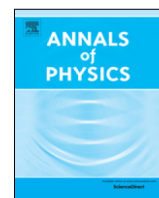




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A contact geometry framework for field theories with dissipation

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ABSTRACT

We develop a new geometric framework suitable for dealing with Hamiltonian field theories with dissipation. To this end we define the notions of k -contact structure and k -contact Hamiltonian system. This is a generalization of both the contact Hamiltonian systems in mechanics and the k -symplectic Hamiltonian systems in field theory. The concepts of symmetries and dissipation laws are introduced and developed. Two relevant examples are analyzed in detail: the damped vibrating string and Burgers' equation.

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1. Introduction

In the last decades, the methods of geometric mechanics and field theory have been widely used in order to give a geometrical description of a large variety of systems in physics and applied mathematics; in particular, those of symplectic and multisymplectic or k -symplectic (polysymplectic) geometry (see, for instance, [1–7] and references therein). All these methods are developed, in

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general, to model systems of variational type; that is, without dissipation or damping, both in the Lagrangian and Hamiltonian formalisms.

Furthermore, in recent years, there has been a growing interest in studying a geometric framework to describe dissipative or damped systems, specifically using contact geometry [8–10]. The efforts have been focused in the study of mechanical systems [11–17]. All of them are described by ordinary differential equations to which some terms that account for the dissipation or damping have been added. Contact geometry has other physical applications, as for instance thermodynamics [18].

Nevertheless, up to our knowledge, the analysis of systems of partial differential equations with dissipation terms, that is in field theory, has not yet been done geometrically. Our basic geometrical model for classical field theories without dissipation is the k -symplectic framework [4], which is the simplest extension of the symplectic formulation of autonomous mechanics to field theory. In this way, the aim of this paper is to develop an extension of the contact geometry in order to create a geometrical framework to deal with these kinds of systems when dissipation is present.

As it is well-known, a simple contact structure can be defined starting from an exact symplectic manifold $(N, \omega = -d\theta)$, taking the product $M = N \times \mathbb{R}$, and endowing this manifold with the contact form $\eta = ds - \theta$ (where s is the cartesian coordinate of \mathbb{R}). So, the *contactification* of the symplectic structure is obtained by the addition of a contact variable s [2, appendix 4]. Given a Hamiltonian function $H: M \rightarrow \mathbb{R}$, one defines the contact Hamilton equations, analogous to the usual Hamilton equations but with a dissipation term originating from the new variable.

Now we move to field theory, and more specifically to the De Donder–Weyl covariant formulation of Hamiltonian field theory. Aiming to introduce dissipation terms in the Hamilton–De Donder–Weyl equations, one realizes that we need to introduce k contact variables s^α , where k is the number of independent variables (for instance the space–time dimension). In the autonomous case, the De Donder–Weyl formulation of Hamiltonian field theory can be geometrically modeled with the notion of k -symplectic structure, that is, a family of k differential 2-forms ω^α satisfying some conditions.

These considerations lead us to define the concept of k -contact structure on a manifold M , as a family of k differential 1-forms η^α satisfying certain properties. This structure implies the existence of two special tangent distributions; one of them spanned by k Reeb vector fields which will be instrumental in the formulation of field equations. Then a k -contact Hamiltonian system is defined as a k -contact manifold endowed with a Hamiltonian function \mathcal{H} . This structure allows us to define k -contact Hamilton equations, which are a generalization of the De Donder–Weyl Hamiltonian formalism, and enables us to describe field theories with dissipation. After that, we can study the symmetries for these Hamiltonian systems and, associated to them, the notion of dissipation law, which is characteristic of dissipative systems, and is analogous to the conservation laws of conservative systems.

The relevance of our framework is illustrated with two noteworthy examples: the vibrating string with damping, and Burgers' equation. The Lagrangian formulation of the (undamped) vibrating string is well known, and from its Hamiltonian counterpart and a contactification procedure we obtain its field equation with a damping term. The case of Burgers' equation is more involved. First we consider the heat equation; although this equation is not variational, the introduction of an auxiliary variable allows to describe the heat equation in Lagrangian terms. From this we have provided a Hamiltonian field theory that still describes the heat equation. Finally, an appropriate contactification of this equation yields Burgers' equation. In both cases, there are different k -contact structures (with $k = 2$), hidden in the standard treatment, but uncovered in this geometric formulation.

The paper is organized as follows. Section 2 is devoted to review briefly several preliminary concepts on contact geometry and contact Hamiltonian systems in mechanics, as well as on k -symplectic manifolds and k -symplectic Hamiltonian systems in field theory. In Section 3 we introduce the definition of k -contact structure, we give the basic definitions and properties of k -contact manifolds, and we include a version of the Darboux theorem for a particular type of these manifolds. In Section 4 we set a geometric framework for Hamiltonian field theories with dissipation on a k -contact manifold and state the geometric form of the contact Hamilton–De Donder–Weyl

equations in several equivalent ways. Section 5 is devoted to study some relevant examples, in particular, the damped vibrating string and the Burgers equation. Finally, in Section 6 we introduce two concepts of symmetry and the relations between them, and the notions of dissipation laws for these kinds of systems.

Throughout the paper all the manifolds and mappings are assumed to be smooth. Sum over crossed repeated indices is understood.

2. Preliminaries

In Section 3 we will introduce the notion of k -contact structure. It is based on the notions of contact and k -symplectic structures, which we review in this section.

2.1. Contact manifolds and contact Hamiltonian systems

The geometry of contact manifolds is described in several books. We are interested, in particular, in contact Hamiltonian systems, see for instance [9,11–14,17,18].

Definition 2.1. Let M be a $(2n + 1)$ -dimensional manifold. A **contact form** in M is a differential 1-form $\eta \in \Omega^1(M)$ such that $\eta \wedge (d\eta)^{\wedge n}$ is a volume form in M . Then, (M, η) is said to be a **contact manifold**.

Remark 2.2. Notice that the condition that $\eta \wedge (d\eta)^{\wedge n}$ is a volume form is equivalent to demand that

$$TM = \ker \eta \oplus \ker d\eta .$$

Proposition 2.3. Let (M, η) be a contact manifold. Then there exists a unique vector field $\mathcal{R} \in \mathfrak{X}(M)$, which is called the **Reeb vector field**, such that

$$i(\mathcal{R})\eta = 1, \quad i(\mathcal{R})d\eta = 0 .$$

The local structure of contact manifolds is given by the following theorem:

Theorem 2.4 (Darboux Theorem for Contact Manifolds ([19] p. 118)). Let (M, η) be a contact manifold. Then around each point of M there exist an open set with local coordinates (q^i, p_i, s) with $1 \leq i \leq n$ such that

$$\eta = ds - p_i dq^i .$$

These are the so-called **Darboux** or **canonical coordinates** of the contact manifold (M, η) .

In Darboux coordinates, the Reeb vector field is $\mathcal{R} = \frac{\partial}{\partial s}$.

The canonical model for contact manifolds is the manifold $T^*Q \times \mathbb{R}$. In fact, if $\theta \in \Omega^1(T^*Q)$ and $\omega = -d\theta \in \Omega^2(T^*Q)$ are the canonical forms in T^*Q , and $\pi_1: T^*Q \times \mathbb{R} \rightarrow T^*Q$ is the canonical projection, then $\eta = ds - \pi_1^*\theta$ is a contact form in $T^*Q \times \mathbb{R}$, with $d\eta = \pi_1^*\omega$.

Finally, given a contact manifold (M, η) , we have the following $\mathcal{C}^\infty(M)$ -module isomorphism [20]

$$\begin{aligned} \flat: \mathfrak{X}(M) &\longrightarrow \Omega^1(M) \\ X &\longmapsto i(X)d\eta - (i(X)\eta)\eta \end{aligned}$$

and as a straightforward consequence we have:

Theorem 2.5. If (M, η) is a contact manifold, for every $H \in \mathcal{C}^\infty(M)$, there exists a unique vector field $X_H \in \mathfrak{X}(M)$ such that

$$i(X_H)d\eta = dH - (\mathcal{R}(H))\eta, \quad i(X_H)\eta = -H . \quad (1)$$

Definition 2.6. The vector field X_H defined by (1) is called the **contact Hamiltonian vector field** associated with H and Eqs. (1) are the **contact Hamilton equations**. The triple (M, η, H) is a **contact Hamiltonian system**.

Remark 2.7. Notice that the contact Hamiltonian equations are equivalent to

$$\mathcal{L}_{X_H}\eta = -(\mathcal{L}_R\mathcal{H})\eta, \quad i(X_H)\eta = -\mathcal{H}.$$

Furthermore, the contact Hamiltonian vector field is such that $\flat(X_H) = i(X_H)d\eta - H\eta$.

Taking Darboux coordinates (q^i, p_i, s) , the contact Hamiltonian vector field is

$$X = \frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial}{\partial q^i} - \left(\frac{\partial \mathcal{H}}{\partial q^i} + p_i \frac{\partial \mathcal{H}}{\partial s} \right) \frac{\partial}{\partial p_i} + \left(p_i \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H} \right) \frac{\partial}{\partial s}.$$

Hence, an integral curve of this vector field satisfies the contact Hamilton equations:

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = - \left(\frac{\partial \mathcal{H}}{\partial q^i} + p_i \frac{\partial \mathcal{H}}{\partial s} \right), \quad \dot{s} = p_i \frac{\partial \mathcal{H}}{\partial p_i} - \mathcal{H}.$$

2.2. k -vector fields and integral sections

The definition of k -vector field and its usage in the geometric study of partial differential equations can be found in [4,21], for instance.

Let M be a manifold, and consider the direct sum of k copies of its tangent bundle, $\oplus^k TM$. It is endowed with the natural projections to each direct summand and to the base manifold:

$$\tau^\alpha: \oplus^k TM \rightarrow TM, \quad \tau_M^1: \oplus^k TM \rightarrow M.$$

Definition 2.8. A k -**vector field** on a manifold M is a section $\mathbf{X}: M \rightarrow \oplus^k TM$ of the projection τ_M^1 .

