Tema 5: INTEGRACIÓ:

① Calculeu les primitives següents:

(a) $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \int \frac{1/a^2}{1+\left(\frac{x}{a}\right)^2}\,dx = \frac{1}{a}\int \frac{1/a}{1+\left(\frac{x}{a}\right)^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

(b) $\displaystyle\int \frac{1}{\sqrt{2x-x^2}}\,dx \underset{\substack{\uparrow \\ \text{completem} \\ \text{quadrats}}}{=} \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx = \arcsin(x-1) + C$

Obs: $\arcsin(-x) = -\arcsin(x)$

$\arccos(-x) = \pi - \arccos(x)$

$\arccos(x) = \frac{\pi}{2} - \arcsin(x)$

(c) $\displaystyle\int e^{e^x} e^x\,dx = e^{e^x} + C$

(d) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{2x}{\sqrt{1-(x^2)^2}}\,dx = \frac{1}{2}\arcsin(x^2) + C$

(e) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx = 2e^{\sqrt{x}} + C$

(f) $\displaystyle\int \frac{\arctan(x)}{1+x^2}\,dx = \frac{1}{2}\arctan^2(x) + C$

(g) $\displaystyle\int \frac{1+\sin x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x}\,dx + \int \frac{\sin x}{\cos^2 x}\,dx = \tan x + \frac{1}{\cos x} + C = \frac{1+\sin x}{\cos x} + C$

(h) $\displaystyle\int \frac{1}{x\ln x}\,dx = \ln(\ln x) + C$

(i) $\displaystyle\int \ln(\cos x)\cdot\tan x\,dx = \frac{-1}{2}\int 2\ln(\cos x)\,\frac{-\sin x}{\cos x}\,dx = -\frac{1}{2}\ln^2(\cos x) + C$

(j) $\displaystyle\int \frac{x}{\cos^2(x^2)}\,dx = \frac{1}{2}\int \frac{2x}{\cos^2(x^2)}\,dx = \frac{1}{2}\tan(x^2) + C$

(k) $\displaystyle\int \frac{1}{\cosh x}\,dx = \int \frac{\cosh x}{\cosh^2 x}\,dx = \int \frac{\cosh x}{1+\sinh^2 x}\,dx = \operatorname{arctg}(\sinh x) + C.$

② Calculeu les primitives següents, pel mètode d'integració per parts:

(a) $\int x \arctan x \, dx = \frac{x^2}{2} \operatorname{arctg} x - \int \frac{x^2}{2(1+x^2)} dx = \frac{x^2}{2} \arctan x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx =$

$f' = x$, $g = \arctan x$

$f = \frac{x^2}{2}$, $g' = \frac{1}{1+x^2}$

$= \frac{x^2}{2} \arctan x - \frac{1}{2}\left(x - \arctan x\right) + C = \frac{x^2+1}{2} \arctan x - \frac{x}{2} + C.$

(b) $\int x \ln^2 x \, dx = \frac{x^2}{2} \ln^2 x - \int \frac{x^2}{2} \cdot 2 \ln x \cdot \frac{1}{x} dx = \frac{x^2}{2} (\ln x)^2 - \underbrace{\int x \ln x \, dx}_{= I}$

$f'(x) = x$, $g(x) = \ln^2 x$

$f(x) = \frac{x^2}{2}$, $g'(x) = \frac{2 \ln x}{x}$

$I = \int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C.$

$f'(x) = x$, $g(x) = \ln x$

$f(x) = \frac{x^2}{2}$, $g'(x) = \frac{1}{x}$

Per tant, $\int x \ln^2 x \, dx = \frac{x^2}{2}\left(\ln^2 x - \ln x + \frac{1}{2}\right) + C$

(c) $\int \ln\left(\sqrt{1+x^2}\right) dx = x \ln \sqrt{1+x^2} - \int \frac{x^2}{1+x^2} dx = x \ln \sqrt{1+x^2} - x + \operatorname{atan} x + C$

apartat (a)

$f'(x) = 1$, $g(x) = \ln \sqrt{1+x^2}$

$f(x) = x$, $g'(x) = \frac{x}{1+x^2}$

(d) $\int x^3 e^{x^2} dx = \frac{1}{2} \int x^2 \cdot 2x e^{x^2} dx = \frac{1}{2}\left(x^2 e^{x^2} - \int 2x e^{x^2} dx\right) =$

$f(x) = x^2$, $g'(x) = 2x e^{x^2}$

$f'(x) = 2x$, $g(x) = e^{x^2}$

$= \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C = \frac{x^2 - 1}{2} e^{x^2} + C.$

(e) $\int e^{ax}\cos(bx)\,dx$

- $a=0$: $\int \cos(bx)\,dx = \frac{1}{b}\sin(bx) + C$

- $b=0$: $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

- $a,b \neq 0$: $\int e^{ax}\cos bx\,dx = \frac{1}{a}e^{ax}\cos bx + \int \frac{b}{a}e^{ax}\sin bx\,dx =$

$f'(x)=e^{ax}$, $g(x)=\cos bx$ $\qquad$ $f'(x)=e^{ax}$, $g(x)=\sin bx$

$f(x)=\frac{1}{a}e^{ax}$, $g'(x)=-b\sin bx$ $\qquad$ $f(x)=\frac{1}{a}e^{ax}$, $g'(x)=b\cos bx$

$$= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\left(\frac{1}{a}e^{ax}\sin bx - \int \frac{b}{a}e^{ax}\cos bx\,dx\right)$$

Anomenant $I=\int e^{ax}\cos bx\,dx$, tenim que

$$I = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}I \Rightarrow$$

$$\Rightarrow \left(\frac{a^2+b^2}{a^2}\right)I = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx \Rightarrow$$

$$\Rightarrow I = \frac{ae^{ax}\cos bx + be^{ax}\sin bx}{a^2+b^2}$$

Notem que fent $a=0$ o $b=0$ recuperem els dos casos anteriors.

③ Calculeu les primitives següents, mitjançant un canvi de variable.

(a) $\int \frac{1}{\sqrt{1+e^x}}\,dx = \int \frac{2}{u^2-1}\,du = -2\int \frac{1}{1-u^2}\,du = -2\,\mathrm{argtanh}(u) + C =$

Canvi de variable:

$u=\sqrt{1+e^x}$

$du = \frac{1}{2u}e^x\,dx = \frac{u^2-1}{2u}\,dx$

$dx = \frac{2u\,du}{u^2-1}$

$$= -2\,\mathrm{argtanh}\left(\sqrt{1+e^x}\right) + C = \ln\left(\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right) + C'$$

$\mathrm{argtanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ amb domini $(-1,1)$

(b) $\displaystyle\int \frac{\sqrt{1-x}}{1-\sqrt{x}}\,dx = \int \frac{\sqrt{1-x}}{1-\sqrt{x}}\,\frac{1+\sqrt{x}}{1+\sqrt{x}}\,dx = \int \frac{\sqrt{1-x}\cdot(1+\sqrt{x})}{1-x}\,dx = \int \frac{1+\sqrt{x}}{\sqrt{1-x}}\,dx =$

$$= \int \frac{1}{\sqrt{1-x}}\,dx + \int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx = -2\sqrt{1-x} + \underbrace{\int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx}_{= I}$$

Ara calculem I:

$$I = \int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx = \int \frac{\sqrt{1-t^2}}{t}(-2t)\,dt = -2\int \sqrt{1-t^2}\,dt \underset{\text{apartat (e)}}{=}$$

$$\begin{cases} t = \sqrt{1-x} \\ t^2 = 1-x \Rightarrow \sqrt{x} = \sqrt{1-t^2} \\ dx = -2t\,dt \end{cases}$$

$$= -2\left(\frac{-\arccos t}{2} + \frac{t\sqrt{1-t^2}}{2}\right) + C = \arccos t - t\sqrt{1-t^2} + C =$$

$$= \arccos\sqrt{1-x} - \sqrt{1-x}\sqrt{x} + C.$$

Aleshores,

$$\int \frac{\sqrt{1-x}}{1-\sqrt{x}}\,dx = \arccos\sqrt{1-x} - (2+\sqrt{x})\sqrt{1-x} + C$$

(c) $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx \underset{\substack{u=\sqrt[6]{x} \\ dx = 6u^5du}}{=} \int \frac{6u^5\cdot u^3}{u^2+1}\,du = 6\int \frac{u^8}{u^2+1}\,du = 6\int\left(u^6 - u^4 + u^2 - 1 + \frac{1}{u^2+1}\right)du =$

$$\begin{array}{l|l} u^8 & u^2+1 \\ \hline -(u^8+u^6) & u^6-u^4+u^2-1 = b(x) \\ \quad -u^6 & \\ -(-u^6-u^4) & \\ \quad u^4 & \\ -(u^4+u^2) & \\ \quad -u^2 & \\ -(-u^2-1) & \\ \quad 1 = r(x) & \end{array}$$

$$\frac{p(x)}{q(x)} = b(x) + \frac{r(x)}{q(x)}$$

$$= \frac{6}{7}u^7 - \frac{6}{5}u^5 + \frac{6}{3}u^3 - 6u + 6\arctan(u) + C =$$

$$= \frac{6}{7}x\sqrt[6]{x} - \frac{6}{5}\sqrt[6]{x^5} + 2\sqrt{x} - 6\sqrt[6]{x} + 6\arctan(\sqrt[6]{x}) + C.$$

