7. Use the result of Problem 6 to discover whether each of the following equations can be transformed into an equation with constant coefficients by changing the independent variable, and solve it if this is possible:
   (a) $xy'' + (x^2 - 1)y' + x^3y = 0$;
   (b) $y'' + 3xy' + x^2y = 0$.
8. In this problem we present another way of discovering the second linearly independent solution of (1) when the roots of the auxiliary equation are real and equal.
   (a) If $m_1 \neq m_2$, verify that the differential equation
   $$y'' - (m_1 + m_2)y' + m_1m_2y = 0$$
   has
   $$y = \frac{e^{m_1x} - e^{m_2x}}{m_1 - m_2}$$
   as a solution.
   (b) Think of m_2 as fixed and use l'Hospital's rule to find the limit of the solution in part (a) as $m_1 \to m_2$.
   (c) Verify that the limit in part (b) satisfies the differential equation obtained from the equation in part (a) by replacing m_1 by m_2.

18 The Method of Undetermined Coefficients

In the preceding two sections we considered several ways of finding the general solution of the homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0. \tag{1}$$

As we saw, these methods are effective in only a few special cases: when the coefficients $P(x)$ and $Q(x)$ are constants, and when they are not constants but are still simple enough to enable us to discover one nonzero solution by inspection. Fortunately these categories are sufficiently broad to cover a number of significant applications. However, it should be clearly understood that many homogeneous equations of great importance in mathematics and physics are beyond the reach of these procedures, and can only be solved by the method of power series developed in Chapter 5.

In this and the next section we turn to the problem of solving the nonhomogeneous equation

$$y'' + P(x)y' + Q(x)y = R(X) \tag{2}$$

for those cases in which the general solution $y_g(x)$ of the corresponding homogeneous equation (1) is already known. By Theorem 14-B, if $y_p(x)$ is any particular solution of (2), then

$$y(x) = y_g(x) + y_p(x)$$

is the general solution of (2). But how do we find y_p? This is the practical problem that we now consider.

The method of undetermined coefficients is a procedure for finding y_p when (2) has the form

$$y'' + py' + qy = R(x), \tag{3}$$

where p and q are constants and $R(x)$ is an exponential, a sine or cosine, a polynomial, or some combination of such functions. As an example, we study the equation

$$y'' + py' + qy = e^{ax}. \tag{4}$$

Since differentiating an exponential such as e^{ax} merely reproduces the function with a possible change in the numerical coefficient, it is natural to guess that

$$y_p = Ae^{ax} \tag{5}$$

might be a particular solution of (4). Here A is the *undetermined coefficient* that we want to determine in such a way that (5) will actually satisfy (4). On substituting (5) into (4), we get

$$A(a^2 + pa + q)e^{ax} = e^{ax},$$

so

$$A = \frac{1}{a^2 + pa + q}. \tag{6}$$

This value of A will make (5) a solution of (4) *except when the denominator on the right of* (6) *is zero*. The source of this difficulty is easy to understand, for the exception arises when a is a root of the auxiliary equation

$$m^2 + pm + q = 0, \tag{7}$$

and in this case we know that (5) reduces the left side of (4) to zero and cannot possibly satisfy (4) as it stands, with the right side different from zero.

What can be done to continue the procedure in this exceptional case? We saw in the previous section that when the auxiliary equation has a double root, the second linearly independent solution of the homogeneous equation is obtained by multiplying by x. With this as a hint, we take

$$y_p = Axe^{ax} \tag{8}$$

as a substitute trial solution. On inserting (8) into (4), we get

$$A(a^2 + pa + q)xe^{ax} + A(2a + p)e^{ax} = e^{ax}.$$

The first expression in parentheses is zero because of our assumption that a is a root of (7), so

$$A = \frac{1}{2a + p}. \tag{9}$$

This gives a valid coefficient for (8) except when $a = -p/2$, that is, except when a is a double root of (7). In this case we hopefully continue the successful pattern indicated above and try

$$y_p = Ax^2e^{ax}. \tag{10}$$

Substitution of (10) into (4) yields

$$A(a^2 + pa + q)x^2e^{ax} + 2A(2a + p)xe^{ax} + 2Ae^{ax} = e^{ax}.$$

Since a is now assumed to be a double root of (7), both expressions in parentheses are zero and

$$A = \frac{1}{2}. \tag{11}$$

To summarize: If a is not a root of the auxiliary equation (7), then (4) has a particular solution of the form Ae^{ax}; if a is a simple root of (7), then (4) has no solution of the form Ae^{ax} but does have one of the form Axe^{ax}; and if a is a double root, then (4) has no solution of the form Axe^{ax} but does have one of the form Ax^2e^{ax}. In each case we have given a formula for A, but only for the purpose of clarifying the reasons behind the events. In practice it is easier to find A by direct substitution in the equation at hand.

