Some duality computations for Dedekind sums

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Let F be a totally real field of degree n, with totally positive units U_F and an embedding $\iota : U_F \hookrightarrow \mathrm{SL}_n(\mathbb{Z})$, with an associated equivariant identification $\alpha : \mathfrak{a} \xrightarrow{\sim} \mathbb{Z}^n$ for some ideal $\mathfrak{a} \subset F$. At some point, we will also need to consider the *adjuate embedding* $\star\iota$ given by

$$u \mapsto (\iota(u)^{-1})^T,$$

and we will also abbreviate the inverse transpose operation on a matrix M as $\star M$. It will also be convenient to notate the *adjugate* $\mathrm{adj}(M) := |\det M| M^{-1}$, which is always an integer matrix when M is.

Let $c > 1$ be a prime number completely split in F, as

$$(c) = \mathfrak{c}_1 \dots \mathfrak{c}_n.$$

It will also be convenient to use the notation $\hat{\mathfrak{c}}_i = (c)\mathfrak{c}_i^{-1}$. We also fix an oriented basis $u_1, \dots, u_{n-1}$ of the free group U_F; usually, we will freely identify them freely with their images under ι (though later, when we need to talk about the $\star$ embedding, we will need to be more careful). We also write the line $e_1 := [1 : 0 : \dots : 0] \in \mathbb{P}^{n-1}(\mathbb{Q})$; then denote $\ell_i = u_1 \dots u_i e_1$. The integer matrix

$$L = \begin{pmatrix} \ell_0 & \dots & \ell_{n-1} \end{pmatrix}$$

is then full-rank since U_F is a totally nonsplit torus. We will further insist that c does not divide the determinant of this matrix; this excludes only finitely many c.

We will compute explicitly c-smoothed formulas (as well as $\mathfrak{c}_i$-smoothed formulas for each i, which combine to give the c-smoothing) for the Bernoulli cocycle and toric cocycle, and compare with the Shintani cocycle of [CDG].

We will also compute some examples with the real quadratic field $F = \mathbb{Q}(\sqrt{3})$, for which we distinguish the fundamental unit $2 + \sqrt{3}$ of norm 1, and take $c = 11$, which splits into $\mathfrak{c}_1 = (11, 5 - \sqrt{3})$ and $\mathfrak{c}_2 = (11, 5 + \sqrt{3})$. We pick the identification $\alpha : \mathbb{Z}[\sqrt{3}] \xrightarrow{\sim} \mathbb{Z}^2$ identifying the standard basis with 1 and $\sqrt{3}$, respectively, and the corresponding ι sends

$$2 + \sqrt{3} \mapsto \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}.$$

Thus, in this case,

$$L = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

which is in this case unimodular (so we could have picked any split c).

1. Bernoulli lifts

In this section, we compute the cocycle of [RX1] for U_F.

There is a natural action of $\mathrm{SL}_n(\mathbb{Z})$ on the n-torus $T = (\mathbb{R}/\mathbb{Z})^n$. If we restrict to considering T only as a U_F-space via ι, then via the identification α, we can also identify T with $F/\mathfrak{a}$. It then makes sense to speak of the I-torsion $T[I]$ for any ideal I of $\mathcal{O}_F$, as the kernel of $F/\mathfrak{a} \to F/I\mathfrak{a}$; the latter torus we can thus identify with $T/T[I]$.

Consider the c-torsion cycle $T[c] - c^n\{0\} = ([c]^* - c^n)\{0\}$. The Bernoulli lifts corresponding to this cycle will give us our "c-smoothed" cocycle.

