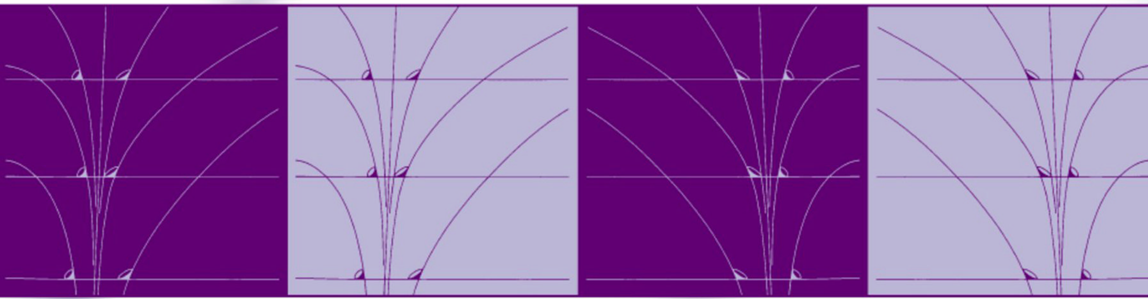
ROBERT H. WASSERMAN



Tensors and Manifolds | SECOND EDITION

With Applications to Physics

OXFORD

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with Applications to Physics

Second edition

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For Liz

PREFACE TO SECOND EDITION

In this edition I have corrected some errors, and expanded and clarified some of the exposition of the first edition. I have added a few problems and I have expanded the section on Notation.

The main change is the addition of 4 chapters extending the application of tensors and manifolds made in Chapters 15-17 on geometry to connections on fiber bundles and culminating with a brief description of gauge theory and the very elegant model of elementary particle physics based on this mathematics. In the new chapters I have tried to keep the mathematical level and style the same as in the first edition.

I would like to thank Prof. Wayne Repko for helpful discussions on elementary particle physics. Again I am very grateful to Cathy Friess for doing a remarkable job with the typing.

East Lansing, Michigan R. H. W.
June, 2003

PREFACE

This book is based on courses taken by advanced undergraduate and beginning graduate students in mathematics and physics at Michigan State University.

The courses were intended to present an introduction to the expanse of modern mathematics and its application in modern physics. The book gives an introductory perspective to young students intending to go into a field of pure mathematics, and who, with the usual "pigeon-holed" graduate curriculum, will not get an overall perspective for several years, much less any idea of application. At the same time, it gives a glimpse of a variety of pure mathematics for applied mathematics and physics students who will have to be carefully selective of the pure mathematics courses they can fit into their curriculum.

Thus, in brief, I have attempted to fill the gap between the basic courses and the highly technical and specialized courses that both mathematics and physics students require in their advanced training, while simultaneously trying to promote, at this early stage, a better appreciation and understanding of each other's discipline.

A third objective is to try to harmonize the two aspects which appear at this level, variously described on the one hand as the "classical," "index," or "local" approach, and on the other hand as the "modern," "intrinsic," or "global" approach.

An underlying theme is an emphasis on mathematical structures. To model a physical phenomenon in general we use some kind of mathematical "space" on which various "physical properties" are defined. For example, in fluid mechanics we make a model in which the fluid consists of points or regions of a space with a certain structure ("ordinary Euclidean space"), and pressure, rate of strain, etc. are mappings into other spaces with certain structures. In general, to model a physical phenomenon, $\mathbb{R}^n$ won't suffice for the domains and images of our mappings - we have to start with manifolds. Moreover, these manifolds have to have additional structures the basic ingredients of which are tensor algebras.

We begin with the algebraic structures we will need and go briefly to $\mathbb{R}^n$ with these in Chapter 7. Manifolds are introduced in Chapter 9 and these structures come together as tensor fields on manifolds in Chapter 11. Chapters 12-14 cover the rudiments of analysis on manifolds, and Chapters 15-17 are devoted to geometry. Finally, a modern treatment of the major ideas of classical analytical mechanics is given in Chapters 18-19, and the remaining chapters are dedicated to an exposition of special and general relativity.

A few words about terminology and notation are needed. I have tried to stick as closely as possible to the most popular current usages, sometimes inserting parenthetically strongly competing alternatives. However, since our text borrows from many different branches of mathematics and physics, we require terminology and notation for a very large variety of concepts and their interrelations. Consequently, the usual problem of how to tread between high precision and readability occurs in aggravated form. Sometimes dropping some notation which is really needed for precision can make it easier to read a given discussion and get the main ideas. On the other hand, sometimes keeping extra terminology and/or notation makes it easier by reminding us of certain important distinctions which might otherwise be temporarily forgotten. A separate section on "Terminology and Notation" is included for convenient reference to some of the conventions used in this book.

Finally, with respect to general style, I have endeavored to steer a safe course between the Scylla of rigor, and the Charybdis of informality. By my not being too heavy-handed with some of the details (including taking some notational liberties as indicated above), hopefully the young student will be able to sail through this passage to enjoy a panorama of interesting mathematics and physics.

This book owes a great deal to the efforts of many classes of students who struggled with earlier classnote versions, and helped hone it into this final form. I was also fortunate to have the services of excellent typists, in particular, Cathy Friess who did the entire final version.

East Lansing, Michigan R. H. W.
November, 1991

CONTENTS

1

VECTOR SPACES

Since tensor algebras are built from vector spaces we will recall some of the theory of the latter. We will review the basic properties of vector spaces, their representations and mappings, and mention a generalization.

1.1 Definitions, properties, and examples

Definition If V is an Abelian group with elements $v, w, \ldots$, if $a, b, \ldots$ are elements of a field, $\mathbb{K}$, and if a mapping $\mathbb{K} \times V \to V$ called "scalar multiplication" and denoted by

$$(a, v) \mapsto av \in V$$

is defined for all elements $a \in \mathbb{K}$ and all $v \in V$ such that

$$\begin{aligned} 1v &= v \\ a(v + w) &= av + aw \\ (a + b)v &= av + bv \\ (ab)v &= a(bv) \end{aligned}$$

then V is a *vector space.*

From these properties one can prove the additional properties

$$\begin{aligned} 0v = 0 \quad &\text{for all} \quad v \in V \\ a0 = 0 \quad &\text{for all} \quad a \in \mathbb{K} \end{aligned}$$

and, conversely, if $av = 0$ then either $a = 0$ or $v = 0$.

Definitions Let S be any set (not necessarily finite) of elements in a vector space, V. Then the intersection of all subspaces of V which contain S and the set of all (finite) linear combinations of elements of S are the same. This set is a subspace, $< S >$, of V called the *linear closure of* S. If a subspace, W, of V is the linear closure of some set $S \subset V$ then we say S *spans*, or *generates* W, and S is *a set of generators* of W.

Definitions Let $\{W_i\}$ be any set (not necessarily finite) of subspaces of a vector space, V. Then the intersection of all subspaces of V containing $\bigcup W_i$, and the set of all finite sums of the form $w_j + w_k + \cdots + w_p$ where $w_j \in W_j$, $w_k \in W_k, \ldots, w_p \in W_p$ are the same. This set is a subspace, $\sum W_i$, of V called the *sum of the subspaces,* W_i *of* V. $\sum W_i$ is *direct* if $W_j \cap \sum_{k \neq j} W_k = 0$ for all j. (Note, there is no restriction on the cardinality of the index set we are using.)

Definitions If for all (finite) sums $\sum_i a_i v_i$ with $v_i \in S$ we have that $\sum_i a_i v_i = 0$ implies that $a_i = 0$ for all i, then S is *a linearly independent set.* This property is equivalent to $0 \notin S$ and every finite sum of subspaces of the form $\ell_v = \{av : a \in \mathbb{K}\}$ is direct. Finally, if W is a subspace of V, and if S spans W, and is a linearly independent set, then S is *a basis* of W. This property is equivalent to the property that for each $w \in W$ corresponding to each $v_i \in S$ there exists a unique element $a_i \in \mathbb{K}$ (with $a_i = 0$ for all but a finite number of i's) such that

$$w = \sum_i a_i v_i \tag{1.1}$$

Clearly, these definitions allow for the possibility that a vector space may contain an infinite linearly independent set, and an infinite basis.

EXAMPLES. (i) $\{(a^1, a^2, \ldots, a^n, \ldots) : a^i \in \mathbb{R}\}$ with "component wise" addition and scalar multiplication with $\mathbb{R}$ is a vector space over $\mathbb{R}$. The set $S = \{(0, 0, \ldots, 1, 0, \ldots) : 1 \text{ is in the } i\text{th place}, i = 1, 2, 3, \ldots\}$ is a linearly independent set but is not a basis.

(ii) $\{(a^1, a^2, \ldots, a^n, \ldots) : \text{only a finite number of the } a\text{'s are nonzero}\}$ with operations as in (i) is a vector space and now S is a basis.

(iii) The set of continuous real-valued functions on $[0, \pi]$ with the usual operations: $\{\sin nx : n = 1, 2, \ldots\}$ is a linearly independent set but not a basis.

(iv) The set of all ordered n-tuples of real numbers, $\mathbb{R}^n$, is an important example of a vector space having a finite basis. $\{(0, 0, \ldots, 1, \ldots, 0) : 1 \text{ in the } i\text{th place}, i = 1, \ldots, n\}$ is called the *natural basis* of $\mathbb{R}^n$.

(v) As generalizations of (i) – (iv) we can consider the set $\mathbb{R}^S$, the set of all (real-valued) functions on an arbitrary set, S, and $\mathbb{R}^S_0$ the set of all (real-valued) functions on an arbitrary set which vanish at all but a finite number of points. Again with pointwise addition and multiplication with $\mathbb{R}$ these are both vector spaces over $\mathbb{R}$. $\mathbb{R}^S_0$ is called *the free vector space over* S. For each $x \in S$ we can define the function f_x by

$$f_x(y) = \begin{cases} 1 & \text{when } y = x \\ 0 & \text{when } y \neq x \end{cases}.$$

Then we have an injection $i : S \to \mathbb{R}^S$ defined by $x \mapsto f_x$. The image set of i, $i(S) \subset \mathbb{R}^S$ is a linearly independent set. It is a basis of $\mathbb{R}^S_0 \subset \mathbb{R}^S$ but not of $\mathbb{R}^S$. Every $f \in \mathbb{R}^S_0$ has the unique representation

$$f = \sum_{x \in S} f(x) f_x$$

(Only a finite number of terms of this sum are nonzero). $\mathbb{R}_0^S$ will be important in the construction of tensor product spaces in Chapter 3.

Theorem 1.1 *If $\boldsymbol{I}$ is a linearly independent set, and $\boldsymbol{G}$ is a set of generators of a vector space, V, and $\boldsymbol{I} \subset \boldsymbol{G}$, then there exists a basis $\boldsymbol{B}$ such that $\boldsymbol{I} \subset \boldsymbol{B} \subset \boldsymbol{G}$. In other words, every linearly independent set can be extended to a basis, and every set of generators contains a basis.*

Proof (1) If $\boldsymbol{G}$ is finite, we can let $\boldsymbol{G} = \{v_1, \dots, v_n\}$ and proceed by induction on n. That is, let v_k be any element of $\boldsymbol{G}$ and consider the subspace, W, generated by $\boldsymbol{G} - v_k$. By the induction hypothesis, W has a basis $\boldsymbol{B}$ with $\boldsymbol{B} \subset \boldsymbol{G} - v_k$. If $v_k \in W$, then $W = V$ and we are done. If $v_k \notin W$, then $\boldsymbol{B} \cup v_k$ is a linearly independent set and spans V and is thus a basis of V.

(2) If $\boldsymbol{G}$ is infinite, we can use Zorn's Lemma, Problem 1.5. □

In particular, since every (nontrivial) vector space has a linearly independent set and a set of generators with $\boldsymbol{I} \subset \boldsymbol{G}$, every vector space has a basis.

