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# Weak Nearly Sasakian and Weak Nearly Cosymplectic Manifolds

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**Abstract:** Weak contact metric structures on a smooth manifold, introduced by V. Rovenski and R. Wolak in 2022, have provided new insight into the theory of classical structures. In this paper, we define new structures of this kind (called weak nearly Sasakian and weak nearly cosymplectic and nearly Kählerian structures) and study their geometry. We introduce weak nearly Kählerian manifolds (generalizing nearly Kählerian manifolds) and characterize weak nearly Sasakian and weak nearly cosymplectic hypersurfaces in such Riemannian manifolds.

**Keywords:** weak nearly Sasakian manifold; weak nearly cosymplectic manifold; killing vector field; hypersurface

**MSC:** 53C15; 53C25; 53D15

## 1. Introduction

Nearly Kähler manifolds $(M, J, g)$ are defined by condition that only the symmetric part of $\nabla J$ vanishes, in contrast to the Kähler case where $\nabla J = 0$. Nearly Sasakian and nearly cosymplectic manifolds $M(\varphi, \xi, \eta, g)$ are defined (see [2,3]) using a similar condition – by a constraint only on the symmetric part of $\varphi$ – starting from Sasakian and cosymplectic manifolds, respectively:

$$(\nabla_X\, \varphi)X = \begin{cases} 2\,g(X,X)\,\xi - \eta(Y)X\,, & \text{nearly Sasakian.} \\ 0\,, & \text{nearly cosymplectic.} \end{cases} \tag{1}$$

These two classes of odd-dimensional counterparts of nearly Kähler manifolds play a key role in the classification of almost contact metric manifolds, see [5]. They also appeared in the study of harmonic almost contact structures: a nearly cosymplectic structure, identified with a section of a twistor bundle, defines a harmonic map, see [8]. In dimensions greater than 5: condition (1) is sufficient for a nearly Sasakian manifold to be Sasakian, see [10], and a nearly cosymplectic manifold $M^{2n+1}$ splits into $\mathbb{R} \times F^{2n}$ or $B^5 \times F^{2n-4}$, where $F$ is a nearly Kähler manifold and $B$ is a nearly cosymplectic manifold, see [4]. Moreover, in dimension 5, any nearly cosymplectic manifold is Einstein with positive scalar curvature, see [4]. In [4,9] it was proved that there are integrable distributions with totally geodesic leaves in a nearly Sasakian manifold, which are either Sasakian or 5-dimensional nearly Sasakian manifolds.

In [12–14], we introduced and studied metric structures on a smooth manifold that generalize the almost contact, Sasakian, cosymplectic, etc. metric structures. Such so-called "weak" structures (the complex structure on the contact distribution is replaced by a nonsingular skew-symmetric tensor) made it possible to take a new look at the theory of classical structures and find new applications.

In this paper we define new structures of this kind, called weak nearly Sasakian and weak nearly cosymplectic structures, and study their geometry. In Section 2, following the introductory Section 1, we recall some results regarding weak almost contact manifolds. In Section 3, we introduce weak nearly Sasakian and weak nearly cosymplectic structures and study their geometry. In Section 4, we introduce weak nearly Kählerian manifolds

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(generalizing nearly Kählerian manifolds) and characterize weak nearly Sasakian and weak nearly cosymplectic hypersurfaces in such Riemannian spaces. The proofs use the properties of new tensors, as well as classical constructions.

## 2. Preliminaries

A *weak almost contact structure* on a smooth manifold $M^{2n+1}$ is a set $(\varphi, Q, \xi, \eta)$, where $\varphi$ is a $(1,1)$-tensor, $\xi$ is a vector field (called Reeb vector field) and $\eta$ is a dual 1-form, satisfying

$$\varphi^2 = -Q + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2}$$

see [12,13], where $Q$ is a nonsingular $(1,1)$-tensor field such that

$$Q\,\xi = \xi.$$

By (2), $\eta$ defines a $2n$-dimensional distribution $\mathcal{D} = \ker \eta$. Assume that $\mathcal{D}$ is $\varphi$-invariant, i.e.,

$$\varphi X \in \mathcal{D}, \quad \forall\, X \in \mathcal{D}, \tag{3}$$

as in the classical theory [1], where $Q = \mathrm{id}_{\,TM}$. By (3) and (2a), $\mathcal{D}$ is invariant for $Q$: $Q(\mathcal{D}) = \mathcal{D}$. A "small" (1,1)-tensor $\widetilde{Q} = Q - \mathrm{id}_{\,TM}$ is a measure of the difference between a weakly contact structure and a contact one. Note that

$$[\widetilde{Q}, \varphi] = 0, \quad \eta \circ \widetilde{Q} = 0, \quad \widetilde{Q}\,\xi = 0.$$

