An energy-momentum method for ordinary differential equations with an underlying k-symplectic structure

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Abstract

This work reviews the known k-polysymplectic Marsden–Weinstein reduction theory showing previous mistakes and inaccuracies in the literature, providing new results, and stressing the real practical relevance of only apparently minor technical issues. Next, we introduce a new k-symplectic energy-momentum method and apply it to Hamiltonian systems of ordinary differential equations. Several examples of physical and mathematical relevance are detailed.

Keywords: energy-momentum method, k-polysymplectic manifold, Marsden–Weinstein reduction, relative equilibrium point, stability.

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Contents

1 Introduction

The classical energy-momentum method is a technique for the analysis of a Hamiltonian system on a symplectic manifold in the region close to solutions whose evolution is induced by the Lie symmetries of the Hamiltonian system (see [6] for a historical introduction and [34] for one of its foundational works). More specifically, it investigates whether, as time passes by, solutions get closer or move away from the solutions given by Lie symmetries of the Hamiltonian system. The classical energy-momentum method is based on the symplectic Marsden–Weinstein reduction theorem and the use of stability analysis techniques.

The main ideas behind the energy-momentum method can be traced back to Routh, Poincaré, Lyapunov, Arnold, Lewis and Smale, among others (see [6, Section 3.14]). Then, the classical energy-momentum method, devised and developed mainly by J. C. Simo and J. E. Marsden [34], was successfully applied to many problems by numerous researchers [1, 32, 33, 35, 39, 44, 51]. Over the years, the energy-momentum method was extended to deal with more general differential equations, e.g. Hamiltonian systems on Poisson manifolds [34, 48], discrete systems [33, 45], etc. In this work, we develop a new energy-momentum method for Hamiltonian systems related to k-symplectic manifolds [3, 11].

k-Symplectic geometry is a generalisation of symplectic geometry introduced by A. Awane [3, 4], and used later by M. de León et al. [10, 11, 12] and L. K. Norris [36, 38], to describe first-order field theories [7, 18, 42]. k-Symplectic geometry coincides with the *polysymplectic geometry* described by G. C. Günther [23], but it is different from the polysymplectic geometry introduced by G. Sardanashvily et al. [21, 43] and I. V. Kanatchikov [26]. k-Symplectic manifolds have been widely used to study physical systems governed by systems of partial differential equations. In particular, it gives a geometric description of the Euler–Lagrange and the Hamilton–de Donder–Weyl field equations and the systems described by them. For instance, k-symplectic geometry enables us to describe their symmetries, conservation laws, reductions, et cetera [3, 23, 30, 42]. As there are many k-symplectic-like definitions with related but mainly different and even contradictory meanings, it

is relevant to fix properly the terminology. Hereafter, we will deal with *k-polysymplectic manifolds*, i.e. manifolds endowed with a closed nondegenerate differential two-form taking values in a finite-dimensional vector space.

It is remarkable that k-polysymplectic geometry has proved to be useful in the analysis of systems of ordinary differential equations and their so-called superposition rules [16]. It is worth stressing that the study of systems of ordinary differential equations via k-polysymplectic geometry differs substantially from the standard framework, focused on systems of partial differential equations, and opens new fields of research.

More specifically, this work focuses on studying systems of first-order differential equations describing the integral curves of a vector field. Moreover, we assume that the vector field is Hamiltonian relative to a k-polysymplectic manifold, which here amounts to the fact that it is Hamiltonian relative to a series of presymplectic forms whose kernels have zero intersection. We aim to develop an energy-momentum method for such systems of ordinary differential equations with an underlying k-polysymplectic geometry. With this goal, we first review and improve previous works studying k-polysymplectic Marsden–Weinstein reductions [5, 14, 19, 30], which is one of the basis of our k-polysymplectic energy-momentum method.

The first k-symplectic reduction was developed by Awane [3, 13, 23]. Unfortunately, his work was flawed due to the improper analysis of the double orthogonal relative to a k-polysymplectic form. This mistake was fixed in [30], where sufficient conditions to accomplish a k-polysymplectic reduction were established. Despite that, [30, Lemma 3.4] claimed that the Sard's Theorem justifies that the momentum map for k-polysymplectic Hamiltonian systems is generally a submersion. Our work proves that this claim is false as the use of Sard's theorem in [30, Lemma 3.4] is based on a superficial analysis of the relevance of a subset of zero-measure in the image of a smooth map between manifolds. Moreover, practical examples showing that it is convenient to assume that the momentum map in k-polysymplectic geometry is not a submersion are provided. Instead, we stress that it is convenient to use a formalism with k-polysymplectic momentum maps that admit weak regular points, as accomplished in [14]. We highlight that this provides a significant generalisation of the k-polysymplectic Marsden–Weinstein reduction and it completes the analysis performed in [14].

Necessary and sufficient conditions for a k-polysymplectic Marsden–Weinstein reduction were described by Blacker in [5]. Unfortunately, the main theorem, [5, Theorem] has a small misleading typo in the statement of the conditions, as noted in [19], as well as other minor technical issues concerning the existence of certain submanifold structures. This last fact is shown and explained in this work for the first time.

The use of Ad-equivariant momentum maps in k-polysymplectic reductions was removed in [14] by extending to the k-polysymplectic realm the classical theory of affine Lie group actions on symplectic manifolds [40]. Next, García Toraño and Mestdag reviewed in [19] the sufficient conditions for the k-polysymplectic reduction given in [14]. They claimed that one of the sufficient conditions for k-polysymplectic reduction given in [30, Theorem 3.17] is enough to ensure the existence of a k-polysymplectic reduction. In this work, we show a mistake in the proof of the main theorem [19]. Indeed, we here show that [19, Lemma 3.1] is false via several counterexamples. Moreover, our work explains the relations between the sufficient conditions for the k-polysymplectic reduction given in [30, Theorem 3.17].

Next, an energy-momentum method for Hamiltonian k-polysymplectic manifolds is developed. This entails the definition and characterisation of a relative equilibrium notion for k-polysymplectic Hamiltonian systems. Then, some applications are developed. In particular, the so-called automorphic Lie systems [8, 9, 49] related to a quantum harmonic oscillator with a magnetic term are studied. The theory of Lie systems is also used to convert an automorphic Lie system into a Hamiltonian system relative to a k-polysymplectic manifold. It is worth noting that a method to extend this idea to non-automorphic Lie systems is accomplished, which extends the ideas applied to k-

polysymplectic Hamiltonian automorphic Lie systems devised in [16]. A k-polysymplectic form is used to study a complex Schwarz equation, which has been studied through the theory of Lie systems and k-polysymplectic geometry for the first time (see [15] for the analysis of the real, much simpler, case). It is worth noting that the Schwarz equation is related to the Schwarz derivative, which has applications in string theory, modular forms, hypergeometric functions, etcetera [24, 29, 22]. Then, a particular Lie system appearing in the description of a control system is used to illustrate certain aspects of our k-polysymplectic energy-momentum method. Finally, other examples related to differential equations are given and analysed.

The structure of the paper goes as follows. Section 2.1 reviews the basics on Lyapunov stability. In Section 2.2 we review the theory of k-polysymplectic structures and the notion of a Hamiltonian vector field on a k-polysymplectic manifold. The notion of momentum map is extended to k-polysymplectic manifolds. Section 3 is devoted to improve the previous Marsden–Weinstein reduction procedures for k-polysymplectic systems. Section 4 is devoted to develop an energy-momentum method for systems of ODEs with an underlying polysymplectic structure. In particular, we define and characterise the notion of relative equilibria for such systems. Finally, in Section 5 several relevant examples are studied in detail, including the cotangent bundle of k-velocities, the complex Schwarz equation, the product of several symplectic manifolds, a control system, and the quantum harmonic oscillator.

2 Fundamentals

Let us set some general assumptions and notation to be used throughout this work. It is hereafter assumed that all structures are smooth. Manifolds are real, Hausdorff, connected, paracompact, and finite-dimensional. Differential forms are assumed to have constant rank unless otherwise stated. Sum over crossed repeated indices is understood. All our considerations are local to stress our main ideas and to avoid technical problems concerning the global existence of quotient manifolds and similar issues. Hereafter, $\mathfrak{X}(P)$ and $\Omega^k(P)$ stand for the $\mathscr{C}^\infty(P)$-modules of vector fields and differential k-forms on a manifold P.

2.1 Lyapunov stability

Let us establish some fundamental notions and theorems on the stability of dynamical systems used in our k-polysymplectic formulation of the energy-momentum method [17, 50].

Since all manifolds considered in this work are paracompact, they admit a Riemannian metric $\mathbf{g}$ [27]. The topology induced by $\mathbf{g}$ is the one of the manifold [27, 28, 50]. From now on, P stand for manifold. The metric $\mathbf{g}$ induces a distance in P so that the distance between two points $x_1, x_2 \in P$ is given by

$$d(x_1, x_2) := \inf \{\ell_{\mathbf{g}}(\gamma) | \gamma : [0,1] \to P, \gamma(0) = x_1, \gamma(1) = x_2\},$$

where $\ell_{\mathbf{g}}(\gamma)$ is the length of the smooth curve $\gamma : [0,1] \to P$ relative to the metric $\mathbf{g}$. Moreover, consider

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(x), \quad \forall x \in P. \tag{1}$$

A point $x_e \in P$ is an *equilibrium point* of (1), or indistinctly X, if $X(x_e) = 0$. Moreover, x_e is *stable* from $t_0 \in \mathbb{R}$ if, for every ball $B_{x_e,\varepsilon} := \{x \in P | d(x, x_e) < \epsilon\}$, there exists a radius $\delta(\varepsilon, x_e)$ such that every solution $x(t)$ of (1) with initial condition $x(t_0) = x_0 \in B_{x_e,\delta(\varepsilon,x_e)}$ is contained in $B_{x_e,\varepsilon}$ for $t > t_0$. An equilibrium point $x_e \in P$ is *unstable* if it is not stable.

Lyapunov theory studies the stability of equilibrium points of first-order differential equations.

Let $\dot{\mathcal{M}} : P \to \mathbb{R}$ be defined as follows

$$\dot{\mathcal{M}}(x) := \sum_{i=1}^{\dim P} \frac{\partial \mathcal{M}}{\partial x^i}(x) X^i(x) , \qquad \forall x \in P ,$$

where $\{x^1, \ldots, x^{\dim P}\}$ is a local coordinate system around a neighbourhood of the point $x \in P$ and $X = \sum_{i=1}^{\dim P} X^i \frac{\partial}{\partial x^i}$.

Let us recall the basic Lyapunov theorem for autonomous systems (1).

Theorem 2.1. *Let x_e be an equilibrium point of (1) and let $\mathcal{M} : P \to \mathbb{R}$ be a continuous function such that $\mathcal{M}(x_e) = 0$, $\mathcal{M}(x) > 0$, and $\dot{\mathcal{M}}(x) \leq 0$ for every $x \in B_{x_e,r}$ and some $r \in \mathbb{R}^+$. Then, x_e is stable.*

In the literature, the function $\mathcal{M}$ is called a *Lyapunov function* [47].

2.2 On k-polysymplectic momentum maps

This section recalls the basic notions in k-polysymplectic geometry to be used later on, which are consistent in terminology with those in [13]. The latter detail is relevant as a single term may refer to different not equivalent geometric concepts in the literature.

Hereafter, we work with differential ℓ-forms on P that take values in $\mathbb{R}^k$. The space of such forms is denoted by $\Omega^\ell(P, \mathbb{R}^k)$, while its elements will be written in bold. Moreover, $\mathbb{R}^k$ has a fixed basis $\{e_1, \ldots, e_k\}$ giving rise to a dual basis $\{e^1, \ldots, e^k\}$ in $\mathbb{R}^{k*}$. Hence, an element $\boldsymbol{\omega} \in \Omega^\ell(P, \mathbb{R}^k)$ can always be written as $\boldsymbol{\omega} = \omega^\alpha \otimes e_\alpha$ for some uniquely defined differential ℓ-forms $\omega^1, \ldots, \omega^k$ on P. A differential ℓ-form on P taking values in $\mathbb{R}^k$, let us say $\boldsymbol{\omega}$, is nondegenerate if

$$\ker \boldsymbol{\omega} = \ker(\omega^\alpha \otimes e_\alpha) = \bigcap_{\alpha=1}^{k} \ker \omega^\alpha = 0 .$$

Let us introduce the following definition that will be useful to simplify the notation of our further work. Let $\boldsymbol{\vartheta} = \vartheta^\alpha \otimes e_\alpha \in \Omega^\ell(P, \mathbb{R}^k)$ be a $\mathbb{R}^k$-valued differential ℓ-form on P. Then, the contraction of $\boldsymbol{\vartheta}$ with a vector field $X \in \mathfrak{X}(P)$ is defined as

$$\iota_X \boldsymbol{\vartheta} = (\iota_X \vartheta^\alpha) \otimes e_\alpha \in \Omega^{\ell-1}(P, \mathbb{R}^k) = \langle \boldsymbol{\vartheta}, X \rangle .$$

In short, the exterior differential, the Lie derivative with respect to vector fields, and many other operations on differential forms can be naturally extended to ℓ-differential forms taking values in vector spaces by considering their natural action on their components and extending them to $\Omega^\ell(P, \mathbb{R}^k)$ by linearity.

k-Vector fields on a manifold P are vector fields taking values in $\mathbb{R}^k$. We write $\mathfrak{X}(P, \mathbb{R}^k)$ for the space of k-vector fields on P and its elements will be written in bold. Moreover, a k-vector field, let us say $\mathbf{X}$, can always be written in a unique manner as $\mathbf{X} = X^\alpha \otimes e_\alpha$ for a family $X_1, \ldots, X_k$ of vector fields on P. The contraction of a k-vector field $\mathbf{X} = X^\alpha \otimes e_\alpha$ with a k-differential forms $\boldsymbol{\omega} = \omega^\alpha \otimes e_\alpha$ is defined as follows

$$\iota_{\mathbf{X}} \boldsymbol{\omega} = \iota_{X_\alpha} \omega^\alpha = \langle \boldsymbol{\omega}, \mathbf{X} \rangle.$$

Now, let us turn to the main fundamental notion to be studied in this paper.

Definition 2.2. A *k-polysymplectic form* on P is a closed nondegenerate $\mathbb{R}^k$-valued differential two-form $\boldsymbol{\omega}$. The pair $(P, \boldsymbol{\omega})$ is called a *k-polysymplectic manifold*.

Consider a k-polysymplectic manifold $(P,\boldsymbol{\omega})$, and let $W_p \subset \mathrm{T}_pP$. The *$k$-polysymplectic orthogonal complement* of W at $p\in P$ with respect to $(P,\boldsymbol{\omega})$ is

$$W_p^{\perp,k}=\{v_p\in T_pP\,|\,\boldsymbol{\omega}(w_p,v_p)=0\,,\quad \forall w_p\in W_p\}.$$

k-Polysymplectic manifolds are called, for simplicity, polysymplectic manifolds in the literature [30]. Nevertheless, the latter term may be misleading as refers here to a different concept shown below. Hence, to avoid confusion, we will use the full term *k-polysymplectic manifold*. Let us define polysymplectic, k-polysymplectic manifolds, and related notions.

Definition 2.3. Let P be an $n(k+1)$-dimensional manifold. Then,

- A *polysymplectic form* on P is a nondegenerate differential two-form, $\boldsymbol{\omega}$, taking values in $\mathbb{R}^k$. We call $(P,\boldsymbol{\omega})$ a *polysymplectic manifold*.
- A *k-symplectic structure* on P is a pair $(\boldsymbol{\omega},\mathcal{D})$, where $(P,\boldsymbol{\omega})$ is a polysymplectic manifold and $\mathcal{D}\subset \mathrm{T}P$ is an integrable distribution on P of rank nk such that
$$\boldsymbol{\omega}|_{\mathcal{D}\times\mathcal{D}}=0\,.$$
In this case, $(P,\boldsymbol{\omega},\mathcal{D})$ is a *k-symplectic manifold*. We call $\mathcal{D}$ a *polarisation* of $(P,\boldsymbol{\omega})$.

If the two-form $\boldsymbol{\omega}$ is exact, namely $\boldsymbol{\omega}=\mathrm{d}\boldsymbol{\theta}$ for some $\boldsymbol{\theta}\in\Omega^1(P,\mathbb{R}^k)$, in any of the notions in Definition 2.3, then such concepts are said to be *exact*.

Note that the difference between polysymplectic and k-polysymplectic manifolds relies on the fact that in the k-polysymplectic case the dimension of the manifold is proportional to $k+1$.

2.3 On $\boldsymbol{\omega}$-Hamiltonian functions and vector fields

Let us survey the basic theory on k-polysymplectic vector fields and functions. Recall that we will not be concerned with the local or global character of the structures to be defined next.

Definition 2.4. Given a k-polysymplectic manifold $(P,\boldsymbol{\omega}=\omega^\alpha\otimes e_\alpha)$, a vector field $Y\in\mathfrak{X}(P)$ is *$\boldsymbol{\omega}$-Hamiltonian* if it is Hamiltonian with respect to all the presymplectic forms $\omega^1,\dots,\omega^k$, namely $\iota_Y\omega^\alpha$ is closed for $\alpha=1,\dots,k$. Let us denote by $\mathfrak{X}_{\boldsymbol{\omega}}(P)$ the space of $\boldsymbol{\omega}$-Hamiltonian vector fields in a k-polysymplectic manifold $(P,\boldsymbol{\omega})$.

Note that if $\iota_Y\omega^\alpha$ is closed, then it admits, in general only locally, a potential function. Anyhow, this work is mainly concerned with local aspects and the local character of the potential function will not have any repercussions in what follows.

Every $\boldsymbol{\omega}$-Hamiltonian vector field for a k-polysymplectic manifold can be associated with a family $h^1,\dots,h^k$ of Hamiltonian functions (each one relative to a different coordinate presymplectic form of the k-symplectic form $\boldsymbol{\omega}$). It is convenient for the study of k-polysymplectic Hamiltonian vector fields to introduce some generalisation of the Hamiltonian function notion for presymplectic forms to deal simultaneously with all $h^1,\dots,h^k$ (see [3, 16] for details).