Then restricted to U_F, this cocycle also breaks up into a telescoping sum

$$T[c] - c^n\{0\} = (T[c] - cT[\mathfrak{c}_{\geqslant 1}]) + c(T[\mathfrak{c}_{\geqslant 1}] - cT[\mathfrak{c}_{\geqslant 2}]) + \ldots + c^{n-1}(T[\mathfrak{c}_{\geqslant n-1}] - cT[\mathfrak{c}_n]).$$

where we write $\mathfrak{c}_{\geqslant i} := \mathfrak{c}_i \ldots \mathfrak{c}_n$. Each of these individual terms in parentheses gives a U_F-fixed degree-zero torsion cycle, which corresponds to stabilization along "one line": for example, we will just focus on the first term

$$T[c] - cT[\hat{\mathfrak{c}}_1],$$

because it lines up with the smoothing of [CDG], as we will see later.

In the simplest case, where $\mathfrak{c}_1, \ldots, \mathfrak{c}_n$ are principal[1] each $[\mathfrak{c}_i]$ can actually be viewed as an endomorphism of T for each i (once one fixes choices of generators, which we do implicitly), so this is equal to

$$([c]^* - c^n)\{0\} = [([c]^* - c[\mathfrak{c}_{\geqslant 1}]^*) + (c[\mathfrak{c}_{\geqslant 1}]^* - c^2[\mathfrak{c}_{\geqslant 2}]^*) + \ldots + (c^{n-1}[\mathfrak{c}_n]^* - c^n)]\{0\}$$

Then the ith term can be written as $c^{i-1}[\mathfrak{c}_{\geqslant i}]_*([\mathfrak{c}_i]^* - c\{0\})$; thus, they are all "the same" (for the different factors $\mathfrak{c}_i$) except for the constant factor c^{i-1} and the pullback $[\mathfrak{c}_{\geqslant i}]_*$, both of which will have invertible, easily understandable actions on the distributions which we will ultimately specialize to. In general, even when the $\mathfrak{c}_i$ are not principal, something analogous happens allowing us to make a similar reduction, but one has to consider combinations of cocycles corresponding to different ideal classes. This adds a bit of technical complication, but does not fundamentally change the situation. The point is that stabilizing at c a split prime is expressible in a simple way as a combination of stabilizations at each of its factors, so we will focus on the latter.

The cycle

$$\mathcal{C}_1 := T[c] - cT[\hat{\mathfrak{c}}_1]$$

not only has total degree zero, but *also* has degree zero when restricted to any line of $T[c] = (\mathbb{Z}/c)^n$ not contained in the hyperplane (codimension-1 subspace) $T[\hat{\mathfrak{c}}_1]$.

Note that because $T[\hat{\mathfrak{c}}_1]$ is U_F-stable, no U_F-translate of e_1 can be contained in it, since the U_F-span of e_1 is the entirety of $F \cong \mathbb{Z}^n$, and this holds even c-integrally by our earlier determinant assumption. In particular, the 1-dimensional subtorus $T_{ue_1} := \mathbb{R}/\mathbb{Z} \hookrightarrow T = (\mathbb{R}/\mathbb{Z})^n$ corresponding to ue_1, for any $u \in U_F$, is such that the restriction of $\mathcal{C}_1$ to $T_{ue_1}[c]$ has degree zero.

Given a c-torsion cycle $\mathcal{C}$ and a function or current β on T, let us write

$$\beta[\mathcal{C}] := \sum_{x \in T[c]} \mathcal{C}[x] t_x^* \beta$$

where $t_x : T \to T, z \mapsto z - x$ is the pullback map and $\mathcal{C}[x]$ denotes the coefficient of x in $\mathcal{C}$. Then the U_F-invariant n-current

$$\delta_{\mathcal{C}_1}$$

is the differential of the $(n-1)$-current

$$\left\{ \left(e_1 \right)_* B_1(z) \right\} [\mathcal{C}_1] \tag{1.1}$$

[1]This is always possible, by picking c to split completely in the Hilbert class field of F, which occurs with positive density by the Cheboratev density theorem.

where the matrix (e_1) represents the inclusion $S^1 \hookrightarrow T$ associated to the column vector e_1. This $(n-1)$-current is essentially a locally constant stabilization of $B_1(z)$ on the subtorus corresponding to e_1, times the current of integration along that subtorus: the stabilization is locally constant precisely because $\mathcal{C}_1$ has degree zero along e_1.