A weak almost contact structure $(\varphi, Q, \xi, \eta)$ on a manifold $M$ will be called *normal* if the following tensor $N^{\,(1)}$ is identically zero:

$$N^{\,(1)}(X,Y) = [\varphi, \varphi](X,Y) + 2\, d\eta(X,Y)\,\xi, \quad X, Y \in \mathfrak{X}_M.$$

Here, $d\eta(X,Y) = \frac{1}{2}\,\{X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])\}$, and the Nijenhuis torsion $[\varphi, \varphi]$ of $\varphi$ is given by

$$[\varphi, \varphi](X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y], \quad X, Y \in \mathfrak{X}_M. \tag{4}$$

If there is a Riemannian metric $g$ on $M$ such that

$$g(\varphi X, \varphi Y) = g(X, Q\,Y) - \eta(X)\,\eta(Y), \quad X, Y \in \mathfrak{X}_M, \tag{5}$$

then $(\varphi, Q, \xi, \eta, g)$ is called a *weak almost contact metric structure* on $M$.

A weak almost contact manifold $M(\varphi, Q, \xi, \eta)$ endowed with a compatible Riemannian metric is said to be a *weak almost contact metric manifold* and is denoted by $M(\varphi, Q, \xi, \eta, g)$. Setting $Y = \xi$ in (5), we obtain as in the classical theory, $\eta(X) = g(X, \xi)$. By (5), we get $g(X, Q\,X) = g(\varphi X, \varphi X) > 0$ for any nonzero vector $X \in \mathcal{D}$; thus, $Q$ is positive definite.

Using the Levi-Civita connection $\nabla$ of $g$, (4) can be written as

$$[\varphi, \varphi](X,Y) = (\varphi \nabla_Y \varphi - \nabla_{\varphi Y} \varphi)X - (\varphi \nabla_X \varphi - \nabla_{\varphi X} \varphi)Y. \tag{6}$$

**Definition 1.** A *weak contact metric structure* is defined as a weak almost contact metric structure satisfying

$$d\,\eta = \Phi$$

where

$$\Phi(X,Y) = g(X, \phi Y) \quad (X, Y \in \mathfrak{X}_M)$$

is called the fundamental 2-form. A normal weak contact metric manifold is called a *weak Sasakian manifold*. A weak almost contact metric structure is said to be *weak almost cosymplectic*, if it is normal and both $\Phi$ and $\eta$ are closed. If a weak almost cosymplectic structure is normal, then it is called *weak cosymplectic*.

A weak almost contact manifold is weak Sasakian if and only if it is Sasakian, see [12, Theorem 4.1]. For any weak almost cosymplectic manifold, the $\xi$-curves are geodesics, see [12, Corollary 1], and if $\nabla\varphi = 0$, then the manifold is weak cosymplectic, see [12, Theorem 5.2].

**Remark 1.** If an almost contact metric structure is normal and contact metric, then it is called *Sasakian*, equivalently

$$(\nabla_X \varphi)Y = -g(X,Y)\xi + \eta(Y)X. \tag{7}$$

Three tensors $N^{(2)}, N^{(3)}$ and $N^{(4)}$ are well known in the classical theory, see [1]:

$$\begin{aligned} N^{(2)}(X,Y) &= (\pounds_{\varphi X}\,\eta)(Y) - (\pounds_{\varphi Y}\,\eta)(X), \\ N^{(3)}(X) &= (\pounds_{\xi}\,\varphi)X = [\xi, \varphi X] - \varphi[\xi, X], \\ N^{(4)}(X) &= (\pounds_{\xi}\,\eta)(X) = 2\,d\eta(\xi, X). \end{aligned}$$

Note that for a weak contact metric structure $(\varphi, Q, \xi, \eta, g)$, the tensors $N^{(2)}$ and $N^{(4)}$ vanish; moreover, $N^{(3)}$ vanishes if and only if $\xi$ is a Killing vector field, see [12, Theorem 2.2]. Moreover, on a weak Sasakian manifold, $\xi$ is a Killing vector field, see [12, Proposition 4.1].