(d) $\displaystyle\int \frac{\ln(3x)}{x\ln(6x)}\,dx = \int \frac{u}{e^u/3\cdot(u+\ln 2)}\,\frac{e^u}{3}\,du = \int \frac{u}{u+\ln 2}\,du = \int\left(1-\frac{\ln 2}{u+\ln 2}\right)du =$

$$\begin{cases} u=\ln 3x \Rightarrow x=\dfrac{e^u}{3} \\ du=\dfrac{1}{x}\,dx \\ dx=\dfrac{e^u}{3}\,du \end{cases}$$

$$= u-\ln 2\cdot\int\frac{1}{u+\ln 2}\,du = u-\ln 2\cdot\ln(u+\ln 2)+C =$$

$$= \ln 3x - \ln 2\cdot\ln(\ln 6x)+C.$$

(e) $\displaystyle\int\sqrt{1-x^2}\,dx = -\int\sin^2 u\,du = -\int\left(\frac{1}{2}-\frac{\cos(2u)}{2}\right)du =$

$$\left.\begin{aligned} x&=\cos u \\ dx&=-\sin u\,du \end{aligned}\right\}$$

$\cos 2u = 1-2\sin^2 u$

$\sin^2 u = \frac{1}{2}(1-\cos 2u)$

$$= \frac{-u}{2}+\frac{1}{2}\int\cos 2u\,du = \frac{-u}{2}+\frac{\sin 2u}{4}+C = \frac{-u}{2}+\frac{\sin u\cos u}{2}+C =$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{-\arccos x}{2}+\frac{\sin(\arccos x)\cdot x}{2}+C = \frac{-\arccos x}{2}+\frac{x\sqrt{1-x^2}}{2}+C$$

$\sin(\arccos x)=\sqrt{1-x^2}$

(fent el canvi $x=\cos y$)

(f) $\displaystyle\int\sqrt{1+x^2}\,dx = \int\sqrt{1+\tan^2 u}\,\frac{1}{\cos^2 u}\,du = \int\frac{1}{\cos^3 u}\,du = \int\frac{1}{\cos^4 u}\,dt =$

$x=\tan u$

$dx=\dfrac{1}{\cos^2 u}\,du$

$t=\sin u$

$dt=\cos u\,du$

$$= \int\frac{1}{(1-\sin^2 u)^2}\,dt = \int\frac{1}{(1-t^2)^2}\,dt$$

Descomposem en fraccions simples:

$$\frac{1}{(1-t^2)^2} = \frac{A}{1-t}+\frac{B}{(1-t)^2}+\frac{C}{1+t}+\frac{D}{(1+t)^2} \Rightarrow \begin{cases} A+B+C+D=1 \\ -A+C=0 \Rightarrow A=C \\ -A+B-C+D=0 \\ A+2B-C-2D=0 \end{cases}$$

$2A+B+D=1$

$-2A+B+D=0$

$2B-2D=0 \longrightarrow B=D$

Aleshores, $A+B=\frac{1}{2}$; $A=B \Rightarrow A=B=C=D=\frac{1}{4}$.

Seguint amb la integral,

$$\int \frac{1}{(1-t^2)^2}\,dt = \int \frac{1/4}{1-t} + \frac{1/4}{(1-t)^2} + \frac{1/4}{1+t} + \frac{1/4}{(1+t)^2}\,dt =$$

$$= \frac{-1}{4}\ln(1-t) + \frac{1}{4}\frac{1}{1-t} + \frac{1}{4}\ln(1+t) - \frac{1}{4}\frac{1}{1+t} + C_1 =$$

$$= \frac{1}{4}\left(\ln\left(\frac{1+t}{1-t}\right) + \frac{1}{1-t} - \frac{1}{1+t}\right) + C_1 =$$

$$= \frac{1}{4}\left(\ln\left(\frac{1+t}{1-t}\right) + \frac{2t}{1-t^2}\right) + C_1 \underset{t=\sin u}{=} \frac{1}{4}\left(\ln\left(\frac{1+\sin u}{1-\sin u}\right) + \frac{2\sin u}{1-\sin^2 u}\right) + C_1 =$$

$$= \frac{1}{4}\left(\ln\left(\frac{1+\sin u}{1-\sin u}\right) + \frac{2\sin u}{\cos^2 u}\right) + C_1 \underset{u=\operatorname{atan} x}{=} \frac{1}{4}\left(\ln\left(\frac{1+\sin(\operatorname{atan} x)}{1-\sin(\operatorname{atan} x)}\right) + \frac{2\sin(\operatorname{atan} x)}{\cos^2(\operatorname{atan} x)}\right) +$$

$$= \frac{1}{4}\left(\ln\left(\frac{1+\frac{x}{\sqrt{1+x^2}}}{1-\frac{x}{\sqrt{1+x^2}}}\right) + \frac{2\frac{x}{\sqrt{1+x^2}}}{\frac{1}{1+x^2}}\right) + C_1 = \frac{1}{4}\left(\ln\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right) + 2x\sqrt{1+x^2}\right) + C_1 =$$

$\sin(\operatorname{atan} x) = \frac{x}{\sqrt{1+x^2}}$

$\cos(\operatorname{atan} x) = \frac{1}{\sqrt{1+x^2}}$

$$= \frac{1}{4}\left(\ln\left(\left(x+\sqrt{1+x^2}\right)^2\right) + 2x\sqrt{1+x^2}\right) = \frac{1}{2}\left(\ln\left(x+\sqrt{1+x^2}\right) + x\sqrt{1+x^2}\right) + C_1$$

$$= \frac{1}{2}\left(\sinh^{-1}(x) + x\sqrt{1+x^2}\right) + C_1.$$

$\operatorname{argsinh}(x) = \ln\left(x+\sqrt{1+x^2}\right)$

Alternativa: Usant les funcions hiperbòliques.

$$\int \sqrt{1+x^2}\,dx = \int \cosh^2 u\,du = \int \left(\frac{1}{2}\cosh(2u) + \frac{1}{2}\right) du =$$

$$\left.\begin{array}{l} x = \sinh u \\ dx = \cosh u \end{array}\right\}$$

$$\cosh^2 u = \frac{1}{2}\cosh(2u) + \frac{1}{2}$$

$$= \frac{u}{2} + \frac{1}{4}\sinh(2u) + C = \frac{1}{2}\left(u + \sinh(u)\cosh(u)\right) + C =$$

$$\sinh(2u) = 2\sinh(u)\cosh(u) \qquad u = \sinh^{-1}(x)$$

$$= \frac{1}{2}\left(\sinh^{-1}(x) + x\cdot\cosh(\operatorname{argsinh}(x))\right) + C =$$

$$\cosh(\sinh^{-1}(x)) = \sqrt{1+x^2}$$

$$= \frac{1}{2}\left(\sinh^{-1}(x) + x\sqrt{1+x^2}\right) + C$$

(g) $\int \sqrt{x^2-1}\,dx$

$$\int \sqrt{x^2-1}\,dx = \int \sinh^2 t\,dt = \frac{1}{2}\int(\cosh(2t) - 1)\,dt = \frac{1}{2}\int \cosh 2t\,dt - \frac{t}{2} + C =$$

$$\left.\begin{array}{l} x = \cosh t \\ dx = \sinh t\,dt \end{array}\right\} \qquad \sinh^2 x = \frac{\cosh(2x) - 1}{2}$$

$$= \frac{1}{2}\,\frac{1}{2}\sinh(2t) - \frac{t}{2} + C = \frac{1}{2}\underbrace{\sinh(\operatorname{argcosh} x)}_{\sqrt{x^2-1}}\cdot x - \frac{\operatorname{argcos} x}{2} + C =$$

$$\sinh(2x) = 2\sinh x\cosh x$$

$$= \frac{1}{2}\left(x\sqrt{x^2-1} - \operatorname{argcosh} x\right) + C$$

(h) $$\int \frac{x^2}{\sqrt{x^2-16}}\,dx = \int \frac{\frac{4^2}{\cos^2 t}}{\frac{4\sin t}{\cos t}}\cdot\frac{4\tan t}{\cos t}\,dt = 16\int \frac{1}{\cos^3 t}\,dt. =$$

Apartat (f)

$$\left\{\begin{array}{l} x = \dfrac{4}{\cos t} \\ dx = \dfrac{4\tan t}{\cos t}\,dt \end{array}\right.$$

$$= 16\cdot\frac{1}{4}\left(\ln\left(\frac{1+\sin t}{1-\sin t}\right) + \frac{2\sin t}{\cos^2 t}\right) + C = 4\left(\ln\left(\frac{1+\sin(\arccos(\frac{4}{x}))}{1-\sin(\arccos(\frac{4}{x}))}\right) + \frac{2\sin(\arccos(\frac{4}{x}))}{(\frac{4}{x})^2}\right) + C$$

$$t = \arccos\left(\frac{4}{x}\right)$$

$$= 4\left(\ln\left(\frac{1+\sqrt{1-\left(\frac{4}{x}\right)^2}}{1-\sqrt{1-\left(\frac{4}{x}\right)^2}}\right) + \frac{2\sqrt{1-\left(\frac{4}{x}\right)^2}}{\frac{4^2}{x^2}}\right) + C =$$

$\sin(\arccos x) = \sqrt{1-x^2}$

$$= 4\left(\ln\left(\frac{1+\frac{1}{x}\sqrt{x^2-16}}{1-\frac{1}{x}\sqrt{x^2-16}}\right) + \frac{\frac{2}{x}\sqrt{x^2-16}}{\frac{4^2}{x^2}}\right) + C =$$

$$= 4\left(\ln\left(\frac{x+\sqrt{x^2-16}}{x-\sqrt{x^2-16}}\right) + \frac{x}{8}\sqrt{x^2-16}\right) + C =$$

$$= 4\ln\left(\frac{\left(x+\sqrt{x^2-16}\right)^2}{16}\right) + \frac{x}{2}\sqrt{x^2-16} + C =$$

$$= \underbrace{8\ln\left(\frac{x}{4}+\sqrt{\left(\frac{x}{4}\right)^2-1}\right)} + \frac{x}{2}\sqrt{x^2-16} + C = 8\,\mathrm{argcosh}\left(\frac{x}{4}\right) + \frac{x}{2}\sqrt{x^2-16} + C.$$

(i) $$\int\sqrt{-x^2+2x}\,dx = \int\sqrt{1-(x-1)^2}\,dx = \int\sqrt{1-u^2}\,du = \int\cos^2 t\,dt =$$

completem quadrats $\begin{cases} u = x-1 \\ du = dx \end{cases}$ $\begin{cases} u = \sin t \\ du = \cos t\,dt \end{cases}$ $\cos^2 t = \frac{\cos(2t)+1}{2}$

$$= \int\left(\frac{1}{2} + \frac{\cos(2t)}{2}\right)dt = \frac{t}{2} + \frac{\sin(2t)}{4} + C =$$

$$= \frac{\arcsin u}{2} + \frac{1}{2}\cos t\sin t + C =$$

$$= \frac{\arcsin u}{2} + \frac{1}{2}u\cos(\arcsin u) + C =$$

$\cos(\arcsin u) = \sqrt{1-u^2}$

$$= \frac{\arcsin u}{2} + \frac{u}{2}\sqrt{1-u^2} + C =$$

$u = x-1$

$$= \frac{\arcsin(x-1)}{2} + \frac{x-1}{2}\sqrt{2x-x^2} + C.$$

④ Calculeu les primitives racionals següents.

