Hamilton–Jacobi theory for non-conservative field theories in the k-contact framework

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October 2, 2023

Abstract

This article inspects the dynamics of classical field theories described via k-contact structures. A generalisation of the evolutionary contact vector field to the k-contact realm is provided and its Hamilton–De Wonder Weyl equations are defined and analysed. In particular, we develop a Hamilton–Jacobi theory for non-conservative Hamiltonian field theories. We shall introduce two non-equivalent approaches: one based on the reconstruction of the dynamics on E from a vector field defined on the base manifold M of a fiber bundle $\rho : E \to M$, and other relying in the reconstruction of the dynamics from a vector field defined on $M \times \mathbb{R}^k$, where $\mathbb{R}^k$ corresponds with the space of k-parameters describing the k-contact forms that are compatible with our Hamiltonian system. Our results are applied to a damped vibrating string.

Keywords: Hamilton–Jacobi equation, k-contact structure.
MSC 2020 codes:

Contents

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1 Introduction

Classical field theories have a long history in physics, since they predict how physical fields (electromagnetism or the gravitational field) interact with matter. One realizes that the classical Hamiltonian framework of mechanics needed to be extended to these objects called fields, which are quantities assigned at each point of space and time, or more geometrically, in points of a manifold. Later, field theory was also adapted to the quantum realm [29, 24], but we here will only focus on classical theories. Nonetheless, establishing the geometric foundations of classical field theory is by no means trivial, and that is why the first inklings of geometric field theory and variational calculus were not properly established until roughly 50 years ago, with the advent of gauge theories (see [1, 13, 14, 23]).

The k-polysymplectic ($k \geq 1$) formalism is the simplest geometric framework for describing classical field theories. It is a generalization to field theories of the standard symplectic formalism in autonomous mechanics [21], and is used to describe geometrically field theories with associated Lagrangian and Hamiltonian functions that do not depend on the spatial or time coordinates [3, 4, 5]. The generalisation of Hamilton equations to field theory receives the name of *Hamilton–De Donder–Weyl* (HDW) equations.

One specific feature of field theory is that the field equations, namely HDW equations, can be written in a geometrical way using k-vector fields. However, although integral sections of k-vector fields (i.e., integrable distributions) that are solutions to the geometrical field equations are proved to be solutions to the HDW equations, the converse is not always true. This also occurs with other geometric descriptions of classical field theories in terms of multivector fields, which receive the name of multisymplectic theories [15, 16]. What is more, it also happens in the case with which are concerned in this paper, that is, the case of k-contact structures [18, 20]. A k-contact structure is a tuple $(N, \eta^1, \ldots, \eta^k)$, where N is a manifold and $\eta^1, \ldots, \eta^k$ is a family of differential one-forms with certain compatibility conditions. The case $k = 1$ corresponds to contact structures, which have been widely studied [8, 11, 12, 17, 19, 30].

In this work we extend the evolutionary Hamiltonian vector field notion appearing in contact geometry that were relevant to the description of thermodynamical systems. The new concept, the so-called evolutionary k-contact Hamiltonian vector field, is employed to study its associated Hamilton-De Wonder-Weyl equations. As in the typical case in k-contact geometry, the relation is given by considering the partial differential equations determining the integral curves of evolutionary k-contact Hamiltonian vector fields.

Formalizing the intertwining of contact structures and the k-symplectic formalism, one is able to describe geometrically some physical problems, for example, a damped vibrating string of the type

$$\rho u_{tt} - \tau u_{xx} + \gamma \rho u_t = 0, \tag{1}$$

where ρ, τ, γ are scalar parameters and $u = u(x, t)$ represents the wave propagating in time, t, along one spatial dimension, x, of the string. Indeed, this will be our main example throughout the development of our geometric fundamentals for a Hamilton–Jacobi equation on k-contact manifolds. Also, the fact that several different k-contact vector fields give rise to the same HDW equations will be a keypoint, since this is an important issue to take into account when one develops a Hamilton–Jacobi theory. Let us introduce why. Long story short, the geometric Hamilton–Jacobi theory that we shall perform here is based on the integrability of vector fields and their projectability to lower dimensional manifolds.

Roughly speaking, in the geometric Hamilton–Jacobi set up, one describes the full dynamics of a mechanical system as the integral curves of a Hamiltonian vector field defined on a certain manifold. This Hamiltonian vector field usually has a nontrivial form, and its integration is not straightforward. Therefore, one may picture this vector field as the tangent lift of another simpler vector field defined on a lower dimensional manifold. This means, that in order to relate the vector field on the lower dimensional manifold and the vector field describing the full dynamics, we need to find a section of a certain vector bundle for which the vector field on the lower dimensional manifold is a projection of the dynamics on the section. This section is what we understand as the solution of the Hamilton–Jacobi equation, and in the classical approach, it is the differential of a transformation function that renders the dynamics simpler. To make this clearer, let us represent it through the following classic commutative diagram

$$\begin{tikzcd} E \arrow[r, "X_h"] \arrow[d, "\rho"] & \mathrm{T}E \arrow[d, "\mathrm{T}\rho"'] \\ M \arrow[r, "X_h^{\gamma}"] \arrow[u, bend left, "\gamma"] & \mathrm{T}M \arrow[u, bend right, "\mathrm{T}\gamma"'] \end{tikzcd} \tag{2}$$

In the diagram above, the section $\gamma : M \to E$ of the bundle $\rho : E \to M$ is called a solution of the Hamilton–Jacobi equation, and it corresponds with an exact differential of $\gamma = \mathrm{d}S$, where S is a transformation function (canonical in the symplectic case) than renders the dynamics trivial. Notice that the vector fields X_h and X_h^{γ} are related if the diagram is commutative, i.e., we can reconstruct the dynamics X_h on $\gamma(M)$ mediated through X_h^{γ} and the relation

$$\mathrm{T}\rho(X_h \circ \gamma) = X_h^{\gamma}.$$

Realize that this is a very general diagram and that it can be applied to several scenarios. In classical mechanics, $E = \mathrm{T}^*Q$ for a configuration manifold Q, which corresponds with the canonical projection, i.e., $\rho = \pi_Q$ and $\pi_Q : \mathrm{T}^*Q \to Q$. In the case of k-symplectic manifolds [14], one has $E = \oplus^k \mathrm{T}^*Q$.

Concerning the case of k-contact structures, $E = \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$, where $\mathbb{R}^k$ corresponds with the k-contact parameters that describe the k-compatible contact forms. It is important to define here which will be our base space M in (2). There are two possibilities: the first one is that $M = Q$, which leads to a k-parameter dependent Hamilton–Jacobi theory. Considering the k-parameters named z^α with $\alpha = 1, \ldots, k$, we will say that this first approach is the z^α-independent approach. The second approach is the z^α-dependent approach in which $M = Q \times \mathbb{R}^k$. We will study both cases and solve the dynamics of a particular example bearing in mind the theoretical particularities. In this way, the outline of the paper goes as follows: Section 2 contains the fundamentals of k-contact structures and k-contact Hamiltonian systems, as well as the HDW equations for k-contact field theory. Section 3 develops a Hamilton–Jacobi field theory for k-contact structures in two different ways, by defining two different projections and introduces the geometry of integrable k-contact Hamiltonian systems. Throughout the section we will display examples of application of our novel Hamilton–Jacobi theory for k-contact structures. A good example of application is that of a damped vibrating string described in (1). In Section 4 we conclude with some remarks and future research lines.

Note: use labels and refs for sections once the paper's structure is decided.

2 k-contact structures and k-contact Hamiltonian systems

2.1 k-vector fields and integral sections

The notion of a k-vector field is of great use in the geometric study of partial differential equations (see for instance [14, 26]). In particular, the dynamics of the so-called k-polycosymplectic Hamiltonian systems is determined by k-vector fields. Hence, the reduction of the dynamics is correlated with the reduction of the so-called Hamiltonian k-vector field. Let us present some relevant details.

Let M be an n-dimensional manifold and $\mathrm{T}M$ its tangent bundle. Consider the Whitney sum of k copies of its tangent bundle as: $\bigoplus^k \mathrm{T}M = \mathrm{T}M \oplus_M \stackrel{(k)}{\cdots} \oplus_M \mathrm{T}M$, and the natural projections

$$\mathrm{pr}^\alpha \colon \bigoplus^k \mathrm{T}M \to \mathrm{T}M\,, \qquad \mathrm{pr}^1_M \colon \bigoplus^k \mathrm{T}M \to M\,,$$

where pr^α denotes the projection to the α-th component of the Whitney sum.

Definition 2.1. A *k-vector field* on a manifold M is a section $\mathbf{X}\colon M \to \oplus^k \mathrm{T}M$ of the projection pr^1_M. Throughout the paper, $\mathfrak{X}^k(M)$ will denote the set of all k-vector fields on M.

$$\begin{tikzcd} & \bigoplus^k \mathrm{T}M \arrow[d, "\mathrm{pr}^\alpha"] \\ M \arrow[ur, "\mathbf{X}"] \arrow[r, "X_\alpha"'] & \mathrm{T}M \end{tikzcd}$$

Taking into account the diagram above, a k-vector field $\mathbf{X} \in \mathfrak{X}^k(M)$ amounts to a number k of vector fields $X_1, \dots, X_k \in \mathfrak{X}(M)$, given by $X_\alpha = \mathrm{pr}^\alpha \circ \mathbf{X}$ with $\alpha = 1, \dots, k$. With this in mind, one can denote $\mathbf{X} = (X_1, \dots, X_k)$. A k-vector field $\mathbf{X}$ induces a decomposable contravariant skew-symmetric tensor field, $X_1 \wedge \cdots \wedge X_k$, which is a section of the bundle $\bigwedge^k \mathrm{T}M \to M$. This also induces a tangent distribution on M.

Definition 2.2. Given a map $\phi\colon U \subset \mathbb{R}^k \to M$, we define its *first prolongation* to $\oplus^k \mathrm{T}M$ as the map $\phi'\colon U \subset \mathbb{R}^k \to \bigoplus^k \mathrm{T}M$, defined by

$$\phi'(t) = \left(\phi(t); \mathrm{T}_t\phi\left(\frac{\partial}{\partial t^1}\bigg|_t\right), \dots, \mathrm{T}_t\phi\left(\frac{\partial}{\partial t^k}\bigg|_t\right)\right) \equiv (\phi(t); \phi'_\alpha(t))\,,$$

where $t = (t^1, \dots, t^k)$ are the canonical coordinates on $\mathbb{R}^k$.

As in the same case of integral curves of vector fields, we can define integral sections of a k-vector field as follows.

Definition 2.3. Let $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k(M)$ be a k-vector field. An *integral section* of $\mathbf{X}$ is a map $\phi\colon U \subset \mathbb{R}^k \to M$ such that

$$\phi' = \mathbf{X} \circ \phi\,,$$

that is, $\mathrm{T}_t\phi\left(\frac{\partial}{\partial t^\alpha}\big|_t\right) = X_\alpha \circ \phi(t)$ for $\alpha = 1, \dots, k$. A k-vector field $\mathbf{X} \in \mathfrak{X}^k(M)$ is *integrable* if every point of M is in the image of an integral section of $\mathbf{X}$.

Consider a k-vector field $\mathbf{X} = (X_1, \dots, X_k)$ with local expression $X_\alpha = X^i_\alpha \frac{\partial}{\partial x^i}$ for $\alpha = 1, \dots, k$, and $i = 1, \dots, n$. Then, $\phi\colon U \subset \mathbb{R}^k \to M$ is an integral section of $\mathbf{X}$ if, and only if, it is a solution of the system of partial differential equations

$$\frac{\partial \phi^i}{\partial t^\alpha} = X^i_\alpha(\phi)\,, \qquad i = 1, \dots, n, \quad \alpha = 1, \dots, k\,.$$

Then, $\mathbf{X}$ is integrable if, and only if, $[X_\alpha, X_\beta] = 0$ for $\alpha, \beta = 1, \dots, k$. These are precisely the necessary and sufficient conditions for the integrability of the above systems of partial differential equations [25].

2.2 k-contact geometry

Definition 2.4. Let M be an m-dimensional smooth manifold.

- A *generalized distribution* on M is a subset $D \subset \mathrm{T}M$ such that $D_x = D \cap \mathrm{T}_x M$ is a vector subspace of $\mathrm{T}_x M$ for every $x \in M$. We call $\dim D_x$ the *rank* of D at $x \in M$.
- A distribution D is *smooth* if it can be locally spanned by a family of vector fields, i.e. if there exists, for every $x \in M$, a family of vector fields $X_1, \ldots, X_r$ defined in a neighbourhood U of x such that $D_{x'} = \langle X_1(x'), \ldots, X_r(x') \rangle$ for every $x' \in U$.
- A distribution D is *regular* if it is smooth and of locally constant rank.
- A *codistribution* on M is a subset $C \subset \mathrm{T}^*M$ such that $C_x = C \cap \mathrm{T}^*_x M$ is a vector subspace of $\mathrm{T}^*_x M$ for every $x \in M$.

If D has a constant rank k, we will write $\operatorname{rank} D = k$. If D has constant corank p, we will denote it by $\operatorname{corank} D = p$.

The *annihilator* of a distribution D is the codistribution $D^\circ = \bigsqcup_{x \in M} D^\circ_x$. If D is not regular, D° may not be smooth. Using the usual identification $E^{**} = E$ for a finite-dimensional vector space E, it follows that $(D^\circ)^\circ = D$.

Consider a differential one-form $\eta \in \Omega^1(M)$. Then, η spans a smooth codistribution $C = \langle \eta \rangle = \{\langle \eta_x \rangle \mid x \in M\} \subset \mathrm{T}^*M$. This codistribution has rank one at every point where η does not vanish. Its annihilator is the distribution $\ker \eta \subset \mathrm{T}M$. This codistribution has corank one at every point where η does not vanish.

