Geometry of k-contact manifolds

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k-contact forms

Definition

A **k-contact form** on an open subset $U \subset M$ is a differential one-form on U taking values in $\mathbb{R}^k$, let us say $\boldsymbol{\eta} \in \Omega^1(U, \mathbb{R}^k)$, such that

(1) $\ker \boldsymbol{\eta} \subset \mathrm{T}U$ is a regular non-zero distribution of corank k,

(2) $\ker \mathrm{d}\boldsymbol{\eta} \subset \mathrm{T}U$ is a regular distribution of rank k,

(3) $\ker \boldsymbol{\eta} \cap \ker \mathrm{d}\boldsymbol{\eta} = 0$.

If the k-contact form $\boldsymbol{\eta}$ is defined on M, the pair $(M, \boldsymbol{\eta})$ is called a **co-oriented k-contact manifold** and $\ker \mathrm{d}\boldsymbol{\eta}$ is called the **Reeb distribution** of $(M, \boldsymbol{\eta})$.

If, in addition, $\dim M = n + nk + k$ for some $n, k \in \mathbb{N}$ and there exists an integrable distribution $\mathcal{V}$ contained in $\ker \boldsymbol{\eta}$ with $\operatorname{rank} \mathcal{V} = nk$, we say that $(M, \boldsymbol{\eta}, \mathcal{V})$ is a **polarised co-oriented k-contact manifold**. We call the distribution $\mathcal{V}$ a **polarisation** of $(M, \boldsymbol{\eta})$.

The case $k = 1$

The condition $\ker \boldsymbol{\eta} \neq 0$ is assumed so as to avoid non-interesting examples and retrieve standard contact geometry in the case $k = 1$.

Consider η to be any non-vanishing one-form on $M = \mathbb{R}$. Thus, $\mathrm{d}\eta = 0$ and $\ker \mathrm{d}\eta = \mathrm{T}\mathbb{R}$. Hence, all conditions (1), (2), and (3) are satisfied. Nevertheless, we want a one-contact form to be a contact form and a non-vanishing η on $\mathbb{R}$ is not a contact form since its kernel is not a contact distribution.

Dimension of polarised k-contact manifolds

If $(M, \boldsymbol{\eta})$ admits a polarisation $\mathcal{V}$, then $\dim M = n + nk + k$ for some $n, k \in \mathbb{N}$. Note that

$$n + nk + k = (n+1)(k+1) - 1\,,$$

so $\dim M + 1$ must be a composite number. Thus, if $(M, \boldsymbol{\eta}, \mathcal{V})$ is a polarised k-contact manifold, $\dim M + 1$ cannot be a prime number.

Reeb vector fields

Theorem

Let $(M, \boldsymbol{\eta} = \sum_{\alpha=1}^{k} \eta^\alpha \otimes e_\alpha)$ *be a co-oriented* k*-contact manifold. There exists a unique family of vector fields* $R_1, \ldots, R_k \in \mathfrak{X}(M)$*, such that*

$$\iota_{R_\alpha}\eta^\beta = \delta_\alpha^\beta\,, \qquad \iota_{R_\alpha}\mathrm{d}\eta^\beta = 0\,,$$

for $\alpha, \beta = 1, \ldots, k$*. The vector fields* $R_1, \ldots, R_k$ *commute between themselves, i.e.*

$$[R_\alpha, R_\beta] = 0\,, \qquad \alpha, \beta = 1, \ldots, k\,.$$

In addition, $\ker \mathrm{d}\boldsymbol{\eta}$ *is spanned by those vector fields, namely* $\ker \mathrm{d}\boldsymbol{\eta} = \langle R_1, \ldots, R_k \rangle$*.*

We call **Reeb k-vector field** of $(M, \boldsymbol{\eta})$ the integrable k-vector field $\mathbf{R} = (R_1, \ldots, R_k)$.

Note that $\mathscr{L}_{R_\alpha}\boldsymbol{\eta} = 0$ for every α.

