$$J[\tau] := c_{U_F} \cap \mathrm{ev}_{[\tau]}(J) \in \mathbb{C}_p^\times.$$

If J is installed valued in $\mathcal{A}^\times/p^{\mathbb{Z}}$, we get an evaluation in $\mathbb{C}_p^\times/p^{\mathbb{Z}}$.

We will be interested in the evaluation of the cocycles J_E defined in 4.4. We note that it is clear from the formula for the Schneider-Teitelbaum transform that if we denote by F_p the completion of F at p, we in fact have $J_{\tilde{\mu}}[\tau] \in F_p^\times/p^{\mathbb{Z}}$.

We also have the a more refined kind of "evaluation at τ" map starting from a U_F-cohomology class valued in distributions; we call the following map D_τ for its similarity to p-adic integrals considered in the work of Dasgupta (e.g. [Das08]).

Lemma 4.8. *Let $\tau \in F^n$ be as above. Then, the morphism*

$$\mathrm{D}_\tau : \mathbb{D}_0(\mathbb{X}, \mathbb{Z}) \to F_p^\times,\ \mu \mapsto \times\!\!\!\!\!\!\int_{\mathbb{X}} \tau^t \cdot x\, d\mu(x)$$

is U_F-equivariant.

Proof. Let $\gamma \in U_F$ and $\mu \in \mathbb{D}_0(\mathbb{X}, \mathbb{Z})$. Proceeding as in the proof of Proposition 4.3, we obtain

$$\times\!\!\!\!\!\!\int_{\mathbb{X}} \tau^t \cdot x\, d(\gamma\cdot\mu)(x) = \times\!\!\!\!\!\!\int_{\mathbb{X}} (\gamma^t\tau)^t \cdot x\, d\mu(x).$$

Since $\gamma \in U_F$, it follows from (38) that $\gamma^t\tau = \varepsilon\tau$ for $\varepsilon \in F$ a fundamental unit. Hence, the right hand side of the equation can be written as

$$\times\!\!\!\!\!\!\int_{\mathbb{X}} (\gamma^t\tau)^t \cdot x\, d\mu(x) = \varepsilon^{\mu(\mathbb{X})} \cdot \times\!\!\!\!\!\!\int_{\mathbb{X}} \tau^t \cdot x\, d\mu(x) = \times\!\!\!\!\!\!\int_{\mathbb{X}} \tau^t \cdot x\, d\mu(x).$$

□

Note that it is clear from the formulas that $\mathrm{D}_\tau(\mu)$ modulo $p^{\mathbb{Z}}$ is equal to $\mathrm{ST}(\mu)[\tau]$, so this is a refinement of the evaluation of rigid analytic functions/cocycles.

5. Values of rigid cocycles and unit formulas

We remain in the setting of our totally real field F in which p is inert, with $\mathfrak{a}$, τ, etc. as before. In this section, we prove that the value

$$\log \mathrm{N}_{F_p/\mathbb{Q}_p} J_E[\tau] \in F_p$$

is equal to a local norm of a Gross–Stark unit[7] in the narrow Hilbert class field of F, using the (known) Gross–Stark conjecture. We will also conjecture a similar comparison without the norms using the map D_τ, which we will be able to prove modulo $p^{\mathbb{Z}}$ (i.e. for the rigid analytic cocycles) in special cases.

5.1. p-adic L-functions and Stark-type conjectures.

We first briefly recall the statements of the Gross–Stark and Brumer–Stark conjectures over F, specified to the setting we will consider in our applications. To do so, we first need some background on (p-adic) L-functions.