Every two-form $\omega \in \Omega^2(M)$ induces a linear morphism $\widehat{\omega} \colon \mathrm{T}M \to \mathrm{T}^*M$ defined by $\widehat{\omega}(v_x) = \omega_x(v_x, \cdot) \in \mathrm{T}^*_x M$ for every $v_x \in \mathrm{T}_x M$ and $x \in M$. The kernel of $\widehat{\omega}$ is a distribution $\ker \omega \subset \mathrm{T}M$.

Given a family of differential one-forms $\eta^1, \ldots, \eta^k \in \Omega^1(M)$, we denote

- $\mathcal{C}^{\mathrm{C}} = \langle \eta^1, \ldots, \eta^k \rangle \subset \mathrm{T}^*M$,
- $\mathcal{D}^{\mathrm{C}} = \left(\mathcal{C}^{\mathrm{C}}\right)^\circ = \displaystyle\bigcap_{\alpha=1}^{k} \ker \eta^\alpha \subset \mathrm{T}M$,
- $\mathcal{D}^{\mathrm{R}} = \displaystyle\bigcap_{\alpha=1}^{k} \ker \mathrm{d}\eta^\alpha \subset \mathrm{T}M$,
- $\mathcal{C}^{\mathrm{R}} = \left(\mathcal{D}^{\mathrm{R}}\right)^\circ \subset \mathrm{T}^*M$.

We call $\mathcal{C}^{\mathrm{C}}$ the *contact codistribution*, $\mathcal{D}^{\mathrm{C}}$ the *contact distribution*, $\mathcal{D}^{\mathrm{R}}$ the *Reeb distribution* and $\mathcal{C}^{\mathrm{R}}$ the *Reeb codistribution*. Note that $\mathcal{D}$ stands for "distribution" and $\mathcal{C}$ means "codistribution". Meanwhile, C and R represent "contact" and "Reeb", respectively. This will help to recall the notation. Let us introduce k-contact manifolds.

Definition 2.5. A *k-polycontact structure* on a manifold M is a family of differential one-forms $\eta^1, \ldots, \eta^k \in \Omega^1(M)$ such that

(1) $\mathcal{D}^{\mathrm{C}} \subset \mathrm{T}M$ is a regular distribution of corank k,

(2) $\mathcal{D}^{\mathrm{R}} \subset \mathrm{T}M$ is a regular distribution of rank k,

(3) $\mathcal{D}^{\mathrm{C}} \cap \mathcal{D}^{\mathrm{R}} = \{0\}$.

A manifold M endowed with a k-polycontact structure $\eta^1, \ldots, \eta^k \in \Omega^1(M)$ is a *k-polycontact manifold*.

Remark 2.6. Note that $\mathcal{D}^C$ has a constant corank k, i.e. condition (1) in Definition 2.5, if and only if anyone of the following two conditions hold

(1′) $\mathcal{C}^{\mathrm{C}} \subset \mathrm{T}^*M$ is a regular codistribution of rank k,

(1″) $\eta^1 \wedge \cdots \wedge \eta^k \neq 0$ everywhere.

Moreover, $\mathcal{D}^C \cap \mathcal{D}^R = \{0\}$, i.e. condition (3) in Definition 2.5, can be rewritten as

(3′) $\bigcap\limits_{\alpha=1}^{k} (\ker \eta^\alpha \cap \ker \mathrm{d}\eta^\alpha) = \{0\}$.

If, $\operatorname{rank} \mathcal{D}^R = k$ and $\operatorname{corank} \mathcal{D}^C = k$, i.e. conditions (1) and (2) in Definition 2.5 hold, then condition (3), namely $\mathcal{D}^R \cap \mathcal{D}^C = 0$, is equivalent to each of the following two conditions:

(3″) $\mathrm{T}M = \mathcal{D}^{\mathrm{C}} \oplus \mathcal{D}^{\mathrm{R}}$,

(3‴) $\mathrm{T}^*M = \mathcal{C}^{\mathrm{C}} \oplus \mathcal{C}^{\mathrm{R}}$.

Furthermore, using the definition of $\mathcal{D}^{\mathrm{C}}$, one can prove that, in Definition 2.5, conditions (2) and (3) imply (1). (This is not true...).

Remark 2.7. A one-contact structure is given by a one-form η. Then Definition 2.5 implies that: (1) $\eta \neq 0$ everywhere, (3) $\ker \eta \cap \ker \mathrm{d}\eta = \{0\}$, which yields that $\ker \mathrm{d}\eta$ has rank zero or one, and (2) means that $\ker \mathrm{d}\eta$ has rank one. Note that (2) is equivalent to saying that $\dim M$ is odd. This recovers the notion of a *co-oriented contact manifold*, which is standardly called a contact manifold [14]. Every k-contact manifold gives rise to the so-called *Reeb vector fields*, which are a key element in k-contact geometry [14].

Lemma 2.8. *The Reeb distribution $\mathcal{D}^{\mathrm{R}}$ is involutive and, therefore, integrable.*

Theorem 2.9 (Reeb vector fields)**.** *Let (M, η^α) be a k-contact manifold. There exists a unique family of k vector fields $R_1, \ldots, R_k \in \mathfrak{X}(M)$, called the Reeb vector fields of (M, η^α), such that*

$$\begin{cases} \iota_{R_\alpha} \eta^\beta = \delta_\alpha^\beta \,, \\ \iota_{R_\alpha} \mathrm{d}\eta^\beta = 0 \,, \end{cases}$$

for $\alpha, \beta = 1, \ldots, k$. The Reeb vector fields $R_1, \ldots, R_k$ commute between themselves, i.e.

$$[R_\alpha, R_\beta] = 0 \,, \qquad \alpha, \beta = 1, \ldots, k \,.$$

In addition, the Reeb vector fields give a basis of the Reeb distribution , i.e. $\mathcal{D}^{\mathrm{R}} = \langle R_1, \ldots, R_k \rangle$ and $R_1 \wedge \ldots \wedge R_k$ is non-vanishing.

Example 2.10. The manifold $M = \bigoplus^k \mathrm{T}^*Q \times \mathbb{R}^k$ has a canonical k-contact structure given by the one-forms $\eta^1, \ldots, \eta^k \in \Omega^1(M)$ defined as

$$\eta^\alpha = \mathrm{d}z^\alpha - \theta^\alpha \,, \quad \alpha = 1, \ldots, k$$

where $\{z^1, \ldots, z^k\}$ are standard linear coordinates in $\mathbb{R}^k$ pull-backed to M in the standard way, while each θ^α is the pull-back of the Liouville one-form θ of the cotangent bundle T^*Q with respect to the projection $M \to \mathrm{T}^*Q$ onto the α-th component of $\bigoplus^k \mathrm{T}^*Q$ within M.

Every set of local coordinates $\{q^i\}$ on Q, along with some linear coordinates on $\mathbb{R}^k$, induce natural coordinates $\{q^i, p_i^\alpha, z^\alpha\}$ on M. Then,

$$\eta^\alpha = \mathrm{d}z^\alpha - p_i^\alpha \mathrm{d}q^i \,, \quad \alpha = 1, \ldots, k.$$

Hence,

$$\mathrm{d}\eta^\alpha = \mathrm{d}q^i \wedge \mathrm{d}p_i^\alpha\,, \quad R_\alpha = \frac{\partial}{\partial z^\alpha}\,, \qquad \alpha = 1,\dots,k.$$

Meanwhile, the Reeb distribution $\mathcal{D}^{\mathrm{R}}$ becomes

$$\mathcal{D}^{\mathrm{R}} = \left\langle \frac{\partial}{\partial z^1},\dots,\frac{\partial}{\partial z^k} \right\rangle .$$

Example 2.11 (Contactification of a k-symplectic manifold)**.** Consider a k-symplectic manifold (P,ω^α) such that $\omega^\alpha = -\mathrm{d}\theta^\alpha$ and the product manifold $M = P\times\mathbb{R}^k$. Let (z^α) be the pull-back to M of some cartesian coordinates of $\mathbb{R}^k$ and also denote by θ^α the pull-back of θ^α on P to the product manifold M. Consider the one-forms $\eta^\alpha = \mathrm{d}z^\alpha - \theta^\alpha \in \Omega^1(M)$, with $\alpha = 1,\dots,k$.

Then, (M,η^α) is a k-contact manifold, because $\mathcal{C}^{\mathrm{C}} = \langle\eta^1,\dots,\eta^k\rangle$ has rank k, while $\mathrm{d}\eta^\alpha = -\mathrm{d}\theta^\alpha$, and $\mathcal{D}^{\mathrm{R}} = \bigcap_{\alpha=1}^k \ker\mathrm{d}\theta^\alpha = \langle\partial/\partial z^1,\dots,\partial/\partial z^k\rangle$ has rank k since (P,ω^α) is k-symplectic. Condition (3) follows from imposing each η^α to vanish in some X taking values in D^{R}.

Note that the so-called canonical k-contact structure described in the previous example is just the contactification of the k-symplectic manifold $P = \bigoplus^k \mathrm{T}^*Q$.

Consider now the particular case $k=1$. Let P be a manifold with a one-form and consider the product manifold $M = P\times\mathbb{R}$ with the one-form $\eta = \mathrm{d}z - \theta \in \Omega^1(M)$. In this case, $\mathcal{C}^{\mathrm{C}} = \langle\eta\rangle$ has rank one, $\mathrm{d}\eta = -\mathrm{d}\theta$, and $\mathcal{D}^{\mathrm{R}} = \ker\mathrm{d}\theta$ has rank one if, and only if, $\mathrm{d}\theta$ is a symplectic form on P. Under these hypotheses, M along with η becomes a one-contact manifold.

Example 2.12. Consider the manifold $M=\mathbb{R}^6$ with linear global coordinates $\{x,y,p,q,z,t\}$. Then,

$$\eta^1 = \mathrm{d}z - \frac{1}{2}(y\mathrm{d}x - x\mathrm{d}y)\ , \quad \eta^2 = \mathrm{d}t - p\mathrm{d}x - q\mathrm{d}y$$

define a 2-contact structure on M. Let us verify that $\eta^1\wedge\eta^2$ is a 2-contact structure on M by using the conditions in Definition 2.5. First, the forms $\eta^1\wedge\eta^2 \neq 0$. Moreover,

$$\mathrm{d}\eta^1 = \mathrm{d}x\wedge\mathrm{d}y\ , \quad \mathrm{d}\eta^2 = \mathrm{d}x\wedge\mathrm{d}p + \mathrm{d}y\wedge\mathrm{d}q\,, \quad \mathcal{D}^{\mathrm{R}} = \left\langle\frac{\partial}{\partial z},\frac{\partial}{\partial t}\right\rangle ,$$

and $\mathcal{D}^{\mathrm{R}}$ has rank two. Moreover $\mathcal{D}^{\mathrm{R}}\cap\ker\mathrm{d}\eta^1\cap\ker\mathrm{d}\eta^2 = 0$, which is the third condition in Definition 2.5. The Reeb vector fields are

$$R_1 = \frac{\partial}{\partial z}\ , \quad R_2 = \frac{\partial}{\partial t}\,.$$

Theorem 2.13 (k-contact Darboux Theorem [18])**.** *Consider a k-polycontact manifold (M,η^α) of dimension $\dim M = n+kn+k$ such that there exists an integrable subdistribution $\mathcal{V}$ of $\mathcal{D}^{\mathrm{C}}$ with $\operatorname{rank}\mathcal{V} = nk$. We call $(M,\eta^\alpha,\mathcal{V})$ a k-contact manifold. Then, around every point of M, there exist local coordinates $\{q^i,p_i^\alpha,z^\alpha\}$, with $1\le\alpha\le k$ and $1\le i\le n$, such that*

$$\eta^\alpha = \mathrm{d}z^\alpha - p_i^\alpha\mathrm{d}q^i\,, \quad \mathcal{D}^{\mathrm{R}} = \left\langle R_\alpha = \frac{\partial}{\partial z^\alpha}\right\rangle\,, \quad \mathcal{V} = \left\langle\frac{\partial}{\partial p_i^\alpha}\right\rangle .$$

These coordinates are called Darboux coordinates of the k-contact manifold (M,η^α).

This theorem allows us to consider the manifold introduced in Example 2.10 as the canonical model of k-contact manifolds. Moreover, every k-polycontact manifold that is the contactification of a k-symplectic manifold (see Example 2.11) has Darboux coordinates.

2.3 Hamiltonian formalism for k-contact systems

After introducing the geometric framework of k-contact geometry, let us deal with the Hamiltonian formulation of field theories with dissipation.

