Morse Families and Lagrangian Submanifolds

Marco Castrillón López and Tudor S. Ratiu

Dedicated to Jaime Muñoz Masqué on the occasion of his 65th birthday.

Abstract This short note presents a comprehensive and pedagogical study of the results in [6] on Lagrangian submanifolds in cotangent bundles T^*X defined by Morse families $S : B \to \mathbb{R}$ for arbitrary submersions $B \to X$.

Keywords Symplectic manifold · Lagrangian submanifold · Morse family

1 Introduction

Lagrangian submanifolds play an essential role in the study of symplectic manifolds, either from a pure mathematical point of view or, in geometric mechanics, when applied to the Hamiltonian formulation of the equations of motion. With respect to the latter, Lagrangian submanifolds naturally appear, for example, as singularities

M. Castrillón López (✉)
ICMAT(CSIC-UAM-UC3M-UCM), Departamento de Geometría y Topología,
Facultad de CC. Matemáticas, Universidad Complutense de Madrid,
28040 Madrid, Spain
e-mail: mcastri@mat.ucm.es

T.S. Ratiu
Department of Mathematics, Shanghai Jiao Tong University,
800 Dongchuan Road, Minhang District, 200240 Shanghai, China
e-mail: ratiu@sjtu.edu.cn

T.S. Ratiu
Séction de Mathématiques, École Polythechnique Fédérale de Lausanne,
1015 Lausanne, Switzerland
e-mail: tudor.ratiu@epfl.ch

M. Castrillón López et al. (eds.), *Geometry, Algebra and Applications: From Mechanics to Cryptography*, Springer Proceedings in Mathematics & Statistics 161, DOI 10.1007/978-3-319-32085-4_6

in ray optics or the level sets of the functions in involution for integrable systems in the Liouville sense. With respect to the former, there are many concepts that can be regarded as Lagrangian submanifolds of cotangent bundles endowed with the canonical symplectic form, such as symplectomorphisms or closed 1-forms, identified with their graphs. From this point of view, the specific case of exact 1-forms is particularly interesting and is exploited in the geometric formulation of the Hamilton-Jacobi theory.

Recent renewed interest in the Cotangent Bundle Reduction Theorem and its applications to mechanical systems (see, e.g., [1, Theorem 4.3.3 and Sect. 4.5]) and the structure of coadjoint orbits (see, e.g., [4]), a reduction theorem for Hamilton-Jacobi theory by a group of point transformations under certain invariance hypotheses (see [4]), and our own attempts to find a general reduction theory for the Hamilton-Jacobi equations, led us to review some very interesting old ideas of Alan Weinstein presented in his well-known lectures [6]. The most general Cotangent Bundle Reduction Theorem can be found in Lecture 6 of these notes; it is a vast generalization of the usual Cotangent Bundle Reduction Theorem at the zero value of the momentum map. He also describes there families $S : N \times X \to \mathbb{R}$ of functions, where N is a manifold of labels, when studying Lagrangian submanifolds of T^*X. In fact, the setup is completely general and is given by a function S is defined on a manifold B and an arbitrary submersion $B \to X$. Under suitable topological conditions, he calls these functions S Morse families and studies their properties. Undoubtedly, the results of [6] have been the inspiring source for many subsequent works and are a classical contribution to symplectic geometry.

The goal of this short note is to give a comprehensive and pedagogical presentation of the results in [6, Lecture 6] concerning reduction by a coisotropic regular foliation and Morse families. We believe that a more elaborate, complete, and self-contained presentation of these, apparently forgotten, ideas and proofs in [6], is helpful and may turn out to be crucial for future investigations in Lagrangian submanifolds and, in particular, the Hamilton-Jacobi theory in the context of reduction by a group of symmetries.

Notations and conventions. Unless otherwise indicated, all objects are smooth. The Einstein summation convention on repeated sub- and super-indices is used. If $E \to Q$ is a vector bundle over the smooth manifold Q and $E^* \to Q$ its dual, $\langle \cdot, \cdot \rangle : E^* \times E \to \mathbb{R}$ denotes the standard fiberwise duality pairing. Given a smooth manifold Q, $\tau_Q : TQ \to Q$ and $\pi_Q : T^*Q \to Q$ denote its tangent and cotangent bundles. If $(q^1, \ldots, q^n)$ are local coordinates on Q, the naturally induced coordinates on TQ and T^*Q are denoted by $(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$ and $(q^1, \ldots, q^n, p_1, \ldots, p_n)$, respectively, i.e., any tangent vector $v_q \in T_qQ$ is written locally as $v_q = \dot{q}^i \frac{\partial}{\partial q^i}$ and any covector $\alpha_q \in T_q^*Q$ is written locally as $\alpha_q = p_i dq^i$. If Q and P are manifolds and $f : Q \to P$ a smooth map, $Tf : TQ \to TP$ denotes its tangent map, or derivative.

The canonical, or Liouville, one-form θ_Q on T^*Q is defined by $\theta_Q(\alpha_q)\left(V_{\alpha_q}\right) := \left\langle \alpha_q, T_{\alpha_q}\pi_Q\left(V_{\alpha_q}\right)\right\rangle$ for any $q \in Q$, $\alpha_q \in T_q^*Q$, and $V_{\alpha_q} \in T_{\alpha_q}(T^*Q)$. The canonical symplectic two-form on T^*Q is defined by $\omega_Q := -\mathbf{d}\theta_Q$, were $\mathbf{d}$ denotes the exterior derivative. In standard cotangent bundle coordinates, we have $\theta_Q = p_i dq^i$ and $\omega_Q = dq^i \wedge dp_i$, where $\wedge$ is the exterior product on forms (with Bourbaki conventions).

2 Coisotropic Reduction of Conormal Bundles

We recall standard terminology from symplectic geometry. If (P, ω) is a symplectic manifold (i.e., the two-form ω on P is closed and non-degenerate) and $E \subset TP$ is a vector subbundle, its *ω-orthogonal vector subbundle* is defined by $E^\perp := \left\{ v \in T_pP \mid \omega(p)(u, v) = 0,\ \forall u \in E_p,\ \forall p \in P \right\}$. The vector subbundle $E \subset TP$ is called *isotropic* (*coisotropic*), if $E \subseteq E^\perp$ ($E \supseteq E^\perp$). The vector subbundle E is called *Lagrangian*, if it is isotropic and has an isotropic complementary vector subbundle F, i.e., $F \subseteq F^\perp$ and $TP = E \oplus F$. Thus, E is Lagrangian if and only if $E = E^\perp$ if and only if E is isotropic and its rank is half the dimension of P. In addition, its isotropic complement F is actually Lagrangian. The vector subbundle E is *symplectic*, if ω restricted to $E \times E$ is nondegenerate. Thus, E is symplectic if and only if $E \oplus E^\perp = TP$. The same terminology is used for vector subbundles of TP restricted to a submanifold of P.