Then the U_F-coboundary of (1.1) is

$$\gamma_1 \mapsto \left\{ \begin{pmatrix} \gamma_1 e_1 \end{pmatrix}_* B_1(z) \right\} [\mathcal{C}_1] - \left\{ \begin{pmatrix} e_1 \end{pmatrix}_* B_1(z) \right\} [\mathcal{C}_1]$$

which is the differential of the 1-cochain valued in $(n-2)$-currents

$$\gamma_1 \mapsto \left\{ \begin{pmatrix} e_1 & \gamma_1 e_1 \end{pmatrix}_* B_1(z_1) B_1(z_2) \right\} [\mathcal{C}_1].$$

Continuing inductively in this fashion, at the last stage we get that the final lift is the $(n-1)$-*cochain* valued in 0-currents

$$(\gamma_1, \dots, \gamma_{n-1}) \mapsto \left\{ \begin{pmatrix} e_1 & \dots & \gamma_1 \dots \gamma_{n-1} e_1 \end{pmatrix}_* B_1(z_1) \dots B_n(z_n) \right\} [\mathcal{C}_1] = \{ L_* B_1(z_1) \dots B_n(z_n) \} [\mathcal{C}_1].$$

At each stage, the differential computation works because the corresponding stabilization of $B_1 \dots B_1$'s is locally constant, because $\mathcal{C}_1$ is degree-zero along any line of the form ue_1, $u \in U_F$.

1.1. **Worked example.** We have, in our example case,

$$\mathcal{C}_1 = T[11] - 11T[(11, 5 - \sqrt{3})]$$

where $T[(11, 5-\sqrt{3})]$ is, more concretely, the subspace of $(\mathbb{Z}/11)^2$ spanned by $(5, 1)$. This is because $(5, 1)$ corresponds to $5+\sqrt{3}$ under α, and

$$\left[\frac{5+\sqrt{3}}{11} \right] \in \mathbb{Q}(\sqrt{3})/\mathbb{Z}[\sqrt{3}]$$

is annihilated by $5 - \sqrt{3}$. Correspondingly, $(5, 1)$ is annihilated modulo 11 by the matrix

$$\begin{pmatrix} 5 & -3 \\ -1 & 5 \end{pmatrix}$$

corresponding to the action of $5 - \sqrt{3}$ on the basis associated to α. Thus, we can also equivalently write

$$\mathcal{C}_1 = T[11] - 11\{\tfrac{1}{11}\mathbb{Z}/\mathbb{Z} \cdot (5, 1)\}.$$

We take the associated 2-current of integration for this torsion cycle; then the first lift is

$$\left\{ \begin{pmatrix} e_1 \end{pmatrix}_* B_1(z) \right\} [\mathcal{C}_1] = \begin{pmatrix} e_1 \end{pmatrix}_* \left(-11 B_1(z) + \sum_{i=0}^{10} B_1(z - i/11) \right)$$

because the restriction of $\mathcal{C}_1$ to $\mathbb{R}/\mathbb{Z}$, included via e_1, is simply $S_1[11] - 11\{0\}$: this is simply the statements that

$$\mathcal{C}_1[(0,0)] = -10$$
$$\mathcal{C}_1[(1,0)] = 1$$
$$\dots$$
$$\mathcal{C}_1[(10,0)] = 1$$

becasue the line e_1 modulo 11 intersects the line $[5:1]$ modulo 11 only at the identity.

