

Structural aspects of Hamilton–Jacobi theory

J. F. Cariñena*

*Departamento de Física Teórica and IUMA, Universidad de Zaragoza
Pedro Cerbuna 12, 50.009 Zaragoza, Spain
jfc@unizar.es*

X. Gràcia

*Departamento de Matemáticas
Edificio C-3, Campus Norte UPC
C/Jordi Girona 1, E-08034 Barcelona, Spain
xavier.gracia@upc.edu*

G. Marmo

*Dipartimento di Fisica, Università di Napoli Federico II
Complesso Universitario di Monte Sant'Angelo
Via Cintia, 80126 Napoli, Italy
INFN, Sezione di Napoli, Complesso Universitario di Monte Sant'Angelo
Via Cintia, 80126 Napoli, Italy
marmo@na.infn.it*

E. Martínez

*Departamento de Matemática Aplicada and IUMA, Universidad de Zaragoza
Pedro Cerbuna 12, 50.009 Zaragoza, Spain
emf@unizar.es*

M. C. Muñoz-Lecanda[†] and N. Román-Roy[‡]

*Departamento de Matemáticas
Edificio C-3, Campus Norte UPC
C/Jordi Girona 1, E-08034 Barcelona, Spain
[†]miguel.carlos.munoz@upc.edu
[‡]narciso.roman@upc.edu*

Received 21 October 2015

Accepted 30 October 2015

Published 28 December 2015

In our previous papers [J. F. Cariñena, X. Gràcia, G. Marmo, E. Martínez, M. C. Muñoz-Lecanda and N. Román-Roy, Geometric Hamilton–Jacobi theory, *Int. J. Geom. Meth. Mod. Phys.* **3** (2006) 1417–1458; Geometric Hamilton–Jacobi theory for nonholonomic dynamical systems, *Int. J. Geom. Meth. Mod. Phys.* **7** (2010) 431–454] we showed that

*Corresponding author.

the Hamilton–Jacobi problem can be regarded as a way to describe a given dynamics on a phase space manifold in terms of a family of dynamics on a lower-dimensional manifold. We also showed how constants of the motion help to solve the Hamilton–Jacobi equation. Here we want to delve into this interpretation by considering the most general case: a dynamical system on a manifold that is described in terms of a family of dynamics (slicing vector fields) on lower-dimensional manifolds. We identify the relevant geometric structures that lead from this decomposition of the dynamics to the classical Hamilton–Jacobi theory, by considering special cases like fibered manifolds and Hamiltonian dynamics, in the symplectic framework and the Poisson one. We also show how a set of functions on a tangent bundle can determine a second-order dynamics for which they are constants of the motion.

Keywords: Hamilton–Jacobi equation; slicing vector field; complete solution; constant of the motion.

Mathematics Subject Classification 2010: 70H20, 70G45

1. Introduction

Hamilton–Jacobi theory originated with Hamilton to deal with what nowadays is called Hamiltonian optics, i.e. to describe the ray propagation of light, and with Jacobi who was interested in devising a procedure to integrate equations of motions when they are given in canonical form. In Jacobi’s own words: “After we have reduced the problems of mechanics to the integration of a nonlinear first-order partial differential equation, we must concern ourselves with the integration of the same, i.e. with the search for a complete solution” [25, p. 183]. Hadamard [22, 23] and Volterra [53] derived the Hamilton–Jacobi equations by considering the short-wave limit of wave equations. It was this association which paved the way for de Broglie to introduce the relation

$$pdx - Hdt = \hbar(kdx - \omega dt)$$

relating wave concepts with particle concepts [2]. Using this analogy, Schrödinger proposed the evolutionary equation for wave mechanics, opening the route to a formalism able to describe physical phenomena at atomic scale. (A geometrical description of the quantum-to-classical transition on space-time was elaborated by Synge [47].)

Concerning the role of solutions to the Hamilton–Jacobi equation, providing a family of solutions for Hamilton’s equations, Dirac wrote [18]: “The family does not have any importance from the point of view of Newtonian mechanics; but it is a family which corresponds to one state of motion in the quantum theory, so presumably the family has some deep significance in nature, not yet properly understood”. These general comments are aimed at contextualizing the role of the Hamilton–Jacobi theory in theoretical physics. To enter the *raison d’être* of this paper, let us recall how Hamilton–Jacobi theory is usually dealt with in textbooks and works on analytical mechanics [3, 21, 39, 44, 26, 57].

Hamilton–Jacobi theory is usually considered when dealing with canonical transformations to define them by means of generating functions. Specifically,

by using canonical coordinates, say $(p, q; t)$ and $(\bar{p}, \bar{q}; t)$, one looks for a function $S : Q \times Q \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$pdq - Hdt = \bar{p}d\bar{q} - Kdt + dS(q, \bar{q}; t),$$

with H and K Hamiltonian functions on phase space. The associated transformation is defined by means of the implicit equations

$$p = \frac{\partial S}{\partial q}, \quad \bar{p} = -\frac{\partial S}{\partial \bar{q}}, \quad K - H = \frac{\partial S}{\partial t},$$

and this canonical transformation, if it exists, converts the Hamiltonian system described by H into the one described by K . By further requiring that K is a constant or that it is a function depending only on \bar{p} , one relates the original system to another one which is completely integrable, and therefore integrable by quadratures.

The short-wave limit point of view starts from a second-order partial differential equation of hyperbolic type and derives what is known as the *eikonal equation*

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 = n^2(x, y, z)$$

with n denoting the refractive index [9, p. 119; 19, p. 108]. The function S is usually called the *eikonal function* or the characteristic function. As a matter of fact, Hamilton introduced two functions, $S(t, x, y, z)$, called the *principal function*, and putting $W(x, y, z) - tE = S(t, x, y, z)$, W was called the *characteristic function* [52].

From the point of view of Jacobi [25], the integration of Hamilton's equations is achieved by solving first the first-order differential equation on configuration space

$$\frac{dq^j}{dt} = \frac{\partial H}{\partial p_j} \Big|_{p_j = \frac{\partial S}{\partial q^j}};$$

then, setting

$$p_j = \frac{\partial S}{\partial q^j}(t, q^j(t)),$$

one finds a full solution of Hamilton's equations with initial condition $(q^j(0), p_j(0), t = 0)$. Thus, from this point of view, the Hamilton–Jacobi equation is instrumental to define a *family of first-order differential equations on configuration space* whose solutions will eventually produce solutions for the Hamilton equations on phase space. In the first-order differential equation (*fode*, in the sequel)

$$\frac{dq^j}{dt} = \frac{\partial H}{\partial p_j} \Big|_{p_j = \frac{\partial S}{\partial q^j}}$$

one changes the values of the arbitrary constants appearing in a complete integral function S and obtains a family of *fodes*. The solutions of each one of these equations are the solutions alluded by Dirac and correspond to a given S , related to the phase of the wave function in quantum mechanics.

Therefore a complete solution to the Hamilton–Jacobi equation gives rise to a family of first-order differential equations on the configuration space, say Q , which are sufficient to recover all the solutions to Hamilton’s equations on T^*Q . From the geometrical point of view, a complete solution amounts to an invariant foliation of T^*Q , with leaves diffeomorphic to Q and transverse to fibers of the cotangent bundle projection. A family of first-order differential equations is obtained by restricting the Hamiltonian vector field to each leaf of the invariant foliation.

From all that we have said about Hamilton–Jacobi theory it is clear that we may identify two main aspects in the Hamilton–Jacobi theory. The first one is to solve a *fode* in a manifold P (usually T^*Q) by solving an associated family of *fode*’s on a lower dimensional manifold Q ; when all the solutions may be found in this manner, the family is said to be *complete*. The second one consists of finding this complete family by solving an associated partial differential equation for a single function S , this would be the analog of the *eikonal equation*.

To analyze these problems we introduce a general scheme by means of a vector field Z on a manifold P , along with a fibration $P \rightarrow M$. We consider all the integral curves of Z on P and project them onto M . Having all these curves on M , we would like to “group” them into coherent sets of integral curves for vector fields on M . In other terms, we would like to put together all those integral curves of Z which may be obtained as integral curves of a certain vector field X on M . If all integral curves of Z may be grouped into families such that each family, after projection, arises as integral curves of a vector field X on M , we say that the family of vector fields X is a *complete slicing* of the dynamics Z , or that it is a complete solution to the *generalized Hamilton–Jacobi problem*. This paper deals mostly with the first aspect, i.e. to solve a differential equation on P by means of a family of differential equations on Q .

A similar problem, i.e. going from trajectories to vector fields on M , was discussed in [40, Chap. 6]. It is shown there that, in this generality, by no means the problem will have solutions. Thus, the existence of a family of vector fields on M sufficient to reproduce all integral curves of Z on P will put quite strong conditions on Z . We have already remarked that for Hamiltonian systems on $P = T^*Q$ the existence of the family would require Z to be a completely integrable system. Of course a kind of inverse problem could be posed: given a family of vector fields on M , is it possible to find a vector field Z on P such that it would be possible to represent the whole family of integral curves of the various vector fields on M as projections of integral curves of the vector field Z on P ? Let us stress that these problems would arise in particular physical problems like motion of particles with internal structure and in general in problems with restricted allowed Cauchy data, for instance, gauge terms. It would also occur in quantum mechanics when we consider a composite system and we would like to describe it in terms of the evolution of subsystems (entanglement would be an obstruction to the solution of the posed inverse problem). A single case where the inverse problem has a nice solution is

provided by a second-order vector field on TQ , completely determined by a suitable family of functionally independent constants of the motion, as we show at the end of the paper.

To pin-point the geometrical contents of the standard Hamilton–Jacobi equation, first we shall consider the usual Hamilton–Jacobi theory from a more geometric point of view. In the usual approach $P = T^*Q$, and $\pi : P \rightarrow Q$ is the usual cotangent bundle projection. The dynamical vector field $\Gamma = Z$ solves the equation $i_\Gamma \omega = dH$, where ω is the canonical symplectic structure in T^*Q and H is the Hamiltonian function. By using the symplectic potential for ω , say $\omega = -d\theta_0$, we define a vector field Δ , $i_\Delta \omega = \theta_0$, which represents the linear structure along the fibers, and the Hamilton–Jacobi equation for S becomes

$$(dS)^*\theta_0 = dS, \quad (dS)^*H = E,$$

where E is a “parameter”. When S is a complete integral, we have that $dS : Q \times N \rightarrow T^*Q$ is a diffeomorphism for “most initial conditions” for Γ . It provides a $\dim Q$ -foliation of T^*Q (or some open dense submanifold of it) transversal to the fibers. The vector field Γ , restricted to each leaf, being tangent to it, defines a vector field which projects onto a vector field X defined on Q . There would be a vector field X for each leaf. In this manner the invariant foliation defines a family of first-order differential equations on Q , each one of them being the projection of the restriction of Γ to the invariant leaf. This means that Γ may be replaced by the family of vector fields that we obtain by restricting Γ to a family of leaves transversal to the fibers. Thus the issue becomes how to find an invariant foliation transversal to fibers.