A submanifold $M \subset P$ is called *isotropic* (*coisotropic, Lagrangian, symplectic*) if its tangent bundle is isotropic (coisotropic, Lagrangian, symplectic) in the restriction $(TP)_M$ of the tangent bundle TP to M. For example, a submanifold M is isotropic if and only if $\iota^*\omega = 0$, where $\iota : M \hookrightarrow P$ is the inclusion. A submanifold M of P is Lagrangian if and only if $\iota^*\omega = 0$ and $\dim M = \frac{1}{2}\dim P$.

Let X be a manifold and $\pi : B \to X$ a smooth submersion (possibly a fiber bundle). Since π is a submersion, it is open and hence $\pi(B)$ is an open subset of X. Let $\pi_B : T^*B \to B$ and $\pi_X : T^*X \to X$ be the cotangent bundle projections. The *conormal bundle to the fibers* $\pi_B|_{N_\pi} : N_\pi := (\ker T\pi)^\circ \to B$ is the vector subbundle of T^*B consisting of all covectors annihilating $\ker T\pi$, i.e., the fiber $(N_\pi)_b$ of N_π at $b \in B$ is the vector subspace $(N_\pi)_b := \{\alpha_b \in T_b^*B \mid \langle \alpha_b, v_b \rangle = 0, \forall v_b \in \ker T_b\pi\}$. The upper circle on a vector subspace denotes its annihilator in the dual of the ambient vector space.

Lemma 1 *The conormal bundle to the fibers $N_\pi \subset T^*B$ is a coisotropic submanifold with respect to the canonical symplectic form on T^*B.*

Proof Let $n = \dim X$, $n + k = \dim B$. Since π is a surjective submersion, it is locally expressed as a projection, i.e., around every point $b \in B$ there are coordinates $(x^1, \ldots, x^n, a^1, \ldots, a^k)$ on B such that π has the expression

$$\pi(x^1, \ldots, x^n, a^1, \ldots, a^k) = (x^1, \ldots, x^n).$$

In these coordinates, we express an arbitrary covector $\alpha_b \in T_b^*B$ as

$$\alpha_b = p_1 dx^1 + \cdots + p_n dx^n + \alpha_1 da^1 + \cdots + \alpha_k da^k, \quad p_1, \ldots, p_n, \alpha_1, \ldots, \alpha_k \in \mathbb{R}.$$

In these coordinates,

$$\ker T\pi = \operatorname{span}\left\{\frac{\partial}{\partial a^1}, \dots, \frac{\partial}{\partial a^k}\right\},$$

and hence $\alpha_b \in N_\pi$ if and only if

$$\alpha_b = p_1 dx^1 + \cdots + p_n dx^n,$$

that is, N_π is locally defined by the equations $\alpha_1 = \cdots = \alpha_k = 0$, i.e., the local expression of N_π is

$$N_\pi = \left\{(x^1, \dots, x^n, a^1, \dots, a^k, p_1, \dots, p_n, 0, \dots, 0) \in \mathbb{R}^{n+k}\right\}. \tag{1}$$

This shows that N_π is a submanifold of T^*B of dimension $2n + k$, whose tangent space at any $\alpha_b \in N_\pi$ is expressed locally as

$$T_{\alpha_b} N_\pi = \operatorname{span}\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial a^1}, \dots, \frac{\partial}{\partial a^k}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}\right\}. \tag{2}$$

Let $\omega_B \in \Omega^2(T^*B)$ be the canonical cotangent bundle symplectic form which, in these local coordinates, has the expression

$$\omega_B = dx^1 \wedge dp_1 + \cdots + dx^n \wedge dp_n + da^1 \wedge d\alpha_1 \wedge + \cdots + da^k \wedge d\alpha_k.$$

Using (2), the ω_B-orthogonal complement of $T_{\alpha_b} N_\pi$ (taken fiber-wise) is easily calculated in these local coordinates to be

$$(T_{\alpha_b} N_\pi)^\perp = \operatorname{span}\left\{\frac{\partial}{\partial a^1}, \dots, \frac{\partial}{\partial a^k}\right\}. \tag{3}$$

Thus, $(T_{\alpha_b} N_\pi)^\perp \subset T_{\alpha_b} N_\pi$, which shows that N_π is coisotropic in T^*B. $\square$

The local expression of $(TN_\pi)^\perp \subset T(T^*B)$ implies that the vector subbundle $(TN_\pi)^\perp$ is integrable. This is a general fact, namely, the tangent bundle of a coisotropic submanifold is an integrable subbundle of the tangent bundle of the ambient symplectic manifold (see, e.g., [6, Lecture 3] or [1, Proposition 5.3.22]). Denote by $\mathscr{N}_\pi^\perp$ the foliation in T^*B defined by the integrable vector subbundle $(TN_\pi)^\perp$. From now on we assume that $\mathscr{N}_\pi^\perp$ is a *regular foliation*, i.e., its space of leaves $N_\pi/\mathscr{N}_\pi^\perp$ is a smooth manifold and the canonical projection $\rho : N_\pi \to N_\pi/\mathscr{N}_\pi^\perp$ is a smooth submersion; this uniquely determines the manifold structure on the space of leaves, assuming it exists.

By (3), in local coordinates, the leaf of the foliation $\mathscr{N}_\pi^\perp$ containing the point

$$(x_0^1, \dots, x_0^n, a_0^1, \dots, a_0^k, (p_1)_0, \dots, (p_n)_0, 0, \dots, 0) \in N_\pi$$

is given by

$$\{(x_0^1, \ldots, x_0^n, a^1, \ldots, a^k, (p_1)_0, \ldots, (p_n)_0, 0, \ldots, 0) \mid a^1, \ldots, a^k \in \mathbb{R}\}. \tag{4}$$

Therefore, the projection $\rho : N_\pi \to N_\pi / \mathscr{N}_\pi^\perp$ has the local expression

$$\rho(x^1, \ldots, x^n, a^1, \ldots, a^k, p_1, \ldots, p_n, 0, \ldots, 0) = (x^1, \ldots, x^n, p_1, \ldots, p_n) \tag{5}$$

and hence $(x^1, \ldots, x^n, p_1, \ldots, p_n)$ are local coordinates on $N_\pi / \mathscr{N}_\pi^\perp$ (remember that $N_\pi / \mathscr{N}_\pi^\perp$ is, by assumption, a manifold).

By the Coisotropic Reduction Theorem (see, e.g., [6, Lecture 3] or [1, Theorem 5.3.33]), the quotient $N_\pi / \mathscr{N}_\pi^\perp$ has a canonical symplectic form ω_π, uniquely characterized by $\rho^*\omega_\pi = \iota^*\omega_B$, where $\iota : N_\pi \hookrightarrow T^*B$ is the inclusion.