In this case, since $n = 2$, there is only one more lift: the final 1-cocycle

$$\gamma_1 \mapsto \left\{ \begin{pmatrix} e_1 & \gamma_1 e_1 \end{pmatrix}_* B_1(z_1) B_1(z_2) \right\} [\mathcal{C}_1]$$

when evaluated at $\gamma_1 = \iota(2+\sqrt{3})$, is

$$\begin{aligned}
\left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}_* B_1(z_1)B_1(z_2) \right\} [\mathcal{C}_1] &= \{B_1(z_1 - 2z_2)B_1(z_2)\}[\mathcal{C}_1] \\
&= \sum_{i,j=0}^{10} \mathcal{C}_1[(i,j)] B_1\left(z_1 - 2z_2 - \frac{i-2j}{11}\right) B_1(z_2 - j/11) \\
&= \sum_{a,b=0}^{10} \kappa(a,b) B_1(z_1 - 2z_2 - a/11) B_1(z_2 - b/11)
\end{aligned}$$

where

$$\kappa(a,b) = \begin{cases} -10 & \text{if } (a,b) \in [3:1] \subset (\mathbb{Z}/11)^2, \\ 1 & \text{otherwise} \end{cases}$$

since one can compute that the image of $[5:1]$ is $[3:1]$ under the change of coordinates from i, j to a, b. One can easily see that for fixed a, or fixed b, the total degree is zero, which implies that this sum of Bernoulli polynomials is indeed locally constant, as expected.

2. Eisenstein duality, Shintani functions

In this section, we use a "duality" identity of Eisenstein to show that the formulas for the cocycles of [RX1], computed above, can also be written in terms of Shintani-style generating series: we show that our Bernoulli cocycles compute (almost) all the same values as the smoothed Shintani cocycle of [CDG]

The cocycles valued in smoothed Shintani generating series are also essentially the same ones which were shown in [X2] to come from a *toric* equivariant polylogarithm, but we do not go into the details of this here.

The identity of Eisenstein in question is as follows: for any integer $k \geqslant 1$, we have

$$B_1(a/k) = \frac{1}{2k} \sum_{\zeta \in \mu_k - \{1\}} \zeta^{-a} \left(\frac{\zeta}{\zeta - 1} + \frac{\zeta^{-1}}{1 - \zeta^{-1}} \right). \tag{2.1}$$

On the right-hand side, we have a kind of symmetrized finite Fourier transform over torsion values of a function like

$$\frac{z}{1-z} \in \mathcal{O}(\mathbb{G}_m - \{1\})^\times$$

which appear in Shintani's method, and are used in [X2]. On the left, you have the Bernoulli numbers used in the first section.

2.1. **Fourier duality on distributions over torsion cycles.**

2.1.1. *The case $n = 1$.* Write

$$s(z) = \frac{1}{2}\left(\frac{z}{z-1} + \frac{z^{-1}}{1-z^{-1}} \right) = \frac{1}{2} \cdot \frac{z+1}{z-1} = \frac{1}{2} + \frac{1}{z-1} = -\frac{1}{2} + \frac{z}{z-1}$$

for the basic building block of the symmetrized Shintani functions. Let us phrase things in terms of distributions on Schwarz functions: we can view both B_1 and the function

$$S : t \mapsto s(\exp(2\pi i t))$$

as functions on the 1-torus $\mathbb{R}/\mathbb{Z}$; we are interested in evaluation just at the torsion points $\mathbb{Q}/\mathbb{Z}$. We then can view the function B_1 as a distribution μ_{B_1} against Schwarz functions on $\mathbb{Q}/\mathbb{Z}$, sending

$$1_{t+\mathbb{Z}} \mapsto B_1(t). \tag{2.2}$$

Notice that this is basically just an alternate notation for our earlier evaluations at torsion cycles (in the case $n = 1$).

This becomes a less trivial change of notation when one notes that as a function, B_1 satisfies the trace-fixed property

$$[a]_* B_1 = B_1$$

for any $a \in \mathbb{N}$, meaning that we can view actually view μ_{B_1} as a distribution against Schwarz functions not just on $\mathbb{Q}/\mathbb{Z}$, but on $\mathbb{Q}$ (with the profinite topology), characterized by (2.2) together with the property that $\mu_{B_1}(U) = \mu_{B_1}(aU)$ for any $a \in \mathbb{N}$.