For a given integral ideal $\mathfrak{f}$, recall that $G_{\mathfrak{f}}$ denotes the ray class group attached to $\mathfrak{f}$. Then, for ε a $\bar{\mathbb{Q}}$-valued function on $G_{\mathfrak{f}}$, we define

$$L(\varepsilon, s) = \sum_{(\mathfrak{b},\mathfrak{f})=1} \varepsilon(\mathfrak{b}) \mathrm{N}\mathfrak{b}^{-s},$$

where the sum is over integral ideals which are coprime to $\mathfrak{f}$. This sum converges for $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$ and it can be extended via analytic continuation to a meromophic function at $\mathbb{C}$ with at most a pole at $s = 1$. Let c be a positive integer and denote by ε_c the function on $G_{\mathfrak{f}}$ given by $\varepsilon_c(\mathfrak{b}) = \varepsilon(\mathfrak{b}c)$. Consider then, for $k \geq 1$,

$$\Delta_c(\varepsilon, 1-k) = L(\varepsilon, 1-k) - c^{nk} L(\varepsilon_c, 1-k).$$

Recall that $\mathfrak{a}$ is an integral ideal of F which is coprime to p. Let

$$1_{[\mathfrak{a}],p} : G_p \to \mathbb{Z}$$

be the characteristic function of the pre-image of $[\mathfrak{a}] \in G_1$ by the natural map $G_p \to G_1$ and consider the L-function $L(1_{[\mathfrak{a}]_1,p}, s)$. It is a partial zeta function with the Euler factor corresponding to p removed, which vanishes at $s = 0$.

By the work of Deligne–Ribet, there exists a p-adic analytic function $L_p(1_{[\mathfrak{a}]_1,p}, s)$ defined on $\mathbb{Z}_p - \{1\}$ such that

$$L_p(1_{[\mathfrak{a}],p}, 1-k) = \Delta_c(1_{[\mathfrak{a}],p}, 1-k)$$

for every $k \geq 1$. The following congruence, which is used to prove the existence of the p-adic L-function $L_p(1_{[\mathfrak{a}]_1,p}, 1-k)$, will be useful for later calculations.

---

[7]The "units" involved in the Gross–Stark and Brumer–Stark conjectures which we will consider are actually *p-units* in general, but we will colloquially call them "units" without comment.

Theorem 5.1. *Consider the same notation as above. Then, we have:*

(1) *For all* $\varepsilon : G_{\mathfrak{f}} \to \bar{\mathbb{Z}}_p$ *and* $k \geq 1$*, we have* $\Delta_c(1-k, \varepsilon) \in \bar{\mathbb{Z}}_p$.
(2) *Let* $\mathfrak{f}$ *be divisible by* p^m*, and let* $k \geq 1$ *be given. Suppose that* $\eta : G_{\mathfrak{f}} \to \bar{\mathbb{Z}}_p$ *is such that*

$$\eta \equiv \mathrm{N}^{k-1} \mod p^m,$$

the two functions considered as functions on the set of prime to $\mathfrak{f}$ *ideals. Then, for all* $\varepsilon : G_{\mathfrak{f}} \to \bar{\mathbb{Z}}_p$

$$\Delta_c(1-k, \varepsilon) \equiv \Delta_c(0, \varepsilon\eta) \mod p^m.$$

We now state the Gross–Stark conjecture. Let H be the narrow Hilbert class field of F, and consider the following subgroup of the p-units

$$U_p := \{u \in H^\times \mid |x|_{\mathfrak{Q}} = 1 \ \forall \mathfrak{Q} \nmid p\},$$

where $\mathfrak{Q}$ runs over places (archimedean and nonarchimedean). Fix $\mathfrak{P}$ a prime of H dividing p.

Proposition 5.2. *There exists a unique element* $u \in U_p \otimes \mathbb{Q}$ *satisfying*

$$\mathrm{ord}_{\mathfrak{P}}(u^{\sigma_{\mathfrak{a}}}) = L(1_{[\mathfrak{a}]_1}, 0) \text{ for all } \mathfrak{a} \text{ coprime to } p,$$

where $1_{[\mathfrak{a}]_1}$ *denotes the characteristic function of* $[\mathfrak{a}]$ *on* G_1 *and* $\sigma_{\mathfrak{a}} \in \mathrm{Gal}\,(H/F)$ *denotes the Frobenius element associated to*

Note that, since p splits completely on H, we have $H \subset H_{\mathfrak{P}} \simeq F_p$.