Proposition 1 ([6], Lecture 6) *The following statements hold.*

(i) *Let*

$$\pi^*T^*X = \left\{\left(b, \beta_{\pi(b)}\right) \,\middle|\, b \in B, \beta_{\pi(b)} \in T^*_{\pi(b)}X\right\} \ni (b, \beta_{\pi(b)}) \longmapsto b \in B$$

be the pull-back bundle to B of the cotangent bundle $T^*X \to X$ by π. The map $\Xi : N_\pi \to \pi^*T^*X$ given by $\Xi(\alpha_b) := \left(b, \beta_{\pi(b)}\right)$, where $\alpha_b \in (N_\pi)_b = N_\pi \cap T_b^*B$ and $\beta_{\pi(b)} \in T^*_{\pi(b)}X$ is defined by $\left\langle \beta_{\pi(b)}, T_b\pi(v_b)\right\rangle := \langle \alpha_b, v_b\rangle$ for all $v_b \in T_bB$, is a vector bundle isomorphism. Its inverse $\Xi^{-1} : \pi^*T^*X \to N_\pi$ is $\Xi^{-1}(b, \beta_{\pi(b)}) = \beta_{\pi(b)} \circ T_b\pi$.

(ii) *Define the submersion $\widetilde{\pi} : \pi^*T^*X \to T^*X$ by $\widetilde{\pi}(b, \beta_{\pi(b)}) := \beta_{\pi(b)}$. Then, for any $\alpha_b \in N_\pi$, we have $T_{\alpha_b}\Xi\left(\left(T_{\alpha_b}N_\pi\right)^\perp\right) = \ker T_{\alpha_b}\widetilde{\pi}$; recall that $\left(T_{\alpha_b}N_\pi\right)^\perp$ is the tangent space at α_b to the fiber of the foliation $\mathscr{N}_\pi^\perp$ containing α_b.*

(iii) *The integral leaves of the foliation $\mathscr{N}_\pi^\perp$ in N_π are images under Ξ^{-1} of the connected components of the fibers of $\widetilde{\pi} : \pi^*T^*X \to T^*X$.*

(iv) *The map $\Phi : N_\pi / \mathscr{N}_\pi^\perp \to T^*X$ defined by $\Phi([\alpha_b]) := \widetilde{\pi}(\Xi(\alpha_b))$ is well-defined, a local diffeomorphism, and a symplectic map.*

(v) *The map Φ is surjective if and only if π is surjective. The map Φ is injective if and only if the fibers of π over $\pi(B)$ are connected.*

(vi) *If π is surjective and has connected fibers, then $\Phi : \left(N_\pi / \mathscr{N}_\pi^\perp, \omega_\pi\right) \to (T^*X, \omega_B)$ is a symplectic diffeomorphism.*

The spaces and maps involved in this proposition are summarized in the commutative diagram below. The first vertical arrow is only a surjective submersion, whereas the second, third, and fourth are vector bundle projections.

$$
\begin{tikzcd}
N/\mathscr{N}_\pi^\perp \arrow[d] & N_\pi = (\ker T\pi)^\circ \arrow[l, "\rho"'] \arrow[r, "\Xi"] \arrow[d, "\pi_B|_{N_\pi}"'] \arrow[rrr, bend left, "\Phi", phantom] & \pi^* T^* X \arrow[r, "\widetilde{\pi}"] \arrow[d] & T^*X \arrow[d, "\pi_X"'] \\
X \supseteq \pi(B) & B \arrow[l, "\pi"] \arrow[r, equal] & B \arrow[r, "\pi"'] & X
\end{tikzcd}
\tag{6}
$$

Point (vi) of this proposition is a vast generalization of the Cotangent Bundle Reduction Theorem ([1, Theorem 4.3.3], [4, Chap. 2]) at the zero value of the momentum map. Finding a similar generalization of this theorem for any value of the momentum map would be very interesting and relate to the general reduction procedure for Hamilton–Jacobi theory.

Proof (i) The map $\Xi : N_\pi \to \pi^* T^* X$ is well defined. Indeed, any tangent vector in $T_{\pi(b)}X$ is necessarily of the form $T_b\pi(v_b)$ for some $v_b \in T_bB$ because π is a submersion. If $T_b\pi(v_b) = T_b\pi(v_b')$ for $v_b, v_b' \in T_bB$, i.e., $v_b - v_b' \in \ker T_b\pi$, then $\left\langle \alpha_b, v_b - v_b' \right\rangle = 0$ for any $\alpha_b \in (\ker T_b\pi)^\circ \subset N_\pi$. This shows that $\langle \alpha_b, v_b \rangle = \left\langle \alpha_b, v_b' \right\rangle$ thus proving that Ξ is well defined.

We compute the local expression of Ξ. If

$$\alpha_b = (x^1, \ldots, x^n, a^1, \ldots, a^n, p_1, \ldots, p_n, 0, \ldots, 0) = p_1 dx^1 + \cdots + p_n dx^n \in N_\pi,$$

and $\Xi(\alpha_b) = (b, \beta_{\pi(b)})$, with $\beta_{\pi(b)} = (x^1, \ldots, x^n, r_1, \ldots, r_n) = r_1 dx^1 + \cdots + r_n dx^n$, then choosing and arbitrary vector

$$v_b = u^1 \frac{\partial}{\partial x^1} + \cdots + u^n \frac{\partial}{\partial x^n} + v^1 \frac{\partial}{\partial a^1} + \cdots + v^k \frac{\partial}{\partial a^k} \in T_bB,$$

we have

$$r_1 u^1 + \cdots + r_n u^n = \left\langle \beta_{\pi(b)}, T_b\pi(v_b) \right\rangle = \langle \alpha_b, v_b \rangle = p_1 u^1 + \cdots + p_n u^n$$

for any $u^1, \ldots, u^n \in \mathbb{R}$, i.e., $\beta_{\pi(b)} = p_1 dx^1 + \cdots + p_n dx^n$. This shows that, choosing the standard cotangent bundle coordinates on T^*X,

$$\Xi(x^1, \ldots, x^n, a^1, \ldots, a^n, p_1, \ldots, p_n) = (x^1, \ldots, x^n, a^1, \ldots, a^n, p_1, \ldots, p_n), \tag{7}$$

that is, Ξ is the identity map in these coordinate systems. Thus, Ξ is smooth and, from the very definition of Ξ, it is a vector bundle morphism.

The smooth vector bundle morphism $\pi^* T^* X \ni (b, \beta_{\pi(b)}) \longmapsto \beta_{\pi(b)} \circ T_b\pi \in N_\pi$ is easily verified to equal $\Xi^{-1} : \pi^* T^* X \to N_\pi$, i.e., $\Xi^{-1}(b, \beta_{\pi(b)}) = \beta_{\pi(b)} \circ T_b\pi$, as claimed in the statement.