A first example

The manifold $M = (\bigoplus^k \mathrm{T}^*Q) \times \mathbb{R}^k$ has a natural k-contact form

$$\boldsymbol{\eta}_Q = \sum_{\alpha=1}^{k} (\mathrm{d}z^\alpha - \theta^\alpha) \otimes e_\alpha = \sum_{\alpha=1}^{k} (\mathrm{d}z^\alpha - p_i^\alpha \mathrm{d}q^i) \otimes e_\alpha \,,$$

where θ^α is the pull-back to M of the Liouville one-form θ of the cotangent bundle T^*Q. Thus, $(M, \boldsymbol{\eta}_Q, \mathcal{V})$ is a polarised co-oriented k-contact manifold, where $\mathcal{V} \subset \ker \boldsymbol{\eta}$ is the vertical distribution or the projection $M \longrightarrow Q \times \mathbb{R}^k$ and $\operatorname{rank} \mathcal{V} = nk$. In local coordinates,

$$\ker \boldsymbol{\eta}_Q = \left\langle \frac{\partial}{\partial p_i^\alpha} \,,\ \frac{\partial}{\partial q^i} + p_i^\alpha \frac{\partial}{\partial z^\alpha} \right\rangle \,, \qquad \mathcal{V} = \left\langle \frac{\partial}{\partial p_i^\alpha} \right\rangle \,.$$

Hence, $\mathrm{d}\boldsymbol{\eta}_Q = (\mathrm{d}q^i \wedge \mathrm{d}p_i^\alpha) \otimes e_\alpha$, the Reeb vector fields are $R_\alpha = \partial/\partial z^\alpha$ for $\alpha = 1, \dots, k$, and

$$\ker \mathrm{d}\boldsymbol{\eta}_Q = \left\langle \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^k} \right\rangle \,.$$

Jet bundles I

Consider the first-order jet bundle $J^1 = J^1(M, E)$ of a fibre bundle $E \to M$ of rank k with adapted coordinates $\{x^i, y^\alpha, y_i^\alpha\}$ on an open set U. Its Cartan distribution

$$\mathcal{C} = \left\langle \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{k} y_i^\alpha \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y_i^\alpha} \right\rangle$$

has rank $m(k+1)$ and it is globally defined. Note that $[\mathcal{C}, \mathcal{C}] = \mathrm{T}J^1$.

On the open subset U of J^1, there exists a k-contact form

$$\boldsymbol{\eta} = \sum_{\alpha=1}^{k} \left(\mathrm{d}y^\alpha - \sum_{i=1}^{m} y_i^\alpha \mathrm{d}x^i \right) \otimes e_\alpha \,,$$

such that $\ker \boldsymbol{\eta} = \mathcal{C}|_U$. Moreover,

$$\left\langle \frac{\partial}{\partial y^\alpha} \right\rangle = \ker \mathrm{d}\boldsymbol{\eta} \,.$$

Jet bundles II

This example is quite interesting due to the fact that, given another set of adapted coordinates $\{\bar{x}^i, \bar{y}^\alpha, \bar{y}^\alpha_i\}$ on $\bar{U}$ to J^1, the $\mathbb{R}^k$-valued differential one-form

$$\bar{\boldsymbol{\eta}} = (\mathrm{d}\bar{y}^\alpha - \bar{y}^\alpha_i \mathrm{d}\bar{x}^i) \otimes e_\alpha$$

is different from $\boldsymbol{\eta}$, but

$$\ker \boldsymbol{\eta}|_{U\cap\bar{U}} = \ker \bar{\boldsymbol{\eta}}|_{U\cap\bar{U}} = \mathcal{C}|_{U\cap\bar{U}} \,,$$
$$\left\langle \frac{\partial}{\partial y^\alpha} \right\rangle = \ker \mathrm{d}\boldsymbol{\eta} \neq \ker \mathrm{d}\bar{\boldsymbol{\eta}} = \left\langle \frac{\partial}{\partial \bar{y}^\alpha} \right\rangle \,,$$

and $\ker \bar{\boldsymbol{\eta}} \oplus \ker \mathrm{d}\bar{\boldsymbol{\eta}} = \mathrm{T}\bar{U}$.

Consequently, locally defined k-contact forms associated with adapted coordinates to J^1 and their differentials do not need to be globally defined, but they share the same kernel given by the Cartan distribution.