Definition 2.14. A *k-polycontact Hamiltonian system* is a family (M, η^α, h), where (M, η^α) is a k-contact manifold and $h \in \mathscr{C}^\infty(M)$ is called a *Hamiltonian function*. Consider a map $\psi\colon D \subset \mathbb{R}^k \to M$. The *$k$-polycontact Hamilton–De Donder–Weyl equations* are

$$\begin{cases} \iota_{\psi'_\alpha} \mathrm{d}\eta^\alpha = (\mathrm{d}h - (\mathscr{L}_{R_\alpha} h)\eta^\alpha) \circ \psi \,, \\ \iota_{\psi'_\alpha} \eta^\alpha = -h \circ \psi \,. \end{cases} \tag{3}$$

In *Darboux coordinates*, for a k-contact manifold $\psi(t) = (q^i(t), p_i^\alpha(t), z^\alpha(t))$ and equations (3) read

$$\begin{cases} \dfrac{\partial q^i}{\partial t^\alpha} = \dfrac{\partial h}{\partial p_i^\alpha} \circ \psi \,, \\ \dfrac{\partial p_i^\alpha}{\partial t^\alpha} = -\left(\dfrac{\partial h}{\partial q^i} + p_i^\alpha \dfrac{\partial h}{\partial z^\alpha} \right) \circ \psi \,, \\ \dfrac{\partial z^\alpha}{\partial t^\alpha} = \left(p_i^\alpha \dfrac{\partial h}{\partial p_i^\alpha} - h \right) \circ \psi \,. \end{cases}$$

Definition 2.15. Consider a k-polycontact Hamiltonian system (M, η^α, h). The *k-contact Hamilton–De Donder–Weyl equations* for a k-vector field $\mathbf{X} = (X_\alpha) \in \mathfrak{X}^k(M)$ are

$$\begin{cases} \iota_{X_\alpha} \mathrm{d}\eta^\alpha = \mathrm{d}h - (\mathscr{L}_{R_\alpha} h)\eta^\alpha \,, \\ \iota_{X_\alpha} \eta^\alpha = -h \,. \end{cases} \tag{4}$$

A k-vector field $\mathbf{X}$ solution to these equations is called a *k-contact Hamiltonian k-vector field.*

Consider a k-vector field $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k(M)$ with local expression in *Darboux coordinates* given by

$$X_\alpha = (X_\alpha)^i \frac{\partial}{\partial q^i} + (X_\alpha)_i^\beta \frac{\partial}{\partial p_i^\beta} + (X_\alpha)^\beta \frac{\partial}{\partial z^\beta} \,.$$

Now, equation (7) implies that

$$\begin{cases} (X_\alpha)^i = \dfrac{\partial h}{\partial p_i^\alpha} \,, \\ (X_\alpha)_i^\alpha = -\left(\dfrac{\partial h}{\partial q^i} + p_i^\alpha \dfrac{\partial h}{\partial z^\alpha} \right) \,, \\ (X_\alpha)^\alpha = p_i^\alpha \dfrac{\partial h}{\partial p_i^\alpha} - h \,. \end{cases} \tag{5}$$

Proposition 2.16. *The k-contact Hamilton–De Donder–Weyl equations (7) admit solutions. They are not unique if $k > 1$.*

It is worth noting that the components $(X_\alpha)^\beta$ and $(X_\alpha)_i^\beta$ for $\alpha \neq \beta$ do not play a role in the equations (8) in the sense that given a Hamiltonian k-vector field for a function H, the value of its components $(X_\alpha)^\beta$ and $(X_\alpha)_i^\beta$ with $\alpha \neq \beta$ can be changed arbitrarily and $\mathbf{X}$ will be still Hamiltonian. Nevertheless, these arbitrary coefficients may play a role in ensuring that $\mathbf{X}$ is integrable, which is still relevant. Moreover, as long as the sums over α $(X_\alpha)^\alpha$ and $(X_\alpha)_i^\alpha$ are the same, $\mathbf{X}$ is Hamiltonian with Hamiltonian relative to the same Hamiltonian function.

Proposition 2.17. *Given an integrable k-vector field $\mathbf{X} \in \mathfrak{X}^k(M)$, every integral section $\psi\colon D \subset \mathbb{R}^k \to M$ of $\mathbf{X}$ satisfies the k-contact Hamilton–De Donder–Weyl equation (3) if, and only if, $\mathbf{X}$ is a solution to (7).*

Recall that given a Hamiltonian function h, its Hamiltonian k-vector field $\mathbf{X}$ is not fully determined, but several k-vector fields related to the same Hamiltonian function h verify that their differences are vertical k-vector fields relative to $\mathrm{pr} : \bigoplus^k \mathrm{T}^*Q \times \mathbb{R}^k \to Q \times \mathbb{R}^k$ and Hamiltonian relative to the zero function. It is also worth noting that the integrability of equations (7) does not imply the integrability of its associated k-vector fields in general.

Remark 2.18. As in the k-symplectic case, equations (3) and (7) are not fully equivalent because a solution to (3) may not be an integral section of an integrable k-vector field solution to (7). This fact is of interest when studying symmetries and dissipation laws [18].

Proposition 2.19. *The k-contact Hamilton–De Donder–Weyl equations (7) are equivalent to*

$$\begin{cases} \mathscr{L}_{X_\alpha}\eta^\alpha = -(\mathscr{L}_{R_\alpha}H)\eta^\alpha\,, \\ \iota_{X_\alpha}\eta^\alpha = -h\,. \end{cases}$$

Example 2.20 (The damped wave equation)**.** A vibrating string can be described using a k-contact Hamiltonian formalism [**?**] Xavier's PhD. Consider the coordinates $\{t, x\}$ for $\mathbb{R}^2$ and set $Q = \mathbb{R}$. The phase space becomes $\oplus^2\mathrm{T}^*\mathbb{R} \times \mathbb{R}^2$ and admits coordinates (u, p^t, p^x, s^t, s^x). Denote by u the separation of a point in the string from its equilibrium point, while p^t and p^x will denote the momenta of u with respect to the two independent variables. The Hamiltonian function for this system is $h \in C^\infty\left(\bigoplus^2 T^*\mathbb{R} \times \mathbb{R}^2\right)$ of the form

$$h(u, p^t, p^x, s^t, s^x) = \frac{1}{2\rho}(p^t)^2 - \frac{1}{2\tau}(p^x)^2 + ks^t,$$

where ρ is the linear mass density of the string, τ is the tension of the string and $k > 0$. We will assume that ρ, τ, k are constant. The Hamilton–de Donder–Weyl equations for u are

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{\rho}p^t\,, \\ \dfrac{\partial u}{\partial x} = -\dfrac{1}{\tau}p^x\,, \\ \dfrac{\partial p^t}{\partial t} + \dfrac{\partial p^x}{\partial x} = -kp^t\,, \\ \dfrac{\partial s^t}{\partial t} + \dfrac{\partial s^x}{\partial x} = \dfrac{1}{2\rho}(p^t)^2 - \dfrac{1}{2\tau}(p^x)^2 - ks^t\,. \end{cases}$$

Hence,

$$u_{tt} = \frac{\tau}{\rho}u_{xx} - \frac{k}{\rho}p^t\,,$$

3 Evolutionary k-contact vector fields

Let us develop a generalisation of contact evolutionary vector fields to the real of k-contact geometry.

Definition 3.1. Given a k-polycontact Hamiltonian system (M,η^α,h), the *evoultionary k-polycontact Hamilton–De Donder–Weyl equations* are

$$\begin{cases}\iota_{\psi'_\alpha}\mathrm{d}\eta^\alpha = (\mathrm{d}h - (\mathscr{L}_{R_\alpha}h)\eta^\alpha)\circ\psi\,,\\ \iota_{\psi'_\alpha}\eta^\alpha = 0\,,\end{cases}\tag{6}$$

where solutions are maps $\psi\colon D\subset\mathbb{R}^k\to M$.

In *Darboux coordinates* for a k-contact manifold, $\psi(t)=(q^i(t),p_i^\alpha(t),z^\alpha(t))$ and equations (6) read

$$\begin{cases}\dfrac{\partial q^i}{\partial t^\alpha} = \dfrac{\partial h}{\partial p_i^\alpha}\circ\psi\,,\\ \dfrac{\partial p_i^\alpha}{\partial t^\alpha} = -\left(\dfrac{\partial h}{\partial q^i}+p_i^\alpha\dfrac{\partial h}{\partial z^\alpha}\right)\circ\psi\,,\\ \dfrac{\partial z^\alpha}{\partial t^\alpha} = p_i^\alpha\dfrac{\partial h}{\partial p_i^\alpha}\circ\psi\,.\end{cases}$$

Definition 3.2. Consider a k-contact Hamiltonian system (M,η^α,h). The *evolutionary k-contact Hamilton–De Donder–Weyl equations* for a k-vector field $\mathbf{X}=(X_\alpha)\in\mathfrak{X}^k(M)$ are

$$\begin{cases}\iota_{X_\alpha}\mathrm{d}\eta^\alpha = \mathrm{d}h - (\mathscr{L}_{R_\alpha}h)\eta^\alpha\,,\\ \iota_{X_\alpha}\eta^\alpha = 0\,.\end{cases}\tag{7}$$

A k-vector field $\mathbf{X}$ solution to these equations is an *evolutionary k-contact Hamiltonian k-vector field.*

Consider a k-vector field $\mathbf{X}=(X_1,\dots,X_k)\in\mathfrak{X}^k(M)$ with local expression in *Darboux coordinates*

$$X_\alpha = (X_\alpha)^i\frac{\partial}{\partial q^i}+(X_\alpha)_i^\beta\frac{\partial}{\partial p_i^\beta}+(X_\alpha)^\beta\frac{\partial}{\partial z^\beta}\,.$$

Now, equation (7) implies that

$$\begin{cases}(X_\alpha)^i = \dfrac{\partial h}{\partial p_i^\alpha}\,,\\ (X_\alpha)_i^\alpha = -\left(\dfrac{\partial h}{\partial q^i}+p_i^\alpha\dfrac{\partial h}{\partial z^\alpha}\right)\,,\\ (X_\alpha)^\alpha = p_i^\alpha\dfrac{\partial h}{\partial p_i^\alpha}\,.\end{cases}\tag{8}$$

Proposition 3.3. *The evolutionary k-contact Hamilton–De Donder–Weyl equations* (7) *admit solutions. They are not unique if $k>1$.*

Comment relating the result with the contact case.
Write comment...

Theorem 3.4. *(z-independent evolutionary k-contact Hamilton–Jacobi Theorem) Let $\mathbf{Z}$ be an evolutionary k-contact Hamiltonian k-vector field for some $h\in\mathscr{C}^\infty(\oplus^k\mathrm{T}^*Q\times\mathbb{R}^k)$ and let γ be a closed section of π_Q such that $\frac{\partial S^\alpha}{\partial q^i}=\gamma_i^\alpha$. If $\mathbf{Z}$ is integrable, the following conditions are equivalent:*

1. *Every integral section, $\sigma: U\subset\mathbb{R}^k\to Q$, of $\mathbf{Z}^\gamma$ gives rise to a solution $\gamma\circ\sigma$ to HDW equations,*

2. $\mathrm{d}(h\circ\gamma)=0$.

Proof. Suppose that $(\gamma\circ\sigma)(t)=(\sigma^i(t),\gamma_i^\alpha(\sigma(t)),S^\alpha(\sigma(t)))$ is a solution to the HDW equations. Then,

$$\begin{cases} \left.\dfrac{\partial\sigma^i}{\partial t^\alpha}\right|_t = \left.\dfrac{\partial h}{\partial p_i^\alpha}\right|_{\gamma\circ\sigma(t)}, \\ \left.\dfrac{\partial(\gamma_i^\alpha\circ\sigma)}{\partial t^\alpha}\right|_t = -\left(\left.\dfrac{\partial h}{\partial q^i}\right|_{\gamma\circ\sigma(t)} + (p_i^\alpha\circ\gamma\circ\sigma(t))\left.\dfrac{\partial h}{\partial z^\alpha}\right|_{\gamma\circ\sigma(t)}\right), \qquad \alpha=1,\dots,k, \quad i=1,\dots,n. \\ \left.\dfrac{\partial(S^\alpha\circ\sigma)}{\partial t^\alpha}\right|_t = (\gamma_i^\alpha\circ\sigma)(t)\left.\dfrac{\partial h}{\partial p_i^\alpha}\right|_{\gamma\circ\sigma(t)}. \end{cases}$$

On the other hand,

$$\mathrm{d}(h\circ\gamma)=\left(\frac{\partial h}{\partial q^i}\circ\gamma+\left(\frac{\partial h}{\partial p_j^\alpha}\circ\gamma\right)\frac{\partial\gamma_j^\alpha}{\partial q^i}+\left(\frac{\partial h}{\partial z^\alpha}\circ\gamma\right)\frac{\partial S^\alpha}{\partial q^i}\right)\mathrm{d}q^i.$$

In local coordinates in Q, one has $\gamma^\alpha=\gamma_i^\alpha\mathrm{d}q^i$ and, since γ is a closed section, it follows that

$$\frac{\partial\gamma_i^\beta}{\partial q^j}=\frac{\partial\gamma_j^\beta}{\partial q^i}, \qquad \beta=1,\dots,k, \quad i=1,\dots,n. \tag{9}$$

Using (9) and HDW equations, we have

$$\begin{aligned} \mathrm{d}(h\circ\gamma)|_{(\sigma(t))} &= \left(\left.\frac{\partial h}{\partial q^i}\right|_{\gamma(\sigma(t))}+\left.\frac{\partial h}{\partial p_j^\alpha}\right|_{\gamma(\sigma(t))}\left.\frac{\partial\gamma_j^\alpha}{\partial q^i}\right|_{\sigma(t)}+\left.\frac{\partial h}{\partial z^\alpha}\right|_{\gamma(\sigma(t))}\left.\frac{\partial S^\alpha}{\partial q^i}\right|_{\sigma(t)}\right)\mathrm{d}q^i(\sigma(t)) \\ &= \left(-\left(\gamma_i^\alpha\circ\sigma\right)(t)\left.\frac{\partial h}{\partial z^\alpha}\right|_{\gamma(\sigma(t))}-\left.\frac{\partial\left(\gamma_i^\alpha\circ\sigma\right)}{\partial t^\alpha}\right|_t+\left.\frac{\partial\sigma^j}{\partial t^\alpha}\right|_t\left.\frac{\partial\gamma_j^\alpha}{\partial q^i}\right|_{\sigma(t)}+\left.\frac{\partial h}{\partial z^\alpha}\right|_{\gamma(\sigma(t))}\left.\frac{\partial S^\alpha}{\partial q^i}\right|_{\sigma(t)}\right)\mathrm{d}q^i(\sigma(t)) \\ &= \left(-\gamma_i^\alpha(\sigma(t))+\left.\frac{\partial S^\alpha}{\partial q^i}\right|_{\sigma(t)}\right)\left.\frac{\partial h}{\partial z^\alpha}\right|_{\gamma(\sigma(t))}\mathrm{d}q^i|_{\sigma(t)}=0\,, \end{aligned}$$

where in the last line we used the fact that $\dfrac{\partial S^\alpha}{\partial q^i}=\gamma_i^\alpha$. We showed that $\mathrm{d}(h\circ\gamma)(q)=0$ for every $q\in\operatorname{Im}\sigma$. Since $\mathbf{Z}$ is integrable, $\mathbf{Z}^\gamma$, as its projection, is also integrable. This means that for every $q\in Q$ there exists an integral section σ such that $\sigma(0)=q$. Therefore, $\mathrm{d}(h\circ\gamma)(q)=0$, for every $q\in Q$.