(ii) Work with the same coordinate systems $(x^1, \ldots, x^n)$ on X, $(x^1, \ldots, x^n, a^1, \ldots, a^k)$ on B, and $(x^1, \ldots, x^n, a^1, \ldots, a^n, p_1, \ldots, p_n)$ on both $N_\pi \subset T^*B$ and $\pi^* T^* X$, as in the proof of (i). In these coordinate systems, the projection $\widetilde{\pi}$ reads

$$(x^1, \ldots, x^n, a^1, \ldots, a^k, p_1, \ldots, p_n) \mapsto (x^1, \ldots, x^n, p_1, \ldots, p_n).$$

On the other hand, the local expression of $\varXi$ is the identity, as was shown in the proof of (i). Thus, taking into account formula (3) for the tangent space of a leaf of $\mathscr{N}_\pi^\perp$ at a point $\alpha_b \in N_\pi$, we conclude that its image under $T_{\alpha_b}\varXi$ is span$\{\partial/\partial a^1, \ldots, \partial/\partial a^k\}$, which is exactly the kernel of $T_{\alpha_b}\widetilde{\pi}$.

(iii) Recall that $\varXi$ is a vector bundle isomorphism. By (ii), its tangent map is an isomorphism of the vector subbundle $TN_\pi^\perp \subset TN_\pi$, defining the foliation $\mathscr{N}_\pi^\perp$, and the vector subbundle $\ker T\widetilde{\pi} \subset T(\pi^*T^*X)$, defining the foliation $\mathscr{F}_{\widetilde{\pi}}$, whose leaves are the connected components of the fibers of $\widetilde{\pi}$. Therefore, the leaves of the two foliations are mapped onto each other by $\varXi$.

(iv) By (iii), the smooth map $\widetilde{\pi} \circ \varXi$ is constant on the leaves of the foliation $\mathscr{N}_\pi^\perp$ and thus it drops to a map $\varPhi : N_\pi/\mathscr{N}_\pi^\perp \to T^*X$ uniquely characterized by the relation $\varPhi \circ \rho = \widetilde{\pi} \circ \varXi$, i.e., $\varPhi(\rho(\alpha_b)) := \widetilde{\pi}(\varXi(\alpha_b)) = \beta_{\pi(b)}$.

We prove that $\varPhi$ is a local diffeomorphism by working in the local coordinates considered earlier. If $\alpha_b \in N_\pi$, $\alpha_b = (x^1, \ldots, x^n, a^1, \ldots, a^k, p_1, \ldots, p_n, 0, \ldots, 0)$, then $\rho(\alpha_b) = (x^1, \ldots, x^n, p_1, \ldots, p_n)$. From (7) and the definition of $\varXi$, it follows that $\varPhi(x^1, \ldots, x^n, p_1, \ldots, p_n) = (x^1, \ldots, x^n, p_1, \ldots, p_n)$, i.e., $\varPhi$ is the identity map in these charts. Hence $\varPhi$ is a local diffeomorphism.

Let $\omega_X \in \varOmega^2(T^*X)$ and $\omega_B \in \varOmega^2(T^*B)$ be the canonical symplectic forms. Then $\varPhi^*\omega_X = \omega_\pi$ if and only if $\iota^*\omega_B = \rho^*\omega_\pi = \rho^*\varPhi^*\omega_X = \varXi^*\widetilde{\pi}^*\omega_X$ by the definition of the reduced symplectic form ω_π and of the map $\varPhi$. The identity $\iota^*\omega_B = \varXi^*\widetilde{\pi}^*\omega_X$ is proved in the local coordinates considered above, in which $\varXi$ is the identity and $\widetilde{\pi}(x^1, \ldots, x^n, a^1, \ldots, a^k, p_1, \ldots, p_n) = (x^1, \ldots, x^n, p_1, \ldots, p_n)$. Therefore,

$$\varXi^*\widetilde{\pi}^*(dx^1 \wedge dp_1 + \cdots + dx^n \wedge dp_n) = dx^1 \wedge dp_1 + \cdots + dx^n \wedge dp_n.$$

On the other hand, $\iota^*\omega_B = \iota^*(dx^1 \wedge dp_1 + \cdots + dx^n \wedge dp_n + da^1 \wedge d\alpha_1 \wedge + \cdots + da^k \wedge d\alpha_k) = dx^1 \wedge dp_1 + \cdots + dx^n \wedge dp_n$, which proves the required identity. Thus, $\varPhi$ is a symplectic map.

(v) Since $\varPhi \circ \rho = \widetilde{\pi} \circ \varXi$ and ρ is surjective, $\varPhi$ is onto if and only if $\widetilde{\pi}$ is onto, because $\varXi$ is bijective by (i). Since $\widetilde{\pi} : \pi^*T^*X \to T^*X$ is surjective when restricted to every fiber of the vector bundle $\pi^*T^*X \to B$, it follows that $\widetilde{\pi}$ is surjective if and only if π is surjective.

We now study injectivity of ϕ. We first prove that the fiber $\pi^{-1}(x), x \in \pi(B) \subset X$, of π is connected if and only if the fiber $\widetilde{\pi}^{-1}(\beta_x^0)$, $\beta_x^0 \in T_x^*X$ of $\widetilde{\pi}$ is connected. On one hand, the restriction of the smooth map $\pi^*T^*X \ni \left(b, \beta_{\pi(b)}\right) \mapsto b \in B$ to the submanifold $\widetilde{\pi}^{-1}(\beta_x^0)$ of π^*T^*X is a bijective smooth map onto the submanifold $\pi^{-1}(x)$ of B. On the other hand, choose a 1-form β on X such that $\beta(x) = \beta_x^0$ and define the smooth map $B \ni b \mapsto (b, \beta(\pi(b))) \in \pi^*T^*X$. The restriction of this smooth map to the submanifold $\pi^{-1}(x)$ of B maps onto the submanifold $\widetilde{\pi}^{-1}(\beta_x^0)$ of π^*T^*X. These two maps are clearly inverses of each other. Thus, the fibers $\pi^{-1}(x)$

and $\widetilde{\pi}^{-1}(\beta_x^0)$ are diffeomorphic. In particular, $\pi^{-1}(x)$ is connected if and only if $\widetilde{\pi}^{-1}(\beta_x^0)$ is connected for any $x \in \pi(B)$.

We assume now that the fibers of $\widetilde{\pi}$ are connected. Take two classes $[\alpha_b]$, $[\alpha'_{b'}] \in N_\pi/\mathscr{N}_\pi^\perp$ such that $\Phi([\alpha_b]) = \Phi([\alpha'_{b'}])$. The identity $\Phi \circ \rho = \widetilde{\pi} \circ \Xi$ implies that $\widetilde{\pi}(\Xi(\alpha_b)) = \widetilde{\pi}(\Xi(\alpha'_{b'}))$, that is, $\Xi(\alpha_b)$ and $\Xi(\alpha'_{b'})$ are in the same fiber of $\widetilde{\pi}$, which is connected. By (iii), its image by the diffeomorphism Ξ coincides with a leaf of the foliation $\mathscr{N}_\pi^\perp$ and hence α_b and $\alpha'_{b'}$ lie on the same leaf, which means that $[\alpha_b] = [\alpha'_{b'}]$.