Jet bundles III

Let us analyse a simple example given by a fibre bundle $(x, y) \in \mathbb{R}^2 \mapsto x \in \mathbb{R}$. Consider the first-order jet bundle $J^1(\mathbb{R}, \mathbb{R}^2)$ with induced variables $\{x, y, \dot{y}\}$. The new adapted coordinates given by

$$\bar{x} = x\,, \qquad \bar{y} = (1 + x^2)y\,,$$

lead to a new adapted coordinate system on $J^1(\mathbb{R}, \mathbb{R}^2)$ of the form

$$\bar{x} = x\,, \qquad \bar{y} = (1 + x^2)y\,, \qquad \dot{\bar{y}} = (1 + x^2)\dot{y} + 2xy\,.$$

Hence,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \bar{x}} + 2\frac{\bar{x}\bar{y}}{1 + \bar{x}^2}\frac{\partial}{\partial \bar{y}} + \frac{2}{1 + \bar{x}^2}\left(\bar{y} + \bar{x}\dot{\bar{y}} - \frac{2\bar{x}^2\bar{y}}{1 + \bar{x}^2}\right)\frac{\partial}{\partial \dot{\bar{y}}}\,,$$

$$\frac{\partial}{\partial y} = (1 + \bar{x}^2)\frac{\partial}{\partial \bar{y}} + 2\bar{x}\frac{\partial}{\partial \dot{\bar{y}}}\,, \qquad \frac{\partial}{\partial \dot{y}} = (1 + \bar{x}^2)\frac{\partial}{\partial \dot{\bar{y}}}\,.$$

As above, $\langle \partial/\partial y \rangle \neq \langle \partial/\partial \bar{y} \rangle$ at a generic point, while $\bar{\eta} = (1 + x^2)\eta$.

For a contact form η on M, one has that $[\ker\eta, \ker\eta] = \mathrm{T}M$.

For k-contact forms, only the following result can be ensured.

Proposition

Given a k-contact form $\boldsymbol{\eta}$ on M, one has that $\ker \mathrm{d}\boldsymbol{\eta}$ is an integrable distribution and $[\ker\boldsymbol{\eta}, \ker\boldsymbol{\eta}]_x \supsetneq \ker\boldsymbol{\eta}_x$ at every point $x \in M$.

In some particular cases, one has $[\ker\boldsymbol{\eta}, \ker\boldsymbol{\eta}] = \mathrm{T}M$ for a k-contact form $\boldsymbol{\eta}$, but it is not needed in general as shown in the following example. Notwithstanding, $[\ker\boldsymbol{\eta}, \ker\boldsymbol{\eta}] = \mathrm{T}M$ is satisfied for k-contact manifolds with a polarisation.

Example

Let us provide a k-contact form on M such that $[\ker \boldsymbol{\eta}, \ker \boldsymbol{\eta}] \neq \mathrm{T}M$. Consider $\mathbb{R}^2_\times \times \mathbb{R}^2$, where $\mathbb{R}_\times = (0, \infty)$, and a global coordinate system $\{x, p, z_1, z_2\}$ in $\mathbb{R}^2_\times \times \mathbb{R}^2$, namely $x, p \in \mathbb{R}_\times, z_1, z_2 \in \mathbb{R}$. If

$$\boldsymbol{\eta} = (\mathrm{d}z_1 - p\,\mathrm{d}x) \otimes e_1 + (\mathrm{d}z_2 - x\,\mathrm{d}p\,) \otimes e_2\,,$$

it follows that

$$\ker \boldsymbol{\eta} = \langle X_1 = x\partial_{z_2} + \partial_p\,,\;\; X_2 = p\partial_{z_1} + \partial_x \rangle\,,$$

$$\mathrm{d}\boldsymbol{\eta} = \mathrm{d}x \wedge \mathrm{d}p \otimes e_1 - \mathrm{d}x \wedge \mathrm{d}p \otimes e_2\,, \qquad \ker \mathrm{d}\boldsymbol{\eta} = \langle \partial_{z_1}, \partial_{z_2} \rangle.$$

Then, $\ker \boldsymbol{\eta} \cap \ker \mathrm{d}\boldsymbol{\eta} = 0$ with $\ker \boldsymbol{\eta} \neq 0$, $\operatorname{corank} \ker \boldsymbol{\eta} = 2$, $\operatorname{rank} \ker \mathrm{d}\boldsymbol{\eta} = 2$, and

$$[\ker \boldsymbol{\eta}, \ker \boldsymbol{\eta}] = \ker \boldsymbol{\eta} \oplus \langle \partial_{z_1} - \partial_{z_2} \rangle \subsetneq \mathrm{T}(\mathbb{R}^2_\times \times \mathbb{R}^2)\,.$$

Note that $[X_1, X_2]$ does not take values in $\ker \boldsymbol{\eta}$ at any point of M, but $\boldsymbol{\eta}$ is a two-contact form.