(a) $$\int \frac{2x+1}{x^3-3x^2+3x-1}\,dx = \int \frac{2x+1}{(x-1)^3}\,dx = \int\left(\frac{2}{(x-1)^2}+\frac{3}{(x-1)^3}\right)dx =$$

$$\frac{2x+1}{(x-1)^3} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3} \Rightarrow \begin{cases} A=0 \\ B=2 \\ C=3 \end{cases}$$

$$= \frac{-2}{x-1} - \frac{3}{2}\,\frac{1}{(x-1)^2} + C$$

(b) $$\int \frac{x^2-x+12}{(x-1)^2(x+2)}\,dx$$

Descomposem en fraccions simples:

$$\frac{x^2-x+12}{(x-1)^2(x+2)} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+2} = \frac{A(x-1)(x+2)+B(x+2)+C(x-1)^2}{(x-1)^2(x+2)} =$$

$$= \frac{(A+C)x^2+(A+B-2C)x-2A+2B+C}{(x-1)^2(x+2)} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} A+C=1 \\ A+B-2C=-1 \\ -2A+2B+C=12 \end{array}\right\} \Rightarrow \begin{cases} A=-1 \\ B=4 \\ C=2 \end{cases}$$

Aleshores

$$\int \frac{x^2-x+12}{(x-1)^2(x+2)}\,dx = \int\left(\frac{-1}{x-1}+\frac{4}{(x-1)^2}+\frac{2}{x+2}\right)dx =$$

$$= -\ln(x-1) - 4\,\frac{1}{x-1} + 2\ln(x+2) + C.$$

(c) $$\int \frac{x+3}{x^2+x+2}\,dx = \int \frac{1}{2}\left(\frac{2x+1}{x^2+x+2}+\frac{5}{x^2+x+2}\right)dx =$$

$$= \frac{1}{2}\underbrace{\int \frac{2x+1}{x^2+x+2}\,dx}_{I_1} + \frac{5}{2}\underbrace{\int \frac{1}{x^2+x+2}\,dx}_{I_2}$$

$$I_1 = \ln(x^2+x+2)$$

$$I_2 = \int \frac{1}{x^2+x+2}\,dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}\,dx = \frac{4}{7}\int \frac{1}{\left(\frac{x+\frac{1}{2}}{\sqrt{7}/2}\right)^2+1}\,dx =$$

completent quadrats.

$$= \frac{4}{7}\int \frac{1}{\left(\frac{2x+1}{\sqrt{7}}\right)^2+1}\,dx = \frac{4}{7}\,\frac{\sqrt{7}}{2}\int \frac{2/\sqrt{7}}{\left(\frac{2x+1}{\sqrt{7}}\right)^2+1}\,dx = \frac{4}{7}\,\frac{\sqrt{7}}{2}\arctan\left(\frac{2x+1}{\sqrt{7}}\right)+C' =$$

$$= \frac{2\sqrt{7}}{7}\arctan\left(\frac{2x+1}{\sqrt{7}}\right)+C.$$

Aleshores,

$$\int \frac{x+3}{x^2+x+2}\,dx = \frac{\ln(x^2+x+2)}{2} + \frac{5\sqrt{7}}{7}\operatorname{atan}\left(\frac{2x+1}{\sqrt{7}}\right)+C$$

(d) $$\int \frac{x^3}{x^3+1}\,dx = \int \frac{x^3+1-1}{x^3+1}\,dx = \int\left(1-\frac{1}{x^3+1}\right)dx = x - \underbrace{\int \frac{1}{x^3+1}\,dx}_{I_1}$$

Descomposem el denominador de I_1.

$$x^3+1 = (x+1)\underbrace{(x^2-x+1)}_{\Delta<0 \Rightarrow \text{ no té arrels reals.}}$$

Ara descomposem en fraccions simples:

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} = \frac{A(x^2-x+1)+(Bx+C)(x+1)}{(x+1)(x^2-x+1)} =$$

$$= \frac{Ax^2-Ax+A+Bx^2+Cx+Bx+C}{(x+1)(x^2-x+1)} =$$

$$= \frac{(A+B)x^2+(-A+B+C)x+A+C}{(x+1)(x^2-x+1)} \Rightarrow \left\{\begin{array}{l} A+B=0 \\ -A+B+C=0 \\ A+C=1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} A=\frac{1}{3} \\ B=\frac{-1}{3} \\ C=2/3 \end{array}\right.$$

Aleshores,

$$I_1 = \int \frac{1}{1+x^3}\,dx = \int \frac{1}{3}\,\frac{1}{x+1}\,dx + \int \frac{-1/3\,x + 2/3}{x^2-x+2}\,dx =$$

$$= \frac{1}{3}\ln(x+1) - \frac{1}{3}\underbrace{\int \frac{x-2}{x^2-x+1}\,dx}_{I_2}$$

$$I_2 = \int \frac{x-2}{x^2-x+1}\,dx = \frac{1}{2}\int \frac{2x-4}{x^2-x+1}\,dx = \frac{1}{2}\int\left(\frac{2x-1}{x^2-x+1} - \frac{3}{x^2-x+1}\right)dx =$$

$$= \frac{1}{2}\ln(x^2-x+1) - \frac{3}{2}\underbrace{\int \frac{1}{x^2-x+1}\,dx}_{I_3}$$

Finalment trobem I_3:

$$I_3 = \int \frac{1}{x^2-x+1}\,dx = \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx = \frac{4}{3}\int \frac{1}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1}\,dx =$$

completant quadrats,

$$x^2-x+1 = \left(x-\frac{1}{2}\right)^2+\frac{3}{4}$$

$$= \frac{4}{3}\,\frac{\sqrt{3}}{2}\int \frac{2/\sqrt{3}}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1}\,dx = \frac{2\sqrt{3}}{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)+C_1.$$

Recopilant,

$$\int \frac{x^3}{x^3+1}\,dx = x - I_1 = x - \frac{1}{3}\ln(x+1) + \frac{1}{3}I_2 =$$

$$I_1 = \frac{1}{3}\ln(x+1) - \frac{1}{3}I_2 \qquad I_2 = \frac{1}{2}\ln(x^2-x+1) - \frac{3}{2}I_3$$

$$= x - \frac{1}{3}\ln(x+1) + \frac{1}{6}\ln(x^2-x+1) - \frac{1}{2}I_3 =$$

$$I_3 = \frac{2\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right)+C_1$$

$$= x - \frac{1}{3}\ln(x+1) + \frac{1}{6}\ln(x^2-x+1) - \frac{\sqrt{3}}{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)+C_1.$$

(5) Calculeu les primitives següents, sense fer cap canvi de variable:

(a) $$\int \cos^3 x\,dx = \frac{3}{4}\int \cos x\,dx + \frac{1}{4}\int \cos 3x\,dx = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x + C_1.$$

$$\cos^3 x = \frac{3\cos x + \cos 3x}{4}.$$

Aquest tipus de fórmules trigonomètriques es poden deduir de la fórmula de De Moivre: $e^{i\theta} = \cos\theta + i\sin\theta$. La fórmula de De Moivre es deduïx fàcilment de les definicions de sinus i cosinus:

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$(e^{ix})^n = e^{inx}$$

$$(\cos x + i\sin x)^n = \cos(nx) + i\sin(nx)$$

En el cas $n=3$, tenim

$$(\cos x + i\sin x)^3 = \cos 3x + i\sin 3x \Rightarrow$$

$$\Rightarrow \cos^3 x + 3\cos^2 x\, i\sin x - 3\cos x \sin^2 x - i\sin^3 x = \cos 3x + i\sin 3x \Rightarrow$$

$$\Rightarrow \begin{cases} \cos 3x = \cos^3 x - 3\cos x \sin^2 x \Rightarrow \cos 3x = 4\cos^3 x - 3\cos x. \\ \sin 3x = 3\cos^2 x \sin x - \sin^3 x \Rightarrow \sin 3x = 3\sin x - 4\sin^3 x. \end{cases}$$

<u>Alternativa:</u>

$$\int \cos^3 x\, dx = \int \cos^2 x \cdot \cos x\, dx = \int (1-\sin^2 x)\cos x\, dx = \int \cos x\, dx - \int \sin^2 x \cos x\, dx =$$

$$= \sin x - \frac{1}{3}\sin^3 x + C.$$

(b) $$\int \sin^2 x\, dx \underset{\sin^2 x = \frac{1-\cos 2x}{2}}{=} \int \frac{1}{2}\, dx - \frac{1}{2}\int \cos 2x\, dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C \underset{\sin 2x = 2\sin x \cos x.}{=} \frac{x}{2} - \frac{1}{2}\sin x \cos x + C$$