Conversely, assume that Φ is injective. Take any two points $\Xi(\alpha_b)$ and $\Xi(\alpha'_{b'})$ in the same fiber of $\widetilde{\pi}$, i.e., $\widetilde{\pi}(\Xi(\alpha_b)) = \widetilde{\pi}(\Xi(\alpha'_{b'}))$, or, equivalently, $\Phi([\alpha_b]) = \Phi([\alpha'_{b'}])$. As Φ injective, we have $[\alpha_b] = [\alpha'_{b'}]$, that is, α_b and $\alpha'_{b'}$, are in the same (connected) leaf of $\mathscr{N}_\pi^\perp$, which is equivalent, by (iii), to $\Xi(\alpha_b)$ and $\Xi(\alpha'_{b'})$ being in the same connected component of the fiber of $\widetilde{\pi}$. Thus, the fibers of $\widetilde{\pi}$, are necessarily connected.

(vi) This is a direct consequence of (iv) and (v). □

3 Morse Families and Transverse Intersections

Let $b \in B$ and $\iota_{\pi^{-1}(\pi(b))} : \pi^{-1}(\pi(b)) \hookrightarrow B$ be the inclusion. For a smooth function $S : B \to \mathbb{R}$ and $b' \in \pi^{-1}(\pi(b))$, denote by $\mathbf{d}^v S(b') := \mathbf{d}S(b')|_{\ker T_{b'}\pi} = \mathbf{d}\left(S \circ \iota_{\pi^{-1}(\pi(b))}\right) : T_{b'}\pi^{-1}(\pi(b)) = \ker T_{b'}\pi \to \mathbb{R}$ the *vertical derivative* at any $b' \in B$. Let

$$\Sigma_S := \{b \in B \mid \mathbf{d}^v S(b) = 0\} \tag{8}$$

be the set of critical points with respect to the projection π. Locally, this states that Σ_S is characterized by points in B with coordinates $(x^1, \ldots, x^n, a^1, \ldots, a^k)$ for which

$$\frac{\partial S}{\partial a^1} = 0, \ldots, \frac{\partial S}{\partial a^k} = 0.$$

Proposition 2 *We have*

$$\mathbf{d}S(B) \cap N_\pi = \mathbf{d}S(\Sigma_S). \tag{9}$$

Proof Indeed, $\mathbf{d}S(b) \in N_\pi = (\ker T\pi)^\circ$ if and only if $\langle \mathbf{d}S(b), v\rangle = 0$, for all $v \in \ker T\pi$, which is equivalent to $b \in \Sigma_S$ by (8).

Definition 1 A function $S : B \to \mathbb{R}$ such that the graph $\mathbf{d}S(B) \subset T^*B$ intersects N_π transversally (i.e., $T_{\mathbf{d}S(b)}\mathbf{d}S(B) + T_{\mathbf{d}S(b)}N_\pi = T_{\mathbf{d}S(b)}(T^*B)$ for any $b \in \Sigma_S$, denoted $\mathbf{d}S(B) \pitchfork N_\pi$) is called a *Morse family*.

Proposition 3 *Given a fibered local system of coordinates* $(x^1, \ldots x^n, a^1, \ldots, a^k)$ *on* B *(i.e., a chart on* B *in which* π *is the projection* $(x^1, \ldots x^n, a^1, \ldots, a^k) \mapsto (x^1, \ldots x^n)$*),* $S : B \to X$ *is a Morse family if and only if the* $k \times (n+k)$*-matrix*

$$\begin{pmatrix} \dfrac{\partial^2 S}{\partial a^i \partial a^j}(b) & \dfrac{\partial^2 S}{\partial a^i \partial x^l}(b) \end{pmatrix} \tag{10}$$

has rank k at every point $b \in \Sigma_S$.

Proof We express the transversality condition $T_{\alpha_b}\mathbf{d}S(B) + T_{\alpha_b}N_\pi = T_{\alpha_b}(T^*B)$ for any $\alpha_b \in \mathbf{d}S(B) \cap N_\pi$ in these local coordinates. Recall that with respect to the standard induced cotangent bundle coordinate system

$$(x^1, \ldots, x^n, a^1, \ldots, a^k, p_1, \ldots, p_n, \alpha_1, \ldots, \alpha_k) \tag{11}$$

on T^*B induced by a fibered system on B, the expression of N_π is $\alpha_1 = \cdots = \alpha_k = 0$ (see (1)). Thus,

$$T_{\alpha_b}N_\pi = \operatorname{span}\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial a^1}, \ldots, \frac{\partial}{\partial a^k}, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n} \right\}.$$

On the other hand, the tangent space to the graph $\mathbf{d}S(B)$ is generated by the vectors

$$\frac{\partial}{\partial x^i} + \frac{\partial^2 S}{\partial x^i \partial x^j}\frac{\partial}{\partial p_j} + \frac{\partial^2 S}{\partial x^i \partial a^r}\frac{\partial}{\partial \alpha_r}, \quad i = 1, \ldots, n, \tag{12}$$

$$\frac{\partial}{\partial a^l} + \frac{\partial^2 S}{\partial a^l \partial x^j}\frac{\partial}{\partial p_j} + \frac{\partial^2 S}{\partial a^l \partial a^r}\frac{\partial}{\partial \alpha_r}, \quad l = 1, \ldots, k. \tag{13}$$

Therefore, for any $\alpha_b \in \mathbf{d}S(B) \cap N_\pi$,

$$T_{\alpha_b}N_\pi + T_{\alpha_b}\mathbf{d}S(B) = \operatorname{span}\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial a^l}, \frac{\partial}{\partial p_i}, \frac{\partial^2 S}{\partial x^i \partial a^r}\frac{\partial}{\partial \alpha_r}, \frac{\partial^2 S}{\partial a^l \partial a^r}\frac{\partial}{\partial \alpha_r} \right\}_{i=1,\ldots,n;\ l=1,\ldots,k},$$

which shows that $T_{\alpha_b}\operatorname{im}(\mathbf{d}S) + T_{\alpha_b}N_\pi = T_{\alpha_b}(T^*B)$ if and only if the coefficient matrix in the statement has maximal rank k. □

Corollary 1 *If S is a Morse family, then the set $\mathbf{d}S(B) \cap N_\pi$ is a submanifold of N_π and $T_{\alpha_b}(\mathbf{d}S(B) \cap N_\pi) = T_{\alpha_b}\mathbf{d}S(B) \cap T_{\alpha_b}N_\pi$, for any $\alpha_b \in \mathbf{d}S(B) \cap N_\pi$. In addition,* $\dim(\mathbf{d}S(B) \cap N_\pi) = n$.