Another important case where the method of undetermined coefficients can be applied is that in which the right side of equation (4) is replaced by $\sin bx$:

$$y'' + py' + qy = \sin bx. \tag{12}$$

Since the derivatives of sin bx are constant multiples of sin bx and cos bx, we take a trial solution of the form

$$y_p = A \sin bx + B \cos bx. \tag{13}$$

The undetermined coefficients A and B can now be computed by substituting (13) into (12) and equating the resulting coefficients of sin bx and cos bx on the left and right. These steps work just as well if the right side of equation (12) is replaced by cos bx or any linear combination of sin bx and cos bx, that is, any function of the form $\alpha \sin bx + \beta \cos bx$. As before, the method breaks down if (13) satisfies the homogeneous equation corresponding to (12). When this happens, the procedure can be carried through by using

$$y_p = x(A \sin bx + B \cos bx) \tag{14}$$

as our trial solution instead of (13).

Example 1. Find a particular solution of

$$y'' + y = \sin x. \tag{15}$$

The reduced homogeneous equation $y'' + y = 0$ has $y = c_1 \sin x + c_2 \cos x$ as its general solution, so it is useless to take $y_p = A \sin x + B \cos x$ as a trial solution for the complete equation (15). We therefore try $y_p = x(A \sin x + B \cos x)$. This yields

$$y_p' = A \sin x + B \cos x + x(A \cos x - B \sin x)$$

and

$$y_p'' = 2A \cos x - 2B \sin x + x(-A \sin x - B \cos x),$$

and by substituting in (15) we obtain

$$2A \cos x - 2B \sin x = \sin x.$$

This tells us that the choice $A = 0$ and $B - \frac{1}{2}$ satisfies our requirement, so $y_p = -\frac{1}{2} x \cos x$ is the desired particular solution.

Finally, we consider the case in which the right side of equation (4) is replaced by a polynomial:

$$y'' + py' + qy = a_0 + a_1 x + \cdots + a_n x^n. \tag{16}$$

Since the derivative of a polynomial is again a polynomial, we are led to seek a particular solution of the form

$$y_p = A_0 + A_1 x + \cdots + A_n x^n. \tag{17}$$

When (17) is substituted into (16), we have only to equate the coefficients of like powers of x to find the values of the undetermined coefficients $A_0, A_1, \ldots, A_n$. If the constant q happens to be zero, then this procedure gives x^{n-1} as the highest power of x on the left of (16), so in this case we take our trial solution in the form

$$\begin{aligned} y_p &= x(A_0 + A_1 x + \cdots + A_n x^n) \\ &= A_0 x + A_1 x^2 + \cdots + A_n x^{n+1} \end{aligned} \tag{18}$$

If p and q are both zero, then (16) can be solved at once by direct integration.

Example 2. Find the general solution of

$$y'' - y' - 2y = 4x^2. \tag{19}$$

The reduced homogeneous equation $y'' - y' - 2y = 0$ has $m^2 - m - 2 = 0$ or $(m - 2)(m + 1) = 0$ as its auxiliary equation, so the general solution of the reduced equation is $y_g = c_1 e^{2x} + c_2 e^{-x}$.

Since the right side of the complete equation (19) is a polynomial of the second degree, we take a trial solution of the form $y_p = A + Bx + Cx^2$ and substitute it into (19):

$$2C - (B + 2Cx) - 2(A + Bx + Cx^2) = 4x^2.$$

Equating coefficients of like powers of x gives the system of linear equations

$$\begin{aligned} 2C - B - 2A &= 0, \\ -2C - 2B &= 0, \\ -2C &= 4. \end{aligned}$$

We now easily see that $C = -2$, $B = 2$, and $A = -3$, so our particular solution is $y_p = -3 + 2x - 2x^2$ and

$$y = c_1 e^{2x} + c_2 e^{-x} - 3 + 2x - 2x^2$$

is the general solution of the complete equation (19).

The above discussions show that the form of a particular solution of equation (3) can often be inferred from the form of the right-hand member $R(x)$.