Let us prove the converse. Assume that $\mathrm{d}(h\circ\gamma)=0$, and let σ be an integral section of $\mathbf{Z}^\gamma$. Let us prove that $\gamma\circ\sigma$ is a solution to HDW equations. From $\mathrm{d}(h\circ\gamma)=0$, we have

$$0=\left(\frac{\partial h}{\partial q^i}\circ\gamma+\left(\frac{\partial h}{\partial p_j^\alpha}\circ\gamma\right)\frac{\partial\gamma_j^\alpha}{\partial q^i}+\left(\frac{\partial h}{\partial z^\alpha}\circ\gamma\right)\frac{\partial S^\alpha}{\partial q^i}\right)\mathrm{d}q^i\,. \tag{10}$$

Since $\mathbf{Z}$ is a k-contact Hamiltonian k-vector field, equations (8) and (12) yield

$$Z_\alpha^\gamma=\left(\frac{\partial h}{\partial p_i^\alpha}\circ\gamma\right)\frac{\partial}{\partial q^i}\,, \qquad \alpha=1,\dots,k.$$

On the other hand, since σ is an integral section of $\mathbf{Z}^\gamma$, we have

$$\left.\frac{\partial\sigma^i}{\partial t^\alpha}\right|_t=\frac{\partial h}{\partial p_i^\alpha}\circ\gamma\circ\sigma(t)\,. \tag{11}$$

From (9), (10), and (11), it follows that

$$\left.\frac{\partial(\gamma_i^\alpha\circ\sigma)}{\partial t^\alpha}\right|_t=\left.\frac{\partial\gamma_i^\alpha}{\partial q^j}\right|_{\sigma(t)}\left.\frac{\partial\sigma^j}{\partial t^\alpha}\right|_t=\left.\frac{\partial\gamma_i^\alpha}{\partial q^j}\right|_{\sigma(t)}\left.\frac{\partial h}{\partial p_j^\alpha}\right|_{\gamma(\sigma(t))}=\left.\frac{\partial\gamma_j^\alpha}{\partial q^i}\right|_{\sigma(t)}\left.\frac{\partial h}{\partial p_j^\alpha}\right|_{\gamma(\sigma(t))}$$
$$=-\left.\frac{\partial h}{\partial q^i}\right|_{\gamma(\sigma(t))}-\left.\frac{\partial S^\alpha}{\partial q^i}\right|_{\sigma(t)}\left.\frac{\partial h}{\partial z^\alpha}\right|_{\gamma(\sigma(t))}=-\left.\frac{\partial h}{\partial q^i}\right|_{\gamma(\sigma(t))}-\gamma_i^\alpha\circ\sigma(t)\left.\frac{\partial h}{\partial z^\alpha}\right|_{\gamma(\sigma(t))}.$$

And finally,

$$\left.\frac{\partial(S^\alpha\circ\sigma)}{\partial t^\alpha}\right|_t=\left.\frac{\partial S^\alpha}{\partial q^i}\right|_{\sigma(t)}\left.\frac{\partial\sigma^i}{\partial t^\alpha}\right|_t=\gamma_i^\alpha\circ\sigma(t)\left.\frac{\partial h}{\partial p_i^\alpha}\right|_{\gamma\circ\sigma(t)}=\gamma_i^\alpha\circ\sigma(t)\left.\frac{\partial h}{\partial p_i^\alpha}\right|_{\gamma\circ\sigma(t)},$$

where we used (11), the Hamilton–Jacobi equation, and the fact that $\mathrm{d}S^\alpha=\gamma^\alpha$. This concludes the proof.

□

4 The Hamilton–Jacobi equation

Let us develop two types of Hamilton-Jacobi theories that offer different possibilities. The main idea is to analyse the HDW equations for a k-contact Hamiltonian system $(\bigoplus^k \mathrm{T}^*Q\times\mathbb{R}^k,\eta^\alpha,h)$ by means of the projections of a related k-Hamiltonian vector field on a submanifold of $\bigoplus^k\mathrm{T}^*Q\times\mathbb{R}^k$ that is projectable in a certain sense.

4.1 z^α-independent approach

Consider a section of the bundle $\pi_Q\colon\bigoplus^k\mathrm{T}^*Q\times\mathbb{R}^k\to Q$, locally given by the coordinate expression in adapted coordinates

$$\begin{array}{rccc}\gamma\colon & Q & \longrightarrow & \bigoplus^k\mathrm{T}^*Q\times\mathbb{R}^k\\ & q=(q^i) & \longmapsto & \left(q^i,\gamma_i^\alpha(q),S^\alpha(q)\right).\end{array}$$

Recall that an arbitrary map $f:M\to N$ defines a map $\bigoplus^k\mathrm{T}f:\bigoplus^k\mathrm{T}M\to\bigoplus^k\mathrm{T}N$ by $\left(\bigoplus^k\mathrm{T}f\right)_x(v_1,\dots,v_k)=(\mathrm{T}_xf(v_1),\dots,\mathrm{T}_xf(v_k))$, for $x\in M$ and $v_1,\dots,v_k\in T_xM$. Since γ is a section of $\bigoplus^k T^*Q\times\mathbb{R}^k\to Q$, one can define the k-vector field $\mathbf{Z}^\gamma$ on Q given by

$$\mathbf{Z}^\gamma=\left(\bigoplus^k\mathrm{T}\pi_Q\right)\circ\mathbf{Z}\circ\gamma\,,$$

where $\mathbf{Z}$ is a Hamiltonian k-vector field relative to $(\bigoplus^k\mathrm{T}^*Q\times\mathbb{R}^k,\eta^\alpha,h)$. Suppose that $\mathbf{Z}^\gamma$ and $\mathbf{Z}$ are γ-related, namely

$$\mathbf{Z}\circ\gamma=\left(\bigoplus^k\mathrm{T}\gamma\right)\circ\mathbf{Z}^\gamma\,.$$

so that the following diagram commutes

$$\begin{array}{ccc}\bigoplus^k\mathrm{T}^*Q\times\mathbb{R}^k & \xrightarrow{Z} & \bigoplus^k\mathrm{T}(\bigoplus^k\mathrm{T}^*Q\times\mathbb{R}^k)\\ {\scriptstyle\gamma}\uparrow\downarrow{\scriptstyle\pi_Q} & & {\scriptstyle\bigoplus^k\mathrm{T}\pi_Q}\downarrow\uparrow{\scriptstyle\bigoplus^k\mathrm{T}\gamma}\\ Q & \xrightarrow{Z^\gamma} & \bigoplus^k\mathrm{T}Q\end{array}$$

Thus, if $\mathbf{Z} = (Z_\alpha)$ reads in local coordinates $Z_\alpha = Z_\alpha^i \frac{\partial}{\partial q^i} + (Z_\alpha)_i^\beta \frac{\partial}{\partial p_i^\beta} + (Z_\alpha)^\beta \frac{\partial}{\partial z^\beta}$ for $\alpha = 1, \dots, k$, then $\mathbf{Z}^\gamma = (Z_\alpha^\gamma)$ and

$$Z_\alpha^\gamma = (Z_\alpha^i \circ \gamma) \frac{\partial}{\partial q^i}\,, \qquad \alpha = 1, \dots, k. \tag{12}$$

If $\mathbf{Z}$ is integrable, then $\mathbf{Z}^\gamma$ is integrable too. In fact, if $\mathbf{Z}$ is integrable, then $[Z_\alpha, Z_\beta] = 0$. Since Z_α, Z_β are tangent to $\mathrm{Im}\gamma$, they are projectable onto Q and $\pi_Q Z_\alpha|_{T\gamma} = Z_\alpha^\gamma$ and $[Z_\alpha^\gamma, Z_\beta^\gamma] = 0$.

Definition 4.1. The *orthogonal subbundle* $W^{\perp_k}$ of a vector subbundle $W^k \subset \mathrm{T}_x M$ is given by

$$W^{\perp_k} \subset \mathrm{T}_x M := \{ v_x \in \mathrm{T}_x M \mid \omega_1(v, v_x) = \dots = \omega_k(v, v_x) = 0\,,\ \forall v \in W \}\,.$$

Definition 4.2. A *k-Legendrian symplectic subspace* of $\mathrm{T}_x M$ is a linear subspace $W \subset \mathrm{T}_x M$ such that $W^{\perp_k} = W$. A subspace $W \subset T_x M$ is said to be *k-isotropic* if $W \subset W^{\perp_k}$ and *k-coisotropic* if $W^{\perp_k} \subset W$. A *k-Legendrian symplectic submanifold* of M is a submanifold S of M such that the tangent space at each point of M is k-Legendrian. The equivalent definitions for isotropic and coisotropic manifolds follow.

Consider a coordinate system on M and a k-Legendrian submanifold of $\bigoplus^k \mathrm{T}^* Q \times \mathbb{R}^k$ relative to its natural k-contact structure. Let us describe the conditions for $\mathrm{Im}\,\gamma$ to be a k-Legendrian symplectic submanifold.

Theorem 4.3. *Given a section* $\gamma\colon Q \longrightarrow \bigoplus^k \mathrm{T}^* Q \times \mathbb{R}^k : \left(q^i\right) \longmapsto \left(q^i, \gamma_i^\alpha(q), S^\alpha(q)\right)$ *in adapted coordinates, the tangent space to* $\mathrm{Im}\,\gamma$ *is*

$$\mathrm{T}\,\mathrm{Im}\,\gamma = \left\langle \frac{\partial}{\partial q^i} + \frac{\partial \gamma_j^\alpha}{\partial q^i} \frac{\partial}{\partial p_j^\alpha} + \frac{\partial S^\alpha}{\partial q^i} \frac{\partial}{\partial z^\alpha} \right\rangle .$$

The k-coisotropic condition $(\mathrm{T}\,\mathrm{Im}\,\gamma)^{\perp_k} \subset \mathrm{T}\,\mathrm{Im}\,\gamma$, *amounts to* $\mathrm{d}\gamma^\mu = 0$, *or, in other words*

$$\frac{\partial \gamma_i^\mu}{\partial q^j} = \frac{\partial \gamma_j^\mu}{\partial q^i}\,, \qquad i, j = 1, \dots, n.$$

Proof. (De lucas will do it) Since

$$\mathrm{d}\eta^\mu \left(\frac{\partial}{\partial q^i} + \frac{\partial \gamma_j^\alpha}{\partial q^i} \frac{\partial}{\partial p_j^\alpha} + \frac{\partial S^\alpha}{\partial q^i} \frac{\partial}{\partial z^\alpha}\,, \ \frac{\partial}{\partial q^l} + \frac{\partial \gamma_j^\beta}{\partial q^l} \frac{\partial}{\partial p_j^\beta} + \frac{\partial S^\beta}{\partial q^l} \frac{\partial}{\partial z^\beta} \right) = \frac{\partial \gamma_i^\mu}{\partial q^l} - \frac{\partial \gamma_l^\mu}{\partial q^i}\,.$$

$\square$

Theorem 4.4. *(z-independent k-contact Hamilton–Jacobi Theorem) Let* $\mathbf{Z}$ *be a k-contact Hamiltonian k-vector field for some* $h \in \mathscr{C}^\infty(\oplus^k \mathrm{T}^* Q \times \mathbb{R}^k)$ *and let* γ *be a closed section of* π_Q *such that* $\frac{\partial S^\alpha}{\partial q^i} = \gamma_i^\alpha$. *If* $\mathbf{Z}$ *is integrable, the following conditions are equivalent:*

1. *Every integral section,* $\sigma : U \subset \mathbb{R}^k \to Q$, *of* $\mathbf{Z}^\gamma$ *gives rise to a solution* $\gamma \circ \sigma$ *to HDW equations,*

2. $h \circ \gamma = 0$.

Proof. Suppose that $(\gamma\circ\sigma)(t)=(\sigma^i(t),\gamma_i^\alpha(\sigma(t)),S^\alpha(\sigma(t)))$ is a solution to the HDW equations. Then,

$$\begin{cases}\left.\dfrac{\partial\sigma^i}{\partial t^\alpha}\right|_t=\left.\dfrac{\partial h}{\partial p_i^\alpha}\right|_{\gamma\circ\sigma(t)}\,,\\ \left.\dfrac{\partial(\gamma_i^\alpha\circ\sigma)}{\partial t^\alpha}\right|_t=-\left(\left.\dfrac{\partial h}{\partial q^i}\right|_{\gamma\circ\sigma(t)}+(p_i^\alpha\circ\gamma\circ\sigma(t))\left.\dfrac{\partial h}{\partial z^\alpha}\right|_{\gamma\circ\sigma(t)}\right)\,, \quad \alpha=1,\dots,k,\quad i=1,\dots,n.\\ \left.\dfrac{\partial(S^\alpha\circ\sigma)}{\partial t^\alpha}\right|_t=(\gamma_i^\alpha\circ\sigma)(t)\left.\dfrac{\partial h}{\partial p_i^\alpha}\right|_{\gamma\circ\sigma(t)}-h\circ\gamma\circ\sigma(t)\,.\end{cases}$$

Given the assumption that $\frac{\partial S^\alpha}{\partial q^i}=\gamma_i^\alpha$ and using the first and third equation above, one has

$$\frac{\partial S^\alpha\circ\sigma}{\partial t^\alpha}=\frac{\partial S^\alpha}{\partial q^i}\frac{\partial\sigma^i}{\partial t^\alpha}\dots$$

we obtain that $h\circ\gamma=0$.