These and other intrinsic considerations about the Hamilton–Jacobi equation can be found in [1, 37, 38]. In addition, in [11] a general geometric framework for the Hamilton–Jacobi theory was presented and the Hamilton–Jacobi equation in the Lagrangian and in the Hamiltonian formalisms was formulated for autonomous and nonautonomous mechanics, recovering the usual Hamilton–Jacobi equation as a special case in this generalized framework. The relationship between the Hamilton–Jacobi equation and some geometric structures of mechanics was analyzed also in [7, 12]. A similar generalization of the Hamilton–Jacobi formalism was outlined in [27]. Later on, these geometric frameworks were used to develop the Hamilton–Jacobi theory in many different situations. Thus, in [8, 13, 30, 24, 41, 42] this is done for holonomic and nonholonomic mechanical systems, in [20, 29, 32, 33] the theory is extended for singular systems, in [4, 28, 34] for geometric mechanics on Lie algebroids and almost-Poisson manifolds respectively, in [6, 54, 55] for control theory, in [10, 31, 35, 36, 51] for different formulations of classical field theories (and in [50] for partial differential equations in general), and in [15–17, 49] for higher order dynamical systems and higher order field theories. Finally, the geometric discretization of the Hamilton–Jacobi equation is also considered in [5, 43].

In particular, in our previous papers [11, 13] we saw that the Hamilton–Jacobi problem can be regarded as a way to describe a given dynamics on a phase space

manifold in terms of a family of dynamics on a lower dimensional manifold. Moreover, we saw that the existence of many constants of the motion for the given dynamics helps to solve the Hamilton–Jacobi problem. The aim of this paper is to look more deeply into this interpretation by considering the most general case and identifying what are the relevant geometric structures.

We should remark that our framework allows to handle dynamical vector fields which cannot be handled with classical approaches to Hamilton–Jacobi equation. For instance, suppose we have a completely integrable Hamiltonian system given by a Hamiltonian vector field Z_H ; its Hamilton–Jacobi equation has a complete solution, and therefore we have a complete slicing of the dynamics. Then consider a new dynamics given by $Z' = fZ_H$, where f is a generic function — this leads to a reparametrization of the integral curves. Our procedure allows to construct a complete slicing for Z' , although Z' may not be Hamiltonian. (An instance where this reparametrization may be required is when Z_H is not a complete vector field.)

The paper is organized as follows: In Sec. 2 we present the general concepts and results needed to state a more general framework for the Hamilton–Jacobi problem. The study of constants of the motion and complete solutions and their relationship for this general setting is done in Sec. 3, by introducing the concept of *slicing vector fields* and complete slicings. Section 4 is devoted to discuss some particular situations deriving from this general framework, such as Hamiltonian systems defined on symplectic and Poisson manifolds. The slicing problem is discussed again in Sec. 5 in the case where the dynamical system, either general or Hamiltonian, is defined on a generic fibered manifold. Finally, in Sec. 6 we show how our previous results in [11] are recovered from here, and we also study how the knowledge of enough constants of the motion determines a second-order dynamics. Along the work, different examples are also introduced to illustrate our results. All the manifolds and maps are assumed to be C^∞ .

2. Dynamical Systems, Invariant Submanifolds and Constants of the Motion

2.1. Dynamical systems

A dynamical system is a pair (P, Z) given by a manifold P and a vector field Z on P . This defines a (first-order, autonomous) differential equation on P , $\gamma' = Z \circ \gamma$, for a path $\gamma : I \rightarrow P$. This dynamics may possess several features. For the purposes of this work we are especially interested in invariant submanifolds and constants of the motion.

A submanifold $M \subset P$ is said to be invariant by Z when the flow of Z leaves M locally invariant, or, in other words, when every integral curve of Z meeting M is contained in M at least for some time (if M is not closed then this integral curve may eventually leave it). These conditions are equivalent to saying that Z is

tangent to M . The preceding definition is applicable to regular submanifolds but also to immersed submanifolds.

A particular instance of invariant submanifolds is provided by constants of the motion. In its most elementary form a constant of the motion for Z is a function $f : P \rightarrow \mathbb{R}$ such that, along every integral curve γ of Z , the function $f \circ \gamma$ is constant. This is equivalent to saying that the Lie derivative of f with respect to Z is zero, $\mathcal{L}_Z f = 0$. In the same way one can consider a vector-valued constant of the motion $F : P \rightarrow \mathbb{R}^n$, whose components are scalar constants of the motion, or, more generally, a manifold-valued map $F : P \rightarrow N$ such that for every integral curve γ the map $F \circ \gamma$ is constant. If $c \in N$, then the closed subset $F^{-1}(c) \subset P$ is clearly invariant by Z . So, those of the sets $F^{-1}(c)$ that are not empty constitute a partition of P . In some cases we can ensure that they are also submanifolds, for instance when F is a submersion. In this case, constants of the motion provide a whole family of invariant submanifolds.

Of course, not all invariant submanifolds are levels sets of constants of the motion. A very simple example is given by the planar system $\dot{x} = -y + x(1 - x^2 - y^2)$, $\dot{y} = x + y(1 - x^2 - y^2)$, that reads in polar coordinates $\dot{r} = r(1 - r^2)$, $\dot{\phi} = 1$; it has an equilibrium point (the origin), a limit cycle ($r = 1$), and no nontrivial *global* constants of the motion. More interesting examples are provided by Liénard's equation and the particular case given by van der Pol's equation. For instance, the system $\dot{x} = -y + x \sin(x^2 + y^2)$, $\dot{y} = x + y \sin(x^2 + y^2)$, has a countable number of limit cycles.

2.2. A general framework for the Hamilton–Jacobi theory: Slicing vector fields

One of the distinctive facts of the Hamilton–Jacobi equation is that it allows to describe the dynamics given by the Hamilton equation on the cotangent bundle in terms of a family of first-order dynamics on the configuration space (as for instance in [1, Theorem 5.2.4]). According to this general principle, to describe the dynamics Z on P in terms of other dynamics on lower dimensional manifolds, we consider another manifold M , a vector field X on M , and a map $\alpha : M \rightarrow P$. The following diagram captures the situation:

$$\begin{array}{ccc} TM & \xrightarrow{T\alpha} & TP \\ X \left(\downarrow \right. & & \left. \downarrow \right) Z \\ M & \xrightarrow{\alpha} & P \end{array}$$

What can be said about the relation between X , α and Z ? The following results are well-known.

Proposition 1. *Given the preceding data, the following properties are equivalent:*

- (1) *For every integral curve ξ of X , $\zeta = \alpha \circ \xi$ is an integral curve of Z .*

(2) X and Z are α -related ($X \sim_{\alpha} Z$), that is to say,

$$T\alpha \circ X = Z \circ \alpha. \quad (1)$$

Suppose moreover that α is an injective immersion, thus inducing a diffeomorphism $\alpha_{\circ} : M \rightarrow \alpha(M)$ of M with an immersed submanifold $\alpha(M) \subset P$. Then the preceding properties are also equivalent to:

(3) Z is tangent to $\alpha(M)$, and, if Z_{\circ} is the restriction of Z to $\alpha(M)$, X is given by the pullback $X = \alpha_{\circ}^*(Z_{\circ})$.

In this case, the map $\xi \mapsto \alpha \circ \xi$ is a bijection between integral curves of X and integral curves of Z passing through $\alpha(M)$.

When these conditions hold, we can regard X as a “partial dynamics”, or a “slice” of the dynamics given by Z . Eventually, if we knew enough of these slices, we could recover the whole dynamics of Z .

Definition. Given a dynamical system (P, Z) , we will call a *slicing* of it a triple (M, α, X) satisfying the *slicing equation* (1).

When α is an immersion the vector field X , if it exists, is uniquely determined by α and Z ; so, in this case, we can speak of (M, α) being a solution of the slicing equation for (P, Z) . This hypothesis will hold in many applications, in particular for the sections α of a bundle $P \rightarrow M$ (as a matter of fact, they are embeddings).

As we will see later on in this paper, Eq. (1) may be thought of as a generalization of the Hamilton–Jacobi equation. One of our main purposes is to identify the precise conditions that take us from the slicing equation to the Hamilton–Jacobi equation.

Coordinate expression. Let us express Eq. (1) in coordinates. Consider coordinates (x^i) in M , (z^k) in P , and use them to express the map $\alpha(x) = (a^k(x))$ and the vector fields $X = X^i \partial / \partial x^i$, and $Z = Z^k \partial / \partial z^k$. Then the difference $T\alpha \circ X - Z \circ \alpha$ reads

$$(x^i) \mapsto \left(a^k(x), \frac{\partial a^k}{\partial x^i} X^i - Z^k(\alpha(x)) \right),$$

and so (M, α, X) is a solution of the slicing equation if and only if $\frac{\partial a^k}{\partial x^i} X^i(x) = Z^k(\alpha(x))$.

2.3. Gauge freedom of the solutions

The notion of a slicing of Z has a certain “gauge freedom”, in the sense that with a given solution (M, α, X) there exist many associated solutions that are *equivalent* to it: if $\varphi : M' \rightarrow M$ is a diffeomorphism then $(M', \alpha \circ \varphi, \varphi^*(X))$ is also a solution of the slicing equation. There are two situations where this freedom can be easily removed.

One, to be studied later on, occurs when P is assumed to be fibered over a manifold and one only deals with maps α that are sections of this projection.

The other one is provided by invariant submanifolds of P . Indeed, this is an immediate consequence of Proposition 1.

Corollary 1. *Let $P_\circ \subset P$ be a regular submanifold. The canonical inclusion $j : P_\circ \hookrightarrow P$ is a solution of the slicing equation if and only if Z is tangent to P_\circ .*

Every other solution given by an embedding α with $\alpha(M) = P_\circ$ is equivalent to it.

3. Constants of the Motion and Complete Solutions

3.1. Constants of the motion

We still deal with our dynamical system (P, Z) . A (generalized) *constant of the motion* of it is a map $F : P \rightarrow N$ into another manifold N satisfying the following property: for any integral curve $\zeta : I \rightarrow P$ of Z , $F \circ \zeta$ is constant.

