An energy-momentum method for ordinary differential equations with an underlying k-symplectic structure

October 11, 2023

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Abstract

This work reviews the known k-polysymplectic reduction theory showing previous mistakes in the literature, providing new results, and showing the real practical relevance of only apparently minor technical generalisations in the previous literature. After that, we introduce a new k-symplectic energy-momentum method and apply it to the study of Hamiltonian systems of ordinary differential equations. Several examples of physical and mathematical relevance are detailed.

Keywords: k-polysymplectic manifold, energy-momentum method, stability, relative equilibrium point, Marsden–Weinstein reduction

MSC2020 codes: code1, code2, code3, code4

Contents

1 Introduction (new)

The energy-momentum method is a technique for the study of the stability of solutions close to equilibrium points of reduced Hamiltonian systems with symmetries on symplectic manifolds [28] based on the classical Marsden–Weinstein reduction theorem. The classical energy-momentum method studies the solutions to the Hamiltonian system on a symplectic manifold (M, ω, h) given by a Hamiltonian function h close to points, the so-called *relative equilibrium points*, that project onto the equilibrium points of one of its reduced Hamiltonian systems $(M_\mu, \omega_\mu, h_\mu)$. The energy-momentum method also analyses the stability of equilibrium points of the Hamilton equations for h_μ relative to the symplectic structure (M_μ, ω_μ), namely whether solutions get closer or escape away from equilibrium points of the Hamilton equations of h_μ as time passes by. It is remarkable that an equilibrium point $x_e \in M_\mu$ of h_μ is called stable whether particular solutions close enough to x_e get even closer as time passes by. Otherwise, the point x_e is called unstable. The energy-momentum method, devised and developed mainly by the Spanish researcher J. C. Simo in collaboration with J. E. Marsden [28], has been successfully applied to many problems by numerous researchers [1, 36, 27, 29, 26, 33, 42]. One of its main advantages is the fact that it allows one to determine the stability of equilibrium points in the reduced Hamiltonians $(M_\mu, \omega_\mu, h_\mu)$ without obtaining the explicit form of the reduced Hamiltonian systems, namely it makes use of structures defined on M, which is much simpler. Over the years, the energy-momentum method was also extended to deal with more general Hamiltonian systems, e.g. Hamiltonian systems on Poisson manifolds [28, 39], discrete systems [27, 37], etc.

There exist many generalizations of the notion of symplectic structure. One of those is *k-symplectic geometry* introduced by A. Awane [3, 4], and used later by M. de León et al. [9, 10, 11], and L. K. Norris [30, 32] to describe first-order field theories. They coincide with the *polysymplectic manifolds* described by G. C. Günther [20] (although these last ones are different from those introduced by G. Sardanashvily et al. [19, 35] and I. V. Kanatchikov [22], that are also called *polysymplectic*). This structure is applied to first-order regular autonomous field theories [6, 16, 34].

Although k-symplectic structures have been widely used to study physical systems governed by systems of partial differential equations, they have also proved to be very useful in the study of systems of ordinary differential equations [14].

The aim of this work is to develop an energy-momentum method for systems of ordinary differential equations with an underlying polysymplectic or k-symplectic structure [14]. In order to do so, we rely on previous works studying Marsden–Weinstein reduction procedures for these kinds of structures [5, 17, 13, 25], improving some of the previous results by dropping unnecessary hypotheses. The energy-momentum method is applied to study different relevant physical systems described by Hamiltonian vector fields relative to k-polysymplectic manifolds. In particular, the so-called automorphic Lie systems [8, 40, 7] related to a quantum harmonic oscillator with a magnetic term is studied. The theory of Lie systems is also used to convert a certain dynamical system into a Hamiltonian system relative to a k-symplectic manifold. Moreover, a particular Lie system appearing in the description of control theory is used to study our k-polysymplectic energy-momentum method.

The structure of the paper goes as follows. Section 3 reviews the basics on Lyapunov stability. In Section 4 we review the theory of k-polysymplectic structures and the notion of Hamiltonian vector field on a k-polysymplectic manifold. The notion of momentum map is extended to k-polysymplectic manifolds. Section 5 is devoted to improve the previous Marsden–Weinstein reduction procedures for k-polysymplectic systems. Section 6 is devoted to develop an energy-momentum method for systems of ODEs with an underlying polysymplectic structure. In particular, we define and characterize the notion of relative equilibria for such systems. Finally, in Section 7 several relevant examples are studied in detail, including the cotangent bundle of k-velocities, the complex Schwarz equation, the product of several symplectic manifolds, a control system, and the quantum harmonic oscillator.

Let us set some general assumptions to be used throughout this work. It is hereafter assumed that all structures are smooth. Manifolds are real, Hausdorff, connected, second countable, and finite-dimensional. Differential forms are assumed to have constant rank unless otherwise stated. Sum over crossed repeated indices is understood. All our considerations are local, to avoid technical problems concerning the global existence of quotient manifolds and similar issues. Hereafter, $\mathfrak{X}(M)$ and $\Omega^k(M)$ stand for the $\mathscr{C}^\infty(M)$-modules of vector fields and differential k-forms on M.

2 Introduction

The first k-polysymplectic reduction was developed by Awane [3, 20, 12]. Nevertheless, his work contained a mistake that had to be corrected in [?], where some sufficient conditions were obtained. Necessary and sufficient conditions for the k-polysymplectic reduction were obtained by Blacker in [?]. Next, García Toraño and Mestdag reviewed conditions in [?] and proved that one of the conditions is enough to ensured the k-symplectic reduction. In this work, we review the work in [?] and we show a mistake in the main theorem [?]. Indeed, we show that the result is false and we provide several counter examples of their result. Moreover, there is a technical result concerning the existence of a submanifold given by the momentum method. This results is standard in many Marsden–Weinstein reductions and, as claimed in [?], it is usually stated that it general enough to be practical. Nevertheless, in this work we show that this is not the case and we point out a misunderstanding about its generality and the Sard's theorem accomplished in [?].

After the above theoretical remarks about the k-polysymplectic reduction, we use the theory to study systems of ordinary differential equations. This significantly differ from the standard literature on k-polysymplectic manifolds, which is mainly devoted to partial differential equations.

More specifically, we use k-polysymplectic reduction to develop a new k-polysymplectic energy-momentum method. This in turn is applied to study different physical systems described by Hamiltonian vector fields relative to k-polysymplectic manifolds. In particular, the so-called automorphic Lie systems[8, 40, 7] related to a quantum harmonic oscillator with a magnetic term is studied. The theory of Lie systems is also used to convert a certain dynamical system into a Hamiltonian system relative to a k-symplectic manifold. Moreover, a particular Lie system appearing in the description of Control Theory is used to study our k-polysymplectic energy-momentum method.

Nowadays, the k-symplectic geometry is mainly applied to the study of first-order classical field theories. In particular, it gives a geometric description of the Euler–Lagrange and the Hamilton–de Donder–Weyl field equations and the systems described by them. For instance, k-symplectic geometry enables us to describe their symmetries, conservation laws, reductions, et cetera [3, 20, 25, 34]. Meanwhile, we consider k-symplectic structures for studying systems of differential equations, which opens a new setting of application of these geometrical structures.

3 Fundamentals on Lyapunov stability

Let us introduce basic notions and theorems on the stability of dynamical systems used in our cosymplectic formulation of the energy-momentum method [15, 41].

As we assume that all manifolds considered in this work are paracompact and Hausdorff, they, therefore, admit a Riemannian metric $\mathbf{g}$ [23]. The topology induced by $\mathbf{g}$ is the same as the topology of the manifold P [41]. The metric $\mathbf{g}$ induces a distance in P so that the distance between two points $x_1, x_2 \in P$ is given by

$$d(x_1,x_2) := \inf_{\substack{\gamma:[0,1]\to P\\ \gamma(0)=x_1\\ \gamma(1)=x_2}} l_{\mathbf{g}}(\gamma)\,,$$

where $\ell_{\mathbf{g}}(\gamma)$ is the length of a smooth curve $\gamma : [0,1] \to P$.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(x)\,, \quad \forall x \in P \tag{1}$$

A point $x_e \in P$ is an *equilibrium point* of (1) if $X(x_e) = 0$. In that case, it is also said that $x_e \in P$ is an *equilibrium point of the vector field* X. Let us hereafter assume, when talking about stability, that $T = \mathbb{R}$. An equilibrium point x_e is *stable* from t^0 if, for every ball $B_{x_e,\varepsilon} := \{x \in P : d(x, x_e) < \epsilon\}$, there exists a radius $\delta(\varepsilon)$ such that every solution $x(t)$ of (1) with initial condition $x(t_0) \in B_{x_e,\delta(\varepsilon)}$ is contained in $B_{x_e,\varepsilon}$ for $t > t_0$. In further applications, we assume $t^0 = -\infty$, if not otherwise stated. An equilibrium point x_e is *unstable* if it is not stable.

An equilibrium point x_e is said to be *asymptotically stable* if it is stable and there exists a radius r such that every solution $x(t)$ of (1) with initial condition $x(t_0) \in B_{x_e,r}$ converges to x_e when t tends to infinity.

Let us introduce notions that are necessary to formulate the Basic Lyapunov Theorem on manifolds (see [15] and references therein). Let $\dot{\mathcal{M}} : P \to \mathbb{R}$ be defined by

$$\dot{\mathcal{M}}(x) := \sum_{i=1}^{\dim P} \frac{\partial \mathcal{M}}{\partial x^i}(x) X^i(x), \qquad \forall x \in P,$$

where $\{x^1, \dots, x^{\dim P}\}$ is a local coordinate system around the neighbourhood of the point $x \in P$ and $X = \sum_{i=1}^{\dim P} X^i \frac{\partial}{\partial x^1}$.

Then, let us recall the basic Lyapunov theorem for autonomous systems (1).

Theorem 3.1 (Autonomous Lyapunov stability theory)**.** *Let x_e be an equilibrium point of* (1) *and let $\mathcal{M} : P \to \mathbb{R}$ be a continuous function such that $\mathcal{M}(x_e) = 0$, $\mathcal{M}(x) > 0$, and $\dot{\mathcal{M}}(x) \leq 0$ for every $x \in B_{x_e,r}$ for some $r \in \mathbb{R}^+$. Then, x_e is stable.*

4 On k-polysymplectic momentum maps

4.1 Basics on k-polysymplectic geometry

This section recalls the basic notions from k-polysymplectic geometry and introduces definitions to be used later on, that are consistent with those in [12]. This explanation is particularly important since the terminology used in the literature is not consistent and a single term may refer to different not equivalent geometric concepts.

Hereafter, we work with differential ℓ-forms on a smooth manifold P taking values in $\mathbb{R}^k$. The set of such forms is denoted by $\Omega^\ell(P,\mathbb{R}^k)$, and $\mathbb{R}^k$-valued differential forms will be written in bold. Moreover, $\mathbb{R}^k$ has a fixed basis $\{e_1,\dots,e_k\}$ giving rise to a dual basis $\{e^1,\dots,e^k\}$ in $\mathbb{R}^{k*}$.

Definition 4.1. A *k-polysymplectic form* on P is a closed non-degenerate $\mathbb{R}^k$-valued differential two-form

$$\boldsymbol{\omega} = \omega^\alpha \otimes e_\alpha \in \Omega^2(P,\mathbb{R}^k)\,.$$

The pair $(P,\boldsymbol{\omega})$ is called a *k-polysymplectic manifold.*

Note that P has a k-polysymplectic form $\boldsymbol{\omega}$ if and only if there exists a family of k closed two-forms $\omega^1,\dots,\omega^k \in \Omega^2(P)$ such that

$$\ker \boldsymbol{\omega} = \ker(\omega^\alpha \otimes e_\alpha) = \bigcap_{\alpha=1}^{k} \ker \omega^\alpha = 0\,.$$

k-polysymplectic manifolds are called, for simplicity, polysymplectic manifolds in the literature (see [25] for instance). Nevertheless, the latter term refers to a different notion that is shown below. Hence we will not shorten the term *k-polysymplectic manifold* and other related ones so as to avoid misunderstandings.

Let us define polysymplectic, k-polysymplectic manifolds, and related structures as follows.

Definition 4.2. Let P be an $n(k+1)$-dimensional manifold. Then,

- A *polysymplectic structure* on P is a differential two-form taking values in $\mathbb{R}^k$ given by $\boldsymbol{\omega} = \omega^\alpha \otimes e_\alpha \in \Omega^2(P,\mathbb{R}^k)$ for certain $\omega^1,\dots,\omega^k \in \Omega^2(P)$ such that

$$\ker \boldsymbol{\omega} = \bigcap_{\alpha=1}^{k} \ker \omega^\alpha = 0\,.$$

  We call $(P,\boldsymbol{\omega})$ a *polysymplectic manifold.*

- A *k-symplectic structure* on P is a pair $(\boldsymbol{\omega}, V)$, where $(P,\boldsymbol{\omega})$ is a polysymplectic manifold and $V \subset \mathrm{T}P$ is an integrable distribution on P of rank nk such that

$$\boldsymbol{\omega}|_{V\times V} = 0\,.$$

  In this case, $(P,\boldsymbol{\omega},V)$ is a *k-symplectic manifold.* We call V a *polarisation* of the k-symplectic manifold.

If the two-form $\boldsymbol{\omega}$ is exact, namely $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\theta}$ for some $\boldsymbol{\theta} \in \Omega^1(M,\mathbb{R}^k)$, in any of the notions in Definition 4.2, then such concepts are said to be *exact.*

Note that the difference between polysymplectic and k-polysymplectic manifolds is the fact that in the k-polysymplectic case the dimension of the manifold is fixed.

4.2 On $\boldsymbol{\omega}$-Hamiltonian functions and $\boldsymbol{\omega}$-Hamiltonian vector fields

Let us survey the basic theory on k-polysymplectic vector fields and functions.

Definition 4.3. Given a k-polysymplectic manifold $(N, \boldsymbol{\omega})$, we say that a vector field $Y \in \mathfrak{X}(N)$ is $\boldsymbol{\omega}$-*Hamiltonian* if it is Hamiltonian with respect to all the presymplectic forms $\omega^1, \ldots, \omega^k$, namely $\iota_Y \omega^\alpha$ is exact (at least locally) for every $\alpha = 1, \ldots, k$.

We denote by $\mathfrak{X}_{\boldsymbol{\omega}}(N)$ the space of $\boldsymbol{\omega}$-Hamiltonian vector fields in a k-polysymplectic manifold $(N, \boldsymbol{\omega})$.

Every $\boldsymbol{\omega}$-Hamiltonian vector field can be associated with a family $h^1, \ldots, h^k$ of Hamiltonian functions (each one relative to a different presymplectic form of a k-symplectic structure). It is convenient for the study k-polysymplectic Hamiltonian vector fields to introduce some generalisation of the Hamiltonian function notion for presymplectic forms to deal simultaneously with all $h^1, \ldots, h^k$ (see [3, 14] for details and related topics).

Definition 4.4. Given a polysymplectic structure $\boldsymbol{\omega} = \sum_{\alpha=1}^k \omega^\alpha \otimes e_\alpha$ on N, we say that $\boldsymbol{h} = h^1 \otimes e_1 + \cdots + h^k \otimes e_k$ is an $\boldsymbol{\omega}$-*Hamiltonian function* if there exists a vector field $X_{\boldsymbol{h}}$ on N such that $\iota_{X_{\boldsymbol{h}}} \omega^\alpha = \mathrm{d}h^\alpha$ for $\alpha = 1, \ldots, k$. In this case, we call $\boldsymbol{h}$ an $\boldsymbol{\omega}$-*Hamiltonian function* for $X_{\boldsymbol{h}}$. We write $\mathscr{C}^\infty_{\boldsymbol{\omega}}(N)$ for the space of $\boldsymbol{\omega}$-Hamiltonian functions.

Note that an $\boldsymbol{\omega}$-Hamiltonian function is a certain type of $\mathbb{R}^k$-valued Hamiltonian function. In [31], the author called k-Hamiltonian system associated to the $\mathbb{R}^k$-valued Hamiltonian function $\boldsymbol{h}$ the vector field $X_{\boldsymbol{h}}$ of the above definition. Moreover, A. Awane [3] called $\boldsymbol{h}$ a Hamiltonian map of X when X is additionally an infinitesimal automorphism of a certain distribution on which it is assumed that the presymplectic forms of the k-symplectic distribution vanish.