Note, however, that for many important spaces the bases are uncountable. In particular, every basis of the vector space of example (i) is uncountable (for method of proof, see Problem 1.12), every basis of the vector space of example (iii) is uncountable (it is an infinite-dimensional Banach space with norm $\max\limits_{[0,\pi]} f$), and if X is an infinite set, then the bases of example (v) are uncountable.

In all the examples above, the field, $\mathbb{K}$, of scalars was $\mathbb{R}$, the field of real numbers. Many of the subsequent results are valid for an arbitrary field (of characteristic 0), in particular for the field of complex numbers, $\mathbb{C}$. It is often useful to bring in complex vector spaces in physical applications. Relations between real and complex vector spaces are indicated in Problems 1.7, 1.8, and 1.14. However, with rare exceptions all our vector spaces in the following will be real.

There is a generalization of the concept of a vector space which we will need when we get to vector fields in Chapter 11. In the definition of vector space we simply replace the field, $\mathbb{K}$, by a ring, $\mathbb{A}$ (with a unit). Such a structure is called an $\mathbb{A}$-*module.* All the definitions above in their various equivalent forms are valid for $\mathbb{A}$-modules. However, there are $\mathbb{A}$-modules that do not have a basis. For example, the module consisting of the elements $0, v_1, v_2, v_3$ over the integers, and with addition given by $v_i + v_i = 0, i = 1, 2, 3$ and $v_i + v_j = v_k$ for i, j, k all different, contains no linearly independent set, and hence has no basis. This example also shows that the property stated right after the definition of a vector space is not valid for modules.

An $\mathbb{A}$-module with a basis is called *a free module.* The $\mathbb{A}$-module we will be studying has a basis. There, $\mathbb{A}$ will be the algebra of (real-valued) functions on a certain set M; i.e., $\mathbb{R}^S$ with the structure of an algebra and with $S = M$.

Definition If a vector space V has a linearly independent set of n elements and no linearly independent set of $n + 1$ elements, then V *has dimension* n, or $\dim V = n$.

Theorem 1.2 *If V has dimension n, then every linearly independent set of n elements is a basis, and every basis has exactly n elements.*

Proof Problem 1.6. □

(Note: Theorem 1.2 can be extended to the "infinite-dimensional" case. In particular, every two bases of V have the same cardinality (cf. Lang, 1965, p. 86).)

PROBLEM 1.1. In the definition of the linear closure of S we actually gave two different characterizations of this concept. Prove they are the same.

PROBLEM 1.2. The same as Problem 1.1 for the definition of the sum of subspaces.

PROBLEM 1.3. The same as Problem 1.1 for the definition of linearly independent set.

PROBLEM 1.4. The same as Problem 1.1 for the definition of basis.

PROBLEM 1.5. Prove Theorem 1.1 for the case $\boldsymbol{G}$ is an infinite set (cf., Greub, 1981, pp. 12 - 13).

PROBLEM 1.6. Prove Theorem 1.2.

PROBLEM 1.7. The set of all ordered n-tuples of complex numbers, $\mathbb{C}^n$, with "component-wise" addition and scalar multiplication with $\mathbb{C}$ is an n-dimensional vector space with the natural basis $S = \{(0, \ldots, 1, 0, \ldots, 0) : 1$ is in the kth place, $k = 1, \ldots, n\}$ (cf., Example (iv)). Show that the *same set* of n-tuples is a real vector space with basis $S \cup \{(0, \ldots, i, 0, \ldots, 0) : i = \sqrt{-1}$ is in the kth place, $k = 1, \ldots, n\}$.

PROBLEM 1.8. Starting with a real vector space, V, we can construct a complex vector space, V_c, *the complexification of* V, whose elements are the elements of $V \times V$ with "component-wise" addition and with scalar multiplication defined by $z(v, w) = (av - bw, aw + bv)$ where $z = a + ib$. Show that $\dim V = \dim V_c$ (cf., Problem 1.14).

1.2 Representation of vector spaces

For an n-dimensional vector space, V^n (over $\mathbb{R}$), Theorem 1.2 implies that in the representation (1.1) we can choose the same finite set of basis elements v_i for all $v \in V^n$ so that (1.1) defines a $1-1$ correspondence between elements $v \in V^n$ and ordered n-tuples $(a^1, \dots, a^n) \in \mathbb{R}^n$. In particular, $v_i \mapsto (0, \dots, 1, \dots, 0)$ with 1 in the ith place. With different bases we have different correspondences. That is, once a basis is chosen we can represent any n-dimensional vector space by the particular space, $\mathbb{R}^n$, of example (iv) above.

If $\{e_1, \dots, e_n\}$ and $\{\bar{e}_1, \dots, \bar{e}_n\}$ are two bases for V^n, the relation between them can be written in the form

$$\bar{e}_j = \sum_i a^i_j e_i \tag{1.2}$$

For any $v \in V^n$ we have $v = \sum v^i e_i$ and $v = \sum \bar{v}^j \bar{e}_j$. Substituting (1.2) into the second representation for v, and then comparing the two expressions for v, we see that

$$v^i = \sum_j \bar{v}^j a^i_j \tag{1.3}$$

Let

$$(a^i_j) = \begin{pmatrix} a^1_1 & a^1_2 & \cdots & a^1_n \\ & & & \\ a^2_1 & & & \vdots \\ \vdots & & & \\ a^n_1 & \cdots & & a^n_n \end{pmatrix}$$

Then (1.3) can be written in matrix notation as either

$$\begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} = (a^i_j) \begin{pmatrix} \bar{v}^1 \\ \vdots \\ \bar{v}^n \end{pmatrix}$$

or

$$\begin{pmatrix} \bar{v}^1 \\ \vdots \\ \bar{v}^n \end{pmatrix} = (a^i_j)^{-1} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \tag{1.4}$$

Writing (1.2) in "matrix notation" as

$$\begin{pmatrix} \bar{e}_1 \\ \vdots \\ \bar{e}_n \end{pmatrix} = (a^i_j)^{tr} \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \tag{1.5}$$

we see that the systems of basis vectors transform with the inverse transpose of the matrix of the transformation of the *components* of a vector. We call (a^i_j) *the change-of-basis* matrix.

Observe that we are using superscripts as well as subscripts in our notation. This enables us to use the "summation convention", which we will do from now on. That is, if in products, such as on the right side of (1.2), the same index occurs as a superscript and a subscript we will sum on that index omitting the Σ notation. Also, note that though in the matrix notation introduced above the superscript is the row index and the subscript is the column index this will not always be the case.

PROBLEM 1.9. Show that the matrix (a^i_j) defined above is nonsingular.

PROBLEM 1.10. If V^n is the direct sum of two subspaces and a change of basis is made in each of the subspaces, what is the form of the change of basis matrix for V^n?

1.3 Linear mappings

Definitions If V and W are two vector spaces, a mapping $\phi : V \to W$ is a *linear mapping* if $\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$ and $\phi(av) = a\phi(v)$ for all v_1, v_2, and v in V and a in the common field of V and W. For such a mapping we write $\phi(v) = \phi \cdot v$. $\phi(V)$ is the *image space* of V under ϕ, $\dim \phi(V)$ is the *rank* of ϕ, and $\phi^{-1}(0)$ is the inverse image of $0 \in W$, or the *null space*, N_ϕ, of ϕ, or the *kernel*, $\ker \phi$, of ϕ. $\phi(V)$ is a subspace of W and $\phi^{-1}(0)$ is a subspace of V. If $W = \mathbb{R}$, then ϕ is called *a linear function*, or *a linear functional*, or *a linear form*, or *a covector*. If $W = V$ then ϕ is called *a linear transformation* or *a linear operator*.

Again, we can generalize the preceding and much of the following to $\mathbb{A}$-modules.

We can construct many different linear mappings between two vector spaces (or, $\mathbb{A}$-modules) as the following theorem shows.

Theorem 1.3 *Given vector spaces V and W and a basis S of V, then there exists a unique linear mapping, $\phi : V \to W$ such that $\phi \cdot v_i = w_i$ where $v_i \in S$ and w_i are arbitrarily chosen elements of W.*

Proof (i) Define ϕ by "extending the given conditions linearly", that is, let $\phi(v) = a^i w_i$ for $v = a^i v_i$. Then it is easy to verify the properties required by the definition. (ii) If ψ is another linear mapping such that $\psi \cdot v_i = w_i$, then by linearity $\psi \cdot v = a^i w_i$ for $v = a^i v_i$. Hence $\psi \cdot v = \phi \cdot v$ for all v so $\psi = \phi$. □

Corollary *If* $\dim V = \dim W$, *then V and W are isomorphic.*

The converse of this corollary is also true; if ϕ is an isomorphism of V and W then they have the same dimensions. More generally, we have the following result for the dimension of V.

Theorem 1.4 *If $\phi : V \to W$ is linear, then*

$$\dim V = \text{rank } \phi + \dim \ker \phi$$

Proof Let $\{e_1, \dots, e_p\}$ be a basis of $\phi^{-1}(0)$, and let $\{e_1, \dots, e_p, \bar{e}_1, \dots, \bar{e}_q\}$ be a basis of V. (See Problem 1.5.) Then $\dim \phi^{-1}(0) = p$ and $\dim V = p + q$. Now we need only to show that $\{\phi \cdot \bar{e}_1, \dots, \phi \cdot \bar{e}_q\}$ is a basis of $\phi(V)$, that is that $\dim \phi(V) = q$. A direct calculation shows that $\phi \cdot \bar{e}_1, \dots, \phi \cdot \bar{e}_q$ is a linearly independent set and spans $\phi(V)$. □

If W is a subspace of V, and $v \in V$, we can form the subset $\{v + w : w \in W\}$. The set of all such subsets (as v ranges over V) is a *factor*, or *quotient space*, V/W. v_1 and v_2 are both in the same subset if and only if $v_1 - v_2 \in W$. The set (i.e., equivalence class) containing v is denoted by $[v]$. The dimension of V/W is called *the codimension of* W. The linear mapping $\pi : V \to V/W$ given by $v \mapsto [v]$ is onto, has the property $\pi^{-1}(0) = W$, and is called *the natural projection* of V onto V/W. From this and Theorem 1.4 we have codim $W = \dim V - \dim W$.

Conversely, starting with a linear mapping, $\phi : V \to W$, we have $\phi^{-1}(0)$ and we can construct a factor space $V/\phi^{-1}(0)$ and an induced isomorphism $V/\phi^{-1}(0) \cong \phi(V)$. There is also the factor space $W/\phi(V)$ called *the cokernel of* ϕ, and rank $\phi + \dim \text{coker } \phi = \dim W$.

PROBLEM 1.11. Let V and W be vector spaces. (i) Define operations in $V \times W$ so that it becomes a vector space (called *the exterior direct sum of* V *and* W and denoted by $V \oplus W$). (ii) Show that the *projections* given by $p_1 : (v, w) \mapsto (v, 0)$ and $p_2 : (v, w) \mapsto (0, w)$ are linear. (iii) Show that $V \oplus W = \ker p_1 \oplus (p_1(V \oplus W))$, where the first $\oplus$ on the right is the direct sum defined in Section 1.1.

PROBLEM 1.12. The set of linear functions, $\mathcal{L}(V, \mathbb{R})$, on V is a vector space (this is a special case of Theorem 2.1). Use Theorem 1.3 to show that if V has a countably infinite basis, then the basis of $\mathcal{L}(V, \mathbb{R})$ is uncountably infinite.

(Hint: By Theorem 1.3, if V has a countably infinite basis, then every sequence of real numbers determines a unique element $f \in \mathcal{L}(V, \mathbb{R})$. For each positive number, r, let f_r be the element determined by $(1, r, r^2, r^3, \dots)$. Consider the set of all such functions, and show that for any finite subset $\{f_1, \dots, f_n\}$ $a^1 f_1 + a^2 f_2 + \cdots + a^n f_n = 0$ implies $a^1 = a^2 = \cdots = a^n = 0$.)

PROBLEM 1.13. If $(a, b) \in \mathbb{R}^2$, let $W = \{t(a, b) : t \in \mathbb{R}\}$, a subset of the vector space $\mathbb{R}^2$. Give a geometrical interpretation of $\mathbb{R}^2/W$.