Example. We consider the isotropic harmonic oscillator with two degrees of freedom (with phase space \mathbb{R}^4),

$$\dot{x} = -y,$$

$$\dot{y} = x.$$

All its integral curves are a foliation of $\mathbb{R}^4 - \{0\} \cong \mathbb{S}^3 \times \mathbb{R}^+$ onto $\mathbb{R}^3 - \{0\} \cong \mathbb{S}^2 \times \mathbb{R}^+$ and the projection $\mathbb{R}^4 - \{0\} \rightarrow \mathbb{R}^3 - \{0\}$ (Kustaanheimo–Stiefel map), or $\mathbb{S}^3 \rightarrow \mathbb{S}^2$, is a constant of the motion.

We have several characterizations of this property:

Proposition 2. *The following properties are equivalent:*

- (1) F is a (manifold-valued) constant of the motion.
- (2) Each integral curve η of Z is contained in a level set $F^{-1}(c)$ of F .
- (3) Z is F -related with the zero vector field of N : $(Z \sim_F 0)$.

Suppose moreover that F is a submersion (thus $\text{Ker } TF \subset TP$ is an integrable tangent subbundle whose associated foliation has as leaves the level sets $F^{-1}(c)$, which are closed submanifolds of P). Then the preceding properties are also equivalent to:

- (4) Z takes its values in $\text{Ker } TF$.
- (5) Z is tangent to every level set $F^{-1}(c)$.

The following diagram summarizes the situation:

$$\begin{array}{ccccc} & & TP & \xrightarrow{TF} & TN \\ & Z \updownarrow & & & \downarrow 0 \\ I & \xrightarrow{\eta} & P & \xrightarrow{F} & N \end{array}$$

The tangency of Z to a certain submanifold shows up in Propositions 1 and 2. This comparison suggests that a constant of the motion is related to a whole family of solutions of the slicing equation, as we are going to show.

3.2. Complete solutions

A single solution $\alpha : M \rightarrow P$, $X : M \rightarrow TM$, of the slicing equation allows to describe the integral curves of Z contained in $\alpha(M) \subset P$. To describe *all* of its integral curves we need a *complete* solution. This can be defined as a family of solutions indexed by some parameter space N .

Definition. Given a dynamical system (P, Z) , a *complete slicing* of it is given by

- a map $\bar{\alpha} : M \times N \rightarrow P$ and
- a vector field $\bar{X} : M \times N \rightarrow TM$ along the projection $M \times N \rightarrow M$

(that is, smooth families of maps $\alpha_c \equiv \bar{\alpha}(\cdot, c) : M \rightarrow P$ and vector fields $X_c \equiv \bar{X}(\cdot, c) : M \rightarrow TM$, both indexed by the points $c \in N$) such that:

- $\bar{\alpha}$ is surjective (or at least its image is an open dense subset), and
- for each $c \in N$, the map $\alpha_c : M \rightarrow P$ and the vector field $X_c : M \rightarrow TM$ constitute a slicing of Z :

$$\begin{array}{ccc} TM \times N & \xrightarrow{T_1 \bar{\alpha}} & TP \\ \bar{X} \left(\begin{array}{c} \downarrow \\ \downarrow \end{array} \right) & & \left(\begin{array}{c} \downarrow \\ \downarrow \end{array} \right) Z \\ M \times N & \xrightarrow{\bar{\alpha}} & P \end{array}$$

Since (almost) every $z \in P$ is the image by $\bar{\alpha}$ of a point $(x, c) \in M \times N$, the integral curve of Z through z can be described as the integral curve of X_c through x by means of the map α_c .

When each α_c is an immersion (for instance, when $\bar{\alpha}$ is a diffeomorphism) the vector fields X_c are determined by the α_c , so in this case we do not need to specify \bar{X} to define the complete solution.

Example. The simplest example of a solution of the slicing equation for a vector field Z is just given by its integral curves $\alpha : I \rightarrow P$. Indeed, consider the following diagram:

$$\begin{array}{ccc} TI & \xrightarrow{T\alpha} & TP \\ \frac{d}{dt} \left(\begin{array}{c} \downarrow \\ \downarrow \end{array} \right) & \nearrow \alpha' & \left(\begin{array}{c} \downarrow \\ \downarrow \end{array} \right) Z \\ I & \xrightarrow{\alpha} & P \end{array}$$

The commutativity of its upper triangle is the definition of the velocity α' , whereas the commutativity of the lower one is the assertion that α being an integral curve

of Z . When this holds, $\frac{d}{dt} \underset{\alpha}{\sim} Z$, which means that α is a solution of the slicing equation for Z .

Let $z \in P$ be a *noncritical point* of Z . Then one can build a *local* complete slicing around z . To this end, consider a hypersurface $N \subset P$ containing z , and such that $Z(z)$ is transversal to N . Then the restriction of the flow F of Z to a smaller product $I_o \times N_o$ gives a diffeomorphism $F_o : I_o \times N_o \rightarrow P_o$ with an open neighborhood P_o of z , such that $\frac{\partial}{\partial t} \underset{F_o}{\sim} Z$. So, F_o with $\frac{\partial}{\partial t}$ is a complete slicing for Z restricted to P_o . Indeed, this is the usual procedure to prove the straightening theorem for vector fields.

3.3. Local existence of complete slicings

The preceding example can be extended to prove a general existence theorem for complete slicings. Indeed, we are going to prove that, under some regularity conditions, *any* given slicing can be *locally* embedded in a regular local complete slicing.

Theorem 1. *Let (P, Z) be a dynamical system, and $z_o \in P$ a noncritical point of Z . Let (M, α, X) be a solution of the slicing equation for Z , with $z_o = \alpha(x_o)$, and such that α is an immersion at x_o .*

There exist an open neighborhood M_o of x_o , an open neighborhood N_o of 0 in \mathbb{R}^n (where $n = \dim P - \dim M$), and a diffeomorphism $\bar{\alpha} : M_o \times N_o \rightarrow P_o$ with an open neighborhood P_o of z_o , such that

- $\bar{\alpha}$ is a complete slicing for $Z|_{P_o}$, and
- $\bar{\alpha}(\cdot, 0) = \alpha|_{M_o}$.

Proof. Since the result is a local one, and every immersion is locally an embedding, the gauge freedom of the solutions of the slicing equation allows us to suppose that M is a regular submanifold of P and that α is the inclusion. The hypothesis is that Z is tangent to M .

The proof of the straightening theorem for vector fields can be adapted to construct coordinates $(z_1, \dots, z_m, \dots, z_p)$ around z_o such that M is locally described by $z_{m+1} = \dots = z_p = 0$, and that $Z = \partial/\partial z_1$. Then, in a small product $M_o \times N_o$, define $\bar{\alpha}(x; s_1, \dots, s_n) = (z_1(x), \dots, z_m(x), s_1, \dots, s_n)$, where the right-hand side is expressed in terms of these coordinates. In a small neighborhood of $(x_o, 0)$ this is a diffeomorphism, and for every $s \in N_o$ the vector field Z is tangent to the submanifold $\alpha_s(M_o)$. Therefore $\bar{\alpha}$ is a complete slicing of Z . \square

3.4. Relation between complete slicings, constants of the motion and connections

Now we are going to see that, under some regularity hypotheses, there is a close relationship between complete slicings and constants of the motion.

Theorem 2. Let (P, Z) be a dynamical system, and $\bar{\alpha} : M \times N \rightarrow P$ a diffeomorphism. Then $\bar{\alpha}$ is a complete slicing for Z if and only if $F = \text{pr}_2 \circ \bar{\alpha}^{-1} : P \rightarrow N$ is a constant of the motion for Z :

$$\begin{array}{ccc} P & \xrightarrow{F} & N \\ \bar{\alpha} \uparrow & \nearrow \text{pr}_2 & \\ M \times N & & \end{array}$$

Proof. If $\bar{\alpha}$ is a complete slicing, for each $c \in N$, $\bar{\alpha}$ restricts to a map $M \times \{c\} \rightarrow \alpha_c(M)$ which is a diffeomorphism, and all the integral curves of Z in $\alpha_c(M)$ correspond to a common value of c . This means the map $F = \text{pr}_2 \circ \bar{\alpha}^{-1} : P \rightarrow N$ is a constant of the motion.

Conversely, from $F = \text{pr}_2 \circ \bar{\alpha}^{-1}$ we have that, for every c , $F(\bar{\alpha}(x, c)) = c$, or $\alpha_c(M) \subset F^{-1}(c)$. Both submanifolds have the same dimension, and, since F is a constant of the motion, Z is tangent to $F^{-1}(c)$; therefore Z is tangent to $\alpha_c(M)$, which proves that the α_c are solutions to slicing equation for Z . \square

This result shows that there is a *bijection* between complete slicings and constants of the motion, but these are being assumed to satisfy a very strong regularity condition, which essentially requires that all the level sets $F^{-1}(c)$ are diffeomorphic to a common manifold M , in such a way that gluing the collection of diffeomorphisms $M \rightarrow F^{-1}(c)$ yields a diffeomorphism $\bar{\alpha} : M \times N \rightarrow P$. Of course, these conditions are very restrictive, but in practice they may hold in a generic way. We will see this in some examples.

Example. Consider the manifold \mathbb{R}^2 with the radial vector field $Z = z_1 \partial / \partial z_1 + z_2 \partial / \partial z_2$, whose integral curves are the equilibrium at the origin and the paths $\zeta(t) = e^t(a_1, a_2)$, $(a_1, a_2) \neq (0, 0)$, running along the half-lines from the origin.

To illustrate the preceding theorem we have to exclude the origin: $P = \mathbb{R}^2 - \{0\}$. The map $F : P \rightarrow N = \mathbb{S}^1$ given by $F(z) = z / \|z\|$ is clearly a constant of the motion for Z . Its level sets $F^{-1}(u)$ (for $u \in \mathbb{S}^1$) are diffeomorphic to the real line $M = \mathbb{R}$; for instance, by $\alpha_u : \mathbb{R} \rightarrow P$, $\alpha_u(x) = e^x u$. All together yield a diffeomorphism $\bar{\alpha} : \mathbb{R} \times \mathbb{S}^1 \rightarrow \mathbb{R}^2 - \{0\} : \bar{\alpha}(x, u) = e^x u$. This is a complete solution of the slicing equation for Z . The corresponding vector fields on M are $X_u = \partial / \partial x$.

The relationship between slicings and constants of the motion is lost when we do not consider complete slicings. A solution of the slicing equation does not need to preserve any given constant of the motion, and the preservation of a constant of the motion does not guarantee that a map is a slicing of the dynamics. The simplest way to show all this is by an example.

Example. We consider the manifold $P = \mathbb{R}^3$, with coordinates (x, y, z) , and the simple dynamics given by the vector field $Z = \partial / \partial x$. The function $F = z$ is obviously a constant of the motion with values in \mathbb{R} .