Example 4.5. Consider the two-polysymplectic manifold $(\mathbb{R}^3, \boldsymbol{\omega})$, where $\{u, v, w\}$ are the coordinates of $\mathbb{R}^3$ and $\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2$, where

$$\omega^1 = \frac{4w}{v^2}\mathrm{d}u \wedge \mathrm{d}w + \frac{1}{v}\mathrm{d}v \wedge \mathrm{d}w + \frac{4w^2}{v^3}\mathrm{d}u \wedge \mathrm{d}v\,, \qquad \omega^2 = \frac{4}{v^2}\mathrm{d}u \wedge \mathrm{d}w + \frac{8w}{v^3}\mathrm{d}u \wedge \mathrm{d}v$$

(see [14] for details on this structure). The vector fields

$$X_1 = 4u^2\frac{\partial}{\partial u} + 4uv\frac{\partial}{\partial v} + v^2\frac{\partial}{\partial w}\,, \qquad X_2 = \frac{\partial}{\partial u}\,, \qquad X_3 = 2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$$

are $\boldsymbol{\omega}$-Hamiltonian with $\boldsymbol{\omega}$-Hamiltonian functions

$$\boldsymbol{f} = \left(4uw - 8\frac{u^2w^2}{v^2} - \frac{v^2}{2}\right) \otimes e_1 + \left(4u - 16\frac{u^2w}{v^2}\right) \otimes e_2\,,$$
$$\boldsymbol{g} = -2\frac{w^2}{v^2} \otimes e_1 - 4\frac{w}{v^2} \otimes e_2\,, \qquad \boldsymbol{h} = \left(w - 4\frac{uw^2}{v^2}\right) \otimes e_1 - 8\frac{uw}{v^2} \otimes e_2\,,$$

respectively, relative to the polysymplectic structure $\boldsymbol{\omega}$. △

Proposition 4.6. *Let $\boldsymbol{\omega} = \sum_{\alpha=1}^k \omega^\alpha \otimes e_\alpha$ be a polysymplectic structure. Then, every $\boldsymbol{\omega}$-Hamiltonian vector field is associated, at least, to one $\boldsymbol{\omega}$-Hamiltonian function. Conversely, every $\boldsymbol{\omega}$-Hamiltonian function induces a unique $\boldsymbol{\omega}$-Hamiltonian vector field.*

Proposition 4.7. *The space $\mathscr{C}^\infty_{\boldsymbol{\omega}}(N)$ is a linear space over $\mathbb{R}$ with the natural operations:*

$$\boldsymbol{h} + \boldsymbol{g} \equiv \sum_{\alpha=1}^k (h^\alpha + g^\alpha) \otimes e_\alpha\,, \qquad \lambda \cdot \boldsymbol{h} \equiv \sum_{\alpha=1}^k \lambda h^\alpha \otimes e_\alpha\,,$$

where $\boldsymbol{h} = \sum_{\alpha=1}^k h^\alpha \otimes e_\alpha$, $\boldsymbol{g} = \sum_{\alpha=1}^k g^\alpha \otimes e_\alpha \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(N)$ and $\lambda \in \mathbb{R}$.

Proposition 4.8. *The space $\mathscr{C}^\infty_{\boldsymbol{\omega}}(N)$ becomes a Lie algebra when endowed with the bracket $\{\cdot,\cdot\}_{\boldsymbol{\omega}} : \mathscr{C}^\infty_{\boldsymbol{\omega}}(N) \times \mathscr{C}^\infty_{\boldsymbol{\omega}}(N) \to \mathscr{C}^\infty_{\boldsymbol{\omega}}(N)$ of the form*

$$\{h^1 \otimes e_1 + \cdots + h^k \otimes e_k, g^1 \otimes e_1 + \cdots + g^k \otimes e_k\}_{\boldsymbol{\omega}} = \{h^1, g^1\}_{\omega^1} \otimes e_1 + \cdots + \{h^k, g^k\}_{\omega^k} \otimes e_k,$$

where $\{\cdot,\cdot\}_{\omega^\alpha}$ is the Poisson bracket naturally induced by the presymplectic form ω_i, with $i = 1, \ldots, k$.

Since we cannot ensure that $(\mathscr{C}^\infty_{\boldsymbol{\omega}}(N), \cdot, \{\cdot,\cdot\}_{\boldsymbol{\omega}})$ is a Poisson algebra [14], we cannot neither say that $\{\boldsymbol{h}, \cdot\}_{\boldsymbol{\omega}} : \boldsymbol{g} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(N) \mapsto \{\boldsymbol{g}, \boldsymbol{h}\}_{\boldsymbol{\omega}} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(N)$, with $\boldsymbol{h} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(N)$, is a derivation with respect to the product of $\boldsymbol{\omega}$-Hamiltonian functions given by

$$\boldsymbol{h} \cdot \boldsymbol{g} = (h^1 g^1) \otimes e_1 + \cdots + (h^k g^k) \otimes e_k \,.$$

This shows that k-polysymplectic geometry is quite different from Poisson and presymplectic geometry, where an equivalent of this result holds. Nevertheless, we can still ensure that $\{h, g\}_\Omega = 0$ for every locally constant function g and, moreover, we can still prove other properties of this Lie algebra. For instance, let us consider the following result.

Proposition 4.9. *Consider a polysymplectic manifold $(N, \boldsymbol{\omega})$. Every $\boldsymbol{\omega}$-Hamiltonian vector field X acts as a derivation on the Lie algebra $(\mathscr{C}^\infty_{\boldsymbol{\omega}}(N), \{\cdot,\cdot\}_{\boldsymbol{\omega}})$ in the form*

$$X\boldsymbol{f} = \{\boldsymbol{f}, \boldsymbol{h}\}_{\boldsymbol{\omega}} \,, \qquad \forall \boldsymbol{f} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(N) \,,$$

with $\boldsymbol{h}$ being an $\boldsymbol{\omega}$-Hamiltonian function for X.

4.3 k-polysymplectic momentum maps

Next, we survey k-polysymplectic momentum maps associated with a k-polysymplectic Lie group action Φ on a k-polysymplectic manifold $(P, \boldsymbol{\omega})$. It is worth noting that the presented results are not restricted to $(\mathrm{Ad}^*)^k$-equivariant momentum maps. Complete proofs with details can be found in [13].

Definition 4.10. A Lie group action $\Phi \colon G \times P \to P$ on a k-polysymplectic manifold $(P, \boldsymbol{\omega})$ is a *k-polysymplectic Lie group action* if $\Phi_g^* \boldsymbol{\omega} = \boldsymbol{\omega}$ for each $g \in G$.

Definition 4.11. A *k-polysymplectic momentum map* for a Lie group action $\Phi : G \times P \to P$ with respect to a k-polysymplectic manifold $(P, \boldsymbol{\omega})$ is a map $\mathbf{J}^\Phi : P \to (\mathfrak{g}^*)^k$ such that

$$\iota_{\xi_P} \boldsymbol{\omega} := (\iota_{\xi_P} \omega^\alpha) \otimes e_\alpha = \mathrm{d} \left\langle \mathbf{J}^\Phi, \xi \right\rangle \,, \qquad \forall \xi \in \mathfrak{g} \,. \tag{2}$$

Equation (2) implies that $\mathbf{J}^\Phi : P \to (\mathfrak{g}^*)^k$ satisfies

$$\iota_{\boldsymbol{\xi}_P} \boldsymbol{\omega} = \mathrm{d} \left\langle \mathbf{J}^\Phi, \boldsymbol{\xi} \right\rangle \,, \qquad \forall \boldsymbol{\xi} \in \mathfrak{g}^k \,, \tag{3}$$

where $\boldsymbol{\xi}_P$ is the k-vector field on P whose k vector field components are the fundamental vector fields of Φ related to the k components of $\boldsymbol{\xi} \in \mathfrak{g}^k$. If we denote $\boldsymbol{\xi} = (0, \ldots, \overset{(\alpha)}{\xi}, \ldots, 0) \in \mathfrak{g}^k$ for any $\xi \in \mathfrak{g}$ and $\alpha = 1, \ldots, k$ and require (3) to hold for a basis $\{\xi_1, \ldots, \xi_r\}$ of $\mathfrak{g}$, we obtain kr conditions. These conditions uniquely determine the value of the kr coordinates of $\mathbf{J}^\Phi$. Conversely, equation (2) evaluated on the previously mentioned basis of $\mathfrak{g}$ imposes r conditions for each of the k components of $\mathbf{J}^\Phi$, resulting in a total of kr conditions.

The following definition has been widely used in the literature (see [25]). However, we have changed the notation of Coad^k to Ad^{*k} to shorten it. Nevertheless, we will see that the Ad^{*k}-equivariance condition is no longer necessary.

Definition 4.12. A k-polysymplectic momentum map $\mathbf{J}^\Phi : P \to (\mathfrak{g}^*)^k$ is Ad^{*k}-*equivariant* if

$$\mathbf{J}^\Phi \circ \Phi_g = \mathrm{Ad}^{*k}_{g^{-1}} \circ \mathbf{J}^\Phi \,, \quad \forall g \in G \,,$$

where $\mathrm{Ad}^{*k}_{g^{-1}} = \mathrm{Ad}^*_{g^{-1}} \otimes \overset{(k)}{\cdots} \otimes \mathrm{Ad}^*_{g^{-1}}$ and

$$\begin{array}{rccc} \mathrm{Ad}^{*k} & : G \times (\mathfrak{g}^*)^k & \longrightarrow & (\mathfrak{g}^*)^k \\ & (g, \boldsymbol{\mu}) & \longmapsto & \mathrm{Ad}^{*k}_{g^{-1}} \boldsymbol{\mu} \,. \end{array}$$

In other words, for every $g \in G$, the following diagram commutes

$$\begin{tikzcd} P \arrow[r, "\mathbf{J}^\Phi"] \arrow[d, "\Phi_g"] & (\mathfrak{g}^*)^k \arrow[d, "\mathrm{Ad}^{*k}_{g^{-1}}"] \\ P \arrow[r, "\mathbf{J}^\Phi"] & (\mathfrak{g}^*)^k. \end{tikzcd}$$

To simplify the notation, let us introduce the following definition.

Definition 4.13. A *G-invariant k-polysymplectic Hamiltonian system* is a tuple $(P, \boldsymbol{\omega}, h, \mathbf{J}^\Phi)$, where $(P, \boldsymbol{\omega})$ is a k-polysymplectic manifold, $\Phi : G \times P \to P$ is a k-polysymplectic Lie group action satisfying $\Phi_g^* h = h$ for every $g \in G$, and $\mathbf{J}^\Phi$ is a k-polysymplectic momentum map associated with Φ. An Ad^{*k}-*equivariant G-invariant k-polysymplectic Hamiltonian system* is a G-invariant k-polysymplectic Hamiltonian system whose k-polysymplectic momentum map is Ad^{*k}-equivariant.

Proposition 4.14. *Let $(P, \boldsymbol{\omega}, h, \mathbf{J}^\Phi)$ be a G-invariant k-polysymplectic Hamiltonian system and let us define the functions on P given by*

$$\boldsymbol{\psi}_{g,\boldsymbol{\xi}} : P \ni x \longmapsto \mathbf{J}^\Phi_{\boldsymbol{\xi}}(\Phi_g(x)) - \mathbf{J}^\Phi_{\mathrm{Ad}^k_{g^{-1}} \boldsymbol{\xi}}(x) \in \mathbb{R} \,, \quad \forall g \in G \,, \quad \forall \boldsymbol{\xi} \in \mathfrak{g}^k \,.$$

Then, $\boldsymbol{\psi}_{g,\boldsymbol{\xi}}$ is constant on P for every $g \in G$ and $\boldsymbol{\xi} \in \mathfrak{g}^k$. Moreover, the function $\boldsymbol{\sigma} : G \ni g \mapsto \boldsymbol{\sigma}(g) \in (\mathfrak{g}^*)^k$ univocally determined by the condition $\langle \boldsymbol{\sigma}(g), \boldsymbol{\xi} \rangle = \boldsymbol{\psi}_{g,\boldsymbol{\xi}}$ satisfies

$$\boldsymbol{\sigma}(g_1 g_2) = \boldsymbol{\sigma}(g_1) + \mathrm{Ad}^{*k}_{g_1^{-1}} \boldsymbol{\sigma}(g_2) \,, \quad \forall g_1, g_2 \in G \,.$$

The map $\boldsymbol{\sigma} : G \to (\mathfrak{g}^*)^k$ of the form

$$\boldsymbol{\sigma}(g) = \mathbf{J}^\Phi \circ \Phi_g - \mathrm{Ad}^{*k}_{g^{-1}} \mathbf{J}^\Phi, \qquad g \in G,$$

is called the *co-adjoint cocycle* associated with the k-polysymplectic momentum map $\mathbf{J}^\Phi$ on P. A map $\boldsymbol{\sigma} : G \to (\mathfrak{g}^*)^k$ is a *coboundary* if there exists $\boldsymbol{\mu} \in (\mathfrak{g}^*)^k$ such that

$$\boldsymbol{\sigma}(g) = \boldsymbol{\mu} - \mathrm{Ad}^{*k}_{g^{-1}} \boldsymbol{\mu} \,, \qquad \forall g \in G \,.$$

If $\mathbf{J}^\Phi$ is an Ad^{*k}-equivariant k-polysymplectic momentum map, then $\boldsymbol{\sigma} = 0$.

Proposition 4.15. *Let $\mathbf{J}^\Phi : P \to (\mathfrak{g}^*)^k$ be a k-polysymplectic momentum map associated with a k-polysymplectic action $\Phi : G \times P \to P$ with co-adjoint cocycle $\boldsymbol{\sigma}$. Then,*

(1) *the map*

$$\boldsymbol{\Delta} : G \times (\mathfrak{g}^*)^k \ni (g, \boldsymbol{\mu}) \mapsto \boldsymbol{\sigma}(g) + \mathrm{Ad}^{*k}_{g^{-1}} \boldsymbol{\mu} = \boldsymbol{\Delta}_g(\boldsymbol{\mu}) \in (\mathfrak{g}^*)^k \,,$$

is a Lie group action of G on $(\mathfrak{g}^*)^k$,

(2) *the k-polysymplectic momentum map $\mathbf{J}^\Phi$ is equivariant with respect to $\boldsymbol{\Delta}$, in other words, for every $g \in G$, one has a commutative diagram*

$$\begin{tikzcd} P \arrow[r, "\mathbf{J}^\Phi"] \arrow[d, "\Phi_g"] & (\mathfrak{g}^*)^k \arrow[d, "\boldsymbol{\Delta}_g"] \\ P \arrow[r, "\mathbf{J}^\Phi"] & (\mathfrak{g}^*)^k . \end{tikzcd}$$

Proposition 4.15 ensures that a general k-polysymplectic momentum map $\mathbf{J}^\Phi$ gives rise to an equivariant k-polysymplectic momentum map relative to a new action $\boldsymbol{\Delta} : G\times(\mathfrak{g}^*)^k \to (\mathfrak{g}^*)^k$, called a *$k$-polysymplectic affine Lie group action*.

It should be noted that an affine Lie group action can also be expressed by writing $\boldsymbol{\Delta}(g,\boldsymbol{\mu}) = (\Delta^1_g\mu^1,\dots,\Delta^k_g\mu^k) \in (\mathfrak{g}^*)^k$, where the mappings $\Delta^1,\dots,\Delta^k$ take the form $\Delta^\alpha : G\times\mathfrak{g}^* \ni (g,\vartheta) \mapsto \mathrm{Ad}^*_{g^{-1}}\vartheta + \sigma^\alpha(g) = \Delta^\alpha_g(\vartheta) \in \mathfrak{g}^*$ and $\boldsymbol{\sigma}(g) = (\sigma^1(g),\dots,\sigma^k(g))$, where $\sigma^\alpha(g) = \mathbf{J}^\Phi_\alpha\circ\Phi_g - \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi_\alpha$ for $\alpha = 1,\dots,k$.

5 k-Polysymplectic Marsden–Weinstein reduction

Let us recall several definitions that are useful for what it follows. They can be considered as technical assumptions that generalise many standard technical requirements used in the literature. First, a weak regular value of a mapping $\phi : M\to N$ is a point $x_0\in N$ such that $\phi^{-1}(x_0)$ is a submanifold and $\ker \mathrm{T}_p\phi = \mathrm{T}_p[\phi^{-1}(x_0)]$ for every $p\in\phi^{-1}(x_0)$. In particular, regular values represent a specific type of weak regular values. Moreover, a Lie group action $\phi : G\times M\to M$ is quotientable when the space of orbits G acting on M, let us say G/M is a manifold and the projection $\pi : M\to M/G$ is an open surjective submersion.

Let us review the k-polysymplectic Marsden–Weinstein reduction theorem, which will be crucial in the k-polysymplectic energy-momentum method, and correct a relevant mistake in the literature [17]. The original proof of this theorem can be found in [25]. The necessary and sufficient conditions to perform a reduction were given by C. Blacker in [5], although there was a relevant typo in his proof as commented in [17]. The original k-polysymplectic Marsden–Weinstein reduction theorem was originally proved under the assumption that the k-polysymplectic momentum map $\mathbf{J} : P\to(\mathfrak{g}^*)^k$ is $(\mathrm{Ad}^*)^k$-equivariant. This condition was eliminated in [13]. The theorem, in its correct and most modern form, reads as follows (see [13] for details).

Theorem 5.1 (k-polysymplectic Marsden–Weinstein reduction theorem)**.** *Consider a G-invariant k-polysymplectic Hamiltonian system $(P,\boldsymbol{\omega},h,\mathbf{J}^\Phi)$. Assume that $\boldsymbol{\mu}\in(\mathfrak{g}^*)^k$ is a weak regular value of $\mathbf{J}^\Phi$ and $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ acts in a quotientable manner on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Let $G^{\Delta^\alpha}_{\mu^\alpha}$ denote the isotropy group at μ^α of the Lie group action $\Delta^\alpha : (g,\vartheta)\in G\times\mathfrak{g}^*\mapsto\Delta^\alpha(g,\vartheta)\in\mathfrak{g}^*$ for $\alpha=1,\dots,k$. Moreover, let the following (sufficient) conditions hold*

$$\ker(\mathrm{T}_p\mathbf{J}^\Phi_\alpha) = \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) + \ker\omega^\alpha_p + \mathrm{T}_p(G^{\Delta^\alpha}_{\mu^\alpha}p)\,, \tag{4}$$

$$\mathrm{T}_p(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}p) = \bigcap_{\alpha=1}^k\left(\ker\omega^\alpha_p + \mathrm{T}_p(G^{\Delta^\alpha}_{\mu^\alpha}p)\right)\cap\mathrm{T}_p(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}))\,, \tag{5}$$

for every $p\in\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ and all $\alpha=1,\dots,k$. Then, $(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}},\boldsymbol{\omega}_{\boldsymbol{\mu}})$ is a k-polysymplectic manifold, where $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ is univocally determined by

$$\pi^*_{\boldsymbol{\mu}}\boldsymbol{\omega}_{\boldsymbol{\mu}} = \jmath^*_{\boldsymbol{\mu}}\boldsymbol{\omega}\,,$$

where $\jmath_{\boldsymbol{\mu}} : \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\hookrightarrow P$ is the canonical immersion and $\pi_{\boldsymbol{\mu}} : \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\to\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ is the canonical projection.