PROBLEM 1.14. Corresponding to each $a \in \mathbb{K}$ there is a linear operator ϕ on V given by $\phi : V \to aV$. In particular, for the space $\mathbb{C}^n$, corresponding to $i = \sqrt{-1}$ we have a linear operator, J with the property that $J^2 v = -v$. On the real vector space $\mathbb{R}^{2n}$, the linear operator J defined by

$$J : \begin{cases} e_k \mapsto e_{k+n} & k = 1, \dots, n \\ e_k \mapsto e_{k-n} & k = n+1, \dots, 2n \end{cases}$$

where $\{e_1, \dots, e_{2n}\}$ is the natural basis of $\mathbb{R}^{2n}$, has this property. Show that for any real vector space which admits such an operator, called *a complex structure*, the same set of elements can be made into a complex vector space by defining multiplication by a complex scalar by $(a + bi)v = av + bJv$ (cf., Problems 1.7 and 1.8).

PROBLEM 1.15. A set, S, on which there is defined a vector space, V, of transformations is called *an affine space* of V, if (i) $0 \in V$ is the identity mapping, if (ii) for $v \neq 0$ and $p \in S, v(p) \neq p$, if (iii) $u + v$ is the composition, and if (iv) for each ordered pair (p, q) of elements of S, there is an element $v \in V$ such that $v(p) = q$. Show that $(-v)(q) = p$ if $v(p) = q$. Show that (p, q) determines a unique vector, so we can write it as $\overrightarrow{pq}$ and call it a *free vector*, or *translation of* S. If $S = \{(a, b, c) \in \mathbb{R}^3 : c = a + b + 1\}$ and V is the vector space $\mathbb{R}^2$, show that S is an affine space of $\mathbb{R}^2$ (cf., Problem 1.13).

1.4 Representation of linear mappings

Just as we have a $1-1$ correspondence between vectors in a real n-dimensional vector space, V^n, and elements of $\mathbb{R}^n$ once a basis is chosen in V^n, we can now, after choosing bases $\{e_1, \dots, e_n\}$ in V^n and $\{E_1, \dots, E_r\}$ in V^r, set up a $1-1$ correspondence between the set of linear mappings from V^n to V^r and the set of r by n matrices. Thus, given a linear mapping, ϕ, then for each $i = 1, \dots, n$, there is a unique set of components, $\phi_i^j, j = 1, \dots, r$, of the image of e_i under ϕ. That is,

$$\phi \cdot e_i = \phi_i^j E_j \tag{1.6}$$

If $w = \phi \cdot v$, then putting $v = v^i e_i$ and $w = w^j E_j$ we have $w^j E_j = w = \phi \cdot v^i e_i = v^i \phi \cdot e_i = v^i \phi_i^j E_j$, so

$$w^j = v^i \phi_i^j \qquad \begin{matrix} i = 1, \dots, n \\ j = 1, \dots, r \end{matrix} \tag{1.7}$$

or, in matrix notation, writing

$$(\phi_i^j) = \begin{pmatrix} \phi_1^1 & \phi_2^1 & \cdots & \phi_n^1 \\ \phi_1^2 & & & \\ \vdots & & & \\ \phi_1^r & & \cdots & \phi_n^r \end{pmatrix}$$

we have

$$\begin{pmatrix} w^1 \\ \vdots \\ w^r \end{pmatrix} = (\phi_i^j) \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \tag{1.8}$$

(Notice that the matrix of coefficients of the system (1.6), as it stands, is the transpose of (ϕ_i^j). Also, compare eqs. (1.6) - (1.8) respectively with eqs. (1.2) - (1.4).)

If $n = r$, then the mapping can be represented by a square matrix. For example, if $V^n = V^r = \mathbb{R}^{2n}$, and ϕ is the linear operator J in Problem 1.14, then in the natural basis of $\mathbb{R}^{2n}$ $\phi = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ where I is the $m \times m$ identity matrix.

If $r = 1$, then the mapping can be represented by a $1 \times n$ matrix, or a row vector. Thus, linear functions, or covectors, can be represented by row vectors.

Conversely, starting with an arbitrary r by r matrix, (ϕ_i^j), we can let the columns be the components, in the $\{E_i\}$ basis of V^r, of the images of the basis elements, e_i, of V^n, i.e., $e_i \mapsto \phi_i^1 E_1 + \phi_i^2 E_2 + \cdots + \phi_i^r E_r$. An arbitrary choice of these images determines a unique linear mapping by Theorem 1.3.

With the representation of vectors by elements of cartesian spaces and the representation of linear mappings by matrices, we see that the specific "concrete" examples of linear mappings of the form $\phi : \mathbb{R}^n \to \mathbb{R}^r$ given by

$$(v^1, \ldots, v^n) \mapsto (\phi_i^1 v^i, \ldots, \phi_i^r v^i) \tag{1.9}$$

or, in matrix notation

$$\begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \mapsto \begin{pmatrix} \phi_1^1 & \cdots & \phi_n^1 \\ \vdots & & \vdots \\ \phi_1^r & \cdots & \phi_n^r \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$$

are really all there are.

Just as a vector will have two different sets of components in two different bases, a linear mapping will have different matrices if different bases are chosen in V^n and V^r.

Definition Two r by n matrices are *equivalent* if they represent the same linear mapping ϕ, relative to different bases in V^n and V^r.

From matrix theory we know that two matrices are equivalent if and only if they have the same rank.

An important special case of the ideas above occurs when $V^r = V^n$, i.e., ϕ is a linear transformation. In this case, two n by n matrices representing ϕ are called *similar*, and matrices (ϕ^i_j) and (ψ^i_j) are similar if and only if there exists a nonsingular matrix, M such that $(\psi^i_j) = M^{-1}(\phi^i_j)M$. M is the change-of-basis matrix. For a linear transformation we also have the important concepts of eigenvectors, eigenvalues, invariant subspaces, etc. We will come back to these ideas later when we need them in Chapter 5.

PROBLEM 1.16. (i) Show that the correspondence $e_i \mapsto (0, \ldots 1, \ldots, 0)$ between V^n and $\mathbb{R}^n$ described in Section 1.2 is an isomorphism. (ii) What is the form of (1.8) for this linear mapping if we choose in $\mathbb{R}^n$ (a) the natural basis, (b) an arbitrary basis $\{(a_{i1}, \ldots, a_{in}) : i = 1, \ldots, n\}$?

2

MULTILINEAR MAPPINGS AND DUAL SPACES

We will discuss the important space, V^*, of linear functions on a vector space, V. We will describe isomorphisms between spaces of multilinear mappings, and, finally, we will focus on special properties of bilinear functions.

2.1 Vector spaces of linear mappings

In Section 1.3 we discussed briefly the idea of a linear mapping between vector spaces V and W. Now we consider the set of all linear mappings from V to W.

Theorem 2.1 *The set, $\mathcal{L}(V, W)$, of all linear mappings from V to W with operations $\phi + \psi$ and $a\phi$ defined by*

$$(\phi + \psi) \cdot v = \phi \cdot v + \psi \cdot v$$
$$(a\phi) \cdot v = a(\phi \cdot v)$$

is a vector space.

Proof $\phi + \psi$ is a linear mapping since

$$\begin{aligned}(\phi + \psi) \cdot (av + bw) &= \phi \cdot (av + bw) + \psi \cdot (av + bw)\\ &= a\phi \cdot v + b\phi \cdot w + a\psi \cdot v + b\psi \cdot w\\ &= a(\phi + \psi) \cdot v + b(\phi + \psi) \cdot w.\end{aligned}$$

Similarly for $a\phi$. Each of the vector space properties of $\mathfrak{L}(V, W)$ comes from the corresponding property of W. □

Theorem 2.2 *With the standard definitions of addition and scalar multiplication, the set of r by n matrices is a vector space, and the $1 - 1$ correspondence described in Section 1.4 between $\mathfrak{L}(V^n, V^r)$ and the vector space of r by n matrices is an isomorphism.*

Proof Problem 2.2. □

Thus, in particular, the set of $n \times n$ matrices, $\mathcal{M}_n$, forms a vector space isomorphic to the vector space of linear transformations of V^n.

Theorem 2.3 *Suppose $\{e_i\}$ and $\{E_j\}$ are bases of finite-dimensional spaces V and W respectively. Then the elements e^i_j of $\mathcal{L}(V, W)$ defined by $e^i_j : v \mapsto v^i E_j$ form a basis for $\mathcal{L}(V, W)$. Moreover, for $\phi \in \mathfrak{L}(V, W)$,*

$$\phi = \phi^j_i e^i_j$$

where ϕ^i_j are given by $\phi \cdot e_k = \phi^j_k E_j$.

Proof First of all, note that the definition of e^i_j is equivalent to $e^i_j : e_k \mapsto \delta^i_k E_j$ with e^i_j extended to V by linearity.

(i) $a^j_i e^i_j(e_k) = a^j_i \delta^i_k E_j = a^j_k E_j$ so $a^j_i e^i_j = 0$ implies that $a^j_k E_j = 0$ for all k, and since $\{E_i\}$ is linearly independent, $a^j_i = 0$.

(ii) Given ϕ in $\mathcal{L}(V, W)$, let $\phi \cdot e_k = \phi^j_i E_j$. Then $\phi^j_i e^i_j(e_k) = \phi^j_i \delta^i_k E_j = \phi^j_k E_j = \phi(e_k)$ so $\phi = \phi^j_i e^i_j$. □

There are two important special cases of $\mathcal{L}(V, W)$; namely, $\mathcal{L}(\mathbb{R}, W)$ and $\mathcal{L}(V, \mathbb{R})$. We will devote our attention exclusively to the second case after giving one result for the first.

Theorem 2.4 *$\mathfrak{L}(\mathbb{R}, W)$ is isomorphic with W.*

Proof Note that every nonzero element of $\mathbb{R}$ is a basis of $\mathbb{R}$. In particular, 1 is the natural basis of $\mathbb{R}$. By Theorem 1.3 for each $w \in W$, there is an element $\bar{w}$ of $\mathcal{L}(\mathbb{R}, W)$ given by $\bar{w} : 1 \mapsto w$. Two different w's must clearly lead to two different mappings, so the correspondence $W \to \mathcal{L}(\mathbb{R}, W)$ is one-to-one. On the other hand, given any $\phi \in \mathcal{L}(\mathbb{R}, W), \phi \cdot 1 \in W$ determines a mapping ψ given by $1 \mapsto \phi \cdot 1$. By the "uniqueness" part of Theorem 1.3 $\psi = \phi$ so the correspondence is onto. The linearity follows from the definition of the operations in $\mathcal{L}(\mathbb{R}, W)$. □

This isomorphism will be invoked to make a certain "identification" when we study tangent maps in Section 7.3.

The space, $\mathfrak{L}(V, \mathbb{R})$, of linear functions on V is also denoted by V^*. As noted in Section 1.3, its elements, f, are also called linear forms (or linear functionals, or covectors).

As a concrete *example* of a space of linear forms we have $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$, whose elements, f, are given by $f : (a^1, \ldots, a^n) \mapsto f_i a^i$, or, in matrix form,

$$\begin{pmatrix} a^1 \\ \vdots \\ a^n \end{pmatrix} \mapsto (f_1 \cdots f_n) \begin{pmatrix} a^1 \\ \vdots \\ a^n \end{pmatrix}$$

Clearly the elements of $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ are just special cases of the linear mappings, (1.9). As there, this example can be thought of as being quite general, in the sense that once we choose a basis, $\{e_i\}$, in any n-dimensional space, V^n, and a basis, $E \in \mathbb{R}$, then $v \in V^n$ can be represented by $\begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$, $f \in V^{n*}$ can be represented by a $1 \times n$ matrix (or row vector), $(f_1, \cdots f_n)$, where the f_i are given by $f \cdot e_i = f_i E$ (cf., eq. (1.6)), and

$$(f_1 \cdots f_n) \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$$

represents the image of v under f.