The map $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by $\alpha(u, v) = (u, v, 0)$ satisfies $F \circ \alpha = 0$, constant. On the other hand, $\bar{\alpha}(u, v) = (u, 0, v)$ satisfies $(F \circ \bar{\alpha})(u, v) = v$, not constant. Both α and $\bar{\alpha}$ are solutions of the slicing equation for (P, Z) , since Z is tangent both to the planes $\alpha(\mathbb{R}^2)$ and $\bar{\alpha}(\mathbb{R}^2)$.

Now consider $\beta : \mathbb{R} \rightarrow \mathbb{R}^3$ given by $\beta(v) = (0, v, 0)$. Obviously $F \circ \beta = 0$ but β is not a solution of the slicing equation since Z is not tangent to the line $\beta(\mathbb{R})$.

3.5. Invariant foliations

The notion of complete solution is close to that of invariant foliation. Roughly speaking, a foliation of P consists in describing it as the disjoint union of immersed submanifolds. This defines an integrable tangent distribution on P , and conversely. The leaves of the foliation are solutions of the slicing equation for Z if and only if Z is tangent to the foliation (or, in other words, if the foliation is invariant by the flow of Z). Equivalently, if and only if Z is a section of the associated tangent distribution.

So, if $\bar{\alpha} : M \times N \rightarrow P$ is a complete slicing, bijective, and with every partial map α_c an immersion, then the submanifolds $\alpha_c(M)$ are a foliation of P invariant by Z . However, not every invariant foliation can be defined by a *global* diffeomorphism in this way.

Example. Consider the “irrational linear flow” on the two-dimensional torus \mathbb{T}^2 : $\dot{x} = 1, \dot{y} = ry$, with r an irrational number. Its integral curves are dense immersions $\mathbb{R} \rightarrow \mathbb{T}^2$. These immersed submanifolds constitute a foliation of the torus invariant by the flow. However, there is no diffeomorphism $\mathbb{R} \times N \rightarrow \mathbb{T}^2$, as well as no nontrivial constants of the motion.

In the usual Hamilton–Jacobi theory a family of vector fields is usually determined by solving an associated partial differential equation of first-order. This requires the use of a skew-symmetric $(0, 2)$ -tensor field which relates a vector field, say Z , with a 1-form. The skew-symmetry ensures that the contraction of Z with the corresponding 1-form identically vanishes.

4. Slicing of Hamiltonian Systems

In the standard Hamilton–Jacobi theory the skew-symmetric $(0, 2)$ -tensor is assumed to be the natural symplectic structure of the cotangent bundle. The classical Hamilton–Jacobi theory makes an essential use of a symplectic structure. In view of this, we still consider the most general slicing problem but now for a Hamiltonian system. Thus P is endowed with a symplectic form ω , which defines a vector bundle isomorphism $\hat{\omega} : TP \rightarrow T^*P$; and $Z = Z_H$ is the Hamiltonian vector field of a Hamiltonian function $H : P \rightarrow \mathbb{R}$: $Z = \hat{\omega}^{-1} \circ dH$.

Lemma 1. *Consider a Hamiltonian dynamical system (P, ω, H) and $Z = Z_H$ its Hamiltonian dynamical vector field. Let $\alpha : M \rightarrow P$ be a map, and X an arbitrary*

vector field on M . We have the following relations:

$$\begin{aligned} {}^t(\mathrm{T}\alpha) \circ \widehat{\omega} \circ \mathrm{T}\alpha \circ X &= i_X \alpha^*(\omega), \\ {}^t(\mathrm{T}\alpha) \circ \widehat{\omega} \circ Z \circ \alpha &= \mathrm{d}\alpha^*(H), \end{aligned}$$

where all the vector bundle sections and maps are understood to be over the base space M .

These relations are expressed in the following diagram (we insist that, since we have to work with the transpose morphism ${}^t(\mathrm{T}\alpha)$, all the involved vector bundles are considered over the base space M):

$$\begin{array}{ccccccc} \mathrm{T}M & \xrightarrow{\mathrm{T}\alpha} & M \times_{\alpha} \mathrm{T}P & \xrightarrow{\widehat{\omega}} & M \times_{\alpha} \mathrm{T}^*P & \xrightarrow{{}^t(\mathrm{T}\alpha)} & \mathrm{T}^*M \\ \downarrow & \nearrow \mathrm{T}\alpha \circ X & \nearrow Z \circ \alpha & \nearrow i_X \alpha^*(\omega) & \nearrow \mathrm{d}\alpha^*(H) & & \\ M & & & & & & \end{array}$$

Proof. The map $\mathrm{T}\alpha \circ X$ is a vector field along α , $\widehat{\omega} \circ \mathrm{T}\alpha \circ X$ is a differential 1-form along α , and finally its composition with the transpose morphism ${}^t(\mathrm{T}\alpha)$ (along M), ${}^t(\mathrm{T}\alpha) \circ \widehat{\omega} \circ \mathrm{T}\alpha \circ X$, is the differential 1-form on M $i_X \alpha^*(\omega)$, since ${}^t(\mathrm{T}\alpha) \circ \widehat{\omega} \circ \mathrm{T}\alpha = \widehat{\alpha^*(\omega)}$.

On the other hand, since Z is the Hamiltonian vector field of H , $\widehat{\omega} \circ Z \circ \alpha = \mathrm{d}H \circ \alpha$, a differential 1-form along α , and its composition with the transpose morphism ${}^t(\mathrm{T}\alpha)$ is just the pull-back by α of $\mathrm{d}H$, ${}^t(\mathrm{T}\alpha) \circ \widehat{\omega} \circ Z \circ \alpha = \alpha^*(\mathrm{d}H) = \mathrm{d}\alpha^*(H)$. \square

Proposition 3. *With the preceding notations, if (M, α, X) is a solution of the slicing equation for (P, Z) , $\mathrm{T}\alpha \circ X - Z \circ \alpha = 0$, then*

$$i_X \alpha^*(\omega) - \mathrm{d}\alpha^*(H) = 0. \quad (2)$$

Proof. From the preceding lemma we have

$${}^t(\mathrm{T}\alpha) \circ \widehat{\omega} \circ (\mathrm{T}\alpha \circ X - Z \circ \alpha) = i_X \alpha^*(\omega) - \mathrm{d}\alpha^*(H). \quad (3)$$

\square

Notice by the way that, if $\alpha^*(\omega)$ were a symplectic form on M , then Eq. (2) would mean that X is the Hamiltonian vector field associated with the Hamiltonian function $\alpha^*(H)$.

Coordinate expressions. It is interesting to reproduce the proof of the previous equations in coordinates. Again we have local charts (x^i) in M , (z^k) in P , and use them to express $\alpha(x) = (a^k(x))$ and $X = X^i \partial / \partial x^i$. The symplectic form reads $\omega = \frac{1}{2} \omega_{k\ell} \mathrm{d}z^k \wedge \mathrm{d}z^{\ell}$, where $\Omega = (\omega_{k\ell})$ is skew-symmetric. The matrix of $\widehat{\omega}$ is Ω^{\top} . And the Hamiltonian vector field $Z = Z_H = \frac{\partial H}{\partial z^{\ell}} \omega^{\ell k} \frac{\partial}{\partial z^k}$, where $(\omega^{k\ell}) = \Omega^{-1}$.

Then $T\alpha \circ X - Z \circ \alpha$ in coordinates reads

$$(x^i) \mapsto \left(a^k(x); \frac{\partial a^k}{\partial x^i} X^i - \frac{\partial H}{\partial z^\ell}(\alpha(x)) \omega^{\ell k}(\alpha(x)) \right).$$

And $\alpha^*(\omega) = \frac{1}{2} \omega_{k\ell}(\alpha(x)) \frac{\partial a^k}{\partial x^i} \frac{\partial a^\ell}{\partial x^j} dx^i \wedge dx^j$, $i_X \alpha^*(\omega) = X^i \omega_{k\ell}(\alpha(x)) \frac{\partial a^k}{\partial x^i} \frac{\partial a^\ell}{\partial x^j} dx^j$, $d\alpha^*(H) = \frac{\partial H}{\partial z^k}(\alpha(x)) \frac{\partial a^k}{\partial x^j} dx^j$, so that $i_X \alpha^*(\omega) - d\alpha^*(H)$ reads

$$\left(X^i \omega_{k\ell}(\alpha(x)) \frac{\partial a^k}{\partial x^i} \frac{\partial a^\ell}{\partial x^j} - \frac{\partial H}{\partial z^k}(\alpha(x)) \frac{\partial a^k}{\partial x^j} \right) dx^j.$$

We see that multiplying the local expression of $T\alpha \circ X - Z \circ \alpha$ by $(\omega_{k\ell})$ and then by $(\partial a^\ell / \partial x^j)$ we obtain the local expression of $i_X \alpha^*(\omega) - d\alpha^*(H)$.

In general the morphism ${}^t(T\alpha)$ is not bijective, therefore the implication in the previous proposition cannot be inverted. This is easily seen in an example.

Example. Consider a Hamiltonian system (P, ω, H) . Let $\alpha : I \rightarrow P$ be any path that is *not* a solution of the Hamilton's equation, but such that $H \circ \alpha = \text{const}$, and consider the vector field $X = \frac{d}{dt}$ on $I \subset \mathbb{R}$. Then $i_X \alpha^*(\omega) - d\alpha^*(H)$ vanishes trivially, whereas, of course, $T\alpha \circ X - Z \circ \alpha = \alpha' - Z \circ \alpha \neq 0$.

The preceding proposition gives a link between the slicing problem and the usual formulation of the Hamilton–Jacobi equation, $d\alpha^*(H) = 0$. However, we need to revert the direct implication, and this can be done in some cases, as it was already shown in our paper [11]. There are at least two ways for doing this, according to whether we have an isotropy condition, as below, or a fibered structure, as in the next section.

4.1. Isotropic and Lagrangian embeddings

In this subsection we are going to study slicings satisfying a geometric property with respect to the symplectic form. First we need to recall that a submanifold $M \subset P$ is called isotropic, coisotropic or Lagrangian [56] when all the tangent spaces at each point are, which means:

- isotropic: $T_z M \subset (T_z M)^\perp$;
- coisotropic: $(T_z M)^\perp \subset T_z M$;
- Lagrangian: isotropic and coisotropic: $T_z M = (T_z M)^\perp$.

(Here the orthogonality is taken with respect to the symplectic form.)