Darboux coordinates

Theorem

Consider a polarised co-oriented k-contact manifold $(M, \boldsymbol{\eta}, \mathcal{V})$. Then, there exists around every $x \in M$ a local coordinate system $\{x^i, y^\alpha, y_i^\alpha\}$ such that

$$\boldsymbol{\eta} = (\mathrm{d}y^\alpha - y_i^\alpha \mathrm{d}x^i) \otimes e_\alpha \,, \quad R_\beta = \frac{\partial}{\partial y^\beta} \,, \quad \mathcal{V} = \left\langle \frac{\partial}{\partial y_i^\alpha} \right\rangle , \quad \beta = 1, \ldots, k \,.$$

Example

The jet bundle $J^1(M, E)$ is a polarised k-contact manifold with the polarisation

$$\mathcal{V} = \ker \mathrm{T}\pi_1 \cap \ker \boldsymbol{\eta} = \left\langle \frac{\partial}{\partial y_i^\alpha} \right\rangle ,$$

where $\pi_1 \colon J^1(M, E) \longrightarrow M$ is the natural projection.

Outline

k-contact distributions

Definition

A **k-contact distribution** on M is a distribution $\mathcal{D} \subset \mathrm{T}M$ such that, for each point $x \in M$, there is an open neighbourhood $U \ni x$ and a k-contact form $\boldsymbol{\eta}$ on U such that $\mathcal{D}|_U = \ker \boldsymbol{\eta}$. We say that $(M, \mathcal{D})$ is a **k-contact manifold**.

Since a one-contact form is a contact form, a one-contact manifold $(M, \mathcal{D})$ is a contact manifold.

This means that every associated contact form η, i.e. $\mathcal{D}|_U = \ker \eta$ on an open subset $U \subset M$, satisfies that $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form for a unique $n \in \mathbb{N}$. As a consequence, it is said that $\mathcal{D}$ is maximally non-integrable.

Maximal non-integrability

Definition

Let $\mathcal{D}$ be a regular distribution on M and let $\pi\colon \mathrm{T}M \to \mathrm{T}M/\mathcal{D}$ be the natural vector bundle projection. Then, $\mathcal{D}$ is *maximally non-integrable* in a *distributional sense* if $\mathcal{D} \neq 0$ and the vector bundle mapping $\rho\colon \mathcal{D} \times_M \mathcal{D} \to \mathrm{T}M/\mathcal{D}$ over M given by

$$\rho(v, v') = \pi([X, X']_x)\,, \qquad \forall v, v' \in \mathcal{D}_x\,, \qquad \forall x \in M\,,$$

where X, X' are vector fields taking values in $\mathcal{D}$ locally defined around x such that $X_x = v$ and $X'_x = v'$, is non-degenerate.

Note that $\rho(v, v') = -\mathrm{d}\boldsymbol{\eta}(v, v')$.

Proposition

Let $\mathcal{D}$ be a distribution of corank one. Then, $\mathcal{D}$ is maximally non-integrable in a distributional sense if, and only if, $\mathcal{D}$ is maximally non-integrable in the contact sense.

Proposition

A regular distribution $\mathcal{D}$ on M is maximally non-integrable if, and only if, for every $x \in M$ there exists an $\boldsymbol{\zeta} \in \Omega^1(U, \mathbb{R}^k)$, where U is an open neighbourhood of x, associated with $\mathcal{D}$ such that $\mathrm{d}\boldsymbol{\zeta}$ is non-degenerate when restricted to $\mathcal{D}|_U$.

Proposition

If $(M, \boldsymbol{\eta})$ is a co-oriented k-contact manifold, then $\mathrm{d}\boldsymbol{\eta}$ is non-degenerate when restricted to $\ker \boldsymbol{\eta}$. In other words, $\ker \boldsymbol{\eta}$ is maximally non-integrable.

Since every k-contact distribution is on a neighbourhood U of every point $x \in M$ of the form $(U, \mathcal{D}|_U = \ker \boldsymbol{\eta})$, it follows that it is maximally non-integrable.

Corollary

Every k-contact distribution is maximally non-integrable.

The maximal non-integrability was introduced by L. Vitagliano. We will see that it is not equivalent be k-contact.

Lie symmetries

Definition

A **Lie symmetry** of a distribution $\mathcal{D}$ on M is a vector field X on M such that $[X, \mathcal{D}] \subset \mathcal{D}$.