It was claimed in [17] that condition (5) implies that there exists a polysymplectic reduction. Let us show this is not true. First, the proof for Proposition 1 in [17] contains a mistake as there is an inclusion written in the opposite way. In particular, since $\mathrm{T}_pJ^{-1}(\mu) \subset \mathrm{T}_pJ_\alpha^{-1}(\mu_\alpha)$ for $\alpha = 1, \ldots, k$ and every $p \in J^{-1}(\mu)$, it should be

$$\left\{v \in \mathrm{T}_pM : \omega^1(v, \mathrm{T}_pJ_1^{-1}(\mu_1)) = \cdots = \omega^k(v, \mathrm{T}_pJ_k^{-1}(\mu_k)) = 0\right\} \subset \left\{v \in \mathrm{T}_pM : \omega^1(v, \mathrm{T}_pJ^{-1}(\mu)) = \cdots = \omega^k(v, \mathrm{T}_pJ^{-1}(\mu)) = 0\right\}$$

instead of

$$\left\{v \in \mathrm{T}_pM : \omega^1(v, \mathrm{T}_pJ_1^{-1}(\mu_1)) = \cdots = \omega^k(v, \mathrm{T}_pJ_k^{-1}(\mu_k)) = 0\right\} \supset \left\{v \in \mathrm{T}_pM : \omega^1(v, \mathrm{T}_pJ^{-1}(\mu)) = \cdots = \omega^k(v, \mathrm{T}_pJ^{-1}(\mu)) = 0\right\}$$

as stated at the end of page 8 in the proof of Proposition 1 in [17]. In other words, if v is perpendicular to $\mathrm{T}_xJ^{-1}(\mu)$ relative to each ω^α, one cannot infer that v is perpendicular to each $\mathrm{T}_xJ_\alpha^{-1}(\mu_\alpha)$ relative to ω^α for $\alpha = 1, \ldots, k$, since the latter conditions are far more restrictive. Then, the proof of Proposition 1 only gives

$$\bigcap_{a=1}^{k} (\ker j_{\mu_a}^* \omega^a|_p) \cap \mathrm{T}_pJ^{-1}(\mu) \subset \widetilde{\mathfrak{g}}_p^\omega \cap \widetilde{\mathfrak{g}}_p^{\omega\omega}\,, \qquad \forall p \in J^{-1}(\mu)\,,$$

instead of the claimed

$$\bigcap_{a=1}^{k} (\ker j_{\mu_a}^* \omega^a|_p) \cap \mathrm{T}_pJ^{-1}(\mu) \supset \widetilde{\mathfrak{g}}_p^\omega \cap \widetilde{\mathfrak{g}}_p^{\omega\omega}\,, \qquad \forall p \in J^{-1}(\mu)\,,$$

which makes the proof of Proposition 1 to fail in proving $\widetilde{\mathfrak{g}}_\mu|_p = \widetilde{\mathfrak{g}}_p^\omega \cap \widetilde{\mathfrak{g}}_p^{\omega\omega}$, namely the polysymplectic reduction conditions, and, therefore, the statement of Proposition 1. Indeed, the above mistake just illustrates the fact that Proposition 1 is false and the comments that follow contain some inaccuracies and problems. Let us provide a counterexample illustrating that Proposition 1 does not hold in two different manners.

Consider $\mathbb{R}^4$ with global coordinates $\{x, y, z, t\}$ and the presymplectic forms

$$\omega^1 = \mathrm{d}x \wedge \mathrm{d}y\,, \qquad \omega^2 = \mathrm{d}x \wedge \mathrm{d}t + \mathrm{d}y \wedge \mathrm{d}z\,,$$

which give rise to a two-polysymplectic structure since ω^2 is a symplectic form and therefore $\ker\omega^1 \cap \ker\omega^2 = 0$. Consider the Lie group action $\Phi : (\lambda; x, y, z, t) \in \mathbb{R} \times \mathbb{R}^4 \mapsto (x + \lambda, y, z, t) \in \mathbb{R}^4$. The Lie algebra of fundamental vector fields of Φ is $V = \langle \partial_x \rangle \simeq \mathbb{R}$. Moreover, Φ admits a momentum map relative to $(\mathbb{R}^4, \omega^1, \omega^2)$ given by

$$J : (x, y, z, t) \in \mathbb{R}^4 \mapsto (y, t) \in (\mathbb{R}^*)^2,$$

which is clearly $(\mathrm{Ad}^*)^2$-equivariant. Additionally, J is regular for every value[1] of $\mathbb{R}^2$. Hence, $J^{-1}(y, t) = \{(x, y, z, t) \in \mathbb{R}^4 : x, z \in \mathbb{R}\} \simeq \mathbb{R}^2$ is a manifold for every $(y, t) \in \mathbb{R}^2$ and

$$\mathrm{T}_p[J^{-1}(y, t)] = \langle \partial_x, \partial_z \rangle\,, \qquad \forall p \in J^{-1}(y, t)\,.$$

Moreover, for every $\mu = (y, t) \in \mathbb{R}^2$, one has that $G_\mu = \mathbb{R}$. Note that G_μ acts freely and properly on $J^{-1}(\mu)$. Let us prove that condition (5) does not imply (4) in this particular example.

It was proved in [25] that $\ker\omega_p^\alpha \subset \ker \mathrm{T}_pJ_\alpha$, which allows one to define the following commutative diagram

[1] Note that Sard's theorem [24] states that the set of critical points of J has zero measure in the codomain of J, but if the image of J has zero measure, which frequently occurs, then J will not be regular at any point and new conditions of regularity for J will be needed. This shows that the comments in [25] concerning the appropriateness of the regularity of J are not well founded.

$$
\begin{tikzcd}
\mathrm{T}_p J^{-1}(\mu) \arrow[r, "\iota"] \arrow[rr, bend left, "\pi_p^\alpha"] & \ker \mathrm{T}_p J_\alpha \arrow[r, "\pi"] & \frac{\ker \mathrm{T}_p J_\alpha}{\ker \omega_p^\alpha}
\end{tikzcd}
$$

where ι and π are the canonical injection and projections, respectively. For simplicity, the equivalence class of an element v in a quotient manifold will be denoted by $[v]$. According to Proposition 3.12 in [25], the above diagram induces the maps

$$\widetilde{\pi}_p^\alpha : \frac{\mathrm{T}_p J^{-1}(\mu)}{\mathrm{T}_p(G_\mu p)} \to \frac{\frac{\ker \mathrm{T}_p J_\alpha}{\ker \omega_p^\alpha}}{\{[(\xi_M)_p] : \xi \in \mathfrak{g}_{\mu_\alpha}\}}, \qquad \alpha = 1, \ldots, k,$$

where $\mathfrak{g}_\mu$ is the Lie algebra of G_{μ_α} and $\{[(\xi_M)_p] : \xi \in \mathfrak{g}_{\mu_\alpha}\} = \mathrm{pr}_\alpha^M(\{(\xi_M)_p : \xi \in \mathfrak{g}_{\mu_\alpha}\})$.

The conditions (4) at $p \in M$ are equivalent to each $\widetilde{\pi}_p^\alpha$ being surjective, while (5) amounts to $0 = \bigcap_{\alpha=1}^k \ker \widetilde{\pi}_p^\alpha$.

In our example, one has $\mu = (y, t)$ with $\mu_1 = y, \mu_2 = t$, while

$$\ker \mathrm{T}_p J_1 = \langle \partial_x, \partial_z, \partial_t \rangle, \quad \ker \omega^1 = \langle \partial_t, \partial_z \rangle, \quad \ker \mathrm{T}_p J_2 = \langle \partial_x, \partial_y, \partial_z \rangle, \quad \ker \omega^2 = 0$$

and

$$\{[(\xi_M)_p] : \xi \in \mathfrak{g}_{\mu_1}\} = \langle \partial_x \rangle, \qquad \{[(\xi_M)_p] : \xi \in \mathfrak{g}_{\mu_2}\} = \langle [\partial_x] \rangle.$$

Then, we have the mappings

$$\widetilde{\pi}_p^1 : \langle [\partial_z] \rangle = \mathrm{T}_p J^{-1}(\mu)/\mathrm{T}_p(G_\mu p) \to \langle 0 \rangle = (\ker \mathrm{T}_p J_1 / \ker \omega_p^1)/\langle [\partial_x] \rangle$$

and

$$\widetilde{\pi}_p^2 : \langle [\partial_z] \rangle = \mathrm{T}_p J^{-1}(\mu)/\mathrm{T}_p(G_\mu p) \to \langle [\partial_y], [\partial_z] \rangle = (\ker \mathrm{T}_p J_2 / \ker \omega_p^2)/\langle [\partial_x] \rangle.$$

As $\widetilde{\pi}_p^2([\partial_z]) = [\partial_z]$, we have

$$\ker \widetilde{\pi}_p^1 = \langle \partial_z \rangle, \qquad \ker \widetilde{\pi}_p^2 = \langle 0 \rangle.$$

Hence, $\ker \widetilde{\pi}_p^1 \cap \ker \widetilde{\pi}_p^2 = 0$ and condition (5) is satisfied. But $\mathrm{Im}\, \widetilde{\pi}_p^2 = \langle \partial_z \rangle$ and $\widetilde{\pi}_p^2$ is not surjective. Hence, (4) does not hold for $\alpha = 2$ in our example. In fact, ω^1, ω^2 become isotropic when restricted to $J^{-1}(\mu)$ and give rise to two zero differential forms on $J^{-1}(\mu)/G_\mu$ which is a one-dimensional manifold. Hence, no two-symplectic structure is induced on $J^{-1}(\mu)/G_\mu$ despite that condition (5) is satisfied.

One can directly prove that condition (5) is satisfied in the previous example, but condition (4) is not. This shows more easily that Proposition 1 in [18] is false, but our previous approach illustrates how we obtained our counterexample and it can also be used to investigate the independence between (4) and (5). In fact, in our actual counterexample, the fact that $\widetilde{\pi}_p^2$ is not surjective implies that (4) does not hold. Indeed, recall that

$$\ker \mathrm{T}_p J_2 = \langle \partial_x, \partial_y, \partial_z \rangle, \qquad \forall p \in J^{-1}(\mu),$$

while

$$\mathrm{T}_p J^{-1}(\mu) + \ker \omega_p^2 + \mathrm{T}_p(G_t p) = \langle \partial_x, \partial_z \rangle + \{0\} + \langle \partial_x \rangle = \langle \partial_x, \partial_z \rangle, \qquad \forall p \in J^{-1}(\mu).$$

On the other hand, condition (5) is satisfied since

$$\mathrm{T}_p(G_\mu p) = \langle \partial_x \rangle$$

and

$$(\ker \omega_p^1 + \mathrm{T}_p(G_y p)) \cap (\ker \omega_p^2 + \mathrm{T}_p(G_t p)) \cap \mathrm{T}_p(J^{-1}(\mu)) = \langle \partial_x \rangle$$

reads

$$(\langle\partial_t,\partial_z\rangle+\langle\partial_x\rangle)\cap(\langle 0\rangle+\langle\partial_x\rangle)\cap\langle\partial_x,\partial_z\rangle=\langle\partial_x\rangle.$$

Let us illustrate the independence of conditions (4) and (5). In particular, we aim to show condition (4) can be satisfied without (5). Consider a two-polysymplectic manifold $(\mathbb{R}^6,\boldsymbol{\omega})$. Let $\{x_1,x_2,x_3,x_4,x_5,x_6\}$ be global linear coordinates on $\mathbb{R}^6$ and define

$$\boldsymbol{\omega}=\omega^1\otimes e_1+\omega_2\otimes e_2=(\mathrm{d}x_1\wedge\mathrm{d}x_2+\mathrm{d}x_5\wedge\mathrm{d}x_6)\otimes e_1+(\mathrm{d}x_3\wedge\mathrm{d}x_4+\mathrm{d}x_5\wedge\mathrm{d}x_6)\otimes e_2\,.$$

Then, $\ker\omega^1_p=\langle\partial_3,\partial_4\rangle$, $\ker\omega^2_p=\langle\partial_1,\partial_2\rangle$, and $\ker\omega^1_p\cap\ker\omega^2_p=0$ for any $p\in\mathbb{R}^6$. This turns $\boldsymbol{\omega}$ into a two-polysymplectic form.

Let us provide now a Lie group action proving our initial claim. Given the Lie group action $\Phi:(\lambda;x_1,x_2,x_3,x_4,x_5,x_6)\in\mathbb{R}\times\mathbb{R}^6\mapsto(x_1+\lambda,x_2,x_3+\lambda,x_4,x_5,x_6)\in\mathbb{R}^6$, its Lie algebra of fundamental vector fields reads $\langle\partial_1+\partial_3\rangle$. The momentum map associated with Φ is given by

$$J:(x_1,x_2,x_3,x_4,x_5,x_6)\in\mathbb{R}^6\mapsto(x_2,x_4)=\mu\in(\mathbb{R}^*)^2,$$

which is $(\mathrm{Ad}^*)^2$-equivariant. Moreover, every $\mu=(x_2,x_4)\in\mathbb{R}^2$ is a regular value of J. Therefore, $\mathbf{J}^{-1}(\mu)=\{(x_1,x_2,x_3,x_4,x_5,x_6)\in\mathbb{R}^6\,:\,x_1,x_3,x_5,x_6\in\mathbb{R}\}\simeq\mathbb{R}^4$ is a submanifold of $\mathbb{R}^6$ for every $\mu\in\mathbb{R}^2$ and

$$\mathrm{T}_pJ^{-1}(\mu)=\langle\partial_1,\partial_3,\partial_5,\partial_6\rangle\,,\qquad\forall p\in J^{-1}(\mu)\,.$$

Hence, $\ker\mathrm{T}_pJ_1^{-1}(x_2)=\langle\partial_1,\partial_3,\partial_4,\partial_5,\partial_6\rangle$ while $\ker\mathrm{T}_pJ_2^{-1}(x_4)=\langle\partial_1,\partial_2,\partial_3,\partial_5,\partial_6\rangle$. Condition (4) holds because both sides are equal to

$$\begin{aligned}\langle\partial_1,\partial_3,\partial_4,\partial_5,\partial_6\rangle&=\langle\partial_1,\partial_3,\partial_5,\partial_6\rangle+\langle\partial_3,\partial_4\rangle+\langle\partial_1+\partial_3\rangle\,,\\ \langle\partial_1,\partial_2,\partial_3,\partial_5,\partial_6\rangle&=\langle\partial_1,\partial_3,\partial_5,\partial_6\rangle+\langle\partial_1,\partial_2\rangle+\langle\partial_1+\partial_3\rangle\,,\end{aligned}$$

respectively, for J_1,J_2. However, condition (5) is not satisfied, namely

$$\begin{aligned}\bigcap_{i=1}^{2}\left(\ker\omega^i_p+\mathrm{T}_p(G_{\mu_i}p)\right)\cap\mathrm{T}_pJ^{-1}(\mu)&=(\langle\partial_3,\partial_4\rangle+\langle\partial_1+\partial_3\rangle)\cap(\langle\partial_1,\partial_2\rangle+\langle\partial_1+\partial_3\rangle)\\ &\cap\langle\partial_1,\partial_3,\partial_5,\partial_6\rangle=\langle\partial_1,\partial_3\rangle\neq\langle\partial_1+\partial_3\rangle=\mathrm{T}_p(G_\mu p)\,,\end{aligned}$$

for any $p\in J^{-1}(\mu)$. Therefore, $\widetilde{\pi}^1,\widetilde{\pi}^2$ are surjective but $\ker\widetilde{\pi}^1\cap\ker\widetilde{\pi}^2\neq 0$. In fact, one can also verify this fact by computing $\widetilde{\pi}^i_p$ for $i=1,2$. Note that

$$\begin{aligned}\widetilde{\pi}^1_p&:\langle[\partial_1],[\partial_5],[\partial_6]\rangle\in\mathrm{T}_pJ^{-1}(\mu)/\mathrm{T}_p(G_\mu p)\mapsto\langle[\partial_5],[\partial_6]\rangle=(\ker\mathrm{T}_pJ_1/\ker\omega^1_p)/\langle[\partial_1+\partial_3]\rangle\,.\\ \widetilde{\pi}^2_p&:\langle[\partial_1],[\partial_5],[\partial_6]\rangle\in\mathrm{T}_pJ^{-1}(\mu)/\mathrm{T}_p(G_\mu p)\mapsto\langle[\partial_5],[\partial_6]\rangle=(\ker\mathrm{T}_pJ_2/\ker\omega^2_p)/\langle[\partial_1+\partial_3]\rangle\,.\end{aligned}$$