(c) $$\int \cos^6\left(\frac{x}{2}\right) dx \underset{\cos^2 x = \frac{1+\cos 2x}{2}}{=} \int \left(\frac{1}{2} + \frac{\cos x}{2}\right)^3 dx = \int \left(\frac{1}{8} + \frac{3}{8}\cos x + \frac{3}{8}\cos^2 x + \frac{1}{8}\cos^3 x\right) dx =$$

$$= \frac{x}{8} + \frac{3}{8}\sin x + \frac{3}{8}\int \cos^2 x\, dx + \frac{1}{8}\int \cos^3 x\, dx \underset{\text{apartat (a)}}{=}$$

$$= \frac{x}{8} + \frac{3}{8}\sin x + \frac{3}{8}\int \left(\frac{1}{2} + \frac{\cos 2x}{2}\right) dx + \frac{1}{8}\left(\sin x - \frac{1}{3}\sin^3 x\right) + C =$$

$$= \frac{x}{8} + \frac{3}{8}\sin x + \frac{3x}{16} + \frac{3}{8}\,\frac{1}{4}\sin 2x + \frac{1}{8}\sin x - \frac{1}{24}\sin^3 x + C =$$

$$= \frac{5}{16}x + \frac{1}{2}\sin x + \frac{3}{32}\sin 2x - \frac{1}{24}\sin^3 x + C.$$

(d) $$\int \sin^3 x \cos^2 x\, dx \underset{\sin^2 x = 1-\cos^2 x}{=} \int \sin x\,(1-\cos^2 x)\cos^2 x\, dx = \int (\cos^2 x \sin x - \cos^4 x \sin x)\, dx =$$

$$= \frac{1}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C.$$

(e) $\displaystyle\int \sinh^2 x\,dx = \int \frac{1}{2}\cosh(2x)\,dx - \frac{x}{2} + C = \frac{1}{4}\sinh 2x - \frac{x}{2} + C =$

$\sinh^2 x = \dfrac{\cosh(2x)-1}{2}$ $\qquad$ $\sinh 2x = 2\sinh x\cosh x$

$$= \frac{1}{2}(\sinh x\cosh x - x) + C.$$

(6) Canvis de variable per a integrals de funcions racionals de funcions trigonomètriques.

Sigui $R(z_1, z_2)$ una funció racional (quocient de polinomis) en dues variables.

Els canvis de variable següents transformen integrals de funcions trigonomètriques, del tipus $R(\cos x, \sin x)$, en integrals racionals:

- $\sin x = t$, quan $R(-\cos x, \sin x) = -R(\cos x, \sin x)$.
- $\cos x = t$, quan $R(\cos x, -\sin x) = -R(\cos x, \sin x)$.
- $\tan x = t$, quan $R(-\cos x, -\sin x) = R(\cos x, \sin x)$.

Utilitzeu algun d'aquests canvis de variable per calcular les primitives següents:

(a) $\displaystyle\int \frac{dx}{\sin x} = \int \frac{-dt}{\sin^2 x} = -\int \frac{dt}{1-t^2} = -\int \frac{1}{2}\left(\frac{1}{1-t} + \frac{1}{1+t}\right)dt =$

$t = \cos x$

$dt = -\sin x\,dx$

$\dfrac{1}{1-t^2} = \dfrac{A}{1-t} + \dfrac{B}{1+t} \longrightarrow A = B = \dfrac{1}{2}$

$$= -\frac{1}{2}\left(-\ln(1-t) + \ln(1+t)\right) = \frac{1}{2}\ln\left(\frac{1-t}{1+t}\right) + C =$$

$t = \cos x$

$$= \frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right) + C.$$

(b) $\displaystyle\int \frac{dx}{\cos x} = \int \frac{dt}{\cos^2 x} = \int \frac{dt}{1-t^2} = \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right) + C.$

$t = \sin x$ $\qquad$ apartat (a)

$dt = \cos x\,dx$

(c) $\displaystyle\int \tan^4 x\,dx = \int \frac{t^4}{t^2+1}\,dt = \int\left(t^2 - 1 + \frac{1}{t^2+1}\right)dt = \frac{t^3}{3} - t + \operatorname{atan} t + C =$

$t = \tan x$ $\qquad$ dividint els polinomis, $\qquad$ $= \dfrac{\tan^3 x}{3} - \tan x + x + C.$

$dt = (1+t^2)\,dx$ $\qquad$ $\dfrac{t^4}{t^2+1} = t^2 - 1 + \dfrac{1}{t^2+1}$ $\qquad$ desfent el canvi

(d) $\displaystyle\int \frac{\sin x \cos x}{1-\cos x}\,dx = \int \frac{\sin x \cdot t}{1-t}\,\frac{dt}{-\sin x} = -\int \frac{t}{1-t}\,dt =$

$t = \cos x$

$dt = -\sin x\, dx$

$$= -\int\left(\frac{1}{1-t} - 1\right)dt = -\left(-\ln(1-t) - t\right) + C = \ln(1-t) + t + C =$$

$$= \cos x + \ln(1-\cos x) + C.$$

(7) Sigui $f(x) = \dfrac{e^x \sin x}{x}$. Justifiqueu que f és integrable a $[0,1]$ i es compleix la desigualtat

$$0 \le \int_0^1 f(x)\,dx \le e-1.$$

Notem que f és contínua en $(0,1]$ i fitada a $[0,1]$:

$$0 \le f(x) \le e^x$$

$\sin x \le x$ en $[0,1]$.

Aleshores, f és integrable en $[0,1]$ i

$$0 \le \int_0^1 f(x)\,dx \le \int_0^1 e^x\,dx = e-1$$

(8) Sigui f una funció contínua i positiva a l'interval $[a,b]$ tal que $\int_a^b f(x)\,dx = 0$. Proveu que $f(x) = 0$ per tot $x \in [a,b]$.

Usarem el Teorema Fonamental del Càlcul. Com que $f(x) \ge 0$ $\forall x \in [a,b]$, aleshores,

$$0 \le \int_a^x f(t)\,dt \le \int_a^b f(t)\,dt = 0 \Rightarrow F(x) = \int_a^x f(t)\,dt = 0$$

Ara, pel Teorema Fonamental del Càlcul, com que f és contínua en $[a,b]$, F és derivable en (a,b). Aleshores, $F'(x) = f(x) = 0$ $\forall x \in (a,b)$. Per continuïtat $f(x) = 0$ $\forall x \in [a,b]$, com volíem veure.

<u>Alternativa</u>: sense usar el Teorema Fonamental del càlcul:

Suposem que f no és idènticament 0 en $[a,b]$. Aleshores $\exists x_0 \in (a,b)$ tal que $f(x_0) > 0$. Pertant $\exists \varepsilon > 0$ tal que $f(x) > 0$ $\forall x \in (x_0-\varepsilon, x_0+\varepsilon)$ ja que f és contínua. Ara,

$$\int_a^b f(t)\,dt \ge \int_{x_0-\varepsilon}^{x_0+\varepsilon} f(t)\,dt > 0$$

i arribem a una contradicció. Aleshores, $f(x) = 0$ $\forall x \in [a,b]$.

(9) Feint servir que, si $x>0$, $\int_1^x \frac{dt}{t} = \ln x$, justifiqueu les desigualtats

$$\frac{1}{n+1} < \ln \frac{n+1}{n} < \frac{1}{n},$$

i deduiu que $e = \lim_n \left(1+\frac{1}{n}\right)^n$

Notem que $\frac{1}{n+1} = \int_n^{n+1} \frac{dt}{n+1}$ i $\frac{1}{n} = \int_n^{n+1} \frac{dt}{n}$.

Aleshores,

$$\frac{1}{n+1} = \int_n^{n+1} \frac{dt}{n+1} < \int_n^{n+1} \frac{dt}{t} = \ln\left(\frac{n+1}{n}\right) < \int_n^{n+1} \frac{dt}{n} = \frac{1}{n}$$

com volíem veure. Provem ara que $\left(1+\frac{1}{n}\right)^n \xrightarrow[n]{} e$.

$$\ln\left(\frac{n+1}{n}\right) < \frac{1}{n} \Rightarrow \ln\left(\left(1+\frac{1}{n}\right)^n\right) < 1 \Rightarrow \left(1+\frac{1}{n}\right)^n < e.$$

per altra banda,

$$\frac{1}{n+1} < \ln\left(\frac{n+1}{n}\right) \Rightarrow 1 < \ln\left(\left(1+\frac{1}{n}\right)^{n+1}\right) \Rightarrow e < \left(\frac{n+1}{n}\right)\left(1+\frac{1}{n}\right)^n \Rightarrow$$

$$\Rightarrow e\frac{n}{n+1} < \left(1+\frac{1}{n}\right)^n.$$

Aleshores,

$$e\frac{n}{n+1} < \left(1+\frac{1}{n}\right)^n < e$$

Passant al límit, $e \leq \lim_n \left(1+\frac{1}{n}\right)^n \leq e \Rightarrow \lim_n \left(1+\frac{1}{n}\right)^n = e$

(10) Calculeu els límits de les successions següents expressant-les com a sumes de Riemann.

(a) $x_n = \frac{1^\alpha + 2^\alpha + \dots + n^\alpha}{n^{\alpha+1}}$, $\alpha > 0$.