Definition 2.5. Given a k-polysymplectic manifold $(P,\boldsymbol{\omega}=\sum_{\alpha=1}^k\omega^\alpha\otimes e_\alpha)$, we say that $\boldsymbol{h}=h^1\otimes e_1+\cdots+h^k\otimes e_k$ is an *$\boldsymbol{\omega}$-Hamiltonian function* if there exists a vector field $X_{\boldsymbol{h}}$ on P such that $\iota_{X_{\boldsymbol{h}}}\boldsymbol{\omega}=d\boldsymbol{h}$, namely $\iota_{X_{\boldsymbol{h}}}\omega^\alpha=\mathrm{d}h^\alpha$ for $\alpha=1,\dots,k$. In this case, we call $\boldsymbol{h}$ an *$\boldsymbol{\omega}$-Hamiltonian function* for $X_{\boldsymbol{h}}$. We write $\mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$ for the space of $\boldsymbol{\omega}$-Hamiltonian functions of $(P,\boldsymbol{\omega})$.

An $\boldsymbol{\omega}$-Hamiltonian function is a certain type of $\mathbb{R}^k$-valued Hamiltonian function. In [37], the author defined the k-Hamiltonian system associated with the $\mathbb{R}^k$-valued Hamiltonian function $\boldsymbol{h}$ as the vector field $X_{\boldsymbol{h}}$ of the above definition. Moreover, A. Awane [3] called $\boldsymbol{h}$ a Hamiltonian map of X when X is additionally an infinitesimal automorphism of a certain distribution on which it is assumed that the presymplectic forms of the k-symplectic distribution vanish.

Example 2.6. Consider the 2-polysymplectic manifold $(\mathbb{R}^3, \boldsymbol{\omega})$, where $\{u, v, w\}$ are linear coordinates on $\mathbb{R}^3$ and $\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2$, where (see [16] for details)

$$\omega^1 = \frac{4w}{v^2}\mathrm{d}u \wedge \mathrm{d}w + \frac{1}{v}\mathrm{d}v \wedge \mathrm{d}w + \frac{4w^2}{v^3}\mathrm{d}u \wedge \mathrm{d}v\,, \qquad \omega^2 = \frac{4}{v^2}\mathrm{d}u \wedge \mathrm{d}w + \frac{8w}{v^3}\mathrm{d}u \wedge \mathrm{d}v,$$

is a two-polysymplectic form. The vector fields

$$X_1 = 4u^2\frac{\partial}{\partial u} + 4uv\frac{\partial}{\partial v} + v^2\frac{\partial}{\partial w}\,, \qquad X_2 = \frac{\partial}{\partial u}\,,$$

are $\boldsymbol{\omega}$-Hamiltonian with $\boldsymbol{\omega}$-Hamiltonian functions

$$\boldsymbol{f} = \left(4uw - 8\frac{u^2w^2}{v^2} - \frac{v^2}{2}\right) \otimes e_1 + \left(4u - 16\frac{u^2w}{v^2}\right) \otimes e_2\,, \quad \boldsymbol{g} = -2\frac{w^2}{v^2} \otimes e_1 - 4\frac{w}{v^2} \otimes e_2\,,$$

respectively, relative to the two-polysymplectic form $\boldsymbol{\omega}$. △

Every $\boldsymbol{\omega}$-Hamiltonian vector field is associated with at least one $\boldsymbol{\omega}$-Hamiltonian function. Conversely, every $\boldsymbol{\omega}$-Hamiltonian function induces a unique $\boldsymbol{\omega}$-Hamiltonian vector field.

Proposition 2.7. *The space $\mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$ becomes a Lie algebra when endowed with the natural operations*

$$\boldsymbol{h} + \boldsymbol{g} \equiv \sum_{\alpha=1}^{k}(h^\alpha + g^\alpha) \otimes e_\alpha\,, \qquad \lambda \cdot \boldsymbol{h} \equiv \sum_{\alpha=1}^{k} \lambda h^\alpha \otimes e_\alpha\,,$$

where $\boldsymbol{h} = \sum_{\alpha=1}^k h^\alpha \otimes e_\alpha$, $\boldsymbol{g} = \sum_{\alpha=1}^k g^\alpha \otimes e_\alpha \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$, $\lambda \in \mathbb{R}$, and the Lie bracket $\{\cdot,\cdot\}_{\boldsymbol{\omega}} : \mathscr{C}^\infty_{\boldsymbol{\omega}}(P) \times \mathscr{C}^\infty_{\boldsymbol{\omega}}(P) \to \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$ of the form

$$\{h^1 \otimes e_1 + \cdots + h^k \otimes e_k, g^1 \otimes e_1 + \cdots + g^k \otimes e_k\}_{\boldsymbol{\omega}} = \{h^1, g^1\}_{\omega^1} \otimes e_1 + \cdots + \{h^k, g^k\}_{\omega^k} \otimes e_k,$$

where $\{\cdot,\cdot\}_{\omega^\alpha}$ is the Poisson bracket naturally induced by the presymplectic form ω^α, with $\alpha = 1, \ldots, k$.

Since $(\mathscr{C}^\infty_{\boldsymbol{\omega}}(P), \cdot, \{\cdot,\cdot\}_{\boldsymbol{\omega}})$ is not in general a Poisson algebra [16], the map $\{\boldsymbol{h}, \cdot\}_{\boldsymbol{\omega}} : \boldsymbol{g} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P) \mapsto \{\boldsymbol{g}, \boldsymbol{h}\}_{\boldsymbol{\omega}} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$, with $\boldsymbol{h} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$, is not in general a derivation with respect to the product of $\boldsymbol{\omega}$-Hamiltonian functions given by

$$\boldsymbol{h} \cdot \boldsymbol{g} = (h^1 g^1) \otimes e_1 + \cdots + (h^k g^k) \otimes e_k\,.$$

Hence, k-polysymplectic geometry is quite different from Poisson and presymplectic geometry, where an equivalent of this result holds. Nevertheless, $\{h, g\}_\Omega = 0$ for every locally constant function g and any $h \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$. Moreover, this Lie algebra admits other properties, as shown next.

Proposition 2.8. *Consider a k-polysymplectic manifold $(P, \boldsymbol{\omega})$. Every $\boldsymbol{\omega}$-Hamiltonian vector field $X_{\boldsymbol{h}}$ acts as a derivation on the Lie algebra $(\mathscr{C}^\infty_{\boldsymbol{\omega}}(P), \{\cdot,\cdot\}_{\boldsymbol{\omega}})$ in the form*

$$X_{\boldsymbol{h}}\boldsymbol{f} = \{\boldsymbol{f}, \boldsymbol{h}\}_{\boldsymbol{\omega}}\,, \qquad \forall \boldsymbol{f} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)\,,$$

where $\boldsymbol{h}$ is an $\boldsymbol{\omega}$-Hamiltonian function for X.

2.4 k-polysymplectic momentum maps

Let us survey the theory of k-polysymplectic momentum maps. Note that the presented results are not restricted to Ad^{*k}-equivariant momentum maps (see [14] for further details).

Definition 2.9. A Lie group action $\Phi\colon G\times P\to P$ on a k-polysymplectic manifold $(P,\boldsymbol{\omega})$ is a *k-polysymplectic Lie group action* if $\Phi_g^*\boldsymbol{\omega}=\boldsymbol{\omega}$ for each $g\in G$. In other words,

$$\mathscr{L}_{\xi_P}\boldsymbol{\omega}=0,\qquad \forall\xi\in\mathfrak{g},$$

where ξ_M is the fundamental vector field of Φ related to $\xi\in\mathfrak{g}$, namely $\xi_P(p)=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Phi(\exp(t\xi),p)$ for any $p\in P$.

Definition 2.10. A *k-polysymplectic momentum map* for a Lie group action $\Phi:G\times P\to P$ with respect to a k-polysymplectic manifold $(P,\boldsymbol{\omega})$ is a map $\mathbf{J}^\Phi:P\to(\mathfrak{g}^*)^k$ such that

$$\iota_{\xi_P}\boldsymbol{\omega}:=(\iota_{\xi_P}\omega^\alpha)\otimes e_\alpha=\mathrm{d}\left\langle\mathbf{J}^\Phi,\xi\right\rangle\,,\qquad\forall\xi\in\mathfrak{g}\,.\tag{2}$$

Equation (2) implies that $\mathbf{J}^\Phi:P\to(\mathfrak{g}^*)^k$ satisfies

$$\iota_{\boldsymbol{\xi}_P}\boldsymbol{\omega}=\mathrm{d}\left\langle\mathbf{J}^\Phi,\boldsymbol{\xi}\right\rangle\,,\qquad\forall\boldsymbol{\xi}\in\mathfrak{g}^k\,.$$

and conversely.

Before continuing studying k-polysymplectic momentum maps, recall that every Lie group G gives rise to Lie group action $I:(g,h)\in G\times G\mapsto I_g(h)=ghg^{-1}\in G$, such that $I_g:h\in G\mapsto I(g,h)\in G$, with $g\in G$. Then, the *adjoint action* of G on its Lie algebra $\mathfrak{g}$ reads $\mathrm{Ad}:(g,v)\in G\times\mathfrak{g}\mapsto\mathrm{Ad}_g(v)=\mathrm{T}_eI_g(v)\in\mathfrak{g}$. In turn, the *co-adjoint action* becomes $\mathrm{Ad}^*:(g,\vartheta)\in G\times\mathfrak{g}^*\mapsto\mathrm{Ad}^*_{g^{-1}}\vartheta=\vartheta\circ\mathrm{Ad}_{g^{-1}}\in\mathfrak{g}^*$.

The following definition has been widely used in the literature [30], although we will see that the Ad^{*k}-equivariance condition is no longer necessary (see [14] for details). Moreover, we have changed the standard notation Coad^k to Ad^{*k} to shorten it.

Definition 2.11. A k-polysymplectic momentum map $\mathbf{J}^\Phi:P\to(\mathfrak{g}^*)^k$ is Ad^{*k}*-equivariant* if

$$\mathbf{J}^\Phi\circ\Phi_g=\mathrm{Ad}^{*k}_{g^{-1}}\circ\mathbf{J}^\Phi\,,\quad\forall g\in G\,,$$

where $\mathrm{Ad}^{*k}_{g^{-1}}=\mathrm{Ad}^*_{g^{-1}}\otimes\overset{(k)}{\cdots}\otimes\mathrm{Ad}^*_{g^{-1}}$ and

$$\begin{array}{rccc}\mathrm{Ad}^{*k} & :G\times(\mathfrak{g}^*)^k & \longrightarrow & (\mathfrak{g}^*)^k\\ & (g,\boldsymbol{\mu}) & \longmapsto & \mathrm{Ad}^{*k}_{g^{-1}}\boldsymbol{\mu}\end{array}\,,\qquad \begin{tikzcd} P \arrow[r, "\mathbf{J}^\Phi"] \arrow[d, "\Phi_g"] & (\mathfrak{g}^*)^k \arrow[d, "\mathrm{Ad}^{*k}_{g^{-1}}"] \\ P \arrow[r, "\mathbf{J}^\Phi"] & (\mathfrak{g}^*)^k, \end{tikzcd}$$

where the above diagram is commutative for every $g\in G$.

To simplify the notation, let us introduce the following definition.

Definition 2.12. A *G-invariant $\boldsymbol{\omega}$-Hamiltonian system* is a tuple $(P,\boldsymbol{\omega},\boldsymbol{h},\mathbf{J}^\Phi)$, where $(P,\boldsymbol{\omega})$ is a k-polysymplectic manifold, $\boldsymbol{h}$ is a $\boldsymbol{\omega}$-Hamiltonian function associated with $X_{\boldsymbol{h}}$, the map $\Phi:G\times P\to P$ is a k-polysymplectic Lie group action satisfying $\Phi_g^*\boldsymbol{h}=\boldsymbol{h}$ for every $g\in G$, and $\mathbf{J}^\Phi$ is a k-polysymplectic momentum map related to Φ. An *Ad^{*k}-equivariant G-invariant $\boldsymbol{\omega}$-polysymplectic Hamiltonian system* is a G-invariant $\boldsymbol{\omega}$-Hamiltonian system $(P,\boldsymbol{\omega},\boldsymbol{h},\mathbf{J}^\Phi)$ such that $\mathbf{J}^\Phi$ is Ad^{*k}-equivariant.

Let us provide the formalism needed to avoid the Ad^{*k}-equivariantness.

Proposition 2.13. *Let $(P,\boldsymbol{\omega},\boldsymbol{h},\mathbf{J}^\Phi)$ be a G-invariant $\boldsymbol{\omega}$-Hamiltonian system. If*

$$\boldsymbol{\psi}_{g,\boldsymbol{\xi}}:P\ni x\longmapsto\mathbf{J}^\Phi_{\boldsymbol{\xi}}(\Phi_g(x))-\mathbf{J}^\Phi_{\mathrm{Ad}^k_{g^{-1}}\boldsymbol{\xi}}(x)\in\mathbb{R}\,,\quad\forall g\in G\,,\quad\forall\boldsymbol{\xi}\in\mathfrak{g}^k\,,$$

then $\boldsymbol{\psi}_{g,\boldsymbol{\xi}}$ is constant on P for every $g\in G$ and $\boldsymbol{\xi}\in\mathfrak{g}^k$. Moreover, $\boldsymbol{\sigma}:G\ni g\mapsto\boldsymbol{\sigma}(g)\in(\mathfrak{g}^*)^k$ uniquely determined by the condition $\langle\boldsymbol{\sigma}(g),\boldsymbol{\xi}\rangle=\boldsymbol{\psi}_{g,\boldsymbol{\xi}}$ and satisfies

$$\boldsymbol{\sigma}(g_1g_2)=\boldsymbol{\sigma}(g_1)+\mathrm{Ad}^{*k}_{g_1^{-1}}\boldsymbol{\sigma}(g_2)\,,\quad\forall g_1,g_2\in G\,.$$

The map $\boldsymbol{\sigma} : G \to (\mathfrak{g}^*)^k$ of the form

$$\boldsymbol{\sigma}(g) = \mathbf{J}^\Phi \circ \Phi_g - \mathrm{Ad}^{*k}_{g^{-1}} \mathbf{J}^\Phi, \qquad g \in G,$$

is called the *co-adjoint cocycle* associated with the k-polysymplectic momentum map $\mathbf{J}^\Phi$ on P. Moreover, $\mathbf{J}^\Phi$ is an Ad^{*k}-equivariant k-polysymplectic momentum map if and only if $\boldsymbol{\sigma} = 0$.

A map $\boldsymbol{\sigma} : G \to (\mathfrak{g}^*)^k$ is a *coboundary* if there exists $\boldsymbol{\mu} \in (\mathfrak{g}^*)^k$ such that

$$\boldsymbol{\sigma}(g) = \boldsymbol{\mu} - \mathrm{Ad}^{*k}_{g^{-1}} \boldsymbol{\mu}, \qquad \forall g \in G.$$

Proposition 2.14. *Let $\mathbf{J}^\Phi : P \to (\mathfrak{g}^*)^k$ be a k-polysymplectic momentum map related to a k-polysymplectic action $\Phi : G \times P \to P$ with co-adjoint cocycle $\boldsymbol{\sigma}$. Then,*

(1) *there exists a Lie group action of G on $(\mathfrak{g}^*)^k$ of the form*

$$\boldsymbol{\Delta} : G \times (\mathfrak{g}^*)^k \ni (g, \boldsymbol{\mu}) \mapsto \boldsymbol{\sigma}(g) + \mathrm{Ad}^{*k}_{g^{-1}} \boldsymbol{\mu} = \boldsymbol{\Delta}_g(\boldsymbol{\mu}) \in (\mathfrak{g}^*)^k,$$

(2) *the k-polysymplectic momentum map $\mathbf{J}^\Phi$ is equivariant with respect to $\boldsymbol{\Delta}$, in other words, for every $g \in G$, one has the commutative diagram aside.*

$$\begin{array}{ccc} P & \xrightarrow{\mathbf{J}^\Phi} & (\mathfrak{g}^*)^k \\ \big\downarrow{\scriptstyle \Phi_g} & & \big\downarrow{\scriptstyle \boldsymbol{\Delta}_g} \\ P & \xrightarrow{\mathbf{J}^\Phi} & (\mathfrak{g}^*)^k . \end{array}$$

Proposition 2.14 ensures that every k-polysymplectic momentum map $\mathbf{J}^\Phi$ gives rise to an equivariant k-polysymplectic momentum map relative to a new action $\boldsymbol{\Delta} : G \times (\mathfrak{g}^*)^k \to (\mathfrak{g}^*)^k$, called a *$k$-polysymplectic affine Lie group action*. Note that an affine Lie group action can also be expressed by writing $\boldsymbol{\Delta}(g, \boldsymbol{\mu}) = (\Delta^1_g \mu^1, \ldots, \Delta^k_g \mu^k) \in (\mathfrak{g}^*)^k$, where the mappings $\Delta^1, \ldots, \Delta^k$ take the form $\Delta^\alpha : G \times \mathfrak{g}^* \ni (g, \vartheta) \mapsto \mathrm{Ad}^*_{g^{-1}} \vartheta + \sigma^\alpha(g) = \Delta^\alpha_g(\vartheta) \in \mathfrak{g}^*$ and $\boldsymbol{\sigma}(g) = (\sigma^1(g), \ldots, \sigma^k(g))$, where $\sigma^\alpha(g) = \mathbf{J}^\Phi_\alpha \circ \Phi_g - \mathrm{Ad}^*_{g^{-1}} \mathbf{J}^\Phi_\alpha$ for $\alpha = 1, \ldots, k$ and $\mathbf{J}^\Phi_1, \ldots, \mathbf{J}^\Phi_k$ are the coordinates of $\mathbf{J}^\Phi$.