Finally, let us prove that the reduction theorem in [25] gives sufficient, but not necessary conditions for the reduction to hold. In this respect, there are cases where the reduction is possible, condition (5) holds, while condition (4) does not. To illustrate this, let us consider a two-polysymplectic manifold $(\mathbb{R}^7,\boldsymbol{\omega})$, where $\{x_1,\ldots,x_7\}$ are global linear coordinates and

$$\begin{aligned}\boldsymbol{\omega}&=\omega_1\otimes e_1+\omega_2\otimes e_2\\ &=(\mathrm{d}x_1\wedge\mathrm{d}x_2+\mathrm{d}x_5\wedge\mathrm{d}x_7+\mathrm{d}x_3\wedge\mathrm{d}x_6)\otimes e_1+(\mathrm{d}x_3\wedge\mathrm{d}x_4+\mathrm{d}x_5\wedge\mathrm{d}x_6)\otimes e_2\,.\end{aligned}$$

This give rise to a two-polysymplectic structure on $\mathbb{R}^7$ since $\ker\omega^1_p=\langle\partial_4\rangle$, $\ker\omega^2_p=\langle\partial_1,\partial_2,\partial_7\rangle$ and $\ker\omega^1_p\cap\ker\omega^2_p=0$ for any $p\in\mathbb{R}^7$. Consider the Lie group action $\Phi:\mathbb{R}\times\mathbb{R}^7\to\mathbb{R}^7$ corresponding to translations along the x_5 coordinate. Then, its Lie algebra of fundamental vector fields is $\langle\partial_5\rangle$. A momentum map associated with Φ reads

$$J:(x_1,x_2,x_3,x_4,x_5,x_6,x_7)\in\mathbb{R}^7\mapsto(x_7,x_6)=\mu\in(\mathbb{R}^2)^*\,.$$

Note that J is $(\mathrm{Ad}^*)^2$-equivariant and every $\mu \in \mathbb{R}^{2*}$ is a regular value of J. Then,

$$J^{-1}(\mu) = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in \mathbb{R}^7 \,:\, x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}\} \simeq \mathbb{R}^5$$

is a submanifold of $\mathbb{R}^7$ for every $\mu = (x_6, x_7) \in \mathbb{R}^2$ and

$$\mathrm{T}_p J^{-1}(\mu) = \langle \partial_1, \partial_2, \partial_3, \partial_4, \partial_5 \rangle \,, \qquad \forall p \in J^{-1}(\mu)\,.$$

Condition (5) is satisfied while (4) for J_1 is not since

$$\widetilde{\pi}^1_p : \langle [\partial_1], [\partial_2], [\partial_3], [\partial_4] \rangle \in \mathrm{T}_p J^{-1}(\mu)/\mathrm{T}_p(G_\mu p) \mapsto \langle [\partial_1], [\partial_2], [\partial_3], [\partial_6] \rangle = (\ker \mathrm{T}_p J_1 / \ker \omega^1_p)/\langle [\partial_5] \rangle\,.$$

Therefore, $\widetilde{\pi}^1_p$ is not surjective. However, the reduced manifold $P_\mu = \mathrm{T}_p J^{-1}(\mu)/\mathrm{T}_p(G_\mu p) \simeq \mathbb{R}^4$ inherits the two-polysymplectic structure, namely

$$\boldsymbol{\omega}_\mu = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes e_1 + \mathrm{d}x_3 \wedge \mathrm{d}x_4 \otimes e_2\,.$$

We have taken into account such since Proposition 4.15 guarantees that the momentum map $\mathbf{J}^\Phi$ can be made equivariant with respect to a k-polysymplectic affine Lie group action $\boldsymbol{\Delta} : G \times (\mathfrak{g}^*)^k \to (\mathfrak{g}^*)^k$, see [13].

Example 5.2. Let us examine the k-polysymplectic reduction of a product of k symplectic manifolds. Let $P = P_1 \times \cdots \times P_k$ for some symplectic manifolds $(P_\alpha, \omega^\alpha)$ with $\alpha = 1, \ldots, k$. If $\mathrm{pr}_\alpha : P \to P_\alpha$ is the canonical projection onto the α-th component, $(P, \sum_{\alpha=1}^k \mathrm{pr}^*_\alpha \omega^\alpha \otimes e_\alpha)$ is a k-polysymplectic manifold. To simplify the notation, we will write $\mathrm{pr}^*_\alpha \omega^\alpha$ as ω^α. Moreover, assume that a Lie group action $\Phi^\alpha : G_\alpha \times P_\alpha \to P_\alpha$ admits a symplectic momentum map $\mathbf{J}^{\Phi^\alpha} : P_\alpha \to \mathfrak{g}^*_\alpha$ for each $\alpha = 1, \ldots, k$ and each Φ^α acts in a quotientable manner on the level sets given by regular values of $\mathbf{J}^{\Phi^\alpha}$.

Let us define the Lie group action of $G = G_1 \times \cdots \times G_k$ on P as

$$\Phi : G \times P \ni (g_1, \ldots, g_k, x_1, \ldots, x_k) \longmapsto (\Phi^1_{g_1}(x_1), \ldots, \Phi^k_{g_k}(x_k)) \in P\,.$$

Moreover, let $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$ be the Lie algebra of G. Then, we have the k-polysymplectic momentum map

$$\mathbf{J} : P \ni (x_1, \ldots, x_k) \longmapsto (\mathbf{J}^{\Phi_1}(x_1), \ldots, \mathbf{J}^{\Phi_k}(x_k)) \in \mathfrak{g}^*,$$

where $\mathfrak{g}^* = \mathfrak{g}^*_1 \times \cdots \times \mathfrak{g}^*_k$ is dual space to $\mathfrak{g}$. Suppose, that $\mu^\alpha \in \mathfrak{g}^*_\alpha$ is a regular value of $\mathbf{J}^{\Phi^\alpha} : P_\alpha \to \mathfrak{g}^*_\alpha$ for each $\alpha = 1, \ldots, k$. Hence, $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k) \in (\mathfrak{g}^*)^k$ is a regular value of $\mathbf{J}$. Then, Φ acts in a quotientable on the associated level sets of $\mathbf{J}$.

Therefore, if $x = (x_1, \ldots, x_k) \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, it follows that

$$\begin{aligned}
\ker \mathrm{T}_x \mathbf{J}^{\Phi^\alpha} &= \mathrm{T}_{x_1} P_1 \oplus \cdots \oplus \ker \mathrm{T}_{x_\alpha} \mathbf{J}^{\Phi^\alpha} \oplus \cdots \oplus \mathrm{T}_{x_k} P_k\,,\\
\mathrm{T}_x \left(\mathbf{J}^{-1}(\boldsymbol{\mu})\right) &= \ker \mathrm{T}_{x_1} \mathbf{J}^{\Phi^1} \oplus \cdots \oplus \ker \mathrm{T}_{x_k} \mathbf{J}^{\Phi^k}\,,\\
\ker \omega^\alpha_x &= \mathrm{T}_{x_1} P_1 \oplus \cdots \oplus \mathrm{T}_{x_{\alpha-1}} P_{\alpha-1} \oplus \{0\} \oplus \mathrm{T}_{x_{\alpha+1}} P_{\alpha+1} \oplus \cdots \oplus \mathrm{T}_{x_k} P_k\,,\\
\mathrm{T}_x \left(G^{\Delta^\alpha}_{\mu^\alpha} x\right) &= \mathrm{T}_{x_1}\left(G_1 x_1\right) \oplus \cdots \oplus \mathrm{T}_{x_\alpha}\left(G^{\Delta^\alpha}_{\alpha\mu^\alpha} x_\alpha\right) \oplus \cdots \oplus \mathrm{T}_{x_k}\left(G_k x_k\right)\,,\\
\mathrm{T}_x \left(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} x\right) &= \mathrm{T}_{x_1}\left(G^{\Delta^1}_{1\mu^1} x_1\right) \oplus \cdots \oplus \mathrm{T}_{x_k}\left(G^{\Delta^k}_{k\mu^k} x_k\right)\,.
\end{aligned}$$

Then,

$$\begin{aligned}
\ker \mathrm{T}_x \mathbf{J}^{\Phi^\alpha} &= \mathrm{T}_x \left(\mathbf{J}^{-1}(\mu)\right) + \ker \omega^\alpha_x\,,\\
\mathrm{T}_x \left(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} x\right) &= \bigcap_{\beta=1}^k \left(\ker \omega^\beta_x + \mathrm{T}_x\left(G^{\Delta^\beta}_{\mu^\beta} x\right)\right)\,,
\end{aligned}$$

for $\alpha = 1, \ldots, k$ and every regular $\boldsymbol{\mu} \in (\mathfrak{g}^*)^k$ and $x \in \mathbf{J}^{-1}(\boldsymbol{\mu})$. Recall that, by Theorem 5.1, these equations guarantee that the reduced space $\mathbf{J}^{-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}}$ can be endowed with a k-polysymplectic structure, while

$$\mathbf{J}^{-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}} \simeq \mathbf{J}^{\Phi^1-1}(\mu^1)/G_{1\mu^1}^{\Delta^1} \times \cdots \times \mathbf{J}^{\Phi^k-1}(\mu^k)/G_{k\mu^k}^{\Delta^k}\,.$$

$\triangle$

In the standard approach, Theorem 5.1 is applied to reduce a k-polysymplectic Hamiltonian k-vector field $\mathbf{X}^h = (X_1^h, \ldots, X_k^h)$ on the manifold P. In the following theorem, instead of reducing a k-polysymplectic k-vector field $\mathbf{X}^h$, we will reduce a vector field X on P, which will be essential to the k-polysymplectic energy-momentum method.

Theorem 5.3. *Consider a vector field X on P such that $\iota_X \boldsymbol{\omega} = \mathrm{d}\boldsymbol{f}$, where $\boldsymbol{f} = \sum_{\alpha=1}^k f^\alpha \otimes e_\alpha$ with $f^\alpha \in \mathscr{C}^\infty(P)$ and $\alpha = 1, \ldots, k$. Let $\Phi : G \times P \to P$ be a k-polysymplectic Lie group action, let $\mathbf{J}^\Phi : P \to (\mathfrak{g}^*)^k$ be a k-polysymplectic momentum map, and let $\Phi_{g*}X = X$ for each $g \in G$. Then, the flow F_t of the vector field X induces the flow $\mathcal{F}_t$ of the vector field $X_{\boldsymbol{\mu}}$ on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}}$ such that $\iota_{X_{\boldsymbol{\mu}}} \boldsymbol{\omega}_{\boldsymbol{\mu}} = \mathrm{d}\boldsymbol{f}_{\boldsymbol{\mu}}$ and functions $f_{\boldsymbol{\mu}}^\alpha \in \mathscr{C}^\infty(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}})$ are defined by $i_{\boldsymbol{\mu}}^* \boldsymbol{f} = \pi_{\boldsymbol{\mu}}^* \boldsymbol{f}_{\boldsymbol{\mu}}$.*

Proof. First, note that since $\Phi_{g*}X = X$ and $\Phi_g^* \boldsymbol{\omega} = \boldsymbol{\omega}$ for each $g \in G$ by assumptions, one gets $\Phi_g^* \boldsymbol{f} = \boldsymbol{f}$. Therefore,

$$\iota_X \mathrm{d}\mathbf{J}_\xi^\Phi = -\iota_{\xi_P} \iota_X \boldsymbol{\omega} = -\iota_{\xi_P} \mathrm{d}\boldsymbol{f} = 0\,, \qquad \forall \xi \in \mathfrak{g}\,.$$

Hence, X is tangent to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Next, for every $\xi \in \mathfrak{g}$, we have

$$\iota_{[\xi_P, X]} \boldsymbol{\omega} = \mathscr{L}_{\xi_P} \iota_X \boldsymbol{\omega} - \iota_{\xi_P} \mathscr{L}_X \boldsymbol{\omega} = 0\,,$$

so by virtue of $\ker \boldsymbol{\omega} = 0$, we obtain that $[\xi_P, X] = 0$. Thus, the vector field X projects onto the vector field $X_{\boldsymbol{\mu}}$ on the reduced manifold $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}}$. In other words, the flow F_t of X induces the flow $\mathcal{F}_t$ of $X_{\boldsymbol{\mu}}$ such that $\pi_{\boldsymbol{\mu}} \circ F_t = \mathcal{F}_t \circ \pi_{\boldsymbol{\mu}}$ for each $t \in \mathbb{R}$. Then, by Theorem 5.1, one has

$$i_{\boldsymbol{\mu}}^* \mathrm{d}\boldsymbol{f} = i_{\boldsymbol{\mu}}^*(\iota_X \boldsymbol{\omega}) = \iota_X i_{\boldsymbol{\mu}}^* \boldsymbol{\omega} = \iota_X \pi_{\boldsymbol{\mu}}^* \boldsymbol{\omega}_{\boldsymbol{\mu}} = \pi_{\boldsymbol{\mu}}^*(\iota_{X_{\boldsymbol{\mu}}} \boldsymbol{\omega}_{\boldsymbol{\mu}}) = \pi_{\boldsymbol{\mu}}^* \mathrm{d}\boldsymbol{f}_{\boldsymbol{\mu}}, \tag{6}$$

where we denoted by X both a factor field X on P and its restriction to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$.

Finally, due to Theorem 5.1 and (6), the following holds

$$\begin{aligned} \mathrm{d}\boldsymbol{f}_{\boldsymbol{\mu}}(\pi_{\boldsymbol{\mu}}(x)) = i_{\boldsymbol{\mu}}^* \mathrm{d}\boldsymbol{f}(x) = (\iota_X i_{\boldsymbol{\mu}}^* \boldsymbol{\omega})(x) &= (\iota_X \pi_{\boldsymbol{\mu}}^* \boldsymbol{\omega}_{\boldsymbol{\mu}})(x) \\ &= \pi_{\boldsymbol{\mu}}^*(\iota_{X_{\boldsymbol{\mu}}} \boldsymbol{\omega}_{\boldsymbol{\mu}})(x) = (\iota_{X_{\boldsymbol{\mu}}} \boldsymbol{\omega}_{\boldsymbol{\mu}})(\pi_{\boldsymbol{\mu}}(x))\,, \quad \forall x \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,. \end{aligned}$$

$\square$

The following lemma will allow us to characterise the so-called k-polysymplectic relative equilibrium points on P, which will be defined in the next section. The proof of this lemma can be found in [13].

Lemma 5.4. *Let $(P, \boldsymbol{\omega},)$ be a k-polysymplectic manifold and let $\boldsymbol{\mu} \in (\mathfrak{g}^*)^k$ be a weak regular value of a k-polysymplectic momentum map $\mathbf{J}^\Phi : P \to (\mathfrak{g}^*)^k$. Then, for every $x \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, one has*

(1) $\mathrm{T}_x(G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}} x) = \mathrm{T}_x(Gx) \cap \mathrm{T}_x\left(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\right)$,

(2) $\mathrm{T}_x(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = \mathrm{T}_x(Gx)^{\perp,k}$.

6 The k-polysymplectic energy momentum-method

6.1 k-polysymplectic relative equilibrium points

This section introduces the notion of k-polysymplectic relative equilibrium point relative to a Hamiltonian k-polysymplectic vector field X.This notion is devised to analyse the stability of k-polysymplectic Hamiltonian vector fields and extends the relative equilibrium point notion for symplectic manifolds (see [2] for details). In brief, a relative equilibrium point for a dynamical system given by a vector field is a point whose evolution is given by a Lie group symmetry of the vector field. If the vector field is additionally Hamiltonian relative to some geometric structure, then it is usual to demand the Lie group symmetries to leave invariant the same geometric structure [21].

Definition 6.1. A point $z_e \in P$ is a *k-polysymplectic relative equilibrium point* of a k-polysymplectic Hamiltonian system $(P, \boldsymbol{\omega}, h)$, if there exists $\xi \in \mathfrak{g}$ so that

$$(X_h)_{z_e} = (\xi_P)_{z_e} \,.$$

The above definition retrieves, for $k = 1$, the standard relative equilibrium notion for symplectic Hamiltonian systems [2]. Furthermore, Lemma 5.4 shows that $\xi \in \mathfrak{g}$ in Definition 6.1 is, in fact, an element of $\mathfrak{g}^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$, which is a Lie subalgebra of $\mathfrak{g}$.

Note that a k-polysymplectic relative equilibrium point $z_e \in P$ projects onto $\pi_{\boldsymbol{\mu}}(z_e)$, with $\boldsymbol{\mu} = \mathbf{J}(z_e)$, which becomes an equilibrium point of the vector field $X_{\boldsymbol{\mu}}$ on the reduced space $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$.

The following theorem characterises k-polysymplectic relative equilibrium points of a k-polysymplectic Hamiltonian vector field $X_{\mathbf{h}}$ by studying the critical points of a modified $\mathbb{R}^k$-valued function $\boldsymbol{f}_\xi$ on P. This will be an application of the Lagrange multiplier theorem, where the role of the multiplier is played by $\xi \in \mathfrak{g}$.

Theorem 6.2. *Given a G-invariant $\boldsymbol{\omega}$-Hamiltonian system $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi : P \to (\mathfrak{g}^*)^k)$. Then, $z_e \in P$ is a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$ if and only if there exists $\xi \in \mathfrak{g}$ such that z_e is a critical point of the following $\mathbb{R}^k$-valued function*

$$\boldsymbol{f}_\xi := \boldsymbol{f} - \langle \mathbf{J}^\Phi - \boldsymbol{\mu}_e, \xi \rangle \,, \tag{7}$$

where $\boldsymbol{\mu}_e := \mathbf{J}^\Phi(z_e) \in (\mathfrak{g}^*)^k$.