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} \frac{1^\alpha+2^\alpha+\dots+n^\alpha}{n^{\alpha+1}} = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^\alpha = \int_0^1 x^\alpha dx = \frac{1}{\alpha+1}$$

(b) $x_n = \frac{1}{\sqrt{n(n+1)}} + \frac{1}{\sqrt{n(n+2)}} + \dots + \frac{1}{\sqrt{n(n+n)}}$

$$\lim_{n\to\infty} x_n = \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n(n+k)}} = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{1}{\sqrt{1+\frac{k}{n}}} = \int_0^1 \frac{1}{\sqrt{1+x}}\,dx =$$

$$= \left[2\sqrt{1+x}\right]_0^1 = 2(\sqrt{2}-1)$$

(c) $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} = \int_0^1 \frac{1}{1+x}\,dx = \left[\ln(1+x)\right]_0^1 = \ln 2.$$

(d) $x_n = \frac{n}{n^2+1} + \frac{n}{n^2+4} + \dots + \frac{n}{n^2+n^2}$

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2} = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+\frac{k^2}{n^2}} = \int_0^1 \frac{1}{1+x^2}\,dx = \left[\operatorname{atan} x\right]_0^1 = \frac{\pi}{4}$$

(e) $x_n = \frac{n+1}{n^2+1} + \frac{n+2}{n^2+4} + \dots + \frac{n+n}{n^2+n^2}$

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{n+k}{n^2+k^2} = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1+\frac{k}{n}}{1+\frac{k^2}{n^2}} = \int_0^1 \frac{1+x}{1+x^2}\,dx =$$

$$= \left[\arctan x + \frac{1}{2}\ln(1+x^2)\right]_0^1 = \frac{\pi}{4} + \frac{\ln 2}{2}$$

(f) $x_n = \sum_{k=1}^{n} \frac{(n-k)k}{n^3}$

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{(n-k)k}{n^3} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(1-\frac{k}{n}\right)\frac{k}{n} = \int_0^1 (1-x)x\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 =$$

$$= \frac{1}{6}$$

(g) $x_n = \frac{1}{n^2} \sum_{k=1}^{n} k \sin\left(\frac{k}{n}\frac{\pi}{2}\right)$

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} \frac{1}{n^2} \sum_{k=1}^{n} k \sin\left(\frac{k}{n}\frac{\pi}{2}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k}{n} \sin\left(\frac{k}{n}\frac{\pi}{2}\right) = \int_0^1 x \sin\left(\frac{\pi}{2}x\right) dx =$$

$$= \underbrace{\left[-x\cos\left(\frac{\pi}{2}x\right)\frac{2}{\pi}\right]_0^1}_{0-0=0} + \int_0^1 \frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right) dx = \frac{2}{\pi}\frac{2}{\pi}\left[\sin\left(\frac{\pi}{2}x\right)\right]_0^1 = \frac{4}{\pi^2}$$

$u = x \qquad du = dx$

$dv = \sin\left(\frac{\pi}{2}x\right)dx \qquad v = -\cos\left(\frac{\pi}{2}x\right)\cdot\frac{2}{\pi}$

(h) $x_n = \left(\frac{(2n)!}{n!\,n^n}\right)^{1/n}$

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} \left(\frac{(2n)!}{n!\,n^n}\right)^{1/n} = e^{\lim_{n\to+\infty} \frac{1}{n}\ln\left(\frac{(2n)!}{n!\,n^n}\right)} =$$

$$= e^{\lim_{n\to+\infty} \frac{1}{n}\ln\left(\frac{2n(2n-1)\cdots(n+1)}{n^n}\right)} =$$

$$= e^{\lim_{n\to+\infty} \frac{1}{n} \ln\left(\frac{n+1}{n}\cdot\frac{n+2}{n}\cdots\frac{n+n}{n}\right)} = e^{\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \ln\left(1+\frac{k}{n}\right)} =$$

$$= e^{\int_0^1 \ln(1+x)\,dx} = e^{2\ln 2 - 1} = 4/e$$

$$\int_0^1 \ln(1+x)\,dx = \left[x\cdot\ln(1+x)\right]_0^1 - \int_0^1 \frac{x}{1+x}\,dx = \ln 2 - \int_0^1\left(1-\frac{1}{1+x}\right)dx =$$

$u = \ln(1+x) \quad du = \frac{dx}{1+x}$ $\qquad = \ln 2 - 1 + \ln 2 = 2\ln 2 - 1$

$dv = dx \quad v = x$

(i) $x_n = \sum_{k=np+1}^{nq} \frac{1}{k} \quad (p,q\in\mathbb{N},\ p<q)$

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty}\sum_{k=np+1}^{nq}\frac{1}{k} = \lim_{n\to+\infty}\sum_{k=1}^{n(q-p)}\frac{1}{np+k} = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n(q-p)}\frac{1}{p+\frac{k}{n}} = \int_0^{q-p}\frac{1}{p+x}\,dx =$$

$$= \left[\ln(p+x)\right]_0^{q-p} = \ln q - \ln p = \ln\frac{q}{p}$$

(11) Siguin f i g dues funcions definides en $[a,b]$, i $c\in(a,b)$. Suposem que f és integrable en $[a,b]$.

(a) Si $g(x) = f(x)$ per a tot $x\neq c$, i $g(c)\neq f(c)$, proveu que g és integrable i $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$.

Definim $h = f-g$ s'anul·la en tots els punts excepte en el punt $x=c$.
Cal veure que h és integrable i que té integral 0.
Si $h(c)>0$, $s(h,\Pi_\varepsilon)$ per tota partició Π_ε donat $\varepsilon>0$ i,

$$S(h,\Pi_\varepsilon) = f(c)\Delta x_i + f(c)\Delta x_{i+1} < \varepsilon,$$

si $c = x_i$ i $\Delta x_i, \Delta x_{i+1} < \frac{\varepsilon}{2f(c)}$.

Aleshores, h és integrable i té integral 0. Per tant g és integrable i $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$

(b) Suposem que f i g prenen els mateixos valors a $[a,b]$ excepte en un nombre finit de punts. Proveu que g és integrable i $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$.

Dividim l'interval $[a,b]$ en subintervals de manera que cada subinterval contingui exactament un dels punts en f i g són diferents. Ara apliquem l'apartat anterior a cada subinterval.

(12) Sigui f una funció integrable a $[-a,a]$, $a>0$. Demostreu que

(a) si f és una funció parella, aleshores $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \underset{f(x)=f(-x)}{=} \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

(b) si f és una funció senar, aleshores $\int_{-a}^{a} f(x)dx = 0$.

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx = -\int_{-a}^{0} f(-x)dx + \int_0^a f(x)dx =$$

$$= -\int_0^a f(x)dx + \int_0^a f(x)\,dx = 0.$$

(c) si f és una funció parella o senar, aleshores $\int_{-a}^{a} (f(x))^2 \sin x\, dx = 0$.

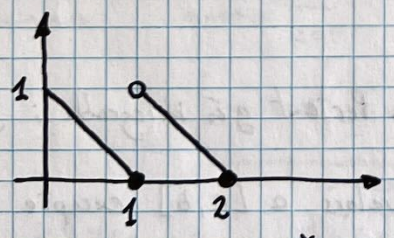
Tant si f és parella com senar, $f\cdot f$ és parella. Com sin és senar, $(f(x))^2 \sin x$ és senar. Per l'apartat anterior, $\int_{-a}^{a} (f(x))^2 \sin x\, dx = 0$.

(13) Considereu la funció real f definida per

$$f(x) = \begin{cases} 1-x, & 0 \le x \le 1, \\ 2-x, & 1 < x \le 2. \end{cases}$$

Trobeu l'expressió analítica i estudieu la continuïtat i la derivabilitat de la funció

$$F(x) = \int_0^x f(t)\,dt, \quad x \in [0,2].$$



$$F(x) = \begin{cases} \int_0^x (1-t)\,dt = \left[t - \frac{t^2}{2}\right]_0^x = x - \frac{x^2}{2} & \text{si } 0 \le x \le 1 \\ \int_0^1 (1-t)\,dt + \int_1^x (2-t)\,dt = \left[t - \frac{t^2}{2}\right]_0^1 + \left[2t - \frac{t^2}{2}\right]_1^x = 2x - \frac{x^2}{2} - 1 & \text{si } 1 < x \le 2. \end{cases}$$

Notem que F és contínua a $[0,1) \cup (1,2]$. Vegem què passa a $x=1$:

$$\left.\begin{array}{l} \lim_{x\to 1^-} F(x) = \lim_{x\to 1^-} \left(x - \frac{x^2}{2}\right) = \frac{1}{2} \\ \lim_{x\to 1^+} F(x) = \lim_{x\to 1^+} \left(2x - \frac{x^2}{2} - 1\right) = \frac{1}{2} \end{array}\right\} \Rightarrow F \text{ és contínua en } x=1.$$

Per tant, F és contínua a $[0,2]$. F és derivable a $[0,2]\setminus\{1\}$ i

$$F'(x)=\begin{cases} 1-x & \text{si } 0<x<1 \\ 2-x & \text{si } 1<x<2 \end{cases}$$

(14) Sigui $f:[0,1]\longrightarrow\mathbb{R}$ la funció definida com $f(x)=0$ si x és irracional, $f(x)=\frac{1}{n}$ si $x=\frac{m}{n}$ és racional de $(0,1]$ amb $\text{mcd}(m,n)=1$. Proveu que f és integrable a $[0,1]$ i que $\int_0^1 f(x)\,dx=0$.

Per veure la integrabilitat de f, cal provar que $\forall\varepsilon>0\ \exists P_\varepsilon$ una partició de $[0,1]$ tal que

$$U(f,P_\varepsilon)-L(f,P_\varepsilon)<\varepsilon,$$

on $U(f,P_\varepsilon)$ i $L(f,P_\varepsilon)$ denoten respectivament les sumes superiors i inferiors de f associades a la partició P_ε.

Tota suma inferior $L(f,P_\varepsilon)$ és 0, ja que a tot interval de la partició hi haurà irracionals i $f(x)\geq 0$.