Remark 2.1. Note that that the Schwarz functions for the profinite topology on $\mathbb{Q}$ can be identified naturally with Schwarz functions on $\mathbb{A}_{\mathbb{Q}}$ in an obvious way; we will freely interchange between thinking of the objects in this section as rational or adelic Schwarz functions/distributions.

On the other hand, the function S satisfies

$$[a]_* S = aS$$

for all $a \in \mathbb{N}$; thus, we may equally view S as giving a distribution μ_S against the subspace of Schwarz functions $\mathcal{S}(\mathbb{Q})^0$ vanishing at zero, by sending

$$1_{t+\mathbb{Z}} \mapsto S(t)$$

and extending by the rule $\mu_S(aU) = a^{-1}\mu_S(U)$.

Write $\mathscr{F} : \mathcal{S}(\mathbb{Q}) \to \mathcal{S}(\mathbb{Q})$ for the Fourier transform

$$\varphi \mapsto \hat{\varphi} := \left(x \mapsto \int_{\mathbb{Q}} \varphi(y) \exp(-2\pi i \langle x, y \rangle)\, dh(y) \right)$$

for $h(y)$ the Haar measure for which $h(\mathbb{Z}) = 1$; one can check the equivariance property

$$\mathscr{F}(M^* \varphi) = |\det M| (\star M)^* \mathscr{F}(\varphi).$$

for any full-rank matrix M.

Then (2.1) can be reformulated as

$$\mu_{B_1}(\varphi) = \mu_S(\hat{\varphi} - \hat{\varphi}(0) \cdot 1_{0+\mathbb{Z}})$$

for any $\varphi \in \mathcal{S}(\mathbb{Q})$. (Note that the Schwarz function on the RHS always is zero at 0.) In particular, if $\hat{\varphi}(0) = 0$ (which corresponds to a degree-zero torsion cycle), $\mu_{B_1}(\varphi) = \mu_S(\hat{\varphi})$.

2.1.2. *General n, and a formula for the cocycle in terms of products of S.* By simply taking products of everything above, and in particular the Eisenstein identity (2.1), we get analogous distributions on $\mathcal{S}(\mathbb{Q}^n)$, respectively the subspace $\mathcal{S}(\mathbb{Q}^n)^0$ of functions vanishing on each standard hyperplane $t_i = 0$ for $1 \leqslant i \leqslant n$, characterized by

$$\mu_{B,n} : 1_{t+\mathbb{Z}^n} \mapsto B_1(t_1) \dots B_1(t_n) \qquad \mu_{B,n}(aU) = \mu_{B,n}(U)$$

$$\mu_{S,n} : 1_{t+\mathbb{Z}^n} \mapsto S(t_1) \dots S(t_n) \qquad \mu_{S,n}(aU) = a^{-n}\mu_{S,n}(U)$$

such that on a Schwarz function of the form $\varphi = \varphi_1 \boxtimes \ldots \boxtimes \varphi_n$, we have

$$\mu_{B,n}(\varphi) = \mu_{S,n}\left[(\hat{\varphi}_1 - \hat{\varphi}_1(0)1_{0+\mathbb{Z}}) \boxtimes \ldots \boxtimes (\hat{\varphi_n} - \hat{\varphi_n}(0)1_{0+\mathbb{Z}})\right]$$

which in the case that $\hat{\varphi}_i(0) = 0$ for $1 \leqslant i \leqslant n$, i.e., φ has "degree zero along each line corresponding to the standard basis of $\mathbb{Q}^n$," simplifies to

$$\mu_{B,n}(\varphi) = \mu_{S,n}(\hat{\varphi}). \tag{2.3}$$

Lemma 2.2. *Any $\varphi \in \mathcal{S}(\mathbb{Q}^n)$ which is in the kernel of each of the pushforwards*

$$\pi_i : \mathcal{S}(\mathbb{Q}^n) \to \mathcal{S}(\mathbb{Q}^{n-1})$$

given by projecting away from the ith coordinate, for $1 \leqslant i \leqslant n$ (i.e. "summing over lines," for the lines corresponding to the standard basis), can be written as a sum of $\varphi_1 \boxtimes \ldots \boxtimes \varphi_n$ with each $\hat{\varphi}_\bullet(0) = 0$.