3 k-Polysymplectic Marsden–Weinstein reduction

Let us now review previous results in the literature for the k-polysymplectic reduction so as to correct previous mistakes and inaccuracies. In particular, this section first reviews the previous k-polysymplectic Marsden–Weinstein reduction theory and explains some, only apparently, minor inaccuracies. After that, we focus on solving a mistake in one of the main results in [19], concerning the conditions to obtain a k-polysymplectic reduction. Finally, in Section , we analyse the relations between the conditions for the k-polysymplectic reduction given in [30].

3.1 A review on k-polysymplectic Marsden–Weinstein reduction

Let us recall several definitions that are useful for what follows. Some technical assumptions will be set to improve the applicability of k-polysymplectic reductions. First, a *weak regular value* of a mapping $\phi : M \to N$ is a point $x_0 \in N$ such that $\phi^{-1}(x_0)$ is a submanifold and $\ker \mathrm{T}_p\phi = \mathrm{T}_p[\phi^{-1}(x_0)]$ for every $p \in \phi^{-1}(x_0)$. In particular, regular values of ϕ are weak regular values too. Moreover, a Lie group action $\phi : G \times M \to M$ is quotientable when the space of orbits G acting on M, let us say M/G is a manifold and the projection $\pi : M \to M/G$ is an open surjective submersion. In particular, this occurs when ϕ is free and proper.

Let us comment on the the regular values of k-polysymplectic momentum maps. The codomain of a k-polysymplectic momentum map $\mathbf{J}^\Phi$ may have a large dimension, even larger than the dimension of P, due to the presence of k copies of $\mathfrak{g}^*$. This implies that it may be impossible for $\mathbf{J}^\Phi$ to be a submersion when k is different from one. Being a submersion is the typical condition used in many types of Marsden–Weinstein reductions [19, 30]. But this property is harder to satisfy in k-polysymplectic geometry.

It is worth noting that it is sometimes claimed in the literature that the Sard's theorem ensures that $\mathbf{J}^\Phi$ is frequently a submersion because the set of singular points of $\mathbf{J}^\Phi$ has zero measure in the image of $\mathbf{J}^\Phi$ (see [31, Lemma 3.4]). Nevertheless, it may happen that the whole image of $\mathbf{J}^\Phi$ is also a zero measure set and every point of P has a singular image by $\mathbf{J}^\Phi$. This always happens when $k \dim \mathfrak{g}^* > \dim P$. That is why using weak regular values employed in [14] is relevant. It is worth noting that Blacker in [6, Theorem 3.22] does not provide any explicit assumption in the structure of $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, although it is implicitly assumed that $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ is a manifold. Blacker also assumes in [6, Theorem 3.28], and somehow implicitly in remaining parts of the work, that the reduced space is an orbifold and the level sets of the momentum map behave as in the case of having an associated weak regular point, but the notion is not mentioned at all. In general, Blacker's work [6] does not analyse the technical conditions needed to make the k-polysymplectic Marsden–Weinstein reduction work in practical cases.

The following lemma allows one to characterise the so-called k-polysymplectic relative equilibrium points on P, which will be defined in the next section. Moreover, it is used to prove k-polysymplectic Marsden–Weinstein reduction theorems. The proof of this lemma can be found in [14].

Lemma 3.1. *Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a G-invariant $\boldsymbol{\omega}$-Hamiltonian system and let $\boldsymbol{\mu} \in (\mathfrak{g}^*)^k$ be a weak regular value of $\mathbf{J}^\Phi : P \to (\mathfrak{g}^*)^k$. Then, for every $p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, one has*

(1) $\mathrm{T}_p(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} p) = \mathrm{T}_p(Gp) \cap \mathrm{T}_p\left(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\right)$,

(2) $\mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = \mathrm{T}_p(Gp)^{\perp,k}$.

Let us review the conditions of the k-polysymplectic Marsden–Weinstein reduction theorem, which will be crucial in the k-polysymplectic energy-momentum method to correct a mistake in one of the main results in [19], in fact, the one, [19, Proposition 1] giving the name to the paper. The first correct proof of the k-polysymplectic Marsden–Weinstein theorem can be found in [30]. The necessary and sufficient conditions to perform a reduction were given by C. Blacker in [5], although there is a relevant typo in his theorem, as commented in [19]. The original k-polysymplectic Marsden–Weinstein reduction theorem was originally proved under the assumption that the k-polysymplectic momentum map $\mathbf{J}^\Phi : P \to (\mathfrak{g}^*)^k$ is Ad^{*k}-equivariant. A version of the k-polysymplectic Marsden–Weinstein theorem without this condition was accomplished in [14]. In its correct and most modern form, the reduction theorem reads as follows (see [14, Theorem 5.10] for details).

Theorem 3.2 (k-polysymplectic Marsden–Weinstein reduction theorem)**.** *Consider a G-invariant $\boldsymbol{\omega}$-Hamiltonian system $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$. Assume that $\boldsymbol{\mu} \in (\mathfrak{g}^*)^k$ is a weak regular value of $\mathbf{J}^\Phi$ and $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ acts in a quotientable manner on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Let $G^{\Delta^\alpha}_{\mu^\alpha}$ denote the isotropy group at μ^α of the Lie group action $\Delta^\alpha : (g, \vartheta) \in G \times \mathfrak{g}^* \mapsto \Delta^\alpha(g, \vartheta) \in \mathfrak{g}^*$ for $\alpha = 1, \ldots, k$. Moreover, let the following (sufficient) conditions hold*

$$\ker(\mathrm{T}_p \mathbf{J}^\Phi_\alpha) = \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) + \ker \omega^\alpha_p + \mathrm{T}_p(G^{\Delta^\alpha}_{\mu^\alpha} p)\,, \tag{3}$$

$$\mathrm{T}_p(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} p) = \bigcap_{\alpha=1}^{k} \left(\ker \omega^\alpha_p + \mathrm{T}_p(G^{\Delta^\alpha}_{\mu^\alpha} p)\right) \cap \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}))\,, \tag{4}$$

for every $p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ and all $\alpha = 1, \ldots, k$. Then, $(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}, \boldsymbol{\omega}_{\boldsymbol{\mu}})$ is a k-polysymplectic manifold, with $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ being uniquely determined by

$$\pi^*_{\boldsymbol{\mu}} \boldsymbol{\omega}_{\boldsymbol{\mu}} = \jmath^*_{\boldsymbol{\mu}} \boldsymbol{\omega}$$

where $\jmath_{\boldsymbol{\mu}} : \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \hookrightarrow P$ is the canonical immersion and $\pi_{\boldsymbol{\mu}} : \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \to \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ is the canonical projection.

Let us recall the sufficient and necessary conditions for a k-polysymplectic reduction given by Blacker in (5). His result is described with our notation and we have corrected the typo in [5, Theroem 3.22] on the k-symplectic reduction. Theorem 3.3 dearly states some technical conditions that were not written in [5, Theorem 3.22].

Theorem 3.3. *Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a Ad^{*k}-equivariant G-invariant $\boldsymbol{\omega}$-polysymplectic Hamiltonian system and let $\boldsymbol{\mu} \in (\mathfrak{g}^*)^k$ be fixed and be a regular value for $\mathbf{J}^\Phi$. If the stabiliser subgroup $G_{\boldsymbol{\mu}}$ of $\boldsymbol{\mu}$ under the Ad^{*k} action is connected, and $P_{\boldsymbol{\mu}} = \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$ is a smooth manifold, then there is a unique $\mathbb{R}^k$-valued two-form $\boldsymbol{\omega}_{\boldsymbol{\mu}} \in \Omega^2(P_{\boldsymbol{\mu}}, \mathbb{R}^k)$ such that $\pi^*_{\boldsymbol{\mu}}\boldsymbol{\omega}_{\boldsymbol{\mu}} = \jmath^*_{\boldsymbol{\mu}}\boldsymbol{\omega}_{\boldsymbol{\mu}}$ where $\jmath_{\boldsymbol{\mu}} : \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \to P$ is the inclusion and $\pi_{\boldsymbol{\mu}} : \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \to M_{\boldsymbol{\mu}}$ is the projection. The form $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ is closed and is nondegenerate if and only if*

$$\mathrm{T}_p(G_{\boldsymbol{\mu}}p) = (\mathrm{T}_p(Gp)^{\perp,k})^{\perp,k} \cap \mathrm{T}_p(Gp)^{\perp,k}. \tag{5}$$

The following theorem reduces a $\boldsymbol{\omega}$-Hamiltonian vector field $X_{\boldsymbol{h}}$ on P, which will be essential for the k-polysymplectic energy-momentum method.

Theorem 3.4. *Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a G-invariant $\boldsymbol{\omega}$-Hamiltonian systems and let $\Phi_{g*}X_{\boldsymbol{h}} = X_{\boldsymbol{h}}$ for each $g \in G$. Then, the flow F_t of the vector field $X_{\boldsymbol{h}}$ induces the flow $\mathcal{F}_t$ of the vector field $X_{\boldsymbol{h}_{\boldsymbol{\mu}}}$ on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ such that $\iota_{X_{\boldsymbol{h}_{\boldsymbol{\mu}}}}\boldsymbol{\omega}_{\boldsymbol{\mu}} = \mathrm{d}\boldsymbol{h}_{\boldsymbol{\mu}}$ and $i^*_{\boldsymbol{\mu}}\boldsymbol{h} = \pi^*_{\boldsymbol{\mu}}\boldsymbol{h}_{\boldsymbol{\mu}}$.*

3.2 On the sufficiency of conditions for the k-polysymplectic reduction

It was claimed in [19, Proposition 1] that condition (4) is enough to ensure that there exists a k-polysymplectic Marsden–Weinstein reduction. Let us show that this is not true. First, the proof for [19, Proposition 1] has a mistake, as there is an inclusion written in the opposite way. In particular, since $\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \subset \mathrm{T}_p\mathbf{J}^{\Phi-1}_\alpha(\mu^\alpha)$ for $\alpha = 1, \ldots, k$ and every $p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ for a regular $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$, one has

$$\begin{aligned}&\left\{v \in \mathrm{T}_pP \,|\, \omega^1(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}_1(\mu^1)) = \cdots = \omega^k(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}_k(\mu^k)) = 0\right\}\\ &\qquad\qquad \subset \left\{v \in \mathrm{T}_pP \,|\, \omega^1(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu)) = \cdots = \omega^k(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = 0\right\}\end{aligned}$$

instead of

$$\begin{aligned}&\left\{v \in \mathrm{T}_pP \,|\, \omega^1(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}_1(\mu^1)) = \cdots = \omega^k(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}_k(\mu^k)) = 0\right\}\\ &\qquad\qquad \supset \left\{v \in \mathrm{T}_pP \,|\, \omega^1(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = \cdots = \omega^k(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = 0\right\}\end{aligned}$$

as claimed at the end of page 8 in the proof of [19, Proposition 1]. In other words, if v is perpendicular to $\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ relative to each ω^α, one cannot infer that v is perpendicular to each $\mathrm{T}_p\mathbf{J}^{\Phi-1}_\alpha(\mu^\alpha)$ relative to ω^α for $\alpha = 1, \ldots, k$, since the latter conditions are more restrictive. Then, the proof of Proposition 1 only gives

$$\bigcap_{a=1}^k (\ker j^*_{\mu_a}\omega^a|_p) \cap \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \subset (\mathrm{T}_p(Gp)^{\perp,k})^{\perp,k} \cap \mathrm{T}_p(Gp)^{\perp,k}\,, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,,$$

instead of the claimed

$$\bigcap_{a=1}^k (\ker j^*_{\mu_a}\omega^a|_p) \cap \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \supset (\mathrm{T}_p(Gp)^{\perp,k})^{\perp,k} \cap \mathrm{T}_p(Gp)^{\perp,k}\,, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,,$$

which makes the proof of Proposition 1 fail in proving (5), namely the k-polysymplectic reduction condition, and, therefore, the statement of Proposition 1. Indeed, the above mistake just illustrates the fact that Proposition 1 is false and the comments that follow in [19] contain some inaccuracies.

Example 3.5. Let us provide a counterexample to show that [19, Proposition 1] does not hold. More specifically, we here describe an an $\mathbb{R}$-invariant $\boldsymbol{\omega}$-Hamiltonian system relative to a two-symplectic form satisfying condition (4) but not giving rise to a k-polysymplectic reduction.

Consider $\mathbb{R}^4$ with linear coordinates $\{x, y, z, t\}$ and the presymplectic forms

$$\omega^1 = \mathrm{d}x \wedge \mathrm{d}y\,, \qquad \omega^2 = \mathrm{d}x \wedge \mathrm{d}t + \mathrm{d}y \wedge \mathrm{d}z\,,$$

which give rise to a 2-polysymplectic form $\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2$ since ω^2 is a symplectic form and therefore $\ker \omega^1 \cap \ker \omega^2 = 0$. Consider the Lie group action $\Phi : (\lambda; x, y, z, t) \in \mathbb{R} \times \mathbb{R}^4 \mapsto (x + \lambda, y, z, t) \in \mathbb{R}^4$. The Lie algebra of fundamental vector fields of Φ is $V = \langle \partial_x \rangle \simeq \mathbb{R}$. Moreover, Φ admits a momentum map relative to $(\mathbb{R}^4, \omega^1, \omega^2)$ given by

$$\mathbf{J}^\Phi : (x, y, z, t) \in \mathbb{R}^4 \longmapsto \boldsymbol{\mu} = (y, t) \in (\mathbb{R}^*)^2,$$

which is clearly Ad^{*2}-equivariant. Additionally, $\mathbf{J}^\Phi$ is regular for every value of $(\mathbb{R}^*)^2$. Hence, $\mathbf{J}^{\Phi-1}(y, t) = \{(x, y, z, t) \in \mathbb{R}^4 : x, z \in \mathbb{R}\} \simeq \mathbb{R}^2$ is a manifold for every $(y, t) \in (\mathbb{R}^*)^2$ and

$$\mathrm{T}_p[\mathbf{J}^{\Phi-1}(y, t)] = \langle \partial_x, \partial_z \rangle\,, \qquad \forall p \in \mathbf{J}^{\Phi-1}(y, t)\,.$$

Moreover, $G_{\boldsymbol{\mu}} = \mathbb{R}$ for each $\boldsymbol{\mu} = (y, t) \in (\mathbb{R}^*)^2$ and $G_{\boldsymbol{\mu}}$ acts freely and properly on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Let us prove that condition (4) does not imply nor the reduction of $\boldsymbol{\omega}$ neither (3).

It was proved in [30] that $\ker \omega^\alpha_p \subset \ker \mathrm{T}_p\mathbf{J}^\Phi_\alpha$, which allows one to define the following commutative diagram

$$\begin{tikzcd} \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \arrow[r, "\iota"] \arrow[rr, bend left, "\pi^\alpha_p"] & \ker \mathrm{T}_p\mathbf{J}^\Phi_\alpha \arrow[r, "\pi"] & \frac{\ker \mathrm{T}_p\mathbf{J}^\Phi_\alpha}{\ker \omega^\alpha_p}, \end{tikzcd} \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$$

where ι and π are the canonical injection and projections, respectively. For simplicity, the equivalence class of an element v in a quotient manifold will be denoted by $[v]$. According to Proposition 3.12 in [30], the above diagram induces the maps

$$\widetilde{\pi}^\alpha_p : \frac{\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})}{\mathrm{T}_p(G_{\boldsymbol{\mu}}p)} \to \frac{\frac{\ker \mathrm{T}_p\mathbf{J}^\Phi_\alpha}{\ker \omega^\alpha_p}}{\{[(\xi_M)_p]\,|\,\xi \in \mathfrak{g}_{\mu^\alpha}\}}\,, \qquad \alpha = 1, \ldots, k\,, \quad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}),$$

where $\mathfrak{g}_{\mu^\alpha}$ is the Lie algebra of G_{μ^α} and $\{[(\xi_P)_p]\,|\,\xi \in \mathfrak{g}_{\mu^\alpha}\} = \mathrm{pr}^P_\alpha(\{(\xi_M)_p\,|\,\xi \in \mathfrak{g}_{\mu^\alpha}\})$.

The conditions (3) at $p \in P$ are equivalent to each $\widetilde{\pi}^\alpha_p$ being surjective, while (4) amounts to $0 = \bigcap_{\alpha=1}^k \ker \widetilde{\pi}^\alpha_p$.

In our example, one has $\boldsymbol{\mu} = (y, t)$ with $\mu^1 = y$ and $\mu^2 = t$, while

$$\ker \mathrm{T}_p\mathbf{J}^\Phi_1 = \langle \partial_x, \partial_z, \partial_t \rangle\,, \quad \ker \omega^1 = \langle \partial_t, \partial_z \rangle\,, \quad \ker \mathrm{T}_p\mathbf{J}^\Phi_2 = \langle \partial_x, \partial_y, \partial_z \rangle\,, \quad \ker \omega^2 = 0$$

and

$$\{[(\xi_M)_p] : \xi \in \mathfrak{g}_{\mu^1}\} = \langle \partial_x \rangle\,, \qquad \{[(\xi_M)_p] : \xi \in \mathfrak{g}_{\mu^2}\} = \langle [\partial_x] \rangle\,.$$

Then, we have the mappings

$$\widetilde{\pi}^1_p : \langle [\partial_z] \rangle = \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_p(G_{\boldsymbol{\mu}}p) \to \langle 0 \rangle = (\ker \mathrm{T}_p\mathbf{J}^\Phi_1 / \ker \omega^1_p)/\langle [\partial_x] \rangle$$

and

$$\widetilde{\pi}^2_p : \langle [\partial_z] \rangle = \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_p(G_{\boldsymbol{\mu}}p) \to \langle [\partial_y], [\partial_z] \rangle = (\ker \mathrm{T}_p\mathbf{J}^\Phi_2 / \ker \omega^2_p)/\langle [\partial_x] \rangle\,.$$

As $\widetilde{\pi}^2_p([\partial_z]) = [\partial_z]$, we have
$$\ker \widetilde{\pi}^1_p = \langle \partial_z \rangle\,, \qquad \ker \widetilde{\pi}^2_p = \langle 0 \rangle\,.$$
Hence, $\ker \widetilde{\pi}^1_p \cap \ker \widetilde{\pi}^2_p = 0$ and condition (4) is satisfied. But $\mathrm{Im}\, \widetilde{\pi}^2_p = \langle \partial_z \rangle$ and $\widetilde{\pi}^2_p$ is not surjective. Hence, (3) does not hold for $\alpha = 2$ in our example. In fact, ω^1, ω^2 become isotropic when restricted to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ and give rise to two zero differential forms on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$ which is a one-dimensional manifold. Hence, no 2-symplectic structure is induced on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$ despite that condition (4) is satisfied.