Proof Standard intersection theory (see, e.g., [2, Corollary 3.5.13]) guarantees the first statement. Thus, by linear algebra, $\dim(\mathbf{d}S(B) \cap N_\pi) = \dim \mathbf{d}S(B) + \dim N_\pi - \dim T^*B = (n + k) - (2n + k) - 2(n + k) = n$. □

Theorem 1 *If S is a Morse family and $\rho : N_\pi \to N_\pi / \mathcal{N}_\pi^\perp$ is the projection, then the restriction $\rho|_{\mathbf{d}S(B) \cap N_\pi} : \mathbf{d}S(B) \cap N_\pi \to N_\pi / \mathcal{N}_\pi^\perp$ is an immersion.*

Proof We need to prove that $\ker T_{\alpha_b}(\rho|_{\mathbf{d}S(B) \cap N_\pi}) = \{0\}$, for any $\alpha_b \in \mathbf{d}S(B) \cap N_\pi$. This is equivalent to checking that $T_{\alpha_b}(\mathbf{d}S(B) \cap N_\pi) \cap \ker T_{\alpha_b}\rho = \{0\}$. Note that

$\ker T_{\alpha_b}\rho$ is just the tangent space to the fiber of the foliation $\mathscr{N}_\pi^\perp$ at α_b. In the coordinate system (11), by (3), every tangent vector to the fibers has the expression

$$X = A^l \frac{\partial}{\partial a^l}, \quad X^l \in \mathbb{R}. \tag{14}$$

Suppose $X \in T_{\alpha_b}(\mathbf{d}S(B) \cap N_\pi)$ so, in particular, $X \in T_{\alpha_b}\mathbf{d}S(B)$ and hence X must be a linear combination of the vectors in (12) and (13). Any linear combination having vectors from (12) necessarily contains a linear combination of the vectors $\{\partial/\partial x^i\}_{i=1,\ldots,n}$. The form of the vector (14) precludes this and hence the vector X can only be a linear combination of vectors in (13), i.e.,

$$X = A^l \left(\frac{\partial}{\partial a^l} + \frac{\partial^2 S}{\partial a^l \partial x^j}\frac{\partial}{\partial p_j} + \frac{\partial^2 S}{\partial a^l \partial a^r}\frac{\partial}{\partial \alpha_r} \right)$$

with the condition

$$A^l \frac{\partial^2 S}{\partial a^l \partial x^j} = 0, \qquad A^l \frac{\partial^2 S}{\partial a^l \partial a^r} = 0, \qquad \forall j = 1, \ldots n, \quad \forall l, r = 1, \ldots, k.$$

In matrix form, these conditions are expressed as

$$(A^l) \begin{pmatrix} \frac{\partial^2 S}{\partial a^l \partial x^j} & \frac{\partial^2 S}{\partial a^l \partial a^r} \end{pmatrix} = (\underbrace{0, \ldots, 0}_{n+k}).$$

This is only possible if $A^l = 0$, for all $l = 1, \ldots, k$, since the matrix of this linear system is (10) which has maximal rank k by Proposition 3. □

Corollary 2 *The set $\rho(\mathbf{d}S(B) \cap N_\pi)$ is "manifold with self intersections". If we denote by $\rho(\mathbf{d}S(B) \cap N_\pi)_0$ the subset where it is a manifold, then it is a Lagrangian submanifold of $N_\pi/\mathscr{N}_\pi^\perp$.*

Proof The symplectic form ω_π on $N_\pi/\mathscr{N}_\pi^\perp$ is uniquely characterized by the property $\rho^*\omega_\pi = \iota^*\omega_B$, where $\iota : N_\pi \hookrightarrow T^*B$ is the inclusion and ω_B is the canonical symplectic form on T^*B. For any two tangent vectors $U_1, U_2 \in T_x\rho(\mathbf{d}S(B) \cap N_\pi)_0$, $x \in \rho(\mathbf{d}S(B) \cap N_\pi)_0$, there are vectors $X_1, X_2 \in T_{\alpha_b}(\mathbf{d}S(B) \cap N_\pi)$, $\rho(\alpha_b) = x$, such that $T_{\alpha_b}\rho(X_1) = U_1$ and $T_{\alpha_b}\rho(X_2) = U_2$. Then

$$\omega_\pi(x)(U_1, U_2) = \left(\rho^*\omega_\pi\right)(\alpha_b)(X_1, X_2) = \omega_B(\alpha_b)(T_{\alpha_b}\iota(X_1), T_{\alpha_b}\iota(X_2)) = 0,$$

because $X_1, X_2 \in T_{\alpha_b}(\mathbf{d}S(B))$ and $\mathbf{d}S(B)$ is a Lagrangian submanifold of T^*B. By Theorem 1, $\dim\left(\rho(\mathbf{d}S(B) \cap N_\pi)_0\right) = \dim\left(\mathbf{d}S(B) \cap N_\pi\right) = n$ (see Corollary 1). Since $\dim\left(N_\pi/\mathscr{N}_\pi^\perp\right) = 2n$ by Proposition 1(iv), this proves that $\rho(\mathbf{d}S(B) \cap N_\pi)_0$ is a Lagrangian submanifold of $N_\pi/\mathscr{N}_\pi^\perp$. □

Example 1 Let $B := \mathbb{R}^2$, $X := \mathbb{R}$, and $\pi : \mathbb{R}^2 \ni (x, a) \mapsto x \in \mathbb{R}$. As in the general theory, coordinates on $T^*B = T^*\mathbb{R}^2 = \mathbb{R}^4$ are denoted by (x, a, p, α) and on $T^*X = T^*\mathbb{R} = \mathbb{R}^2$ by (x, p). The conormal bundle to the fibers is

$$N_\pi = \{(x, a, p, 0) \mid x, a, p \in \mathbb{R}\} \subset T^*\mathbb{R}^2.$$

We regard N_π as Euclidean space $\mathbb{R}^3$, which enables us to describe the objects of the general theory, for this case, concretely.