Proof. Let z_e be a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$, i.e. $X_{\boldsymbol{h}}(z_e) = \xi_P(z_e)$ for some $\xi \in \mathfrak{g}$. Then,

$$\mathrm{d}\boldsymbol{f}_\xi(z_e) = \mathrm{d}(\boldsymbol{f} - \langle \mathbf{J}^\Phi, \xi \rangle)(z_e) = (\iota_{X - \xi_P} \boldsymbol{\omega})(z_e) = 0 \,. \tag{8}$$

Hence, $z_e \in P$ is a critical point of the $\mathbb{R}^k$-valued function $\boldsymbol{f}_\xi$.

Conversely, assume z_e is a critical point of some $\boldsymbol{f}_\xi$ with $\xi \in \mathfrak{g}$. Then, $(\iota_{X - \xi_P} \boldsymbol{\omega})(z_e) = 0$ and $(X - \xi_P)(z_e) \in \ker \boldsymbol{\omega}_{z_e}$, since $\ker \boldsymbol{\omega} = 0$, and (8) yields that $X(z_e) = \xi_P(z_e)$. Hence, z_e is a k-polysymplectic relative equilibrium point of X. □

It is left to understand the role of the constant term $\langle \mu_e, \xi \rangle$ in the definition of $\boldsymbol{f}_\xi$. Note that one can understand that (z_e, ξ) is a critical point of $\boldsymbol{f}$ and ξ is indeed a Lagrange multiplier. Hence, all the components of $\boldsymbol{f}$ must have a critical point on $\mathbf{J}^{-1}(\mu_e)$ as long as it is a submanifold. Note also that the standard Lagrange multiplier method requires the map $\mathbf{J}$ to be a submersion at z_e as, otherwise, one does not know whether

6.2 The k-polysymplectic energy momentum-method

Let us develop the main part of the k-polysymplectic energy-momentum method relative to a k-polysymplectic manifold $(P, \boldsymbol{\omega})$. Recall that Theorem 6.2 characterises relative equilibrium points as critical points of the function (7). However, when studying the stability of relative equilibrium points, due to the symmetry, one has to take into account vectors that are tangent to $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$. In other words, we need to investigate how the second variation of $\boldsymbol{f}_\xi$ in the directions tangent to the isotropy group $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ affects the positive definiteness of $\boldsymbol{f}_\xi$.

Let us define the second variation of each of the components of $\boldsymbol{f}_\xi$ at a relative equilibrium point $z_e \in P$ as the mapping

$$\left(\delta^2 \boldsymbol{f}_\xi\right)_{z_e}(v_1, v_2) = \sum_{\alpha=1}^{k} \iota_Y \left(\mathrm{d}\left(\iota_X \mathrm{d} f^\alpha_\xi\right)\right)_{z_e} \otimes e_\alpha \,,$$

for some vector fields X, Y on P defined on a neighbourhood of $z_e \in P$ and such that $v_1 = X_{z_e}$, $v_2 = Y_{z_e}$.

Proposition 6.3. *Let $z_e \in P$ be a relative equilibrium point of X on a k-polysymplectic manifold $(P, \boldsymbol{\omega})$. If $\{t, x_1, \ldots, x_{2n}\}$ are coordinates on a neighbourhood of $z_e \in P$, then*

$$(\delta^2 f^\alpha_\xi)_{z_e}(w, v) = \sum_{i,j=1}^{2n} \frac{\partial^2 f^\alpha_\xi}{\partial x_i \partial x_j}(z_e) w_i v_j, \quad \forall v, w \in T_{z_e}P, \quad \alpha = 1, \ldots, k,$$

where $w = \sum_{i=1}^{2n} w_i \partial/\partial x_i$ and $v = \sum_{i=1}^{2n} v_i \partial/\partial x_i$.

Proposition 6.4. *Let X be a vector field on P invariant with respect to k-polysymplectic Lie group action $\Phi : G \times P \to P$ and let $z_e \in P$ be a k-polysymplectic relative equilibrium point of X. Then,*

$$(\delta^2 \boldsymbol{f}_\xi)_{z_e}((\zeta_P)_{z_e}, (\nu_P)_{z_e}) = 0 \,, \qquad \forall \zeta, \nu \in \mathfrak{g}^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} \,,$$

Proof. First, since $f^\alpha \in \mathscr{C}^\infty(P)$ is G-invariant and $\mathbf{J}^\Phi$ is equivariant with respect to the k-polysymplectic affine Lie group action $\boldsymbol{\Delta} : G \times (\mathfrak{g}^*)^k \to (\mathfrak{g}^*)^k$, then for every $g \in G$ and $x \in P$, one has

$$\begin{aligned}\boldsymbol{f}_\xi(\Phi_g(x)) &= \boldsymbol{f}(\Phi_g(x)) - \langle \mathbf{J}^\Phi(\Phi_g(x)), \xi\rangle + \langle \mu_e, \xi\rangle \\ &= \boldsymbol{f}(x) - \langle \boldsymbol{\Delta}_g \mathbf{J}^\Phi(x), \xi\rangle + \langle \mu_e, \xi\rangle = \boldsymbol{f}(x) - \langle \mathbf{J}^\Phi(x), \boldsymbol{\Delta}^T_g \xi\rangle + \langle \mu_e, \xi\rangle \,,\end{aligned}$$

where $\boldsymbol{\Delta}^T_g : (\mathfrak{g})^k \to (\mathfrak{g})^k$ is a transpose of $\boldsymbol{\Delta}_g$ for every $g \in G$. Let us substitute $g = \exp(t\zeta)$, with $\zeta \in \mathfrak{g}$, and differentiate with respect to t. Then,

$$(\iota_{\zeta_P} \mathrm{d}\boldsymbol{f}_\xi)_{z_e} = -\left\langle \mathbf{J}^\Phi(x), \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \boldsymbol{\Delta}^T_{\exp t\zeta}\xi \right\rangle = -\left\langle \mathbf{J}^\Phi(x), (\zeta^{\boldsymbol{\Delta}}_{\mathfrak{g}})_\xi \right\rangle \,, \tag{9}$$

where $(\zeta^{\boldsymbol{\Delta}}_{\mathfrak{g}})_\xi$ is the fundamental vector field of $\boldsymbol{\Delta}^T : G \times \mathfrak{g}^k \to \mathfrak{g}^k$ at $\xi \in \mathfrak{g}$. Taking second variation of (9) relative to $x \in P$, evaluating at $z_e \in P$, and contracting with ν_P, one has

$$\left(\delta^2 \boldsymbol{f}_\xi\right)_{z_e}((\zeta_P)_{z_e}, (\nu_P)_{z_e}) = -\left\langle \mathrm{T}_{z_e}\mathbf{J}^\Phi\left((\nu_P)_{z_e}\right), (\zeta^{\boldsymbol{\Delta}}_{\mathfrak{g}})_\xi \right\rangle .$$

Therefore, by Proposition 5.4 the second variation $\left(\delta^2 \boldsymbol{f}_\xi\right)_{z_e}((\zeta_P)_{z_e}, (\nu_P)_{z_e})$ vanish since $(\nu_P)_{z_e} \in \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu_e})$ for $\nu \in \mathfrak{g}^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$. □

From Proposition 6.4, we can conclude that the definiteness of $\left(\delta^2 \boldsymbol{f}_\xi\right)_{z_e}$ in directions tangent to $\mathrm{T}_{z_e}\left(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} z_e\right)$ has no effect. Only directions transverse to the orbit of $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$ are significant. This is an essential advantage of the energy-momentum method.

Proposition 6.5. *Let $z_e \in P$ be a relative equilibrium point of a k-polysymplectic Hamiltonian system $(P, \boldsymbol{\omega}, h)$. Then, z_e is a stable relative equilibrium point if*

$$\left(\delta^2 f^i_\xi\right)_{z_e}(v_{z_e}, v_{z_e}) > 0, \qquad \forall v_{z_e} \in \left(\mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)/\mathrm{T}_{z_e}(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e} z_e)\right) \bigcap (\ker \omega^i_{\boldsymbol{\mu}_e})_{[z_e]}, \quad i = 1, \ldots, k,$$

where $[z_e] = \pi_{\boldsymbol{\mu}_e}(z_e)$.

The stability of the k-polysymplectic Hamiltonian system requires the positive definiteness of $(\delta^2 f^\alpha_\xi)_{z_e}$ on $\mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$ modulo directions along $\mathrm{T}_{z_e}(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e} z_e)$ such that they do not belong to $\ker \omega^\alpha_{z_e}$. Taking that into account, one gets the following theorem

Theorem 6.6. *Let $\mathcal{S}_{z_e}$ be a subspace of P transversal to the orbits of $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$. Then, a relative equilibrium point x_e is stable if*

$$\left(\delta^2 f^\alpha_\xi\right)_{z_e}(v_{z_e}, v_{z_e}) > 0, \qquad \forall v_{z_e} \in \mathrm{T}_{z_e}\mathcal{S}/\left(\mathrm{T}_{z_e}\mathcal{S} \cap \ker \omega^\alpha_{z_e}\right), \quad \alpha = 1, \ldots, k.$$

7 Examples

7.1 Example: The cotangent bundle of two-covelocities of $\mathbb{R}^2$

Our first example is related to the so-called cotangent bundle of k-covelocities of a manifold. Let Q be an n-dimensional manifold and let $\pi_Q : \mathrm{T}^*Q \to Q$ be the cotangent bundle projection. Consider the Whitney sum $\oplus^k \mathrm{T}^*Q = \mathrm{T}^*Q \oplus_Q \overset{(k)}{\cdots} \oplus_Q \mathrm{T}^*Q$ of k copies of T^*Q and the projection $\pi^k_Q : \oplus^k \mathrm{T}^*Q \to Q$. It is well-known that $\oplus^k \mathrm{T}^*Q$ can be identified with the first-jet manifold, $J^1(Q, \mathbb{R}^k)$, of maps $\sigma : Q \to \mathbb{R}^k$ via the diffeomorphism $J^1(Q, \mathbb{R}^k) \ni j^1_q\sigma \mapsto \left(\mathrm{d}\sigma^1(q), \ldots, \mathrm{d}\sigma^k(q)\right)$, where σ^α is the α-th component of σ. Then, $\oplus^k \mathrm{T}^*Q$ is called *the cotangent bundle of k-covelocities of Q*. Moreover, $J^1(Q, \mathbb{R}^k)$ is a k-polysymplectic manifold (see [12] for details).

The following example will illustrate the two-polysymplectic energy-momentum method. Note that $\mathrm{T}^*\mathbb{R}^2 \oplus_{\mathbb{R}^2} \mathrm{T}^*\mathbb{R}^2 \simeq \mathbb{R}^6$, for $x \in \mathbb{R}^2$ is a two-polysymplectic manifold isomorphic to $J^1(\mathbb{R}^2, \mathbb{R}^2)$. Indeed,$(\mathbb{R}^6, \boldsymbol{\omega})$ is a two-polysymplectic manifold relative to

$$\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2 = (\mathrm{d}x_1 \wedge \mathrm{d}x_3 + \mathrm{d}x_2 \wedge \mathrm{d}x_4) \otimes e_1 + (\mathrm{d}x_1 \wedge \mathrm{d}x_5 + \mathrm{d}x_2 \wedge \mathrm{d}x_6) \otimes e_2$$

since

$$\ker \omega^1 = \left\langle \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6} \right\rangle, \quad \ker \omega^2 = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right\rangle,$$

and $\ker \omega^1 \cap \ker \omega^2 = 0$. Let us consider the Lie group action $\Phi : \mathbb{R} \times \mathbb{R}^6 \to \mathbb{R}^6$ given by

$$\Phi : \mathbb{R} \times \mathbb{R}^6 \ni (\lambda; x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_1 + \lambda, x_2 + \lambda, x_3 + \lambda, x_4 + \lambda, x_5 + \lambda, x_6 + \lambda) \in \mathbb{R}^6.$$

The fundamental vector fields associated with the Lie group action Φ are spanned by

$$\xi_P = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}.$$

Note that, the Lie group action Φ is two-polysymplectic since it leaves $\boldsymbol{\omega}$ invariant, namely $\mathscr{L}_{\xi_P}\omega^\alpha = 0$ for $\alpha = 1, 2$. Then, Φ gives rise to a two-polysymplectic momentum map $\mathbf{J}$ given by

$$\mathbf{J}^\Phi : \mathbb{R}^6 \ni (x_1, x_2, x_3, x_4, x_5, x_6) \longmapsto (x_3 + x_4 - x_1 - x_2, x_5 + x_6 - x_1 - x_2) = (\mu_1, \mu_2) = \boldsymbol{\mu} \in \mathbb{R}^2.$$

Therefore, the level set of the two-polysymplectic momentum map $\mathbf{J}^\Phi$ has the following form

$$\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \left\{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 \mid x_3 + x_4 - x_1 - x_2 = \mu_1,\ x_5 + x_6 - x_1 - x_2 = \mu_2\right\}. \tag{10}$$

Note that $\boldsymbol{\mu} = (\mu_1, \mu_2)$ is a weak regular value of a two-polysymplectic momentum map $\mathbf{J}^\Phi$ and the level set $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \simeq \mathbb{R}^4$. Moreover, since a Lie group is a one-dimensional group of translations, we get that $\mathbf{J}^\Phi$ is Ad^{*2}-equivariant. Then,

$$
\begin{aligned}
\mathrm{T}_x(G_{\boldsymbol{\mu}}x) = \mathrm{T}_x(G_{\mu_1}x) = \mathrm{T}_x(G_{\mu_2}x) &= \left\langle \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} \right\rangle , \\
\mathrm{T}_x\mathbf{J}_1^{\Phi-1}(\mu_1) &= \left\langle \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6} \right\rangle , \\
\mathrm{T}_x\mathbf{J}_2^{\Phi-1}(\mu_2) &= \left\langle \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_6} \right\rangle , \\
\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) &= \left\langle \sum_{i=1}^{6} \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right\rangle ,
\end{aligned}
$$

and one can verify that conditions (4) and (5) are fulfilled.

Recall that $\iota_\mu : \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \hookrightarrow P$ is the natural immersion and $\pi_{\boldsymbol{\mu}} : \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \to \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$ is the canonical projection. Then, remembering that our symmetry group consists of translations, Theorem 5.1 yields that the reduced manifold $(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}} \simeq \mathbb{R}^3, \boldsymbol{\omega}_{\boldsymbol{\mu}})$ is a two-polysymplectic manifold with coordinates $(x_1, x_3, x_5) \in \mathbb{R}^3$, where

$$\boldsymbol{\omega}_{\boldsymbol{\mu}} = \omega^1_{\boldsymbol{\mu}} \otimes e_1 + \omega^2_{\boldsymbol{\mu}} \otimes e_2 = 2\mathrm{d}x_1 \wedge \mathrm{d}x_3 \otimes e_1 + 2\mathrm{d}x_1 \wedge \mathrm{d}x_5 \otimes e_2 .$$

Next, let us consider a vector field X, on $P = \mathbb{R}^3$, tangent to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, which in general has the following form

$$X = F_1 \sum_{i=1}^{6} \frac{\partial}{\partial x_i} + F_2 \left(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} \right) + F_3 \left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5} \right) + F_4 \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) ,$$

where $F_i \in \mathscr{C}^\infty(P)$ are G-invariant for $i = 1, \ldots, 4$. Then, one can immediately see that a point $z_e \in P$ is a k-polysymplectic relative equilibrium point of X if and only if $X(z_e) = \xi_P(z_e)$, which holds when $F_1(z_e) = 1$ and $F_2(z_e) = F_3(z_e) = F_4(z_e) = 0$. However, let us verify that we obtain the same result using the k-polysymplectic energy-momentum method.

First, $\mathrm{d}f_1$ and $\mathrm{d}f_2$ read

$$
\begin{aligned}
\mathrm{d}f^1 &= \iota_X \omega^1 = -(F_1 + F_2 + F_3)\,\mathrm{d}x_1 - (F_1 - F_2)\,\mathrm{d}x_2 + (F_1 + F_4)\,\mathrm{d}x_3 + (F_1 + F_3 - F_4)\,\mathrm{d}x_4 , \\
\mathrm{d}f^2 &= \iota_X \omega^2 = -(F_1 + F_3)\,\mathrm{d}x_1 - F_1 \mathrm{d}x_2 + (F_1 + F_4)\,\mathrm{d}x_5 + (F_1 + F_3 - F_4)\,\mathrm{d}x_6 \,.
\end{aligned}
$$

Then, Theorem 6.2 yields that $z_e \in P$ is a two-polysymplectic relative equilibrium point of X if and only if $\mathrm{d}f^1_\xi(z_e) = 0$ and $\mathrm{d}f^2_\xi(z_e) = 0$. Indeed, using (10), one has

$$
\begin{aligned}
\mathrm{d}f^1_\xi = \mathrm{d}f^1 - \mathrm{d}J^1_\xi = &-(F_1 + F_2 + F_3 - \xi)\,\mathrm{d}x_1 - (F_1 - F_2 - \xi)\,\mathrm{d}x_2 \\
&+ (F_1 + F_4 - \xi)\,\mathrm{d}x_3 + (F_1 + F_3 - F_4 - \xi)\,\mathrm{d}x_4 \,,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\mathrm{d}f^2_\xi = \mathrm{d}f^2 - \mathrm{d}J^2_\xi = &-(F_1 + F_3 - \xi)\,\mathrm{d}x_1 - (F_1 - \xi)\,\mathrm{d}x_2 \\
&+ (F_1 + F_4 - \xi)\,\mathrm{d}x_5 + (F_1 + F_3 - F_4 - \xi)\,\mathrm{d}x_6 \,,
\end{aligned} \tag{12}
$$

for $\xi \in \mathfrak{g} = \mathbb{R}$. Since at z_e both (11) and (12) must vanish, one gets that $F_1(z_e) = \xi$ and $F_2(z_e) = F_3(z_e) = F_4(z_e) = 0$. Therefore, $z_e \in P$ is a two-polysymplectic relative equilibrium point.