A k -vector field \mathbf{X} is specified by giving k vector fields $X_1, \dots, X_k \in \mathfrak{X}(M)$, obtained as $X_\alpha = \tau_\alpha \circ \mathbf{X}$. Then it is denoted $\mathbf{X} = (X_1, \dots, X_k)$. A k -vector field $\mathbf{X} = (X_1, \dots, X_k)$ induces a decomposable contravariant skew-symmetric tensor field, $X_1 \wedge \dots \wedge X_k$, which is a section of $\Lambda^k TM \rightarrow M$. This also induces a tangent distribution on M .

Definition 2.9. Given a map $\phi: D \subset \mathbb{R}^k \rightarrow M$, the **first prolongation** of ϕ to $\oplus^k TM$ is the map $\phi': D \subset \mathbb{R}^k \rightarrow \oplus^k TM$ defined by

$$\phi'(t) = \left(\phi(t), T\phi \left(\frac{\partial}{\partial t^1} \Big|_t \right), \dots, T\phi \left(\frac{\partial}{\partial t^k} \Big|_t \right) \right) \equiv (\phi(t); \phi'_\alpha(t)),$$

where $t = (t^1, \dots, t^k)$ are the cartesian coordinates of \mathbb{R}^k .

Definition 2.10. An **integral section** of a k -vector field $\mathbf{X} = (X_1, \dots, X_k)$ is a map $\phi: D \subset \mathbb{R}^k \rightarrow M$ such that

$$\phi' = \mathbf{X} \circ \phi.$$

Equivalently, $T\phi \circ \frac{\partial}{\partial t^\alpha} = X_\alpha \circ \phi$ for every α .

A k -vector field \mathbf{X} is **integrable** if every point of M is in the image of an integral section of \mathbf{X} .

In coordinates, if $X_\alpha = X_\alpha^i \frac{\partial}{\partial x^i}$, then ϕ is an integral section of \mathbf{X} if, and only if, it is a solution to the following system of partial differential equations:

$$\frac{\partial \phi^i}{\partial t^\alpha} = X_\alpha^i(\phi).$$

A k -vector field $\mathbf{X} = (X_1, \dots, X_k)$ is integrable if, and only if, $[X_\alpha, X_\beta] = 0$, for every α, β (see, for instance, [22]), and these are the necessary and sufficient conditions for the integrability of the above system of partial differential equations.

2.3. k -symplectic manifolds and k -symplectic Hamiltonian systems

A simple geometric framework for Hamiltonian field theory can be built upon the notion of k -symplectic structure; see for instance [4,21,23–25].

Definition 2.11. Let M be a manifold of dimension $N = n + kn$. A k -**symplectic structure** on M is a family $(\omega^1, \dots, \omega^k; V)$, where ω^α ($\alpha = 1, \dots, k$) are closed 2-forms, and V is an integrable nk -dimensional tangent distribution on M such that

$$(i) \omega^\alpha|_{V \times V} = 0 \text{ (for every } \alpha), \quad (ii) \cap_{\alpha=1}^k \ker \omega^\alpha = \{0\}.$$

Then (M, ω^α, V) is called a k -**symplectic manifold**.

If (M, ω^α, V) is a k -symplectic manifold, for every point of M there exist a neighborhood U and local coordinates (q^i, p_i^α) ($1 \leq i \leq n, 1 \leq \alpha \leq k$) such that, on U ,

$$\omega^\alpha = dq^i \wedge dp_i^\alpha, \quad V = \left\langle \frac{\partial}{\partial p_i^1}, \dots, \frac{\partial}{\partial p_i^k} \right\rangle.$$

These are the so-called *Darboux* or *canonical coordinates* of the k -symplectic manifold [23].

The canonical model for k -symplectic manifolds is $\oplus^k T^*Q = T^*Q \oplus \dots \oplus^k T^*Q$, with natural projections

$$\pi^\alpha: \oplus^k T^*Q \rightarrow T^*Q, \quad \pi_Q^1: \oplus^k T^*Q \rightarrow Q.$$

As in the case of the cotangent bundle, local coordinates (q^i) in $U \subset Q$ induce induced natural coordinates (q^i, p_i^α) in $(\pi_Q^1)^{-1}(U)$.

If θ and $\omega = -d\theta$ are the canonical forms of T^*Q , then $\oplus^k T^*Q$ is endowed with the canonical forms

$$\theta^\alpha = (\pi^\alpha)^* \theta, \quad \omega^\alpha = (\pi^\alpha)^* \omega = -(\pi^\alpha)^* d\theta = -d\theta^\alpha,$$

and in natural coordinates we have that $\theta^\alpha = p_i^\alpha dq^i$ and $\omega^\alpha = dq^i \wedge dp_i^\alpha$. Thus, the triple $(\oplus^k T^*Q, \omega^\alpha, V)$, where $V = \ker T\pi_Q^1$, is a k -symplectic manifold, and the natural coordinates in $\oplus^k T^*Q$ are Darboux coordinates.

Definition 2.12. A k -**symplectic Hamiltonian system** is a family (M, ω^α, V, H) , where (M, ω^α, V) is a k -symplectic manifold, and $H \in \mathcal{C}^\infty(M)$ is called a *Hamiltonian function*.

The **Hamilton–De Donder–Weyl equation** for a map $\psi: D \subset \mathbb{R}^k \rightarrow M$ is

$$i(\psi'_\alpha) \omega^\alpha = dH \circ \psi. \quad (2)$$

The **Hamilton–De Donder–Weyl equation** for a k -vector field $\mathbf{X} = (X_1, \dots, X_k)$ in M is

$$i(X_\alpha) \omega^\alpha = dH. \quad (3)$$

For k -symplectic Hamiltonian systems, solutions to this equation always exist, although they are not unique. Moreover, solutions are not necessarily integrable.

In canonical coordinates, if $\psi = (\psi^i, \psi_i^\alpha)$, then $\psi'_\beta = \left(\psi^i, \psi_i^\alpha, \frac{\partial \psi^i}{\partial t^\beta}, \frac{\partial \psi_i^\alpha}{\partial t^\beta} \right)$, and Eq. (2) reads

$$\frac{\partial \psi^i}{\partial t^\alpha} = \frac{\partial H}{\partial p_i^\alpha} \circ \psi, \quad \frac{\partial \psi_i^\alpha}{\partial t^\alpha} = -\frac{\partial H}{\partial q^i} \circ \psi.$$

Furthermore, if $\mathbf{X} = (X_\alpha)$ is a k -vector field solution to (3) and $X_\alpha = (X_\alpha)^i \frac{\partial}{\partial q^i} + (X_\alpha)^\beta \frac{\partial}{\partial p_i^\beta}$, then

$$\frac{\partial H}{\partial q^i} = -(X_\alpha)_i^\alpha, \quad \frac{\partial H}{\partial p_i^\alpha} = (X_\alpha)^i.$$

Proposition 2.13 ([4,21]). If \mathbf{X} is an integrable k -vector field on M then every integral section $\psi: D \subset \mathbb{R}^k \rightarrow M$ of \mathbf{X} satisfies Eq. (2) if, and only if, \mathbf{X} is a solution to Eq. (3).

Eqs. (2) and (3) are not, in general, fully equivalent: a solution to Eq. (2) may not be an integral section of some integrable k -vector field in M solution to (3).

3. k -contact structures

Next we develop the general geometric framework of our formalism.

3.1. Definitions and basic properties

Let M be a smooth manifold of dimension m . A (generalized) distribution on M is a subset $D \subset TM$ such that, for every $x \in M$, $D_x \subset T_x M$ is a vector subspace. D is called smooth when it can be locally spanned by a family of smooth vector fields; it is called regular when it is smooth and of locally constant rank. One defines in the same way the notion of codistribution, as a subset $C \subset T^*M$. The annihilator D° of a distribution D is a codistribution, but if D is not regular then D° may not be smooth. Within the usual identification $E^{**} = E$ of finite-dimensional linear algebra, we have $(D^\circ)^\circ = D$.

A (smooth) differential 1-form $\eta \in \Omega^1(M)$ generates a smooth codistribution that we denote by $\langle \eta \rangle \subset T^*M$; it has rank 1 at every point where η does not vanish. Its annihilator is a distribution $\langle \eta \rangle^\circ \subset TM$; it can be described also as the kernel of the linear morphism $\widehat{\eta}: TM \rightarrow M \times \mathbb{R}$ defined by η . This distribution has corank 1 at every point where η does not vanish.

In a similar way, a differential 2-form $\omega \in \Omega^2(M)$ induces a linear morphism $\widehat{\omega}: TM \rightarrow T^*M$, $\widehat{\omega}(v) = i(v)\omega$. Its kernel is a distribution $\ker \widehat{\omega} \subset TM$. Recall that the rank of $\widehat{\omega}$ is an even number.

Now we consider k differential 1-forms $\eta^1, \dots, \eta^k \in \Omega^1(M)$, and introduce the following notations:

- $\mathcal{C} = \langle \eta^1, \dots, \eta^k \rangle \subset T^*M$;
- $\mathcal{D}^C = (\mathcal{C})^\circ = \ker \widehat{\eta^1} \cap \dots \cap \ker \widehat{\eta^k} \subset TM$;
- $\mathcal{D}^R = \ker d\widehat{\eta^1} \cap \dots \cap \ker d\widehat{\eta^k} \subset TM$;
- $\mathcal{C}^R = (\mathcal{D}^R)^\circ \subset T^*M$.

Definition 3.1. A k -contact structure on a manifold M is a family of k differential 1-forms $\eta^\alpha \in \Omega^1(M)$ such that, with the preceding notations,

- (i) $\mathcal{D}^C \subset TM$ is a regular distribution of corank k ;
- (ii) $\mathcal{D}^R \subset TM$ is a regular distribution of rank k ;
- (iii) $\mathcal{D}^C \cap \mathcal{D}^R = \{0\}$.

