k-contact reduction

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(joint work with J. de Lucas, X. Rivas, and S. Vilariño)
in progress

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19.01.2024

1. k-**symplectic manifolds**
2. k-**symplectic reduction**
3. k-**contact manifolds**
4. k-**contact reduction of the geometry**
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k-Contact reduction of the dynamics

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Definition

A *k-vector field* on a manifold M is a section $\mathbf{X} \colon M \to \bigoplus^k TM$ of the projection pr_M. Then, $\mathfrak{X}^k(M)$ denotes the set of all k-vector fields on M.

$$\begin{tikzcd} TM & \bigoplus^k TM \arrow[l, "\mathrm{pr}_\alpha"'] \\ M \arrow[u, "X_\alpha"] \arrow[ur, "\mathbf{X}"'] & \end{tikzcd}$$

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Taking into account the diagram above, a k-vector field $\boldsymbol{X} \in \mathfrak{X}^k(M)$ amounts to some vector fields $X_1, \dots, X_k \in \mathfrak{X}(M)$ given by $X_\alpha = \mathrm{pr}^\alpha \circ \boldsymbol{X}$ with $\alpha = 1, \dots, k$.

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Let $h \in C^\infty(M)$ be a Hamiltonian function. A k-contact Hamiltonian k-vector field is $\boldsymbol{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k(M)$ satisfying

$$\begin{cases} \iota_{X_\alpha} d\eta^\alpha = dh - (R_\alpha h)\eta^\alpha\,, \\ \iota_{X_\alpha} \eta^\alpha = -h\,. \end{cases}$$

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Example

Consider a manifold $M = \bigoplus^2 T^*\mathbb{R} \times \mathbb{R}^2$ with coordinates $(q^1, q^2, p_1^t, p_2^t, p_1^x, p_2^x, s^t, s^x)$.

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$$\boldsymbol{\eta} = \eta^t \otimes e_1 + \eta^x \otimes e_2 = (ds^t - p_1^t dq^1 - p_2^t dq^2) \otimes e_1 + (ds^x - p_1^x dq^1 - p_2^x dq^2) \otimes e_2.$$

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$$\Phi : \mathbb{R}^2 \times M \ni (\lambda_1, \lambda_2; q^1, q^2, p_1^t, p_2^t, p_1^x, p_2^x, s^t, s^x) \mapsto (q^1 + \lambda_1, q^2 + \lambda_1, p_1^t, p_2^t, p_1^x, p_2^x, s^t, s^x + \lambda_2) \in M.$$

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The fundamental vector fields associated with $\Phi : \mathbb{R}^2 \times M \to M$ read

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and the contact momentum map $\mathbf{J}$ is given by

$$\mathbf{J} : M \ni x \longmapsto \boldsymbol{\mu} = \mu^1 \otimes e_1 + \mu^2 \otimes e_2 = (0, 1) \otimes e_1 - (p^t_1 + p^t_2, p^x_1 + p^x_2) \otimes e_2 \in (\mathbb{R}^2)^{*2}.$$

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Let us fix $\boldsymbol{\mu} = \mu_1 \otimes e_1$ and recall that

$$T_x \mathbf{J}^{-1}(\mathbb{R}^\times \boldsymbol{\mu}) = \{v_x \in T_x M \ : \ T_x \mathbf{J}(v_x) = \lambda \boldsymbol{\mu}, \quad \lambda \in \mathbb{R}^\times\},$$

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Then, it follows that the reduced manifold $M_{\boldsymbol{\mu}} = (\mathbf{J}^{-1}(\mathbb{R}^{\times}\boldsymbol{\mu})/K_{\boldsymbol{\mu}}, \boldsymbol{\eta}_{\boldsymbol{\mu}})$ is a 2-contact manifold with

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where $(q := q^1 - q^2, p^t := p_1^t - p_2^t, p^x := p_1^x - p_2^x, s^t, s^x)$ are local coordinates on $M_{\boldsymbol{\mu}}$.

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$$h(q^1, q^2, p_1^t, p_2^t, p_1^x, p_2^x, s^t, s^x) = \frac{1}{2}\left((p_1^t)^2 + (p_2^t)^2 - (p_1^x)^2 - (p_1^x)^2\right) + C(q^1 - q^2) + \gamma s^t \, ,$$

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where $C(q^1 - q^2)$ is a coupling function between the two strings.

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After the k-contact reduction we get

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\frac{\partial^2 q}{\partial t^2} - \frac{\partial^2 q}{\partial x^2} = \gamma \frac{\partial q}{\partial t} - 2\frac{\partial C}{\partial q}\,,
$$

which is the equation of a single damped string with an external force acting on it.

Thank you for your attention

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Literature:

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