Now, assume that $h\circ\gamma=0$ and let σ be an integral section of $\mathbf{Z}^\gamma$. Let us prove that $\gamma\circ\sigma$ is a solution to HDW equations. In local coordinates in Q, one has $\gamma^\alpha=\gamma_i^\alpha\mathrm{d}q^i$ and, since γ is a closed section, it follows that

$$\frac{\partial\gamma_i^\beta}{\partial q^j}=\frac{\partial\gamma_j^\beta}{\partial q^i}\,,\qquad \beta=1,\dots,k,\quad i=1,\dots,n. \tag{13}$$

Since $h\circ\gamma=0$, we have $\mathrm{d}(h\circ\gamma)=0$ as well, which reads

Is it possible with differential

$$0=\left(\frac{\partial h}{\partial q^i}\circ\gamma+\left(\frac{\partial h}{\partial p_j^\alpha}\circ\gamma\right)\frac{\partial\gamma_j^\alpha}{\partial q^i}+\left(\frac{\partial h}{\partial z^\alpha}\circ\gamma\right)\frac{\partial S^\alpha}{\partial q^i}\right)\mathrm{d}q^i\,. \tag{14}$$

Since $\mathbf{Z}$ is a k-contact Hamiltonian k-vector field, equations (8) and (12) yield

$$Z_\alpha^\gamma=\left(\frac{\partial h}{\partial p_i^\alpha}\circ\gamma\right)\frac{\partial}{\partial q^i}\,,\qquad \alpha=1,\dots,k.$$

On the other hand, since σ is an integral section of $\mathbf{Z}^\gamma$, we have

$$\left.\frac{\partial\sigma^i}{\partial t^\alpha}\right|_t=\frac{\partial h}{\partial p_i^\alpha}\circ\gamma\circ\sigma(t)\,. \tag{15}$$

From (13), (14), (15), and the fact that $\frac{\partial S^\alpha}{\partial q^i}=\gamma_i^\alpha$, it follows that

$$\begin{aligned}\left.\frac{\partial(\gamma_i^\alpha\circ\sigma)}{\partial t^\alpha}\right|_t&=\left.\frac{\partial\gamma_i^\alpha}{\partial q^j}\right|_{\sigma(t)}\left.\frac{\partial\sigma^j}{\partial t^\alpha}\right|_t=\left.\frac{\partial\gamma_i^\alpha}{\partial q^j}\right|_{\sigma(t)}\left.\frac{\partial h}{\partial p_j^\alpha}\right|_{\gamma(\sigma(t))}=\left.\frac{\partial\gamma_j^\alpha}{\partial q^i}\right|_{\sigma(t)}\left.\frac{\partial h}{\partial p_j^\alpha}\right|_{\gamma(\sigma(t))}\\&=-\left.\frac{\partial h}{\partial q^i}\right|_{\gamma(\sigma(t))}-\left.\frac{\partial S^\alpha}{\partial q^i}\right|_{\sigma(t)}\left.\frac{\partial h}{\partial z^\alpha}\right|_{\gamma(\sigma(t))}=-\left.\frac{\partial h}{\partial q^i}\right|_{\gamma(\sigma(t))}-\gamma_i^\alpha\circ\sigma(t)\left.\frac{\partial h}{\partial z^\alpha}\right|_{\gamma(\sigma(t))}\,.\end{aligned}$$

And finally,

$$\left.\frac{\partial(S^\alpha\circ\sigma)}{\partial t^\alpha}\right|_t=\left.\frac{\partial S^\alpha}{\partial q^i}\right|_{\sigma(t)}\left.\frac{\partial\sigma^i}{\partial t^\alpha}\right|_t=\gamma_i^\alpha\circ\sigma(t)\left.\frac{\partial h}{\partial p_i^\alpha}\right|_{\gamma\circ\sigma(t)}=\gamma_i^\alpha\circ\sigma(t)\left.\frac{\partial h}{\partial p_i^\alpha}\right|_{\gamma\circ\sigma(t)}\,,$$

where we used (15), the Hamilton–Jacobi equation, and the fact that $\mathrm{d}S^\alpha=\gamma^\alpha$. This concludes the proof, since $h\circ\gamma=0$. $\square$

Definition 4.5. A *complete solution of the action-independent Hamilton–Jacobi problem* for $(\bigoplus^k \mathrm{T}^*Q \times \mathbb{R}^k, \eta^\alpha, h)$ is a local bundle isomorphism $\Phi\colon Q \times \mathbb{R}^n \to \bigoplus^k \mathrm{T}^*Q \times \mathbb{R}^k$ such that, for each $\lambda \in \mathbb{R}^n$, the map

$$\begin{aligned} \Phi_\lambda\colon Q &\longrightarrow \bigoplus^k \mathrm{T}^*Q \times \mathbb{R}^k \\ q^i &\longmapsto \Phi(q^i, \lambda) \end{aligned}$$

is a solution of the action-independent Hamilton–Jacobi problem for $(\bigoplus^k \mathrm{T}^*Q \times \mathbb{R}^k, \eta^\alpha, h)$.

Let $\alpha\colon Q \times \mathbb{R}^n \to \mathbb{R}^n$ be the canonical projection onto $\mathbb{R}^n$ and $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ be such that $\pi_i(t_1, \dots, t_n) = t_i$ for every $t_1, \dots, t_n \in \mathbb{R}$ and $i = 1, \dots, n$. Define the functions $f_i = \pi_i \circ \alpha \circ \Phi^{-1}$ on $\bigoplus^k \mathrm{T}^*Q \times \mathbb{R}^k$ with $i = 1, \dots, n$. Then, the following diagram commutes

$$\begin{tikzcd} Q \times \mathbb{R}^n \arrow[r, shift left, "\Phi"] \arrow[d, "\alpha"] & \bigoplus^k \mathrm{T}^*Q \times \mathbb{R}^k \arrow[l, shift left, "\Phi^{-1}"] \arrow[d, "f_i"] \\ \mathbb{R}^n \arrow[r, "\pi_i"] & \mathbb{R} \end{tikzcd}$$

Thus, $\operatorname{Im}\Phi_\lambda = \bigcap_{i=1}^n f_i^{-1}(\lambda_i)$, where $\lambda = (\lambda_1, \dots, \lambda_n)$. This can be rewritten as

$$\operatorname{Im}\Phi_\lambda = \{x \in \bigoplus^k \mathrm{T}^*Q \times \mathbb{R}^k \mid f_i(x) = \lambda_i\,,\ i = 1, \dots, n\}\,.$$

Let us consider $\cap_{i=1}^n f_i^{-1}(\lambda_i)$. Note that $\operatorname{Im}\Phi_\lambda = \cap_{i=1}^n f_i^{-1}(\lambda_i)$. Then, $(\mathrm{d}f_a)_{\mathrm{T}\operatorname{Im}\Phi_\lambda} = 0$. Assume that $\operatorname{Im}\Phi_\lambda$ is Lagrangian, then

$$\sharp_\ell((\mathrm{T}\operatorname{Im}\Phi)^\circ) \subset \mathrm{T}\operatorname{Im}\Phi \cap \mathcal{C}_\ell\,, \qquad \ell = 1, \dots, k\,.$$

Hence, $X_{f_a} = \sharp_\ell \mathrm{d}f_a$ is in particular tangent to $\operatorname{Im}\Phi_\lambda$. Since the functions f_a are constant over $\operatorname{Im}\Phi$ and X_{f_a} is tangent to it, then $X_{f_a} f_b = 0$. Hence, using the relation for the brackets,

$$X_{f_a} f_b = \{f_a, f_b\}_\ell = 0\,.$$

And the functions f_a, f_b are in involution relative to every bracket.

Conversely, if $\{f_a, f_b\}_\ell = 0$ for every bracket, then $X_{f_a} f_b = 0$, which means that X_{f_i} is tangent to the $\bigcap_{i=1}^n f_i^{-1}(p)$. Hence, $\sharp_\ell(\mathrm{d}f_b) \subset \mathrm{T}\operatorname{Im}\Phi_\lambda \cap \mathcal{C}_\ell$. Therefore, since the $\mathrm{d}f_\alpha$ span the annihilator to $\mathrm{T}\operatorname{Im}\Phi$, we have

$$\sharp_\ell((\mathrm{T}\operatorname{Im}\Phi_\lambda)^\circ) \subset \mathrm{T}\operatorname{Im}\Phi_\lambda \cap \mathcal{C}_\ell\,, \qquad \ell = 1, \dots, k\,.$$

Contact manifolds are related to Jacobi manifolds of a particular type. Let us prove that locally k-contact manifolds have an analogous property.

4.2 z^α-dependent approach

Let us propose an alternative approach to the one given in the previous section by considering solutions of the Hamilton–Jacobi problem depending on the variables $z^1, \dots, z^k$. Let $(\mathbb{R}^k \times \oplus^k \mathrm{T}^*Q, \eta^\alpha, h)$ be a k-contact Hamiltonian system. Consider a section of the bundle $\tilde{\pi}_Q\colon \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k \to Q \times \mathbb{R}^k$ given by

$$\begin{aligned} \gamma : Q \times \mathbb{R}^k &\longrightarrow \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k \\ (q^i, z^\alpha) &\longmapsto \left(q^i, \gamma_i^\alpha(q^i, z^\alpha), z^\alpha\right)\,. \end{aligned}$$

As in Section 4.1, assume that $\mathbf{X}_h^\gamma$ and $\mathbf{X}_h$ are γ-related so that the diagram

$$
\begin{tikzcd}
\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k \arrow[r, "\mathbf{X}_h"] \arrow[d, "\pi^z_Q"] & \oplus^k \mathrm{T}(\bigoplus^k \mathrm{T}^*Q \times \mathbb{R}^k) \arrow[d, "\oplus^k \mathrm{T}\pi^z_Q"'] \\
Q \times \mathbb{R}^k \arrow[r, "\mathbf{X}^\gamma_h"] \arrow[u, bend left, "\gamma"] & \oplus^k \mathrm{T}(Q \times \mathbb{R}^k) \arrow[u, bend right, "\oplus^k \mathrm{T}\gamma"']
\end{tikzcd}
$$

Let us analyse, in coordinates, the meaning of the above diagram. Consider a Hamiltonian k-vector field $\mathbf{X}_h = (X_1, \dots, X_k)$ on $\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$. In local coordinates $\{q^i, p^\beta_i, z^\beta\}$ adapted to the manifold, one has

$$
X_\alpha = (X_\alpha)^i \frac{\partial}{\partial q^i} + (X_\alpha)^\beta_i \frac{\partial}{\partial p^\beta_i} + (X_\alpha)^\beta \frac{\partial}{\partial z^\beta}\,, \tag{16}
$$

where the coefficients satisfy the conditions (8). Moreover,

$$
X^\gamma_\alpha = \mathrm{T}\pi^z_Q \circ X_\alpha = (X_\alpha)^i \frac{\partial}{\partial q^i} + (X_\alpha)^\beta \frac{\partial}{\partial z^\beta}\,.
$$

Then, the k-vector fields $\mathbf{X}^\gamma_h$ and $\mathbf{X}_h$ are γ-related if and only if

$$
\mathrm{T}\gamma \circ X^\gamma_\beta = (X_\beta)^i \frac{\partial}{\partial q^i} + \left((X_\beta)^i \frac{\partial \gamma^\mu_j}{\partial q^i} + (X_\beta)^\alpha \frac{\partial \gamma^\mu_j}{\partial z^\alpha} \right) \frac{\partial}{\partial p^\mu_j} + (X_\beta)^\alpha \frac{\partial}{\partial z^\alpha} = X_\beta \circ \gamma\,.
$$

In other words,

$$
(X_\beta)^\mu_j \circ \gamma = (X_\beta)^i \frac{\partial \gamma^\mu_j}{\partial q^i} + (X_\beta)^\alpha \frac{\partial \gamma^\mu_j}{\partial z^\alpha}\,.
$$

Given h, only the components $(X_\beta)^i$ are uniquely fixed, namely $(X_\beta)^i = \dfrac{\partial h}{\partial p^\beta_i}$.

It can be seen that the previous expression retrieves for $k = 1$, the known expression for contact manifolds [10, eq. 24] given by

$$
\mathrm{d}_Q (H \circ \gamma) + \frac{\partial}{\partial z}(H \circ \gamma)\gamma = (H \circ \gamma)\mathscr{L}_{\frac{\partial}{\partial z}}\gamma\,.
$$

Nevertheless, the idea behind the γ-relation is somehow lost in field theories. In ordinary differential equations, two vector fields are γ-related if and only if the composition of an integral curve of one with γ gives an integral curve of the second. In field theories, where HDW equations is the main object of study, k-vector fields convey more information than their HDW equations as illustrated by the fact that different vector fields may give rise to the same HDW equations. Hence, what is important to us is that the composition of γ with the integral curves of a vector field will give the HDW equations of the second. This is indeed what is important to the Hamilton–Jacobi method.