We call \mathcal{C}^C the **contact codistribution**; \mathcal{D}^C the **contact distribution**; \mathcal{D}^R the **Reeb distribution**; and \mathcal{C}^R the **Reeb codistribution**.

A k -contact manifold is a manifold endowed with a k -contact structure.

Remark 3.2. Condition (i) in Definition 3.1 is equivalent to each one of these two conditions:

- (i') $\mathcal{C}^C \subset T^*M$ is a regular codistribution of rank k ;
- (i'') $\eta^1 \wedge \dots \wedge \eta^k \neq 0$ at every point.

Condition (iii) can be obviously rewritten as

$$(iii') \bigcap_{\alpha=1}^k (\ker \widehat{\eta^\alpha} \cap \ker d\widehat{\eta^\alpha}) = \{0\}.$$

Provided that conditions (i) and (ii) in Definition 3.1 hold, condition (iii) is also equivalent to each one of these two conditions:

- (iii'') $TM = \mathcal{D}^C \oplus \mathcal{D}^R$.
- (iii''') $T^*M = \mathcal{C}^C \oplus \mathcal{C}^R$.

Remark 3.3. For the case $k = 1$, a 1-contact structure is provided by a differential 1-form η , and conditions in Definition 3.1 mean the following: (i) $\eta \neq 0$ at every point; (iii) $\ker \widehat{\eta} \cap \ker \widehat{d\eta} = \{0\}$, which implies that $\ker \widehat{d\eta}$ has rank 0 or 1; (ii) means that $\ker \widehat{d\eta}$ has rank 1. So, provided that (i) and (iii) hold, condition (ii) is equivalent to saying that $\dim M$ is odd. In this way, we recover the definition of contact structure.

Lemma 3.4. The Reeb distribution \mathcal{D}^R is involutive, and therefore integrable.

Proof. We use the relation

$$i([X, X']) = \mathcal{L}_X i(X') - i(X') \mathcal{L}_X = di(X)i(X') + i(X)di(X') - i(X')di(X) - i(X)i(X')d.$$

When X, X' are sections of \mathcal{D}^R and we apply this relation to the closed 2-form $d\eta^\alpha$ the result is zero. \square

Theorem 3.5 (Reeb Vector Fields). On a k -contact manifold (M, η^α) there exist k vector fields $\mathcal{R}_\alpha \in \mathfrak{X}(M)$, the Reeb vector fields, uniquely defined by the relations

$$i(\mathcal{R}_\beta)\eta^\alpha = \delta_\beta^\alpha, \quad i(\mathcal{R}_\beta)d\eta^\alpha = 0. \quad (4)$$

The Reeb vector fields commute:

$$[\mathcal{R}_\alpha, \mathcal{R}_\beta] = 0.$$

In particular, $\mathcal{D}^R = \langle \mathcal{R}_1, \dots, \mathcal{R}_k \rangle$.

Proof. We take $T^*M = C^C \oplus C^R$. The η^α are a global frame for the contact codistribution C^C ; we can find a local frame η^μ for C^R . So, $(\eta^\alpha; \eta^\mu)$ is a local frame for T^*M . The corresponding dual frame for TM is constituted by (smooth) vector fields $(\mathcal{R}_\beta; \mathcal{R}_\nu)$, where the \mathcal{R}_β are uniquely defined by

$$\langle \eta^\alpha, \mathcal{R}_\beta \rangle = \delta_\beta^\alpha, \quad \langle \eta^\mu, \mathcal{R}_\beta \rangle = 0.$$

Notice that the second set of relations does not depend on the choice of the η^μ and simply means that the \mathcal{R}_β are sections of $(C^R)^\circ = \mathcal{D}^R$, the Reeb distribution; in other words, it means that, for every α , $i(\mathcal{R}_\beta)d\eta^\alpha = 0$. Notice finally that, since the η^α are globally defined, \mathcal{R}_α also are.

To prove that the Reeb vector fields commute, notice that

$$i([\mathcal{R}_\alpha, \mathcal{R}_\beta])\eta^\gamma = 0, \quad i([\mathcal{R}_\alpha, \mathcal{R}_\beta])d\eta^\gamma = 0,$$

which is a consequence of their definition and of the above formula for $i([X, X'])$ when applied to them. \square

Proposition 3.6. On a k -contact manifold there exist local coordinates $(x^l; s^\alpha)$ such that

$$\mathcal{R}_\alpha = \frac{\partial}{\partial s^\alpha}, \quad \eta^\alpha = ds^\alpha - f_I^\alpha(x) dx^I,$$

where $f_I^\alpha(x)$ are functions depending only on the x^I .

Proof. The Reeb vector fields commute, so there exist local coordinates $(x^l; s^\alpha)$ where they can be straightened out simultaneously (see for instance [22, p. 234]): $\mathcal{R}_\alpha = \frac{\partial}{\partial s^\alpha}$.

Now we express the contact forms in these coordinates. First, relation $i(\mathcal{R}_\beta)\eta^\alpha = \delta_\beta^\alpha$ implies that $\eta^\alpha = ds^\alpha - f_I^\alpha dx^I$, where the functions f_I^α depend in principle on all the coordinates $(x^l; s^\alpha)$. But then $d\eta^\alpha = dx^I \wedge df_I^\alpha$, and the only way to ensure that $i(\mathcal{R}_\beta)d\eta^\alpha = 0$ is that $\partial f_I^\alpha / \partial s^\beta = 0$. \square

We will say that the coordinates provided by this proposition are **adapted** to the k -contact structure.

Example 3.7. (Canonical k -contact Structure). Given $k \geq 1$, the manifold $M = (\oplus^k T^*Q) \times \mathbb{R}^k$ has a canonical k -contact structure defined by the 1-forms

$$\eta^\alpha = ds^\alpha - \theta^\alpha,$$

where s^α is the α th cartesian coordinate of \mathbb{R}^k , and θ^α is the pull-back of the canonical 1-form of T^*Q with respect to the projection $M \rightarrow T^*Q$ to the α th direct summand.

Using coordinates q^i on Q and natural coordinates (q^i, p_i^α) on the α th copy of T^*Q , the local expressions of the contact forms are

$$\eta^\alpha = ds^\alpha - p_i^\alpha dq^i,$$

from which $d\eta^\alpha = dq^i \wedge dp_i^\alpha$, the Reeb distribution is $\mathcal{D}^R = \langle \partial/\partial s^1, \dots, \partial/\partial s^k \rangle$, and the Reeb vector fields are

$$\mathcal{R}_\alpha = \frac{\partial}{\partial s^\alpha}.$$

Example 3.8 (Contactification of a k -symplectic Manifold). Let (P, ω^α) be a k -symplectic manifold such that $\omega^\alpha = -d\theta^\alpha$, and consider $M = P \times \mathbb{R}^k$. Denoting by (s^1, \dots, s^k) the cartesian coordinates of \mathbb{R}^k , and representing also by θ^α the pull-back of θ^α to the product, we consider the 1-forms $\eta^\alpha = ds^\alpha - \theta^\alpha$ on M . Then (M, η^α) is a k -contact manifold because $\mathcal{C}^C = \langle \eta^1, \dots, \eta^k \rangle$ has rank k , $d\eta^\alpha = -d\theta^\alpha$, and $\mathcal{D}^R = \bigcap_\alpha \ker d\theta^\alpha = \langle \partial/\partial s^1, \dots, \partial/\partial s^k \rangle$ has rank k since (P, ω^α) is k -symplectic, and the last condition is immediate.

In particular, if $k = 1$, let P be a manifold endowed with a 1-form θ , and consider $M = P \times \mathbb{R}$. Denoting by s the cartesian coordinate of \mathbb{R} , and representing again by θ the pull-back of θ to the product, we consider the 1-form $\eta = ds - \theta$ on M . Then $\mathcal{C}^C = \langle \eta \rangle$ has rank 1, $d\eta = -d\theta$, and $\mathcal{D}^R = \ker d\theta$ has rank 1 if, and only if, $d\theta$ is a symplectic form on P . In this case M becomes a 1-contact manifold.

Example 3.9. Let $P = \mathbb{R}^6$ with coordinates (x, y, p, q, s, t) . The differential 1-forms

$$\eta^1 = ds - \frac{1}{2}(ydx - xdy) \quad , \quad \eta^2 = dt - pdx - qdy$$

define a 2-contact structure on P . Let us check the conditions of the definition. First, the 1-forms are clearly linearly independent. Then,

$$d\eta^1 = dx \wedge dy \quad , \quad d\eta^2 = dx \wedge dp + dy \wedge dq \quad ,$$

from which $\mathcal{D}^R = \langle \partial/\partial s, \partial/\partial t \rangle$, which has rank 2. Obviously none of these two vector fields belongs to the kernel of the 1-forms, which is condition (iii). The Reeb vector fields are

$$\mathcal{R}_1 = \frac{\partial}{\partial s} \quad , \quad \mathcal{R}_2 = \frac{\partial}{\partial t} \quad .$$

3.2. A Darboux theorem for k -contact manifolds

The following result ensures the existence of canonical coordinates for a particular kind of k -contact manifolds:

Theorem 3.10 (k -contact Darboux Theorem). Let (M, η^α) be a k -contact manifold of dimension $n + kn + k$ such that there exists an integrable subdistribution \mathcal{V} of \mathcal{D}^C with $\text{rank } \mathcal{V} = nk$. Around every point of M , there exists a local chart of coordinates $(U; q^i, p_i^\alpha, s^\alpha)$, $1 \leq \alpha \leq k$, $1 \leq i \leq n$, such that

$$\eta^\alpha|_U = ds^\alpha - p_i^\alpha dq^i \quad .$$

In these coordinates,

$$\mathcal{D}^R|_U = \left\langle \mathcal{R}_\alpha = \frac{\partial}{\partial s^\alpha} \right\rangle, \quad \mathcal{V}|_U = \left\langle \frac{\partial}{\partial p_i^\alpha} \right\rangle \quad .$$

These are the so-called **canonical** or **Darboux coordinates** of the k -contact manifold.