Finally, let's verify that $\pi_{\boldsymbol{\mu}}(z_e)$ is a critical point of $f^i_{\boldsymbol{\mu}} \in \mathscr{C}^\infty(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}})$. The reduced vector field $X_{\boldsymbol{\mu}}$ has the form

$$X_{\boldsymbol{\mu}} = F_2 \left(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} \right) + F_3 \left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5} \right) + F_4 \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) .$$

Then,

$$\mathrm{d}f^1_{\boldsymbol{\mu}}(\pi_{\boldsymbol{\mu}}(z_e)) = \left(\iota_{X_{\boldsymbol{\mu}}}\omega^1_{\boldsymbol{\mu}}\right)_{\pi_{\boldsymbol{\mu}}(z_e)} = 2\,(F_4(z_e)\mathrm{d}x_3 - (F_2(z_e)+F_3(z_e))\mathrm{d}x_1) = 0\,,$$
$$\mathrm{d}f^2_{\boldsymbol{\mu}}(\pi_{\boldsymbol{\mu}}(z_e)) = \left(\iota_{X_{\boldsymbol{\mu}}}\omega^2_{\boldsymbol{\mu}}\right)_{\pi_{\boldsymbol{\mu}}(z_e)} = 2\,(F_4(z_e)\mathrm{d}x_5 - F_3(z_e)\mathrm{d}x_1) = 0\,,$$

where we denoted by F_2, F_3, F_4 both G-invariant functions on P and functions on the reduced manifold $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$.

7.2 Complex Schwarz derivative

The Schwarz derivative appears in string theory and in the study of linearisation of t-dependent systems. It is related to the t-dependent complex differential equation given by

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v\,,\qquad \frac{\mathrm{d}v}{\mathrm{d}t} = a\,,\qquad \frac{\mathrm{d}a}{\mathrm{d}t} = \frac{3}{2}\frac{a^2}{v} + 2b(t)v\,.$$

Writing it in real coordinates

$$v_1 = \mathfrak{Re}(z)\,,\quad v_2 = \mathfrak{Im}(z)\,,\quad v_3 = \mathfrak{Re}(v)\,,\quad v_4 = \mathfrak{Im}(v)\,,\quad v_5 = \mathfrak{Re}(a)\,,\quad v_6 = \mathfrak{Im}(a)\,,$$

one obtains that the previous system is related to the t-dependent vector field

$$X = X_1 + b_R(t)X_3 + b_I(t)X_4\,,$$

where $b_R(t) = \mathfrak{Re}(b(t))$ and $b_I(t) = \mathfrak{Im}(b(t))$, and

$$X_1 = v_3\frac{\partial}{\partial v_1} + v_4\frac{\partial}{\partial v_2} + v_5\frac{\partial}{\partial v_3} + v_6\frac{\partial}{\partial v_4} + \frac{3}{2}\frac{2v_4v_5v_6 + (v_5^2 - v_6^2)v_3}{v_3^2+v_4^2}\frac{\partial}{\partial v_5} + \frac{3}{2}\frac{2v_3v_5v_6 - v_4(v_5^2-v_6^2)}{v_3^2+v_4^2}\frac{\partial}{\partial v_6}\,,$$
$$X_2 = v_3\frac{\partial}{\partial v_5} + v_4\frac{\partial}{\partial v_6}\,,\qquad X_3 = -v_4\frac{\partial}{\partial v_5} + v_3\frac{\partial}{\partial v_6}\,,$$
$$X_4 = -v_3\frac{\partial}{\partial v_3} - v_4\frac{\partial}{\partial v_4} - 2v_5\frac{\partial}{\partial v_5} - 2v_6\frac{\partial}{\partial v_6}\,,\qquad X_5 = v_4\frac{\partial}{\partial v_3} - v_3\frac{\partial}{\partial v_4} + 2v_6\frac{\partial}{\partial v_5} - 2v_5\frac{\partial}{\partial v_6}\,,$$
$$X_6 = -v_4\frac{\partial}{\partial v_1} + v_3\frac{\partial}{\partial v_2} - v_6\frac{\partial}{\partial v_3} + v_5\frac{\partial}{\partial v_4} - \frac{3}{2}\frac{2v_3v_5v_6 - v_4(v_5^2-v_6^2)}{2(v_3^2+v_4^2)}\frac{\partial}{\partial v_5} + \frac{3}{2}\frac{2v_4v_5v_6 + v_3(v_5^2-v_6^2)}{2(v_3^2+v_4^2)}\frac{\partial}{\partial v_6}\,,$$

satisfy the commutation relations

$$\begin{array}{lllll}
[X_1,X_2] = X_4\,, & [X_1,X_3] = X_5\,, & [X_1,X_4] = X_1\,, & [X_1,X_5] = X_6\,, & [X_1,X_6] = 0\,,\\
 & [X_2,X_3] = 0\,, & [X_2,X_4] = -X_2\,, & [X_2,X_5] = -X_3\,, & [X_2,X_6] = -X_5\,,\\
 & & [X_3,X_4] = -X_3\,, & [X_3,X_5] = X_2\,, & [X_3,X_6] = X_4\,,\\
 & & & [X_4,X_5] = 0\,, & [X_4,X_6] = -X_6\,,\\
 & & & & [X_5,X_6] = X_1\,,
\end{array}$$

which give rise to a Lie algebra isomorphic to $\mathbb{C}\otimes\mathfrak{sl}_2$. Meanwhile, one has the following Lie algebra of symmetries of the problem, $Y_1,\ldots,Y_6$ with

$$Y_1,\ldots,Y_6 =$$

One can choose $\eta_1,\ldots,\eta_6$ are the duals to $Y_1,\ldots,Y_6$. Then, choosing the two forms.

$$\mathrm{d}\eta_1 = \eta_5\wedge\eta_6 + \eta_1\wedge\eta_4\,,\qquad \mathrm{d}\eta_2 = \eta_3\wedge\eta_5 + \eta_4\wedge\eta_2$$

The example follows. Note

7.3 The k-polysymplectic manifold as the product of k symplectic manifolds

This section presents an illustrative example of the k-polysymplectic reduction of a product of k symplectic manifolds.

For simplicity, we will assume some technical conditions. Let $P = P_1 \times \cdots \times P_k$ for some k symplectic manifolds $(P_\alpha, \omega^\alpha)$ where $\alpha = 1, \ldots, k$. If $\mathrm{pr}_\alpha : P \to P_\alpha$ is the canonical projection onto the α-th component, $(P, \sum_{\alpha=1}^k \mathrm{pr}_\alpha^* \omega^\alpha \otimes e_\alpha)$ is a k-polysymplectic manifold. Moreover, assume that a Lie group action $\Phi^\alpha : G_\alpha \times P_\alpha \to P_\alpha$ admits a cosymplectic momentum map $\mathbf{J}^{\Phi^\alpha} : P_\alpha \to \mathfrak{g}_\alpha^*$ for each $\alpha = 1, \ldots, k$ and each Φ^α acts in a quotientable manner on the level sets given by regular values of $\mathbf{J}^{\Phi^\alpha}$.

Then, let us define the Lie group action of $G = G_1 \times \cdots \times G_k$ on P as

$$\Phi : G \times P \ni (g_1, \ldots, g_k, x_1, \ldots, x_k) \longmapsto (\Phi^1_{g_1}(x_1), \ldots, \Phi^k_{g_k}(x_k)) \in P\,.$$

Let $\mathfrak{g} = \mathfrak{g}_1 \times \ldots \times \mathfrak{g}_k$ denotes be the Lie algebra of G. Then, there exists a k-polycosymplectic momentum map

$$\mathbf{J} : P \ni (x_1, \ldots, x_k) \longmapsto (\mathbf{J}^{\Phi_1}(x_1), \ldots, \mathbf{J}^{\Phi_k}(x_k)) \in \mathfrak{g}^*\,,$$

where $\mathfrak{g}^* = \mathfrak{g}_1^* \times \ldots \times \mathfrak{g}_k^*$ is dual space to $\mathfrak{g}$. Suppose, that $\mu^\alpha \in \mathfrak{g}_\alpha^*$ is a regular value of $\mathbf{J}^{\Phi_\alpha} : P_\alpha \to \mathfrak{g}_\alpha^*$ for each $\alpha = 1, \ldots, k$. Hence, $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k) \in \mathfrak{g}^*$ is a regular value of $\mathbf{J}$. Then, Φ acts in a quotientable on the associated level sets of $\mathbf{J}$.

Then, one can check [25, 13] that

$$\ker \mathrm{T}_x \mathbf{J}^{\Phi_\alpha} = \mathrm{T}_x\left(\mathbf{J}^{-1}(\mu)\right) + \ker \omega_x^\alpha,$$
$$\mathrm{T}_x\left(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} x\right) = \bigcap_{\beta=1}^k \left(\ker \omega_x^\beta + \mathrm{T}_x\left(G^{\Delta^\beta}_{\beta\mu^\beta} x\right)\right),$$

for $\alpha = 1, \ldots, k$ and every regular $\boldsymbol{\mu} \in \mathfrak{g}^*$ and $x \in \mathbf{J}^{-1}(\boldsymbol{\mu})$. Recall that, by Theorem 5.1, these equations guarantee that on the reduced space $\mathbf{J}^{-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ there is a k-polysymplectic structure, while

$$\mathbf{J}^{-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} \simeq \mathbf{J}^{\Phi_1 -1}(\mu^1)/G^{\Delta^1}_{1\mu^1} \times \cdots \times \mathbf{J}^{\Phi_k -1}(\mu^k)/G^{\Delta^k}_{k\mu^k}.$$

Next, let us consider a vector field X on P that is G-invariant. By Theorem 5.3 X can be written in the following way

$$X = \sum_{\alpha=1}^k X_\alpha,$$

where X_α is tangent to $\mathbf{J}^{\phi_\alpha -1}(\mu_i)$ for $\alpha = 1, \ldots, k$. Recall, that $\iota_{X_\alpha}\omega^\alpha = \mathrm{d}f^\alpha$. Then,

$$\mathrm{d}\boldsymbol{f} = \sum_{\alpha=1}^k \mathrm{d}f^\alpha \otimes e_\alpha = \sum_{\alpha=1}^k \iota_X \omega^\alpha \otimes e_\alpha\,.$$

Next, one has that $\boldsymbol{f}_\xi$ has the following form $\boldsymbol{f}_\xi = \boldsymbol{f} - \langle \mathbf{J} - \boldsymbol{\mu}_e, \xi\rangle$ and by Theorem 6.2 it follows that $z_e = (z_{1e}, \ldots, z_{ke}) \in P$ is a k-polysymplectic relative equilibrium point if and only if each of $z_{\alpha e}$ is a symplectic relative equilibrium point of a Hamiltonian vector field X_α on a symplectic manifold $(P_\alpha, \omega^\alpha)$, for more details see [15, 38]. Then, a relative equilibrium point z_e is stable if

$$\left(\delta^2 f_\xi^\alpha\right)_{z_e}(v_{z_e}, v_{z_e}) > 0, \qquad \forall v_{z_e} \in \mathrm{T}_{z_e}\mathcal{S}/\left(\mathrm{T}_{z_e}\mathcal{S} \cap \ker \omega_{z_e}^\alpha\right), \quad \alpha = 1, \ldots, k, \tag{13}$$

where $\mathcal{S}$ is a submanifold transversal to $G_{\boldsymbol{\mu}_e} z_e$. Note that, (13) boils down to

$$\left(\delta^2 f_\xi^\alpha\right)_{z_e}(v^\alpha_{z_e}, v^\alpha_{z_e}) > 0, \qquad \forall v^\alpha_{z_e} \in \mathrm{T}_{z_e}\mathcal{S}_\alpha/\left(\mathrm{T}_{z_e}\mathcal{S}_\alpha \cap \ker \omega_{z_e}^\alpha\right), \qquad \alpha = 1, \ldots, k,$$

where S_α is a submanifold of P_α that is transversal to $G_{\alpha\mu_{e\alpha}} z_{\alpha e}$.

7.4 Example

Let us consider the system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sum_{\alpha=1}^{5} b_\alpha(t) X_\alpha \,, \tag{14}$$

where $b_1(t), \ldots, b_5(t)$ are arbitrary t-dependent functions and

$$X_1 = \frac{\partial}{\partial x_1}\,, \quad X_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + x_1^2 \frac{\partial}{\partial x_4} + 2x_1x_2 \frac{\partial}{\partial x_5}\,,$$
$$X_3 = \frac{\partial}{\partial x_3} + 2x_1 \frac{\partial}{\partial x_4} + 2x_2 \frac{\partial}{\partial x_5}\,, \qquad X_4 = \frac{\partial}{\partial x_4}\,, \qquad X_5 = \frac{\partial}{\partial x_5}$$

span a Lie algebra V of vector fields whose non-vanishing commutation relations read

$$[X_1, X_2] = X_3\,, \qquad [X_1, X_3] = 2X_4\,, \qquad [X_2, X_3] = 2X_5\,.$$

Consequently, X is a Lie system as proved in [**?**]. The initial motivation to study (14) comes from the fact that it covers as a particular case the system of differential equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = b_1(t)\,, \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = b_2(t)\,, \qquad \frac{\mathrm{d}x_3}{\mathrm{d}t} = b_2(t)x_1\,, \qquad \frac{\mathrm{d}x_4}{\mathrm{d}t} = b_2(t)x_1^2\,, \qquad \frac{\mathrm{d}x_5}{\mathrm{d}t} = 2b_2(t)x_1x_2\,,$$

where $b_1(t)$ and $b_2(t)$ are arbitrary t-dependent functions and whose interest is due to its relation to certain control problems [**?**, **?**].

Let us show how to consider (14) as a k-symplectic Lie system. Consider the Lie algebra of symmetries of the vector fields $Y_1, \ldots, Y_5$, namely

$$Y_1 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} + 2x_3 \frac{\partial}{\partial x_4} + x_2^2 \frac{\partial}{\partial x_5}\,, \qquad Y_2 = \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial x_5}\,,$$
$$Y_3 = \frac{\partial}{\partial x_3}\,, \qquad Y_4 = \frac{\partial}{\partial x_4}\,, \qquad Y_5 = \frac{\partial}{\partial x_5}$$

admitting the dual forms

$$\Upsilon_1 = \mathrm{d}x_1\,, \qquad \Upsilon_2 = \mathrm{d}x_2\,, \qquad \Upsilon_3 = -x_2\mathrm{d}x_1 + \mathrm{d}x_3\,,$$
$$\Upsilon_4 = -2x_3\mathrm{d}x_1 + \mathrm{d}x_4\,, \qquad \Upsilon_5 = -x_2^2\mathrm{d}x_1 - 2x_3\mathrm{d}x_2 + \mathrm{d}x_5\,.$$

Note that the above dual forms exist because $Y_1 \wedge \cdots \wedge Y_5 \neq 0$. The differential one-forms $\Upsilon_1, \ldots, \Upsilon_5$ are invariant relative to all vector fields $X_1, \ldots, X_5$. Moreover, the differentials of the $\Upsilon_1, \ldots, \Upsilon_5$ depend on the commutation relations of the vector fields $Y_1, \ldots, Y_5$, namely

$$\mathrm{d}\eta^\gamma = c^\gamma_{\alpha\beta}\eta^\alpha \wedge \eta^\beta, \qquad \alpha, \beta, \gamma = 1, \ldots, 5\,,$$

where $c^\gamma_{\alpha\beta}$ are the structure constants of our problem, namely $[X_\alpha, X_\beta] = c^\gamma_{\alpha\beta} X_\gamma$ for $\alpha, \beta, \gamma = 1, \ldots, 5$. In particular, for every differential one-form $\Upsilon \in \langle \Upsilon_1, \ldots, \Upsilon_5 \rangle$, its differential is closed and invariant relative to the vector fields of $X_1, \ldots, X_5$, which become Hamiltonian relative to such presymplectic forms. Using this, it is relatively easy to obtain a k-symplectic structure turning every element of V into a Hamiltonian vector field. Nevertheless, if we want to study the relative equilibrium points of a system of the form (14), the above approach is generally insufficient. In particular, one needs to find a symmetry of the vector fields in V that is also Hamiltonian relative to the k-symplectic structure. Since $\iota_Y\omega^\alpha$ must then be closed, it frequently happens that one has to develop a more general method to study the systems in that way.