In general this is true whenever $R(x)$ is a function with only a finite number of essentially different derivatives. We have seen how this works for exponentials, sines and cosines, and polynomials. In Problem 3 we indicate a course of action for the case in which $R(x)$ is a sum of such functions. It is also possible to develop slightly more elaborate techniques for handling various products of these elementary functions, but for most practical purposes this is unnecessary. In essence, the whole matter is simply a question of intelligent guesswork involving a sufficient number of undetermined coefficients that can be tailored to fit the circumstances.

Problems

1. Find the general solution of each of the following equations:

(a) $y'' + 3y' - 10y = 6e^{4x}$;

(b) $y'' + 4y = 3 \sin x$;

(c) $y'' + 10y' + 25y = 14e^{-5x}$;

(d) $y'' - 2y' + 5y = 25x^2 + 12$;

(e) $y'' - y' - 6y = 20e^{-2x}$;

(f) $y'' - 3y' + 2y = 14 \sin 2x - 18 \cos 2x$;

(g) $y'' + y = 2 \cos x$;

(h) $y'' - 2y' = 12x - 10$;

(i) $y'' - 2y' + y = 6e^x$;

(j) $y'' - 2y' + 2y = e^x \sin x$;

(k) $y'' + y' = 10x^4 + 2$.

2. If k and b are positive constants, find the general solution of

$$y'' + k^2 y = \sin bx.$$

3. If $y_1(x)$ and $y_2(x)$ are solutions of

$$y'' + P(x)y' + Q(x)y = R_1(x)$$

and

$$y'' + P(x)y' + Q(x)y = R_2(x),$$

show that $y(x) = y_1(x) + y_2(x)$ is a solution of

$$y'' + P(x)y' + Q(x)y = R_1(x) + R_2(x).$$

This is called the *principle of superposition*. Use this principle to find the general solution of

(a) $y'' + 4y = 4 \cos 2x + 6 \cos x + 8x^2 - 4x$;

(b) $y'' + 9y = 2 \sin 3x + 4 \sin x - 26e^{-2x} + 27x^3$.

19 The Method of Variation of Parameters

The technique described in Section 18 for determining a particular solution of the nonhomogeneous equation

$$y'' + P(x)y' + Q(x)y = R(x) \tag{1}$$

has two severe limitations: it can be used only when the coefficients $P(x)$ and $Q(x)$ are constants, and even then it works only when the right-hand term $R(x)$ has a particularly simple form. Within these limitations, however, this procedure is usually the easiest to apply.

We now develop a more powerful method that always works—regardless of the nature of P, Q, and R—provided only that the general solution of the corresponding homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{2}$$

is already known. We assume, then, that in some way the general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{3}$$

of (2) has been found. The method is similar to that discussed in Section 16; that is, we replace the constants c_1 and c_2 by unknown functions $v_1(x)$ and $v_2(x)$, and attempt to determine v_1 and v_2 in such a manner that

$$y = v_1 y_1 + v_2 y_2 \tag{4}$$

will be a solution of (1).[7] With two unknown functions to find, it will be necessary to have two equations relating these functions. We obtain one of these by requiring that (4) be a solution of (1). It will soon be clear what the second equation should be. We begin by computing the derivative of (4), arranged as follows:

$$y' = (v_1 y_1' + v_2 y_2') + (v_1' y_1 + v_2' y_2). \tag{5}$$

[7] This is the source of the name *variation of parameters*: we vary the parameters c_1 and c_2.

Another differentiation will introduce second derivatives of the unknowns v_1 and v_2. We avoid this complication by requiring the second expression in parentheses to vanish:

$$v_1'y_1 + v_2'y_2 = 0. \tag{6}$$

This gives

$$y' = v_1y_1' + v_2y_2', \tag{7}$$

so

$$y'' = v_1y_1'' + v_1'y_1' + v_2y_2'' + v_2'y_2'. \tag{8}$$

On substituting (4), (7), and (8) into (1), and rearranging, we get

$$v_1(y_1'' + Py_1' + Qy_1) + v_2(y_2'' + Py_2' + Qy_2) + v_1'y_1' + v_2'y_2' = R(x). \tag{9}$$

Since y_1 and y_2 are solutions of (2), the two expressions in parentheses are equal to 0, and (9) collapses to

$$v_1'y_1' + v_2'y_2' = R(x). \tag{10}$$

Taking (6) and (10) together, we have two equations in the two unknowns v_1' and v_2':

$$v_1'y_1 + v_2'y_2 = 0,$$
$$v_1'y_1' + v_2'y_2' = R(x).$$

These can be solved at once, giving

$$v_1' = \frac{-y_2R(x)}{W(y_1, y_2)} \quad \text{and} \quad v_2' = \frac{y_1R(x)}{W(y_1, y_2)}. \tag{11}$$