Per fitar les sumes superiors, $\forall\varepsilon>0$ considerem les fraccions irreductibles $\frac{m}{n}$ amb denominador n tal que $\frac{1}{n}<\frac{\varepsilon}{2}$. D'aquestes n'hi ha un nombre infinit i, per contra, existeix un nombre finit N de fraccions irreductibles a $[0,1]$ amb denominador més petit o igual a $n_0=\left\lceil\frac{2}{\varepsilon}\right\rceil$.

Considerem $P_\varepsilon=\{x_0,x_1,\dots,x_k\}$ una partició de $[0,1]$ amb

$$\delta=\max_{i=1,\dots,k}\{x_i-x_{i-1}\}<\frac{\varepsilon}{2N}$$

Aleshores,

fita superior de fraccions amb denominador $>\frac{2}{\varepsilon}$

$$U(f,P_\varepsilon)<\underbrace{N}_{\substack{\text{nombre de fraccions}\\ \text{amb denominador}\leq 2/\varepsilon}}\cdot\delta+\frac{1}{n}\cdot 1<\frac{\varepsilon}{2N}\cdot N+\frac{\varepsilon}{2}=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Com que $0\leq U(f,P_\varepsilon)-\underbrace{L(f,P_\varepsilon)}_{=0}=U(f,P_\varepsilon)<\varepsilon$, f és integrable Riemann a $[0,1]$ i $\int_0^1 f(x)\,dx=0$.

(15) Comproveu les següents fórmules recurrents vàlides per $n \geq 2$, $n \in \mathbb{N}$.

(a) $I_n = \int \cos^n x \, dx = \frac{1}{n}\left(\cos^{n-1} x \, \sin x + (n-1) I_{n-2} \right)$.

$$I_n = \int \cos^n x \, dx = \int \cos^{n-1} x \cos x \, dx =$$

$u = \cos^{n-1} x$, $du = -(n-1)\cos^{n-2} x \sin x \, dx$

$dv = \cos x \, dx$, $v = \sin x$

$$= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx =$$

$\sin^2 x = 1 - \cos^2 x$

$$= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n \Longrightarrow$$

$$\Longrightarrow n I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}.$$

(b) $J_n = \int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - J_{n-2}$

$$J_n = \int \tan^n x \, dx = \int \tan^{n-2} x \cdot \tan^2 x \, dx = \int \tan^{n-2} x \left(\frac{1}{\cos^2 x} - 1 \right) dx =$$

$$= \int \tan^{n-2} \frac{1}{\cos^2 x} \, dx - \int \tan^{n-2} x \, dx = \frac{\tan^{n-1} x}{n-1} - J_{n-2}.$$

(16) Definim per tot $n \in \mathbb{N} \cup \{0\}$, $I_n = \int_0^{\pi/2} \sin^n x \, dx$,

(a) Proveu que $I_0 = \frac{\pi}{2}$, i $I_n = (n-1)(I_{n-2} - I_n)$, si $n \geq 2$.

$$I_0 = \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2}$$

Ara, per $n \geq 2$,

$$I_n = \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \sin^{n-2} x \sin^2 x \, dx = \int_0^{\pi/2} \sin^{n-2} x \, (1 - \cos^2 x) \, dx =$$

$$I_{n-2} - \int_0^{\pi/2} \sin^{n-2} x \cos^2 x \, dx = I_{n-2} - \left[\frac{\cos x \sin^{n-1} x}{n-1} \right]_0^{\pi/2} - \frac{1}{n-1} \int_0^{\pi/2} \sin^n x \, dx =$$

$u = \cos x$ $\quad du = -\sin x \, dx$

$dv = \sin^{n-2} x \cos x \, dx$ $\quad v = \frac{\sin^{n-1} x}{n-1}$

$$= I_{n-2} - \frac{1}{n-1} I_n \Longrightarrow I_n \left(1 + \frac{1}{n-1} \right) = I_{n-2} \Longrightarrow I_n \left(\frac{1}{n-1} \right) = I_{n-2} - I_n =$$

$$\Longrightarrow I_n = (n-1)(I_{n-2} - I_n).$$

Com volíem veure.

(b) Proveu que $I_1 = 1$, i si $n \geq 1$,

$$I_{2n} = \frac{1\cdot 3 \cdots (2n-3)(2n-1)}{2\cdot 4 \cdots (2n-2)(2n)} \frac{\pi}{2},$$

i

$$I_{2n+1} = \frac{2\cdot 4 \cdots (2n-2)(2n)}{1\cdot 3 \cdots (2n-1)(2n+1)}.$$

$$I_1 = \int_0^{\pi/2} \sin x \, dx = \left[-\cos x\right]_0^{\pi/2} = -(-\cos 0) = 1.$$

$$I_2 \underset{(a)}{=} 1\cdot(I_0 - I_2) \Rightarrow I_2 = \frac{I_0}{2} = \frac{\pi}{2}$$

$$I_{2n} \underset{(a)}{=} \frac{2n-1}{2n} I_{2n-2} = \frac{2n-1}{2n} \cdot \frac{(2n-3)\cdots 3\cdot 1}{(2n-2)\cdots 4\cdot 2} \frac{\pi}{2}$$

$$I_{2n+1} \underset{(a)}{=} \frac{2n}{2n+1} I_{2n-1} = \frac{2n}{2n+1} \frac{(2n-2)\cdots 4\cdot 2}{(2n-1)\cdots 3\cdot 1}$$

(c) Deduïu que, si $n \geq 1$,

$$\frac{I_{2n}}{I_{2n+1}} = \left(\frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots (2n-2)(2n)}\right)^2 \frac{(2n+1)\pi}{2}$$

$$\frac{I_{2n}}{I_{2n+1}} = \frac{\dfrac{1\cdot 3\cdot 5 \cdots (2n-3)(2n-1)}{2\cdot 4 \cdots (2n-2)(2n)} \dfrac{\pi}{2}}{\dfrac{2\cdot 4 \cdots (2n-2)(2n)}{1\cdot 3 \cdots (2n-1)(2n+1)}} = \left(\frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots (2n-2)(2n)}\right)^2 \frac{(2n+1)\pi}{2}$$

(d) Proveu les desigualtats $0 \leq I_{2n+1} \leq I_{2n} \leq I_{2n-1}$ i que, per tant,

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{I_{2n-1}}{I_{2n+1}} = 1 + \frac{1}{2n}$$

Com que $0 \leq \sin x \leq 1$ per $x \in [0, \pi/2]$, aleshores

$$0 \leq \int_0^{\pi/2} \sin^{2n+1} x \, dx \leq \int_0^{\pi/2} \sin^{2n} x \, dx \leq \int_0^{\pi/2} \sin^{2n-1} x \, dx \quad \text{com volíem veure.}$$

Ara, dividint $0 \leq I_{2n+1} \leq I_{2n} \leq I_{2n-1}$ entre I_{2n+1}, tenim

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{I_{2n-1}}{I_{2n+1}} = \frac{2n+1}{2n} = 1 + \frac{1}{2n}$$

(e) Demostreu que

$$\lim_{n\to+\infty} \frac{2\cdot 4\cdot 6\cdots(2n-2)(2n)}{1\cdot 3\cdot 5\cdots(2n-1)}\frac{1}{\sqrt{2n+1}} = \sqrt{\frac{\pi}{2}}$$

$$\underset{\text{apartat (d)}}{1} = \lim_{n\to+\infty}\frac{I_{2n}}{I_{2n+1}} \underset{\text{apartat (c)}}{=} \lim_{n\to+\infty}\left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n-2)(2n)}\right)^2\frac{2n+1}{2}\pi$$

Aleshores,

$$\lim_{n\to+\infty}\frac{2\cdot 4\cdots(2n-2)(2n)}{1\cdot 3\cdot 5\cdots(2n-1)}\frac{\sqrt{2}}{\sqrt{\pi}\sqrt{2n+1}} = 1 \longrightarrow$$

$$\Longrightarrow \lim_{n\to+\infty}\frac{2\cdot 4\cdots(2n-2)(2n)}{1\cdot 3\cdot 5\cdots(2n-1)}\frac{1}{\sqrt{2n+1}} = \sqrt{\frac{\pi}{2}}.$$

com volíem veure.

(f) Demostreu que

$$\lim_{n\to+\infty}\sqrt{n}\,\frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n+1)} = \frac{\sqrt{\pi}}{2}$$

$$\lim_{n\to+\infty}\frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n+1)}\sqrt{n} = \lim_{n\to+\infty}\frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)}\cdot\frac{\sqrt{n}}{\sqrt{2n+1}\sqrt{2n+1}} =$$

$$\underset{\text{apartat (e)}}{=} \lim_{n\to+\infty}\sqrt{\frac{n}{2n+1}}\sqrt{\frac{\pi}{2}} = \frac{1}{\sqrt{2}}\sqrt{\frac{\pi}{2}} = \frac{\sqrt{\pi}}{2}.$$

(17) Sigui f una funció contínua tal que $\int_0^x t\cdot f(t)\,dt = \sin x - x\cos x$. Calculeu $f(\frac{\pi}{2})$ i $f'(\frac{\pi}{2})$.

Definim $g(x) = \int_0^x t\cdot f(t)\,dt$. $g'(x) = x\cdot f(x)$.

Per altra banda, $g(x) = \sin x - x\cos x$. Aleshores, $g'(x) = x\sin x$ i per tant,

$f(x) = \sin x$ i $f'(x) = \cos x$.

$f(\frac{\pi}{2}) = 1$, $f'(\frac{\pi}{2}) = 0$.

(18) Sigui f una funció contínua i definim $F(x) = \int_1^x \left(t\int_1^t f(s)\,ds\right)dt$. Calculeu $F'(1)$ i $F''(x)$.

$F'(x) = x\int_1^x f(s)\,ds$

$F'(1) = 1\cdot\int_1^1 f(s)\,ds = 0$

$F''(x) = \int_1^x f(s)\,ds + x f(x)$.