## 3. Main Results

**Definition 2.** An weak almost contact metric structure is called *weak nearly Sasakian* if

$$(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = 2\,g(X,Y)\xi - \eta(Y)X - \eta(X)Y. \tag{8}$$

A weak almost contact metric structure is called *weak nearly cosymplectic* if $\varphi$ is Killing,

$$(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = 0, \tag{9}$$

or, equivalently, (1) is satisfied.

**Example 1.** Let a Riemannian manifold $(M^{2n+1}, g)$ admit two nearly Sasakian structures (or, nearly cosymplectic structures) with common Reeb vector field $\xi$ and and one-form $\eta = g(\xi, \cdot)$. Suppose that $\phi_1 \neq \phi_2$ are such that $\psi := \varphi_1\,\varphi_2 + \varphi_2\,\varphi_1 \neq 0$. Then $\varphi := (\cos t)\,\varphi_1 + (\sin t)\,\varphi_2$ for small $t > 0$ satisfies (8) (respectively, (9)) and $\varphi^2 = -\mathrm{id} + (\sin t\cos t)\,\psi + \eta\otimes\xi$. Thus, $(\varphi, Q, \xi, \eta, g)$ is a weak nearly Sasakian (respectively, weak nearly symplectic) structure on $M$ with $Q = \mathrm{id} - (\sin t\cos t)\,\psi$.

We will generalize the result in [3, Proposition 3.1].

**Proposition 1.** *Both on weak nearly Sasakian and weak nearly cosymplectic manifolds with the condition*

$$\nabla Q = 0, \tag{10}$$

*the vector field $\xi$ is Killing.*

**Proof.** Putting $X = Y = \xi$ in (8) or (9), we find $(\nabla_\xi \varphi)\xi = 0$, or $\varphi\nabla_\xi\,\xi = 0$. Applying $\varphi$ to this and using (2) and $\eta(\nabla_\xi\,\xi) = 0$, we obtain

$$0 = \varphi^2\nabla_\xi\,\xi = -Q\nabla_\xi\,\xi + \eta(\nabla_\xi\,\xi)\,\xi = -Q\nabla_\xi\,\xi.$$

Since the (1,1)-tensor $Q$ is nonsingular, we get

$$\nabla_\xi\,\xi = 0.$$

Then we calculate

$$(\nabla_\xi\,\eta)X = \xi(g(\xi, X)) - g(\xi, \nabla_\xi\,X) = g(\nabla_\xi\,\xi, X) = 0.$$

Thus

$$\nabla_\xi\,\eta = 0.$$

Applying the derivative in the $\xi$-direction to (5) and using $\nabla_\xi\,Q = 0$ and $\nabla_\xi\,\eta = 0$, we find

$$\begin{aligned} & g((\nabla_\xi\,\varphi)X, \varphi Y) + g(\varphi X, (\nabla_\xi\,\varphi)Y) = \nabla_\xi\,g(\varphi X, \varphi Y) \\ & = g(X, (\nabla_\xi\,Q)Y) + (\nabla_\xi\,\eta)(X)\,\eta(Y) + \eta(X)(\nabla_\xi\,\eta)(Y) = 0. \end{aligned}$$