It should be noted that these formulas are legitimate, for the Wronskian in the denominators is nonzero by the linear independence of y_1 and y_2. All that remains is to integrate formulas (11) to find v_1 and v_2:

$$v_1 = \int \frac{-y_2R(x)}{W(y_1, y_2)}dx \quad \text{and} \quad v_2 = \int \frac{y_1R(x)}{W(y_1, y_2)}dx. \tag{12}$$

We can now put everything together and assert that

$$y = y_1\int \frac{-y_2R(x)}{W(y_1, y_2)}dx + y_2\int \frac{y_1R(x)}{W(y_1, y_2)}dx \tag{13}$$

is the particular solution of (1) we are seeking.

The reader will see that this method has disadvantages of its own. In particular, the integrals in (12) may be difficult or impossible to work out. Also, of course, it is necessary to know the general solution of (2) before the process can even be started; but this objection is really immaterial because we are unlikely to care about finding a particular solution of (1) unless the general solution of (2) is already at hand.

The method of variation of parameters was invented by the French mathematician Lagrange in connection with his epoch-making work in analytical mechanics (see Appendix A in Chapter 12).

Example 1. Find a particular solution of $y'' + y = \csc x$.

The corresponding homogeneous equation $y'' + y = 0$ has $y(x) = c_1 \sin x + c_2 \cos x$ as its general solution, so $y_1 = \sin x$, $y_1' = \cos x$, $y_2 = \cos x$, and $y_2' = -\sin x$. The Wronskian of y_1 and y_2 is

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = -\sin^2 x - \cos^2 x = -1,$$

so by (12) we have

$$v_1 = \int \frac{-\cos x \csc x}{-1}\, dx = \int \frac{\cos x}{\sin x}\, dx = \log(\sin x)$$

and

$$v_2 = \int \frac{\sin x \csc x}{-1}\, dx = -x.$$

Accordingly,

$$y = \sin x \log(\sin x) - x \cos x$$

is the desired particular solution.

Problems

1. Find a particular solution of

$$y'' - 2y' + y = 2x,$$

first by inspection and then by variation of parameters.

2. Find a particular solution of

$$y'' - y' - 6y = e^{-x},$$

first by undetermined coefficients and then by variation of parameters.

3. Find a particular solution of each of the following equations:

(a) $y'' + 4y = \tan 2x$;

(b) $y'' + 2y' + y = e^{-x} \log x$;

(c) $y'' - 2y' - 3y = 64xe^{-x}$;

(d) $y'' + 2y' + 5y = e^{-x} \sec 2x$;

(e) $2y'' + 3y' + y = e^{-3x}$;

(f) $y'' - 3y' + 2y = (1 + e^{-x})^{-1}$.

4. Find a particular solution of each of the following equations:

(a) $y'' + y = \sec x$;

(b) $y'' + y = \cot^2 x$;

(c) $y'' + y = \cot 2x$;

(d) $y'' + y = x \cos x$;

(e) $y'' + y = \tan x$;

(f) $y'' + y = \sec x \tan x$;

(g) $y'' + y = \sec x \csc x$.

5. (a) Show that the method of variation of parameters applied to the equation $y'' + y = f(x)$ leads to the particular solution

$$y_p(x) = \int_0^x f(t)\sin(x-t)dt.$$

(b) Find a similar formula for a particular solution of the equation $y'' + k^2 y = f(x)$, where k is a positive constant.

6. Find the general solution of each of the following equations:

(a) $(x^2 - 1)y'' - 2xy' + 2y = (x^2 - 1)^2$;

(b) $(x^2 + x)y'' + (2 - x^2)y' - (2 + x)y = x(x + 1)^2$;

(c) $(1 - x)y'' + xy' - y = (1 - x)^2$;

(d) $xy'' - (1 + x)y' + y = x^2 e^{2x}$;

(e) $x^2 y'' - 2xy' + 2y = xe^{-x}$

20 Vibrations in Mechanical and Electrical Systems

Generally speaking, vibrations occur whenever a physical system in stable equilibrium is disturbed, for then it is subject to forces tending to restore its equilibrium. In the present section we shall see how situations of this kind can lead to differential equations of the form