Let us focus on a system like (14) for vector fields X_3. Then, $f_1 = x_1x_5 - x_2x_4$ is the first integral of X_3. Moreover, $\mathrm{d}x_3$ is also invariant relative to X_3. Using this and previous relations, one proposes

$$\omega^1 = \Upsilon_3 \wedge \Upsilon_2 + \mathrm{d}f_1 \wedge \mathrm{d}x_3 + \mathrm{d}x_4 \wedge \mathrm{d}x_1, \qquad \omega^2 = \Upsilon_2 \wedge \Upsilon_5 + \Upsilon_1 \wedge \Upsilon_3 + \mathrm{d}f_1 \wedge \mathrm{d}x_3\,.$$

Note that

$$\mathrm{d}\Upsilon_2 = 0\,,\quad \mathrm{d}\Upsilon_1 = 0\,,\quad \mathrm{d}\Upsilon_3 = \Upsilon_1 \wedge \Upsilon_2\,,\quad \mathrm{d}\Upsilon_4 = 2\Upsilon_1 \wedge \Upsilon_3\,,\quad \mathrm{d}\Upsilon_5 = 2\Upsilon_2 \wedge \Upsilon_3\,.$$

Hence, $\mathrm{d}\omega_1 = \mathrm{d}\omega_2 = 0$ and $\ker\omega_1 \cap \ker\omega_2 = 0$. It is worth noting that X_3 is Hamiltonian relative to (ω_1, ω_2) since

$$\iota_{X_3}\omega_1 = \mathrm{d}(x_2 - f_1 + x_1 + x_1^2), \qquad \iota_{X_3}\omega_2 = \mathrm{d}(-x_1 - f_1 - x_2^2)\,.$$

Since

$$\iota_{Y_3}\omega_1 = \mathrm{d}x_2 - \mathrm{d}f_1 = \mathrm{d}(x_2 - x_1x_5 + x_4x_2)\,, \qquad \iota_{Y_3}\omega_2 = -\mathrm{d}x_1 - \mathrm{d}f_1 = \mathrm{d}(-x_1 - x_1x_5 + x_4x_2)\,,$$

the momentum map associated with Y_3 is

$$J : \mathbb{R}^5 \ni x \mapsto (x_2 - x_1x_5 + x_4x_2, -x_1 - x_1x_5 + x_4x_2) \in \mathbb{R}^2\,.$$

Therefore, $J^{-1}(k_1, k_2)$ is given by the points $(x_1(x_3, x_4, x_5, k_1, k_2), x_2(x_3, x_4, x_5, k_1, k_2), x_3, x_4, x_5)$, where $x_1(x_3, x_4, x_5, k_1, k_2), x_2(x_3, x_4, x_5, k_1, k_2)$ are given by the solutions to the system

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} -1 + x_5 & x_4 \\ -x_5 & 1 + x_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\frac{1}{1 + x_4 + x_5} \begin{pmatrix} 1 + x_4 & -x_4 \\ x_5 & 1 + x_5 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}\,.$$

Using that

$$-\mathrm{d}x_1|_{J^{-1}(\mu)} = \mathrm{d}f_1|_{J^{-1}(\mu)}, \quad \mathrm{d}x_2|_{J^{-1}(\mu)} = \mathrm{d}f_1|_{J^{-1}(\mu)}, \quad (1{+}x_4{+}x_5)\mathrm{d}f_1|_{J^{-1}(\mu)} = (x_1\mathrm{d}x_5{-}x_2\mathrm{d}x_4)|_{J^{-1}(\mu)},$$

the restriction of ω_1, ω_2 to $J^{-1}(\mu)$ is given by

$$\omega_1|_{J^{-1}}(\Upsilon_3 \wedge \mathrm{d}f_1 + \mathrm{d}f_1 \wedge \mathrm{d}x_3 - \mathrm{d}x_4 \wedge \mathrm{d}f_1)|_{J^{-1}}$$
$$= ((-x_2\mathrm{d}x_1 + \mathrm{d}x_2) \wedge \mathrm{d}f_1 + \mathrm{d}f_1 \wedge \mathrm{d}x_3 - \mathrm{d}x_4 \wedge \mathrm{d}f_1)|_{J^{-1}}$$

$$\mathrm{d}f_1 \wedge \mathrm{d}x_4$$

$$\begin{aligned} \omega_2|_{J^{-1}} &= (\mathrm{d}f_1 \wedge \Upsilon_5 + \mathrm{d}f_1 \wedge \Upsilon_3 + \mathrm{d}f_1 \wedge \mathrm{d}x_3)\,|_{J^{-1}} \\ &= \mathrm{d}f_1 \wedge (-x_2^2\mathrm{d}x_1 + \mathrm{d}x_5) - \mathrm{d}f_1 \wedge (\mathrm{d}x_3 - x_2\mathrm{d}f_1) + \mathrm{d}f_1 \wedge \mathrm{d}x_3 \\ &= \mathrm{d}f_1 \wedge \mathrm{d}x_5|_{J^{-1}}. \end{aligned}$$

One can use coordinates x_3, x_4, x_5 on $J^{-1}(\mu)$. Moreover, the action of $\mathbb{R}$ on $J^{-1}(\mu)$ removes the coordinate x_3 and we obtain

$$\widetilde{\omega}_1|_{J^{-1}(\mu)} = \frac{1}{\ldots}\mathrm{d}x_4 \wedge \mathrm{d}x_5 \qquad \widetilde{\omega}_2|_{J^{-1}(\mu)} = \frac{1}{\ldots}\mathrm{d}x_4 \wedge \mathrm{d}x_5$$

The projection of the vector field X_3 to the reduction is given by

Hence, one has that the relative equilibrium points are given by

The projection of X_3 onto the quotient space is given by

$$\widetilde{X}_3 = \frac{-1}{1 + x_5 + x_4}\left[2((1 + x_4)k_1 - x_4k_2)\frac{\partial}{\partial x_4} + 2(x_5k_1 + 1 + x_5k_2)\frac{\partial}{\partial x_5}\right].$$

7.5 Quantum harmonic oscillator

Let us analyse a last example based upon the Wei–Norman equations for the automorphic Lie system related to quantum harmonic oscillators [**?**]: Comparing the system X^{do} in intrinsic form (**??**) with the above, we obtain

In this case, the Vessiot–Guldberg Lie algebra is given by the vector fields

$$
\begin{aligned}
X_1^R &= \frac{\partial}{\partial v_1} + v_5\frac{\partial}{\partial v_4} - \frac{1}{2}v_5^2\frac{\partial}{\partial v_6}\,, & X_2^R &= v_1\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2} + \frac{1}{2}v_4\frac{\partial}{\partial v_4} - \frac{1}{2}v_5\frac{\partial}{\partial v_5}\,,\\
X_3^R &= v_1^2\frac{\partial}{\partial v_1} + 2v_1\frac{\partial}{\partial v_2} + e^{v_2}\frac{\partial}{\partial v_3} - v_4\frac{\partial}{\partial v_5} + \frac{1}{2}v_4^2\frac{\partial}{\partial v_6}\,, & X_4^R &= \frac{\partial}{\partial v_4}\,,\\
X_5^R &= \frac{\partial}{\partial v_5} - v_4\frac{\partial}{\partial v_6}\,, & X_6^R &= \frac{\partial}{\partial v_6}\,.
\end{aligned}
$$

The commutation relations between the above vector fields read

$$
\begin{array}{lllll}
[X_1^R, X_2^R] = X_1^R\,, & & & & \\
[X_1^R, X_3^R] = 2\,X_2^R\,, & [X_2^R, X_3^R] = X_3^R\,, & & & \\
[X_1^R, X_4^R] = 0\,, & [X_2^R, X_4^R] = -\frac{1}{2}\,X_4^R\,, & [X_3^R, X_4^R] = X_5^R\,, & & \\
[X_1^R, X_5^R] = -X_4^R\,, & [X_2^R, X_5^R] = \frac{1}{2}\,X_5^R\,, & [X_3^R, X_5^R] = 0\,, & [X_4^R, X_5^R] = -X_6^R\,, & \\
[X_1^R, X_6^R] = 0\,, & [X_2^R, X_6^R] = 0\,, & [X_3^R, X_6^R] = 0\,, & [X_4^R, X_6^R] = 0\,, & [X_5^R, X_6^R] = 0\,.
\end{array}
$$

It is known that the Lie algebra of symmetries of these vector fields is given by

$$
\begin{aligned}
X_1^L &= e^{v_2}\frac{\partial}{\partial v_1} + 2v_3\frac{\partial}{\partial v_2} + v_3^2\frac{\partial}{\partial v_3}\,, \quad X_2^L = \frac{\partial}{\partial v_2} + v_3\frac{\partial}{\partial v_3}\,, \quad X_3^L = \frac{\partial}{\partial v_3}\,,\\
X_4^L &= e^{-v_2/2}(e^{v_2} - v_1v_3)\frac{\partial}{\partial v_4} - e^{-v_2/2}v_3\frac{\partial}{\partial v_5} - e^{-v_2/2}(e^{v_2} - v_1v_3)v_5\frac{\partial}{\partial v_6}\,,\\
X_5^L &= v_1e^{-v_2/2}\frac{\partial}{\partial v_4} + e^{-v_2/2}\frac{\partial}{\partial v_5} - v_1v_5e^{-v_2/2}\frac{\partial}{\partial v_6}\,, \quad X_6^L = \frac{\partial}{\partial v_6}\,.
\end{aligned}
$$

In particular, let us focus on

$$X_5^R = \frac{\partial}{\partial v_5} - v_4\frac{\partial}{\partial v_6}.$$

Then, a symmetry of our system is given by

$$Y = \frac{\partial}{\partial v_5}.$$

The 2-polysymplectic structure on $\mathbb{R}^6$ can be defined in the following way

$$
\begin{aligned}
\boldsymbol{\omega} &= \omega^1 \otimes e_1 + \omega^2 \otimes e_2\\
&= (\mathrm{d}v_1 \wedge \mathrm{d}v_3 + \mathrm{d}v_2 \wedge \mathrm{d}v_4 + \mathrm{d}v_5 \wedge \mathrm{d}v_1 + \mathrm{d}v_4 \wedge \mathrm{d}v_6) \otimes e_1 + (\mathrm{d}v_4 \wedge \mathrm{d}v_6 - \mathrm{d}v_3 \wedge \mathrm{d}v_5) \otimes e_2
\end{aligned}
$$

Note that

$$\ker\omega^1 = \left\langle \frac{\partial}{\partial v_3} + \frac{\partial}{\partial v_5}, \frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_6} \right\rangle\,, \qquad \ker\omega^2 = \left\langle \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2} \right\rangle,$$

hence $\ker\omega_x^1 \cap \omega_x^2 = 0$ and indeed $(\mathbb{R}^6, \boldsymbol{\omega})$ is a 2-polysymplectic manifold. The vector field Y is also a symmetry of the 2-polysymplectic structure, i.e. $\mathscr{L}_Y = 0$. Then,

$$\iota_{Y_3}\omega^1 = \mathrm{d}v_1\,, \qquad \iota_{Y_3}\omega^2 = \mathrm{d}v_3,$$

and the momentum map $\mathbf{J}^\Phi$ reads

$$\mathbf{J}^\Phi : \mathbb{R}^6 \ni x \longmapsto (v_1, v_3) = \boldsymbol{\mu} \in (\mathfrak{g}^*)^2 \simeq \mathbb{R}^2.$$

Note that $\boldsymbol{\mu}$ is a regular value of $\mathbf{J}^\Phi$, hence $\mathbf{J}$

The vector field X_5^R is $\boldsymbol{\omega}$-Hamiltonian with

$$\iota_X\boldsymbol{\omega} = \iota_{X_5^R}\omega^1 \otimes e_1 + \iota_X\omega^2 \otimes e_2 = \mathrm{d}\left(v_1 + \frac{v_4^2}{2}\right) \otimes e_1 + \mathrm{d}\left(v_3 + \frac{v_4^2}{2}\right) \otimes e_2$$

If we restrict ourselves to the submanifold given by the v_1, v_3 being a constant, then

$$\omega^1|_J = \mathrm{d}v_2 \wedge \mathrm{d}v_4 + \mathrm{d}v_4 \wedge \mathrm{d}v_6\,, \qquad \omega^2|_J = \mathrm{d}v_4 \wedge \mathrm{d}v_6\,.$$

We have

$$\ker\omega^1|_J = \left\langle \frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_6}, \frac{\partial}{\partial v_5} \right\rangle, \qquad \ker\omega^2|_J = \left\langle \frac{\partial}{\partial v_2}, \frac{\partial}{\partial v_5} \right\rangle,$$

and then they do not define a 2-polysymplectic structure.

7.6 Example

Consider the vector field X on $\mathbb{R}^8 \ni (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$, given by

$$X = x_6^a \frac{\partial}{\partial x_2} + x_4^b \frac{\partial}{\partial x_3} - x_3^c \frac{\partial}{\partial x_4} + x_8^d \frac{\partial}{\partial x_7} - x_7^e \frac{\partial}{\partial x_8},$$

where $a, b, c, d, e \in \mathbb{N}$. On $\mathbb{R}^8$ the 2-polysymplectic can be defined as

$$\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2 = (dx_3 \wedge dx_4 + dx_1 \wedge dx_5) \otimes e_1 + (dx_2 \wedge dx_6 + dx_7 \wedge dx_8) \otimes e_2.$$

Then, $\ker\omega_x^1 = \langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \rangle$ and $\ker\omega_x^2 = \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5} \rangle$, hence $\ker\omega_x^1 \cap \ker\omega_x^2 = 0$ for any $x \in \mathbb{R}^8$.

The vector field X admits symmetries, $Y_1 = \frac{\partial}{\partial x_1}$, $Y_2 = \frac{\partial}{\partial x_2}$, $Y_3 = \frac{\partial}{\partial x_5}$. These symmetries correspond to translations along the x_1, x_2, and x_5 coordinates, and they also leave the 2-polysymplectic structure invariant, i.e. $\mathscr{L}_{Y_i}\omega^j = 0$ for $i = 1, 2, 3$ and $j = 1, 2$.

The momentum maps are computed as follows

$$\begin{aligned} &\iota_{Y_1}\omega^1 = 0, \quad \iota_{Y_2}\omega^1 = dx_5, \quad \iota_{Y_3}\omega^1 = -dx_1 \\ &\iota_{Y_1}\omega^2 = dx_6, \quad \iota_{Y_2}\omega^2 = 0, \quad \iota_{Y_3}\omega^2 = 0. \end{aligned}$$

Therefore, the momentum map $\mathbf{J}^\Phi$ is given by

$$\mathbf{J}^\Phi : \mathbb{R}^8 \ni x \mapsto \mathbf{J}^\Phi(x) = \boldsymbol{\mu} = (0, x_5, -x_1; x_6, 0, 0) \in (\mathfrak{g}^*)^2 = (\mathbb{R}^3)^2.$$

Then, $\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \rangle$ for $x \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. The momentum map $\mathbf{J}^\Phi$ is $(\mathrm{Ad}^*)^2$-equivariant and $\boldsymbol{\mu} \in (\mathbb{R}^3)^2$ is a weakly regular value of $\mathbf{J}^\Phi$ since $\mathbf{J}^\Phi(\boldsymbol{\mu}) \simeq \mathbb{R}^5$ is a submanifold of $\mathbb{R}^8$.

Note, that Y_1 and Y_3 do not belong to $\mathrm{T}_x(G_{\boldsymbol{\mu}}x)$ but Y_1 does. The assumptions of Theorem 5.1 are satisfied, and the quotient space $\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_x(G_{\boldsymbol{\mu}}x)$ is a 2-polysymplectic manifold, where

$$\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_x(G_{\boldsymbol{\mu}}x) = \langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \rangle,$$

and

$$\boldsymbol{\omega}_{\boldsymbol{\mu}} = \omega_{\boldsymbol{\mu}}^1 \otimes e_1 + \omega_{\boldsymbol{\mu}}^2 \otimes e_2 = (dx_3 \wedge dx_4) \otimes e_1 + (dx_7 \wedge dx_8) \otimes e_2.$$

The vector field X is $\boldsymbol{\omega}$-Hamiltonian, with

$$\begin{aligned} \mathrm{d}\boldsymbol{f} &= \iota_X\boldsymbol{\omega} = \iota_X\omega^1 \otimes e_1 + \iota_X\omega^2 \otimes e_2 \\ &= \mathrm{d}\left(\frac{1}{1+b}x_4^{b+1} + \frac{1}{c+1}x_3^{c+1}\right) \otimes e_1 + \mathrm{d}\left(\frac{1}{1+a}x_6^{a+1} + \frac{1}{d+1}x_8^{d+1} + \frac{1}{1+e}x_7^{e+1}\right) \otimes e_2. \end{aligned}$$

Then, by Theorem 5.3 the vector field X project onto the quotient manifold and its projection X_μ is given by

$$X_{\boldsymbol{\mu}} = x_4^b \frac{\partial}{\partial x_3} - x_3^c \frac{\partial}{\partial x_4} + x_8^d \frac{\partial}{\partial x_7} - x_7^e \frac{\partial}{\partial x_8}.$$

The vector field $X_{\boldsymbol{\mu}}$ is $\boldsymbol{\omega_\mu}$-Hamiltonian vector field, with

$$\mathrm{d}\boldsymbol{f_\mu} = \iota_{X_\mu}\boldsymbol{\omega_\mu} = \mathrm{d}\left(\frac{1}{1+b}x_4^{b+1} + \frac{1}{c+1}x_3^{c+1}\right) \otimes e_1 + \mathrm{d}\left(\frac{1}{d+1}x_8^{d+1} + \frac{1}{1+e}x_7^{e+1}\right) \otimes e_2.$$

According to Theorem 6.2, a point z_e is a k-polysymplectic relative equilibrium point if it is a critical point of $\boldsymbol{f}_\xi$ for $\xi \in \mathfrak{g} \simeq \mathbb{R}$. Then, one has that

$$\mathrm{d}\boldsymbol{f}_\xi = \mathrm{d}f_\xi^1 \otimes e_1 + \mathrm{d}f_\xi^2 \otimes e_2 = \left(x_4^b \mathrm{d}x_4 + x_3^c dx_3\right) \otimes e_1 + \left((x_6^a - 1)dx_6 + x_8^d + x_7^e\right) \otimes e_2.$$

From this, it follows that points $z_e^{\pm} = (x_1, x_2, 0, 0, x_5, \pm 1, 0, 0)$ are 2-polysymplectic relative equilibrium points of X.