One can directly prove that condition (4) is satisfied in the previous example, but condition (3) is not. This shows more easily that Proposition 1 in [20] is false, but our previous approach illustrates how we obtained our counterexample and it can also be used to investigate the independence between (3) and (4). In fact, in our actual counterexample, the fact that $\widetilde{\pi}^2_p$ is not surjective implies that (3) does not hold. Indeed, recall that

$$\ker \mathrm{T}_p\mathbf{J}^{\Phi}_2 = \langle \partial_x, \partial_y, \partial_z \rangle\,, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,,$$

while

$$\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) + \ker \omega^2_p + \mathrm{T}_p(G_{\mu^2}p) = \langle \partial_x, \partial_z \rangle + \{0\} + \langle \partial_x \rangle = \langle \partial_x, \partial_z \rangle\,, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,.$$

On the other hand, condition (4) is satisfied since

$$\mathrm{T}_p(G_{\boldsymbol{\mu}}p) = \langle \partial_x \rangle$$

and
$$(\ker \omega^1_p + \mathrm{T}_p(G_{\mu^1}p)) \cap (\ker \omega^2_p + \mathrm{T}_p(G_{\mu^2}p)) \cap \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = \langle \partial_x \rangle$$
reads
$$(\langle \partial_t, \partial_z \rangle + \langle \partial_x \rangle) \cap (\langle 0 \rangle + \langle \partial_x \rangle) \cap \langle \partial_x, \partial_z \rangle = \langle \partial_x \rangle.$$

△

3.3 On the relations between conditions for k-polysymplectic reduction

Let us illustrate the independence of conditions (3) and (4). In particular, the following example illustrates that condition (3) can be satisfied without (4) being obeyed too.

Example 3.6. Consider a 2-polysymplectic manifold $(\mathbb{R}^6, \boldsymbol{\omega})$. Let $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ be global linear coordinates on $\mathbb{R}^6$ and define

$$\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2 = (\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_5 \wedge \mathrm{d}x_6) \otimes e_1 + (\mathrm{d}x_3 \wedge \mathrm{d}x_4 + \mathrm{d}x_5 \wedge \mathrm{d}x_6) \otimes e_2\,.$$

Then, $\ker \omega^1_p = \langle \partial_3, \partial_4 \rangle$, $\ker \omega^2_p = \langle \partial_1, \partial_2 \rangle$, and $\ker \omega^1_p \cap \ker \omega^2_p = 0$ for every $p \in \mathbb{R}^6$. This turns $\boldsymbol{\omega}$ into a 2-polysymplectic form.

Let us provide now a Lie group action proving our initial claim. Given the Lie group action $\Phi : (\lambda; x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R} \times \mathbb{R}^6 \mapsto (x_1 + \lambda, x_2, x_3 + \lambda, x_4, x_5, x_6) \in \mathbb{R}^6$, its Lie algebra of fundamental vector fields reads $\langle \partial_1 + \partial_3 \rangle$. The momentum map associated with Φ is given by

$$\mathbf{J}^{\Phi} : (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 \mapsto (x_2, x_4) = \boldsymbol{\mu} \in (\mathbb{R}^*)^2,$$

which is Ad^{*2}-equivariant. Moreover, every $\boldsymbol{\mu} = (x_2, x_4) \in (\mathbb{R}^*)^2$ is a regular value of $\mathbf{J}^{\Phi}$. Therefore, $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 \,|\, x_1, x_3, x_5, x_6 \in \mathbb{R}\} \simeq \mathbb{R}^4$ is a submanifold of $\mathbb{R}^6$ for every $\boldsymbol{\mu} \in (\mathbb{R}^*)^2$ and
$$\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \langle \partial_1, \partial_3, \partial_5, \partial_6 \rangle\,, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,.$$

Hence, $\ker \mathrm{T}_p\mathbf{J}_1^{\Phi-1} = \langle \partial_1, \partial_3, \partial_4, \partial_5, \partial_6\rangle$ while $\ker \mathrm{T}_p\mathbf{J}_2^{\Phi-1} = \langle \partial_1, \partial_2, \partial_3, \partial_5, \partial_6\rangle$. Condition (3) holds because both sides are equal to

$$\begin{aligned}\langle \partial_1, \partial_3, \partial_4, \partial_5, \partial_6\rangle &= \langle \partial_1, \partial_3, \partial_5, \partial_6\rangle + \langle \partial_3, \partial_4\rangle + \langle \partial_1 + \partial_3\rangle\,,\\ \langle \partial_1, \partial_2, \partial_3, \partial_5, \partial_6\rangle &= \langle \partial_1, \partial_3, \partial_5, \partial_6\rangle + \langle \partial_1, \partial_2\rangle + \langle \partial_1 + \partial_3\rangle\,,\end{aligned}$$

for $\mathbf{J}_1^\Phi, \mathbf{J}_2^\Phi$, respectively. However, condition (4) is not satisfied, namely

$$\begin{aligned}\bigcap_{\alpha=1}^{2}\big(\ker\omega_p^\alpha + \mathrm{T}_p(G_{\mu^\alpha}p)\big)\cap \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = (\langle \partial_3, \partial_4\rangle + \langle \partial_1 + \partial_3\rangle) \cap (\langle \partial_1, \partial_2\rangle + \langle \partial_1 + \partial_3\rangle)\\ \cap \langle \partial_1, \partial_3, \partial_5, \partial_6\rangle = \langle \partial_1, \partial_3\rangle \neq \langle \partial_1 + \partial_3\rangle = \mathrm{T}_p(G_{\boldsymbol{\mu}}p)\,,\end{aligned}$$

for any $p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Therefore, $\widetilde{\pi}^1, \widetilde{\pi}^2$ are surjective but $\ker\widetilde{\pi}^1 \cap \ker\widetilde{\pi}^2 \neq 0$. One can also verify this fact by computing $\widetilde{\pi}_p^\alpha$ for $\alpha = 1, 2$. Note that

$$\begin{aligned}\widetilde{\pi}_p^1 : \langle[\partial_1], [\partial_5], [\partial_6]\rangle \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_p(G_{\boldsymbol{\mu}}p) \mapsto \langle[\partial_5], [\partial_6]\rangle = (\ker \mathrm{T}_p\mathbf{J}_1^\Phi/\ker\omega_p^1)/\langle[\partial_1 + \partial_3]\rangle\,.\\ \widetilde{\pi}_p^2 : \langle[\partial_1], [\partial_5], [\partial_6]\rangle \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_p(G_{\boldsymbol{\mu}}p) \mapsto \langle[\partial_5], [\partial_6]\rangle = (\ker \mathrm{T}_p\mathbf{J}_2^\Phi/\ker\omega_p^2)/\langle[\partial_1 + \partial_3]\rangle\,.\end{aligned}$$

for all $p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. △

Example 3.7. Let us prove that the reduction theorem in [30] gives sufficient, but not necessary conditions for the reduction to hold. In this respect, there are cases where the reduction is possible, condition (4) holds, while condition (3) does not. To illustrate this, let us consider a 2-polysymplectic manifold $(\mathbb{R}^7, \boldsymbol{\omega})$, where $\{x_1, \ldots, x_7\}$ are global linear coordinates and

$$\begin{aligned}\boldsymbol{\omega} &= \omega^1 \otimes e_1 + \omega^2 \otimes e_2\\ &= (\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_5 \wedge \mathrm{d}x_7 + \mathrm{d}x_3 \wedge \mathrm{d}x_6) \otimes e_1 + (\mathrm{d}x_3 \wedge \mathrm{d}x_4 + \mathrm{d}x_5 \wedge \mathrm{d}x_6) \otimes e_2\,.\end{aligned}$$

This give rise to a 2-polysymplectic structure on $\mathbb{R}^7$ since $\ker\omega^1 = \langle\partial_4\rangle$, $\ker\omega^2 = \langle \partial_1, \partial_2, \partial_7\rangle$ and $\ker\omega^1 \cap \ker\omega^2 = 0$. Consider the Lie group action $\Phi : \mathbb{R} \times \mathbb{R}^7 \to \mathbb{R}^7$ corresponding to translations along the x_5 coordinate. Then, its Lie algebra of fundamental vector fields is $\langle\partial_5\rangle$. A momentum map associated with Φ reads

$$\mathbf{J}^\Phi : (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in \mathbb{R}^7 \mapsto (x_7, x_6) = \boldsymbol{\mu} \in (\mathbb{R}^2)^*\,.$$

Note that $\mathbf{J}^\Phi$ is Ad^{*2}-equivariant and every $\boldsymbol{\mu} \in \mathbb{R}^{2*}$ is a regular value of $\mathbf{J}^\Phi$. Then,

$$\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in \mathbb{R}^7 \,|\, x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}\} \simeq \mathbb{R}^5$$

is a submanifold of $\mathbb{R}^7$ for every $\boldsymbol{\mu} = (\mu^1, \mu^2) = (x_6, x_7) \in \mathbb{R}^2$ and

$$\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \langle \partial_1, \partial_2, \partial_3, \partial_4, \partial_5\rangle\,, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,.$$

Condition (4) is satisfied, while (3) for $\mathbf{J}_1^\Phi$ is not since

$$\widetilde{\pi}_p^1 : \langle[\partial_1], [\partial_2], [\partial_3], [\partial_4]\rangle \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_p(G_{\boldsymbol{\mu}}p) \mapsto \langle[\partial_1], [\partial_2], [\partial_3], [\partial_6]\rangle = (\ker \mathrm{T}_p\mathbf{J}_1^\Phi/\ker\omega_p^1)/\langle[\partial_5]\rangle\,.$$

Therefore, $\widetilde{\pi}_p^1$ is not surjective. However, the reduced manifold $P_{\boldsymbol{\mu}} = \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_p(G_{\boldsymbol{\mu}}p) \simeq \mathbb{R}^4$ inherits the 2-polysymplectic structure, namely

$$\boldsymbol{\omega}_{\boldsymbol{\mu}} = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes e_1 + \mathrm{d}x_3 \wedge \mathrm{d}x_4 \otimes e_2\,.$$

We have taken into account that Proposition 2.14 guarantees that the momentum map $\mathbf{J}^\Phi$ can be made equivariant with respect to a k-polysymplectic affine Lie group action $\boldsymbol{\Delta} : G \times (\mathfrak{g}^*)^k \to (\mathfrak{g}^*)^k$, see [14]. △

Example 3.8. Let us examine the k-polysymplectic reduction of a product of k symplectic manifolds. Let $P = P_1 \times \cdots \times P_k$ for some symplectic manifolds $(P_\alpha, \omega^\alpha)$ with $\alpha = 1, \ldots, k$. If $\mathrm{pr}_\alpha : P \to P_\alpha$ is the canonical projection onto the α-th component, P_α, in P, then $(P, \sum_{\alpha=1}^k \mathrm{pr}_\alpha^* \omega^\alpha \otimes e_\alpha)$ is a k-polysymplectic manifold. To simplify the notation, we will write $\mathrm{pr}_\alpha^* \omega^\alpha$ as ω^α. Moreover, assume that a Lie group action $\Phi^\alpha : G_\alpha \times P_\alpha \to P_\alpha$ admits a symplectic momentum map $\mathbf{J}^{\Phi^\alpha} : P_\alpha \to \mathfrak{g}_\alpha^*$ for each $\alpha = 1, \ldots, k$ and each Φ^α acts in a quotientable manner on the level sets given by weak regular values of $\mathbf{J}^{\Phi^\alpha}$.

Define the Lie group action

$$\Phi : G \times P \ni (g_1, \ldots, g_k, x_1, \ldots, x_k) \longmapsto (\Phi^1_{g_1}(x_1), \ldots, \Phi^k_{g_k}(x_k)) \in P\,.$$

Then, $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$ is the Lie algebra of G and we have the k-polysymplectic momentum map

$$\mathbf{J} : P \ni (x_1, \ldots, x_k) \longmapsto (\mathbf{J}^{\Phi^1}(x_1), \ldots, \mathbf{J}^{\Phi^k}(x_k)) \in \mathfrak{g}^{*k},$$

where $\mathfrak{g}^* = \mathfrak{g}_1^* \times \cdots \times \mathfrak{g}_k^*$ is dual space to $\mathfrak{g}$. Suppose, that $\mu^\alpha \in \mathfrak{g}_\alpha^*$ is a weak regular value of $\mathbf{J}^{\Phi^\alpha} : P_\alpha \to \mathfrak{g}_\alpha^*$ for each $\alpha = 1, \ldots, k$. Hence, $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k) \in (\mathfrak{g}^*)^k$ is a weak regular value of $\mathbf{J}$. Then, Φ acts in a quotientable on the level sets of $\mathbf{J}$.

Therefore, if $x = (x_1, \ldots, x_k) \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, it follows that

$$\begin{aligned}
\ker \mathrm{T}_p \mathbf{J}^{\Phi^\alpha} &= \mathrm{T}_{x_1} P_1 \oplus \cdots \oplus \ker \mathrm{T}_{x_\alpha} \mathbf{J}^{\Phi^\alpha} \oplus \cdots \oplus \mathrm{T}_{x_k} P_k\,,\\
\mathrm{T}_p \left(\mathbf{J}^{-1}(\boldsymbol{\mu})\right) &= \ker \mathrm{T}_{x_1} \mathbf{J}^{\Phi^1} \oplus \cdots \oplus \ker \mathrm{T}_{x_k} \mathbf{J}^{\Phi^k}\,,\\
\ker \omega^\alpha_p &= \mathrm{T}_{x_1} P_1 \oplus \cdots \oplus \mathrm{T}_{x_{\alpha-1}} P_{\alpha-1} \oplus \{0\} \oplus \mathrm{T}_{x_{\alpha+1}} P_{\alpha+1} \oplus \cdots \oplus \mathrm{T}_{x_k} P_k\,,\\
\mathrm{T}_p \left(G^{\Delta^\alpha}_{\mu^\alpha} p\right) &= \mathrm{T}_{x_1} \left(G_1 x_1\right) \oplus \cdots \oplus \mathrm{T}_{x_\alpha} \left(G^{\Delta^\alpha}_{\alpha\mu^\alpha} x_\alpha\right) \oplus \cdots \oplus \mathrm{T}_{x_k} \left(G_k x_k\right)\,,\\
\mathrm{T}_p \left(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} p\right) &= \mathrm{T}_{x_1} \left(G^{\Delta^1}_{1\mu^1} x_1\right) \oplus \cdots \oplus \mathrm{T}_{x_k} \left(G^{\Delta^k}_{k\mu^k} x_k\right)\,.
\end{aligned}$$

Then,

$$\ker \mathrm{T}_p \mathbf{J}^{\Phi^\alpha} = \mathrm{T}_p \left(\mathbf{J}^{-1}(\mu)\right) + \ker \omega^\alpha_x\,, \quad \mathrm{T}_p \left(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} p\right) = \bigcap_{\beta=1}^{k} \left(\ker \omega^\beta_p + \mathrm{T}_p \left(G^{\Delta^\beta}_{\mu^\beta} p\right)\right)\,,$$

for $\alpha = 1, \ldots, k$, every regular $\boldsymbol{\mu} \in (\mathfrak{g}^*)^k$ and $p \in \mathbf{J}^{-1}(\boldsymbol{\mu})$. Recall that, by Theorem 3.2, these equations guarantee that the reduced space $\mathbf{J}^{-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ can be endowed with a k-polysymplectic structure, while

$$\mathbf{J}^{-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} \simeq \mathbf{J}^{\Phi^1-1}(\mu^1)/G^{\Delta^1}_{1\mu^1} \times \cdots \times \mathbf{J}^{\Phi^k-1}(\mu^k)/G^{\Delta^k}_{k\mu^k}\,.$$

△

4 The k-polysymplectic energy momentum-method

4.1 k-polysymplectic relative equilibrium points

This section introduces the notion of k-polysymplectic relative equilibrium point relative to a Hamiltonian $\boldsymbol{\omega}$-Hamiltonian vector field X. This notion is devised to analyse the stability of $\boldsymbol{\omega}$-Hamiltonian vector fields and extends the relative equilibrium point notion for symplectic manifolds (see [2] for details). In brief, a relative equilibrium point for a dynamical system given by a vector field is a point whose evolution is given by a Lie group symmetry of the vector field. If the vector field is additionally Hamiltonian relative to some geometric structure, then it is usual to demand the Lie group symmetries to leave invariant the same geometric structure [25].

Definition 4.1. Let $(P, \omega, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a G-invariant $\boldsymbol{\omega}$-Hamiltonian system. A point $z_e \in P$ is a *k-polysymplectic relative equilibrium point* of a $\boldsymbol{\omega}$-Hamiltonian vector field $X_{\boldsymbol{h}}$ if there exists $\xi \in \mathfrak{g}$ so that

$$(X_{\boldsymbol{h}})(z_e) = (\xi_P)(z_e)\,.$$

The above definition retrieves, for $k = 1$, the standard relative equilibrium notion for symplectic Hamiltonian systems [2]. Furthermore, Lemma 3.1 shows that $\xi \in \mathfrak{g}$ in Definition 4.1 is, in fact, an element of $\mathfrak{g}^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$, which is a Lie subalgebra of $\mathfrak{g}$.

Note that a k-polysymplectic relative equilibrium point $z_e \in P$ projects onto $\pi_{\boldsymbol{\mu}}(z_e)$, with $\boldsymbol{\mu}_e = \mathbf{J}(z_e)$, which becomes an equilibrium point of the vector field $X_{\boldsymbol{\mu}}$ on the reduced space $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$.