From the fact that V^{n*} is isomorphic to the set of $1 \times n$ matrices, it is clear the $\dim V^{n*} = n$; i.e., $\dim V^{n*} = \dim V^n$. It follows as a special case of Theorem 2.3 that the elements $\varepsilon^i \in V^{n*}$ defined by $\varepsilon^i \cdot e_k = \delta^i_k E \quad i, k = 1, \ldots, n, \quad E \in \mathbb{R}$, form a basis for V^{n*} and

$$f = f_i \varepsilon^i$$

for any f in V^{n*}.

Definition If we choose $E = 1$, then the basis $\{\varepsilon_i\}$ of V^{n*} given by

$$\varepsilon^i \cdot e_k = \delta^i_k, \qquad i, k = 1, \ldots, n$$

is called *the dual basis* of $\{e_i\}$.

Since, when V is finite-dimensional, $\dim V = \dim V^*$, it follows that we have a result for $\mathfrak{L}(V, \mathbb{R})$ analogous to Theorem 2.4 for $\mathfrak{L}(\mathbb{R}, W)$; namely, $\mathfrak{L}(V, \mathbb{R})$ is isomorphic to V.

In general, however, if V is not finite-dimensional, V^* is not necessarily isomorphic with V. It is generally larger than V. See Problem 1.12. However, there is a subspace of V^* which is isomorphic with V.

Definition Let $\{e_i\}$ be a basis of V. We denote the set $\{f \in V^* : f \cdot e_i = 0$ for all but a finite number of the e_i's$\}$ by V_0^*.

Theorem 2.5 *The set, $\{\varepsilon^i\}$, of elements of V^* defined by $\varepsilon_i \cdot e_k = \delta^i_k$ is a basis for V_0^*, and $V_0^* \cong V$.*

Proof (i) Consider any linear combination $a_i \varepsilon^i$. Then

$$(a_i \varepsilon^i) \cdot e_k = a_i(\varepsilon^i \cdot e_k) = a_k$$

so $a_i \varepsilon^i = 0$ implies that $a_k = 0$. (ii) Given f in V_0^*, let $f \cdot e_i = f_i$, and look at $f_i \varepsilon^i$. $(f_i \varepsilon^i) \cdot e_j = f_i \delta^i_j = f_j = f \cdot e_j$ for all e_j, so $f = f_i \varepsilon^i$. (i) and (ii) show that $\{\varepsilon^i\}$ is a basis for V_0^*. (iii) Notice that the condition $\varepsilon^i \cdot e_k = \delta^i_k$ defines a $1-1$ correspondence between the basis sets $\{e_i\}$ and $\{\varepsilon^i\}$; namely, $e_k \mapsto \varepsilon^i$, the basis element of V_0^* which takes e_i to 1 and all the others to zero. If we extend this correspondence by linearity, then the resulting mapping is $1-1$ and onto V_0^*. □

Having constructed a new vector space V^* from V we can now ask about linear transformations on V^*; in particular, we can study the set $\mathfrak{L}(V^*, \mathbb{R}) = V^{**}$.

Theorem 2.6 *Let v be any arbitrarily chosen (fixed) element of V. Then the map $\bar{v} : V^* \to \mathbb{R}$ defined by $\bar{v} : f \mapsto f \cdot v$ is in V^{**}.*

Proof Problem 2.4. □

Since for each $v \in V$ we have a $\bar{v} \in V^{**}$ by Theorem 2.6, that theorem gives us a mapping $\mathcal{I} : V \to V^{**}$. With the notation of Theorem 2.6, $\mathcal{I}(v) = \bar{v}$ and the map defined in that theorem can be written

$$\mathcal{I}(v) \cdot f = f \cdot v \tag{2.1}$$

Theorem 2.7 *The mapping $\mathcal{I} : V \to \mathcal{I}(V) \subset V^{**}$ is an isomorphism.*

Proof

$$\text{(i)} \qquad \begin{aligned} \mathcal{I}(av_1 + bv_2) \cdot f &= f \cdot (av_1 + bv_2) && \text{by (2.1)} \\ &= af \cdot v_1 + bf \cdot v_2 && \text{by linearity of } f \\ &= a\mathcal{I}(v_1) \cdot f + b\mathcal{I}(v_2) \cdot f && \text{by (2.1)} \\ &= (a\mathcal{I}(v_1) + b\mathcal{I}(v_2)) \cdot f \end{aligned}$$

by definition of operations in V^{**}. Hence $\mathcal{I}(av_1 + bv_2) = a\mathcal{I}(v_1) + b\mathcal{I}(v_2)$, so $\mathcal{I}$ is linear.

(ii) $\mathcal{I}(v_1) = \mathcal{I}(v_2)$ implies $f \cdot v_1 = f \cdot v_2$ for all $f \in V^*$, so $v_1 = v_2$ and $\mathcal{I}$ is $1 - 1$. (See Problem 2.7.) □

Corollary *If V is finite-dimensional, then V^{**} is isomorphic with V. Thus, if we identify V and V^{**} we can think of V as the space of linear functions on V^*. (See Section 2.2.)*

Definition $\mathcal{I}$ is called *the natural injection of V into V^{**}.*

PROBLEM 2.1. (i) Prove Theorem 2.1 with $\mathcal{L}(V, W)$ replaced by W^V, the set of all maps from V to W. (ii) Prove Theorem 2.1 with $\mathcal{L}(V, W)$ replaced by W^S where S is an arbitrary set. (Compare with $\mathbb{R}^S$ in Section 1.1.)

PROBLEM 2.2. Prove Theorem 2.2.

PROBLEM 2.3. Suppose $\{e_i\}$ and $\{\bar{e}_i\}$ are bases of $V, i = 1, \ldots, n$, and suppose $\{\varepsilon^i\}$ and $\{\bar{\varepsilon}^i\}$ are corresponding dual bases of V^*. Show that if

$$(\bar{e}_1 \cdots \bar{e}_n) = (e_1 \cdots e_n) \begin{pmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & & \vdots \\ a_1^n & \cdots & a_n^n \end{pmatrix}$$

$$\text{then}\quad \begin{pmatrix} \bar{\varepsilon}^1 \\ \vdots \\ \bar{\varepsilon}^n \end{pmatrix} = \begin{pmatrix} b_1^1 & \cdots & b_n^1 \\ \vdots & & \vdots \\ b_1^n & \cdots & b_b^n \end{pmatrix} \begin{pmatrix} \varepsilon^1 \\ \vdots \\ \varepsilon^n \end{pmatrix}$$

where $a_j^i b_k^j = \delta_k^i$. That is, the change-of-basis matrix (b_j^i) is the inverse of the change-of-basis matrix (a_j^i). Write the relation between the components, f_i and $\bar{f}_i$, of an element of V^* in the two bases in terms of (a_j^i).

PROBLEM 2.4. Write out the matrix of the linear transformation e_j^i in Theorem 2.3.

PROBLEM 2.5. In Theorem 2.5 we got a basis for a space of linear forms on a vector space which was permitted to be either finite- or infinite-dimensional. In Theorem 2.3 we got bases for spaces of mappings, and the proof given for Theorem 2.3 resembles that of Theorem 2.5. If in Theorem 2.3 V and/or W are infinite-dimensional we can define e_j^i as in the finite case. Precisely where will the proof fail if we try to prove Theorem 2.3 if V and/or W are infinite-dimensional?

PROBLEM 2.6. Prove Theorem 2.6.

PROBLEM 2.7. In the proof of part (ii) of Theorem 2.7 we stated that $f \cdot v_1 = f \cdot v_2$ for all $f \in V^*$ implies that $v_1 = v_2$. This is equivalent to the statement that if $v_1 \neq v_2$, then there is an f in V^* such that $f \cdot v_1 \neq f \cdot v_2$, or if $v \neq 0$, then there is an f in V^* such that $f \cdot v \neq 0$. Prove this.

PROBLEM 2.8. Prove that two matrices (ϕ_j^i) and (ψ_j^i) represent (with respect to two bases in V and their duals in V^*) the same linear mapping of V to V^* if and only if there is a matrix M such that $(\psi_j^i) = M^{tr}(\phi_j^i)M$, and M is the change of basis matrix, (a_j^i). Two such matrices are called *congruent* (cf., definition of similar matrices in Section 1.4).

2.2 Vector spaces of multilinear mappings

Definition Given vector spaces $V_1, V_2, \ldots, V_p, W$. A mapping $\phi : V_1 \times \cdots \times V_p \to W$ from the cartesian product $V_1 \times \cdots \times V_p$ to W is *multilinear* if it is linear in each argument.

Examples (i) We get an important example of a bilinear function ($p = 2$, $W = \mathbb{R}$) if we choose $V_1 = \mathbb{R}^n, V_2 = \mathbb{R}^m$. Then, given any a_{ij} $i = 1, \ldots, n$, $j = 1, \ldots, m$, the map $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ defined by

$$((v^1, \ldots, v^n), (w^1, \ldots, w^m)) \mapsto a_{ij} v^i w^j \tag{2.2}$$

is bilinear. Moreover, every bilinear function $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ can be written in this form. For, given ϕ, then

$$\phi : ((0, \ldots, \underset{i\text{th place}}{1}, \ldots, 0)(0, \ldots, \underset{j\text{th place}}{1}, \ldots, 0)) \mapsto a_{ij}$$

for some a_{ij} and then by bilinearity $\phi((v^1, \ldots, v^n), (w^1, \ldots, w^m)) = a_{ij}v^i w^j$. Clearly, these bilinear functions are generalizations of the dot product of vector analysis.

(ii) Another example of a bilinear function (in a certain sense both more and less general than the previous one) which we will see again in Chapter 3 is obtained if V_1 and V_2 are two vector spaces and we are given $f \in \mathcal{L}(V_1, \mathbb{R})$ and $g \in \mathcal{L}(V_2, \mathbb{R})$. Then $\phi : V_1 \times V_2 \to \mathbb{R}$ given by $(v_1, v_2) \mapsto (f \cdot v_1)(g \cdot v_2)$ is bilinear.

(iii) Finally, the "oriented volume" of a parallelepiped in "ordinary" space (a normed determinant function) is an example of a trilinear function.

The bilinear functions of example (i) are analogous to the "concrete" examples given by eq. (1.9) in the sense that every linear map is represented by the latter when bases are chosen in V and W, and every bilinear function is represented by the former when bases are chosen in V, W, and $\mathbb{R}$.

Theorem 2.8 *Given vector spaces $V_1, \ldots, V_p, W$ and bases S_i of V_i, $i = 1, \ldots, p$, then there exists a unique multi linear mapping $\phi : V_1 \times \cdots \times V_p \to W$ such that $\phi(v_{k_1}, v_{k_2}, \ldots, v_{k_p}) = w_{k_1 \cdots k_p}$ where $v_{k_i} \in S_i$, and $w_{k_1 \cdots k_p}$ are arbitrary elements of W.*

Proof See Theorem 1.3. □

Theorem 2.9 *The set, $\mathfrak{L}(V_1, V_2, \ldots, V_p; W)$ of all p-linear maps from $V_1 \times \cdots \times V_p$ to W (with the obvious definitions for the operations) is a vector space.*

Proof Problem 2.9. □

If the vector spaces $V_1, \ldots, V_p, W$ are finite-dimensional, then $\mathfrak{L}(V_1, \ldots, V_p; W)$ has a basis in terms of those of $V_1, \ldots, V_p, W$ and the dimension of $\mathfrak{L}(V_1, \ldots, V_p; W)$ is the product of the dimensions of $V_1, \ldots, V_p, W$. More explicitly, we have the following special case.

Theorem 2.10 *Suppose $\{e_i\}$ and $\{E_i\}$ are bases of V and W respectively. Then the functions $f^{ij} : V \times W \to \mathbb{R}$ defined by $f^{ij} : (v, w) \mapsto v^i w^j$ where v^i and w^i are the components of v and w in the chosen bases, form a basis for $\mathfrak{L}(V, W; \mathbb{R})$. Moreover, for $b \in \mathfrak{L}(V, W; \mathbb{R})$*

$$b = b_{ij} f^{ij}$$

where $b_{ij} = b(e_i, E_j)$ (cf. Theorem 2.3).