Let $\psi : \mathbb{R}^k \to Q \times \mathbb{R}^k$ be an integral section of $\mathbf{X}^\gamma_h$, such that $\psi(t) = (q^i(t), z^\alpha(t)) \equiv (\psi^i(t), \psi^\alpha(t))$. In this case,

$$
\frac{\partial \psi^i}{\partial t^\alpha} = (X^\gamma_\alpha)^i = (X_\alpha)^i\,, \quad \frac{\partial \psi^\beta}{\partial t^\alpha} = (X^\gamma_\alpha)^\beta = (X_\alpha)^\beta\,, \qquad i = 1, \dots, \dim Q\,, \quad \alpha = 1, \dots, k\,.
$$

Let us require $\gamma \circ \psi$ to be a solution to the HDW equations for $\gamma : Q \times \mathbb{R}^k \to \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$, such that $\gamma(q, z) = (q^i, \gamma^\alpha_i(q, z), z^\alpha) \equiv (\gamma^i, \gamma^\alpha_i(q, z), \gamma^\alpha)$. Then,

$$
\begin{cases}
\dfrac{\partial(\gamma^i \circ \psi)}{\partial t^\alpha} = \dfrac{\partial \psi^i}{\partial t^\alpha} = (X^\gamma_\alpha)^i = \dfrac{\partial h}{\partial p^\alpha_i} \circ \gamma \circ \psi \\
\dfrac{\partial(\gamma^\beta_i \circ \psi)}{\partial t^\beta} = \dfrac{\partial \gamma^\beta_i}{\partial z^\sigma}(X^\gamma_H)^\sigma_\beta + \dfrac{\partial \gamma^\beta_i}{\partial q^j}(X^\gamma_H)^j_\beta = -\left(\dfrac{\partial H}{\partial q^i} + p^\alpha_i \dfrac{\partial H}{\partial z^\alpha} \right) \circ \gamma \circ \psi \\
\dfrac{\partial(\gamma^\mu \circ \psi)}{\partial t^\mu} = \dfrac{\partial \psi^\mu}{\partial t^\mu} = (X^\gamma_H)^\mu_\mu = \left(p^\mu_i \dfrac{\partial H}{\partial p^\mu_i} - H \right) \circ \gamma \circ \psi
\end{cases}
$$

Recall that a k-vector field $\mathbf{X}=(X_1,\dots,X_k)$ has a local form in Darboux coordinates given by

$$X_\alpha=(X_\alpha)^i\frac{\partial}{\partial q^i}+(X_\alpha)_i^\beta\frac{\partial}{\partial p_i^\beta}+(X_\alpha)^\beta\frac{\partial}{\partial z_\beta}\,.$$

Remember also, that $(X_\alpha)^\beta=0$, for $\alpha\neq\beta$, do not play a role in the determination of the HDW equations for $\mathbf{X}$. Consider the k-vector field given by

$$Y_1=X_\alpha^\alpha\frac{\partial}{\partial z_1}\,,\qquad Y_2=X_2^2\frac{\partial}{\partial z_2}\,,\quad \dots\,,\qquad Y_k=X_k^k\frac{\partial}{\partial z_k}\,.$$

Note that

$$\iota_{Y^\alpha}\mathrm{d}\eta^\alpha=0\,,\qquad \iota_{Y^\alpha}\eta^\alpha=0\,.$$

In other words, $Y=(Y_1,\dots,Y_k)$ is a Hamiltonian k-vector field associated with a zero Hamiltonian function. Hence, the k-vector field $\mathbf{Z}$ given by

$$Z_\alpha=X_\alpha+Y_\alpha\,,\qquad \alpha=1,\dots,k\,,$$

satisfies that

$$\iota_{Z_\alpha}\mathrm{d}\eta^\alpha=\mathrm{d}H+(R_\alpha H)\eta_\alpha\,,\qquad \iota_{Z_\alpha}\eta^\alpha=-h\,.$$

In other words, it is a k-Hamiltonian vector field related to the function H. Moreover, it satisfies that

$$Z_\alpha^\beta=0$$

for all coefficients apart from $\alpha=\beta=1$.

Then, It is possible to introduce a set of vector fields $\{Y_\alpha\}_{\alpha=1,\dots,k}\in\ker\mathrm{d}\eta$:

I am not sure of what this transformation is exactly like. I think that Z2 to ZK lose all the z1,...,zk dependency, if I am not mistaken, and the Z1 can only depend on partial wrt to z1, but the coefficient of this partial is what is missing... I think that Javi commented that we need to include in this coefficient all of the terms we took from Z2 to Zk, but what do we do with the terms in partial z2,...,zk of the own Z1? we put them in that coefficient too?

$$Y_1=\sum_{\beta=1}^k X_\alpha^\beta\frac{\partial}{\partial z_1}-\text{rest its own z2,...zk, I guess?},\qquad Y_\alpha=-X_\alpha^\beta\frac{\partial}{\partial z_\beta},\quad \alpha=2,\dots,k,\quad \beta=1,\dots,k. \tag{17}$$

such that a k-vector field $\mathbf{Z}=(Z_1,\dots,Z_k)$ defined as that $Z_\alpha=X_\alpha+Y_\alpha$ for $\alpha=1,\dots,k$ is independent of the contact parameters $(z_1,\dots,z_k)$ in its $Z_2,\dots,Z_k$ components, and only the first component Z_1 depends on one contact parameter z_1.

Roughly speaking, this is an specific kind of rectification theorem, which states that if a vector field X_α does not vanish at a point, then it is possible to find a coordinate chart around that point such that the coordinate representation of X_α in the chart is a constant vector field. Now, if there are more than one vector field, as it is in our case, this rectification theorem holds as long as $[X_\alpha,X_{\hat\alpha}]=0$, being X_α and $X_{\hat\alpha}$ two functionally independent vector fields. In the case of flows, the associated flows $\Phi_\alpha^{s_\alpha}$ and $\Phi_{\hat\alpha}^{\hat s_{\hat\alpha}}$ must commute at every point in $\oplus^k\mathrm{T}^*Q\times\mathbb{R}^k$. With this in mind, the vector fields can even be rectified simultaneously, and it is known as the simultaneous rectification theorem [2]. Also, the commutativity of the vector fields is what we understand as integrability.

We would like to comment a couple of things about integrability of projected and rectified vector fields in the z^α-dependent formalism.

- In the case of integrability of projected vector fields under π_Q^z, it is straightforward to see that if the initial distribution $X_h : \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k \to \oplus^k \mathrm{T}\left(\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k\right)$ is integrable, then $X_h^\gamma : Q \times \mathbb{R}^k \to \oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$ is integrable too. It is easy to compute the brackets of two vector fields with expression in coordinates of the type (16) and realize that, if the integrability condition $[X_\alpha, X_{\hat{\alpha}}] = 0$ holds for the two vector fields X_α and $X_{\hat{\alpha}}$ in $\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$, then, it has be satisfied that:

$$[X^\gamma_{\alpha H}, X^\gamma_{\hat{\alpha} H}] = X_{\hat{\alpha} H}\left((X_\alpha)^\beta_i\right)\frac{\partial}{\partial p^\beta_i} - X_{\alpha H}\left((X_{\hat{\alpha}})^{\hat{\beta}}_{\hat{i}}\right)\frac{\partial}{\partial p^{\hat{\beta}}_{\hat{i}}}$$

  Realize that the right-hand side of the equation corresponds with the extra terms of the vector fields in $\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$ before the momentum components are projected. Realize also that the right-hand side and the left-hand side can only be equal if both equal zero, since there is no momenta dependency on the left-hand side. Therefore, we conclude that $[X^\gamma_{\alpha H}, X^\gamma_{\hat{\alpha} H}] = 0$ and the projected vector fields to $Q \times \mathbb{R}^k$ are integrable.

  Note: If the vector fields in $\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k$ are not integrable in every point of the manifold, then, the commutator is not equally zero in all of the points of the manifold. There exists a generalization of the commutator for Lipschitz continuous vector fields that commute "almost everywhere" [28]. The commutator is defined on a set $[X_\alpha, X_{\hat{\alpha}}]_{\mathrm{set}}(x) = \bar{\mathrm{co}}\left\{v = \lim_{j\to\infty}[X_\alpha, X_{\hat{\alpha}}](x_j)\right\}$ is equivalent to a Lie bracket commutator when $x \mapsto [X_\alpha, X_{\hat{\alpha}}]_{\mathrm{set}}(x)$ is an upper semi-continous set-valued vector field with compact convex nonempty values, and $x_j \in \mathrm{Diff}(X_\alpha) \cap \mathrm{Diff}(X_{\hat{\alpha}})$, where $\mathrm{Diff}(X_k)$ is the set of points where X_k is differentiable, and it is a full measure set in the manifold, by Radamacher's theorem [27]. In the Appendix 7 we anunciate a weak version of the Frobenius theorem in such a case.

- In order to obtain integrable rectified vector fields $Z = (Z_1, \ldots, Z_k)$ after the transformation (17), we need to impose that $[\mathrm{pr}Z_1, Z_\alpha] = 0$ for $\alpha = 2, \ldots, k$, where $\mathrm{pr}Z_1$ is the projection of Z_1 in (17) to $\mathbb{R}$ with local coordinate z_1.

Therefore, if $\mathbf{X}$ satisfies HDW equations, then the same applies to $\mathbf{Z}$. In particular, summing over indexes $\beta = \sigma$ and using the Hamilton–De Donder–Weyl equations for $\mathbf{Z}$, one obtains

$$-\left(\frac{\partial H}{\partial q^i} + p^\alpha_i \frac{\partial H}{\partial z^\alpha}\right)\circ\gamma\circ\psi = \frac{\partial \gamma^\epsilon_i}{\partial q^j}\frac{\partial H}{\partial p^\epsilon_j}\circ\gamma\circ\psi + \frac{\partial \gamma^1_i}{\partial z^1}\left(p^\alpha_j\frac{\partial H}{\partial p^\alpha_j} - H\right)\circ\gamma\circ\psi. \tag{18}$$

Definition 4.6. A Legendrian k-contact submanifold is a maximal integral submanifold of $\cap_{\alpha=1}^k \ker\eta^\alpha$.

We will assume that $\gamma_z : Q \to \oplus^k_{\alpha=1}\mathrm{T}^*Q$, for every $z \in \mathbb{R}^k$, we have $\pi^\alpha : \oplus^k_{\beta=1}\mathrm{T}^*Q \to \mathrm{T}^*Q$ is a maximal integral isotropic submanifold of each $d\eta^\alpha$. Note that $\gamma^\alpha_z : Q \to \mathrm{T}^*Q$ are such that the condition of being Lagrangian implies that

$$(\gamma^\alpha_z)^*\mathrm{d}\eta^\alpha = 0 \Rightarrow \gamma^\alpha_z(\mathrm{d}p^\alpha_i \wedge \mathrm{d}q^i) = \mathrm{d}\gamma^\alpha_i \wedge \mathrm{d}q^i = 0$$

which amounts to

$$\frac{\partial \gamma^\alpha_i}{\partial q^j} = \frac{\partial \gamma^\alpha_j}{\partial q^i}.$$

Using previous conditions, one has:

$$\gamma^\alpha_j\frac{\partial \gamma^\alpha_i}{\partial z} = \gamma^\alpha_i\frac{\partial \gamma^\alpha_j}{\partial z}.$$

Or something of the short? consider the canonical forms $\theta^\alpha = p_i^\alpha \mathrm{d}q^i$. Note that

$$\iota_{R^\alpha}\mathrm{d}(\gamma^\alpha)^*\theta^\alpha_Q = \frac{\partial \gamma_i^\alpha}{\partial z^\alpha}\mathrm{d}q^j\,.$$

Let us prove the following Hamilton–Jacobi theorem

Theorem 4.7. *Assume that a section $\gamma : Q\times\mathbb{R}^k \to \oplus^k\mathrm{T}^*Q\times\mathbb{R}^k$ of the projection $\oplus_k\mathrm{T}^*Q\times\mathbb{R}^k \to Q\times\mathbb{R}$ is such that $\gamma(Q\times\mathbb{R}^k)$ is coisotropic for $(\oplus^k\mathrm{T}^*Q\times\mathbb{R}^k,\eta^\alpha)$ and $\gamma_Z(Q)$ is a Lagrangian submanifold of $(\oplus^k\mathrm{T}^*Q\times\mathbb{R}^k,\omega^\alpha_Q)$, for every $z\in\mathbb{R}^k$. Then, the k-vector fields $\mathbf{X}_h$ and $\mathbf{X}_h^\gamma$ are γ-related if and only if (18) holds (equivalently, (25) holds).*

Proof. □

Definition 4.8. Given a section $\alpha : Q\times\mathbb{R}^k \to \bigwedge^k\mathrm{T}^*Q\times\mathbb{R}^k$ and $z\in\mathbb{R}^k$, let

$$\begin{aligned}\alpha_z : Q &\longrightarrow \bigwedge^k \mathrm{T}^*Q\\ x &\longmapsto \mathrm{pr}_{\Lambda^k\mathrm{T}^*Q}(\alpha(x,z))\,,\end{aligned}$$

where $\mathrm{pr}_{\bigwedge^k\mathrm{T}^*Q} : \bigwedge^k\mathrm{T}^*Q\times\mathbb{R}\to\bigwedge^k\mathrm{T}^*Q$ is the canonical projection. The *exterior derivative of α at fixed z* is the section of $\bigwedge^{k+1}\mathrm{T}^*Q\times\mathbb{R}\to Q\times\mathbb{R}$ given by

$$\mathrm{d}_Q\alpha(x,z) = (\mathrm{d}\alpha_z(x), z)\,.$$

The coisotropic condition can be written in local coordinates as follows.

Lemma 4.9. *Assume that an $(n+k)$-dimensional submanifold N of a $(2n+2)$-dimensional k-cocontact manifold (M,τ,η) is of the form $N = f^{-1}(0)$ for $f : x\in M\to(f^1(x),\dots,f^{n+2-k}(x))\in\mathbb{R}^{n+2-k}$ and 0 being a regular point of M. Then, N is k-coisotropic if and only if the following equation holds in Darboux coordinates:*

$$\left(\frac{\partial f_a}{\partial q^i} + p_i^\beta\frac{\partial f_a}{\partial z^\beta}\right)\frac{\partial f_b}{\partial p_i^\beta} - \left(\frac{\partial f_b}{\partial q^i} + p_i^\beta\frac{\partial f_b}{\partial z^\beta}\right)\frac{\partial f_a}{\partial p_i^\beta} = 0\,. \tag{19}$$

Proof. Let (M,τ,η) be a $(2n+2)$-dimensional cocontact manifold. Since 0 is a regular point of f, then N is a submanifold of dimension k. In adapted coordinates

$$\mathrm{T}N^\perp = \left\langle \{Z_a\}_{a=1,\dots,2n+2-tk}\right\rangle\,,$$

where

$$Z_a = \hat{\Lambda}(\mathrm{d}\phi_a) = \left(\frac{\partial\phi_a}{\partial q^i} + p_i\frac{\partial\phi_a}{\partial z}\right)\frac{\partial}{\partial p_i} - \frac{\partial\phi_a}{\partial p_i}\left(\frac{\partial}{\partial q^i} + p_i\frac{\partial}{\partial z}\right)\,.$$

Therefore, N is coisotropic if and only if $Z_a(\phi_b) = 0$ for all a,b, which in Darboux coordinates yields Eq. (19).