An important type of solutions $\alpha : M \rightarrow P$ of the slicing equation for Z satisfies the condition

$$\alpha^*(\omega) = 0. \tag{4}$$

When α has constant rank this condition means that, locally, the image $\alpha(M) \subset P$ is an *isotropic submanifold*. When α is an immersion this requires that $\dim M \leq \frac{1}{2} \dim P$. Of course, in this case the preceding proposition takes a simpler form: a

solution of the slicing problem satisfies $d\alpha^*(H) = 0$, whereas its converse is false, as is also shown by the same preceding example.

To go further, we need a couple of lemmas. The notation $F^\circ \subset E^*$ denotes the *annihilator* of a vector subspace $F \subset E$.

Lemma 2. *Suppose that α is an embedding, so that $P_0 = \alpha(M) \subset P$ is a submanifold. Then:*

- $\alpha^*(\omega) = 0$ if and only if $\widehat{\omega}(TP_0) \subset (TP_0)^\circ$, i.e. $P_0 \subset P$ is an isotropic submanifold.
- $\widehat{\omega}(TP_0) = (TP_0)^\circ$ if and only if $\widehat{\omega}(TP_0) \subset (TP_0)^\circ$ and $\dim P = 2 \dim M$, i.e. $P_0 \subset P$ is a Lagrangian submanifold.

Proof. The first statement is a consequence of the fact that $\alpha^*(\omega)$ is essentially the restriction of ω to tangent vectors to $\alpha(M)$; the second one is a matter of dimension counting: $m = p - m$. \square

Lemma 3. *If α is an embedding with $\alpha(M) = P_0$ then $\text{Ker } {}^t(T\alpha) = (TP_0)^\circ$.*

Proof. Basic linear algebra applied to $T_x\alpha$ for every $x \in M$. \square

Consider the following diagram, which contains all of these objects:

$$\begin{array}{ccccc}
 M \times_\alpha TP_0 & & (M \times_\alpha TP_0)^\circ & & \\
 \downarrow & & \downarrow & & \\
 M \times_\alpha TP & \xrightarrow{\widehat{\omega}} & M \times_\alpha T^*P & \xrightarrow{{}^t(T\alpha)} & T^*M \\
 \uparrow Z \circ \alpha & \nearrow dH \circ \alpha & & \nearrow \alpha^*(dH) = d\alpha^*(H) & \\
 M & & & &
 \end{array}$$

Theorem 3. *Let (P, ω, H) be a symplectic Hamiltonian system, with Hamiltonian vector field Z , and let $\alpha : M \rightarrow P$ be an embedding.*

If α is a solution of the slicing equation (1) (that is, Z is tangent to $\alpha(M)$) and satisfies the isotropy condition ($\alpha^(\omega) = 0$) then α satisfies*

$$d\alpha^*(H) = 0. \quad (5)$$

Conversely, if α satisfies this equation and the Lagrangianity condition ($\alpha^(\omega) = 0$ and $\dim P = 2 \dim M$) then it is a solution of the slicing equation.*

Proof. We have already proved the direct implication.

Conversely, if $\alpha^*(dH)$ is zero then $dH \circ \alpha$ takes its values in the kernel, which is $(M \times_\alpha TP_0)^\circ$. When the Lagrangianity condition $\widehat{\omega}(TP_0) = (TP_0)^\circ$ holds we

conclude that $Z \circ \alpha$ is a section of TP_0 , or, in other words, that Z is tangent to P_0 , which is one of the ways of saying that α is slicing of Z . \square

So for Lagrangian embeddings to solve the slicing equation is equivalent to solving Eq. (5). We call these solutions *Lagrangian slicings* of Z .

4.2. Constants of the motion and involutivity

In the preceding section we have observed the close relationship between complete slicings and constants of the motion. So, consider a submersion $F : P \rightarrow \mathbb{R}^n$, with level sets $P_c \equiv F^{-1}(c)$.

Lemma 4. *The functions F^i are in involution, $\{F^i, F^j\} = 0$, if and only if all the level sets P_c are coisotropic submanifolds of P .*

When $\dim P = 2n$ this means that the P_c are Lagrangian submanifolds.

Proof. The proof is easy, see [37, p. 101]. \square

So, a complete slicing given by n constants of the motion *in involution* has coisotropic leaves, and if $\dim P = 2n$ then the leaves are Lagrangian submanifolds, and conversely.

Remark. As all our preliminary analysis has been made without the help of a $(0, 2)$ -tensor field, it is clear that when the vector field Z allows for alternative invariant skew symmetric $(0, 2)$ -tensor fields, it is possible to consider alternative cotangent bundle structures on P and therefore different projections.

Example. We can consider the isotropic harmonic oscillator and if we write, in coordinates $(x, p) \in \mathbb{R}^2$,

$$x \cos \alpha + P \sin \alpha = q, \quad P \cos \alpha - x \sin \alpha = p,$$

we have that $dq \wedge dp = dx \wedge dP$, $d(p dq) = d(P dx)$.

The fibering vector fields $p\partial/\partial p$ and $P\partial/\partial P$ are diffeomorphically related but induce alternative cotangent bundle structures on \mathbb{R}^2 [14].

4.3. Local existence of complete Lagrangian slicings

In the preceding section we have proved a local existence theorem for complete slicings. Now we are going to prove a similar result in the Hamiltonian framework, for solutions satisfying the Lagrangianity condition. See also [46, p. 156].

Theorem 4. *Let (P, ω, H) be a symplectic Hamiltonian system, with Hamiltonian vector field Z , and $z_o \in P$ a noncritical point of H . There exists a Lagrangian slicing of Z passing through z_o . Indeed, this slicing is contained in a local complete Lagrangian slicing of Z .*

Proof. By applying the Carathéodory–Jacobi–Lie theorem — see for instance [37, p. 51] — H can be included in a set of local Darboux coordinates $(q^1, \dots, q^n, H = p_1, \dots, p_n)$ centered at z_o . Then $Z = \partial/\partial q^1$ is tangent to the Lagrangian submanifolds of P defined by $p_1 = c_1, \dots, p_n = c_n$. These submanifolds constitute the complete slicing we sought. \square

4.4. Poisson Hamiltonian systems

The preceding argument can be adapted to the Poisson case. Let P be a manifold endowed with an almost-Poisson tensor field Λ , that is to say, a section of $\Lambda^2 TP$. This defines a vector bundle morphism $\hat{\Lambda} : T^*P \rightarrow TP$ by $\langle \beta, \hat{\Lambda}(\alpha) \rangle = \Lambda(\alpha, \beta)$. The image of this morphism, $C = \text{Im } \hat{\Lambda} \subset TP$, is called the *characteristic tangent distribution* of Λ . If Λ has constant rank then C is a vector subbundle.

We will need a generalization of the concept of Lagrangian submanifold to the Poisson case. A submanifold $P_0 \subset P$ of an almost-Poisson manifold is called *Lagrangian* [48, p. 100] when

$$\hat{\Lambda}((P_0 \times_{P_0} TP_0)^\circ) = TP_0 \cap (P_0 \times_{P_0} C).$$

The almost-Poisson tensor field also defines an almost-Poisson bracket $\{f, g\} = \Lambda(df, dg)$, which is skew-symmetric and a derivation on each of its arguments (it does not necessarily satisfy the Jacobi identity unless the Schouten bracket vanishes, i.e. $[\Lambda, \Lambda] = 0$).

Suppose that we have a Hamiltonian function $H : P \rightarrow \mathbb{R}$, which defines a Hamiltonian vector field $Z = Z_H = \hat{\Lambda} \circ dH$ and the corresponding Hamiltonian dynamics. We want to study the slicing problem for (P, Z) .

As before, we consider the elements in this diagram, but notice that Λ may be degenerate:

$$\begin{array}{ccccc} M \times_\alpha TP_0 & & (M \times_\alpha TP_0)^\circ & & \\ \downarrow & & \downarrow & & \\ M \times_\alpha TP & \xleftarrow{\hat{\Lambda}} & M \times_\alpha T^*P & \xrightarrow{t(T\alpha)} & T^*M \\ & \nwarrow Z \circ \alpha & \uparrow dH \circ \alpha & \nearrow \alpha^*(dH) = d\alpha^*(H) & \\ & & M & & \end{array}$$

Lemma 5. Let E be a finite-dimensional vector space, E^* its dual space, $E_0 \subset E$ a vector subspace, $\lambda : E^* \rightarrow E$ a linear map, $\delta \in E^*$ a covector. Denote by $E_0^\circ \subset E^*$ the annihilator of E_0 and by ${}^t\lambda : E^* \rightarrow E$ the transpose map of λ . Then $\lambda(\delta) \in E_0$ if and only if $\delta \in ({}^t\lambda(E_0^\circ))^\circ$. If moreover λ is symmetric or skew-symmetric (${}^t\lambda = \pm\lambda$), then $\lambda(\delta) \in E_0$ if and only if $\delta \in (\lambda(E_0^\circ))^\circ$.

Theorem 5. Let (P, Λ, H) be an almost-Poisson Hamiltonian system, with Hamiltonian vector field Z . Let $\alpha : M \rightarrow P$ be an embedding with image $\alpha(M) = P_0$.

Then α is a solution of the slicing equation if and only if

$$dH \circ \alpha \text{ is a section of } (\widehat{\Lambda}((M \times_{\alpha} TP_0)^{\circ}))^{\circ}. \quad (6)$$

Suppose that $P_0 \subset P$ is a Lagrangian submanifold. Then α is a solution of the slicing equation if and only if

$$dH \circ \alpha \text{ is a section of } (M \times_{\alpha} TP_0)^{\circ} + (M \times_{\alpha} \text{Ker } \widehat{\Lambda}), \quad (7)$$

that is to say, if and only if

$$\alpha^*(dH) \text{ is a section of } {}^t(T\alpha)(\text{Ker } \widehat{\Lambda}). \quad (8)$$

Proof. The first statement is a consequence of the lemma. As for the second statement, being P_0 Lagrangian means that $\widehat{\Lambda}((P_0 \times_{P_0} TP_0)^{\circ}) = TP_0 \cap (P_0 \times_{P_0} C)$. When restricted this to α and with the annihilator we have $(\widehat{\Lambda}((M \times_{\alpha} TP_0)^{\circ}))^{\circ} = ((M \times_{\alpha} TP_0) \cap (M \times_{\alpha} C))^{\circ} = (M \times_{\alpha} TP_0)^{\circ} + (M \times_{\alpha} C)^{\circ}$, and remember that $C^{\circ} = \text{Ker } {}^t\widehat{\Lambda} = \text{Ker } \widehat{\Lambda}$. \square

The symplectic case is obtained when $C = TP$, or equivalently when $\text{Ker } \widehat{\Lambda} = \{0\}$. Then for the Lagrangian case the last statement in the theorem means that α is a slicing if and only if $\alpha^*(dH) = 0$, as was already given by Theorem 3.