Proof First of all, note that the definition of f^{ij} is equivalent to $f^{ij} : (e_k, E_l) \mapsto \delta^i_k \delta^j_l$ with f^{ij} extended to $V \times W$ by bilinearity.

(i) $b_{ij} f^{ij}(e_k, E_l) = b_{ij} \delta^i_k \delta^j_l = b_{kl}$ so $b_{ij} f^{ij} = 0$ implies that $b_{kl} = 0$ for all k, l. Hence the f^{ij} form a linearly independent set.

(ii) If $b \in \mathcal{L}(V, W; \mathbb{R})$ then $b(v, w) = v^i w^j b(e_i, E_j) = b(e_i, E_j) f^{ij}(v, w)$ for all $v \in V$ and $w \in W$, so $b = b(e_k, E_j) f^{ij}$. □

In the sequel we will be dealing with a variety of special cases of these spaces of multilinear mappings. It will be important to observe that these special cases are not all really different from one another – that certain spaces of mapping can be identified with certain others. What we mean by saying that two spaces can be "identified" with one another is that an isomorphism can be constructed between the two which does not require any choice of bases. Note that the isomorphisms established in Theorem 2.2 and Theorem 2.4 *do* require a choice of a basis. In the finite-dimensional case, isomorphisms between V and V^* require choosing bases, but the isomorphism $\mathcal{I}$ between V and V^{**} in the corollary of Theorem 2.7 does not. Hence we do not identify V and V^*, but we do identify V and V^{**}.

Definition A linear mapping of vector spaces which is independent of the choice of bases is called *natural* (or *canonical*).

Theorem 2.11 *There is a natural isomorphism between* $\mathfrak{L}(V_1, V_2; W)$ *and* $\mathfrak{L}(V_1, \mathfrak{L}(V_2, W))$.

Proof Let b be an element of $\mathfrak{L}(V_1, V_2; W)$. We define a map $\phi : \mathfrak{L}(V_1, V_2; W) \to \mathfrak{L}(V_1, \mathfrak{L}(V_2, W))$ by $b \mapsto \phi(b) \in \mathfrak{L}(V_1, \mathfrak{L}(V_2, W))$ where $\phi(b)$ is the linear map whose values, $\phi(b) \cdot v_1 \in \mathfrak{L}(V_2, W)$ are linear mappings given by

$$\phi(b) \cdot v_1 : v_2 \mapsto b(v_1, v_2) \in W. \tag{2.3}$$

That is, $\phi(b)$ is defined by its values, $\phi(b) \cdot v_1$, on V_1, each of which (i.e., for each fixed v_1) is a linear function from V_2 to W defined by (2.3).

We also write, for given b, and v_1, the partial map, $b(v_1, -) : v_2 \mapsto b(v_1, v_2)$, so with this notation $\phi(b) \cdot v_1$ is defined to be $b(v_1, -)$, that is, $\phi(b) \cdot v_1 = b(v_1, -)$.

We will prove that ϕ is $1-1$ and onto. (The linearity of ϕ comes by following through the definitions.) Let $\mathcal{A} \in \mathfrak{L}(V_1, \mathfrak{L}(V_2, W))$. We define a map $\psi : \mathfrak{L}(V_1, \mathfrak{L}(V_2, W)) \to \mathfrak{L}(V_1, V_2; W)$ by $\psi(\mathcal{A}) : (v_1, v_2) \mapsto (\mathcal{A} \cdot v_1) \cdot v_2$ (Figure 2.1). Now we form the compositions $\psi \circ \phi$ and $\phi \circ \psi$, and evaluate them on their respective domains.

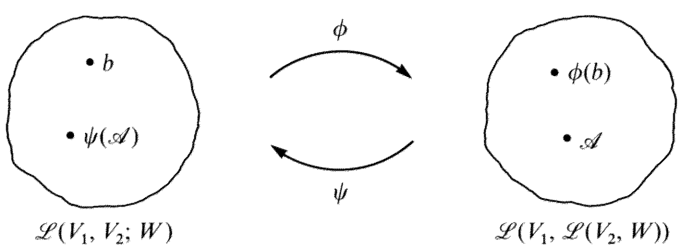

Figure 2.1

(i) For all b, v_1, v_2

$$\begin{aligned}\psi(\phi(b))(v_1, v_2) &= (\phi(b) \cdot v_1) \cdot v_2 \quad &&\text{(by the definition of } \psi)\\ &= b(v_1, -) \cdot v_2 &&\text{(by the definition of } \phi(b))\\ &= b(v_1, v_2)\end{aligned}$$

Hence $\psi(\phi(b))(v_1, v_2) = b(v_1, v_2)$ for all v_1, v_2, and b, so $\psi(\phi(b)) = b$ for all b and $\psi \circ \phi =$ identity on $\mathfrak{L}(V_1, V_2; W)$ which implies that ϕ is $1-1$.

(ii) For all $\mathcal{A}, v_1, v_2$

$$\begin{aligned}\phi(\psi(\mathcal{A})) \cdot v_1) \cdot v_2 &= \psi(\mathcal{A})(v_1, -) \cdot v_2 \quad &&\text{(by the definition of } \phi(b))\\ &= \psi(\mathcal{A})(v_1, v_2)\\ &= (\mathcal{A} \cdot v_1) \cdot v_2 &&\text{(by the definition of } \psi)\end{aligned}$$

Again, since this is valid for all v_1, v_2 and $\mathcal{A}$, we have $\phi \circ \psi =$ identity on $\mathfrak{L}(V_1, \mathfrak{L}(V_2, W))$ which implies that ϕ is onto. □

By Theorem 2.11 to each bilinear function $b \in \mathfrak{L}(V, W; \mathbb{R})$ there corresponds a unique linear map $\mathcal{A}_1 = \phi_1 \cdot b \in \mathfrak{L}(V, \mathfrak{L}(W, \mathbb{R}) \cong \mathfrak{L}(V, W^*)$ where $\phi_1 : \mathfrak{L}(V, W; \mathbb{R}) \to \mathfrak{L}(V, \mathfrak{L}(W, \mathbb{R}))$. Since, $\mathfrak{L}(V, \mathfrak{L}(W, \mathbb{R})) \cong \mathfrak{L}(W, \mathfrak{L}(V, \mathbb{R}))$, (see, Problem 2.14) corresponding to b there is also a unique linear map $\mathcal{A}_2 = \phi_2 \cdot b \in \mathfrak{L}(W, \mathfrak{L}(V, \mathbb{R})) \cong \mathfrak{L}(W, V^*)$ where $\phi_2 : \mathfrak{L}(V, W; \mathbb{R}) \to \mathfrak{L}(W, \mathfrak{L}(V, \mathbb{R}))$. $\phi_1 \cdot b : V \to W^*$ is given by $v \mapsto b(v, -)$ and $\phi_2 \cdot b : W \to V^*$ is given by $w \mapsto b(-, w)$.

If V and W are finite-dimensional and $\{e_i\}$ is a basis of V and $\{E_i\}$ is a basis of W, then $\phi_1 \cdot b : e_i \mapsto a_{ij}\Xi^j$ where $\{\Xi^j\}$ is the basis dual to $\{E_i\}$. But $\phi_1 \cdot b : e_i \mapsto b(e_i, -)$, so $a_{ij}\Xi^j \cdot E_k = b(e_i, E_k)$ and $a_{ik} = b(e_i, E_k) = b_{ik}$. That is the ikth component of b in the $\{f^{ik}\}$ basis is the ikth element of the matrix of the linear map $\mathcal{A}_1$. Similarly, if $\phi_2 \cdot b : E_i \mapsto a_{ij}\varepsilon^j$, then $a_{ik} = b_{ki}$, which says that the ikth component of b in the $\{f^{ik}\}$ basis is the ikth element of the transpose of the matrix of the linear map $\mathcal{A}_2$.

More generally, we have the following.

Theorem 2.12 *There is a natural isomorphism between* $\mathfrak{L}(V_1, V_2, \ldots, V_p; W)$ *and* $\mathfrak{L}(V_i, \mathfrak{L}(V_1, \ldots, \hat{V}_i, \ldots, V_p; W))$. *($\hat{V}_i$ means that argument is to be omitted.)*

Proof Problem 2.13. □

On the basis of these and other similar types of isomorphisms (see, e.g., Problem 2.14), we will be able to describe tensor spaces in a variety of equivalent ways - which can be convenient and also confusing. Moreover, such isomorphisms are used for the construction of "vector-valued forms" which we will encounter in geometry (cf., Section 4.1).

PROBLEM 2.9. Prove Theorem 2.9.

PROBLEM 2.10. Prove the general result stated below Theorem 2.9.

PROBLEM 2.11. We mentioned above Theorem 2.8 that every bilinear function is represented by the mapping (2.2) when bases are chosen. What are the coefficients of (2.2) for the f^{ij} of Theorem 2.10?

PROBLEM 2.12. Verify the statements in Theorem 2.11 that $\psi \circ \phi =$ identity on $\mathfrak{L}(V_1, V_2; W)$ implies that ϕ is $1-1$, and $\phi \circ \psi =$ identity on $\mathfrak{L}(V_1 : \mathfrak{L}(V_2, W))$ implies that ϕ is onto.

PROBLEM 2.13. Prove Theorem 2.12.

PROBLEM 2.14. Prove that there is a natural isomorphism between $\mathfrak{L}(V_1, \ldots, V_p; W)$ and $\mathfrak{L}(V_{i_1}, \ldots, V_{i_s}; \mathfrak{L}(V_{i_{s+1}}, \ldots, V_{i_p}; W))$ where $i_1, \ldots, i_p$ is a permutation of $1, \ldots, p$.

PROBLEM 2.15. The cartesian product $W_1 \times \cdots \times W_n$ of vector spaces with the operations

$$\begin{aligned}(w_1, \ldots, w_n) + (x_1, \ldots, x_n) &= (w_1 + x_1, \ldots, w_n + x_n)\\ a(w_1, \ldots, w_n) &= (aw_1, \ldots, aw_n)\end{aligned}$$

is a vector space denoted by $W_1 \oplus W_2 \oplus \cdots \oplus W_n$ and called *the exterior direct sum* (see Problem 1.11). Notice that the sets $\tilde{W}_i = \{(0, \ldots, w_i, \ldots, 0)\}$ are subspaces of $W_1 \oplus \cdots \oplus W_n$ and $W_1 \oplus \cdots \oplus W_n$ is the (interior) direct sum of $\tilde{W}_i$ as defined in Section 1.1. Show that if $\dim V = n$, then $\mathfrak{L}(V, W) \cong W \oplus \cdots \oplus W$, the direct sum of n copies of W (see Theorem 2.4).

2.3 Nondegenerate bilinear functions

Starting with a vector space V, we constructed a second vector space V^* each of whose elements is a mapping from V to $\mathbb{R}$. Instead of fixing an element of V^* and letting v vary in V to get a mapping with values in $\mathbb{R}$, we can, as we did in Theorem 2.6, fix an element $v \in V$ and let f vary in V^* and get a mapping $f \mapsto f \cdot v$ with values in $\mathbb{R}$. These two mappings appear in more symmetrical roles as partial mappings of a mapping $V^* \times V \to \mathbb{R}$ of the cartesian product of V^* and V to $\mathbb{R}$.

Definition The function $\delta : V^* \times V \to \mathbb{R}$ defined by $(f, v) \mapsto f \cdot v$ is called *the Kronecker delta*, or *the natural pairing of* V^* *and* V *into* $\mathbb{R}$. We write $\delta(f, v) = \langle f, v\rangle$, and $\delta = \langle -, -\rangle$; i.e., $\langle f, v\rangle = f \cdot v$.