Define the function $S : \mathbb{R}^2 \to \mathbb{R}$ by

$$S(x, a) := \frac{a^3}{3} + a(x^2 - 1).$$

Since

$$\Sigma_S = \left\{ (x, a) \in \mathbb{R}^2 \;\middle|\; \frac{\partial S}{\partial a} = a^2 + x^2 - 1 = 0 \right\}$$

and the matrix

$$\left(\frac{\partial^2 S}{\partial x \partial a} \;\; \frac{\partial^2 S}{\partial a^2} \right) = (2a \;\; 2x)$$

never vanishes on Σ_S, it follows by Proposition 3 that S is a Morse family. Therefore, the set

$$\mathbf{d}S(\mathbb{R}^2) \cap N_\pi = \{(x, a, 2ax, 0) \mid x^2 + a^2 = 1\} \subset N_\pi = \mathbb{R}^3$$

is a one-dimensional manifold (see Corollary 1). Since π is surjective with connected fibers, by Proposition 1(vi), $N_\pi / \mathscr{N}_\pi^\perp = T^*\mathbb{R} = \mathbb{R}^2$, as symplectic manifolds. Of course, in this special case, this can be shown directly. In fact, in agreement with the general formula (5), the projection $\rho : N_\pi \to N_\pi / \mathscr{N}_\pi^\perp$ is just $\rho(x, a, p, 0) = (x, p)$. Therefore

$$\rho(\mathbf{d}S(\mathbb{R}^2) \cap N_\pi) = \left\{ (x, 2xa) \mid x^2 + a^2 = 1 \right\} = \left\{ \left(x, \pm 2x\sqrt{1 - x^2} \right) \;\middle|\; x \in [-1, 1] \right\}.$$

This is the Bernoulli Lemniscate which clearly has a self-intersection. Away from this self-intersection point, this is a one-dimensional submanifold of $\mathbb{R}^2$ and hence clearly Lagrangian, in agreement with Corollary 2.

4 The Converse: Construction of a Morse Family for a Lagrangian Submanifold

A reasonable converse to Corollary 2 is Theorem 2 below. We use the following notation. If $\varepsilon : A \to B$ is a locally trivial fiber bundle and $M \subset B$ is a submanifold, $\varepsilon_M : A_M \to M$ denotes the restriction of the fiber bundle ε obtained by shrinking the base B to M. For any manifold X, denote by $\pi_X : T^*X \to X$ the cotangent bundle projection.

In the proof of the theorem below, we need a classical result of Weinstein, sometimes called the *Lagrangian Tubular Neighborhood Theorem* or the *Relative Darboux Theorem* (see [5, Theorem 7.1], [6, Lecture 5], [1, Theorem 5.3.18]). We need a general formulation appropriate for our purposes, as stated and proved with all details in [3, Theorem 31.20]. *Let* (P, ω) *be a symplectic manifold and* $L \subset P$ *a Lagrangian submanifold. Then there exist an open neighborhood* U *of* L *in* P, *a tubular neighborhood* V *of the zero section in* T^*L, *and a symplectic diffeomorphism* $\varphi : (U, \omega|_U) \to (V, \omega_L|_V)$ *such that* $\varphi(x) = 0_x$ *for all* $x \in L$. The proof of the theorem starts by considering a Lagrangian complementary vector subbundle E of TL in $(TP)_L$ which ultimately provides the neighborhood U of L in P by constructing a tubular neighborhood of the zero section in E. If such a complement is given, the theorem just cited has an important refinement. *Suppose that* $E \to L$ *is a given Lagrangian complement to* $TL \to L$ in $(TP)_L$. *Then the symplectic diffeomorphism* φ *can be chosen such that* $T_x\varphi(E_x) = \ker T_{0_x}\pi_L \subset T_{0_x}(T^*L)$ *for all* $x \in L$, *where* $\pi_L : T^*L \to L$ *is the cotangent bundle projection.* In other words, $T_x\varphi$ maps the fiber E_x to the vertical vectors at $0_x \in T^*L$ in $T(T^*L)$.

Theorem 2 *Let* $L \subset T^*X$ *be a Lagrangian submanifold such that*

- *the pull back of the Liouville 1-form* $\theta_X \in \Omega^1(T^*X)$ *to* L *is exact, and*
- *there is a Lagrangian subbundle* $\Lambda \subset T(T^*X)_L$ *which is transversal to both* $(\ker T\pi_X)_L$ *and* TL *in* $T(T^*X)$.

Then there is a locally trivial fiber bundle $\pi : B \to X$ *and a Morse family* $S : B \to \mathbb{R}$ *such that* $L = (\Phi \circ \rho)(\mathbf{d}S(B) \cap N_\pi)$ (*see diagram* (6)).

Proof By hypothesis, $\Lambda \to L$ and $TL \to L$ are complementary Lagrangian subbundles. Thus, by the Relative Darboux Theorem cited above, there are open neighborhoods U of L in T^*X and V of the zero section in T^*L, and a symplectic diffeomorphism $f : U \to V$, such that $T_{\alpha_x} f(\Lambda_{\alpha_x}) = \ker T_{0_{\alpha_x}}\pi_L \subset T_{0_{\alpha_x}}(T^*L)$ for all $\alpha_x \in L$. Without loss of generality, we can assume that the fibers $V \cap T_{0_{\alpha_x}}(T^*L)$ are contractible. Since these are fibers of a locally trivial bundle, they form a foliation of V. Therefore, the collection $\left\{f^{-1}\left(V \cap T_{0_{\alpha_x}}(T^*L)\right) \mid \alpha_x \in L\right\}$ forms a foliation on U, the tangent space to every leaf being Λ_{α_x}; these leaves are simply connected, by construction.

Let $\iota : L \hookrightarrow T^*X$ be the inclusion. The pull back of the canonical Liouville 1-form $\theta_L \in \Omega^1(T^*L)$ to the zero section vanishes. Since $f \circ \iota : L \to T^*L$ is the inclusion of the zero section in its cotangent bundle, we conclude that $\iota^*(\theta_X - f^*\theta_L) = \iota^*\theta_X$ is an exact 1-form on L, by hypothesis. As the de Rham cohomology of U, V, and L are isomorphic, it follows that the closed 1-form $\theta_X - f^*\theta_L \in \Omega^1(U)$ is exact. Therefore, there is a smooth function $S : U \to \mathbb{R}$ such that $\mathbf{d}S = \theta_X - f^*\theta_L$.

The vector subbundle $\ker T\pi_X$ is integrable and its leaves are the fibers T^*_xL. Since the vector subbundles $\Lambda \to L$ and $(\ker T\pi_X)_L \to L$ are transversal along L, it follows that their leaves are also transversal at every point of L. Therefore, there is an open neighborhood $B \subset U$ of L, which we choose to have contractible fibers, such that the leaves of these two foliations are transversal at every point of B. Let $\pi : B \to X$ be the restriction of the cotangent bundle projection $\pi_X : T^*X \to X$ to B. Thus $\pi : B \to X$ is a locally trivial fiber bundle.