Theorem 5.3 (Gross–Stark conjecture)**.** *Let* u *be as above. We have*

$$L_p'(1_{[\mathfrak{a}],p}, 0) = -(1-c^n)\log_p(\mathrm{N}_{F_p/\mathbb{Q}_p} u^{\sigma_{\mathfrak{a}}}) \text{ for all } \mathfrak{a} \text{ coprime to } p.$$

Thus, the first derivative of a p-adic L-function is related to the *norm* of a *virtual* unit - i.e. a formal rational power of a unit. A priori, this may not be an *actual* p-unit; we have only that some integer power of it is. On the other hand, in this same setting, the Brumer–Stark conjecture (as proven in [DK23] and [DKSW23]) tells us that we do have a genuine unit:

Theorem 5.4. *If the auxiliary smoothing integer* c *is such that all of its prime factors are greater than* $n+1$*, then there is an* actual *unit* $u \in \mathcal{O}_H[1/p]^\times_-$ *satisfying the conditions of the Gross–Stark conjecture.*

(See [RX25, Remark 6.3] for how this follows from more standard statements of the Brumer–Stark conjecture.)

Note that the property above defines u only up to is well-defined only up to $p^{\mathbb{Z}}$ and a root of unity.

5.2. **Interpolation of L-values.** In this section, we give the relation of the cocycle $\boldsymbol{\mu}$ with the p-adic L-function $L_p(1_{[\mathfrak{a}]_1,p}, s)$ introduced above.

Let $\chi : \mathbb{Z}_p^\times \to \bar{\mathbb{Q}}_p^\times$ be a continuous function and fix $\tau \in F_p^n$ a representative of $\tau \in \mathbb{P}^{n-1}(F_p)$. Consider the map

$$\varphi_{\tau,\chi} : \mathbb{D}(\mathbb{X}, \mathbb{Z}_p) \to \bar{\mathbb{Q}}_p, \ \lambda \mapsto \int_{\mathbb{X}} \chi\left(\mathrm{N}(c\mathfrak{a})\mathrm{N}_{F_p/\mathbb{Q}_p}(\tau^t \cdot x)\right) d\lambda(x),$$

which is U_F-equivariant. We can then consider

$$\varphi_{\tau,\chi}\left(\mathrm{res}_{U_F}(\tilde{\mu})\right) = \varphi_{\tau,\chi}\left(\mathrm{res}_{U_F}(\mu)\right) \in H^{n-1}(U_F, F_p).$$

Remark 5.5. Note that though λ is only a $\mathbb{Z}_p$-valued distribution on *locally constant functions* a priori, one can easily check that the usual definition of the integral against λ via Riemann sums converges p-adically on any uniformly continuous function so long as the values of λ are p-adically bounded, hence on *any* continuous function on the compact set $\mathbb{X}$.

Recall that c_{U_F} is a fundamental class of $H_{n-1}(U_F, \mathbb{Z}_p)$. We will be interested in the function of one p-adic variable

$$\mathbb{Z}_p \ni s \mapsto c_{U_F} \cap \varphi_{\tau,()^{-s}}(\mathrm{res}_{U_F}(\boldsymbol{\mu})) = \int_{\mathbb{X}} (\mathrm{N}(c\mathfrak{a})\mathrm{N}(\tau^t \cdot x))^{-s}\, d\boldsymbol{\mu}(x) \frown c_{U_F} \in F_p,$$

where $(\cdot)^s : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ denotes the character $x \mapsto x^s$. Observe that this function is a p-adic analytic function on $\mathbb{Z}_p$.