On a weak nearly Sasakian manifold, using (8) and $\eta\circ\widetilde{Q} = 0$, yields

$$\begin{aligned} & g((\nabla_\xi\,\varphi)X, \varphi Y) + g(\varphi X, (\nabla_\xi\,\varphi)Y) \\ & = -g((\nabla_X\,\varphi)\xi, \varphi Y) - g(\varphi X, (\nabla_Y\,\varphi)\xi) \\ & + g(2\,\eta(X)\,\xi - X - \xi, \varphi Y) + g(2\,\eta(Y)\,\xi - Y - \xi, \varphi X) \\ & = -g(\nabla_X\,\xi, \varphi^2 Y) - g(\varphi^2 X, \nabla_Y\,\xi) - g(X, \varphi Y) - g(Y, \varphi X) \\ & = g(\nabla_X\,\xi, QY) + g(QX, \nabla_Y\,\xi) \\ & = g(\nabla_X\,\xi, Y) + g(X, \nabla_Y\,\xi) + g(\nabla_X\,\xi, \widetilde{Q}Y) + g(\widetilde{Q}X, \nabla_Y\,\xi) \\ & = (\pounds_\xi\,g)(X, Y) - g(\xi, (\nabla_X\widetilde{Q})Y) - g((\nabla_Y\widetilde{Q})X, \xi). \end{aligned}$$

Similarly, on a weak nearly cosymplectic manifold, using (9) yields

$$(\pounds_\xi\,g)(X, Y) - g(\xi, (\nabla_X\widetilde{Q})Y) - g((\nabla_Y\widetilde{Q})X, \xi) = 0.$$

From the above, using $\nabla\widetilde{Q} = 0$, for both cases we get $\pounds_\xi\,g = 0$, that is $\xi$ is Killing. $\square$

We will generalize [2, Theorem 5.2].

**Proposition 2.** *There are no weak nearly cosymplectic structures which are weak contact metric structures.*

**Proof.** Suppose that our weak nearly cosymplectic manifold is weak contact metric. Since also $\xi$ is Killing, then $M$ is weak K-contact. By [14, Theorem 2], the following holds:

$$\nabla\xi = -\varphi.$$

Also, by [14, Corollary 2], the $\xi$-sectional curvature is positive, i.e., $K(\xi, X) > 0$ $(X \perp \xi)$. Thus, if $X \neq 0$ is a vector orthogonal to $\xi$, then

$$\begin{aligned} & 0 < K(\xi, X) = g(\nabla_\xi\nabla_X\xi - \nabla_X\nabla_\xi\xi - \nabla_{[\xi,X]}\xi, X) \\ & = g(-(\nabla_\xi\varphi)(X) + \varphi^2 X, X) = g((\nabla_X\varphi)\xi, X) - g(\varphi X, \varphi X) \\ & = -g(\varphi(\nabla_X\xi), X) + g(\varphi^2 X, X) = 2\,g(\varphi^2 X, X). \end{aligned}$$

This contradicts to the following: $g(\varphi^2 X, X) = -g(\varphi X, \varphi X) \leq 0$. $\square$

We will generalize [3, Theorem 5.2] that a normal nearly Sasakian structure is Sasakian.

**Theorem 1.** *For a weak nearly Sasakian structure with the condition* (10)*, normality* ($N^{(1)} = 0$) *is equivalent to weak contact metric* ($d\eta = \Phi$)*. In particular, a normal weak nearly Sasakian structure with condition* (10) *is Sasakian.*

**Proof.** First, we will show that a weak nearly Sasakian structure with conditions (10) and $\eta\circ N^{\,(1)}=0$ is a weak contact metric structure. Applying $\nabla_\xi$ to (2) and using $\nabla_\xi\,Q=0$, $\nabla_\xi\,\eta=0$ and $\nabla_\xi\,\xi=0$, we find

$$(\nabla_\xi\,\varphi)\,\varphi X+\varphi(\nabla_\xi\,\varphi)X=(\nabla_\xi\,\varphi^2)X=-(\nabla_\xi\,Q)X+\nabla_\xi(\eta(X)\,\xi)=0.$$

We compute, using (6),

$$\begin{aligned}\eta([\varphi,\varphi](X,Y))&=\eta((\nabla_{\varphi X}\,\varphi)Y-(\nabla_{\varphi Y}\,\varphi)X)\\&=\eta((\nabla_X\,\varphi)\,\varphi Y-(\nabla_Y\,\varphi)\,\varphi X)-4\,g(X,\varphi Y)\\&=\eta(\varphi\,(\nabla_Y\,\varphi X)-\varphi\,(\nabla_X\,\varphi)\,Y)-4\,g(X,\varphi Y)=-4\,g(X,\varphi Y).\end{aligned}$$

Thus, if $\eta(N^{\,(1)}(X,Y))=0$, then $d\,\eta(X,Y)=2\,g(X,\varphi\,Y)$.