To analyse the stability of these 2-polysymplectic relative equilibrium points, let's look at the second derivative of $\boldsymbol{f}_\xi$. Then,

$$\begin{aligned}\delta^2 \boldsymbol{f}_\xi = \delta^2 f_\xi^1 \otimes e_1 + \delta^2 f_\xi^2 \otimes e_2 = &\left(c x_3^{c_1} dx_3 \otimes dx_3 + b x_4^{b-1} dx_4 \otimes dx_4\right) \otimes e_1 \\ &+ \left(e x_7^{e-1} dx_7 \otimes dx_7 + d x_8^{d-1} dx_8 \otimes dx_8\right) \otimes e_2.\end{aligned}$$

Taking into account that $T_{z_e}\mathcal{S}/T_{z_e}\mathcal{S} \cap \ker \omega_x^1 = \langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \rangle$ and $T_{z_e}\mathcal{S}/T_{z_e}\mathcal{S} \cap \ker \omega_x^2 = \langle \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \rangle$, Theorem 6.5 gives that $z_e^{\pm}$ is a stable k-polysymplectic relative equilibrium point if

$$(\delta^2 f_\xi^1)_{z_e}(v_{z_e}, v_{z_e}) > 0, \quad \forall v_{z_e} \in \mathrm{T}_{z_e}\mathcal{S}/\left(\mathrm{T}_{z_e}\mathcal{S} \cap \ker \omega_{z_e}^1\right),$$

and

$$(\delta^2 f_\xi^2)_{z_e}(v_{z_e}, v_{z_e}) > 0, \quad \forall v_{z_e} \in \mathrm{T}_{z_e}\mathcal{S}/\left(\mathrm{T}_{z_e}\mathcal{S} \cap \ker \omega_{z_e}^2\right).$$

These inequalities hold if and only if $b, c, d, e = 1$. Hence, $z_e^{\pm}$ are stable k-symplectic relative equilibrium points of X. Indeed, $(X_{\boldsymbol{\mu}_e})_{[z_e]} = 0$.

7.7 Another example

Consider the vector field X on $\mathbb{R}^5 \ni (x_1, x_2, x_3, x_4, x_5)$ given by

$$X = \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4}.$$

On $\mathbb{R}^5$ we can define 2-polysymplectic structure as follows

$$\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2 = dx^2 \wedge dx^1 + dx^3 \wedge dx^4 \otimes e_1 + dx^5 \wedge dx^2 + dx^3 \wedge dx^4 \otimes e_3.$$

The Lie group action $\Phi : \mathbb{R} \times \mathbb{R}^5 \to \mathbb{R}^5$ defined as

$$\Phi : (\lambda; x_1, x_2, x_3, x_4, x_5) \longmapsto (x_1, x_2 + \lambda, x_3, x_4, x_5),$$

is a 2-polysymplectic Lie group action. The fundamental vector field associated with Φ is given by $\xi_{\mathbb{R}^5} = \frac{\partial}{\partial x_2}$. Moreover, Φ is a symmetry of the vector field X since $[X, \xi_{\mathbb{R}^5}] = 0$. Then, one has

$$\iota_{\xi_{\mathbb{R}^5}} \omega^1 = dx_1, \qquad \iota_{\xi_{\mathbb{R}^5}} \omega^2 = dx_5.$$

Thus, the momentum map reads

$$\mathbf{J}^{\Phi}: x \in \mathbb{R}^5 \mapsto (x_1, x_5) = \boldsymbol{\mu} \in (\mathfrak{g}^*)^2 \simeq \mathbb{R}^2.$$

Note that, $\mathbf{J}^{\Phi}$ is $(\mathrm{Ad}^*)^2$-equivariant. Since $\boldsymbol{\mu}$ is a regular value of $\mathbf{J}^{\Phi}$, the level set of $\mathbf{J}^{\Phi}$ is a submanifold of P. Therefore,

$$T_x \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \rangle,$$

and

$$T_x \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) / T_x\left(G_{\boldsymbol{\mu}} x\right) = \langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \rangle.$$

Then, according to Theorem 5.1 the reduced manifold $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$ is a 2-polysymplectic manifold with $\omega^1_{\boldsymbol{\mu}} = \omega^2_{\boldsymbol{\mu}} = dx_3 \wedge dx_4$.

The vector field X is a $\boldsymbol{\omega}$-Hamiltonian vector field with

$$\iota_X \boldsymbol{\omega} = \iota_X \omega^1 \otimes e_1 + \iota_X \omega^2 \otimes e_2 = d\left(x_1 + \frac{1}{2}(x_3^2 + x_4^2)\right) \otimes e_1 + d\left(-x_5 + \frac{1}{2}(x_3^2 + x_4^2)\right) \otimes e_2.$$

By Theorem 5.3 X reduces onto the quotient manifold. Its reduction, $X_{\boldsymbol{\mu}}$, is given by

$$X_{\boldsymbol{\mu}} = x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4}.$$

Theorem 6.2 gives that $z_e \in \mathbb{R}^5$ is a 2-polysymplectic relative equilibrium of X if z_e is a critical point of $\boldsymbol{f}_\xi$. Namely,

$$d\boldsymbol{f}_\xi = df^1_\xi \otimes e_1 + df^2_\xi \otimes e_2 = (x_3 dx_3 + x_4 dx_4) \otimes e_1 + (x_3 dx_3 + x_4 dx_4) \otimes e_2.$$

Therefore, $z_e = (x_1, x_2, 0, 0, x_5)$ is 2-polysymplectic relative equilibrium points of X. Indeed, $X_{\boldsymbol{\mu}}(\pi_{\boldsymbol{\mu}}(z_e)) = 0$.

Now, let us study the stability of $z_e \in \mathbb{R}^5$. To do so, we need to compute the second derivative of $\boldsymbol{f}_\xi$. Then,

$$\delta^2 \boldsymbol{f}_\xi = \delta^2 f^1_\xi \otimes e_1 + \delta^2 f^2_\xi \otimes e_2 = (dx_3 \otimes dx_3 + dx_4 \otimes dx_4) \otimes e_1 + (dx_3 \otimes dx_3 + dx_4 \otimes dx_4) \otimes e_2.$$

Since, $T_{z_e}\mathcal{S}/\left(T_{z_e}\mathcal{S} \cap \ker \omega^1_x\right) = T_{z_e}\mathcal{S}/\left(T_{z_e}\mathcal{S} \cap \ker \omega^2_x\right) = \langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \rangle$, by Theorem 6.5, we conclude that z_e is a stable 2-polysymplectic relative equilibrium point of X since $\delta^2 f^1_\xi$ and $\delta^2 f^2_\xi$ are positive definite.

8 Conclusions and Outlook

In the present work we have devised a new energy-momentum method for systems of ordinary differential equations with an underlying k-polysymplectic structure. In order to do so, we have also performed several improvements to previous Marsden–Weinstein reductions for k-polysymplectic systems [17, 25]. In order to illustrate this new energy-momentum method, we have studied several relevant examples are studied in detail, including the cotangent bundle of k-velocities, the complex Schwarz equation, the product of several symplectic manifolds, a control system, and the quantum harmonic oscillator.

A non-autonomous analogue of the methods devised in this paper can be accomplished by using the Lyapunov theory depicted in [15].

Acknowledgements

J. de Lucas, X. Rivas and B.M. Zawora acknowledge financial support from the Novee Idee 2B-POB II project PSP: 501-D111-20-2004310 funded by the "Inicjatywa Doskonałości - Uczelnia Badawcza" (IDUB) program. X. Rivas acknowledges financial support from the Ministerio de Ciencia, Innovación y Universidades (Spain) project D2021-125515NB-21. J. de Lucas acknowledges a CRM-Simons professorship funded by the Simons Foundation and the Centre de Recherches Mathématiques (CRM) of the Université Montreal. X. Rivas acknowledges funding from J. de Lucas's CRM-Simons professorship to accomplish a stay at the CRM, which allowed him to complete some of the main parts of the present work. B.M. Zawora achknowledges funding from and IDUB mikrogrant to accomplish a research stay at the CRM.

Acknowledgements

References

[1] H. Abarbanel and D. Holm. Nonlinear stability analysis of inviscid flows in three dimensions: incompressible fluids and barotropic fluids. *Phys. Fluids*, **30**(11):3369–3382, 1987. 10.1063/1.866469. 2

[2] R. Abraham and J. E. Marsden. *Foundations of mechanics*, volume 364 of *AMS Chelsea publishing*. Benjamin/Cummings Pub. Co., New York, 2nd edition, 1978. 10.1090/chel/364. 15

[3] A. Awane. k-symplectic structures. *J. Math. Phys.*, **33**(12):4046, 1992. 10.1063/1.529855. 2, 3, 4, 6

[4] A. Awane and M. Goze. *Pfaffian systems, k-symplectic systems*. Springer, Dordrecht, 1st edition, 2000. 10.1007/978-94-015-9526-1. 2

[5] C. Blacker. Polysymplectic reduction and the moduli space of flat connections. *J. Phys. A: Math. Theor.*, **52**(33):335201, 2019. 10.1088/1751-8121/ab2eed. 3, 9

[6] L. Búa, I. Bucataru, M. de León, M. Salgado, and S. Vilariño. Symmetries in Lagrangian field theory. *Rep. Math. Phys.*, **75**(3):333–357, 2015. 10.1016/S0034-4877(15)30010-0. 2

[7] J. F. Cariñena, J. Grabowski, and G. Marmo. *Lie–Scheffers systems: a geometric approach*. Napoli Series on Physics and Astrophysics. Bibliopolis, Naples, 2000. 3

[8] J. F. Cariñena, J. Grabowski, and G. Marmo. Superposition rules, lie theorem, and partial differential equations. *Rep. Math. Phys.*, **60**(2):237–258, 2007. 10.1016/S0034-4877(07)80137-6. 3

[9] M. de León, E. Merino, J. A. Oubiña, and M. Salgado. Stable almost cotangent structures. *Bolletino Unione Mat. Ital. B (7)*, **11**(3):509–529, 1997. 2

[10] M. de León, I. Méndez, and M. Salgado. p-almost tangent structures. *Rend. Circ. Mat. Palermo*, **37**(2):282–294, 1988. 10.1007/BF02844526. 2

[11] M. de León, I. Méndez, and M. Salgado. Regular p-almost cotangent structures. *J. Korean Math. Soc.*, **25**(2):273–287, 1988. `https://jkms.kms.or.kr/journal/view.html?spage=273&volume=25&number=2`. 2

[12] M. de León, M. Salgado, and S. Vilariño. *Methods of Differential Geometry in Classical Field Theories*. World Scientific, 2015. 10.1142/9693. 3, 5, 17

[13] J. de Lucas, X. Rivas, S. Vilariño, and B. M. Zawora. On k-polycosymplectic Marsden–Weinstein reductions. *J. Geom. Phys.*, **191**:104899, 2023. 10.1016/j.geomphys.2023.104899. 3, 7, 9, 13, 14, 20

[14] J. de Lucas and S. Vilariño. k-symplectic Lie systems: theory and applications. *J. Differ. Equ.*, **258**(6):2221–2255, 2015. 10.1016/j.jde.2014.12.005. 3, 6, 7

[15] J. de Lucas and B. M. Zawora. A time-dependent energy-momentum method. *J. Geom. Phys.*, **170**:104364, 2021. 10.1016/j.geomphys.2021.104364. 4, 20, 26

[16] A. Echeverría-Enríquez, M. C. Muñoz-Lecanda, and N. Román-Roy. Geometry of Lagrangian first-order classical field theories. *Fortschr. Phys.*, **44**(3):235–280, 1996. 10.1002/prop.2190440304. 2

[17] E. García-Toraño Andrés and T. Mestdag. Conditions for symmetry reduction of polysymplectic and polycosymplectic structures. *J. Phys. A: Math. Theor.*, **56**(33):335202, 2023. 10.1088/1751-8121/ace74c. 3, 9, 10, 26

[18] J. Gaset, A. López-Gordón, and X. Rivas. Symmetries, conservation and dissipation in time-dependent contact systems. *Fortschr. Phys.*, **71**(8–9):2300048, 2023. 10.1002/prop.202300048. 11

[19] G. Giachetta, L. Mangiarotti, and G. A. Sardanashvily. *New Lagrangian and Hamiltonian Methods in Field Theory*. World Scientific, River Edge, 1997. 10.1142/2199. 2

[20] C. Günther. The polysymplectic Hamiltonian formalism in field theory and calculus of variations I: The local case. *J. Differ. Geom.*, **25**(1):23–53, 1987. 10.4310/jdg/1214440723. 2, 3, 4

[21] B. M. Z. J. de Lucas, A. Maskalaniec. A cosymplectic energy-momentum method with applications. 2302.05827, 2023. 15

[22] I. V. Kanatchikov. Canonical structure of classical field theory in the polymomentum phase space. *Rep. Math. Phys.*, **41**(1):49–90, 1998. 10.1016/S0034-4877(98)80182-1. 2

[23] J. M. Lee. *Manifolds and differential geometry*, volume 107 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009. 4

[24] J. M. Lee. *Introduction to Smooth Manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer New York Heidelberg Dordrecht London, 2nd edition, 2012. 10.1007/978-1-4419-9982-5. 10

[25] J. C. Marrero, N. Román-Roy, M. Salgado, and S. Vilariño. Reduction of polysymplectic manifolds. *J. Phys. A: Math. Theor.*, **48**(5):055206, 2015. 10.1088/1751-8113/48/5/055206. 3, 4, 5, 7, 9, 10, 11, 12, 20, 26

[26] J. E. Marsden, T. A. Posbergh, and J. C. Simo. Stability of coupled rigid body and geometrically exact rods: block diagonalization and the energy–momentum method. *Phys. Rep.*, **193**:280–360, 1990. 10.1016/0370-1573(90)90125-L. 2

[27] J. E. Marsden and T. Ratiu. *Introduction to Mechanics and Symmetry. A basic exposition of classical mechanical systems*, volume 17 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1999. 10.1007/978-0-387-21792-5. 2

[28] J. E. Marsden and J. C. Simo. The energy momentum method. *Act. Acad. Sci. Tau.*, **1**(124):245–268, 1988. https://resolver.caltech.edu/CaltechAUTHORS:20101019-093814651. 2

[29] J. E. Marsden, J. C. Simo, D. Lewis, and T. Posbergh. A block diagonalization theorem in the energy-momentum method. In *Dynamics and control of multibody systems*, volume 97 of *Contemp. Math.*, pages 297–313. Amer. Math. Soc., Providence, RI, 1989. 10.1090/conm/097/1021043. 2

[30] M. McLean and L. K. Norris. Covariant field theory on frame bundles of fibered manifolds. *J. Math. Phys.*, **41**(10):6808–6823, 2000. 10.1063/1.1288797. 2

[31] E. E. Merino. *Geometría k-simpléctica y k-cosimpléctica. Aplicaciones a las teorías clásicas de campos.* PhD thesis, Universidade de Santiago de Compostela, 1997. 6

[32] L. K. Norris. Generalized symplectic geometry on the frame bundle of a manifold. In *Proc. Symp. Pure Math.*, volume 54.2, pages 435–465. Amer. Math. Soc., Providence RI, 1993. 10.1090/pspum/054.2. 2

[33] J. Ortega, V. Planas-Bielsa, and T. Ratiu. Asymptotic and Lyapunov stability of constrained and Poisson equilibria. *J. Differ. Equ.*, **214**(1):92–127, 2005. 10.1016/j.jde.2004.09.016. 2

[34] N. Román-Roy, M. Salgado, and S. Vilariño. Symmetries and conservation laws in the Günther k-symplectic formalism of field theory. *Rev. Math. Phys.*, **19**(10):1117–1147, 2007. 10.1142/S0129055X07003188. 2, 4

[35] G. A. Sardanashvily. *Generalized Hamiltonian formalism for field theory. Constraint systems.* World Scientific Publishing Company, River Edge, NJ, 1995. 10.1142/2550. 2

[36] J. Simo, D. Lewis, and J. Marsden. Stability of relative equilibria. Part I: The reduced energy-momentum method. *Arch. Rat. Mech. Anal.*, **115**:15–59, 1991. 10.1007/BF01881678. 2

[37] J. Simo and N. Tarnow. The discrete energy-momentum method. Conserving algorithms for nonlinear elastodynamics. *Z. Angew. Math. Phys.*, **43**:757–792, 1992. 10.1007/BF00913408. 2

[38] J. Simon. On the integrability of representations of finite dimensional real Lie algebras. *Comm. Math. Phys.*, **28**:39–46, 1972. 10.1007/BF02099370. 20

[39] L. Wang and P. Krishnaprasad. Gyroscopic control and stabilization. *J. Nonlinear Sci.*, **2**:367–415, 1992. 10.1007/BF01209527. 2

[40] P. Winternitz. Lie groups and solutions of nonlinear differential equations. In K. B. Wolf, editor, *Nonlinear Phenomena*, volume 189 of *Lecture Notes in Physics*, pages 263–305. Springer, Berlin, Heidelberg, 1983. 10.1007/3-540-12730-5_12. 3

[41] B. M. Zawora. A time-dependent energy-momentum method. Master's thesis, University of Warsaw, Faculty of Physics, 2021. 4

[42] D. V. Zenkov, A. M. Bloch, and J. E. Marsden. The energy-momentum method for the stability of non-holonomic systems. *Dyn. Stab. Syst.*, **13**(2):123–165, 1998. 10.1080/02681119808806257. 2