Proof. (i): By Proposition 3.6, there exists a chart $(y^l; s^\alpha)$ of adapted coordinates around p such that

$$\mathcal{R}_\alpha = \frac{\partial}{\partial s^\alpha}, \quad \eta^\alpha = ds^\alpha - f_l^\alpha(y) dy^l.$$

Therefore, we can locally construct the quotient manifold $M/\mathcal{D}^R \equiv \tilde{M}$, with projection $\tilde{\tau}: M \rightarrow \tilde{M}$, and local coordinates (\tilde{y}^l) .

(ii): The distribution \mathcal{D}^C , with $\text{rank } \mathcal{D}^C = nk + k$, is $\tilde{\tau}$ -projectable because, for every $\mathcal{R}_\alpha \in \mathfrak{X}(\mathcal{D}^R)$, $Z \in \mathfrak{X}(\mathcal{D}^C)$ and $d\eta^\beta$, we have

$$i([\mathcal{R}_\alpha, Z])d\eta^\beta = \mathcal{L}_{\mathcal{R}_\alpha}i(Z)d\eta^\beta - i(Z)\mathcal{L}_{\mathcal{R}_\alpha}d\eta^\beta = -i(Z)di(\mathcal{R}_\alpha)\eta^\beta = -i(Z)d\delta_\alpha^\beta = 0,$$

and so $[\mathcal{R}_\alpha, Z] \in \mathfrak{X}(\mathcal{D}^R)$.

(Observe that this property is also a consequence of the condition (iii) in Definition 3.1).

(iii): For every β , the forms $d\eta^\beta$ are $\tilde{\tau}$ -projectable because, by Theorem 3.5, for every $\mathcal{R}_\alpha \in \mathfrak{X}(\mathcal{D}^R)$, we have that $i(\mathcal{R}_\alpha)d\eta^\beta = 0$; and hence

$$\mathcal{L}_{\mathcal{R}_\alpha}d\eta^\beta = di(\mathcal{R}_\alpha)\eta^\beta = d\delta_\alpha^\beta = 0.$$

The $\tilde{\tau}$ -projected forms $\tilde{\omega}^\beta \in \Omega^2(\tilde{M})$ such that $d\eta^\beta = \tilde{\tau}^*\tilde{\omega}^\beta$ are obviously closed. In coordinates they read $\tilde{\omega}^\beta = d\tilde{f}_l^\beta(\tilde{y}) \wedge d\tilde{y}^l$.

In addition, for every $Z, Y \in \Gamma(\mathcal{V})$ we have that, as \mathcal{V} is involutive,

$$i(Z)i(Y)d\eta^\beta = i(Z)(\mathcal{L}_Y\eta^\beta - di(Y)\eta^\beta) = i(Z)\mathcal{L}_Y\eta^\beta = \mathcal{L}_Yi(Z)\eta^\beta - i([Y, Z])\eta^\beta = 0.$$

Denoting by $\tilde{\mathcal{V}}$ the distribution induced in \tilde{M} by \mathcal{V} (which has $\text{rank } \tilde{\mathcal{V}} = kn$), then, for every $\tilde{Z}, \tilde{Y} \in \Gamma(\tilde{\mathcal{V}})$ if $Z, Y \in \Gamma(\mathcal{V})$ are such that $\tilde{\tau}_*Z = \tilde{Z}$, $\tilde{\tau}_*Y = \tilde{Y}$, we obtain that

$$0 = i(Z)i(Y)d\eta^\beta = i(Z)i(Y)(\tilde{\tau}^*\tilde{\omega}^\beta) = \tilde{\tau}^*i(\tilde{Z})i(\tilde{Y})\tilde{\omega}^\beta, \quad (5)$$

and, as $\tilde{\tau}$ is a submersion, the map $\tilde{\tau}^*$ is injective and, from (5) we conclude that $i(\tilde{Z})i(\tilde{Y})\tilde{\omega}^\beta = 0$. (Observe that this proof is independent of the representative vector fields Y, Z used, because two of them differ in an element of $\ker \tilde{\tau}_* = \Gamma(\mathcal{D}^R)$). Thus we have proved that, for every β , we have that $\tilde{\omega}^\beta|_{\tilde{\mathcal{V}} \times \tilde{\mathcal{V}}} = 0$.

Finally, as a consequence of (ii), we have that

$$\ker \tilde{\omega}^1 \cap \dots \cap \ker \tilde{\omega}^k = \{0\},$$

Thus we conclude that $(\tilde{M}, \tilde{\omega}^\alpha, \tilde{\mathcal{V}})$ is a k -symplectic manifold.

(iv): Therefore, by the Darboux theorem for k -symplectic manifolds [23], there are local charts of coordinates $(\tilde{U}; \tilde{q}^i, \tilde{p}_i^\alpha)$, $1 \leq i \leq n$, in \tilde{M} , such that

$$\tilde{\omega}^\alpha|_{\tilde{U}} = d\tilde{q}^i \wedge d\tilde{p}_i^\alpha \quad ; \quad \tilde{\mathcal{V}}|_{\tilde{U}} = \left\langle \frac{\partial}{\partial \tilde{p}_i^\alpha} \right\rangle.$$

Therefore, in $U = \tilde{\tau}^{-1}(\tilde{U}) \subset M$ we can take the coordinates $(y^l, s^\alpha) = (q^i, p_i^\alpha, s^\alpha)$, with $q^i = \tilde{q}^i \circ \tilde{\tau}$ and $p_i^\alpha = \tilde{p}_i^\alpha \circ \tilde{\tau}$ verifying the conditions of the theorem. \square

This theorem allows us to consider the manifold presented in Example 3.7 as a canonical model for these kinds of k -contact manifolds. Furthermore, if (M, η^α) is a contactification of a k -symplectic manifold (Example 3.8), then there trivially exist Darboux coordinates.

4. k -contact Hamiltonian systems

Using the geometric framework introduced in the previous section, we are ready to deal with Hamiltonian systems with dissipation in field theories.

Definition 4.1. A k -contact Hamiltonian system is a family $(M, \eta^\alpha, \mathcal{H})$, where (M, η^α) is a k -contact manifold, and $\mathcal{H} \in \mathcal{C}^\infty(M)$ is called a **Hamiltonian function**.

The **k -contact Hamilton–De Donder–Weyl equations** for a map $\psi: D \subset \mathbb{R}^k \rightarrow M$ is

$$\begin{cases} i(\psi'_\alpha) d\eta^\alpha = (d\mathcal{H} - (\mathcal{L}_{\mathcal{R}_\alpha} \mathcal{H}) \eta^\alpha) \circ \psi, \\ i(\psi'_\alpha) \eta^\alpha = -\mathcal{H} \circ \psi. \end{cases} \quad (6)$$

Let us express these equations in coordinates. First, consider adapted coordinates $(x^l; s^\alpha)$, with $R_\alpha = \partial/\partial s^\alpha$, $\eta^\alpha = ds^\alpha - f_l^\alpha(x) dx^l$, and $d\eta^\alpha = \frac{1}{2} \omega_{lj}^\alpha dx^l \wedge dx^j$, with $\omega_{lj}^\alpha = \frac{\partial f_l^\alpha}{\partial x^j} - \frac{\partial f_j^\alpha}{\partial x^l}$. The map ψ is expressed as $\psi(t) = (x^l(t), s^\beta(t))$, and $\psi'_\alpha = (x^l, s^\beta; \partial x^l/\partial t^\alpha, \partial s^\beta/\partial t^\alpha)$. Then, Hamilton–De Donder–Weyl equations read

$$\begin{cases} \frac{\partial x^l}{\partial t^\alpha} \omega_{jl}^\alpha = \frac{\partial \mathcal{H}}{\partial x^l} + \frac{\partial \mathcal{H}}{\partial s^\alpha} f_l^\alpha, \\ \frac{\partial s^\alpha}{\partial t^\alpha} - f_l^\alpha \frac{\partial x^l}{\partial t^\alpha} = -\mathcal{H}. \end{cases}$$

Analogously, in canonical coordinates, if $\psi = (q^i, p_i^\alpha, s^\alpha)$, these equations read

$$\begin{cases} \frac{\partial q^i}{\partial t^\alpha} = \frac{\partial \mathcal{H}}{\partial p_i^\alpha} \circ \psi, \\ \frac{\partial p_i^\alpha}{\partial t^\alpha} = - \left(\frac{\partial \mathcal{H}}{\partial q^i} + p_i^\alpha \frac{\partial \mathcal{H}}{\partial s^\alpha} \right) \circ \psi, \\ \frac{\partial s^\alpha}{\partial t^\alpha} = \left(p_i^\alpha \frac{\partial \mathcal{H}}{\partial p_i^\alpha} - \mathcal{H} \right) \circ \psi, \end{cases} \quad (7)$$

In order to give an alternative geometrical interpretation we consider:

Definition 4.2. Let $(M, \eta^\alpha, \mathcal{H})$ be a k -contact Hamiltonian system. The **k -contact Hamilton–De Donder–Weyl equations** for a k -vector field $\mathbf{X} = (X_1, \dots, X_k)$ in M are

$$\begin{cases} i(X_\alpha) d\eta^\alpha = d\mathcal{H} - (\mathcal{L}_{\mathcal{R}_\alpha} \mathcal{H}) \eta^\alpha, \\ i(X_\alpha) \eta^\alpha = -\mathcal{H}. \end{cases} \quad (8)$$

A k -vector field which is solution to these equations is called a **Hamiltonian k -vector field**.