Proposition 5.6. *Let $\chi : \mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}_p/p^r\mathbb{Z}_p)^\times \to \bar{\mathbb{Q}}^\times$ be a finite order character. We have*

$$c_{U_F} \frown \varphi_{\tau,\chi}\left(\mathrm{res}_{U_F}(\boldsymbol{\mu})\right) = \Delta_c(1_{[\mathfrak{a}],p}\tilde{\chi}, 0).$$

Proof. By Theorem 3.16, $c_{U_F} \frown \varphi_{\tau,\chi}(\mathrm{res}_{U_F}(\boldsymbol{\mu}))$ satisfies the same interpolation properties at every finite level $\Gamma_1(p^r)$ as the cocycle μ used in [RX25, Proposition 6.9], from which we derive the same conclusion as in loc. cit. □

Using the existence of the Deligne–Ribet L-function, we also immediately derive the analogue of [RX25, Corollary 6.10]:

Corollary 5.7. *For any $s \in \mathbb{Z}_p$, we have*

$$L_p(1_{[a],p}, s) = \int_{\mathbb{X}} \left\langle \mathrm{N}(\mathfrak{a})\mathrm{N}_{F_p/\mathbb{Q}_p}(c\tau^t \cdot x)\right\rangle^{-s} d\boldsymbol{\mu}(x) \frown c_{U_F}$$

By comparing the derivatives at $s = 0$ of the two sides, we hence deduce the analogue of [RX25, Theorem 1.6]:

Corollary 5.8 (Theorem 1.4)**.** *Let $u \in U_p \otimes \mathbb{Q}$ be the Gross–Stark unit introduced in Proposition 5.2 and denote by $u_\tau := u^{c^n-1} \in U_p \otimes \mathbb{Q}$. We have,*

$$\log_p \mathrm{N}_{F_p/\mathbb{Q}_p} J_E[\tau] \equiv \log_p \mathrm{N}_{F_p/\mathbb{Q}_p}(u_\tau) \in F_p$$

5.3. **Brumer–Stark units and the cyclic Galois case.** The above result gives us evidence to conjecture that $J_E[\tau]$ should in fact be equal to a genuine p-unit $u_\tau \in \mathcal{O}_H[1/p]^\times$ satisfying the Brumer–Stark conjecture. As in [RX25, §7.2], we are able to prove some cases of the weak form of this conjecture in the Galois case: suppose that F is Galois over $\mathbb{Q}$. If the narrow ideal class $[\mathfrak{a}]$ is $\mathrm{Gal}(F/\mathbb{Q})$-stable, we prove that $\sigma_{\mathfrak{a}} u \in \mathbb{Q}_p$ up to roots of unity. If moreover the ideal $\mathfrak{a}$ is $\mathrm{Gal}(F/\mathbb{Q})$-stable, we show that $J_L[\tau] \in \mathbb{Q}_p$. Thus, the norms in Theorem 1.4 simply become nth powers, and we are able to remove them up to an nth root of unity ambiguity.

Observe that under these assumptions, H is Galois over $\mathbb{Q}$. Denote by $D_{\mathfrak{p}} \subset \mathrm{Gal}(H/\mathbb{Q})$ the decomposition group at $\mathfrak{p}$. Note that $\mathrm{Gal}(H/\mathbb{Q})$, and therefore also $D_{\mathfrak{p}}$, act on $\mathcal{O}_H[1/p]_-^\times \otimes \mathbb{Q}$. Then we have from [RX25, Lemma 7.2, Proposition 7.3]:

Lemma 5.9. *Let u be the Gross–Stark unit as above and let $[\mathfrak{a}]$ be a narrow ideal class that is $\mathrm{Gal}\,(F/\mathbb{Q})$-fixed. For every $\eta \in D_{\mathfrak{p}}$, we have $\eta(\sigma_{\mathfrak{a}} u) = \sigma_{\mathfrak{a}} u$ in $\mathcal{O}_H[1/p]_-^{\times} \otimes \mathbb{Q}$. As a result, $\log_p(\sigma_{\mathfrak{a}} u) \in \mathbb{Q}_p$.*

We proceed to study the invariant $J_E[\tau]$ in the case that the ideal $\mathfrak{a}$ is $\mathrm{Gal}\,(F/\mathbb{Q})$-stable. In this setting, we can refine the embedding of U_F into $\mathrm{SL}_n(\mathbb{Z})$ to an embedding

$$\mathcal{O}_F^{\times} \rtimes \mathrm{Gal}\,(F/\mathbb{Q}) \longrightarrow \mathrm{GL}_n(\mathbb{Z})$$

determined by the following equations: for every $x \in \mathbb{Q}^n$, $\alpha \in F^{\times}$ and $\sigma \in \mathrm{Gal}\,(F/\mathbb{Q})$,

$$\alpha(\tau^t \cdot x) = \tau^t A_\alpha x, \ \sigma(\tau^t \cdot x) = \tau^t A_\sigma x.$$