Proof. This follows easily from standard character theory. □

Thus, by linearity, for any such φ as in the lemma, we also have (2.3).

Now, consider the Bernoulli function

$$\{L_* B_1(z_1) \ldots B_1(z_n)\}[\mathcal{C}_1]$$

in the setting of the previous section. If φ_c is the Schwarz function on $\mathbb{Q}_c^n$ corresponding to $\mathcal{C}_1$, then an arbitrary evaluation of the above prime-to-c torsion

$$(L_*\mu_{B,n})(\varphi_c \boxtimes \varphi^c) = \mu_{B,n}(L^*(\varphi_c \boxtimes \varphi^c))$$

where φ^c is a Schwarz function on the prime-to-c adeles $(\mathbb{A}_{\mathbb{Q}}^{(c)})^n$ encoding to the prime-to-c torsion we are evaluating at.

Remark 2.3. For example, to evaluate at $(1/p, 0, \ldots 0)$ for a prime p, φ^c would be the indicator function of the compact open set

$$(p^{-1}\mathbb{Z}_p \times \mathbb{Z}_p \ldots \times \mathbb{Z}_p) \times \prod_{\ell \notin \{p,c\}} \mathbb{Z}_\ell^n \subset (\mathbb{A}_{\mathbb{Q}}^{(c)})^n$$

But the degree-zero condition we noted in the previous section about $\mathcal{C}_1$, leading to the local constancy of our lifts, is exactly synonymous with saying that φ_c is in the kernel of each of the projection maps

$$\mathcal{S}(\mathbb{Q}_c^n) \to \mathcal{S}(\mathbb{Q}_c^{n-1})$$

whose kernels are the lines ℓ_i, the ith column of L, for $1 \leqslant i \leqslant n$. This in turn implies that $L^*\varphi_c$ is in the kernel of each of the maps π_i from the lemma (restricted to the c-adic adelic coordinate), from which it follows that $L^*(\varphi_c \boxtimes \varphi^c)$ satisfies the condition of the lemma. We therefore conclude that

$$\mu_{B,n}(L^*(\varphi_c \boxtimes \varphi^c)) = \mu_{S,n}(\mathscr{F}(L^*(\varphi_c \boxtimes \varphi^c))) = |\det L|^{-1}\mu_{S,n}((\star L)^*(\widehat{\varphi_c} \boxtimes \widehat{\varphi^c})).$$

In concrete terms, take φ^c to be the indicator function of a torsion section

$$x = (x_1, \ldots, x_n) \in \left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^n$$

with $(N, c) = 1$. Then

$$\widehat{\varphi^c}(y) = \widehat{1_{x+\mathbb{Z}^n}}(y) = \exp(-2\pi i\langle x, y\rangle)1_{\mathbb{Z}^n}(y)$$

On the other hand, our φ_c corresponding to $\mathcal{C}_1$ is

$$1_{c^{-1}\mathbb{Z}_c^n} - c1_{\hat{\mathfrak{c}}_1^{-1}\mathfrak{a}},$$

and whose Fourier transform is

$$\widehat{\varphi_c}(y) = c^n(1_{\mathbb{Z}_c^n}(y) - 1_{\mathfrak{c}_1\mathfrak{a}}(y)).$$