Proposition 4.3. The k -contact Hamilton–De Donder–Weyl equations (8) admit solutions. They are not unique if $k > 1$.

Proof. A k -vector field \mathbf{X} can be decomposed as $\mathbf{X} = \mathbf{X}^C + \mathbf{X}^R$ corresponding to the direct sum decomposition $TM = \mathcal{D}^C \oplus \mathcal{D}^R$. If \mathbf{X} is a solution to the k -contact Hamilton–De Donder–Weyl equations, then \mathbf{X}^C is a solution to the first of these equations and \mathbf{X}^R of the second one.

Now we introduce two vector bundle maps:

$$\begin{aligned} \rho: TM &\rightarrow \oplus^k T^*M, & \rho(v) &= (\widehat{d\eta^1}(v), \dots, \widehat{d\eta^k}(v)), \\ \tau: \oplus^k TM &\rightarrow T^*M, & \tau(v_1, \dots, v_k) &= \widehat{d\eta^\alpha}(v_\alpha). \end{aligned}$$

Then, notice the following facts:

- $\ker \rho = \mathcal{D}^R$ is the Reeb distribution.
- With the canonical identification $(E \oplus F)^* = E^* \oplus F^*$, the transposed morphism of τ is ${}^t\tau = -\rho$. The proof uses that ${}^t\widehat{d\eta^\alpha} = -\widehat{d\eta^\alpha}$.
- The first Hamilton–De Donder–Weyl equation for a k -vector field \mathbf{X} can be written as $\tau \circ \mathbf{X} = d\mathcal{H} - (\mathcal{L}_{\mathcal{R}_\alpha} \mathcal{H}) \eta^\alpha$.

A sufficient condition for this linear equation to have solutions \mathbf{X} is that the right-hand side be in the image of τ , that is, to be annihilated by any section of $\ker {}^t\tau = \mathcal{D}^R$. But one easily computes that $i(\mathcal{R}_\beta)(d\mathcal{H} - (\mathcal{L}_{\mathcal{R}_\alpha} \mathcal{H}) \eta^\alpha) = 0$, for any β . So we conclude that the first Hamilton–De Donder–Weyl equation has solutions, and in particular solutions \mathbf{X}^C belonging to the contact distribution.

Finally, the second Hamilton–De Donder–Weyl equation for \mathbf{X} admits solutions belonging to the Reeb distribution; for instance, $\mathbf{X}^R = -\mathcal{H}\mathcal{R}_1$. Non-uniqueness is obvious. \square

If $\mathbf{X} = (X_\alpha)$, is a k -vector field solution to Eqs. (8) and $X_\alpha = (X_\alpha)^\beta \frac{\partial}{\partial s^\beta} + (X_\alpha)^l \frac{\partial}{\partial x^l}$ is its expression in adapted coordinates of M , then we have that

$$\begin{cases} (X_\alpha)^j \omega_{jl}^\alpha = \frac{\partial \mathcal{H}}{\partial x^l} + \frac{\partial \mathcal{H}}{\partial s^\alpha} f_l^\alpha, \\ (X_\alpha)^\alpha - f_l^\alpha (X_\alpha)^l = -\mathcal{H}. \end{cases}$$

In canonical coordinates, if $X_\alpha = (X_\alpha)^\beta \frac{\partial}{\partial s^\beta} + (X_\alpha)^i \frac{\partial}{\partial q^i} + (X_\alpha)^\beta_i \frac{\partial}{\partial p_i^\beta}$, then

$$\begin{cases} (X_\alpha)^i = \frac{\partial \mathcal{H}}{\partial p_i^\alpha}, \\ (X_\alpha)_i^\alpha = -\left(\frac{\partial \mathcal{H}}{\partial q^i} + p_i^\alpha \frac{\partial \mathcal{H}}{\partial s^\alpha} \right), \\ (X_\alpha)^\alpha = p_i^\alpha \frac{\partial \mathcal{H}}{\partial p_i^\alpha} - \mathcal{H}, \end{cases} \quad (9)$$

Proposition 4.4. *Let \mathbf{X} be an integrable k -vector field in M . Then every integral section $\psi: D \subset \mathbb{R}^k \rightarrow M$ of \mathbf{X} satisfies the k -contact equation (6) if, and only if, \mathbf{X} is a solution to (8).*

Proof. This is a direct consequence of Eqs. (6) and (8), and the fact that any point of M is in the image of an integral section of \mathbf{X} . \square

Remark 4.5. As in the k -symplectic case, Eqs. (6) and (8) are not, in general, fully equivalent, since a solution to (6) may not be an integral section of an integrable k -vector field solution to (8). This remark will be relevant in Section 6.

Furthermore, in addition to not being unique, solutions to Eqs. (8) are not necessarily integrable.

One can obtain the following alternative expression for the Hamilton–De Donder–Weyl equations; its proof is immediate (the case $k = 1$ was done in [18]).

Proposition 4.6. *The k -contact Hamilton–De Donder–Weyl equations (8) are equivalent to*

$$\begin{cases} \mathcal{L}_{X_\alpha} \eta^\alpha = -(\mathcal{L}_{\mathcal{R}_\alpha} \mathcal{H}) \eta^\alpha, \\ i(X_\alpha) \eta^\alpha = -\mathcal{H}. \end{cases} \quad (10)$$

Finally, we present a sufficient condition for a k -vector field to be a solution of the Hamilton–De Donder–Weyl equations (8) without using the Reeb vector fields \mathcal{R}_α :

Theorem 4.7. *Let $(M, \eta^\alpha, \mathcal{H})$ be a k -contact Hamiltonian system. Consider the 2-forms $\Omega^\alpha = -\mathcal{H} d\eta^\alpha + d\mathcal{H} \wedge \eta^\alpha$. On the open set $O = \{p \in M; \mathcal{H} \neq 0\}$, if a k -vector field $\mathbf{X} = (X_\alpha)$ in M verifies that*

$$\begin{cases} i(X_\alpha) \Omega^\alpha = 0, \\ i(X_\alpha) \eta^\alpha = -\mathcal{H}, \end{cases} \quad (11)$$

then \mathbf{X} is a solution of the Hamilton–De Donder–Weyl equations (8).

Proof. Suppose that \mathbf{X} satisfies Eqs. (11). Then,

$$0 = i(X_\alpha) \Omega^\alpha = -\mathcal{H} i(X_\alpha) d\eta^\alpha + (i(X_\alpha) d\mathcal{H}) \eta^\alpha + \mathcal{H} d\mathcal{H},$$

and hence,

$$\mathcal{H} i(X_\alpha) d\eta^\alpha = (i(X_\alpha) d\mathcal{H}) \eta^\alpha + \mathcal{H} d\mathcal{H}. \quad (12)$$

Contracting this equation with every Reeb vector field \mathcal{R}_β ,

$$0 = \mathcal{H} i(\mathcal{R}_\beta) i(X_\alpha) d\eta^\alpha = (i(X_\alpha) d\mathcal{H}) i(\mathcal{R}_\beta) \eta^\alpha + \mathcal{H} i(\mathcal{R}_\beta) d\mathcal{H} = (i(X_\alpha) d\mathcal{H}) \delta_\beta^\alpha + \mathcal{H} i(\mathcal{R}_\beta) d\mathcal{H},$$

and then $i(X_\beta) d\mathcal{H} = -\mathcal{H} i(\mathcal{R}_\beta) d\mathcal{H}$, for every β . Using this in Eq. (12), we get

$$\mathcal{H} i(X_\alpha) d\eta^\alpha = \mathcal{H} (d\mathcal{H} - (i(\mathcal{R}_\alpha) d\mathcal{H}) \eta^\alpha) = \mathcal{H} (d\mathcal{H} - (\mathcal{R}_\alpha(\mathcal{H})) \eta^\alpha);$$

and therefore $i(X_\alpha) d\eta^\alpha = d\mathcal{H} - (\mathcal{R}_\alpha(\mathcal{H})) \eta^\alpha$. \square

Bearing in mind Definition 4.1 and Proposition 4.4, we can write the equations for the corresponding integral sections:

Proposition 4.8. *On the open set $O = \{p \in M; \mathcal{H} \neq 0\}$, if $\psi: D \subset \mathbb{R}^k \rightarrow O$ is an integral section of a k -vector field solution to the Eqs. (11), then it is a solution to*

$$\begin{cases} i(\psi'_\alpha) \Omega^\alpha = 0, \\ i(\psi'_\alpha) \eta^\alpha = -\mathcal{H} \circ \psi. \end{cases} \quad (13)$$

5. Examples

5.1. Damped vibrating string

It is well known that a vibrating string can be described within the Lagrangian formalism. Let us use coordinates (t, x) for the time and the space, and let u be the separation of a point of the string from the equilibrium position; we also denote by u_t and u_x the derivatives of u with respect to the independent variables. Let ρ be the linear mass density of the string and τ its tension (they are assumed to be constant). Taking as Lagrangian density

$$L = \frac{1}{2} \rho u_t^2 - \frac{1}{2} \tau u_x^2$$

and defining $c^2 = \tau/\rho$ one obtains as the Euler–Lagrange equation the wave equation

$$u_{tt} = c^2 u_{xx}.$$

We rather need to express this equation within the Hamiltonian formalism. We add the momenta of u as dependent variables p^t, p^x . The Legendre transformation \mathcal{FL} of \mathcal{L} is such that

$$\mathcal{FL}^*(p^t) = \rho u_t, \quad \mathcal{FL}^*(p^x) = -\tau u_x,$$

and from it we obtain the Hamiltonian function

$$H = \frac{1}{2\rho} (p^t)^2 - \frac{1}{2\tau} (p^x)^2.$$

As we have a scalar field u and two independent variables (t, x) , this corresponds to a 2-symplectic theory in the canonical model $\oplus^2 T^*\mathbb{R}$. The Hamilton–De Donder–Weyl equations are

$$\frac{\partial u}{\partial t} = \frac{\partial H}{\partial p^t}, \quad \frac{\partial u}{\partial x} = \frac{\partial H}{\partial p^x}, \quad \frac{\partial p^t}{\partial t} + \frac{\partial p^x}{\partial x} = -\frac{\partial H}{\partial u}.$$

For our Hamiltonian, they read

$$\frac{\partial u}{\partial t} = \frac{p^t}{\rho}, \quad \frac{\partial u}{\partial x} = -\frac{p^x}{\tau}, \quad \frac{\partial p^t}{\partial t} + \frac{\partial p^x}{\partial x} = 0.$$

The last equation yields immediately $\rho \frac{\partial^2 u}{\partial t^2} - \tau \frac{\partial^2 u}{\partial x^2} = 0$, which is the wave equation.