The partial function $\langle f, -\rangle : V \to \mathbb{R}$ given by $v \mapsto f \cdot v$ is the same as f itself. The partial function $\langle -, v\rangle : V^* \to \mathbb{R}$ given by $f \mapsto f \cdot v$ is the function $\bar{v}$ in Theorem 2.6. It is easy to see that δ is a bilinear function, i.e., $\delta \in \mathfrak{L}(V^*, V; \mathbb{R})$.

By Theorem 2.10, if V is finite-dimensional and $\{e_i\}$ and $\{\varepsilon^i\}$ are dual bases, then $f_i^j : (\varepsilon^k, e_l) \mapsto \delta_i^k \delta_l^j$ define a basis for $\mathfrak{L}(V^*, V; R)$ and

$$\delta = \delta_j^i f_i^j = f_1^1 + f_2^2 + \cdots + f_n^n$$

δ has a property which defines an important class of bilinear functions.

Definitions A bilinear function $b : V \times W \to \mathbb{R}$ is (weakly) *nondegenerate* if $b(v, w) = 0$ for all $v \in V$ implies $w = 0$, and $b(v, w) = 0$ for all $w \in W$ implies $v = 0$. If b is a nondegenerate bilinear function on $V \times W$, then V *and* W *are dual spaces with respect to* b.

The Kronecker delta is nondegenerate, for the fact that $\langle f, v\rangle = 0$ for all v implies $f = 0$ is just the definition of $f = 0$. The fact that $\langle f, v\rangle = 0$ for all f implies $v = 0$ was already used in Theorem 2.6. See Problem 2.7. Thus, the natural pairing $\langle -, -\rangle$ of V^* and V is just one particular nondegenerate bilinear function, and V^* and V are dual with respect to $\langle -, -\rangle$.

Recall that in Section 2.2. we gave two examples of bilinear functions. In the first example, the bilinear function is nondegenerate and $\mathbb{R}^n$ and $\mathbb{R}^m$ are dual with respect to this function if and only if $m = n$, and the matrix (a_{ij}) is nonsingular. In this case b is a nondegenerate bilinear function of the form $b : V \times V \to \mathbb{R}$ and V is *self-dual* (with respect to that bilinear function). We will study important examples in Section 5.4.

PROBLEM 2.16. Prove the necessary and sufficient conditions stated for the nondegeneracy of the bilinear function $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ of the example given above.

PROBLEM 2.17. If V and W are dual with respect to b, then the maps $\mathcal{A}_1$ and $\mathcal{A}_2$ in Section 2.2 are injective.

PROBLEM 2.18. Choose a basis in V. (i) Express $\langle f, v\rangle$ in terms of this basis and its dual in V^*. (ii) What are the matrices, relative to these bases, of the linear maps $\phi_1 \cdot b$ and $\phi_2 \cdot b$ in Section 2.2 where b is the natural pairing of V and V^*?

PROBLEM 2.19. Give an example of a vector space with two nonisomorphic dual spaces.

2.4 Orthogonal complements and the transpose of a linear mapping

Definitions Suppose b is a bilinear function on $V \times W$, and $S \subset V$ and $T \subset W$. $S^\perp = \{w \in W : b(v, w) = 0 \text{ for all } v \in S\}$ is called *the orthogonal complement*, or *annihilator of* S *with respect to* b. ${}^\perp T = \{v \in V : b(v, w) = 0 \text{ for all } w \in T\}$ is *the orthogonal complement*, or *annihilator of* T *with respect to* b.

Theorem 2.13 *For any set* $S \subset V, S^\perp$ *is a subspace of* W, *and for any set* $T \subset W, {}^\perp T$ *is a subspace of* V.

Proof Problem 2.20. □

Theorem 2.14 *$N_1 = {}^{\perp}W$ and $N_2 = V^{\perp}$ where $N_1 \subset V$ is the null space of $\phi_1 \cdot b$ and $N_2 \subset W$ is the null space of $\phi_2 \cdot b$.*

Proof Problem 2.21. □

Corollary *b is nondegenerate if and only if $N_1 = 0$ and $N_2 = 0$.*

Theorem 2.15 *If finite-dimensional vector spaces V and W are dual with respect to b, then $W \cong V^*$ and $V \cong W^*$.*

Proof Since $N_1 = N_2 = 0$ the linear maps $\phi_1 \cdot b$ and $\phi_2 \cdot b$ are $1-1$. Hence $\dim V \leq \dim W^* = \dim W \leq \dim V^* = \dim V$. So $\dim V = \dim W^*$ and $\dim W = \dim V^*$ and the maps are onto. □

Corollary *If V and W are dual with respect to b, then $\dim V = \dim W$.*

On the basis of Theorem 2.15 we frequently say that V^* is *the* dual space of V, and $V^{**} = V$ is *the* dual space of V^*.

(Note that in the infinite-dimensional case the conclusion of Theorem 2.15 is not necessarily valid. If it is, we say that b is strongly nondegenerate, or *nonsingular*.)

Theorem 2.16 *If $S \subset V$ and $T \subset W$, $\langle S \rangle \subset {}^{\perp}(S^{\perp})$ and $\langle T \rangle \subset ({}^{\perp}T)^{\perp}$.*

Proof If $v \in \langle S \rangle$, then for all $w \in S^{\perp}, b(v, w) = a_i b(v^i, w) = 0$ so $v \in {}^{\perp}(S^{\perp})$. Similarly for the second part. □

We can strengthen this result, and get other interesting relations in the case $W = V^*$ and, $b = \delta$. Hence, *the orthogonal complement will be with respect to δ from now on.*

Theorem 2.17 *If $S \subset V$, then $\langle S \rangle = {}^{\perp}(S^{\perp})$.*

Proof From Theorem 2.16 we know $\langle S \rangle \subset {}^{\perp}(S^{\perp})$. Now suppose $\tilde{v} \in {}^{\perp}(S^{\perp})$ and $\tilde{v} \notin \langle S \rangle$. We can choose the $\tilde{v}$ to be one of the basis elements of V by Theorem 1.1. Let $\tilde{f}$ be the element of V^* which maps $\tilde{v}$ to 1, and the other basis elements to zero. $\tilde{f}$ exists and is unique by Theorem 1.3. In particular, for all $v \in \langle S \rangle, \langle \tilde{f}, v \rangle = 0$. That is, $\tilde{f} \in \langle S \rangle^{\perp}$. But $S \subset \langle S \rangle$, so $\tilde{f} \in S^{\perp}$. Hence, for all $v \in {}^{\perp}(S^{\perp}), \langle \tilde{f}, v \rangle = 0$. In particular, since by assumption $\tilde{v} \in {}^{\perp}(S^{\perp})$, we must have $\langle \tilde{f}, \tilde{v} \rangle = 0$, but this contradicts $\tilde{f} \cdot \tilde{v} = 1$, so ${}^{\perp}(S)^{\perp} \subset \langle S \rangle$. □

Theorem 2.18 *If V is finite-dimensional and S is a subspace of V then $\dim S^{\perp} = \dim V - \dim S$ ($\perp$ is with respect to δ).*

Proof Problem 2.23. □

Now suppose we have two bilinear functions $b_1 : V_1 \times W_1 \to \mathbb{R}$ and $b_2 : V_2 \times W_2 \to \mathbb{R}$ and a linear mapping $\Phi : V_1 \to V_2$. Φ determines a *dual* linear mapping, Φ^*, from W_2 to W_1 by

$$b_1(v_1, \Phi^* \cdot w_2) = b_2(\Phi \cdot v_1, w_2) \tag{2.4}$$

We will confine ourselves to the special case $b_1 = b_2 = \delta$.

Definition If $\Phi : V \to W$, then the *transpose*, or *dual of* Φ is the map $\Phi^* : W^* \to V^*$ defined by $\Phi^* : w^* \mapsto w^* \circ \Phi$ where $w^* \in W^*$.

We can think of Φ^* as taking functions defined on W and "pulling them back" to functions defined on V.

$$\begin{tikzcd}
& \mathbb{R} & \\
V \arrow[ur, "\Phi^*(w^*) = w^* \circ \Phi"] \arrow[rr, "\Phi"'] & & W \arrow[ul, "w^*"']
\end{tikzcd}$$

Theorem 2.19 Φ^* *is linear.*

Proof We evaluate $\Phi^*(aw_1^* + bw_2^*)$ at an arbitrary $v \in V$. Thus,

$$\begin{aligned}
\Phi^*(aw_1^* + bw_2^*) \cdot v &= ((aw_1^* + bw_2^*) \circ \Phi) \cdot v \qquad \text{by definition of } \Phi^* \\
&= (aw_1^* + bw_2^*) \cdot (\Phi \cdot v) = a(w_1^* \cdot (\Phi \cdot v)) + b(w_2^* \cdot (\Phi \cdot v)) \\
&= a(\Phi^*(w_1^*) \cdot v) + b(\Phi^*(w_2^*) \cdot v) \quad \text{by definition of } \Phi^* \\
&= (a\Phi^*(w_1^*) + b\Phi^*(w_2^*)) \cdot v \\
&\qquad \text{by definition of operations in } V^*
\end{aligned}$$

So $\Phi^*(aw_1^* + bw_2^*) = a\Phi^*(w_1^*) + b\Phi^*(w_2^*)$. □

We discussed an example of a mapping and its transpose near the end of Section 2.2. More precisely, if $\mathcal{A}_1 = \phi_1 \cdot b : V \to W^*$ and $\mathcal{A}_2 = \phi_2 \cdot b : W \to V^*$, then $\mathcal{A}_2 = \mathcal{A}_1^*\big|_W$. That is, $(\mathcal{A}_2 \cdot w) \cdot v = (\mathcal{A}_1^* \cdot w^{**}) \cdot v$ when $w^{**} = w$ (by the isomorphism $\mathcal{I}$). This is seen by rewriting the right side: $(\mathcal{A}_1^* \cdot w^{**}) \cdot v = [w^{**} \circ (\phi_1 \cdot b)](v) = w^{**} \cdot w^* = w^* \cdot w = (\phi_1 \cdot b) \cdot v \cdot w = (\phi_2 \cdot b) \cdot w \cdot v$. Moreover, that discussion (in Section 2.2) shows that the transpose of the matrix of a linear map is the matrix of the transpose of that map.

Theorem 2.20 *If* $\Phi_1 : V \to W$ *and if* $\Phi_2 = \Phi_1^*$, *then* $\Phi_1 = \Phi_2^*\big|_V$.

Proof Problem 2.24. □

Theorem 2.21 Φ^* *is the transpose of* Φ *if and only if*

$$\langle \Phi^* \cdot w^*, v \rangle = \langle w^*, \Phi \cdot v \rangle \tag{2.5}$$

for all $v \in V$ *and* $w^* \in W^*$.

Proof If we evaluate $\Phi^* \cdot w^* = w^* \circ \Phi$ at v we get (2.5) immediately. □

Theorem 2.22 *If* $\Phi : V \to W$ *and* Φ^* *is its transpose, then*

(i) $\ker \Phi^* = (\Phi(V))^\perp$
(ii) $\Phi(V) = {}^\perp(\ker \Phi^*)$

(iii) $\ker \Phi = {}^{\perp}(\Phi^*(W^*))$
(iv) $\Phi^*(W^*) = \ker(\Phi)^{\perp}$
See Figure 2.2.