□

Proposition 4.10. *Let γ be a section of $\oplus_k\mathrm{T}^*Q\times\mathbb{R}^k$ over $Q\times\mathbb{R}^k$. Then $\operatorname{Im}\gamma$ is a coisotropic submanifold if and only if*

$$\frac{\partial\gamma_j^\mu}{\partial q^i} + \gamma_i^\mu\frac{\partial\gamma_j^\mu}{\partial z^\mu} = \frac{\partial\gamma_i^\mu}{\partial q^j} + \gamma_j^\mu\frac{\partial\gamma_i^\mu}{\partial z^\mu}\,,\qquad \frac{\partial\gamma_i^\alpha}{\partial q^j} + \gamma_j^\mu\frac{\partial\gamma_i^\alpha}{\partial z^\mu} = 0 \tag{20}$$

for all $i,j = 1,\dots,n$ and $\mu,\alpha = 1,\dots,k$, where $\alpha\neq\mu$.

Proof. Eq. (20) is obtained by applying the previous result to the submanifold $N = \operatorname{Im}\gamma$, which is locally defined by the constraints $\phi_i^\alpha = p_i^\alpha - \gamma_i^\alpha = 0$, for $i = 1, \dots, n$, $\alpha = 1, \dots, k$. In that case

$$\mathrm{d}\phi_i^\alpha = -\frac{\partial \gamma_i^\alpha}{\partial q^k}\mathrm{d}q^k + \mathrm{d}p_i^\alpha - \frac{\partial \gamma_i^\alpha}{\partial z^\epsilon}\mathrm{d}z^\epsilon ,$$

and

$$Z_i^{\alpha\mu} = \sharp_{\Lambda_\mu}(\mathrm{d}\phi_i^\alpha) = -\delta_\mu^\alpha \frac{\partial}{\partial q^i} - \left(\frac{\partial \gamma_i^\alpha}{\partial q^k} + p_k^\mu \frac{\partial \gamma_i^\alpha}{\partial z^\mu}\right)\frac{\partial}{\partial p_k^\mu} - \delta_\mu^\alpha p_i^\mu \frac{\partial}{\partial z^\mu} ,$$

where μ index is fixed. Now,

$$Z_i^{\alpha\mu}(\mathrm{d}\phi_j^\beta) = \Lambda_\mu(\mathrm{d}\phi_i^\alpha, \mathrm{d}\phi_j^\beta) = \delta_\mu^\alpha \left(\frac{\partial \gamma_j^\beta}{\partial q^i} + p_i^\mu \frac{\partial \gamma_j^\beta}{\partial z^\mu}\right) - \delta_\mu^\beta \left(\frac{\partial \gamma_i^\alpha}{\partial q^j} + p_j^\mu \frac{\partial \gamma_i^\alpha}{\partial z^\mu}\right) . \tag{21}$$

Observe that in case of $\alpha, \beta \neq \mu$, Eq. (21) vanishes. If $\operatorname{Im}\gamma$ is a coisotropic manifold then Eq. (21) must vanish for all $\alpha, \beta, \mu = 1, \dots, k$ and $i, j = 1, \dots, n$. Substituting $p_i^\alpha = \gamma_i^\alpha$, we obtain

$$\begin{cases} \dfrac{\partial \gamma_j^\mu}{\partial q^i} + \gamma_i^\mu \dfrac{\partial \gamma_j^\mu}{\partial z^\mu} - \dfrac{\partial \gamma_i^\mu}{\partial q^j} - \gamma_j^\mu \dfrac{\partial \gamma_i^\mu}{\partial z^\mu} = 0 , \qquad \forall \mu = 1, \dots k , \quad \forall i, j = 1, \dots, n , \\ \dfrac{\partial \gamma_i^\alpha}{\partial q^j} + \gamma_j^\mu \dfrac{\partial \gamma_i^\alpha}{\partial z^\mu} = 0 , \qquad \forall \mu \neq \alpha , \quad \alpha, \mu = 1, \dots k , \quad \forall i, j = 1, \dots, n . \end{cases}$$

□

Now suppose that the γ appearing in Eq. (**??**) is such that $\operatorname{Im}\gamma$ is coisotropic. Then, by means of Eq. (20) we obtain

$$\frac{\partial h}{\partial q^i} + \frac{\partial h}{\partial p_j}\frac{\partial \gamma_j}{\partial q^i} + \gamma_i \left(\frac{\partial h}{\partial p_j}\frac{\partial \gamma_j}{\partial z} + \frac{\partial h}{\partial z}\right) + \frac{\partial \gamma_i}{\partial t} = h \frac{\partial \gamma_i}{\partial z} ,$$

or, globally,

$$\mathrm{d}_Q\,(h \circ \gamma) + \frac{\partial}{\partial z}(h \circ \gamma)\gamma + \mathscr{L}_{R_t}\gamma = (h \circ \gamma)\mathscr{L}_{\frac{\partial}{\partial z}}\gamma . \tag{22}$$

This equation will be called the *action-dependent Hamilton–Jacobi* equation for $(\oplus^k \mathrm{T}^*Q \times \mathbb{R}^k, \tau, \eta, h)$.

Theorem 4.11 (Action-dependent Hamilton–Jacobi Theorem)**.** *Let γ be a section of $\pi_Q^{t,z}$: $\bigoplus^k \mathrm{T}^*Q \times \mathbb{R}^k \to Q \times \mathbb{R}$ such that $\operatorname{Im}\gamma$ is a coisotropic submanifold of $(\bigoplus^k \mathrm{T}^*Q \times \mathbb{R}^k, \tau, \eta)$. Then, X_H^γ and X_H are γ-related if and only if Eq. (22) holds.*

Definition 4.12. Let $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$ be a cocontact Hamiltonian system. A *complete solution of the action-dependent Hamilton–Jacobi problem* for $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$ is a local diffeomorphism $\Phi\colon \mathbb{R} \times Q \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$ such that, for each $\lambda \in \mathbb{R}^n$,

$$\begin{aligned} \Phi_\lambda \colon \mathbb{R} \times Q \times \mathbb{R} &\longrightarrow \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \\ \left(t, q^i, z\right) &\longmapsto \Phi\left(t, q^i, \lambda, z\right) \end{aligned}$$

is a solution of the action-dependent Hamilton–Jacobi problem for $(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}, \tau, \eta, H)$.

Let $\alpha\colon \mathbb{R} \times Q \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, and $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ denote the canonical projections. Let us define the functions $f_i = \pi_i \circ \alpha \circ \Phi^{-1}$ on $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$, so that the following diagram commutes:

$$\begin{tikzcd} \mathbb{R} \times Q \times \mathbb{R}^n \times \mathbb{R} \arrow[r, shift left, "\Phi"] \arrow[d, "\alpha"] & \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \arrow[l, shift left, "\Phi^{-1}"] \arrow[d, "f_i"] \\ \mathbb{R}^n \arrow[r, "\pi_i"] & \mathbb{R} \end{tikzcd}$$

Theorem 4.13. *Let $\Phi\colon \mathbb{R}\times Q\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}\times\mathrm{T}^*Q\times\mathbb{R}$ be a complete solution of the action-dependent Hamilton–Jacobi problem for $(\mathbb{R}\times\mathrm{T}^*Q\times\mathbb{R},\tau,\eta,H)$. Then,*

(i) *For each $i\in\{1,\dots,n\}$, the function $f_i=\pi_i\circ\alpha\circ\Phi^{-1}$ is a constant of the motion. However, these functions are not necessarily in involution, i.e., $\{f_i,f_j\}\neq 0$.*

(ii) *For each $i\in\{1,\dots,n\}$, the function $\hat{f}_i=gf_i$, where g is a dissipated quantity, is also a dissipated quantity. Moreover, if $R_tH=0$ and taking $g=H$, these functions are in involution, i.e., $\{\hat{f}_i,\hat{f}_j\}=0$.*

Proof. Observe that

$$\operatorname{Im}\Phi_\lambda=\bigcap_{i=1}^n f_i^{-1}(\lambda_i)\,,$$

where $\lambda=(\lambda,\dots,\lambda_n)\in\mathbb{R}^n$. In other words,

$$\operatorname{Im}\Phi_\lambda=\{x\in\mathbb{R}\times\mathrm{T}^*Q\times\mathbb{R}\mid f_i(x)=\lambda_i, i=1,\dots,n\}\,.$$

Therefore, since X_H is tangent to any of the submanifolds $\operatorname{Im}\Phi_\lambda$, we deduce that

$$X_H(f_i)=0\,.$$

Moreover, we can compute

$$\{f_i,f_j\}=\Lambda(\mathrm{d}f_i,\mathrm{d}f_j)-f_iR_z(f_j)+f_jR_z(f_i)\,,$$

but

$$\Lambda(\mathrm{d}f_i,\mathrm{d}f_j)=\hat{\Lambda}(\mathrm{d}f_i)(f_j)=0\,,$$

since $(\mathrm{T}\operatorname{Im}\Phi_\lambda)^\perp=\hat{\Lambda}((\mathrm{T}\operatorname{Im}\Phi_\lambda)^\circ)\subset\mathrm{T}\operatorname{Im}\Phi_\lambda$, so

$$\{f_i,f_j\}=-f_iR_z(f_j)+f_jR_z(f_i)\,.$$

By Proposition **??**, we already know that the product of a conserved quantity and a dissipated quantity is a dissipated quantity. Let f_i and f_j be conserved quantities and take $g=H$. Then, by Eq. (**??**), $\{\hat{f}_i,\hat{f}_j\}$ vanishes.

□

From a complete solution of the Hamilton–Jacobi problem one can reconstruct the dynamics of the system. If σ is an integral curve of the vector field X_H^γ, then $\Phi_\lambda\circ\sigma$ is an integral curve of X_H, thus recovering the dynamics of the original system.

4.3 Integrable contact Hamiltonian systems

Let $(\mathrm{T}^*Q\times\mathbb{R},\eta,H)$ be a contact Hamiltonian system. Recall that the action-dependent Hamilton–Jacobi equation for $(\mathrm{T}^*Q\times\mathbb{R},\eta,H)$ is given by [10, 9]

$$\mathrm{d}_Q\left(H\circ\gamma\right)+\frac{\partial}{\partial z}(H\circ\gamma)\gamma=(H\circ\gamma)\mathscr{L}_{\frac{\partial}{\partial z}}\gamma\,.$$

A *complete solution of the action-dependent Hamilton–Jacobi problem* for $(\mathrm{T}^*Q\times\mathbb{R},\eta,H)$ is a local diffeomorphism $\Phi\colon Q\times\mathbb{R}^n\times\mathbb{R}\to\mathrm{T}^*Q\times\mathbb{R}$ such that, for each $\lambda\in\mathbb{R}^n$,

$$\begin{aligned}\Phi_\lambda\colon Q\times\mathbb{R}&\longrightarrow\mathrm{T}^*Q\times\mathbb{R}\\ \left(q^i,z\right)&\longmapsto\Phi\left(q^i,\lambda,z\right)\end{aligned}$$

is a solution of the action-dependent Hamilton–Jacobi problem for $(\mathrm{T}^*Q \times \mathbb{R}, \eta, H)$.

Let $\Phi\colon Q \times \mathbb{R}^n \times \mathbb{R} \to \mathrm{T}^*Q \times \mathbb{R}$ be a complete solution of the Hamilton–Jacobi problem for $(\mathrm{T}^*Q \times \mathbb{R}, \eta, H)$. Then,

$$\mathcal{F} = \{\mathcal{F}_\lambda = \operatorname{Im} \Phi_\lambda \mid \lambda \in \mathbb{R}^n\} \subseteq \mathrm{T}^*Q \times \mathbb{R}$$

is a foliation in coisotropic submanifolds.

In the symplectic case, since solutions of the Hamilton–Jacobi equation are closed one-forms on Q, the images of a complete solution for each choice of parameters λ form a Lagrangian foliation invariant under the action of the Hamiltonian flow. This structure is called an integrable system. In analogy, we introduce the following definition:

Definition 4.14. Let (M, η, H) be a contact Hamiltonian system and let $\mathcal{F}$ be a foliation consisting of $(n+1)$-dimensional coisotropic (with respect to the Jacobi structure of the contact manifold) leaves invariant under the flow of the Hamiltonian vector field X_H. Then we call $(M, \eta, H, \mathcal{F})$ an *integrable system.*

This definition can be compared to the ones given in [7, 22]:

- In [7], Boyer proposes a concept of completely integrable system for the so-called good Hamiltonians, that is, the Hamiltonian function is preserved along the flow of the Reeb vector field. This is a particular case of our definition, in which both the Hamiltonian and the constants of the motion do not depend on z.

- In [22], Khesin and Tabachnikov call a foliation *co-Legendrian* when it is transverse to $\mathcal{H}$ and $\mathrm{T}\mathcal{F} \cap \mathcal{H}$ is integrable. Then they define an integrable system as a particular case of a co-Legendrian foliation with some extra regularity conditions. In the case that the dimension of the leaves is $n+1$, the following proposition shows that co-Legendrian foliations are particular cases of coisotropic foliations.