Example. Consider $P = \mathbb{R}^3$ with coordinates (x, y, z) and the Poisson structure given by the Poisson bracket $\{f, g\} = z(\frac{\partial f}{\partial y} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial y})$ — indeed, this is the Lie–Poisson structure constructed from the Heisenberg Lie algebra [48, p. 153]. The Hamiltonian function $H = \frac{1}{2}z(x^2 + y^2)$ defines the Hamiltonian vector field $Z = z^2(-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y})$.

We have two constants of the motion, $x^2 + y^2$ and z . Excluding the z -axis, all their level sets are diffeomorphic to the unit circle; parametrizing the circle with the natural angle, these diffeomorphisms read $\alpha_{r,c}(\phi) = (r \cos \phi, r \sin \phi, c)$. It is easily checked that $\alpha_{r,c}^*(dH) = 0$.

Since for $c \neq 0$ all these level sets are Lagrangian submanifolds, we conclude from the preceding theorem that the $\alpha_{r,c}$ constitutes a complete Lagrangian slicing for Z on the open set given by $z \neq 0$.

In general this situation prevails for Poisson manifolds and we have to consider Casimir functions and constants of the motion in involution. Casimir functions identify “parameters” (like mass, spin, charge, isospin, colored charge) while the constants of the motion identify the decomposition into vector fields on lower dimensional submanifolds

5. Slicing in Fibered Manifolds

In this section we consider a dynamical system (P, Z) , where the manifold P is fibered over another manifold, that is to say, we work in a *fiber bundle* $\pi : P \rightarrow M$.

We consider the slicing problem as before:

$$\begin{array}{ccc} TM & \xrightarrow{T\alpha} & TP \\ X \downarrow & & \downarrow Z \\ M & \xrightarrow{\alpha} & P \\ & \xleftarrow{\pi} & \end{array}$$

but only for *sections* of π , that is to say, for maps $\alpha : M \rightarrow P$ such that $\pi \circ \alpha = \text{Id}_M$. For this problem there is not a “gauge freedom” as mentioned in Sec. 2: the submanifold $\alpha(M) \subset P$ cannot be expressed as the image of any other section.

Since α is an embedding, we know that Eq. (1) determines X . Nevertheless, composing this equation with the tangent map $T\pi$, we can give an explicit formula for X :

$$X = T\pi \circ Z \circ \alpha. \quad (9)$$

So, from now on we assume that X is defined by this equation from α . In this case, Proposition 1 adopts the following form.

Proposition 4. *A section α of $\pi : P \rightarrow M$ is a solution of the slicing equation for (P, Z) if and only if*

$$T\alpha \circ T\pi \circ Z \circ \alpha = Z \circ \alpha; \quad (10)$$

that is to say, if $T\alpha \circ T\pi \circ Z$ agrees with Z on the submanifold $\alpha(M)$.

Lemma 6. *If α is a slicing section, the vector field along α defined as $T\alpha \circ T\pi \circ Z \circ \alpha - Z \circ \alpha$ is π -vertical.*

Remember that the vertical subbundle of TP is $VP = \text{Ker } T\pi$. Its fibers are $V_z P = \text{Ker } T_z \pi \subset T_z P$ and are naturally identified with the tangent spaces to the fibers of π . Application of $T\pi$ to $T\alpha \circ T\pi \circ Z \circ \alpha - Z \circ \alpha$ yields immediately zero since α is a section of π .

5.1. Sections, projectors, and connections

If α is a section of P , let us have a look at the composition $T\alpha \circ T\pi$. At a given point $z = \alpha(x) \in P$, $T_z(\alpha \circ \pi) : T_z P \rightarrow T_z P$ is an endomorphism, and since $T\pi \circ T\alpha$ is the identity, we note that $T_z(\alpha \circ \pi) \circ T_z(\alpha \circ \pi) = T_z(\alpha \circ \pi)$, therefore it is a projector in $T_z P$. Since $T_x \alpha$ is injective, it is clear that

$$\text{Ker } T_z(\alpha \circ \pi) = \text{Ker } T_z \pi = V_z P.$$

Therefore

$$\text{Im } T_z(\alpha \circ \pi) = T_z \alpha(M)$$

is a complementary subspace to $V_z P$.

So we can write, for every $x \in M$, a direct sum decomposition

$$T_{\alpha(x)}P = V_{\alpha(x)}P \oplus T_{\alpha(x)}\alpha(M).$$

This can be written globally in the pull-back vector bundle:

$$M \times_{\alpha} TP = M \times_{\alpha} VP \oplus M \times_{\alpha} T\alpha(M).$$

Now suppose that we have not only a section but a family of nonoverlapping sections covering the whole manifold P ; this can be defined by a diffeomorphism $\bar{\alpha} : M \times N \rightarrow P$, where each $\alpha_c = \bar{\alpha}(\cdot, c)$ is a section of P , but this diffeomorphism could as well be defined on open sets of P . The preceding study can be performed at every point $z \in P$, therefore the family $\bar{\alpha}$ defines a *horizontal subbundle*, that is, a vector subbundle $H \subset TP$ complementary to the vertical subbundle $VP \subset TP$. A horizontal subbundle of TP is also called a (nonlinear) *connection* on the bundle P . This horizontal subbundle is obviously integrable, its integral manifolds being given by the embeddings α_c .

Conversely, if a connection on the bundle $P \rightarrow M$ has integrable horizontal subbundle (which amounts to saying that its curvature vanishes, see [45, p. 90]), then its integral manifolds are locally the images of sections of the bundle.

5.2. Complete solutions and connections

Still working with the diffeomorphism $\bar{\alpha} : M \times N \rightarrow P$, when is it a complete solution of the slicing equation for sections? In addition to defining an integrable horizontal subbundle, Z has to be tangent to it. Therefore, locally, complete solutions of the slicing equation are equivalent to connections on $\pi : P \rightarrow M$, with zero curvature, and invariant by Z .

5.3. The Hamiltonian case on a fibered manifold

Here we consider both a bundle structure and a Hamiltonian structure on P . So, $\pi : P \rightarrow M$ is a fiber bundle and (P, ω) is a symplectic manifold, and $Z = Z_H$ is a Hamiltonian vector field (with Hamiltonian function H). Let α be a section of π , and let us determine if it is a slicing section for Z . We wish to give a kind of converse to Proposition 3, which relates $T\alpha \circ X - Z \circ \alpha$ with $i_X \alpha^*(\omega) - d\alpha^*(H)$ (where X is given by $X = T\pi \circ Z \circ \alpha$):

$$\begin{array}{ccccccc} TM & \xrightarrow{T\alpha} & M \times_{\alpha} TP & \xrightarrow{\widehat{\omega}} & M \times_{\alpha} T^*P & \xrightarrow{t(T\alpha)} & T^*M \\ \downarrow & \nearrow & & & & \nearrow & \\ M & & & & & & \end{array}$$

$T\alpha \circ X - Z \circ \alpha$ $i_X \alpha^*(\omega) - d\alpha^*(H)$

In this diagram $\widehat{\omega}$ is bijective, and, as we have already noted, the problem is that $t(T\alpha)$ is not injective, since $\text{Ker } t(T\alpha) = (M \times_{\alpha} TP_0)^{\circ}$. However, we have also noted that $T\alpha \circ X - Z \circ \alpha$ is π -vertical. Therefore we only need to impose the injectivity

of the restriction of ${}^t(\mathrm{T}\alpha) \circ \widehat{\omega}$ to the subbundle $M \times_{\alpha} \mathrm{VP}$, and this is equivalent to saying that

$$\widehat{\omega}(M \times_{\alpha} \mathrm{VP}) \cap (M \times_{\alpha} \mathrm{TP}_0)^{\circ} = \{0\}.$$

Lemma 7. *With the preceding hypotheses, the following conditions are equivalent:*

- *The fibers of $\pi : P \rightarrow M$ are isotropic submanifolds (with respect to ω).*
- *For every couple of vertical vectors $w_z, w'_z \in \mathrm{V}_z P \subset \mathrm{T}_z P$ one has $\omega(w_z, w'_z) = 0$.*
- $\widehat{\omega}(\mathrm{VP}) \subset (\mathrm{VP})^{\circ}$.

Proof. The equivalence of the first two is due to the fact that the vertical vectors are those that are tangent to the fibers. \square

Corollary 2. *If α is a section of P and the fibers are isotropic then $\widehat{\omega}(M \times_{\alpha} \mathrm{VP}) \cap (M \times_{\alpha} \mathrm{TP}_0)^{\circ} = \{0\}$.*

Proof. The vertical + horizontal decomposition yields $M \times_{\alpha} \mathrm{T}^*P = (M \times_{\alpha} \mathrm{VP})^{\circ} \oplus (M \times_{\alpha} P_0)^{\circ}$. \square

Theorem 6. *Let (P, ω, H) be a Hamiltonian system on a fiber bundle $\pi : P \rightarrow M$. Let $\alpha : M \rightarrow P$ be a section of π , and define its associated vector field $X = \mathrm{T}\pi \circ Z \circ \alpha$. Suppose that the fibers of π are isotropic. Then α is a slicing section if and only if*

$$i_X \alpha^*(\omega) - \mathrm{d}\alpha^*(H) = 0.$$

Proof. As we have just shown, the isotropy condition implies that ${}^t(\mathrm{T}\alpha) \circ \widehat{\omega}$ is injective when applied to vertical vectors. Therefore if $i_X \alpha^*(\omega) - \mathrm{d}\alpha^*(H)$ is zero then $\mathrm{T}\alpha \circ X - Z \circ \alpha$ also is. \square

Coordinate expressions. Let us understand the proof of the theorem on the light of coordinates. We use coordinates (x^i) in M and adapted coordinates (x^i, y^{μ}) in P . The section takes the form $\alpha(x) = (x, a^{\mu}(x))$ and its tangent map is represented by the matrix $\begin{pmatrix} I \\ A \end{pmatrix}$, where A is the Jacobian matrix of the a^{μ} . The symplectic form ω is represented by a skew-symmetric matrix $\Omega = \begin{pmatrix} \Omega_b & N \\ -N^{\top} & \Omega_f \end{pmatrix}$. The matrix of $\widehat{\omega}$ is Ω^{\top} . Then the linear map ${}^t(\mathrm{T}_x \alpha) \circ \widehat{\omega}_z$ is represented by the matrix $(\Omega_b^{\top} + A^{\top} N^{\top} - N + A^{\top} \Omega_f^{\top})$, and its restriction to the vertical subspace by its second block,

$$-N + A^{\top} \Omega_f^{\top}.$$

Now, the fibers are isotropic if and only if $\Omega_f = 0$, and since Ω is nondegenerate N has to have maximal rank and be injective. So, the only vertical vector sent to 0 by this map is 0.