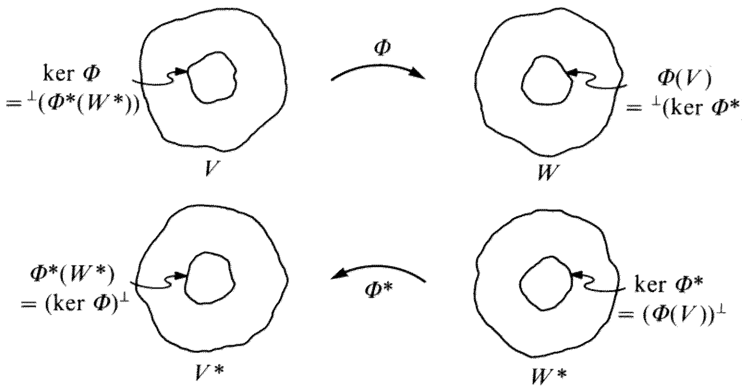

Figure 2.2

Proof (i) If $w^* \in \ker \Phi^*$, then $\Phi^* \cdot w^* = 0$ and by eq. (2.5) $\langle w^*, \Phi \cdot v \rangle = 0$ for all $v \in V$, so $w^* \in \Phi(V)^{\perp}$. On the other hand, if $w^* \in \Phi(V)^{\perp}$, then $\langle w^*, \Phi \cdot v \rangle = 0$ for all $v \in V$. By eq. (2.5) $\langle \Phi^* \cdot w^*, v \rangle = 0$ for all $v \in V$. Since $\langle -, - \rangle$ is nondegenerate $\Phi^* \cdot w^* = 0$ so $W^* \in \ker \Phi^*$.

(ii) By Theorem 2.17 $\Phi(V) = {}^{\perp}(\Phi(V)^{\perp})$ which is ${}^{\perp}(\ker \Phi^*)$ by part (i).

(iii) and (iv): Problem 2.26. $\square$

Corollary *Φ is onto if and only if Φ^* is $1-1$. Φ is $1-1$ if and only if Φ^* is onto.*

Some of the results above are the algebraic abstractions of important results in other areas of mathematics. Thus, e.g., Theorem 2.22(ii) has the interpretation that a nonhomogeneous system of linear equations has a solution if and only if the nonhomogeneous part is orthogonal to every solution of the adjoint homogeneous system. This is known as the "Fredholm Alternative" in the theory of integral equations, or, more generally, in functional analysis (A. Friedman, p. 189 ff).

PROBLEM 2.20. Prove Theorem 2.13.

PROBLEM 2.21. Prove Theorem 2.14.

PROBLEM 2.22. Prove that $\mathcal{I}(^{\perp}S^*) = \mathcal{I}(V) \cap {}^{\perp}(S^*)$ for $S^* \subset V^*$.

PROBLEM 2.23. Prove Theorem 2.18.

PROBLEM 2.24. Prove Theorem 2.20.

PROBLEM 2.25. In the finite-dimensional case with given bases, write (2.4) in matrix notation and find the relation between the matrices of Φ and Φ^*.

PROBLEM 2.26. Prove parts (iii) and (iv) of Theorem 2.22.

PROBLEM 2.27. Using parts (iii) and (iv) of Theorem 2.22 prove that if $T \subset W$, then $\langle T \rangle = (^{\perp}T)^{\perp}$.

PROBLEM 2.28. In the general case of the dual linear mapping, Φ^*, prove that Φ^* satisfies parts (i) and (iii) of Theorem 2.22.

PROBLEM 2.29. (Abstract homology, cf., Whitney, p. 346 ff or Greub, p. 178 ff). Suppose $W = V$ and Φ is a linear transformation on V with the property $\Phi^2 = 0$. (Φ is called *a differential operator*). Show (i) $\Phi(V) \subset \ker \Phi$, (ii) $\Phi^{*2} = 0$, (iii) $\Phi^*(V^*) \subset \ker \Phi^*$. Finally, the factor spaces $\ker \Phi / \Phi(V)$ and $\ker \Phi^* / \Phi^*(V^*)$, called the *homology* and *cohomology* spaces of V, respectively, are isomorphic. These ideas will arise in a more specific context in Chapter 12.

3

TENSOR PRODUCT SPACES

We will describe the space of bilinear functions on a pair of finite-dimensional vector spaces as the tensor product space of the duals of those two vector spaces. We then consider tensor product spaces of more than two vector spaces. Finally we define tensor product spaces in the general case so that they reduce to spaces of multilinear functions in finite dimensions. From now on we will denote vectors by $v, w, \ldots$, which may have subscripts, and linear functions by Greek letters $\sigma, \tau, \ldots$, which may have superscripts.

3.1 The tensor product of two finite-dimensional vector spaces

Consider the set of bilinear functions $\mathcal{L}(V, W; \mathbb{R})$, with V and W finite-dimensional. Notice that this set has certain elements of the following type. Let $\sigma \in V^*$ and $\tau \in W^*$. Then with each fixed pair $(\sigma, \tau) \in V^* \times W^*$ we have the function $V \times W \to \mathbb{R}$ given by

$$(v, w) \mapsto (\sigma \cdot v)(\tau \cdot w) \tag{3.1}$$

By the linearity of σ and τ this function is bilinear, so it is in $\mathcal{L}(V, W; \mathbb{R})$. (See example (ii) in Section 2.2.) Since the values of this function are products of the values of σ and τ, we use the product notation $\sigma \otimes \tau$ for this function. That is, $\sigma \otimes \tau : V \times W \to \mathbb{R}$ is the element of $\mathcal{L}(V, W; \mathbb{R})$ given by (3.1).

For an example, consider the case where $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$. Then $\sigma \in \mathbb{R}^{n*}$ will have the form $\sigma : (v^1, \ldots, v^n) \mapsto \sigma_i v^i$, and $\tau \in \mathbb{R}^{m*}$ will have the form $\tau : (w^1, \ldots, w^m) \mapsto \tau_i w^i$. (See below Theorem 2.4.) According to (3.1) $\sigma \otimes \tau$ will be given by $\sigma \otimes \tau : ((v^1, \ldots, v^n), (w^1, \ldots, w^m)) \mapsto \sigma_i \tau_j v^i w^j$. Note that this is a special case of example (i) in Section 2.2. Note also, that with bases in V and W every element in $\mathfrak{L}(V, W; \mathbb{R})$ of the form $\sigma \otimes \tau$ can be represented by such examples.

Now, for each pair $(\sigma, \tau) \in V^* \times W^*$ we have an element $\sigma \otimes \tau$ of $\mathfrak{L}(V, W; \mathbb{R})$. Thus, $\mathfrak{L}(V, W; \mathbb{R})$ contains a set of "products", the image set of a mapping

$$\otimes : V^* \times W^* \to \mathfrak{L}(V, W; \mathbb{R})$$

given by

$$(\sigma, \tau) \mapsto \sigma \otimes \tau \tag{3.2}$$

One easily verifies that $\otimes$ is bilinear. Thus, $\sigma \otimes \tau$ plays two roles: it is a bilinear function given by (3.1), and it is in the image of a bilinear map, $\otimes$, given by (3.2).

The image set of $\otimes$ is not a subspace of $\mathfrak{L}(V, W; \mathbb{R})$ and, in particular, not $\mathfrak{L}(V, W; \mathbb{R})$, since $\sigma^1 \otimes \tau^1 + \sigma^2 \otimes \tau^2$ is not always of the form $\sigma \otimes \tau$. However, when V and W are finite-dimensional spaces, if $\{\varepsilon^i\}$ is a basis of V^* and $\{\Xi^i\}$ is a basis of W^*, then $\{\varepsilon^i \otimes \Xi^j\} = \{f^{ij}\}$ is a basis of $\mathfrak{L}(V, W; \mathbb{R})$ (cf., Theorem 2.10), so the set of products does span $\mathfrak{L}(V, W; \mathbb{R})$. This accounts for the following terminology and notation.

Definition If V and W are finite-dimensional spaces, then the vector space $\mathfrak{L}(V, W; \mathbb{R})$ is called *the tensor* (or *Kronecker*) *product* of V^* and W^*. We write $\mathfrak{L}(V, W; \mathbb{R}) = V^* \otimes W^*$. (Note that we are using the notation $\otimes$ in three different ways.)

Similarly, since in $\mathfrak{L}(V^*, W^*; \mathbb{R})$ we have the "product" elements

$$v \otimes w : (\sigma, \tau) \mapsto (\sigma \cdot v)(\tau \cdot w),$$

(cf., corollary of Theorem 2.7), the vector space $\mathfrak{L}(V^*, W^*; \mathbb{R})$ is the *tensor* (or *Kronecker*) *product* of V and W, and we write $\mathfrak{L}(V^*, W^*; \mathbb{R}) = V \otimes W$.

$\mathfrak{L}(V, V^*; \mathbb{R}) = V^* \otimes V$ is an example of a tensor product space. So is $\mathfrak{L}(V^*, V; \mathbb{R}) = V \otimes V^*$. The natural pairing, $\langle -, - \rangle$ (Section 2.3) is an element of $V \otimes V^*$.

Theorem 3.1 *Given two finite-dimensional vector spaces V and W. Form $V \otimes W = \mathfrak{L}(V^*, W^*; \mathbb{R})$ and the map $\otimes : V \times W \to V \otimes W$ with $\otimes(v, w)$ defined as above. The pair $(V \otimes W, \otimes)$ has the following properties:*

(i) $\otimes$ is bilinear, and $\otimes(V \times W)$ spans $V \otimes W$

(ii) If Z is any vector space, and b is any bilinear map, $b : V \times W \to Z$, then there exists a unique linear map, $\phi : V \otimes W \to Z$, onto $\langle b(V \times W) \rangle$, the linear closure of the range of b, and such that $b = \phi \circ \otimes$; i.e., any bilinear map, b, has a unique factorization into the product of $\otimes$ and a linear map. (The fact that every bilinear map, b, can be factored into a linear map and a particular unique map is referred to as the universal property of bilinear maps.)

Proof (i) These two properties of $\otimes : V \times W \to V \otimes W$ are precisely analogous to those of the mapping given by eq. (3.2). (ii) Let $\{v_i \otimes w_j\}$ be a basis of $V \otimes W$. Define ϕ by mapping $v_i \otimes w_j \mapsto b(v_i, w_j)$ and then extending this correspondence to all of $V \otimes W$ by linearity. Every $z \in \langle b(V \times W) \rangle$ can be written $z = a^{ij} b(v_i, w_j)$. But this is the image of $a^{ij} v_i \otimes w_j$ under ϕ, so ϕ is onto $\langle b(V \times W) \rangle$. If $\phi_1 \circ \otimes = \phi_2 \circ \otimes$, then $(\phi_1 - \phi_2) \cdot \otimes(v, w) = 0$ for all v, w. But, since $\{\otimes(v, w) = v \otimes w : v \in V, w \in W\}$ spans $V \otimes W, (\phi_1 - \phi_2) \cdot A = 0$ for all $A \in V \otimes W$, which implies that $\phi_1 - \phi_2$. $\square$

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "b"'] & V \otimes W \arrow[d, "\phi"] \\ & Z \end{tikzcd}$$

Corollary *For a bilinear map, b, in Theorem 3.1 (ii) with the additional property that the dimension of the linear closure $\langle b(V \times W) \rangle$ of its range is* $\dim V \times W$, *the corresponding linear map, ϕ, is an isomorphism of $V \times W$ with $\langle b(V \times W) \rangle$.*

Proof By the theorem, ϕ is onto $\langle b(V \times W) \rangle$. This plus the fact that $\dim \langle b(V \times W) \rangle = \dim V \times W$ makes ϕ an isomorphism. □

Corollary *Given spaces V, W, and Z, there is a natural isomorphism between the space $\mathfrak{L}(V, W; Z)$ of bilinear mappings and the space $\mathfrak{L}(V \otimes W, Z)$ of linear mappings.*

Proof Problem 3.4. □

At the beginning of this section we saw that if $\{\varepsilon^i\}$ is a basis of V^* and $\{\Xi^i\}$ is a basis of W^*, then $\{\varepsilon^i \otimes \Xi^j\}$ form a basis of $V^* \otimes W^*$, so for any $A \in V^* \otimes W^*$ we can write

$$A = A_{ij} \varepsilon^i \otimes \Xi^j \tag{3.3}$$

and since, by eq. (3.1), $\varepsilon^i \otimes \Xi^j (v, w) = v^i w^j$, A has values

$$A(v, w) = A_{ij} v^i w^j. \tag{3.4}$$

Similarly $\{e_i \otimes E_j\}$ form a basis of $V \otimes W$, so for any $A \in V \otimes W$,

$$A = A^{ij} e_i \otimes E_j \tag{3.5}$$

and, since $e_i \otimes E_j(\sigma, \tau) = \sigma_i \tau_j$, A has values

$$A(\sigma, \tau) = A^{ij} \sigma_i \tau_j. \tag{3.6}$$

PROBLEM 3.1. Write the explicit form of the product $v \otimes w \in V \otimes W$ as a bilinear function if $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$.