A simple model of a vibrating string with an external damping can be obtained by adding to the wave equation a dissipation term proportional to the speed of an element of the string. So this is given by the equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + k \frac{\partial u}{\partial t} = 0,$$

where $k > 0$ is the damping constant [26, p.284]. Now we will show that this equation can be formulated as a contact Hamiltonian system. To this end, according to Example 3.8, we add two additional variables s^t and s^x , and define an extended Hamiltonian $\mathcal{H}(u, p^t, p^x, s^t, s^x)$ by

$$\mathcal{H} = H + h ,$$

where $H = \frac{1}{2\rho}(p^t)^2 - \frac{1}{2\tau}(p^x)^2$ is the Hamiltonian of the undamped vibrating string and

$$h = ks^t .$$

Then the contact Hamilton–De Donder–Weyl equations for H read

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{\rho} p^t , \\ \frac{\partial u}{\partial x} = -\frac{1}{\tau} p^x , \\ \frac{\partial p^t}{\partial t} + \frac{\partial p^x}{\partial x} = -kp^t , \\ \frac{\partial s^t}{\partial t} + \frac{\partial s^x}{\partial x} = \frac{1}{2\rho}(p^t)^2 - \frac{1}{2\tau}(p^x)^2 - ks^t . \end{cases}$$

Using the first and second equations within the third we obtain

$$\rho \frac{\partial^2 u}{\partial t^2} - \tau \frac{\partial^2 u}{\partial x^2} + k\rho \frac{\partial u}{\partial t} = 0 ,$$

which is the equation of the damped string.

5.2. Burgers' equation

Burgers' equation (Bateman, 1915) is a remarkable nonlinear partial differential equation that appears in several areas of applied mathematics. There is one dependent variable u and two independent variables (t, x) , and reads

$$u_t + uu_x = k u_{xx} ,$$

where $k \geq 0$ is a diffusion coefficient [26, p.217]. Notice that it looks quite similar to the heat equation $u_t = k u_{xx}$; indeed we will show that Burgers' equation can be formulated as a contactification of the heat equation. This will be performed in several steps.

Lagrangian formulation of the heat equation. We will need a Hamiltonian formulation of the heat equation. This can be obtained from a Lagrangian formulation of it. Although the heat equation is not variational, it can be made variational by considering an auxiliary dependent variable v , and taking as Lagrangian – see for instance [27]

$$L = -k u_x v_x - \frac{1}{2} (v u_t - u v_t) ,$$

whose Euler–Lagrange equations are

$$[L]_u = k v_{xx} + v_t = 0 , \quad [L]_v = k u_{xx} - u_t = 0 .$$

The first equation is linear homogeneous, therefore always has solutions (for instance $v = 0$). So, there is a correspondence between solutions of the heat equation and solutions of the Euler–Lagrange equations of L with $v = 0$.

Hamiltonian formulation of the heat equation. Now we apply the Donder–Weyl Hamiltonian formalism to L . Its Legendre map (fiber derivative) is a map $\mathcal{FL}: \oplus^2 \mathbb{T}\mathbb{R}^2 \rightarrow P = \oplus^2 \mathbb{T}^* \mathbb{R}^2$. The phase space is $P \approx \mathbb{R}^6$, where the fields are u, v and their respective momenta p^t, p^x and q^t, q^x with respect to the

independent variables. The Legendre map \mathcal{FL} of \mathcal{L} relates these momenta with the configuration fields and their velocities:

$$\begin{aligned}\mathcal{FL}^*(p^x) &= -k v_x, & \mathcal{FL}^*(p^t) &= -\frac{1}{2} v, \\ \mathcal{FL}^*(q^x) &= -k u_x, & \mathcal{FL}^*(q^t) &= \frac{1}{2} u.\end{aligned}$$

So the image $P_0 \subset P$ of the Legendre map is defined by the two constraints

$$p^t \approx -\frac{1}{2} v, \quad q^t \approx \frac{1}{2} u.$$

We will use coordinates (u, v, p^x, q^x) on it. Finally, the Hamiltonian function on P_0 is

$$H = -\frac{1}{k} p^x q^x.$$

The manifold P is endowed with an exact 2-symplectic structure defined by the canonical 1-forms

$$p^t du + q^t dv, \quad p^x du + q^x dv.$$

Their pullbacks to P_0 are not anymore a 2-symplectic structure (under the standard definition), but nevertheless we have two differential 1-forms

$$\theta^t = \frac{1}{2}(-v du + u dv), \quad \theta^x = p^x du + q^x dv$$

with

$$\omega^t = -d\theta^t = -du \wedge dv, \quad \omega^x = -d\theta^x = du \wedge dp^x + dv \wedge dq^x.$$

Now, let $\psi: \mathbb{R}^2 \rightarrow P_0$ be a map, $\psi = (u, v, p^x, q^x)$. It is readily computed that the Hamilton–De Donder–Weyl equation

$$i(\psi'_t)\omega^t + i(\psi'_x)\omega^x = dH \circ \psi$$

for ψ is equivalent to

$$\partial_t v - \partial_x p^x = 0, \quad -\partial_t u - \partial_x q^x = 0, \quad \partial_x u = -\frac{1}{k} q^x, \quad \partial_x v = -\frac{1}{k} p^x.$$

Using the last two equations in the first two ones, we obtain the heat equation for u , and its complementary equation for v :

$$\partial_t u = k \partial_x^2 u, \quad \partial_t v = -k \partial_x^2 v.$$

Notice again that the equation for v is linear homogeneous, therefore there is a bijection between solutions of this system with $v = 0$, and solutions of the heat equation. So this is the Hamiltonian formulation of the heat equation we sought.

Contact Hamiltonian formulation of the Burgers' equation. Now we take again the manifold P_0 and its two differential 1-forms to construct a 2-contact structure. To this end we consider the product manifold $M = P_0 \times \mathbb{R}^2 = \mathbb{R}^6$, with the cartesian coordinates s^t, s^x of \mathbb{R}^2 , and construct the contact forms

$$\eta^t = ds^t - \theta^t, \quad \eta^x = ds^x - \theta^x,$$

where we keep the same notation for θ^t, θ^x as 1-forms on M . Their differentials are the same 2-forms ω^t, ω^x written before. With the notations of Section 3, since η^t, η^x are linearly independent at each point, $\mathcal{C}^C = \langle \eta^t, \eta^x \rangle$ is a regular codistribution of rank 2, and $\mathcal{D}^R = \langle \mathcal{R}_t, \mathcal{R}_x \rangle$, with $\mathcal{R}_t = \partial/\partial s^t$ and $\mathcal{R}_x = \partial/\partial s^x$, is a regular distribution of rank 2. Moreover, $\mathcal{D}^C \cap \mathcal{D}^R = \{0\}$, since no nonzero linear combination of the $\partial/\partial s^t, \partial/\partial s^x$ is annihilated by the contact forms. Therefore, $(M; \eta^t, \eta^x)$ is a 2-contact manifold. This is indeed [Example 3.9](#).

Finally, we take as a contact Hamiltonian the function

$$\mathcal{H} = -\frac{1}{k} p^x q^x + \gamma u s^x.$$

In this way we obtain a 2-contact Hamiltonian system $(M, \eta^\alpha, \mathcal{H})$.

Let us compute the contact Hamilton–De Donder–Weyl equations for this system,

$$i(\psi'_t)\omega^t + i(\psi'_x)\omega^x = d\mathcal{H} - (\mathcal{L}_{\mathcal{R}_t}\mathcal{H})\eta^t - (\mathcal{L}_{\mathcal{R}_x}\mathcal{H})\eta^x,$$

$$i(\psi'_t)\eta^t + i(\psi'_x)\eta^x = -\mathcal{H}.$$

The first equation is similar to the contactless Hamilton–De Donder–Weyl equation, with just some additional terms:

$$\partial_t v - \partial_x p^x = \gamma(s^x + up^x), \quad -\partial_t u - \partial_x q^x = \gamma u q^x, \quad \partial_x u = -\frac{1}{k} q^x, \quad \partial_x v = -\frac{1}{k} p^x.$$

Again, putting the latter two equations inside the former ones, we obtain

$$\partial_t u - \gamma k u \partial_x u = k \partial_x^2 u, \quad \partial_t v + \gamma k u \partial_x v = -k \partial_x^2 v + \gamma s^x.$$

Setting the value of the constant $\gamma = -1/k$, the first equation is Burgers' equation for u .

Finally, the second contact Hamilton–De Donder–Weyl equation reads:

$$\partial_t s^t - \frac{1}{2}(-v \partial_t u + u \partial_t v) + \partial_x s^x - p^x \partial_x u - q^x \partial_x v = \frac{1}{k} p^x q^x - \gamma u s^x.$$

Again, notice that these equations admit solutions $(u, v, p^x, q^x, s^t, q^t)$ with $v, p^x, s^t, s^x = 0$, $q^x = -k \partial_x u$, and u an arbitrary solution of Burgers' equation. Therefore, we conclude that the Burgers' equation can be described by the 2-contact Hamiltonian system $(M, \eta^\alpha, \mathcal{H})$.