Corollary

Every k-contact distribution $\mathcal{D}$ on M admits, on an open neighbourhood U of each point $x \in M$, an integrable k-vector field $\mathbf{S} = (S_1, \dots, S_k) \in \mathfrak{X}^k(U)$, whose components are Lie symmetries of $\mathcal{D}$ such that

$$\mathcal{D}|_U \oplus \langle S_1, \dots, S_k \rangle = \mathrm{T}U\,. \tag{1}$$

There exist distributions that have k commuting Lie symmetries but are not k-contact distributions (because they are involutive, for instance). We will see that maximal non-integrability is not a sufficient condition for a regular distribution to become k-contact by giving a maximally non-integrable distributions that do not admit k commuting Lie symmetries satisfying (1).

Lie flags

Definition

The **Lie flag** of a distribution $\mathcal{D} \subset \mathrm{T}M$ is a series of distributions $\mathcal{D}, \mathcal{D}^{1)}, \mathcal{D}^{2)}, \ldots$ on M, where $\mathcal{D}^{\ell)}$ is the distribution spanned by the Lie brackets of the vector fields taking values in $\mathcal{D}$ and $\mathcal{D}^{\ell-1)}$, namely

$$\mathcal{D}^{\ell)} = [\mathcal{D}, \mathcal{D}^{\ell-1)}]\,, \qquad \ell \in \mathbb{N}\,,$$

where we denote $\mathcal{D}^{0)} = \mathcal{D}$. We call **small growth function** of the Lie flag of $\mathcal{D}$ the vector function

$$\mathcal{G}_{\mathcal{D}}(x) = (\dim \mathcal{D}_x, \dim \mathcal{D}_x^{1)}, \ldots)\,, \qquad \forall x \in M\,.$$

Lie flags

Since $\mathcal{D}$ is smooth by assumption, the distributions $\mathcal{D}^{\ell)}$ are smooth and $\mathcal{D}^{\ell-1)} \subset \mathcal{D}^{\ell-1)}$ for every $\ell \in \mathbb{N}$.

The sequence $\mathcal{G}_{\mathcal{D}}(x)$ is (not strictly) increasing and stabilises since it is upper-bounded by $\dim M$ at every $x \in M$.

Proposition

Let us define $\mathcal{D} = \ker \boldsymbol{\eta}$ for a k-contact form $\boldsymbol{\eta}$, and let $R_1, \ldots, R_k$ be the associated Reeb vector fields. Then, the one-parameter groups of diffeomorphisms of the Reeb vector fields leave invariant the rank of every $\mathcal{D}^{\ell)}$ for $\ell \in \mathbb{N} \cup \{0\}$.

The next example shows that maximal non-integrability is not equivalent to be k-contact.

Example (part 1)

Consider $\mathbb{R}^4$ endowed with linear coordinates $\{x, y, z, t\}$ and the regular distribution given by

$$\mathcal{D} = \langle X_1 = \partial_x\,,\ X_2 = \partial_y + (x^3/3 + z^2 x + t^2)\partial_z + x\partial_t \rangle\,.$$

Then,

$$\mathcal{D}^{1)} = \langle X_1\,,\ X_2\,,\ (x^2 + z^2)\partial_z + \partial_t \rangle\,,$$
$$\mathcal{D}^{2)} = \langle X_1\,,\ X_2\,,\ (x^2 + z^2)\partial_z + \partial_t\,,\ 2x\partial_z\,,\ (2x^3 z/3 + t - zt^2)\partial_z \rangle\,.$$

$\mathcal{D}^{1)}$ has rank three everywhere, while $\mathcal{D}^{2)}$ has rank three when $x = 0$ and $t(1 - zt) = 0$, and four elsewhere. The space $S \subset \mathbb{R}^4$ defined by $x = 0$ and $t(1 - zt) = 0$ has to be invariant relative to the action of the Reeb vector fields if $\mathcal{D}$ is a two-contact distribution. Note that S is a regular submanifold whose tangent space is given by the annihilator of the differential forms

$$\theta^1 = \mathrm{d}x\,, \qquad \theta^2 = (1 - 2zt)\mathrm{d}t - t^2\mathrm{d}z\,.$$

Example (part 2)

Indeed, $\theta^1 \wedge \theta^2$ does not vanish at any point of $\mathbb{R}^4$ and the subset $x = 0$ and $t(1 - zt) = 0$ is a regular submanifold. If $\mathcal{D}$ is a two-contact distribution, the Reeb vector fields of any associated k-contact form $\boldsymbol{\eta} \in \Omega^1(U, \mathbb{R}^2)$ must be tangent to S.