Conversely, if a weak nearly Sasakian structure with condition (10) is also a weak contact metric structure, then $\Phi=d\,\eta$, hence $d\,\Phi=0$, where

$$\begin{aligned}d\Phi(X,Y,Z)=\frac{1}{3}\,\{&X\,\Phi(Y,Z)+Y\,\Phi(Z,X)+Z\,\Phi(X,Y)\\&-\Phi([X,Y],Z)-\Phi([Z,X],Y)-\Phi([Y,Z],X)\}.\end{aligned}$$

It is easy to calculate

$$\begin{aligned}3\,d\,\Phi(X,Y,Z)&=-g((\nabla_X\,\varphi)Y,Z)+g((\nabla_Y\,\varphi)X,Z)-g((\nabla_Z\,\varphi)X,Y)\\&=-g((\nabla_X\,\varphi)Y,Z)+g(-(\nabla_X\,\varphi)Y-2\,g(X,Y)\,\xi+\eta(X)Y+\eta(Y)X,Z)\\&\quad-g(-(\nabla_X\,\varphi)Z-2\,g(X,Z)\,\xi+\eta(X)Z+\eta(Z)X,Y)\\&=-3\,g((\nabla_X\,\varphi)Y,Z)-3\,g(X,Y)\,\eta(Z)+3\,g(X,Z)\,\eta(Y).\end{aligned}$$

Hence (7) is true. Using (7) in (6), we find that our structure is normal:

$$[\varphi,\varphi](X,Y)=-2\,d\eta(X,Y)\,\xi,$$

By the above, a weak nearly Sasakian structure with conditions (10) and $N^{\,(1)}=0$ is weak Sasakian (see Definition 1). Using [12, Theorem 4.1] completes the proof of the second assertion. □

## 4. Hypersurfaces of Weak Nearly Kählerian Manifolds

Here, we define weak nearly Kählerian manifolds (generalizing nearly Kählerian manifolds) and study weak nearly Sasakian and weak nearly cosymplectic hypersurfaces in such Riemannian spaces.

**Definition 3.** A Riemannian manifold $(\bar M,\bar g)$ equipped with a skew-symmetric (1,1)-tensor $\bar\varphi$ such that the tensor $\bar\varphi^{\,2}$ is negative definite will be called *weak Hermitian manifold*. Such $(\bar M,\bar\varphi,\bar g)$ will be called *weak nearly Kählerian manifold*, if $(\bar\nabla_X\,\bar\varphi)X=0$ $(X\in T\bar M)$, where $\bar\nabla$ is the Levi-Civita connection of $\bar g$, or equivalently,

$$(\bar\nabla_X\,\bar\varphi)Y+(\bar\nabla_Y\,\bar\varphi)X=0\quad(X,Y\in T\bar M).\tag{11}$$

**Remark 2.** Several authors studied the problem of finding skew-symmetric parallel 2-tensors (different from almost complex structures) on a Riemannian space and classified such tensors (e.g., [6]) or proved that some spaces do not admit them (e.g., [7]).

The scalar second fundamental form $h$ of a hypersurface $M\subset\bar M$ with a unit normal $N$ is related with $\bar\nabla$ and the Levi-Civita connection $\nabla$ of induced metric $g$ by the Gauss equation

$$\bar\nabla_XY=\nabla_XY+h(X,Y)\,N\quad(X,Y\in TM).\tag{12}$$

The Weingarten operator $A_N : X \to -\bar\nabla_X N$ is related with $h$ by the following equality:

$$\bar g(h(X,Y),N) = g(A_N(X),Y) \quad (X,Y\in TM).$$

**Lemma 1.** *A hypersurface $M$ with a unit normal $N$ and induced metric $g$ in a weak Hermitian manifold $(\bar M,\bar\varphi,\bar g)$ inherits a weak almost contact structure $(\varphi,Q,\xi,\eta,g)$ given by*

$$\xi = \bar\varphi N,\quad \eta = \bar g(\bar\varphi N,\,\cdot),\quad \varphi = \bar\varphi + \bar g(\bar\varphi N,\,\cdot)\,N,\quad Q = -\bar\varphi^{\,2} + \bar g(\bar\varphi^{\,2}N,\,\cdot)\,N.$$