The following theorem provides the characterisation of k-polysymplectic relative equilibrium points of a $\boldsymbol{\omega}$-Hamiltonian vector field $X_{\boldsymbol{h}}$ by studying the critical points of a modified $\mathbb{R}^k$-valued function $\boldsymbol{h}_\xi$ on P. This will be an application of the Lagrange multiplier theorem, where the role of the multiplier is played by $\xi \in \mathfrak{g}$.

Theorem 4.2. *Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a G-invariant $\boldsymbol{\omega}$-Hamiltonian system. Then, $z_e \in P$ is a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$ if and only if there exists $\xi \in \mathfrak{g}$ such that z_e is a critical point of the following $\mathbb{R}^k$-valued function*

$$\boldsymbol{h}_\xi := \boldsymbol{h} - \langle \mathbf{J}^\Phi - \boldsymbol{\mu}_e, \xi\rangle\,, \tag{6}$$

where $\boldsymbol{\mu}_e := \mathbf{J}^\Phi(z_e) \in (\mathfrak{g}^*)^k$.

Proof. Let z_e be a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$, i.e. $X_{\boldsymbol{h}}(z_e) = \xi_P(z_e)$ for some $\xi \in \mathfrak{g}$. Then,

$$\mathrm{d}\boldsymbol{h}_\xi(z_e) = \mathrm{d}(\boldsymbol{h} - \langle \mathbf{J}^\Phi, \xi\rangle)(z_e) = (\iota_{X_{\boldsymbol{h}}-\xi_P}\boldsymbol{\omega})(z_e) = 0\,. \tag{7}$$

Hence, $z_e \in P$ is a critical point of the $\mathbb{R}^k$-valued function $\boldsymbol{h}_\xi$.

Conversely, assume z_e is a critical point of some $\boldsymbol{h}_\xi$ with $\xi \in \mathfrak{g}$. Then, $(\iota_{X_{\boldsymbol{h}}-\xi_P}\boldsymbol{\omega})(z_e) = 0$ and $(X_{\boldsymbol{h}} - \xi_P)(z_e) \in \ker \boldsymbol{\omega}_{z_e}$, since $\ker \boldsymbol{\omega} = 0$, and (7) yields that $X_{\boldsymbol{h}}(z_e) = \xi_P(z_e)$. Hence, z_e is a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$. □

It is left to understand the role of the constant term $\langle \mu_e, \xi\rangle$ in the definition of $\boldsymbol{h}_\xi$. Note that one can understand that (z_e, ξ) is a critical point of $\boldsymbol{h}$ and ξ is indeed a Lagrange multiplier. Hence, all the components of $\boldsymbol{h}$ must have a critical point on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$ as long as it is a submanifold. Note also that the standard Lagrange multiplier method requires the map $\mathbf{J}^\Phi$ to be a submersion at z_e as, otherwise, one does not know whether

Example 4.3. The cotangent bundle of two-covelocities of $\mathbb{R}^2$.

Our first example is related to the so-called cotangent bundle of k-covelocities of a manifold. Let Q be an n-dimensional manifold and let $\pi_Q : \mathrm{T}^*Q \to Q$ be the cotangent bundle projection. Consider the Whitney sum $\oplus^k \mathrm{T}^*Q = \mathrm{T}^*Q \oplus_Q \overset{(k)}{\cdots} \oplus_Q \mathrm{T}^*Q$ of k copies of T^*Q and the projection $\pi^k_Q : \oplus^k\mathrm{T}^*Q \to Q$. It is well-known that $\oplus^k\mathrm{T}^*Q$ can be identified with the first-jet manifold, $J^1(Q, \mathbb{R}^k)$, of maps $\sigma : Q \to \mathbb{R}^k$ via the diffeomorphism $J^1(Q, \mathbb{R}^k) \ni j^1_q\sigma \mapsto \big(\mathrm{d}\sigma^1(q), \dots, \mathrm{d}\sigma^k(q)\big)$, where σ^α is the α-th component of σ. Then, $\oplus^k\mathrm{T}^*Q$ is called *the cotangent bundle of k-covelocities of Q*. Moreover, $J^1(Q, \mathbb{R}^k)$ is a k-polysymplectic manifold (see [13] for details).

The following example will illustrate the 2-polysymplectic energy-momentum method. Note that $\mathrm{T}^*\mathbb{R}^2 \oplus_{\mathbb{R}^2} \mathrm{T}^*\mathbb{R}^2 \simeq \mathbb{R}^6$, for $x \in \mathbb{R}^2$ is a 2-polysymplectic manifold isomorphic to $J^1(\mathbb{R}^2, \mathbb{R}^2)$. Indeed,$(\mathbb{R}^6, \boldsymbol{\omega})$ is a 2-polysymplectic manifold relative to

$$\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2 = (\mathrm{d}x_1 \wedge \mathrm{d}x_3 + \mathrm{d}x_2 \wedge \mathrm{d}x_4) \otimes e_1 + (\mathrm{d}x_1 \wedge \mathrm{d}x_5 + \mathrm{d}x_2 \wedge \mathrm{d}x_6) \otimes e_2$$

since

$$\ker\omega^1 = \left\langle \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6} \right\rangle , \qquad \ker\omega^2 = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right\rangle ,$$

and $\ker\omega^1 \cap \ker\omega^2 = 0$. Let us consider the Lie group action $\Phi : \mathbb{R}\times\mathbb{R}^6 \to \mathbb{R}^6$ given by

$$\Phi : \mathbb{R}\times\mathbb{R}^6 \ni (\lambda; x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_1+\lambda, x_2+\lambda, x_3+\lambda, x_4+\lambda, x_5+\lambda, x_6+\lambda) \in \mathbb{R}^6 .$$

The fundamental vector fields associated with the Lie group action Φ are spanned by

$$\xi_P = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} .$$

Note that, the Lie group action Φ is 2-polysymplectic since it leaves $\boldsymbol{\omega}$ invariant, namely $\mathscr{L}_{\xi_P}\omega^\alpha = 0$ for $\alpha = 1, 2$. Then, Φ gives rise to a 2-polysymplectic momentum map $\mathbf{J}$ given by

$$\mathbf{J}^\Phi : \mathbb{R}^6 \ni (x_1, x_2, x_3, x_4, x_5, x_6) \longmapsto (x_3+x_4-x_1-x_2, x_5+x_6-x_1-x_2) = (\mu^1, \mu^2) = \boldsymbol{\mu} \in \mathbb{R}^2 .$$

Therefore, the level set of the 2-polysymplectic momentum map $\mathbf{J}^\Phi$ has the following form

$$\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \left\{(x_1, x_2, x_3, x_4, x_5, x_6)\in\mathbb{R}^6 \mid x_3+x_4-x_1-x_2 = \mu^1,\ x_5+x_6-x_1-x_2 = \mu^2\right\} . \tag{8}$$

Note that $\boldsymbol{\mu} = (\mu^1, \mu^2)$ is a weak regular value of a 2-polysymplectic momentum map $\mathbf{J}^\Phi$ and the level set $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \simeq \mathbb{R}^4$. Moreover, since a Lie group is a one-dimensional group of translations, we get that $\mathbf{J}^\Phi$ is Ad^{*2}-equivariant. Then,

$$\begin{aligned}
\mathrm{T}_x(G_{\boldsymbol{\mu}}x) = \mathrm{T}_x(G_{\mu^1}x) = \mathrm{T}_x(G_{\mu^2}x) &= \left\langle \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} \right\rangle ,\\
\mathrm{T}_x\mathbf{J}_1^{\Phi-1}(\mu^1) &= \left\langle \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6} \right\rangle ,\\
\mathrm{T}_x\mathbf{J}_2^{\Phi-1}(\mu^2) &= \left\langle \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_6} \right\rangle ,\\
\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) &= \left\langle \sum_{i=1}^{6} \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right\rangle ,
\end{aligned}$$

and one can verify that conditions (3) and (4) are fulfilled.

Recall that $\iota_{\boldsymbol{\mu}} : \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \hookrightarrow P$ is the natural immersion and $\pi_{\boldsymbol{\mu}} : \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \to \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$ is the canonical projection. Then, remembering that our symmetry group consists of translations, Theorem 3.2 yields that the reduced manifold $(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}} \simeq \mathbb{R}^3, \boldsymbol{\omega}_{\boldsymbol{\mu}})$ is a 2-polysymplectic manifold with coordinates $(x_1, x_3, x_5) \in \mathbb{R}^3$, where

$$\boldsymbol{\omega}_{\boldsymbol{\mu}} = \omega^1_{\boldsymbol{\mu}} \otimes e_1 + \omega^2_{\boldsymbol{\mu}} \otimes e_2 = 2\mathrm{d}x_1 \wedge \mathrm{d}x_3 \otimes e_1 + 2\mathrm{d}x_1 \wedge \mathrm{d}x_5 \otimes e_2.$$

Next, let us consider a vector field X, on $P = \mathbb{R}^3$, tangent to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, which in general has the following form

$$X_{\boldsymbol{h}} = F_1 \sum_{i=1}^{6} \frac{\partial}{\partial x_i} + F_2\left(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}\right) + F_3\left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}\right) + F_4\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right),$$

where $F_i \in \mathscr{C}^\infty(P)$ are G-invariant for $i = 1, \ldots, 4$. Then, one can immediately see that a point $z_e \in P$ is a 2-polysymplectic relative equilibrium point of X if and only if $X_{\boldsymbol{h}}(z_e) = \xi_P(z_e)$, which holds when $F_1(z_e) = 1$ and $F_2(z_e) = F_3(z_e) = F_4(z_e) = 0$. However, let us verify that we obtain the same result using the 2-polysymplectic energy-momentum method.

First, $\mathrm{d}h_1$ and $\mathrm{d}h_2$ read

$$\begin{aligned}
\mathrm{d}h^1 &= \iota_X\omega^1 = -\left(F_1+F_2+F_3\right)\mathrm{d}x_1 - \left(F_1-F_2\right)\mathrm{d}x_2 + \left(F_1+F_4\right)\mathrm{d}x_3 + \left(F_1+F_3-F_4\right)\mathrm{d}x_4,\\
\mathrm{d}h^2 &= \iota_X\omega^2 = -\left(F_1+F_3\right)\mathrm{d}x_1 - F_1\mathrm{d}x_2 + \left(F_1+F_4\right)\mathrm{d}x_5 + \left(F_1+F_3-F_4\right)\mathrm{d}x_6\,.
\end{aligned}$$

Then, Theorem 4.2 yields that $z_e \in P$ is a 2-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$ if and only if $\mathrm{d}h^1_\xi(z_e) = 0$ and $\mathrm{d}h^2_\xi(z_e) = 0$. Indeed, using (8), one has

$$\begin{aligned}
\mathrm{d}h^1_\xi = \mathrm{d}h^1 - \mathrm{d}J^1_\xi = &-\left(F_1+F_2+F_3-\xi\right)\mathrm{d}x_1 - \left(F_1-F_2-\xi\right)\mathrm{d}x_2\\
&+\left(F_1+F_4-\xi\right)\mathrm{d}x_3 + \left(F_1+F_3-F_4-\xi\right)\mathrm{d}x_4\,,
\end{aligned}\tag{9}$$

$$\begin{aligned}
\mathrm{d}h^2_\xi = \mathrm{d}h^2 - \mathrm{d}J^2_\xi = &-\left(F_1+F_3-\xi\right)\mathrm{d}x_1 - \left(F_1-\xi\right)\mathrm{d}x_2\\
&+\left(F_1+F_4-\xi\right)\mathrm{d}x_5 + \left(F_1+F_3-F_4-\xi\right)\mathrm{d}x_6\,,
\end{aligned}\tag{10}$$

for $\xi \in \mathfrak{g} = \mathbb{R}$. Since at z_e both (9) and (10) must vanish, one gets that $F_1(z_e) = \xi$ and $F_2(z_e) = F_3(z_e) = F_4(z_e) = 0$. Therefore, $z_e \in P$ is a 2-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$.

Finally, let's verify that $\pi_{\boldsymbol{\mu}}(z_e)$ is a critical point of $h^i_{\boldsymbol{\mu}} \in \mathscr{C}^\infty(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}})$. The reduced vector field $X_{\boldsymbol{h}_{\boldsymbol{\mu}}}$ has the form

$$X_{\boldsymbol{h}_{\boldsymbol{\mu}}} = F_2\left(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}\right) + F_3\left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}\right) + F_4\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right).$$

Then,

$$\begin{aligned}
\mathrm{d}h^1_{\boldsymbol{\mu}}(\pi_{\boldsymbol{\mu}}(z_e)) &= \left(\iota_{X_{\boldsymbol{h}_{\boldsymbol{\mu}}}}\omega^1_{\boldsymbol{\mu}}\right)_{\pi_{\boldsymbol{\mu}}(z_e)} = 2\left(F_4(z_e)\mathrm{d}x_3 - (F_2(z_e)+F_3(z_e))\mathrm{d}x_1\right) = 0\,,\\
\mathrm{d}h^2_{\boldsymbol{\mu}}(\pi_{\boldsymbol{\mu}}(z_e)) &= \left(\iota_{X_{\boldsymbol{h}_{\boldsymbol{\mu}}}}\omega^2_{\boldsymbol{\mu}}\right)_{\pi_{\boldsymbol{\mu}}(z_e)} = 2\left(F_4(z_e)\mathrm{d}x_5 - F_3(z_e)\mathrm{d}x_1\right) = 0\,,
\end{aligned}$$

where we denoted by F_2, F_3, F_4 both G-invariant functions on P and functions on the reduced manifold $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$.

4.2 The k-polysymplectic energy momentum-method

Let us develop the main part of the k-polysymplectic energy-momentum method relative to a k-polysymplectic manifold $(P, \boldsymbol{\omega})$. Recall that Theorem 4.2 characterises relative equilibrium points as critical points of the function (6). However, when studying the stability of relative equilibrium points, due to the symmetry, one has to take into account vectors that are tangent to $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$. In other words, we need to investigate how the second variation of $\boldsymbol{h}_\xi$ in the directions tangent to the isotropy group $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ affects the positive definiteness of $\boldsymbol{h}_\xi$.

Let us define the second variation of each of the components of $\boldsymbol{h}_\xi$ at a relative equilibrium point $z_e \in P$ as the mapping

$$\left(\delta^2\boldsymbol{h}_\xi\right)_{z_e}(v_1, v_2) = \sum_{\alpha=1}^{k} \iota_Y\left(\mathrm{d}\left(\iota_X \mathrm{d}h^\alpha_\xi\right)\right)_{z_e} \otimes e_\alpha\,,\tag{11}$$

for some vector fields X, Y on P defined on a neighbourhood of $z_e \in P$ and such that $v_1 = X_{z_e}$, $v_2 = Y_{z_e}$.

Proposition 4.4. *Let $z_e \in P$ be a relative equilibrium point of X on a k-polysymplectic manifold $(P, \boldsymbol{\omega})$. If $\{t, x_1, \ldots, x_{2n}\}$ are coordinates on a neighbourhood of $z_e \in P$, then*

$$(\delta^2 h^\alpha_\xi)_{z_e}(w, v) = \sum_{i,j=1}^{2n} \frac{\partial^2 h^\alpha_\xi}{\partial x_i \partial x_j}(z_e) w_i v_j\,, \qquad \forall v, w \in \mathrm{T}_{z_e}P\,, \quad \alpha = 1, \ldots, k\,,$$

where $w = \sum_{i=1}^{2n} w_i \partial/\partial x_i$ and $v = \sum_{i=1}^{2n} v_i \partial/\partial x_i$.

Proof. From (11) for $\alpha = 1, \ldots, k$, we have

$$
\begin{aligned}
(\delta^2 h^\alpha_\xi)_{z_e}(w,v) &= \iota_Y(\mathrm{d}\iota_X \mathrm{d}h^\alpha_\xi)_{z_e} \\
&= \sum_{i,j=1}^{2n} \frac{\partial^2 h^\alpha_\xi}{\partial x_i \partial x_j}(z_e) w_i v_j + \sum_{i,j=1}^{2n} \frac{\partial h^\alpha_\xi}{\partial x_i}(z_e)\frac{\partial X_i}{\partial x_j}(z_e) v_j \\
&= \sum_{i,j=1}^{2n} \frac{\partial^2 h^\alpha_\xi}{\partial x_i \partial x_j}(z_e) w_i v_j \,,
\end{aligned}
$$

where $X = \sum_{i=1}^{2n} X^i \partial/\partial x^i$, $X(z_e) = w$, and we have used that z_e is a relative equilibrium point. $\square$

Note that the maps $(\delta^2 h^\alpha_\xi)_{z_e}$ are symmetric for $\alpha = 1, \ldots, k$. Therefore, $(\delta^2 \boldsymbol{h}_\xi)_{z_e}$ is a symmetric map. Let us study 11 in more detail.