6. Symmetries and dissipation laws

Finally, we introduce some basic ideas about symmetries and deduce an associated dissipation law for k -contact Hamiltonian field theories.

6.1. Symmetries

There are different concepts of symmetry of a problem, depending on which structure was preserved. One can put the emphasis on the transformations that preserve the geometric structures of the problem, or on the transformations that preserve its solutions [28]. This has been done in particular for the k -symplectic Hamiltonian formalism [29]. We will apply the same idea to k -contact Hamiltonian systems. First we consider symmetries as those transformations that map solutions of the equations into other solutions. So we define:

Definition 6.1. Let $(M, \eta^\alpha, \mathcal{H})$ be a k -contact Hamiltonian system.

- A **dynamical symmetry** is a diffeomorphism $\Phi : M \rightarrow M$ such that, for every solution ψ to the k -contact Hamilton–De Donder–Weyl equations (8), $\Phi \circ \psi$ is also a solution.
- An **infinitesimal dynamical symmetry** is a vector field $Y \in \mathfrak{X}(M)$ whose local flow is made of local dynamical symmetries.

We will give a characterization of symmetries in terms of k -vector fields. First let us recall the following fact about integral sections:

Lemma 6.2. Let $\Phi : M \rightarrow M$ be a diffeomorphism and $\mathbf{X} = (X_1, \dots, X_k)$ a k -vector field in M . If ψ is an integral section of \mathbf{X} , then $\Phi \circ \psi$ is an integral section of $\Phi_* \mathbf{X} = (\Phi_* X_\alpha)$. In particular, if \mathbf{X} is integrable then $\Phi_* \mathbf{X}$ is also integrable.

Then we have:

Proposition 6.3. *If $\Phi: M \rightarrow M$ is a dynamical symmetry then, for every integrable k -vector field \mathbf{X} solution to the k -contact Hamilton–De Donder–Weyl equations (8), $\Phi_*\mathbf{X}$ is another solution.*

On the other side, if Φ transforms every k -vector field \mathbf{X} solution to Eqs. (8) into another solution, then for every integral section ψ of \mathbf{X} , we have that $\Phi \circ \psi$ is a solution to the k -contact Hamilton–De Donder–Weyl equations (6).

Proof. (\Rightarrow) Let $x \in M$ and let ψ be an integral section of \mathbf{X} through the point $\Phi^{-1}(x)$, that is $\psi(t_0) = \Phi^{-1}(x)$. We know that ψ is a solution to the k -contact Hamilton–De Donder–Weyl equations (6) and, as Φ is a dynamical symmetry, $\Phi \circ \psi$ is also a solution to Eqs. (6). But, by the preceding lemma, it is an integral section of $\Phi_*\mathbf{X}$ through the point $\Phi(\psi(t_0)) = \Phi(\Phi^{-1}(x)) = x$ and hence we have that $\Phi_*\mathbf{X}$ must be a solution to Eqs. (8) at the points $(\Phi \circ \psi)(t)$ and, in particular, at the point $(\Phi \circ \psi)(t_0) = x$.

(\Leftarrow) On the other side, let \mathbf{X} be a solution to Eqs. (8) and $\psi: U \subset \mathbb{R}^k \rightarrow M$ an integral section of \mathbf{X} . Then, by hypothesis, $\Phi_*\mathbf{X}$ is also a solution and then, as a consequence of the previous lemma, we have that $\Phi \circ \psi$ is a solution to the k -contact Hamilton–De Donder–Weyl equations (6). \square

In geometrical physics, among the most relevant symmetries there are those that let the geometric structures invariant:

Definition 6.4. Let $(M, \eta^\alpha, \mathcal{H})$ be a k -contact Hamiltonian system.

A **Hamiltonian k -contact symmetry** is a diffeomorphism $\Phi: M \rightarrow M$ such that

$$\Phi^*\eta^\alpha = \eta^\alpha, \quad \Phi^*\mathcal{H} = \mathcal{H}.$$

An **infinitesimal Hamiltonian k -contact symmetry** is a vector field $Y \in \mathfrak{X}(M)$ whose local flow is a local Hamiltonian k -contact symmetry; that is,

$$\mathcal{L}_Y\eta^\alpha = 0, \quad \mathcal{L}_Y\mathcal{H} = 0.$$

Proposition 6.5. *Every (infinitesimal) Hamiltonian k -contact symmetry preserves the Reeb vector fields, that is; $\Phi_*\mathcal{R}_\alpha = \mathcal{R}_\alpha$ (or $[Y, \mathcal{R}_\alpha] = 0$).*

Proof. We have that

$$\begin{aligned} i(\Phi_*^{-1}\mathcal{R}_\alpha)(\Phi^*d\eta^\alpha) &= \Phi^*i(\mathcal{R}_\alpha)d\eta^\alpha = 0, \\ i(\Phi_*^{-1}\mathcal{R}_\alpha)(\Phi^*\eta^\alpha) &= \Phi^*i(\mathcal{R}_\alpha)\eta^\alpha = 1, \end{aligned}$$

and, as $\Phi^*\eta^\alpha = \eta^\alpha$ and the Reeb vector fields are unique, from these equalities we conclude that $\Phi_*\mathcal{R}_\alpha = \mathcal{R}_\alpha$.

The proof for the infinitesimal case is immediate from the definition. \square

Finally, as a consequence of these results, we obtain the relation between Hamiltonian k -contact symmetries and dynamical symmetries:

Proposition 6.6. *(Infinitesimal) Hamiltonian k -contact symmetries are (infinitesimal) dynamical symmetries.*

Proof. Let ψ be a solution to the k -contact-De Donder–Weyl equations (6), and Φ a Hamiltonian k -contact symmetry. Then

$$\begin{aligned} i((\Phi \circ \psi)'_\alpha)\eta^\alpha &= i(\Phi_*(\psi'_\alpha))((\Phi^{-1})^*\eta^\alpha) = (\Phi^{-1})^*i(\psi'_\alpha)\eta^\alpha \\ &= (\Phi^{-1})^*(-\mathcal{H} \circ \psi) = -\mathcal{H} \circ (\Phi \circ \psi), \\ i((\Phi \circ \psi)'_\alpha)d\eta^\alpha &= i(\Phi_*(\psi'_\alpha))((\Phi^{-1})^*d\eta^\alpha) = (\Phi^{-1})^*i(\psi'_\alpha)d\eta^\alpha \\ &= (\Phi^{-1})^*((d\mathcal{H} - (\mathcal{L}_{\mathcal{R}_\alpha}\mathcal{H})\eta^\alpha) \circ \psi) \end{aligned}$$

$$\begin{aligned}
&= \left(d(\Phi^{-1})^* \mathcal{H} - (\mathcal{L}_{(\Phi^{-1})^* \mathcal{R}_\alpha} (\Phi^{-1})^* \mathcal{H}) (\Phi^{-1})^* \eta^\alpha \right) \circ (\Phi \circ \psi) \\
&= \left(d\mathcal{H} - (\mathcal{L}_{\mathcal{R}_\alpha} \mathcal{H}) \eta^\alpha \right) \circ (\Phi \circ \psi) .
\end{aligned}$$

The proof for the infinitesimal case is immediate from the definition. \square

6.2. Dissipation laws

In many mechanical systems without dissipation, we are interested in quantities which are conserved along a solution. Classical examples are the energy or the different momenta. From a physical point of view, if a system has dissipation, these quantities are not conserved. This behavior is explicitly shown for Hamiltonian contact systems in the so called energy dissipation theorem [17] which says that, if $X_{\mathcal{H}}$ is a contact Hamiltonian vector field, then

$$\mathcal{L}_{X_{\mathcal{H}}} \mathcal{H} = -(\mathcal{L}_{\mathcal{R}} \mathcal{H}) \mathcal{H}.$$

This equation shows that, in a contact system, the dissipations are exponential with rate $-(\mathcal{L}_{\mathcal{R}} \mathcal{H})$. In dissipative field theories, a similar structure can be observed in the first equation of (10), which can be interpreted as the dissipation of the contact forms (η^α) . Then, bearing in mind the definition of conservation law for field theories as stated in [30], and taking into account Remark 4.5, this suggests the following definitions of dissipation laws for k -contact Hamiltonian systems:

Definition 6.7. Let $(M, \eta^\alpha, \mathcal{H})$ be a k -contact Hamiltonian system. A map $F: M \rightarrow \mathbb{R}^k$, $F = (F^1, \dots, F^k)$, is said to satisfy:

1. The **dissipation law for sections** if, for every solution ψ to the k -contact Hamilton–De Donder–Weyl equations (6), the divergence of $F \circ \psi = (F^\alpha \circ \psi): \mathbb{R}^k \rightarrow \mathbb{R}^k$, which is defined as usual by $\text{div}(F \circ \psi) = \partial(F^\alpha \circ \psi)/\partial t^\alpha$, satisfies that

$$\text{div}(F \circ \psi) = -[(\mathcal{L}_{\mathcal{R}_\alpha} \mathcal{H}) F^\alpha] \circ \psi . \quad (14)$$

2. The **dissipation law for k -vector fields** if, for every k -vector field \mathbf{X} solution to the k -contact Hamilton–De Donder–Weyl equations (8), the following equation holds:

$$\mathcal{L}_{X_\alpha} F^\alpha = -(\mathcal{L}_{\mathcal{R}_\alpha} \mathcal{H}) F^\alpha . \quad (15)$$

Both concepts are partially related by the following property:

Proposition 6.8. If $F = (F^\alpha)$ satisfies the dissipation law for sections then, for every integrable k -vector field $\mathbf{X} = (X_\alpha)$ which is a solution to the k -contact Hamilton–De Donder–Weyl equations (8), we have that Eq. (15) holds for \mathbf{X} .

On the other side, if (15) holds for a k -vector field \mathbf{X} , then (14) holds for every integral section ψ of \mathbf{X} .