In the preceding section we have already obtained the equation for the Lagrangian slicings. We can combine Theorems 3 and 6 in this way.

Corollary 3. *For a Hamiltonian system (P, ω, H) fibered over M , with isotropic fibers, let $\alpha : M \rightarrow P$ be a section with isotropic image. Then α is a solution of the slicing problem if and only if*

$$d\alpha^*(H) = 0.$$

Proof. The isotropy of the fibers requires $\dim M \geq \dim P/2$ and the isotropy of $\alpha(M)$ requires $\dim M \leq \dim P/2$. Therefore $\dim M = \dim P/2$, which in particular means that $\alpha(M) \subset P$ is a Lagrangian submanifold and then application of Theorem 2 yields the desired result. Otherwise, the isotropy of the image means $\alpha^*(\omega) = 0$ and one can apply Theorem 4 at once. \square

The isotropy of the fibers is necessary to prove this result, as shown by the following example.

Example. Take $P = \mathbb{R}^4$, with coordinates (x, p_x, y, p_y) , with the usual symplectic form $\omega = dx \wedge dp_x + dy \wedge dp_y$, and the Hamiltonian of the isotropic double harmonic oscillator $H = \frac{1}{2}(x^2 + p_x^2 + y^2 + p_y^2)$; its Hamiltonian vector field is $Z = p_x \partial/\partial x - x \partial/\partial p_x + p_y \partial/\partial y - y \partial/\partial p_y$.

Consider the trivial fiber bundle $\pi : P \rightarrow M$ given by $M = \mathbb{R}$, with projection $\pi(x, p_x, y, p_y) = x$. Of course, since M is one-dimensional any section α of π satisfies $\alpha^*(\omega) = 0$.

Then consider the local section $\alpha(x) = (x, x, \sqrt{c^2 - x^2}, \sqrt{c^2 - x^2})$. It satisfies $H \circ \alpha = c^2 = \text{const}$, but one easily checks that it is not a slicing section. The point is that the fibers of π are not isotropic — they cannot be since they are three-dimensional submanifolds of a four-dimensional symplectic manifold.

6. Lagrangian and Hamiltonian Formalisms

In this section we study some features specific to the dynamics on tangent and cotangent bundles, and in particular to Lagrangian and Hamiltonian formalisms.

First, notice that the results of the preceding section apply directly to a canonical Hamiltonian system $(P = T^*Q, \omega, H)$, while T^*Q endowed with its vector bundle structure $\pi : T^*Q \rightarrow Q$ and its canonical symplectic form ω . The dynamical vector field Z is the symplectic gradient Z_H of the Hamiltonian function H .

Then we consider the slicing equation $X \sim_\alpha Z$ for a section α of P , that is to say, a differential 1-form on Q . From (9) we can compute the slicing vector field X , which in this case turns out to be $X = \mathcal{F}H \circ \alpha$, where $\mathcal{F}H : T^*Q \rightarrow TQ$ is the fiber derivative of H .

Now, notice that the fibers of T^*Q , that is to say, the cotangent spaces T_q^*Q , are *isotropic* submanifolds of the cotangent bundle with respect to its canonical symplectic structure. So, we are under the hypotheses of Theorem 6 and its corollary, which give a special form for the slicing equation.

In particular, the classical *Hamilton–Jacobi equation* is nothing but the slicing equation for a *closed* 1-form α . This means that α is locally exact, $\alpha = dW$, and the slicing equation has the well-known form

$$H \circ dW = \text{const.} \quad (11)$$

The same applies to the Lagrangian formulation of mechanics when it is defined by a *regular* Lagrangian function $L : TQ \rightarrow \mathbb{R}$. In this case the fibered manifold is $P = TQ$; now we do not have a canonical symplectic form, but the 2-form ω_L defined from the Lagrangian. The Hamiltonian vector field is the symplectic gradient of the energy E_L [1]. Then all proceeds as in the Hamiltonian case.

Within this framework we recover some of our previous results. In fact Theorem 6 has, as particular cases, Theorems 1 and 2 in our paper [11], corresponding to the Lagrangian and the Hamiltonian formulations, respectively. In the same way, Corollary 3 corresponds to Propositions 3 and 7 of the same paper. There it is also proved (Theorem 3) the equivalence between the Hamilton–Jacobi theories for the Lagrangian and the Hamiltonian dynamics for regular systems. The relationship between constants of the motion and complete slicings (Theorem 2) was also established for these particular cases in [11].

6.1. *Determination of a second-order dynamics from constants of the motion*

Suppose we have a foliation $\{M_c\}$ of a manifold P . A vector field Z on P tangent to the foliation defines a vector field X_c on every leaf M_c of the foliation. Conversely, a vector field X_c on every M_c defines a vector field Z on P (though *a priori* one cannot guarantee it to be continuous).

This is what happens when we have a complete slicing $\{(M, \alpha_c, X_c)\}$ of a dynamics (P, Z) , as discussed in Sec. 3. Now, suppose that the hypotheses of Theorem 2 are satisfied, so that the complete slicing is equivalent to a (manifold-valued) constant of the motion $F : P \rightarrow N$. Then it could seem that the dynamics Z is determined by F . But of course this is not true: the conditions of the theorem assume that Z is *already* given, otherwise the vector fields X_c could not be determined.

However, there is a very special instance where the knowledge of some constants of the motion suffices to determine the dynamics. Recall that a vector field Z defined on the tangent bundle TM of a manifold is said to satisfy the second-order condition when its integral curves are the velocities of their projections to the base space M . It is easily proved that this is equivalent to saying that, besides being a section of the tangent bundle of TM , $\tau_{TM} : T(TM) \rightarrow TM$, Z is also a section of the other

vector bundle structure of $T(TM)$, the one given by $T\tau_M : T(TM) \rightarrow TM$. In brief, this means that $T\tau_M \circ Z = \text{Id}$.

Lemma 8. *Consider a dynamical system (P, Z) where $P \subset TM$ is an open subset projecting over M .*

If (M, α, X) is a slicing of Z by a section α of P , and Z satisfies the second-order condition, then $X = \alpha$.

Conversely, suppose we have a complete slicing (α_c, X_c) of Z by sections α_c of P . If $X_c = \alpha_c$ for every c , then Z satisfies the second-order condition.

Proof. If Z satisfies this condition, then the slicing equation $T\alpha \circ X = Z \circ \alpha$, composed with $T\tau_M$, yields $X = \alpha$.

Conversely, when $X = \alpha$ the slicing equation reads $T\alpha \circ \alpha = Z \circ \alpha$, and composition with $T\tau_M$ yields $\alpha = T\tau_M \circ Z \circ \alpha$. This means that Z satisfies the second-order condition on every point of $\alpha(M)$. \square

Theorem 7. *Let $P \subset TM$ be an open subset projecting onto M . Suppose we have $m = \dim M$ functions $f^\alpha : P \rightarrow \mathbb{R}$ whose fiber derivatives $\mathcal{F}f^\alpha : P \rightarrow T^*M$ are linearly independent at each point.*

Then around any point $v \in P$ there exists a unique local vector field Z , satisfying the second-order condition, and for which the f^α are constants of the motion.

Proof. Put $F = (f^1, \dots, f^m) : P \rightarrow \mathbb{R}^m$. For every $c \in \mathbb{R}^m$ we have a submanifold $P_c = F^{-1}(c) \subset P$. The hypotheses imply that the restriction of the projection to this submanifold, $\tau|_{P_c} : P_c \rightarrow M$, is a diffeomorphism in a neighborhood of any point $v \in P_c$. Let $\alpha_c : M \rightarrow P_c$ be its inverse. This α_c is also a vector field on M , so it defines a vector field $Z|_{P_c}$, and this satisfies the second-order condition by the preceding lemma. All of these together yield Z .

To complete the proof we need to show that Z is smooth, and we will do this by an explicit computation in coordinates. Let $v_o \in P$ be an arbitrary point, and use natural coordinates (q^i, v^i) around it. Write $Z = v^i \frac{\partial}{\partial q^i} + Z^i(q, v) \frac{\partial}{\partial v^i}$. Imposing that the f^α are constants of the motion for Z we obtain

$$\mathcal{L}_Z f^\alpha = \frac{\partial f^\alpha}{\partial q^i} v^i + \frac{\partial f^\alpha}{\partial v^i} Z^i = 0.$$

The linear independence of the fiber derivatives means in coordinates that the matrix $(\frac{\partial f^\alpha}{\partial v^i})$ is invertible. Hence, we determine the last coefficients of Z as

$$Z^i = - \left(\left(\frac{\partial f}{\partial v} \right)^{-1} \right)^i_{\beta} \frac{\partial f^\beta}{\partial q^j} v^j.$$

\square

Remark. It is known that from a vector field X on M one can construct its canonical lift X^T to the tangent bundle. This vector field does not satisfy the second-order condition in the whole TM , but in the points of $X(M)$ it does. Indeed,

in the first part of the preceding proof, what we are defining is $Z|_{X(M)} = X^T|_{X(M)}$, where $\alpha \equiv X$. Since we have a whole family of α 's covering the whole space, the vector field Z constructed in this way satisfies the second-order condition at every point.

Example. We will use the free particle to show that working in the Lagrangian or in the Hamiltonian formalisms is philosophically different.

If we take $P = T^*\mathbb{R}^n$ and the Hamiltonian $H = \frac{1}{2}(p_1^2 + \cdots + p_n^2)$ then the dynamical vector field is $Z = \sum p_i \frac{\partial}{\partial q_i}$, and its constants of the motion are the functions $f(p)$. But notice that these functions are constants of the motion for H and also for any Hamiltonian of the form $H(p)$; the corresponding dynamical vector field is $Z = \sum \partial H / \partial p_i \partial / \partial q_i$.

Now take $P = T\mathbb{R}^n$ and consider the functions $f_i = v_i$. Following the preceding theorem, we can look for a second-order vector field Z having the f_i as constants of the motion. There is a *unique* such a vector field, and it is $Z = \sum v_i \partial / \partial q_i$. By the way, the same vector field would be obtained if one considered, instead of the v_i , any set of m independent functions $f_i(v)$.

Acknowledgments

J. F. Cariñena and E. Martínez acknowledge the financial support of the Ministerio de Economía y Competitividad (Spain) project MTM-2012-33575 and the DGA (Aragon) project DGA E24/1. X. Gràcia, M. C. Muñoz-Lecanda and N. Román-Roy acknowledge the financial support of the Ministerio de Ciencia e Innovación (Spain) projects MTM2014-54855-P and MTM2011-22585.