Proposition 4.5. *Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a $\boldsymbol{\omega}$-Hamiltonian system and let $z_e \in P$ be a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$. Then,*

$$
(\delta^2 \boldsymbol{h}_\xi)_{z_e}((\zeta_P)_{z_e}, (\nu_P)_{z_e}) = 0 \,, \qquad \forall \zeta, \nu \in \mathfrak{g}^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} \,,
$$

Proof. First, since $h^\alpha \in \mathscr{C}^\infty(P)$ is G-invariant and $\mathbf{J}^\Phi$ is equivariant with respect to the k-polysymplectic affine Lie group action $\boldsymbol{\Delta} : G \times (\mathfrak{g}^*)^k \to (\mathfrak{g}^*)^k$, then for every $g \in G$ and $x \in P$, one has

$$
\begin{aligned}
\boldsymbol{h}_\xi(\Phi_g(x)) &= \boldsymbol{h}(\Phi_g(x)) - \langle \mathbf{J}^\Phi(\Phi_g(x)), \xi \rangle + \langle \mu_e, \xi \rangle \\
&= \boldsymbol{h}(x) - \langle \boldsymbol{\Delta}_g \mathbf{J}^\Phi(x), \xi \rangle + \langle \mu_e, \xi \rangle = \boldsymbol{h}(x) - \langle \mathbf{J}^\Phi(x), \boldsymbol{\Delta}^T_g \xi \rangle + \langle \mu_e, \xi \rangle \,,
\end{aligned}
$$

where $\boldsymbol{\Delta}^T_g : (\mathfrak{g})^k \to (\mathfrak{g})^k$ is a transpose of $\boldsymbol{\Delta}_g$ for every $g \in G$. Let us substitute $g = \exp(t\zeta)$, with $\zeta \in \mathfrak{g}$, and differentiate with respect to t. Then,

$$
(\iota_{\zeta_P} \mathrm{d}\boldsymbol{h}_\xi)_{z_e} = - \left\langle \mathbf{J}^\Phi(x), \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \boldsymbol{\Delta}^T_{\exp t\zeta} \xi \right\rangle = - \left\langle \mathbf{J}^\Phi(x), (\zeta^{\boldsymbol{\Delta}}_{\mathfrak{g}})_\xi \right\rangle \,, \tag{12}
$$

where $(\zeta^{\boldsymbol{\Delta}}_{\mathfrak{g}})_\xi$ is the fundamental vector field of $\boldsymbol{\Delta}^T : G \times \mathfrak{g}^k \to \mathfrak{g}^k$ at $\xi \in \mathfrak{g}$. Taking second variation of (12) relative to $x \in P$, evaluating at $z_e \in P$, and contracting with ν_P, one has

$$
\left(\delta^2 \boldsymbol{h}_\xi\right)_{z_e} ((\zeta_P)_{z_e}, (\nu_P)_{z_e}) = - \left\langle \mathrm{T}_{z_e} \mathbf{J}^\Phi \left((\nu_P)_{z_e}\right), (\zeta^{\boldsymbol{\Delta}}_{\mathfrak{g}})_\xi \right\rangle .
$$

Therefore, by Proposition 3.1 the second variation $\left(\delta^2 \boldsymbol{h}_\xi\right)_{z_e} ((\zeta_P)_{z_e}, (\nu_P)_{z_e})$ vanish since $(\nu_P)_{z_e} \in \mathrm{T}_{z_e} \mathbf{J}^{\Phi-1}(\boldsymbol{\mu_e})$ for $\nu \in \mathfrak{g}^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$. $\square$

From Proposition 4.5, we can conclude that the definiteness of $\left(\delta^2 \boldsymbol{h}_\xi\right)_{z_e}$ in directions tangent to $\mathrm{T}_{z_e}\left(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} z_e\right)$ has no effect. Only directions transverse to the orbit of $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$ are significant. This is an essential advantage of the energy-momentum method.

Proposition 4.6. *Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a $\boldsymbol{\omega}$-Hamiltonian system and let $z_e \in P$ be a relative equilibrium point of $X_{\boldsymbol{h}}$. Then, z_e is a stable relative equilibrium point if*

$$
\left(\delta^2 h^\alpha_\xi\right)_{z_e} (v_{z_e}, v_{z_e}) > 0 \,, \qquad \forall v_{z_e} \in \left(\mathrm{T}_{z_e} \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e) / \mathrm{T}_{z_e}(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e} z_e)\right) \cap (\ker \omega^\alpha_{\boldsymbol{\mu}_e})_{[z_e]} \,, \quad \alpha = 1, \ldots, k \,,
$$

where $[z_e] = \pi_{\boldsymbol{\mu}_e}(z_e)$.

The stability of the k-polysymplectic Hamiltonian system requires the positive definiteness of $(\delta^2 h^\alpha_\xi)_{z_e}$ on $\mathrm{T}_{z_e} \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$ modulo directions along $\mathrm{T}_{z_e}(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e} z_e)$ such that they do not belong to $\ker \omega^\alpha_{z_e}$. Taking that into account, one gets the following theorem

Theorem 4.7. *Let $\mathcal{S}_{z_e}$ be a subspace of P transversal to the orbits of $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$. Then, a relative equilibrium point x_e is stable if*

$$
\left(\delta^2 h^\alpha_\xi\right)_{z_e} (v_{z_e}, v_{z_e}) > 0 \,, \qquad \forall v_{z_e} \in \mathrm{T}_{z_e}\mathcal{S} / \left(\mathrm{T}_{z_e}\mathcal{S} \cap \ker \omega^\alpha_{z_e}\right) \,, \quad \alpha = 1, \ldots, k \,.
$$

5 Examples and applications

This section illustrates how the theory and applications of the previous sections can be applied to relevant examples with physical and mathematical applications.

5.1 Complex Schwarz derivative

Let us illustrate how the structure of the so-called locally automorphic Lie systems can be used to transform them into Hamiltonian systems relative to a k-symplectic structure.

The Schwarz derivative appears in the study of linearisation of t-dependent systems, projective systems, the theory of mathematical functions, etc [22, 24, 29]. The Schwarz derivative is related to the t-dependent complex differential equation given by

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v\,, \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = a\,, \qquad \frac{\mathrm{d}a}{\mathrm{d}t} = \frac{3}{2}\frac{a^2}{v} + 2b(t)v\,, \qquad z, v, a \in \mathbb{C},$$

for a certain complex t-dependent function $b(t)$. The system (5.1) is the complex analogue of the Lie system on $T^2\mathbb{R}$ studied in [16] and represents the complex equation $\{z, b(t)\}_{Sc} = 0$, where $\{\cdot,\cdot\}_{sc}$ is the so-called Schwarz derivative. Writing (5.1) in real coordinates

$$v_1 = \mathfrak{Re}(z)\,, \quad v_2 = \mathfrak{Im}(z)\,, \quad v_3 = \mathfrak{Re}(v)\,, \quad v_4 = \mathfrak{Im}(v)\,, \quad v_5 = \mathfrak{Re}(a)\,, \quad v_6 = \mathfrak{Im}(a)\,,$$

it follows that (5.1) is related to the t-dependent vector field

$$X = X_1 + b_R(t)X_3 + b_I(t)X_4\,,$$

where $b_R(t) = \mathfrak{Re}(b(t))$, $b_I(t) = \mathfrak{Im}(b(t))$, and

$$X_1 = v_3\frac{\partial}{\partial v_1} + v_4\frac{\partial}{\partial v_2} + v_5\frac{\partial}{\partial v_3} + v_6\frac{\partial}{\partial v_4} + \frac{3}{2}\frac{2v_4v_5v_6 + (v_5^2 - v_6^2)v_3}{v_3^2 + v_4^2}\frac{\partial}{\partial v_5} + \frac{3}{2}\frac{2v_3v_5v_6 - v_4(v_5^2 - v_6^2)}{v_3^2 + v_4^2}\frac{\partial}{\partial v_6}\,,$$

$$X_2 = v_3\frac{\partial}{\partial v_5} + v_4\frac{\partial}{\partial v_6}\,, \qquad X_3 = -v_4\frac{\partial}{\partial v_5} + v_3\frac{\partial}{\partial v_6}\,,$$

$$X_4 = -v_3\frac{\partial}{\partial v_3} - v_4\frac{\partial}{\partial v_4} - 2v_5\frac{\partial}{\partial v_5} - 2v_6\frac{\partial}{\partial v_6}\,, \qquad X_5 = v_4\frac{\partial}{\partial v_3} - v_3\frac{\partial}{\partial v_4} + 2v_6\frac{\partial}{\partial v_5} - 2v_5\frac{\partial}{\partial v_6}\,,$$

$$X_6 = -v_4\frac{\partial}{\partial v_1} + v_3\frac{\partial}{\partial v_2} - v_6\frac{\partial}{\partial v_3} + v_5\frac{\partial}{\partial v_4} - \frac{3}{2}\frac{2v_3v_5v_6 - v_4(v_5^2 - v_6^2)}{2(v_3^2 + v_4^2)}\frac{\partial}{\partial v_5} + \frac{3}{2}\frac{2v_4v_5v_6 + v_3(v_5^2 - v_6^2)}{2(v_3^2 + v_4^2)}\frac{\partial}{\partial v_6}\,,$$

satisfy the commutation relations

$$\begin{array}{lllll}
[X_1, X_2] = X_4\,, & [X_1, X_3] = X_5\,, & [X_1, X_4] = X_1\,, & [X_1, X_5] = X_6\,, & [X_1, X_6] = 0\,,\\
 & [X_2, X_3] = 0\,, & [X_2, X_4] = -X_2\,, & [X_2, X_5] = -X_3\,, & [X_2, X_6] = -X_5\,,\\
 & & [X_3, X_4] = -X_3\,, & [X_3, X_5] = X_2\,, & [X_3, X_6] = X_4\,,\\
 & & & [X_4, X_5] = 0\,, & [X_4, X_6] = -X_6\,,\\
 & & & & [X_5, X_6] = X_1\,,
\end{array}$$

which give rise to a Lie algebra isomorphic to $\mathbb{C} \otimes \mathfrak{sl}_2$ as a real vector space. Meanwhile, one has the following Lie algebra of symmetries of the problem,

$$Y_1 = \frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2}, Y_2 = v_1\frac{\partial}{\partial v_1} + \ldots + v_6\frac{\partial}{\partial v_6}, Y_3 = (v_1^2 - v_2^2)\frac{\partial}{\partial v_2} + 2v_1v_2\frac{\partial}{\partial v_2} + (2v_1v_3 - 2v_2v_4)\frac{\partial}{\partial v_3} + 2(v_3$$

$$Y_4 = \frac{\partial}{\partial v_1}, Y_5 = -2v_1v_2\frac{\partial}{\partial v_1} + (v_1^2 - v_2^2)\frac{\partial}{\partial v_2} - 2(v_2v_3 + v_1v_4)\frac{\partial}{\partial v_3} + (2v_1v_2 - 2v_2v_4)\frac{\partial}{\partial v_4} - 2(v_3v_4 + v_2v_5 + v_1v_6)\frac{\partial}{\partial v}$$

One can choose $\eta_1, \ldots, \eta_6$ to be the dual forms to $Y_1, \ldots, Y_6$. Then,

$$\mathrm{d}\eta_1 = \eta_5 \wedge \eta_6 + \eta_1 \wedge \eta_4\,, \qquad \mathrm{d}\eta_2 = \eta_3 \wedge \eta_5 + \eta_4 \wedge \eta_2$$

The example follows. Note

5.2 The k-polysymplectic manifold as the product of k symplectic manifolds

This section presents an illustrative example of the k-polysymplectic reduction of a product of k symplectic manifolds (see Example 3.8).

For simplicity, we will assume some technical conditions. Let $P = P_1 \times \cdots \times P_k$ for some k symplectic manifolds $(P_\alpha, \omega^\alpha)$ where $\alpha = 1, \ldots, k$. If $\mathrm{pr}_\alpha : P \to P_\alpha$ is the canonical projection onto the α-th component, $(P, \sum_{\alpha=1}^k \mathrm{pr}_\alpha^* \omega^\alpha \otimes e_\alpha)$ is a k-polysymplectic manifold. Moreover, assume that a Lie group action $\Phi^\alpha : G_\alpha \times P_\alpha \to P_\alpha$ admits a cosymplectic momentum map $\mathbf{J}^{\Phi^\alpha} : P_\alpha \to \mathfrak{g}_\alpha^*$ for each $\alpha = 1, \ldots, k$ and each Φ^α acts in a quotientable manner on the level sets given by regular values of $\mathbf{J}^{\Phi^\alpha}$.

Then, let us define the Lie group action of $G = G_1 \times \cdots \times G_k$ on P as

$$\Phi : G \times P \ni (g_1, \ldots, g_k, x_1, \ldots, x_k) \longmapsto (\Phi^1_{g_1}(x_1), \ldots, \Phi^k_{g_k}(x_k)) \in P\,.$$

Let $\mathfrak{g} = \mathfrak{g}_1 \times \ldots \times \mathfrak{g}_k$ denotes be the Lie algebra of G. Then, there exists a k-polycosymplectic momentum map

$$\mathbf{J} : P \ni (x_1, \ldots, x_k) \longmapsto (\mathbf{J}^{\Phi^1}(x_1), \ldots, \mathbf{J}^{\Phi^k}(x_k)) \in \mathfrak{g}^*\,,$$

where $\mathfrak{g}^* = \mathfrak{g}_1^* \times \ldots \times \mathfrak{g}_k^*$ is dual space to $\mathfrak{g}$. Suppose, that $\mu^\alpha \in \mathfrak{g}_\alpha^*$ is a weak regular value of $\mathbf{J}^{\Phi^\alpha} : P_\alpha \to \mathfrak{g}_\alpha^*$ for each $\alpha = 1, \ldots, k$. Hence, $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k) \in \mathfrak{g}^*$ is a regular value of $\mathbf{J}$. Then, Φ acts in a quotientable on the associated level sets of $\mathbf{J}$.

Then, one can check [14, 30] that

$$\begin{aligned}
\ker \mathrm{T}_x \mathbf{J}^{\Phi^\alpha} &= \mathrm{T}_x \left(\mathbf{J}^{-1}(\mu) \right) + \ker \omega_x^\alpha, \\
\mathrm{T}_x \left(G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}} x \right) &= \bigcap_{\beta=1}^{k} \left(\ker \omega_x^\beta + \mathrm{T}_x \left(G^{\Delta^\beta}_{\beta\mu^\beta} x \right) \right),
\end{aligned}$$

for $\alpha = 1, \ldots, k$ and every regular $\boldsymbol{\mu} \in \mathfrak{g}^*$ and $x \in \mathbf{J}^{-1}(\boldsymbol{\mu})$. Recall that, by Theorem 3.2, these equations guarantee that on the reduced space $\mathbf{J}^{-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}}$ there is a k-polysymplectic structure, while

$$\mathbf{J}^{-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}} \simeq \mathbf{J}^{\Phi^1 -1}(\mu^1)/G^{\Delta^1}_{1\mu^1} \times \cdots \times \mathbf{J}^{\Phi^k -1}(\mu^k)/G^{\Delta^k}_{k\mu^k}.$$

Next, let us consider a vector field X on P that is G-invariant. By Theorem 3.4 X can be written in the following way

$$X = \sum_{\alpha=1}^{k} X_\alpha,$$

where X_α is tangent to $\mathbf{J}^{\Phi^\alpha -1}(\mu_\alpha)$ for $\alpha = 1, \ldots, k$. Recall that $\iota_{X_\alpha} \omega^\alpha = \mathrm{d} f^\alpha$. Then,

$$\mathrm{d}\boldsymbol{f} = \sum_{\alpha=1}^{k} \mathrm{d} f^\alpha \otimes e_\alpha = \sum_{\alpha=1}^{k} \iota_X \omega^\alpha \otimes e_\alpha\,.$$

Next, one has that $\boldsymbol{f}_\xi$ has the following form $\boldsymbol{f}_\xi = \boldsymbol{f} - \langle \mathbf{J} - \boldsymbol{\mu}_e, \xi \rangle$ and by Theorem 4.2 it follows that $z_e = (z_{1e}, \ldots, z_{ke}) \in P$ is a k-polysymplectic relative equilibrium point if and only if each of $z_{\alpha e}$ is a symplectic relative equilibrium point of a Hamiltonian vector field X_α on a symplectic manifold $(P_\alpha, \omega^\alpha)$, for more details see [17, 46]. Then, a relative equilibrium point z_e is stable if

$$\left(\delta^2 f_\xi^\alpha \right)_{z_e} (v_{z_e}, v_{z_e}) > 0, \qquad \forall v_{z_e} \in \mathrm{T}_{z_e} \mathcal{S} / \left(\mathrm{T}_{z_e} \mathcal{S} \cap \ker \omega_{z_e}^\alpha \right), \quad \alpha = 1, \ldots, k, \tag{13}$$

where $\mathcal{S}$ is a submanifold transversal to $G_{\boldsymbol{\mu}_e} z_e$. Note that, (13) boils down to

$$\left(\delta^2 f_\xi^\alpha \right)_{z_e} (v^\alpha_{z_e}, v^\alpha_{z_e}) > 0, \qquad \forall v^\alpha_{z_e} \in \mathrm{T}_{z_e} \mathcal{S}_\alpha / \left(\mathrm{T}_{z_e} \mathcal{S}_\alpha \cap \ker \omega_{z_e}^\alpha \right), \qquad \alpha = 1, \ldots, k,$$

where S_α is a submanifold of P_α that is transversal to $G_{\alpha\mu_{e\alpha}} z_{\alpha e}$.

5.3 A control Lie system

Let us consider the system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sum_{\alpha=1}^{5} b_\alpha(t) X_\alpha \,, \tag{14}$$

where $b_1(t), \ldots, b_5(t)$ are arbitrary t-dependent functions and

$$X_1 = \frac{\partial}{\partial x_1}\,, \quad X_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + x_1^2 \frac{\partial}{\partial x_4} + 2x_1x_2 \frac{\partial}{\partial x_5}\,,$$
$$X_3 = \frac{\partial}{\partial x_3} + 2x_1 \frac{\partial}{\partial x_4} + 2x_2 \frac{\partial}{\partial x_5}\,, \qquad X_4 = \frac{\partial}{\partial x_4}\,, \qquad X_5 = \frac{\partial}{\partial x_5}.$$

Not that the above vector fields span a Lie algebra V of vector fields whose non-vanishing commutation relations read

$$[X_1, X_2] = X_3\,, \qquad [X_1, X_3] = 2X_4\,, \qquad [X_2, X_3] = 2X_5\,.$$

Consequently, X is a Lie system as proved in [41]. The initial motivation to study (14) comes from the fact that it covers as a particular case the system of differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = b_1(t)\,, \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = b_2(t)\,, \qquad \frac{\mathrm{d}x_3}{\mathrm{d}t} = b_2(t)x_1\,, \qquad \frac{\mathrm{d}x_4}{\mathrm{d}t} = b_2(t)x_1^2\,, \qquad \frac{\mathrm{d}x_5}{\mathrm{d}t} = 2b_2(t)x_1x_2\,,$$

where $b_1(t)$ and $b_2(t)$ are arbitrary t-dependent functions whose interest is due to their relation to certain control problems [**?**, 41].