Proof. If $F = (F^\alpha)$ satisfies the dissipation law for sections, $\mathbf{X} = (X_\alpha)$ is an integrable k -vector field which is a solution to the k -contact Hamilton–De Donder–Weyl equations (8), and $\psi: \mathbb{R}^k \rightarrow M$ an integral section of \mathbf{X} , then by Proposition 4.4, ψ is a solution to the k -contact Hamilton–De Donder–Weyl equations (6); therefore

$$(\mathcal{L}_{X_\alpha} F^\alpha) \circ \psi = \frac{d}{dt^\alpha} (F^\alpha \circ \psi) = \text{div}(F \circ \psi) = -[(\mathcal{L}_{\mathcal{R}_\alpha} \mathcal{H}) F^\alpha] \circ \psi ,$$

and, as \mathbf{X} is integrable, there exists an integral section through every point, hence the result follows.

On the other side, if (15) holds, then the statement is a straightforward consequence of the above expression. \square

Lemma 6.9. If Y is an infinitesimal dynamical symmetry then, for every solution $\mathbf{X} = (X_\alpha)$ to the k -contact Hamilton–De Donder–Weyl equations (8), we have that

$$i([Y, X_\alpha]) \eta^\alpha = 0 , \quad i([Y, X_\alpha]) d\eta^\alpha = 0 .$$

Proof. Let F_ε be the local 1-parameter groups of diffeomorphisms generated by Y . As Y is an infinitesimal dynamical symmetry, $i(F_\varepsilon^* X_\alpha) \eta^\alpha = i(X_\alpha) \eta^\alpha$, because both are solutions to the Hamilton–De Donder–Weyl equations (8). Then, as the contraction is a continuous operation,

$$i([Y, X_\alpha]) \eta^\alpha = i \left(\lim_{\varepsilon \rightarrow 0} \frac{F_\varepsilon^* X_\alpha - X_\alpha}{\varepsilon} \right) \eta^\alpha = \lim_{\varepsilon \rightarrow 0} \frac{i(F_\varepsilon^* X_\alpha) \eta^\alpha - i(X_\alpha) \eta^\alpha}{\varepsilon} = 0.$$

The equation involving $d\eta^\alpha$ is proved in the same way. \square

Then we have the following fundamental result which associates dissipation laws for k -vector fields with symmetries:

Theorem 6.10 (Dissipation Theorem). *If Y is an infinitesimal dynamical symmetry, then $F^\alpha = -i(Y) \eta^\alpha$ satisfies the dissipation law for k -vector fields.*

Proof. Consider a solution X to the k -contact Hamilton–De Donder–Weyl equations (8). From Lemma 6.9, we have that $i([Y, X_\alpha]) \eta^\alpha = 0$, therefore

$$\mathcal{L}_{X_\alpha} i(Y) \eta^\alpha = i([X_\alpha, Y]) \eta^\alpha + i(Y) \mathcal{L}_{X_\alpha} \eta^\alpha = -(\mathcal{L}_{\mathcal{R}_\alpha} \mathcal{H}) i(Y) \eta^\alpha. \quad \square$$

6.3. Examples

6.3.1. Mechanics: energy dissipation

In this case $k = 1$. Let X_h be the Hamiltonian contact vector field. Then, as $[X_h, X_h] = 0$, the vector field X_h is a dynamical symmetry. Then, applying the dissipation theorem we have that $-i(X_h) \eta = \mathcal{H}$ satisfies the dissipation law

$$\mathcal{L}_{X_h} \mathcal{H} = -(\mathcal{L}_{\mathcal{R}} \mathcal{H}) \mathcal{H},$$

which is the energy dissipation theorem [17].

6.3.2. Damped vibrating string

We resume the example discussed in Section 5.1. The vector field $\frac{\partial}{\partial q}$ is a contact symmetry. Hence it induces the map

$$F = \left(-i \left(\frac{\partial}{\partial q} \right) \eta^t, -i \left(\frac{\partial}{\partial q} \right) \eta^x \right) = (p^t, p^x),$$

which follows the dissipation law (15):

$$\mathcal{L}_{X_t} p^t + \mathcal{L}_{X_x} p^x = -(\mathcal{L}_{\mathcal{R}_t} \mathcal{H}) p^t - (\mathcal{L}_{\mathcal{R}_x} \mathcal{H}) p^x = -2\gamma p^t.$$

Over a solution (X_t, X_x) , this law is

$$p_t^t + p_x^x = -2\gamma p^t.$$

6.3.3. Burgers' equation

Now we take up the example discussed in Section 5.2.

The vector field $\frac{\partial}{\partial v}$ is not a contact symmetry but a dynamical symmetry. Hence it induces the map

$$F = \left(-i \left(\frac{\partial}{\partial v} \right) \eta^t, -i \left(\frac{\partial}{\partial v} \right) \eta^x \right) = \left(\frac{1}{2k} u, p_v^x \right).$$

which follows the dissipation law (15):

$$\mathcal{L}_{X_t} \frac{1}{2k} u + \mathcal{L}_{X_x} p_v^x = -(\mathcal{L}_{\mathcal{R}_t} \mathcal{H}) \frac{1}{2k} u - (\mathcal{L}_{\mathcal{R}_x} \mathcal{H}) p_v^x.$$

Over a solution (X_t, X_x) this law is

$$\frac{1}{2k} \frac{\partial u}{\partial t} + \frac{\partial p_v^x}{\partial x} = \gamma u p_v^x,$$

which is the Burgers' equation again.

6.3.4. A model of two coupled vibrating strings with damping

Consider $M = \oplus^2 T^* \mathbb{R}^2 \times \mathbb{R}^2$, with coordinates $(t, x, q^1, q^2, p_1^t, p_2^t, p_1^x, p_2^x, s^t, s^x)$, where q^1 and q^2 represent the displacements of both strings. When it is endowed with the forms

$$\eta^t = ds^t - p_1^t dq^1 - p_2^t dq^2, \quad \eta^x = ds^x - p_1^x dq^1 - p_2^x dq^2,$$

we have the 2-contact manifold (M, η^t, η^x) . Now consider the Hamiltonian function

$$\mathcal{H} = \frac{1}{2}((p_1^t)^2 + (p_2^t)^2 + (p_1^x)^2 + (p_2^x)^2) + G(z) + \gamma s^t,$$

where G is a function that represents a coupling of the two strings, and which we assume to depend only on $z = ((q^2)^2 + (q^1)^2)^{1/2}$.

A simple computation shows that the following vector field is an infinitesimal contact symmetry

$$Y = q^1 \frac{\partial}{\partial q^2} - q^2 \frac{\partial}{\partial q^1} + p_1^t \frac{\partial}{\partial p_2^t} - p_2^t \frac{\partial}{\partial p_1^t} + p_1^x \frac{\partial}{\partial p_2^x} - p_2^x \frac{\partial}{\partial p_1^x},$$

and it induces the map

$$F = (-i(Y)\eta^t, -i(Y)\eta^x) = (q^1 p_2^t - q^2 p_1^t, q^1 p_2^x - q^2 p_1^x),$$

which satisfies the dissipation equation along a solution (X^t, X^x)

$$\begin{aligned} \mathcal{L}_{X^x}(q^1 p_2^t - q^2 p_1^t) + \mathcal{L}_{X^t}(q^1 p_2^x - q^2 p_1^x) &= q^1 \left(\frac{\partial p_2^t}{\partial t} + \frac{\partial p_2^x}{\partial x} \right) - q^2 \left(\frac{\partial p_1^t}{\partial t} + \frac{\partial p_1^x}{\partial x} \right) \\ &= -\gamma(q^1 p_2^t - q^2 p_1^t). \end{aligned}$$

7. Conclusions and outlook

In this paper we have introduced a Hamiltonian formalism for field theories with dissipation. Using techniques from contact geometry and the k -symplectic Hamiltonian formalism, we have developed a new geometric framework, defining the concepts of k -contact manifold and k -contact Hamiltonian system. In the same way that a contact structure allows to describe dissipative mechanics, a k -contact structure gives a transparent description of dissipative systems in field theory over a k -dimensional parameter space. It is important to stress that, to our knowledge, this is the first time these geometric structures are presented.

In more detail, we have stated the definition of k -contact structure on a manifold, as a family of k differential 1-forms satisfying certain properties (Definition 3.1). This implies the existence of two special tangent distributions, in particular the Reeb distribution, which is spanned by k Reeb vector fields. We have proved the existence of special systems of coordinates, and a Darboux-type theorem for a particular kind of these manifolds.

Using this structure, the notion of k -contact Hamiltonian system is defined. The corresponding field equations (Definition 4.1) are a generalization of both the contact Hamilton equations of dissipative mechanics, and the Hamilton–De Donder–Weyl equations of Hamiltonian field theory.

We have analyzed the concept of symmetry for dissipative Hamiltonian field theories. We study two natural types of symmetries: those preserving the solutions to the field equations, and those preserving the geometric k -contact structure and the Hamiltonian function. We have also defined the notion of dissipation law in order to extend the energy dissipation theorem of contact mechanics, stating a dissipation theorem which relates symmetries and dissipation laws which is analogous to the conservation theorems in the case of conservative field theories.

Two relevant examples show the significance of our framework: the vibrating string with damping, and the Burgers equation. In our presentation, Burgers' equation is obtained as a contactification of the heat equation; to this end, we have also provided, for the first time, a Hamiltonian field theory describing the heat equation.

The results of this work open some future lines of research. The first one would be the definition of the Lagrangian formalism for dissipative field theories and the associated Hamiltonian formalism. The case of a singular Lagrangian would require to define the notion of k -precontact structure on a manifold, that is, a family of k 1-forms η^α that do not meet all the conditions of Definition 3.1. (The case $k = 1$ has been recently analyzed in [14].) Another interesting issue would be to deepen the study of symmetries for k -contact Lagrangian and Hamiltonian systems.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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