References

- [1] R. Abraham and J. E. Marsden, *Foundations of Mechanics*, 2nd edn. (Addison-Wesley, Reading, MA, 1978).
- [2] S. Antoci and L. Mihich, The issue of photons in dielectrics: Hamiltonian viewpoint, *Nuovo Cimento B* **122** (2007) 413–424.
- [3] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, NY, 1989).
- [4] P. Balseiro, J. C. Marrero, D. Martín de Diego and E. Padrón, A unified framework for mechanics: Hamilton–Jacobi equation and applications, *Nonlinearity* **23**(8) (2010) 1887–1918.
- [5] M. Barbero Liñán, F. Barbero and D. Martín de Diego, A geometric framework for discrete Hamilton–Jacobi equation, in *Proc. Int. Fall Workshop on Geometry Physics*, AIP Conference Proceedings, Vol. 1460, 2012, pp. 164–168.
- [6] M. Barbero-Liñán, M. de León, D. Martín de Diego, J. C. Marrero and M. C. Muñoz-Lecanda, Kinematic reduction and the Hamilton–Jacobi equation, *J. Geom. Mech.* **4**(3) (2012) 207–237.
- [7] M. Barbero-Liñán, M. de León and D. Martín de Diego, Lagrangian submanifolds and Hamilton–Jacobi equation, *Monatsh. Math.* **171**(3–4) (2013) 269–290.
- [8] L. M. Bates, F. Fassò and N. Sansonetto, The Hamilton–Jacobi equation, integrability, and nonholonomic systems, *J. Geom. Mech.* **6**(4) (2014) 441–449.

- [9] M. Born and E. Wolf, *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light*, 7th edn. (Cambridge University Press, Cambridge, 1999).
- [10] C. M. Campos, M. de León, D. Martín de Diego and M. Vaquero, Hamilton–Jacobi theory in Cauchy data space, preprint (2014), arXiv: 1411.3959 [math-ph].
- [11] J. F. Cariñena, X. Gràcia, G. Marmo, E. Martínez, M. C. Muñoz-Lecanda and N. Román-Roy, Geometric Hamilton–Jacobi theory, *Int. J. Geom. Meth. Mod. Phys.* **3** (2006) 1417–1458.
- [12] J. F. Cariñena, X. Gràcia, G. Marmo, E. Martínez, M. C. Muñoz-Lecanda and N. Román-Roy, Geometric Hamilton–Jacobi theory and the evolution operator, in *Mathematical Physics and Field Theory*, eds. M. Asorey *et al.* (Prensas Universitarias de Zaragoza, Zaragoza, 2009), pp. 177–186.
- [13] J. F. Cariñena, X. Gràcia, G. Marmo, E. Martínez, M. C. Muñoz-Lecanda and N. Román-Roy, Geometric Hamilton–Jacobi theory for nonholonomic dynamical systems, *Int. J. Geom. Meth. Mod. Phys.* **7** (2010) 431–454.
- [14] J. F. Cariñena, A. Ibort, G. Marmo and G. Morandi, *Geometry from Dynamics, Classical and Quantum* (Springer, Berlin, 2015).
- [15] L. Colombo, M. de León, P. D. Prieto-Martínez and N. Román-Roy, Geometric Hamilton–Jacobi theory for higher-order autonomous systems, *J. Phys. A: Math. Theor.* **47** (2014) 235203.
- [16] L. Colombo, M. de León, P. D. Prieto-Martínez and N. Román-Roy, Unified formalism for the generalized k th-order Hamilton–Jacobi problem, *Int. J. Geom. Meth. Mod. Phys.* **11**(9) (2014) 1460037.
- [17] G. C. Constantelos, On the Hamilton–Jacobi theory with derivatives of higher order, *Nuovo Cimento B* (11) **84**(1) (1984) 91–101.
- [18] P. A. M. Dirac, The Hamiltonian form of field dynamics, *Canad. J. Math.* **3** (1951) 1–23.
- [19] G. Esposito, G. Marmo and G. Sudarshan, *From Classical to Quantum Mechanics* (Cambridge University Press, 2004).
- [20] D. Dominici, J. Gomis, G. Longhi and J. M. Pons, Hamilton–Jacobi theory for constrained systems, *J. Math. Phys.* **25** (1984) 2439–2460.
- [21] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, 1951).
- [22] J. Hadamard, Théorie des équations aux dérivées partielles lineaires hyperbolyques et du problème de Cauchy, *Acta Math.* **31** (1908) 333–380.
- [23] J. Hadamard, *Lectures on Cauchy’s Problem in Linear Partial Differential Equations* (Yale University Press, New Haven, 1923).
- [24] D. Iglesias-Ponte, M. de León and D. Martín de Diego, Towards a Hamilton–Jacobi theory for nonholonomic mechanical systems, *J. Phys. A: Math. Theor.* **41** (2008) 015205.
- [25] C. G. J. Jacobi, *Jacobi’s Lectures on Dynamics*, 2nd rev. edn., ed. A. Clebsch (Hindustan Book Agency, New Delhi, 1884/2009).
- [26] J. V. José and E. J. Saletan, *Classical Dynamics: A Contemporary Approach* (Cambridge University Press, 1998).
- [27] O. Krupková and A. Vondra, On some integration methods for connections on fibered manifolds, in *Differential Geometry and Its Applications* (Silesian University, Opava, 1993), pp. 89–101.
- [28] M. Leok and D. Sosa, Dirac structures and Hamilton–Jacobi theory for Lagrangian mechanics on Lie algebroids, *J. Geom. Mech.* **4**(4) (2012) 421–442.
- [29] M. Leok, T. Ohsawa and D. Sosa, Hamilton–Jacobi theory for degenerate Lagrangian systems with holonomic and nonholonomic constraints, *J. Math. Phys.* **53**(7) (2012) 072905.

- [30] M. de León, J. C. Marrero and D. Martín de Diego, Linear almost Poisson structures and Hamilton–Jacobi equation: Applications to nonholonomic mechanics, *J. Geom. Mech.* **2**(2) (2010) 159–198.
- [31] M. de León, D. Martín de Diego, J. C. Marrero, M. Salgado and S. Vilariño, Hamilton–Jacobi theory in k -symplectic field theories, *Int. J. Geom. Meth. Mod. Phys.* **7**(8) (2010) 1491–1507.
- [32] M. de León, D. Martín de Diego and M. Vaquero, A Hamilton–Jacobi theory for singular Lagrangian systems in the Skinner and Rusk setting, *Int. J. Geom. Meth. Mod. Phys.* **9**(8) (2012) 1250074.
- [33] M. de León, D. Martín de Diego and M. Vaquero, On the Hamilton–Jacobi theory for singular Lagrangian systems, *J. Math. Phys.* **54**(3) (2013) 032902.
- [34] M. de León, D. Martín de Diego and M. Vaquero, A Hamilton–Jacobi theory on Poisson manifolds, *J. Geom. Mech.* **6**(1) (2014) 121–140.
- [35] M. de León, P. D. Prieto-Martínez, N. Román-Roy and S. Vilariño, Hamilton–Jacobi theory in multisymplectic classical field theories, preprint (2015), arXiv: 1504.02020 [math-ph].
- [36] M. de León and S. Vilariño, Hamilton–Jacobi theory in k -cosymplectic field theories, *Int. J. Geom. Meth. Mod. Phys.* **11**(1) (2014) 1450007.
- [37] P. Libermann and C. M. Marle, *Symplectic Geometry and Analytical Mechanics* (D. Reidel, Dordrecht, 1987).
- [38] G. Marmo, G. Morandi and N. Mukunda, A geometrical approach to the Hamilton–Jacobi form of dynamics and its generalizations, *Riv. Nuovo Cimento* **13** (1990) 1–74.
- [39] G. Marmo, G. Morandi and N. Mukunda, The Hamilton–Jacobi theory and the analogy between classical and quantum mechanics, *J. Geom. Mech.* **1**(3) (2009) 317–355.
- [40] G. Marmo, E. J. Saletan, A. Simoni and B. Vitale, *Dynamical Systems: A Differential Geometric Approach to Symmetry and Reduction* (Wiley, 1985).
- [41] T. Ohsawa and A. M. Bloch, Nonholonomic Hamilton–Jacobi equation and integrability, *J. Geom. Mech.* **1**(4) (2011) 461–481.
- [42] T. Ohsawa, A. M. Bloch and M. Leok, Nonholonomic Hamilton–Jacobi theory via Chaplygin Hamiltonization, *J. Geom. Phys.* **61**(8) (2011) 1263–1291.
- [43] T. Ohsawa, A. M. Bloch and M. Leok, Discrete Hamilton–Jacobi theory, *SIAM J. Control Optim.* **49**(4) (2011) 1829–1856.
- [44] E. J. Saletan and A. H. Cromer, *Theoretical Mechanics* (Wiley, New York, 1971).
- [45] D. J. Saunders, *The Geometry of Jet Bundles* (Cambridge University Press, Cambridge, 1989).
- [46] F. Scheck, *Mechanics: From Newton’s Laws to Deterministic Chaos*, 5th edn. (Springer, Berlin, 2010).
- [47] J. L. Synge, *Geometrical Mechanics and de Broglie Waves* (Cambridge University Press, Cambridge, 1954).
- [48] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds* (Birkhäuser, Basel, 1994).
- [49] L. Vitagliano, The Hamilton–Jacobi formalism for higher-order field theories, *Int. J. Geom. Meth. Mod. Phys.* **7**(8) (2010) 1413–1436.
- [50] L. Vitagliano, Hamilton–Jacobi diffieties, *J. Geom. Phys.* **61**(2011) 1932–1949.
- [51] L. Vitagliano, Geometric Hamilton–Jacobi field theory, *Int. J. Geom. Meth. Mod. Phys.* **9**(2) (2012) 1260008.
- [52] L. Vitagliano, Characteristics, bicharacteristics and geometric singularities of solutions of PDEs, *Int. J. Geom. Meth. Mod. Phys.* **11** (2014) 1460039.
- [53] V. Volterra, *Theory of Functionals and of Integral and Integro-Differential Equations* (Dover Phoenix Editions, NY, 1959).

- [54] H. Wang, Symmetric Reduction and Hamilton–Jacobi equation of rigid spacecraft with a rotor, *J. Geom. Symmetry Phys.* **32** (2013) 87–111.
- [55] H. Wang, Hamilton–Jacobi theorems for regular controlled Hamiltonian system and its reductions, preprint (2013), arXiv: 1305.3457 [math.SG].
- [56] A. Weinstein, Lagrangian submanifolds and Hamiltonian systems, *Ann. Math.* **98** (1973) 377–410.
- [57] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge University Press, Cambridge, 1937).