In particular, at points where $x = 0$ and $t = 0$, one has that $\mathrm{T}S = \langle \theta^1, \theta^2 \rangle^\circ = \langle \partial_y, \partial_z \rangle$. Note that the two Reeb vector fields must span such a subspace, i.e. $\ker \mathrm{d}\boldsymbol{\eta} = \langle \partial_y, \partial_z \rangle$ when $x = 0$ and $t = 0$. But $\mathcal{D} = \langle \partial_x, \partial_y \rangle$ at such points of S, thus if $\mathcal{D}|_U = \ker \boldsymbol{\eta}$, then $\ker \mathrm{d}\boldsymbol{\eta} \cap \ker \boldsymbol{\eta} = \langle \partial_y \rangle$.

This is a contradiction and $\mathcal{D}$ is not the kernel of any two-contact form. On the other hand, it is immediate that $\rho\colon \mathcal{D} \times_{\mathbb{R}^4} \mathcal{D} \to \mathrm{T}\mathbb{R}^4 / \mathcal{D}$ is non-degenerate and $\mathcal{D}$ is maximally non-integrable.

Main result

Previous results imply that a k-contact distribution is a maximally non-integrable distribution admitting around each point k commuting Lie symmetries giving a supplementary to the k-contact distribution.

The converse is also true and, in fact, the maximal non-integrability and the existence of k commuting Lie symmetries spanning a supplementary distribution characterise k-contact distributions.

Theorem

A distribution $\mathcal{D}$ on M is a k-contact distribution if, and only if,

- *$\mathcal{D}$ is maximally non-integrable and,*
- *for every $x \in M$, there exists an open neighbourhood $U \ni x$ such that there is an integrable k-vector field $\mathbf{S} = (S_1, \ldots, S_k)$ of Lie symmetries of $\mathcal{D}|_U$ such that*

$$\langle S_1, \ldots, S_k \rangle \oplus \mathcal{D}|_U = \mathrm{T}U\,.$$

Lie groups

Example

Consider a Lie group G and a basis $X_1^L, \ldots, X_r^L$ of left-invariant vector fields on G and a basis $X_1^R, \ldots, X_r^R$ of right-invariant vector fields.

Let $\mathcal{D}$ be the distribution spanned by X_1^L and X_2^L. Assume that $[X_1^L, X_2^L] \notin \mathcal{D}$ and $X_3^R, \ldots, X_r^R$ commute among themselves.

Then $\mathcal{D} = \langle X_1^L, X_2^L \rangle$ is a $(r-2)$-contact distribution of rank two and Reeb vector fields $X_3^R, \ldots, X_r^R$.

Example

One can prove that the Lie group $\mathrm{SU}(n)$ is a coorientable n-contact manifold.

Outline

Representing k-contact distributions

Definition

Two differential one-forms $\boldsymbol{\zeta}, \bar{\boldsymbol{\zeta}} \in \Omega^1(U, \mathbb{R}^k)$ are **compatible** if $\ker \boldsymbol{\zeta} = \ker \bar{\boldsymbol{\zeta}}$ is a regular distribution.

We will now examine whether a form $\boldsymbol{\zeta} \in \Omega^1(U, \mathbb{R}^k)$ admits an associated k-contact form $\boldsymbol{\eta} \in \Omega^1(U, \mathbb{R}^k)$.

Example

Consider $\mathbb{R}^4$ with Cartesian coordinates $\{x, y, z, p\}$ and $\boldsymbol{\eta} = (\mathrm{d}x - y\,\mathrm{d}p) \otimes e_1 + (\mathrm{d}z - p\,\mathrm{d}y) \otimes e_2$. Then,

$$\ker \boldsymbol{\eta} = \left\langle y\frac{\partial}{\partial x} + \frac{\partial}{\partial p}, p\frac{\partial}{\partial z} + \frac{\partial}{\partial y} \right\rangle, \qquad \ker \mathrm{d}\boldsymbol{\eta} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial z} \right\rangle$$

and $\ker \mathrm{d}\boldsymbol{\eta} \cap \ker \boldsymbol{\eta} = 0$. If follows that $\boldsymbol{\eta}$ is a two-contact form. Consider now $\boldsymbol{\zeta} = e^z \boldsymbol{\eta}$. In this case, $\ker \boldsymbol{\eta} = \ker \boldsymbol{\zeta}$, but $\mathrm{d}\boldsymbol{\zeta} = \mathrm{d}\zeta^1 \otimes e_1 + \mathrm{d}\zeta^2 \otimes e_2$ has zero kernel since $\mathrm{d}(e^z \eta^1)$ is a symplectic form on $\mathbb{R}^4$. Thus, $\ker \mathrm{d}\boldsymbol{\zeta} = 0$ and $\boldsymbol{\zeta}$ is not a two-contact form.