Denote $\mathbb{D}_0 := \mathbb{D}_0(\mathbb{X})$. Recall that $\mathrm{GL}_n(\mathbb{Z})$ acts on $\mathbb{D}_0$ as follows: for $g \in \mathrm{GL}_n(\mathbb{Z})$, $\lambda \in \mathbb{D}_0$, and $U \subset \mathbb{X}$ compact open

$$(g \cdot \lambda)(U) := \lambda(g^{-1}U).$$

Consider also the $\mathrm{GL}_n(\mathbb{Z})$-module $\mathbb{D}_0(\det) := \mathbb{D}_0 \otimes_{\mathbb{Z}} \mathbb{Z}(\det)$. We use these actions and the embedding above to describe an action of $\mathcal{O}_F^{\times}$ and of $\mathrm{Gal}\,(F/\mathbb{Q})$, on $\mathbb{D}_0$ and $\mathbb{D}_0(\det)$. In particular, since $\{1\} \rtimes \mathrm{Gal}\,(F/\mathbb{Q})$ normalizes $U_F \rtimes \{1\}$, we have natural actions of $\mathrm{Gal}\,(F/\mathbb{Q})$ on $H^{n-1}(U_F, \mathbb{D}_0(\det))$ as well as on the coinvariants $(\mathbb{D}_0)_{U_F}$. Under these, Lemma actually tells us that $\boldsymbol{\mu} = (\varphi_p)_* \Psi_* \theta(n)_{\mathcal{C}}$ actually lifts to

$$\in H^{n-1}(\mathrm{GL}_n(\mathbb{Z}[1/p]), \mathbb{D}_0(V_{\mathbb{Q}_p}, \mathbb{Z})^{(0)}(\det))$$

Lemma 5.10. *The element $c_{U_F} \frown \boldsymbol{\mu} \in (\mathbb{D}_0)_{U_F}$ is fixed by $\mathrm{Gal}\,(F/\mathbb{Q})$.*

Proof. The proof is identical to [RX25, Lemma 7.4], following immediately from the fact that the restriction

$$H^{n-1}(\mathcal{O}_F^{\times} \rtimes \mathrm{Gal}\,(F/\mathbb{Q}), \mathbb{D}_0(V_{\mathbb{Q}_p}, \mathbb{Z})^{(0)}(\det)) \to H^{n-1}(\mathcal{O}_F^{\times}, \mathbb{D}_0(V_{\mathbb{Q}_p}, \mathbb{Z})^{(0)}(\det))$$

lands in the $\mathrm{Gal}\,(F/\mathbb{Q})$-invariants. □

Likewise, the following can be deduced in exactly the same way as [RX25, Theorem 7.5]:

Theorem 5.11. *Suppose that the coordinates of $\tau \in F^n$ given an oriented $\mathbb{Z}$-basis of a $\mathrm{Gal}\,(F/\mathbb{Q})$-stable ideal $\mathfrak{a}^{-1}$. Then $J_E[\tau] \in \mathbb{Q}_p$.*

We hence deduce:

Corollary 5.12. *Suppose that F is a totally real field that is Galois over $\mathbb{Q}$ and where p is inert. Let $\tau \in F^n$ with coordinates generating $\mathfrak{a}^{-1}$, where $\mathfrak{a}$ is a $\mathrm{Gal}\,(F/\mathbb{Q})$-stable ideal, and let $u \in \mathcal{O}[1/p]_-^{\times} \otimes \mathbb{Q}$ be the Gross–Stark unit of Proposition 5.2. We have*

$$J_E[\tau] = u^{\sigma_{\mathfrak{a}}}$$

up to $p^{\mathbb{Z}}$ and roots of unity.