Proposition 4.15. *Let $i\colon N \hookrightarrow M$ be a submanifold of a $(2n+1)$-dimensional contact manifold (M, η). If N is an $(n+1)$-dimensional co-Legendrian submanifold, then it is also a coisotropic submanifold.*

Proof. Let us write $\mathrm{T}N = \mathcal{D}_{\mathcal{H}} \oplus \mathcal{E}$, where $\mathcal{D}_{\mathcal{H}} = \mathrm{T}N \cap \mathcal{H}$. Then, $\mathrm{T}N^\perp = \mathcal{D}_{\mathcal{H}}^\perp \cap \mathcal{E}^\perp$. Obviously, η vanishes in $\mathrm{T}N \cap \mathcal{H}$. Moreover, since $\mathcal{D}_{\mathcal{H}}$ is integrable,

$$0 = \eta([v, w]) = \iota_{[v,w]}\eta = \mathscr{L}_v \iota_w \eta - \iota_w \mathscr{L}_v \eta = -\iota_w \iota_v \mathrm{d}\eta - \iota_w \mathrm{d}\iota_v \eta = -\iota_w \iota_v \mathrm{d}\eta\,,$$

for any $v, w \in \mathcal{D}_{\mathcal{H}}$, so $\mathrm{d}\eta_{|\mathcal{D}_{\mathcal{H}}} = 0$. Observe that $\hat{\Lambda}_{|\mathcal{H}} = \sharp_{|\mathcal{H}}$, and $\sharp_{|\mathcal{H}}\colon \mathcal{H} \to \langle R\rangle^\circ$, $\sharp_{|\mathcal{H}}^{-1}(v) = \iota_v \mathrm{d}\eta$ is an isomorphism. Since $\mathrm{d}\eta_{|\mathcal{D}_{\mathcal{H}}} = 0$, $\sharp_{|\mathcal{H}}^{-1}(\mathcal{D}_{\mathcal{H}}) \subseteq \mathcal{D}_{\mathcal{H}}^\circ$. Thus, $\mathcal{D}_{\mathcal{H}} \subseteq \hat{\Lambda}(\mathcal{D}_{\mathcal{H}}^\circ) = \mathcal{D}_{\mathcal{H}}^\perp$. By a dimension counting argument, we can see that both spaces are equal and, thus, $\mathcal{D}_{\mathcal{H}} = \mathcal{D}_{\mathcal{H}}^\perp$.

□

We also note that a foliation $\tilde{\mathcal{F}}$ by Legendrian submanifolds can never be invariant by the Hamiltonian flow. Indeed, let $\tilde{F} \in \tilde{\mathcal{F}}$. The leaves of $\tilde{\mathcal{F}}$ are Lagrangian, thus $\mathrm{T}\tilde{F}_0 \subseteq \ker \eta$. Since $\eta(X_H) = -H$, X_H can only be tangent to the leaves at the zero set of H, hence its flow cannot leave invariant the whole foliation.

Observe that Definition 4.14 can be naturally extended to cocontact Hamiltonian systems.

Definition 4.16. Let (M, τ, η, H) be a cocontact Hamiltonian system and let $\mathcal{F}$ be a foliation consisting of $(n+2)$-dimensional coisotropic leaves (with respect to the Jacobi structure of the cocontact manifold) invariant under the flow of the cocontact Hamiltonian vector field X_H. Then we call $(M, \tau, \eta, H, \mathcal{F})$ an *integrable cocontact system.*

$$\begin{array}{rccc} \gamma\colon & Q\times\mathbb{R}^k & \longrightarrow & \bigoplus^k \mathrm{T}^*Q\times\mathbb{R}^k \\ & (q^i, z^\alpha) & \longmapsto & (q^i, \gamma_i^\alpha(q^i,z^\alpha), z^\alpha) \end{array}.$$

Let us introduce the k-vector field Z_H^γ on $Q\times\mathbb{R}^k$ given by

$$Z_H^\gamma = \mathrm{T}_k\tilde{\pi}_Q\circ Z_H\circ\gamma\,,$$

where Z_H is the Hamiltonian k-vector field of $(\bigoplus^k \mathrm{T}^*Q\times\mathbb{R}^k, \eta^\alpha, H)$. Suppose that Z_H^γ and Z_H are γ-related, i.e.,

$$Z_H\circ\gamma = \mathrm{T}_k\gamma\circ Z_H^\gamma\,,$$

so that the following diagram commutes

$$\begin{tikzcd} \bigoplus^k \mathrm{T}^*Q\times\mathbb{R}^k \arrow[r, "Z_H"] \arrow[d, "\tilde{\pi}_Q"'] & \bigoplus^k \mathrm{T}(\bigoplus^k \mathrm{T}^*Q\times\mathbb{R}^k) \arrow[d, "\mathrm{T}_k\tilde{\pi}_Q"'] \\ Q\times\mathbb{R}^k \arrow[r, "Z_H^\gamma"] \arrow[u, bend left, "\gamma"] & \bigoplus^k \mathrm{T}(Q\times\mathbb{R}^k) \arrow[u, bend right, "\mathrm{T}_k\gamma"'] \end{tikzcd}$$

Theorem 4.17. *(Hamilton–Jacobi Theorem) Let Z be a k-contact vector field for some $h\in\mathscr{C}^\infty(\bigoplus^k T^*Q\times\mathbb{R}^k)$ and γ a closed section of $\tilde{\pi}_Q$ such that $\frac{\partial S^\alpha}{\partial q^i}=\gamma_i^\alpha$. If Z is integrable, then the following conditions are equivalent:*

1. *Every integral section, $\sigma:\mathcal{U}\subset\mathbb{R}^k\to Q\times\mathbb{R}^k$, of Z^γ gives rise to a solution $\gamma\circ\sigma$ to HDW equations,*
2. $\mathrm{d}(h\circ\gamma)=0$.

Proof. □

5 Non-equilibrium thermodynamics

[6]

6 Conclusions and further research

Check the Hamilton–Jacobi theorem for the z-independent case Check the structure of all k-vector fields having the same HDW equations, studying the complete solutions in the z-independent case.

Study the integrability of the Hamiltonian rectified and not rectified k-vector fields in the z-dependent case. Check the Hamilton–Jacobi equation for the z-dependent case.

Acknowledgements

We thank prof. M. de León for fruitful discussions and comments. X. Rivas acknowledges the financial support of the Ministerio de Ciencia, Innovación y Universidades (Spain), project PGC2018-098265-B-C33. J. de Lucas, C. Sardón, and X. Rivas acknowledge partial financial support from the Novee Idee 2B-POB II project PSP: 501-D111-20-2004310 funded by the "Inicjatywa Doskonałości - Uczelnia Badawcza" (IDUB) program. Cristina Sardón acknowledges funding from the department of Applied Mathematics at ETSII, Universidad Politécnica de Madrid.

7 Appendix

The weak Frobenius theorem

Theorem 7.1. *Let X_α and $X_{\hat{\alpha}}$ be vector fields of a manifold of class C^2. Then the flows $\Phi_\alpha^{s_\alpha}$ and $\Phi_{\hat{\alpha}}^{\hat{s}_{\hat{\alpha}}}$ commute if and only if $[X_\alpha, X_{\hat{\alpha}}]_{set}(x) = \{0\}$ for every x in the manifold, if and only if $[X_\alpha, X_{\hat{\alpha}}](x)$ equals zero for almost every x.*

In these circumstances, it is also possible to write a rectification theorem [28].

Theorem 7.2. *Let $X_1, \ldots, X_k$ be locally Lipschitz vector fields on a manifold of class C^2 and x^* be a point in the manifold. Also, we assume that the vector fields are linearly independent in x^*, i.e., their Lie brackets equal zero almost everywhere in a neighborhood of x^*, or equivalently, their "set" brackets equal zero in every point of the manifold. Then, there exists a coordinate chart near x^* in which the representations of $X_1, X_2, \ldots, X_k$ are constant.*

So, we can enunciate a modified version of the Frobenius theorem in the case of a nonintegrable distribution in a small number of points [27].

Theorem 7.3. *Let Δ_x be the distribution spanned by a set of locally Lipschitz and linearly independent vector fields $X_1, \ldots, X_k$ on a manifold of class C^2. The following statements are equivalent:*

1. *Delta is completely integrable, i.e., for each x^* there exists a submanifold N of class C^1, passing through x^* such that $T_xN = \Delta(x)$ for every $x \in N$.*
2. *Δ is set-involutive, i.e., $[X_i, X_j]_{set}(x) \subset \Delta_x$*
3. *Δ is almost everywhere involutive, for almost every x in the manifold.*

References

[1] V. Aldaya and J. A. de Azcárraga. Geometric formulation of classical mechanics and field theory. *Riv. Nuovo Cim.*, **3**(1):1–66, 1980. 10.1007/BF02906204.

[2] V. I. Arnold. *Mathematical Methods of Classical Mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer, New York, 2nd edition, 1989. 10.1007/978-1-4757-1693-1.

[3] A. Awane. k-symplectic structures. *J. Math. Phys.*, **33**(12):4046, 1992. 10.1063/1.529855.

[4] A. Awane. G-espaces K-symplectiques homogènes. *J. Geom. Phys.*, **13**(2):139–157, 1994. 10.1016/0393-0440(94)90024-8.

[5] A. Awane and M. Goze. *Pfaffian systems, k-symplectic systems.* Springer, Dordrecht, 1st edition, 2000. 10.1007/978-94-015-9526-1.

[6] Z. Banach and S. Pierkarski. Non-equilibrium thermodynamics and dissipative fluid theories. *Il Nuovo Cimento D*, **15**:1063–1086, 1993. 10.1007/BF02451879.

[7] C. P. Boyer. Completely Integrable Contact Hamiltonian Systems and Toric Contact Structures on $S^2 \times S^3$. *Symmetry Integr. Geom.: Methods Appl. (SIGMA)*, **7**:058, 2011. 10.3842/SIGMA.2011.058.

[8] M. de León, J. Gaset, M. Lainz-Valcázar, X. Rivas, and N. Román-Roy. Unified Lagrangian-Hamiltonian formalism for contact systems. *Fortschr. Phys.*, **68**(8):2000045, 2020. 10.1002/prop.202000045.

[9] M. de León, M. Lainz, A. López-Gordón, and X. Rivas. Hamilton–Jacobi theory and integrability for autonomous and non-autonomous contact systems. *J. Geom. Phys.*, **187**:104787, 2023. 10.1016/j.geomphys.2023.104787.

[10] M. de León, M. Lainz, and A. Muñiz-Brea. The Hamilton—Jacobi Theory for Contact Hamiltonian Systems. *Mathematics*, **9**(16):1993, 2021. 10.3390/math9161993.

[11] M. de León and M. Lainz-Valcázar. Contact Hamiltonian systems. *J. Math. Phys.*, **60**(10):102902, 2019. 10.1063/1.5096475.

[12] M. de León and M. Lainz-Valcázar. Infinitesimal symmetries in contact Hamiltonian systems. *J. Geom. Phys.*, **153**:103651, 2020. 10.1016/j.geomphys.2020.103651.

[13] M. de León and P. R. Rodrigues. *Generalized Classical Mechanics and Field Theory*, volume 112 of *Mathematics Studies*. North-Holland, 1985.

[14] M. de León, M. Salgado, and S. Vilariño. *Methods of Differential Geometry in Classical Field Theories*. World Scientific, 2015. 10.1142/9693.

[15] A. Echeverría-Enríquez, M. C. Muñoz-Lecanda, and N. Román-Roy. Multivector field formulation of Hamiltonian field theories: equations and symmetries. *J. Phys. A: Math. Gen.*, **32**(48):8461, 1999. 10.1088/0305-4470/32/48/309.

[16] O. Esen, M. de León, C. Sardón, and M. Zając. Cauchy data space and multisymplectic formulation of conformal classical field theories. *Ann. Phys.*, **434**:168616, 2021. 10.1016/j.aop.2021.168616.

[17] O. Esen, M. Lainz Valcázar, M. de León, and J. C. Marrero. Contact Dynamics: Legendrian and Lagrangian Submanifolds. *Mathematics*, **9**(21):2704, 2021. 10.3390/math9212704.

[18] J. Gaset, X. Gràcia, M. C. Muñoz-Lecanda, X. Rivas, and N. Román-Roy. A contact geometry framework for field theories with dissipation. *Ann. Phys.*, **414**:168092, 2020. 10.1016/j.aop.2020.168092.

[19] J. Gaset, X. Gràcia, M. C. Muñoz-Lecanda, X. Rivas, and N. Román-Roy. New contributions to the Hamiltonian and Lagrangian contact formalisms for dissipative mechanical systems and their symmetries. *Int. J. Geom. Methods Mod. Phys.*, **17**(6):2050090, 2020. 10.1142/S0219887820500905.

[20] J. Gaset, X. Gràcia, M. C. Muñoz-Lecanda, X. Rivas, and N. Román-Roy. A k-contact Lagrangian formulation for nonconservative field theories. *Rep. Math. Phys.*, **87**(3):347–368, 2021. 10.1016/S0034-4877(21)00041-0.

[21] C. Günther. The polysymplectic Hamiltonian formalism in field theory and calculus of variations I: The local case. *J. Differ. Geom.*, **25**(1):23–53, 1987. 10.4310/jdg/1214440723.

[22] B. Khesin and S. Tabachnikov. Contact complete integrability. *Regul. Chaotic Dyn.*, **15**:504–520, 2010. 10.1134/S1560354710040076.

[23] J. Kijowski and W. M. Tulczyjew. *A symplectic framework for field theories*, volume 107 of *Lecture Notes in Physics*. Springer-Verlag, Berlin Heidelberg, 1st edition, 1979. 10.1007/3-540-09538-1.

[24] R. Kleiss. *Quantum Field Theory: A Diagrammatic Approach*. Cambridge University Press, 2021. 10.1017/9781108665209.

[25] J. M. Lee. *Introduction to Smooth Manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer New York Heidelberg Dordrecht London, 2nd edition, 2013. 10.1007/978-1-4419-9982-5.

[26] J. C. Marrero, N. Román-Roy, M. Salgado, and S. Vilariño. Reduction of polysymplectic manifolds. *J. Phys. A: Math. Theor.*, **48**(5):055206, 2015. 10.1088/1751-8113/48/5/055206.

[27] F. Rampazzo. Frobenius-type theorems for Lipschitz distributions. *J. Differ. Equ.*, **243**(2):270–300, 2007. 10.1016/j.jde.2007.05.040.

[28] F. Rampazzo and H. J. Sussmann. Commutators of flow maps of nonsmooth vector fields. *J. Differ. Equ.*, **232**(1):134–175, 2007. 10.1016/j.jde.2006.04.016.

[29] L. H. Ryder. *Quantum Field Theory*. Cambridge University Press, 2nd edition, 1996. 10.1017/CBO9780511813900.

[30] A. A. Simoes, D. M. de Diego, M. L. Valcázar, and M. de León. On the Geometry of Discrete Contact Mechanics. *J. Nonlinear Sci.*, **31**(53), 2021. 10.1007/s00332-021-09708-2.