Then we have

$$\begin{aligned}\{L_* B_1(z_1)\dots B_1(z_n)\}[\mathcal{C}_1]\big|_{z=t} &= \mu_{B,n}(L^*(\varphi_c \boxtimes 1_{t+\mathbb{Z}^n})) \\ &= |\det L|\mu_{S,n}((\star L)^*(\widehat{\varphi_c} \boxtimes \widehat{\varphi^c})) \\ &= \mu_{S,n}((|\det L| \cdot \star L)^*(\widehat{\varphi_c} \boxtimes \widehat{\varphi^c})) \\ &= \frac{1}{N^n} \sum_{y \in (\mathbb{Z}/N\mathbb{Z})^n} \exp(-2\pi i \langle x, y\rangle)\{(\mathrm{adj}(L^T))_*(S(t_1)\dots S(t_n))\}[\widehat{\mathcal{C}}_1]\big|_{x=t}\end{aligned}$$

where here

$$\widehat{\mathcal{C}}_1 = T[c] - T[\hat{\mathfrak{c}}_1].$$

We therefore have an expression for our lift in terms of a smoothed sum of Shintani-type functions.

2.2. **Shintani functions and unit formulas.** Let us briefly recall the Shintani approach to L-functions for F, and how Dasgupta and collaborators obtain Gross–Stark units formulas from it: as before, we have an identification $\mathfrak{a} \cong \mathbb{Z}^n$, making $\mathfrak{a}$ a lattice inside $\mathfrak{a} \otimes \mathbb{R} \cong \mathbb{R}^n$.

Inside $\mathfrak{a} \otimes \mathbb{R}$ there is the closed convex domain $(\mathfrak{a} \otimes \mathbb{R})_+$ of totally positive elements, preserved by the action of U_F. For any collection of linearly independent rays $r_1, \dots, r_k \subset \mathbb{R}^n$, $k \leqslant n$, consider the relatively open simplicial cone

$$C^\circ(r_1, \dots, r_n) := \{a_1 r_1 + \dots + a_k r_k; a_1, \dots, a_k \in (0, \infty)\}.$$

Then Shintani showed that there is always a finite collection of cones $\mathcal{S}$ in $(\mathfrak{a} \otimes \mathbb{R})_+$ whose disjoint union is a fundamental domain for the action of U_F on this convex domain; we call this $\mathcal{S}$ a *Shintani decomposition* and their union a *Shintani domain*.

Following [CDG] (after an idea of Colmez), one can generalize this by formally putting coefficients (allowing ± 1 suffices) on the cones, so that the formal "sum" of all the cones in $\mathcal{S}$ is $(\mathfrak{a} \otimes \mathbb{R})_+$ after cancellations. In such a case, loc. cit. also shows that one can pick $\mathcal{S}$ such that its collection of n-dimensional cones is precisely

$$C^\circ(e_1, u_{\sigma(1)} e_1, \dots, u_{\sigma(1)} \dots u_{\sigma(n-1)} e_1)$$

as σ ranges over S_{n-1}, and all the other cones are faces one or more of these. (Recall that $u_1, \dots, u_{n-1}$ is a basis of U_F.)

For each cone $C = C^\circ(r_1, \dots, r_k)$, the *Shintani generating function* is

$$f_C(z_1, \dots, z_n) := \sum_{y \in R_C} \frac{z_y}{\prod_{r_i}(1 - z_r)}$$

where

$$R_C : \mathbb{Z}^n \cap \{c_1 r_1 + \dots + c_k r_k, \forall i, c_i \in \mathbb{Q}, 0 < c_i \leqslant 1\}$$

where we (a bit abusively) identify the rays $r_1, \dots, r_k$ with their integral generators,and we write z_x for the monomial $z_1^{x_1} \dots z_n^{x_n}$, for any $x \in \mathfrak{a} \cong \mathbb{Z}^n$. This generating function is equivalently

$$\mathrm{adj}\begin{pmatrix} r_1 & \dots & r_n \end{pmatrix}_*^T \frac{z_1 \dots z_k}{(1 - z_1)\dots(1 - z_k)}$$

(where we interpret this matrix as map from $\mathbb{G}_m^n \to \mathbb{G}_m^n$ in the obvious way). (If $k < n$, the result does not actually depend on anything beyond $r_1, \ldots, r_k$.)