This implies Theorem 1.5 of the introduction. Note that the denominator $o_p(c)c^n$ occurring in the definition of J_E does not matter, since we tolerate a root of unity ambiguity.

References

[AR79] Avner Ash and Lee Rudolph. The modular symbol and continued fractions in higher dimensions. *Inventiones mathematicae.*, 55:241–250, 1979.

[BCG20] Nicolas Bergeron, Pierre Charollois, and Luis E. Garcia. Transgressions of the Euler class and Eisenstein cohomology of $\mathrm{GL}_N(\mathbf{Z})$. *Jpn. J. Math.*, 15(2):311–379, 2020.

[BCG23] Nicolas Bergeron, Pierre Charollois, and Luis Garcia. *Cocycles de de groupe pour GL_n et arrangements d'hyperplans.*, volume 39 of *CRM Monographs.* Centre de recherches mathématiques., 2023.

[Bjo90] Anders Bjorner. *Homology and shellability of matroids and geometric lattices.* Cambridge University Press, 1990.

[BKL18] Alexander Beilinson, Guido Kings, and Andrey Levin. Topological polylogarithms and p-adic interpolation of l-values of totally real fields. *Mathematische Annalen*, 371:1449–1495, 2018.

[CDG15] P. Charollois, S. Dasgupta, and M. Greenberg. Integral Eisenstein cocycles on GL_n, II: Shintani's method. *Comment. Math. Helv.*, 90(2):443–488, 2015.

[Das08] Samit Dasgupta. Shintani zeta functions and Gross-Stark units for totally real fields. *Duke Math. J.*, 143:225–279, 2008.

[DD06] Henri Darmon and Samit Dasgupta. Elliptic units for real quadratic fields. *Ann. of Math.*, 163:301–46, 2006.

[DK23] Samit Dasgupta and Mahesh Kakde. On the Brumer–Stark conjecture. *Ann. of Math.*, 197:289–388, 2023.

[DKSW23] Samit Dasgupta, Mahesh Kakde, Jesse Silliman, and Jun Wang. The Brumer–Stark Conjecture over $\mathbb{Z}$, 2023. Preprint.

[DPV] Henri Darmon, Alice Pozzi, and Jan Vonk. The values of the Dedekind-Rademacher cocycle at real multiplication points. *Journal of the European Mathematical Society*, 26(10):3987â4032.

[DPV21] Henri Darmon, Alice Pozzi, and Jan Vonk. Diagonal restrictions of p-adic Eisenstein families. *Mathematische Annalen*, 379:503–548, 2021.

[DV21] Henri Darmon and Jan Vonk. Singular moduli for real quadratic fields: a rigid analytic approach. *Duke Math Journal*, 170(1):23–93, 2021.

[GS24] A. Galanakis and M. Spieß. Adelic Eisenstein classes and divisibility properties of Stickelberger elements. *arXiv preprint arXiv:2402.11583*, 2024.

[OT92] Peter Orlik and Hiroaki Terao. *Arrangements of Hyperplanes.*, volume 300 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, 1992.

[RT10] Michael Ruzhansky and Ville Turunen. *Pseudo-Differential Operators and Symmetries.*, volume 2. Birkhäuser, 2010.

[RX25] Marti Roset and Peter Xu. Eisenstein class of a torus bundle and log-rigid cocycles for $\mathrm{SL}_n(\mathbb{Z})$. 2025. Preprint.

[Scz93] Robert Sczech. Eisenstein group cocycles for GL_n and values of L-functions. *Invent. Math.*, 113:581–616, 1993.

[SV24] Romyar Sharifi and Akshay Venkatesh. Eisenstein cocycles in motivic cohomology. *Compositio Mathematica*, 160(10):2407–2479, 2024.

[Xu23] Peter Xu. *Arithmetic Eisenstein theta lifts.* PhD thesis, McGill University, 2023.

[Xu24] Peter Xu. Symbols for toric Eisenstein cocycles and arithmetic applications., 2024. Preprint.

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