Representing k-contact distributions

Definition

Every $\boldsymbol{\zeta} \in \Omega^1(U, \mathbb{R}^k)$ associated with a regular distribution gives rise to a non-vanishing differential k-form $\Omega_{\boldsymbol{\zeta}} = \zeta^1 \wedge \cdots \wedge \zeta^k$ on U. A **conformal Lie symmetry of Ω_{ζ}** is a vector field X on U such that $\mathscr{L}_X \Omega_{\boldsymbol{\zeta}} = f_X \Omega_{\boldsymbol{\zeta}}$ for some function $f_X \in \mathscr{C}^\infty(U)$.

Proposition

Let $(M, \boldsymbol{\eta})$ be a k-contact manifold and let $\Omega_{\boldsymbol{\eta}}$ be its associated differential k-form, then the Reeb vector fields are Lie symmetries of $\Omega_{\boldsymbol{\eta}}$. The space of conformal symmetries of $\Omega_{\boldsymbol{\zeta}}$, where ζ is compatible with $\boldsymbol{\eta}$, is a Lie algebra.

Representing k-contact distributions

Proposition

One has that $\boldsymbol{\zeta}, \bar{\boldsymbol{\zeta}} \in \Omega^1(U, \mathbb{R}^k)$ with regular kernels are compatible if, and only if, $\Omega_{\boldsymbol{\zeta}}$ and $\Omega_{\bar{\boldsymbol{\zeta}}}$ are proportional. Moreover, $\ker \boldsymbol{\zeta} = \ker \Omega_{\boldsymbol{\zeta}}$ and a vector field X on U is a Lie symmetry of $\ker \boldsymbol{\zeta}$ if, and only if, X is a conformal Lie symmetry of $\Omega_{\boldsymbol{\zeta}}$.

This result implies that X is a conformal Lie symmetry of $\Omega_{\boldsymbol{\eta}}$, if, and only if, $\mathscr{L}_X \eta^\alpha = \sum_{\beta=1}^k f_\beta^\alpha \eta^\beta$ with $\alpha = 1, \ldots, k$ for certain functions $f_\beta^\alpha \in \mathscr{C}^\infty(U)$ with $\alpha, \beta = 1, \ldots, k$.

Representing k-contact distributions

Proposition

Let $\boldsymbol{\zeta} \in \Omega^1(M, \mathbb{R}^k)$ have a regular kernel different from zero and suppose that $\mathrm{d}\boldsymbol{\zeta}|_{\ker \boldsymbol{\zeta} \times_U \ker \boldsymbol{\zeta}}$ is non-degenerate. Assume that $S_1, \ldots, S_k$ are conformal commuting Lie symmetries of $\Omega_{\boldsymbol{\zeta}}$ with $S_1 \wedge \cdots \wedge S_k \neq 0$ such that $\langle S_1, \ldots, S_k \rangle$ is a supplementary to $\ker \boldsymbol{\zeta}$. Then, $\boldsymbol{\zeta}$ admits a compatible k-contact form $\boldsymbol{\eta}$.

Theorem

An $\boldsymbol{\zeta} \in \Omega^1(U, \mathbb{R}^k)$ associated with a regular distribution $\mathcal{D} \neq 0$ of corank k is compatible with a k-contact form if, and only if, $\ker \boldsymbol{\zeta} \cap \ker \mathrm{d}\boldsymbol{\zeta} = 0$, while $\ker \mathrm{d}\boldsymbol{\zeta}|_{\ker \boldsymbol{\zeta} \times \ker \boldsymbol{\zeta}} = 0$, and $\Omega_{\boldsymbol{\zeta}}$ has k commuting conformal Lie symmetries spanning a supplementary to $\ker \boldsymbol{\zeta}$.

Outline

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