(19) Provar que per tot $x \in [0, \pi/2]$ es verifica la igualtat:

$$\int_0^{\cos^2 x} \arccos(\sqrt{t})\,dt + \int_0^{\sin^2 x} \arcsin(\sqrt{t})\,dt = \frac{\pi}{4}.$$

Sigui $G(x) = \int_0^{\cos^2 x} \underbrace{\arccos(\sqrt{t})}_{f(t)}\,dt + \int_0^{\sin^2 x} \underbrace{\arcsin(\sqrt{t})}_{h(t)}\,dt$

Calculem $G'(x) = \frac{d}{dx}\Big[F(t)\Big]_0^{\cos^2 x} + \frac{d}{dx}\Big[H(t)\Big]_0^{\sin^2 x}$ on $F'(t) = f(t)$; $H'(t) = h(t)$.

Aleshores,

$$G'(x) = \arccos(\cos x)\cdot 2\cos x\cdot(-\sin x) + \arcsin(\sin x)\cdot 2\sin x\cos x =$$
$$= -2x\sin x\cos x + 2x\sin x\cos x = 0.$$

Per tant $G(x)$ és constant.

$$G\left(\frac{\pi}{4}\right) = \int_0^{\left(\frac{\sqrt{2}}{2}\right)^2} \arccos(\sqrt{t})\,dt + \int_0^{\left(\frac{\sqrt{2}}{2}\right)^2} \arcsin(\sqrt{t})\,dt =$$

$$= \int_0^{\left(\frac{\sqrt{2}}{2}\right)^2} \underbrace{\left(\arccos(\sqrt{t}) + \arcsin(\sqrt{t})\right)}_{\frac{\pi}{2}}\,dt = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$$

problema 21 del tema 1

Aleshores $G(x) = \frac{\pi}{4}$ $\forall x \in [0, \frac{\pi}{2}]$.

(20) Sigui $f: \mathbb{R} \longrightarrow \mathbb{R}$ una funció contínua. Es defineix $F: \mathbb{R} \longrightarrow \mathbb{R}$ tal que

$$F(x) = \begin{cases} \frac{1}{2x}\int_{-x}^{x} f(t)\,dt, & x \neq 0 \\ f(0), & x = 0. \end{cases}$$

Demostrar que

(a) F és contínua a $\mathbb{R}$ i derivable a $\mathbb{R}\setminus\{0\}$.

$$\lim_{x\to 0} F(x) = \lim_{x\to 0} \frac{\int_{-x}^{x} f(t)\,dt}{2x} \underset{\text{l'Hôpital ja que és } \frac{0}{0}}{=} \lim_{x\to 0} \frac{f(x) - (-f(-x))}{2} = \lim_{x\to 0} \frac{f(x) + f(-x)}{2} = f(0).$$

Així, F és contínua a $\mathbb{R}$, ja que per $x \neq 0$, F és contínua ja que f és contínua.

Ara, per $x \neq 0$,

$$F'(x) = \frac{1}{2}\,\frac{(f(x) - (-f(-x)))x - \int_{-x}^{x} f(t)\,dt}{x^2} = \frac{1}{2}\,\frac{(f(x) + f(-x))x - \int_{-x}^{x} f(t)\,dt}{x^2}$$

i per tant, F és derivable.

(b) si existeix $f'(0)$, aleshores existeix $F'(0)$.

$$F'(0) = \lim_{x\to 0} \frac{F(x)-F(0)}{x} = \lim_{x\to 0} \frac{\frac{1}{2x}\int_{-x}^{x} f(t)\,dt - F(0)}{x} =$$

$$= \lim_{x\to 0} \frac{\int_{-x}^{x} f(t)\,dt - f(0)\cdot 2x}{2x^2} \underset{\text{l'Hôpital } \frac{0}{0}}{=} \lim_{x\to 0} \frac{f(x)+f(-x)-2f(0)}{4x} =$$

$$= \lim_{x\to 0} \frac{f(x)-f(0)}{4x} + \lim_{x\to 0} \frac{f(-x)-f(0)}{4x} \underset{\exists f'(0).}{=} \frac{f'(0)}{4} - \frac{f'(0)}{4} = 0$$

(21) Sigui $f: \mathbb{R} \longrightarrow \mathbb{R}$ una funció contínua i derivable tal que $f(0)=0$; $f'(0)=1$.

Demostreu que $$F(x) = \int_{-2x+1}^{4x^2-1} f(t)\,dt$$

té un mínim al punt $x=\frac{1}{2}$.

$$F'(x) = 8x\cdot f(4x^2-1) - (-2)f(-2x+1) = 8x\cdot f(4x^2-1) + 2f(1-2x)$$

$$F'(\tfrac{1}{2}) = 4\cdot f(0) + 2\cdot f(0) = 0$$

$$F''(x) = (8x)^2 f'(4x^2-1) + 8\cdot f(4x^2-1) + 2f'(1-2x)\cdot(-2)$$

$$F''(\tfrac{1}{2}) = 4^2 f'(0) + 8f(0) - 4f'(0) = 12\cdot f'(0) = 12 > 0.$$

Aleshores, F té un mínim en $x=\frac{1}{2}$.

(22) Estudieu la monotonia de la funció $f(x) = \int_x^{2x} \frac{dt}{\sqrt{t^4+t^2+2}}$.

$$f'(x) = \frac{2}{\sqrt{(2x)^4+(2x)^2+2}} - \frac{1}{\sqrt{x^4+x^2+2}} = \frac{2\sqrt{x^4+x^2+2} - \sqrt{16x^4+4x^2+2}}{\sqrt{16x^4+4x^2+2}\,\sqrt{x^4+x^2+2}} = 0 \Leftrightarrow$$

$$\Leftrightarrow 2\sqrt{x^4+x^2+2} = \sqrt{16x^4+4x^2+2} \Leftrightarrow 4(x^4+x^2+2) = 16x^4+4x^2+2 \Leftrightarrow$$

$$\Leftrightarrow 12x^4 - 6 = 0 \Leftrightarrow x^4 = \tfrac{1}{2} \Leftrightarrow x = \pm\sqrt[4]{\tfrac{1}{2}}.$$

Aleshores, f és estrictament creixent a $(-\infty, -\sqrt[4]{\tfrac{1}{2}}) \cup (+\sqrt[4]{\tfrac{1}{2}}, +\infty)$; estrictament decreixent a $(-\sqrt[4]{\tfrac{1}{2}}, +\sqrt[4]{\tfrac{1}{2}})$.

(23) Calculeu els límits següents.

(a) $$\lim_{x\to 0^+} \frac{\int_0^{x^2} \sin\sqrt{t}\,dt}{x^3} \underset{\text{l'Hôpital } \frac{0}{0}}{=} \lim_{x\to 0^+} \frac{\sin x \cdot 2x}{3x^2} = \lim_{x\to 0^+} \frac{2}{3}\,\frac{\sin x}{x} = \frac{2}{3}.$$

(b) $$\lim_{x\to +\infty} \frac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt} \underset{\text{l'Hôpital } \frac{\infty}{\infty}}{=} \lim_{x\to +\infty} \frac{2\left(\int_0^x e^{t^2}dt\right)\cdot e^{x^2}}{e^{2x^2}} = \lim_{x\to +\infty} \frac{2\int_0^x e^{t^2}dt}{e^{x^2}} \underset{\text{l'Hôpital } \frac{\infty}{\infty}}{=}$$

$$= \lim_{x\to +\infty} \frac{2e^{x^2}}{2xe^{x^2}} = \lim_{x\to +\infty} \frac{1}{x} = 0.$$

(c) $$\lim_{x\to 0} \frac{\int_1^{x^2+1} e^{-t}/t\,dt}{x^2} \underset{\text{l'Hôpital } \frac{0}{0}}{=} \lim_{x\to 0} \frac{\frac{e^{-(x^2+1)}}{x^2+1}\cdot 2x}{2x} = \lim_{x\to 0} \frac{e^{-(x^2+1)}}{x^2+1} = e^{-1} = \frac{1}{e}$$

(24) Calculeu totes les funcions de classe C^1 en $\mathbb{R}$ tals que

$$f^2(x) = \int_0^x \left(f(t)^2 + f'(t)^2\right) dt + 2018.$$

Derivant a banda i banda,

$$2\cdot f(x)\cdot f'(x) = f^2(x) + f'(x)^2 \iff (f(x) - f'(x))^2 = 0 \iff$$

$$\iff f = f' \Rightarrow \ln f(x) = x + k \to f(x) = Ce^x$$

Com que $(f(0))^2 = 2018 \Rightarrow f(0) = \pm\sqrt{2018}$

Aleshores, $f(x) = \pm\sqrt{2018}\,e^x$.

(25) Sigui $f: [a,b] \longrightarrow [f(a), f(b)]$ una funció derivable tal que $f'(x) > 0$, per tot $x \in [a,b]$. Demostreu l'equació següent de les dues maneres que s'indiquen:

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(x)\,dx = b f(b) - a f(a)$$

(a) Treballant directament el membre de l'esquerra.

$$\int_{f(a)}^{f(b)} f^{-1}(x)\,dx = \int_a^b y\cdot f'(y)\,dy = \left[y\cdot f(y)\right]_a^b - \int_a^b f(x)\,dx = bf(b) - af(a) - \int_a^b f(x)\,dx.$$

$f^{-1}(x) = y$

$x = f(y)$

$dx = f'(y)\,dy$

$u = y \quad dv = f'(y)\,dy$

$du = dy \quad v = f(y)$

(b) Derivant ambdós costats respecte b (i considerant a fix).

$$G(x) = \int_a^x f(t)\,dt + \int_{f(a)}^{f(x)} f^{-1}(t)\,dt$$

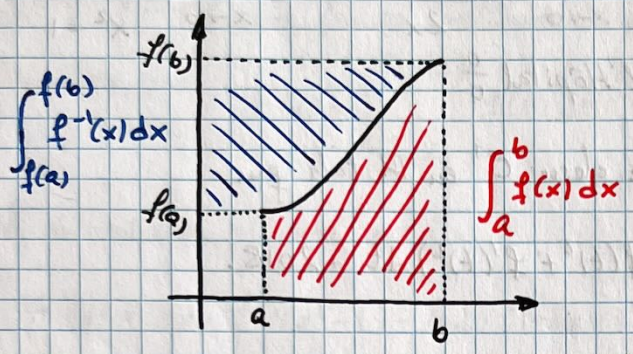
$$G'(x) = f(x) + f^{-1}(f(x)) \cdot f'(x) = f(x) + x f'(x).$$

Per altra banda, si $H(x) = x f(x) - a f(a)$,

$H'(x) = f(x) + x f'(x)$.