**Proof.** Using the skew-symmetry of $\bar\varphi$ (e.g., $\bar g(\bar\varphi N,N)=0$), we verify (2) for $X\in TM$:

$$\begin{aligned}\varphi^2 X &= \varphi(\bar\varphi X - \bar g(\bar\varphi X,N)\,N)\\ &= \bar\varphi(\bar\varphi X - \bar g(\bar\varphi X,N)\,N) - \bar g(\bar\varphi(\bar\varphi X - \bar g(\bar\varphi X,N)\,N),N)\,N\\ &= \bar\varphi^{\,2}X - \bar g(\bar\varphi^{\,2}N,X) + \bar g(\bar\varphi N,X)\,\bar\varphi N + \bar g(\bar\varphi X,N)\,\bar g(\bar\varphi N,N)\,N\\ &= -QX + \eta(X)\,\xi.\end{aligned}$$

Since $\bar\varphi^{\,2}$ is negative definite,

$$g(QX,X) = \bar g(-\bar\varphi^{\,2}X + \bar g(\bar\varphi^{\,2}N,X)\,N,\,X) = -\bar g(\bar\varphi^{\,2}X,X) > 0$$

for $X\in TM$, i.e., the tensor $Q$ is positive definite. □

A hypersurface is called *quasi-umbilical* if its 2nd fundamental form has the view

$$h(X,Y) = A\,g(X,Y) + B\,\mu(X)\,\mu(Y),$$

where $A,B$ are smooth functions on $M$ and $\mu$ is a non-vanishing one-form.

The following theorem generalizes the fact (see [2,3]) that a hypersurface of a nearly Kähler manifold is nearly Sasakian or nearly cosymplectic if and only if it is quasi-umbilical with respect to the (almost) contact form.

**Theorem 2.** *Let $M^{2n+1}$ be a hypersurface of a weak nearly Kählerian manifold $(\bar M^{2n+2},\bar\varphi,\bar g)$. Then the induced structure $(\varphi,Q,\xi,\eta,g)$ on $M$ is*

*(i) weak nearly Sasakian, (ii) weak nearly cosymplectic,*

*if and only if the hypersurface $M$ is quasi-umbilical with the following second fundamental form:*

$$(i)\ h(X,Y) = g(X,Y) + (h(\xi,\xi)-1)\,\eta(X)\,\eta(Y),\ \ (ii)\ h(X,Y) = h(\xi,\xi)\,\eta(X)\,\eta(Y). \tag{13}$$

*Moreover, if $(\bar\nabla_X\bar\varphi^2)^\top = 0$ $(X\in TM)$ then condition $\nabla_X Q = 0$ $(X\in TM)$ holds.*

**Proof.** Substituting $\bar\varphi\,Y = \varphi\,Y - \bar g(\bar\varphi\,N,Y)\,N$ in $(\bar\nabla_X\bar\varphi)Y$, and using (12) and Lemma 1, we get

$$\begin{aligned}(\bar\nabla_X\bar\varphi)Y &= \bar\nabla_X(\bar\varphi\,Y) - \bar\varphi(\bar\nabla_X Y)\\ &= (\nabla_X\varphi)Y + \eta(Y)A_N(X) - h(X,Y)\,\xi + [X(\eta(Y)) - \eta(\nabla_X Y) + h(X,\varphi Y)]\,N.\end{aligned}$$

Thus, the $TM$-component of the weak nearly Kählerian condition (11) takes the form

$$(\nabla_X\varphi)Y + (\nabla_Y\varphi)X - 2\,h(X,Y)\,\xi + \eta(X)A_N(Y) + \eta(Y)A_N(X) = 0. \tag{14}$$