Now, let us show how to approach (14) as a k-symplectic Lie system. Consider the Lie algebra of symmetries given by the following vector fields $Y_1, \ldots, Y_5$, where

$$Y_1 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} + 2x_3 \frac{\partial}{\partial x_4} + x_2^2 \frac{\partial}{\partial x_5}\,, \qquad Y_2 = \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial x_5}\,,$$
$$Y_3 = \frac{\partial}{\partial x_3}\,, \qquad Y_4 = \frac{\partial}{\partial x_4}\,, \qquad Y_5 = \frac{\partial}{\partial x_5}.$$

These vector fields admit dual forms that are in the following form

$$\Upsilon_1 = \mathrm{d}x_1\,, \qquad \Upsilon_2 = \mathrm{d}x_2\,, \qquad \Upsilon_3 = -x_2\mathrm{d}x_1 + \mathrm{d}x_3\,,$$
$$\Upsilon_4 = -2x_3\mathrm{d}x_1 + \mathrm{d}x_4\,, \qquad \Upsilon_5 = -x_2^2\mathrm{d}x_1 - 2x_3\mathrm{d}x_2 + \mathrm{d}x_5\,.$$

Note that the above dual forms mentioned above exist due to the condition that $Y_1 \wedge \cdots \wedge Y_5 \neq 0$. The differential one-forms $\Upsilon_1, \ldots, \Upsilon_5$ are invariant relative to all vector fields $X_1, \ldots, X_5$, namely $\mathscr{L}_{Y_i} = \Upsilon_j = 0$ for $i, j = 1, \ldots, 5$. Furthermore, the differentials of $\Upsilon_1, \ldots, \Upsilon_5$ depend on the commutation relations of the vector fields $Y_1, \ldots, Y_5$, namely

$$\mathrm{d}\eta^\gamma = c_{\alpha\beta}^{\gamma} \eta^\alpha \wedge \eta^\beta\,, \qquad \alpha, \beta, \gamma = 1, \ldots, 5\,,$$

where $c_{\alpha\beta}^{\gamma}$ are the structure constants of our problem, namely $[X_\alpha, X_\beta] = c_{\alpha\beta}^{\gamma} X_\gamma$ for $\alpha, \beta, \gamma = 1, \ldots, 5$. In particular, for every differential one-form $\Upsilon \in \langle \Upsilon_1, \ldots, \Upsilon_5 \rangle$, its differential is both closed and invariant relative to the vector fields $X_1, \ldots, X_5$. Consequently, these vector fields become $\boldsymbol{\omega}$-Hamiltonian relative to such presymplectic forms. Then, it is straightforward to establish a k-polysymplectic structure turning every element of V into a $\boldsymbol{\omega}$-Hamiltonian vector field. Nevertheless, when it comes to study the relative equilibrium points of a system of the form (14), the above approach is generally insufficient. In particular, one needs to find a symmetry of the vector fields in V that is also $\boldsymbol{\omega}$-Hamiltonian relative to the k-polysymplectic structure. Since $\iota_Y \omega^\alpha$ must then be closed, it frequently forces the development of a more general method to study the systems in that way.

Let us focus on a system of the form (14) determined by the vector field X_3. We observe that $f^1 = x_1x_5 - x_2x_4$ is the first integral of X_3. Moreover, $\mathrm{d}x_3$ is also invariant relative to X_3. Therefore, one proposes

$$\omega^1 = \Upsilon_3 \wedge \Upsilon_2 + \mathrm{d}f^1 \wedge \mathrm{d}x_3 + \mathrm{d}x_4 \wedge \mathrm{d}x_1 \,, \qquad \omega^2 = \Upsilon_2 \wedge \Upsilon_5 + \Upsilon_1 \wedge \Upsilon_3 + \mathrm{d}f^1 \wedge \mathrm{d}x_3 \,.$$

The following relations

$$\mathrm{d}\Upsilon_2 = 0\,, \quad \mathrm{d}\Upsilon_1 = 0\,, \quad \mathrm{d}\Upsilon_3 = \Upsilon_1 \wedge \Upsilon_2\,, \quad \mathrm{d}\Upsilon_4 = 2\Upsilon_1 \wedge \Upsilon_3\,, \quad \mathrm{d}\Upsilon_5 = 2\Upsilon_2 \wedge \Upsilon_3\,.$$

yield that $\mathrm{d}\omega^1 = \mathrm{d}\omega^2 = 0$ and $\ker \omega^1 \cap \ker \omega^2 = 0$. Therefore, $(\mathbb{R}^5, \boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2)$ defines a 2-polysymplectic manifold. It is worth noting that X_3 is $\boldsymbol{\omega}$-Hamiltonian relative since

$$\iota_{X_3}\omega^1 = \mathrm{d}(x_2 - f^1 + x_1 + x_1^2)\,, \qquad \iota_{X_3}\omega^2 = \mathrm{d}(-x_1 - f^1 - x_2^2)\,.$$

Since

$$\iota_{Y_3}\omega^1 = \mathrm{d}x_2 - \mathrm{d}f^1 = \mathrm{d}(x_2 - x_1x_5 + x_4x_2)\,, \qquad \iota_{Y_3}\omega^2 = -\mathrm{d}x_1 - \mathrm{d}f^1 = \mathrm{d}(-x_1 - x_1x_5 + x_4x_2)\,,$$

the momentum map associated with Y_3 is

$$\mathbf{J} : \mathbb{R}^5 \ni x \mapsto (x_2 - x_1x_5 + x_4x_2, -x_1 - x_1x_5 + x_4x_2) \in \mathbb{R}^2\,.$$

Therefore, $\mathbf{J}^{-1}(\mu^1, \mu^2)$ is given by the points $(x_1(x_3, x_4, x_5, \mu^1, \mu^2), x_2(x_3, x_4, x_5, \mu^1, \mu^2), x_3, x_4, x_5)$, where $x_1(x_3, x_4, x_5, \mu^1, \mu^2), x_2(x_3, x_4, x_5, \mu^1, \mu^2)$ are given by the solutions to the following system

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} -1 + x_5 & x_4 \\ -x_5 & 1 + x_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\frac{1}{1 + x_4 + x_5} \begin{pmatrix} 1 + x_4 & -x_4 \\ x_5 & 1 + x_5 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}\,.$$

Since the following relations hold

$$\mathrm{d}x_1|_{\mathbf{J}^{-1}(\boldsymbol{\mu})} = -\mathrm{d}f^1|_{\mathbf{J}^{-1}(\boldsymbol{\mu})}\,, \quad \mathrm{d}x_2|_{\mathbf{J}^{-1}(\boldsymbol{\mu})} = \mathrm{d}f^1|_{\mathbf{J}^{-1}(\boldsymbol{\mu})}\,,$$

the restriction of ω^1, ω^2 to $\mathbf{J}^{-1}(\boldsymbol{\mu})$ is given by

$$\begin{aligned} \iota_{\boldsymbol{\mu}}^*\omega^1 &= \iota_{\boldsymbol{\mu}}^* \left(\Upsilon_3 \wedge \Upsilon_2 + \mathrm{d}f^1 \wedge \mathrm{d}x_3 + \mathrm{d}x_4 \wedge \mathrm{d}x_1\right) \\ &= (f^1\mathrm{d}f^1 + \mathrm{d}x_3) \wedge \mathrm{d}f^1 + \mathrm{d}f^1 \wedge \mathrm{d}x_3 - \mathrm{d}x_4 \wedge \mathrm{d}f^1 = \mathrm{d}f^1 \wedge x_4\,, \end{aligned}$$

and

$$\begin{aligned} \iota_{\boldsymbol{\mu}}^*\omega^2 &= \iota_{\boldsymbol{\mu}}^* \left(\Upsilon_2 \wedge \Upsilon_5 + \Upsilon_1 \wedge \Upsilon_3 + \mathrm{d}f^1 \wedge \mathrm{d}x_3\right) \\ &= \mathrm{d}f^1 \wedge (-x_2^2\mathrm{d}x_1 + \mathrm{d}x_5) - \mathrm{d}f^1 \wedge (\mathrm{d}x_3 - x_2\mathrm{d}f^1) + \mathrm{d}f^1 \wedge \mathrm{d}x_3 = \mathrm{d}f^1 \wedge \mathrm{d}x_5\,. \end{aligned}$$

One can use coordinates x_3, x_4, x_5 on $\mathbf{J}^{-1}(\boldsymbol{\mu})$. Moreover, the action of $\mathbb{R}$ on $\mathbf{J}^{-1}(\boldsymbol{\mu})$ removes the coordinate x_3 and we obtain

$$\begin{aligned} \omega_{\boldsymbol{\mu}}^1 &= \frac{1}{x_4 + x_5 + 1} \left((1 + x_4)\mu^1 - x_4\mu^2\right) \mathrm{d}x_4 \wedge \mathrm{d}x_5\,, \\ \omega_{\boldsymbol{\mu}}^2 &= \frac{-1}{1 + x_4 + x_5} \left(\mu^1 x_5 + (1 + x_5)\mu^2\right) \mathrm{d}x_4 \wedge \mathrm{d}x_5\,. \end{aligned}$$

Then, the projection of the vector field X_3 to the reduction is given by

$$\widetilde{X}_3 = \frac{-1}{1 + x_5 + x_4} \left[2((1 + x_4)k_1 - x_4k_2)\frac{\partial}{\partial x_4} + 2(x_5k_1 + 1 + x_5k_2)\frac{\partial}{\partial x_5}\right]\,.$$

Hence, one has that the relative equilibrium points are given by

$$dh_\xi^1 = 2x_1dx_1, \quad dh_\xi^2 = -2x_2dx_2$$

Thus, $z_e = (0, 0, x_3, x_4, x_5)$.

5.4 Quantum harmonic oscillator

Let us analyse a last example based upon the Wei–Norman equations for the automorphic Lie system related to quantum harmonic oscillators [?]: Comparing the system X^{do} in intrinsic form (??) with the above, we obtain

In this case, the Vessiot–Guldberg Lie algebra is given by the vector fields

$$
\begin{aligned}
X_1^R &= \frac{\partial}{\partial v_1} + v_5\frac{\partial}{\partial v_4} - \frac{1}{2}v_5^2\frac{\partial}{\partial v_6}\,, & X_2^R &= v_1\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2} + \frac{1}{2}v_4\frac{\partial}{\partial v_4} - \frac{1}{2}v_5\frac{\partial}{\partial v_5}\,,\\
X_3^R &= v_1^2\frac{\partial}{\partial v_1} + 2v_1\frac{\partial}{\partial v_2} + e^{v_2}\frac{\partial}{\partial v_3} - v_4\frac{\partial}{\partial v_5} + \frac{1}{2}v_4^2\frac{\partial}{\partial v_6}\,, & X_4^R &= \frac{\partial}{\partial v_4}\,,\\
X_5^R &= \frac{\partial}{\partial v_5} - v_4\frac{\partial}{\partial v_6}\,, & X_6^R &= \frac{\partial}{\partial v_6}\,.
\end{aligned}
$$

The commutation relations between the above vector fields read

$$
\begin{array}{lllll}
[X_1^R, X_2^R] = X_1^R\,, & & & & \\
[X_1^R, X_3^R] = 2\,X_2^R\,, & [X_2^R, X_3^R] = X_3^R\,, & & & \\
[X_1^R, X_4^R] = 0\,, & [X_2^R, X_4^R] = -\frac{1}{2}\,X_4^R\,, & [X_3^R, X_4^R] = X_5^R\,, & & \\
[X_1^R, X_5^R] = -X_4^R\,, & [X_2^R, X_5^R] = \frac{1}{2}\,X_5^R\,, & [X_3^R, X_5^R] = 0\,, & [X_4^R, X_5^R] = -X_6^R\,, & \\
[X_1^R, X_6^R] = 0\,, & [X_2^R, X_6^R] = 0\,, & [X_3^R, X_6^R] = 0\,, & [X_4^R, X_6^R] = 0\,, & [X_5^R, X_6^R] = 0\,.
\end{array}
$$

It is known that the Lie algebra of symmetries of these vector fields is given by

$$
\begin{aligned}
X_1^L &= e^{v_2}\frac{\partial}{\partial v_1} + 2v_3\frac{\partial}{\partial v_2} + v_3^2\frac{\partial}{\partial v_3}\,, \quad X_2^L = \frac{\partial}{\partial v_2} + v_3\frac{\partial}{\partial v_3}\,, \quad X_3^L = \frac{\partial}{\partial v_3}\,,\\
X_4^L &= e^{-v_2/2}(e^{v_2} - v_1v_3)\frac{\partial}{\partial v_4} - e^{-v_2/2}v_3\frac{\partial}{\partial v_5} - e^{-v_2/2}(e^{v_2} - v_1v_3)v_5\frac{\partial}{\partial v_6}\,,\\
X_5^L &= v_1e^{-v_2/2}\frac{\partial}{\partial v_4} + e^{-v_2/2}\frac{\partial}{\partial v_5} - v_1v_5e^{-v_2/2}\frac{\partial}{\partial v_6}\,, \quad X_6^L = \frac{\partial}{\partial v_6}\,.
\end{aligned}
$$

In particular, let us focus on

$$X_5^R = \frac{\partial}{\partial v_5} - v_4\frac{\partial}{\partial v_6}\,.$$

Then, a symmetry of our system is given by

$$Y = \frac{\partial}{\partial v_5}\,.$$

The 2-polysymplectic structure on $\mathbb{R}^6$ can be defined in the following way

$$
\begin{aligned}
\boldsymbol{\omega} &= \omega^1\otimes e_1 + \omega^2\otimes e_2\\
&= (\mathrm{d}v_1\wedge\mathrm{d}v_3 + \mathrm{d}v_2\wedge\mathrm{d}v_4 + \mathrm{d}v_5\wedge\mathrm{d}v_1 + \mathrm{d}v_4\wedge\mathrm{d}v_6)\otimes e_1 + (\mathrm{d}v_4\wedge\mathrm{d}v_6 - \mathrm{d}v_3\wedge\mathrm{d}v_5)\otimes e_2\,.
\end{aligned}
$$

Note that

$$\ker\omega^1 = \left\langle \frac{\partial}{\partial v_3} + \frac{\partial}{\partial v_5}, \frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_6}\right\rangle\,, \qquad \ker\omega^2 = \left\langle \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}\right\rangle,$$

hence $\ker\omega_x^1\cap\omega_x^2 = 0$ and indeed $(\mathbb{R}^6, \boldsymbol{\omega})$ is a 2-polysymplectic manifold. The vector field Y is also a symmetry of the 2-polysymplectic structure, i.e. $\mathscr{L}_Y\boldsymbol{\omega} = 0$. Then,

$$\iota_{Y_3}\omega^1 = \mathrm{d}v_1\,, \qquad \iota_{Y_3}\omega^2 = \mathrm{d}v_3\,,$$

and the momentum map $\mathbf{J}^\Phi$ reads

$$\mathbf{J}^\Phi : \mathbb{R}^6 \ni x \longmapsto (v_1, v_3) = \boldsymbol{\mu} \in (\mathfrak{g}^*)^2 \simeq \mathbb{R}^2\,.$$

Note that $\boldsymbol{\mu}$ is a regular value of $\mathbf{J}^\Phi$, hence $\mathbf{J}$

The vector field X_5^R is $\boldsymbol{\omega}$-Hamiltonian with

$$\iota_X\boldsymbol{\omega} = \iota_{X_5^R}\omega^1 \otimes e_1 + \iota_X\omega^2 \otimes e_2 = \mathrm{d}\left(v_1 + \frac{v_4^2}{2}\right) \otimes e_1 + \mathrm{d}\left(v_3 + \frac{v_4^2}{2}\right) \otimes e_2 \,.$$

If we restrict ourselves to the submanifold given by the v_1, v_3 being a constant, then

$$\omega^1|_J = \mathrm{d}v_2 \wedge \mathrm{d}v_4 + \mathrm{d}v_4 \wedge \mathrm{d}v_6 \,, \qquad \omega^2|_J = \mathrm{d}v_4 \wedge \mathrm{d}v_6 \,.$$

We have

$$\ker\omega^1|_J = \left\langle \frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_6}, \frac{\partial}{\partial v_5} \right\rangle \,, \qquad \ker\omega^2|_J = \left\langle \frac{\partial}{\partial v_2}, \frac{\partial}{\partial v_5} \right\rangle \,,$$

and then they do not define a 2-polysymplectic structure.

5.5 Example

Consider the vector field X on $\mathbb{R}^8 \ni (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$, given by

$$X_{\boldsymbol{f}} = x_6^a \frac{\partial}{\partial x_2} + x_4^b \frac{\partial}{\partial x_3} - x_3^c \frac{\partial}{\partial x_4} + x_8^d \frac{\partial}{\partial x_7} - x_7^e \frac{\partial}{\partial x_8} \,,$$

where $a, b, c, d, e \in \mathbb{N}$. On $\mathbb{R}^8$ the 2-polysymplectic can be defined as

$$\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2 = (\mathrm{d}x_3 \wedge \mathrm{d}x_4 + \mathrm{d}x_1 \wedge \mathrm{d}x_5) \otimes e_1 + (\mathrm{d}x_2 \wedge \mathrm{d}x_6 + \mathrm{d}x_7 \wedge \mathrm{d}x_8) \otimes e_2 \,.$$

Then, $\ker\omega_x^1 = \left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \right\rangle$ and $\ker\omega_x^2 = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5} \right\rangle$, hence $\ker\omega_x^1 \cap \ker\omega_x^2 = 0$ for any $x \in \mathbb{R}^8$.

The vector field X admits symmetries, $Y_1 = \frac{\partial}{\partial x_1}$, $Y_2 = \frac{\partial}{\partial x_2}$, $Y_3 = \frac{\partial}{\partial x_5}$. These symmetries correspond to translations along the x_1, x_2, and x_5 coordinates, and they also leave the 2-polysymplectic structure invariant, i.e. $\mathscr{L}_{Y_i}\omega^\alpha = 0$ for $i = 1, 2, 3$ and $\alpha = 1, 2$.