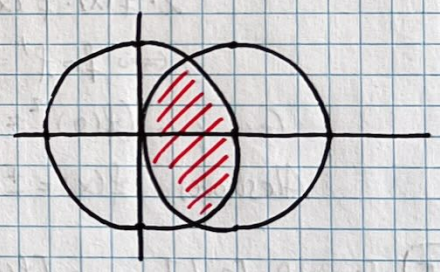
Ara, com $G(a) = H(a) = 0$, tenim que $G(x) = H(x)$ $\forall x \in [a,b]$.

Doneu una interpretació geomètrica d'aquesta fórmula:



(26) Calculeu l'àrea comuna als cercles $x^2+y^2=9$ i $(x-3)^2+y^2=9$.

$$S = 2S_0 = 2\int_0^{3/2} \sqrt{9-(x-3)^2}\,dx + 2\int_{3/2}^{3} \sqrt{9-x^2}\,dx =$$



$$= 4\int_{3/2}^{3} \sqrt{9-x^2}\,dx = 4\int_{\pi/6}^{\pi/2} (3\cos\theta)^2\,d\theta =$$

$y = x-3$

$dy = dx$

$x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$x = \frac{3}{2} \Rightarrow \theta = \frac{\pi}{6}$

$x = 3 \Rightarrow \theta = \frac{\pi}{2}$

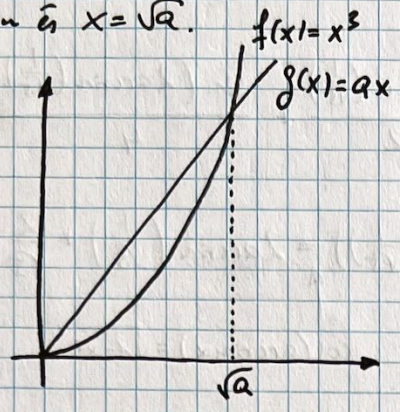
$$= 36\int_{\pi/6}^{\pi/2} \cos^2\theta\,d\theta = 36\int_{\pi/6}^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\,d\theta = 18\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/6}^{\pi/2} =$$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= 18\left[\left(\frac{\pi}{2} - \frac{\pi}{6}\right) + \frac{\sin\pi - \sin\frac{\pi}{3}}{2}\right] = 18\left(\frac{2\pi}{6} - \frac{\sqrt{3}}{4}\right) = 6\pi - \frac{9}{2}\sqrt{3}.$$

(27) Calculeu el valor d'$a>0$ per tal que la funció $f(x)=x^3$ divideixi en dues parts iguals el triangle determinat per la funció $g(x)=ax$, l'eix OX i el punt de tall més gran de les funcions f i g.

$f(x)=g(x) \Leftrightarrow x^3=ax \Leftrightarrow x=0,\ x=-\sqrt{a},\ x=\sqrt{a}$. El punt de tall més gran és $x=\sqrt{a}$.



l'àrea del triangle és $\frac{1}{2}\cdot\sqrt{a}\cdot a\sqrt{a}=\frac{a^2}{2}$.

Ara,

$$\int_0^{\sqrt{a}} x^3\,dx=\left[\frac{x^4}{4}\right]_0^{\sqrt{a}}=\frac{a^2}{4}=\frac{1}{2}\left(\frac{a^2}{2}\right).$$

Aleshores, l'enunciat es verifica per a qualsevol valor d'$a>0$.

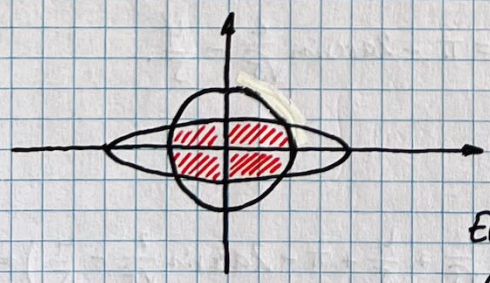
(28) Trobeu l'àrea limitada per la corba $\sqrt{x}+\sqrt{y}=1$ i els eixos de coordenades.

La corba talla els eixos a $(1,0)$ i $(0,1)$.

$$S=\int_0^1(1-\sqrt{x})^2dx=\int_0^1(1-2\sqrt{x}+x)dx=\left[x+\frac{x^2}{2}-2\,x^{3/2}/\tfrac{3}{2}\right]_0^1=$$

$$=\left[x+\frac{x^2}{2}-\frac{4}{3}x^{3/2}\right]_0^1=1+\frac{1}{2}-\frac{4}{3}=\frac{3}{2}-\frac{4}{3}=\frac{9-8}{6}=\frac{1}{6}.$$

(29) Calculeu l'àrea de la regió comú a la circumferència $x^2+y^2=4$ i l'elipse $\frac{x^2}{16}+y^2=1$.



$$\left.\begin{array}{l} x^2+y^2=4 \\ \frac{x^2}{16}+y^2=1 \end{array}\right\} = x^2\left(1-\frac{1}{16}\right)=3 \Rightarrow$$

$$\Rightarrow x^2=\frac{3}{15}16 \Rightarrow x=\pm\frac{4}{\sqrt{5}} \rightarrow y=\frac{\pm 2}{\sqrt{5}}$$

Els quatre punts de tall entre les dues còniques són $\left(\frac{\pm 4}{\sqrt{5}},\frac{\pm 2}{\sqrt{5}}\right)$.

Per simetria, $S=4\cdot S_{\blacksquare}$

$$S_{\blacksquare}=\underbrace{\int_0^{4/\sqrt{5}}\sqrt{1-\frac{x^2}{16}}\,dx}_{I_1}+\underbrace{\int_{4/\sqrt{5}}^{2}\sqrt{4-x^2}\,dx}_{I_2}$$

$$I_1 = \int_0^{4/\sqrt{5}} \sqrt{1-\frac{x^2}{16}}\,dx = \int_0^{\arcsin\left(\frac{1}{\sqrt{5}}\right)} 4\cos^2\theta\,d\theta = 4\int_0^{\arcsin\left(\frac{1}{\sqrt{5}}\right)} \cos^2\theta\,d\theta = 4\int_0^{\arcsin\left(\frac{1}{\sqrt{5}}\right)} \frac{1+\cos 2\theta}{2}\,d\theta =$$

$x = 4\sin\theta$

$dx = 4\cos\theta\,d\theta$

$x = \frac{4}{\sqrt{5}} \Rightarrow \theta = \arcsin\left(\frac{1}{\sqrt{5}}\right)$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= 4\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\arcsin\left(\frac{1}{\sqrt{5}}\right)} = 4\left(\frac{\arcsin\left(\frac{1}{\sqrt{5}}\right)}{2} + \frac{\sin\left(2\arcsin\left(\frac{1}{\sqrt{5}}\right)\right)}{4}\right) =$$

$$= 2\arcsin\left(\frac{1}{\sqrt{5}}\right) + 2\frac{1}{\sqrt{5}}\cos\left(\arcsin\left(\frac{1}{\sqrt{5}}\right)\right) = 2\arcsin\left(\frac{1}{\sqrt{5}}\right) + \frac{2}{\sqrt{5}}\sqrt{1-\frac{1}{5}} =$$

$\cos(\arcsin x) = \sqrt{1-x^2}$

$$= 2\arcsin\left(\frac{1}{\sqrt{5}}\right) + \frac{4}{5}.$$

$$I_2 = \int_{4/\sqrt{5}}^{2} \sqrt{4-x^2}\,dx = \int_{\arcsin\frac{2}{\sqrt{5}}}^{\pi/2} 4\cos^2\theta\,d\theta = 4\int_{\arcsin\frac{2}{\sqrt{5}}}^{\pi/2} \cos^2\theta\,d\theta =$$

$x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$x = \frac{4}{\sqrt{5}} \Rightarrow \theta = \arcsin\frac{2}{\sqrt{5}}$

$x = 2 \Rightarrow \theta = \arcsin 1 = \frac{\pi}{2}$

$$= 4\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{\arcsin\frac{2}{\sqrt{5}}}^{\pi/2} = 4\left(\frac{\pi}{4} - \frac{\arcsin\left(2/\sqrt{5}\right)}{2} - \frac{\sin\left(2\arcsin\frac{2}{\sqrt{5}}\right)}{4}\right) =$$

$$= \pi - 2\arcsin\left(\frac{2}{\sqrt{5}}\right) - 2\frac{2}{\sqrt{5}}\cos\left(\arcsin\frac{2}{\sqrt{5}}\right) = \pi - 2\arcsin\frac{2}{\sqrt{5}} - \frac{4}{5}.$$

Així doncs,

$$S = 4\cdot S = 4(I_1 + I_2) = 4\left(2\arcsin\frac{1}{\sqrt{5}} + \frac{4}{5} + \pi - 2\arcsin\frac{2}{\sqrt{5}} - \frac{4}{5}\right) =$$

$$= 4\pi + 8\left(\arcsin\frac{1}{\sqrt{5}} - \arcsin\frac{2}{\sqrt{5}}\right).$$