We show that $\Sigma_S = L$. Let $\beta_x \in B$ and $V_{\beta_x} \in \ker T_{\beta_x}\pi$ be arbitrary. Then $\beta_x \in \Sigma_S$ if and only if $\mathbf{d}S(\beta_x)|_{\ker T_{\beta_x}\pi} = 0$, i.e.,

$$\begin{aligned} 0 = \mathbf{d}S(\beta_x)\left(V_{\beta_x}\right) &= \left(\theta_X - f^*\theta_L\right)(\beta_x)\left(V_{\beta_x}\right) \\ &= \left\langle \beta_x, T_{\beta_x}\pi\left(V_{\beta_x}\right)\right\rangle - \left\langle f(\beta_x), T_{f(\beta_x)}\pi_L T_{\beta_x} f\left(V_{\beta_x}\right)\right\rangle . \end{aligned}$$

The first term vanishes since $V_{\beta_x} \in \ker T_{\beta_x}\pi$. The point β_x belongs to a unique leaf of the foliation $\left\{f^{-1}\left(V \cap T_{0_{\alpha_x}}(T^*L)\right) \mid \alpha_x \in L\right\}$, i.e., there is a unique $\alpha_{x_0} \in L$ such that $f(\beta_x) \in V \cap T_{0_{\alpha_{x_0}}}(T^*L)$. This defines a smooth map $g : V \to L$, namely $g(\beta_x)$ is the unique point of intersection of the leaf containing β_x and L. Therefore, $\pi_L \circ f = g$ and the identity above becomes $0 = \left\langle f(\beta_x), T_{\beta_x} g\left(V_{\beta_x}\right)\right\rangle$ for all $V_{\beta_x} \in \ker T_{\beta_x}\pi$. However, $T_{\beta_x} g\left(V_{\beta_x}\right) \neq 0$ for any $V_{\beta_x} \neq 0_{\beta_x}$. Indeed, $T_{\beta_x} g\left(V_{\beta_x}\right) = 0$ if and only if $V_{\beta_x} \in \ker T_{\beta_x} g$ which is the tangent space to the leaf of the foliation $\left\{f^{-1}\left(V \cap T_{0_{\alpha_x}}(T^*L)\right) \mid \alpha_x \in L\right\}$ at β_x. However, this vector space is complementary to $\ker T_{\beta_x}\pi_X$, since the leaf is transversal to $T^*_x L$. Therefore $V_{\beta_x} \in \ker T_{\beta_x}\pi \cap \ker T_{\beta_x} g = \left\{0_{\beta_x}\right\}$. We conclude, therefore, that $0 = \left\langle f(\beta_x), W_{f(\beta_x)}\right\rangle$ for all $W_{f(\beta_x)} \in T_{f(\beta_x)}(T^*L)$. This is equivalent to $f(\beta_x) = 0_{f(\beta_x)}$, i.e., $\beta_x \in L$ since only points in L are mapped by f to the zero section of T^*L.

Note that $\mathbf{d}S(B) \cap N_\pi = \mathbf{d}S(L) = \mathbf{d}S(\Sigma_S)$. Indeed, if $\beta_x \in B$, then $\mathbf{d}S(\beta_x) \in N_\pi = (\ker T\pi)^\circ$ if and only if $\mathbf{d}S(\beta_x)\left(V_{\beta_x}\right) = 0$ for all $V_{\beta_x} \in \ker T_{\beta_x}\pi$. But this is exactly the condition considered above and we conclude that this is equivalent to $\beta_x \in L$.

We finally check that $L = (\Phi \circ \rho)(\mathbf{d}S(B) \cap N_\pi) = (\widetilde{\pi} \circ \Xi)(\mathbf{d}S(\Sigma_S))$ (see diagram (6)). The definition of Ξ from Proposition 1(i) yields for $\alpha_x \in L = \Sigma_S$, $\Xi(\mathbf{d}S(\alpha_x)) = (\alpha_x, \beta_x) \in \pi^*T^*X$, where $\beta_x \in T^*_x X$ is defined by

$$\left\langle \beta_x, T_{\alpha_x}\pi\left(V_{\alpha_x}\right)\right\rangle = \left\langle \mathbf{d}S(\alpha_x), V_{\alpha_x}\right\rangle, \quad \forall V_{\alpha_x} \in T_{\alpha_x}B.$$

Therefore, $(\widetilde{\pi} \circ \Xi)(\mathbf{d}S(\alpha_x)) = \widetilde{\pi}(\alpha_x, \beta_x) = \beta_x$.

We show that $\beta_x = \alpha_x$. Indeed, since

$$\left\langle f^*\theta_L(\alpha_x), V_{\alpha_x}\right\rangle = \left\langle \theta_L(f(\alpha_x)), T_{\alpha_x} f\left(V_{\alpha_x}\right)\right\rangle = \left\langle f(\alpha_x), T_{f(\alpha_x)}\pi_L T_{f\alpha_x}\left(V_{\alpha_x}\right)\right\rangle = 0$$

because $f(\alpha_x) = 0_{\alpha_x}$, we get

$$\left\langle \mathbf{d}S(\alpha_x), V_{\alpha_x}\right\rangle = \left\langle \theta_X(\alpha_x) - (f^*\theta_L)(\alpha_x), V_{\alpha_x}\right\rangle = \left\langle \theta_X(\alpha_x), V_{\alpha_x}\right\rangle = \left\langle \alpha_x, T_{\alpha_x}\pi_L\left(V_{\alpha_x}\right)\right\rangle ,$$

which shows that $\alpha_x = \beta_x$ and hence $(\Phi \circ \rho)(\mathbf{d}S(\alpha_x)) = \alpha_x \in L$. □

Acknowledgments Both authors thank the support of the project Edital 061-2011 of the program Ciências sem Fronteiras (CAPES), Brazil, as well as the Pontificia Universidade Católica do Rio de Janeiro for its hospitality in the visit during which this article was written. TSR was partially supported by NCCR SwissMAP of the Swiss National Science Foundation.

References

1. Abraham, R., Marsden, J.E.: Foundations of Mechanics. 2 revised and enlarged edn. Advanced Book Program. Benjamin/Cummings Publishing Co., Inc., Reading (1978). With the assistance of Tudor Ratiu and Richard Cushman
2. Abraham, R., Marsden, J.E., Ratiu, T.S.: Manifolds, Tensor analysis, and Applications. Applied Mathematical Sciences, vol. 75, 2nd edn. Springer, New York (1988)
3. Michor, P.W.: Topics in Differential Geometry. Graduate Studies in Mathematics, vol. 93. American Mathematical Society, Providence (2008)
4. Marsden, J.E., Misiolek, G., Ortega, J.-P., Perlmutter, M., Ratiu, T.S.: Hamiltonian Reduction by Stages. Lecture Notes in Mathematics, vol. 1913. Springer, Berlin (2007)
5. Weinstein, A.: Symplectic manifolds and their Lagrangian submanifolds. Adv. Math. **6**, 329–346 (1971)
6. Weinstein, A.: Lectures on Symplectic Manifolds, Expository lectures from the CBMS Regional Conference held at the University of North Carolina, 8–12 March (1976); Regional Conference Series in Mathematics, vol. 29. American Mathematical Society, Providence (1977)