Then we calculate $(\nabla_X Q)Y$ for $X,Y\in TM$, using Lemma 1, (12) and $\bar\varphi^2 N = -N$,

$$\begin{aligned}(\nabla_X Q)Y &= \nabla_X(QY) - Q(\nabla_X Y)\\ &= (\bar\nabla_X(-\bar\varphi^2 Y + g(\bar\varphi^2 N,Y)N) - h(X,QY)N + \bar\varphi^2(\bar\nabla_X Y - h(X,Y)N)\\ &\quad - g(\bar\varphi^2 N,\bar\nabla_X Y - h(X,Y)N)N)^\top\\ &= (-(\bar\nabla_X(\bar\varphi^2 Y)) + \bar\varphi^2(\bar\nabla_X Y))^\top = -((\bar\nabla_X\bar\varphi^2)Y)^\top,\end{aligned}$$

where $^\top$ is the $TM$-component of a vector.

(i) If the structure is weak nearly Sasakian, see (8), then from (14) we get

$$2\,g(X,Y)\,\xi-\eta(Y)X-\eta(X)Y-2\,h(X,Y)\,\xi+\eta(X)A_N(Y)+\eta(Y)A_N(X)=0,$$

from which, taking the scalar product with $\xi$, we obtain

$$2\,g(X,Y)-2\,\eta(Y)\,\eta(X)-2\,h(X,Y)+\eta(X)\,h(Y,\xi)+\eta(Y)\,h(X,\xi)=0. \tag{15}$$

Setting $Y=\xi$ and taking the scalar product with $\xi$, we obtain

$$h(X,\xi)=h(\xi,\xi)\,\eta(X). \tag{16}$$

Using this in (15), we obtain (13)(i).

Conversely, if (13)(i) is valid, then substituting $Y=\xi$ yields (16). Using (13)(i), we express the Weingarten operator as

$$A_N(X)=X+(h(\xi,\xi)-1)\,\eta(X)\,\xi.$$

Substituting the above expressions of $h(X,Y), h(X,\xi)$ and $A_N$ in (14) gives (8), thus the structure is weak nearly Sasakian.

(ii) If the structure is weak nearly cosymplectic, see (9), then from (14) we get

$$2\,h(X,Y)\,\xi=\eta(X)A_N(Y)+\eta(Y)A_N(X),$$

from which, taking the scalar product with $\xi$, we obtain

$$2\,h(X,Y)=\eta(X)\,h(Y,\xi)+\eta(Y)\,h(X,\xi). \tag{17}$$

Setting $Y=\xi$ and taking the scalar product with $\xi$, we obtain (16). Using this in (17), we obtain (13)(ii).

Conversely, if (13)(ii) is valid, then substituting $Y=\xi$ yields (16). Using (13)(ii), we express the Weingarten operator as

$$A_N(X)=h(\xi,\xi)\,\eta(X)\,\xi.$$

Substituting the above expressions of $h(X,Y), h(X,\xi)$ and $A_N$ in (14) gives (9), thus the structure is weak nearly cosymplectic. □

## 5. Conclusions

We have shown that weak nearly Sasakian and weak nearly cosymplectic structures are useful tools for studying almost contact metric structures and Killing vector fields. Some classical results have been extended in this paper to weak nearly Sasakian and weak nearly cosymplectic structures. Based on the numerous applications of nearly Sasakian and nearly cosymplectic structures, we expect that certain weak structures will also be useful for geometry and physics, e.g., in QFT.

The idea of considering the entire bundle of almost-complex structures compatible with a given metric led to the twistor construction and then to twistor string theory. Thus, it may be interesting to consider the entire bundle of weak Hermitian or weak nearly Kählerian structures (see Definition 3) that are compatible with a given metric.

In conclusion, we ask the following questions for dimensions greater than five: find conditions under which

(i) a weak nearly Sasakian manifold is Sasakian,

(ii) a weak nearly Sasakian manifold has totally geodesic foliations,

(iii) a weak nearly cosymplectic manifold is a Riemannian product.

We also ask the question (inspired by [4, Corollary 6.4]): when a hypersurface in a weak nearly Kähler 6-dimensional manifold has Sasaki-Einstein structure.

These questions can be answered by generalizing some deep results on nearly Sasakian and nearly cosymplectic manifolds (e.g., [4,5,10,11]) to their weak analogues.

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