In any case, the total Shintani generating function for F is then defined by

$$f_{\mathcal{S}}(z) = \sum_{C \in \mathcal{S}} f_C(z).$$

Let

$$f_k(z_1, \ldots, z_k) = \frac{z_1 \ldots z_k}{(1-z_1)\ldots(1-z_k)}$$

be the Shintani generating function for the "standard" k-cone. Analogously to before, define distributions $\mu_{Sh_k,n}$ for $1 \leqslant k \leqslant n$ on $\mathcal{S}(\mathbb{Q}^n)^0$ by

$$\mu_{Sh_k,n} : 1_{t+\mathbb{Z}^n} \mapsto f_k(t); \;\; \mu_{Sh_k,n}(aU) = a^{-n}\mu_{Sh_k,n}(U)$$

(When $k < n$, we don't need all of the vanishing conditions on $\mathcal{S}(\mathbb{Q}^n)^0$, but impose them anyway because it doesn't matter too much for us.)

We note the important property that

$$s(z_1)\ldots s(z_n) = f_n(z_1, \ldots, z_n) + \text{linear combination of } f_k(z_I) \text{ for } k < n, k = |I| \subset [n]$$

Then for any Schwarz function $\varphi \in \mathcal{S}(\mathbb{Q}^n)$ for which its restriction to the hyperplanes

$$H_i := \{a \in \mathbb{Q}^n; \langle a, \ell_i \rangle = 0\}$$

for $1 \leqslant i \leqslant n$ is degree zero, we have

$$f_C[\varphi] = \{s(z_1)\ldots s(z_n)\}[L_*\varphi].$$

2.3. **Worked example.** Recall our previously-computed Bernoulli cocycle lift

$$\left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}_* B_1(z_1)B_1(z_2) \right\} [\mathcal{C}_1] = \sum_{a,b=0}^{10} \kappa(a,b) B_1(z_1 - 2z_2 - a/11) B_1(z_2 - b/11).$$

Here, κ, as a function on $(\mathbb{Z}/11)^2$ embedded in the natural way into $(\mathbb{Q}/\mathbb{Z})^2$, is precisely $L_*\varphi_c$; observe that indeed the sums of κ along "vertical" and "horizontal" lines in the c-torsion is zero.

The Shintani-type expression which is equal to this is, for z a N-torsion point,

$$\frac{1}{N^2} \sum_{y \in (\mathbb{Z}/N)^2} \exp(-2\pi i \langle z, y \rangle) \left\{ \begin{pmatrix} 1 & \\ -2 & 1 \end{pmatrix}_* S(t_1)S(t_2) \right\} [\widehat{\mathcal{C}}_1]\big|_{t=z}$$

which we can write out more explicitly as

$$\frac{1}{N^2} \sum_{\zeta \in \mu_N^2} \zeta_1^{-z_1}\zeta_2^{-z_2} \sum_{i,j=0}^{10} \widehat{\mathcal{C}}_1[(i,j)] S\left(z_1 - \frac{i}{11}\right) S\left(2z_1 + z_2 - \frac{2i+j}{11}\right)$$

$$= \frac{1}{N^2} \sum_{\zeta \in \mu_N^2} \zeta_1^{-z_1}\zeta_2^{-z_2} \sum_{a,b=0}^{10} \tau(a,b) S\left(z_1 - \frac{a}{11}\right) S\left(2z_1 + z_2 - \frac{b}{11}\right)$$

where

$$\tau(a,b) := \begin{cases} 0 & \text{if } (a,b) \in [5:9] \subset (\mathbb{Z}/11)^2 \\ 1 & \text{otherwise.} \end{cases}$$