The momentum maps are computed as follows

$$\begin{aligned} &\iota_{Y_1}\omega^1 = 0\,, \qquad &&\iota_{Y_2}\omega^1 = \mathrm{d}x_5\,, \qquad &&\iota_{Y_3}\omega^1 = -\mathrm{d}x_1\,,\\ &\iota_{Y_1}\omega^2 = \mathrm{d}x_6, \qquad &&\iota_{Y_2}\omega^2 = 0\,, \qquad &&\iota_{Y_3}\omega^2 = 0\,. \end{aligned}$$

Therefore, the momentum map $\mathbf{J}^\Phi$ is given by

$$\mathbf{J}^\Phi : \mathbb{R}^8 \ni x \mapsto \mathbf{J}^\Phi(x) = \boldsymbol{\mu} = (0, x_5, -x_1; x_6, 0, 0) \in (\mathfrak{g}^*)^2 = (\mathbb{R}^3)^2 \,.$$

Then, $\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \right\rangle$ for $x \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. The momentum map $\mathbf{J}^\Phi$ is Ad^{*2}-equivariant and $\boldsymbol{\mu} \in (\mathbb{R}^3)^2$ is a weakly regular value of $\mathbf{J}^\Phi$ since $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \simeq \mathbb{R}^5$ is a submanifold of $\mathbb{R}^8$.

Note, that Y_1 and Y_3 do not belong to $\mathrm{T}_x(G_{\boldsymbol{\mu}}x)$ but Y_1 does. The assumptions of Theorem 3.2 are satisfied, and the quotient space $\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_x(G_{\boldsymbol{\mu}}x)$ is a 2-polysymplectic manifold, where

$$\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_x(G_{\boldsymbol{\mu}}x) = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \right\rangle \,,$$

and

$$\boldsymbol{\omega}_{\boldsymbol{\mu}} = \omega_{\boldsymbol{\mu}}^1 \otimes e_1 + \omega_{\boldsymbol{\mu}}^2 \otimes e_2 = (\mathrm{d}x_3 \wedge \mathrm{d}x_4) \otimes e_1 + (\mathrm{d}x_7 \wedge \mathrm{d}x_8) \otimes e_2 \,.$$

The vector field X is $\boldsymbol{\omega}$-Hamiltonian, with

$$\begin{aligned} \mathrm{d}\boldsymbol{f} &= \iota_X\boldsymbol{\omega} = \iota_X\omega^1 \otimes e_1 + \iota_X\omega^2 \otimes e_2 \\ &= \mathrm{d}\left(\frac{1}{1+b}x_4^{b+1} + \frac{1}{c+1}x_3^{c+1}\right) \otimes e_1 + \mathrm{d}\left(\frac{1}{1+a}x_6^{a+1} + \frac{1}{d+1}x_8^{d+1} + \frac{1}{1+e}x_7^{e+1}\right) \otimes e_2\,. \end{aligned}$$

Then, by Theorem 3.4 the vector field X project onto the quotient manifold and its projection X_μ is given by

$$X_{\boldsymbol{\mu}} = x_4^b\frac{\partial}{\partial x_3} - x_3^c\frac{\partial}{\partial x_4} + x_8^d\frac{\partial}{\partial x_7} - x_7^e\frac{\partial}{\partial x_8}\,.$$

The vector field $X_{\boldsymbol{\mu}}$ is $\boldsymbol{\omega_\mu}$-Hamiltonian vector field, with

$$\mathrm{d}\boldsymbol{f_\mu} = \iota_{X_\mu}\boldsymbol{\omega_\mu} = \mathrm{d}\left(\frac{1}{1+b}x_4^{b+1} + \frac{1}{c+1}x_3^{c+1}\right) \otimes e_1 + \mathrm{d}\left(\frac{1}{d+1}x_8^{d+1} + \frac{1}{1+e}x_7^{e+1}\right) \otimes e_2\,.$$

According to Theorem 4.2, a point z_e is a k-polysymplectic relative equilibrium point if it is a critical point of $\boldsymbol{f}_\xi$ for $\xi \in \mathfrak{g} \simeq \mathbb{R}$. Then, one has that

$$\mathrm{d}\boldsymbol{f}_\xi = \mathrm{d}f_\xi^1 \otimes e_1 + \mathrm{d}f_\xi^2 \otimes e_2 = \left(x_4^b\mathrm{d}x_4 + x_3^c\mathrm{d}x_3\right) \otimes e_1 + \left((x_6^a - 1)\mathrm{d}x_6 + x_8^d\mathrm{d}x_8 + x_7^e\mathrm{d}x_7\right) \otimes e_2\,.$$

From this, it follows that points $z_e^\pm = (x_1, x_2, 0, 0, x_5, \pm1, 0, 0)$ are 2-polysymplectic relative equilibrium points of X.

To analyse the stability of these 2-polysymplectic relative equilibrium points, let's look at the second derivative of $\boldsymbol{f}_\xi$. Then,

$$\begin{aligned} \delta^2\boldsymbol{f}_\xi &= \delta^2 f_\xi^1 \otimes e_1 + \delta^2 f_\xi^2 \otimes e_2 \\ &= \left(cx_3^{c_1}\mathrm{d}x_3 \otimes \mathrm{d}x_3 + bx_4^{b-1}\mathrm{d}x_4 \otimes \mathrm{d}x_4\right) \otimes e_1 + \left(ex_7^{e-1}\mathrm{d}x_7 \otimes \mathrm{d}x_7 + dx_8^{d-1}\mathrm{d}x_8 \otimes \mathrm{d}x_8\right) \otimes e_2\,. \end{aligned}$$

Taking into account that $\mathrm{T}_{z_e}\mathcal{S}/\mathrm{T}_{z_e}\mathcal{S} \cap \ker\omega_x^1 = \langle\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\rangle$ and $\mathrm{T}_{z_e}\mathcal{S}/\mathrm{T}_{z_e}\mathcal{S} \cap \ker\omega_x^2 = \langle\frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8}\rangle$, Theorem 4.6 gives that $z_e^\pm$ is a stable k-polysymplectic relative equilibrium point if

$$(\delta^2 f_\xi^1)_{z_e}(v_{z_e}, v_{z_e}) > 0, \quad \forall v_{z_e} \in \mathrm{T}_{z_e}\mathcal{S}/\left(\mathrm{T}_{z_e}\mathcal{S} \cap \ker\omega_{z_e}^1\right),$$

and

$$(\delta^2 f_\xi^2)_{z_e}(v_{z_e}, v_{z_e}) > 0, \quad \forall v_{z_e} \in \mathrm{T}_{z_e}\mathcal{S}/\left(\mathrm{T}_{z_e}\mathcal{S} \cap \ker\omega_{z_e}^2\right)\,.$$

These inequalities hold if and only if $b, c, d, e = 1$. Hence, $z_e^\pm$ are stable k-symplectic relative equilibrium points of X. Indeed, $(X_{\boldsymbol{\mu}_e})_{[z_e]} = 0$.

6 Conclusions and Outlook

In the present work, we have devised a new energy-momentum method for systems of ordinary differential equations with an underlying k-polysymplectic structure. In order to do so, we have also performed several improvements to previous Marsden–Weinstein reductions for k-polysymplectic systems [19, 30]. In order to illustrate this new energy-momentum method, we have studied several relevant examples in detail, including the cotangent bundle of k-velocities, the complex Schwarz equation, the product of several symplectic manifolds, a control system, and the quantum harmonic oscillator.

A non-autonomous analogue of the methods devised in this paper can be accomplished by using the Lyapunov theory depicted in [17].

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References

[1] H. Abarbanel and D. Holm. Nonlinear stability analysis of inviscid flows in three dimensions: incompressible fluids and barotropic fluids. *Phys. Fluids*, **30**(11):3369–3382, 1987. 10.1063/1.866469.

[2] R. Abraham and J. E. Marsden. *Foundations of mechanics*, volume 364 of *AMS Chelsea publishing*. Benjamin/Cummings Pub. Co., New York, 2nd edition, 1978. 10.1090/chel/364.

[3] A. Awane. k-symplectic structures. *J. Math. Phys.*, **33**(12):4046, 1992. 10.1063/1.529855.

[4] A. Awane and M. Goze. *Pfaffian systems, k-symplectic systems*. Springer, Dordrecht, 1st edition, 2000. 10.1007/978-94-015-9526-1.

[5] C. Blacker. Polysymplectic reduction and the moduli space of flat connections. *J. Phys. A: Math. Theor.*, **52**(33):335201, 2019. 10.1088/1751-8121/ab2eed.

[6] A. M. Bloch. *Nonholonomic mechanics and control*, volume 24 of *Interdisciplinary Applied Mathematics*. Springer, New York, second edition, 2015. With the collaboration of J. Bailieul, P. E. Crouch, J. E. Marsden and D. Zenkov, With scientific input from P. S. Krishnaprasad and R. M. Murray.

[7] L. Búa, I. Bucataru, M. de León, M. Salgado, and S. Vilariño. Symmetries in Lagrangian field theory. *Rep. Math. Phys.*, **75**(3):333–357, 2015. 10.1016/S0034-4877(15)30010-0.

[8] J. F. Cariñena, J. Grabowski, and G. Marmo. *Lie–Scheffers systems: a geometric approach*. Napoli Series on Physics and Astrophysics. Bibliopolis, Naples, 2000.

[9] J. F. Cariñena, J. Grabowski, and G. Marmo. Superposition rules, lie theorem, and partial differential equations. *Rep. Math. Phys.*, **60**(2):237–258, 2007. 10.1016/S0034-4877(07)80137-6.

[10] M. de León, E. Merino, J. A. Oubiña, and M. Salgado. Stable almost cotangent structures. *Bolletino Unione Mat. Ital. B (7)*, **11**(3):509–529, 1997.

[11] M. de León, I. Méndez, and M. Salgado. p-almost tangent structures. *Rend. Circ. Mat. Palermo*, **37**(2):282–294, 1988. 10.1007/BF02844526.

[12] M. de León, I. Méndez, and M. Salgado. Regular p-almost cotangent structures. *J. Korean Math. Soc.*, **25**(2):273–287, 1988. https://jkms.kms.or.kr/journal/view.html?spage=273&volume=25&number=2.

[13] M. de León, M. Salgado, and S. Vilariño. *Methods of Differential Geometry in Classical Field Theories*. World Scientific, 2015. 10.1142/9693.

[14] J. de Lucas, X. Rivas, S. Vilariño, and B. M. Zawora. On k-polycosymplectic Marsden–Weinstein reductions. *J. Geom. Phys.*, **191**:104899, 2023. 10.1016/j.geomphys.2023.104899.

[15] J. de Lucas and C. Sardón. *A Guide to Lie Systems with Compatible Geometric Structures*. World Scientific Publishing Co. Pte. Ltd., Singapore, 2020. 10.1142/q0208.

[16] J. de Lucas and S. Vilariño. k-symplectic Lie systems: theory and applications. *J. Differ. Equ.*, **258**(6):2221–2255, 2015. 10.1016/j.jde.2014.12.005.

[17] J. de Lucas and B. M. Zawora. A time-dependent energy-momentum method. *J. Geom. Phys.*, **170**:104364, 2021. 10.1016/j.geomphys.2021.104364.

[18] A. Echeverría-Enríquez, M. C. Muñoz-Lecanda, and N. Román-Roy. Geometry of Lagrangian first-order classical field theories. *Fortschr. Phys.*, **44**(3):235–280, 1996. 10.1002/prop.2190440304.

[19] E. García-Toraño Andrés and T. Mestdag. Conditions for symmetry reduction of polysymplectic and polycosymplectic structures. *J. Phys. A: Math. Theor.*, **56**(33):335202, 2023. 10.1088/1751-8121/ace74c.

[20] J. Gaset, A. López-Gordón, and X. Rivas. Symmetries, conservation and dissipation in time-dependent contact systems. *Fortschr. Phys.*, **71**(8–9):2300048, 2023. 10.1002/prop.202300048.

[21] G. Giachetta, L. Mangiarotti, and G. A. Sardanashvily. *New Lagrangian and Hamiltonian Methods in Field Theory*. World Scientific, River Edge, 1997. 10.1142/2199.

[22] L. Guieu and C. Roger. *L'algèbre et le groupe de Virasoro*. Les Publications CRM, Montreal, QC, 2007. Aspects géométriques et algébriques, généralisations. [Geometric and algebraic aspects, generalizations], With an appendix by Vlad Sergiescu.

[23] C. Günther. The polysymplectic Hamiltonian formalism in field theory and calculus of variations I: The local case. *J. Differ. Geom.*, **25**(1):23–53, 1987. 10.4310/jdg/1214440723.

[24] E. Hille. *Ordinary differential equations in the complex domain*. Dover Publications, Inc., Mineola, NY, 1997. Reprint of the 1976 original.

[25] B. M. Z. J. de Lucas, A. Maskalaniec. A cosymplectic energy-momentum method with applications. 2302.05827, 2023.

[26] I. V. Kanatchikov. Canonical structure of classical field theory in the polymomentum phase space. *Rep. Math. Phys.*, **41**(1):49–90, 1998. 10.1016/S0034-4877(98)80182-1.

[27] J. M. Lee. *Manifolds and differential geometry*, volume 107 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009.

[28] J. M. Lee. *Introduction to Smooth Manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer New York Heidelberg Dordrecht London, 2nd edition, 2012. 10.1007/978-1-4419-9982-5.

[29] O. Lehto. Remarks on Nehari's theorem about the Schwarzian derivative and schlicht functions. *J. Analyse Math.*, 36:184–190 (1980), 1979.

[30] J. C. Marrero, N. Román-Roy, M. Salgado, and S. Vilariño. Reduction of polysymplectic manifolds. *J. Phys. A: Math. Theor.*, **48**(5):055206, 2015. 10.1088/1751-8113/48/5/055206.

[31] J. C. Marrero, N. Román-Roy, M. Salgado, and S. Vilariño. Symmetries, Noether's theorem and reduction in k-cosymplectic field theories. *AIP Conference Proceedings*, 1260(1):173–179, 2010. 10.1063/1.3479319.

[32] J. E. Marsden, T. A. Posbergh, and J. C. Simo. Stability of coupled rigid body and geometrically exact rods: block diagonalization and the energy–momentum method. *Phys. Rep.*, **193**:280–360, 1990. 10.1016/0370-1573(90)90125-L.

[33] J. E. Marsden and T. Ratiu. *Introduction to Mechanics and Symmetry. A basic exposition of classical mechanical systems*, volume 17 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1999. 10.1007/978-0-387-21792-5.

[34] J. E. Marsden and J. C. Simo. The energy momentum method. *Act. Acad. Sci. Tau.*, **1**(124):245–268, 1988. https://resolver.caltech.edu/CaltechAUTHORS:20101019-093814651.

[35] J. E. Marsden, J. C. Simo, D. Lewis, and T. Posbergh. A block diagonalization theorem in the energy-momentum method. In *Dynamics and control of multibody systems*, volume 97 of *Contemp. Math.*, pages 297–313. Amer. Math. Soc., Providence, RI, 1989. 10.1090/conm/097/1021043.

[36] M. McLean and L. K. Norris. Covariant field theory on frame bundles of fibered manifolds. *J. Math. Phys.*, **41**(10):6808–6823, 2000. 10.1063/1.1288797.

[37] E. E. Merino. *Geometría k-simpléctica y k-cosimpléctica. Aplicaciones a las teorías clásicas de campos*. PhD thesis, Universidade de Santiago de Compostela, 1997.

[38] L. K. Norris. Generalized symplectic geometry on the frame bundle of a manifold. In *Proc. Symp. Pure Math.*, volume 54.2, pages 435–465. Amer. Math. Soc., Providence RI, 1993. 10.1090/pspum/054.2.

[39] J. Ortega, V. Planas-Bielsa, and T. Ratiu. Asymptotic and Lyapunov stability of constrained and Poisson equilibria. *J. Differ. Equ.*, **214**(1):92–127, 2005. 10.1016/j.jde.2004.09.016.

[40] J. P. Ortega and T. S. Ratiu. *Momentum maps and Hamiltonian reduction*, volume 222 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, 2004. 10.1007/978-1-4757-3811-7.

[41] M. Razavy. *Classical and quantum dissipative systems*. Imperial College Press, World Scientific Publishing Co., London, UK, 2006. 10.1142/p376.

[42] N. Román-Roy, M. Salgado, and S. Vilariño. Symmetries and conservation laws in the Günther k-symplectic formalism of field theory. *Rev. Math. Phys.*, **19**(10):1117–1147, 2007. 10.1142/S0129055X07003188.

[43] G. A. Sardanashvily. *Generalized Hamiltonian formalism for field theory. Constraint systems*. World Scientific Publishing Company, River Edge, NJ, 1995. 10.1142/2550.

[44] J. Simo, D. Lewis, and J. Marsden. Stability of relative equilibria. Part I: The reduced energy-momentum method. *Arch. Rat. Mech. Anal.*, **115**:15–59, 1991. 10.1007/BF01881678.

[45] J. Simo and N. Tarnow. The discrete energy-momentum method. Conserving algorithms for nonlinear elastodynamics. *Z. Angew. Math. Phys.*, **43**:757–792, 1992. 10.1007/BF00913408.

[46] J. Simon. On the integrability of representations of finite dimensional real Lie algebras. *Comm. Math. Phys.*, **28**:39–46, 1972. 10.1007/BF02099370.

[47] M. Vidyasagar. *Nonlinear systems analysis*, volume 42 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. 10.1137/1.9780898719185.

[48] L. Wang and P. Krishnaprasad. Gyroscopic control and stabilization. *J. Nonlinear Sci.*, **2**:367–415, 1992. 10.1007/BF01209527.

[49] P. Winternitz. Lie groups and solutions of nonlinear differential equations. In K. B. Wolf, editor, *Nonlinear Phenomena*, volume 189 of *Lecture Notes in Physics*, pages 263–305. Springer, Berlin, Heidelberg, 1983. 10.1007/3-540-12730-5_12.

[50] B. M. Zawora. A time-dependent energy-momentum method. Master's thesis, University of Warsaw, Faculty of Physics, 2021.

[51] D. V. Zenkov, A. M. Bloch, and J. E. Marsden. The energy-momentum method for the stability of non-holonomic systems. *Dyn. Stab. Syst.*, **13**(2):123–165, 1998. 10.1080/02681119808806257.