PROBLEM 3.2. Give a specific example in which $\sigma^1 \otimes \tau^1 + \sigma^2 \otimes \tau^2$ in $V^* \otimes W^*$ is not a product.

PROBLEM 3.3. Prove part (i) of Theorem 3.1.

PROBLEM 3.4. Prove the second corollary of Theorem 3.1.

3.2 Generalizations, isomorphisms, and a characterization

We can generalize the development of the last section and construct tensor products of any finite number of vector spaces. For example, we note that with every triple $(\sigma, \tau, \omega) \in V^* \times W^* \times X^*$ we have an element of $\mathfrak{L}(V, W, X; \mathbb{R})$

$$(v, w, x) \mapsto (\sigma \cdot v)(\tau \cdot w)(\omega \cdot x)$$

and we call this function $\sigma \otimes \tau \otimes \omega$. That is, we have a map $\otimes : V^* \times W^* \times X^* \to \mathfrak{L}(V, W, X; \mathbb{R})$ whose images are of the form $\sigma \otimes \tau \otimes \omega$. We have a theorem analogous to Theorem 3.1. In particular, $\otimes$ is trilinear and $\otimes(V^* \times W^* \times X^*)$ spans $\mathfrak{L}(V, W, X; \mathbb{R})$, which we now denote by $V^* \otimes W^* \otimes X^*$. Also, any trilinear map has a unique factorization into the product of $\otimes$ and a linear map.

We can generalize eqs. (3.3) - (3.6). In particular, if $\{\varepsilon_1^i\}$, $\{\varepsilon_2^i\}$, and $\{\varepsilon_3^i\}$ are bases respectively of V^*, W^*, and X^*, then $\{\varepsilon_1^i \otimes \varepsilon_2^j \otimes \varepsilon_3^k\}$ is a basis of $V^* \otimes W^* \otimes X^*$ and for any $A \in V^* \otimes W^* \otimes X^*$,

$$A = A_{ijk}\varepsilon_1^i \otimes \varepsilon_2^j \otimes \varepsilon_3^k \tag{3.7}$$

and A has values

$$A(v, w, x) = A_{ijk} v^i w^j x^k \tag{3.8}$$

Generally, given m vector spaces $V_1, \ldots, V_m$, we can construct $V_1 \otimes \cdots \otimes V_m$. If $\{e_{i_1}^1\}, \ldots, \{e_{i_m}^m\}$ are bases of $V_1, \ldots, V_m (i_k = 1, \ldots, n_k$ where $n_k = \dim V_k)$ then $\{e_{i_1}^1 \otimes \cdots \otimes e_{i_m}^m\}$ is a basis of $V_1 \otimes \cdots \otimes V_m$ and for any $A \in V_1 \otimes \cdots \otimes V_m$,

$$A = A^{i_1 \cdots i_m} e_{i_1}^1 \otimes \cdots \otimes e_{i_m}^m \tag{3.9}$$

and A has values

$$A(\sigma^1, \ldots, \sigma^m) = A^{i_1 \cdots i_m} \sigma_{i_1}^1 \cdots \sigma_{i_m}^m \tag{3.10}$$

Note that $\dim V_1 \otimes \cdots \otimes V_m = n_1 \cdots n_m$. This is a special case of the general result stated above Theorem 2.10. (See Problem 2.10.)

Since we can construct the tensor product of any two spaces, we can form the tensor product of tensor products. In general, the operations of taking tensor products and/or duals can be iterated, resulting in an apparently bewildering proliferation of vector spaces. The following theorems help to keep things under control.

Theorem 3.2 *The vector spaces $V \otimes W \otimes X$, $(V \otimes W) \otimes X$, and $V \otimes (W \otimes X)$ are naturally isomorphic.*

Proof A bilinear map $b : (V \otimes W) \times X \to V \otimes W \otimes X$ is determined by $(v \otimes w, x) \mapsto v \otimes w \otimes x$. (For each fixed $v \otimes w$ this prescription determines a linear map from X to $V \otimes W \otimes X$, and for each fixed x, by Theorem 3.1, since $(v, w) \mapsto v \otimes w \otimes x$ is bilinear, it determines a linear map from $V \otimes W$ to

$V \otimes W \otimes X$.) Then by Theorem 3.1 we have the linear map $\phi : v(\otimes w) \otimes x \mapsto v \otimes w \otimes x$.

$$\begin{tikzcd}
(V \otimes W) \times X \arrow[r, "\otimes"] \arrow[dr, "b"'] & (V \otimes W) \otimes X \arrow[d, "\phi"] \\
& V \otimes W \otimes X
\end{tikzcd}$$

From the trilinear map $t : V \times W \times X \to (V \otimes W) \otimes X$ given by $(v, w, x) \mapsto (v \otimes w) \otimes x$ we have the linear map $\psi : v \otimes w \otimes x \mapsto (v \otimes w) \otimes x$

$$\begin{tikzcd}
V \times W \times X \arrow[r, "\otimes"] \arrow[dr, "t"'] & V \otimes W \otimes X \arrow[d, "\psi"] \\
& (V \otimes W) \otimes X
\end{tikzcd}$$

But $\phi \circ \psi$ and $\psi \circ \phi$ are both identities so ϕ is an isomorphism. A similar argument shows that $V \otimes (W \otimes X)$ and $V \otimes X \otimes X$ are isomorphic. □

Theorem 3.3 *$V \otimes W$ and $W \otimes V$ are naturally isomorphic.*

Proof Problem 3.7. □

We saw that $\otimes$ is bilinear, and, in particular, distributive with respect to $+$. We might be tempted to infer from Theorems 3.2 and 3.3 that $\otimes$ is also associative and commutative. However, in general, these properties are not even defined for $\otimes$ and, when they are, they are not generally true (see Problem 3.8). When we define a closely related mapping in Section 4.3 this situation will be partially rectified (see Theorem 4.6).

Theorem 3.4 *$(V \otimes W)^*$ is naturally isomorphic to $V^* \otimes W^*$.*

Proof The result follows immediately from the second corollary of Theorem 3.1 for the case $Z = \mathbb{R}$. □

Corollary *$(V^* \otimes W^*)^*$ is naturally isomorphic to $V \otimes W$.*

Theorem 3.4 implies that $V \otimes W$ and $V^* \otimes W^*$ are dual spaces with respect to the natural pairing. It is an example of "duality" in tensor product spaces (see Section 4.1). It is important to note that Theorem 3.4 is not necessarily valid if V and W are not both finite-dimensional.

We have seen that when V and W are finite-dimensional, the pair $(V \otimes W, \otimes)$ has the properties listed in Theorem 3.1, which with its corollaries, in turn, lead to the isomorphisms described above. Now suppose we abstract this situation a bit; suppose V, W, and X are any vector spaces, and $\mathfrak{F}$ is any map $V \times W \to X$, and suppose the pair $(X, \mathfrak{F})$ satisfies the properties listed in Theorem 3.1. It turns out that $(X, \mathfrak{F})$ is really no more general than $(V \otimes W, \otimes)$; that is, the properties of Theorem 3.1 essentially uniquely determine the pair $(X, \mathfrak{F})$.

Theorem 3.5 *If $(X, \mathfrak{F})$ has the properties of Theorem 3.1, then $X \cong V \otimes W$ and $\mathfrak{F} = \phi \circ \otimes$ where ϕ is the isomorphism $V \otimes W \to X$.*

Proof Letting $\mathfrak{F}$ have the role of b in Theorem 3.1 we have $\mathfrak{F} = \phi \circ \otimes$ where $\phi : V \otimes W \to X$. Then letting $\otimes$ have the role of b in Theorem 3.1 we have $\otimes = \psi \circ \mathfrak{F}$ where $\psi : X \to V \otimes W$. Combining these two equalities we get $\mathfrak{F} = \phi \circ \psi \circ \mathfrak{F}$ and $\otimes = \psi \circ \phi \circ \otimes$. Since $\mathfrak{F}(V \times W)$ spans $X, \phi \circ \psi$ is the identity on X, so ϕ is onto. Since $\otimes(V \times W)$ spans $V \otimes W, \psi \circ \phi$ is the identity on $V \otimes W$ so ϕ is $1 - 1$. Hence ϕ is an isomorphism. □

On the basis of Theorem 3.5 we can now say that given two finite-dimensional vector spaces, there is one and only one tensor product space with the properties of Theorem 3.1, and that tensor product space is the space $V \otimes W$ defined in Section 3.1. The major significance, however, of the characterization of Theorem 3.1 is that it can be extended to any two vector spaces, not necessarily finite-dimensional. The uniqueness requirement is shown exactly as in the proof of Theorem 3.5, as in that proof the nature of $V \otimes W$ is irrelevant. The existence requirement will be satisfied by construction, in the next section. Clearly, the constructed space will have to reduce to $V \otimes W = \mathfrak{L}(V^*, W^*; \mathbb{R})$ in the finite-dimensional case.

PROBLEM 3.5. Prove that $\mathfrak{L}(V_1, \ldots, V_n, W^*; \mathbb{R}) \cong \mathfrak{L}(V_1 \otimes \cdots \otimes V_n, W)$.

PROBLEM 3.6. The proof of Theorem 3.2 requires only that the tensor products satisfy Theorem 3.1 and its generalizations. Hence it is valid for arbitrary vector spaces (not necessarily finite-dimensional). Give a simpler proof of Theorem 3.2 by using Theorem 3.4. (Note, however, that Theorem 3.4 is only valid in the finite-dimensional case.)

PROBLEM 3.7. Prove Theorem 3.3 for arbitrary vector spaces (not necessarily finite-dimensional).

PROBLEM 3.8. If V and W are the same, then $V \otimes W$ and $W \otimes V$ are the same space, but in general $v \otimes w \neq w \otimes v$.

PROBLEM 3.9. Generalize Theorem 3.4 to $(V_1 \otimes V_2 \otimes \cdots \otimes V_p)^* \cong V_1^* \otimes V_2^* \otimes \cdots \otimes V_p^*$.

3.3 Tensor products of infinite-dimensional vector spaces

In Section 3.1, we defined tensor products of finite-dimensional vector spaces. In the general case where V and/or W may not be finite-dimensional we still have the maps $\otimes : V^* \times W^* \to \mathfrak{L}(V, W; \mathbb{R})$ given by eq. (3.2) and $\otimes : V \times W \to \mathfrak{L}(V^*, W^*; \mathbb{R})$ in Theorem 3.1. However, now the sets $\otimes(V^* \times W^*)$ and $\otimes(V \times W)$ do not necessarily span the spaces $\mathfrak{L}(V, W; \mathbb{R})$ and $\mathfrak{L}(V^*, W^*; \mathbb{R})$. These latter spaces may be too large.

It turns out that there are a space and a mapping which satisfy Theorem 3.1. As we have already pointed out, this can be the only such pair. We will denote it by $(V \otimes W, \otimes)$. That is, the tensor product of V and W, $(V \otimes W, \otimes)$, will be defined by the properties of Theorem 3.1.

Given V and W to get a space $V \otimes W$, and a mapping, $\otimes$ which satisfy Theorem 3.1, we start out by considering the vector space, $\mathbb{R}_0^{V \times W}$, of all real-valued functions with domain $V \times W$ which vanish at all but a finite number of points. As we saw in Section 1.1, there is a $1-1$ correspondence between elements $(v, w) \in V \times W$ and functions $f_{(v,w)} \in \mathbb{R}_0^{V \times W}$ which are 1 on (v, w) and 0 otherwise, and $\{f_{(v,w)} : (v, w) \in V \times W\}$ is the basis of $\mathbb{R}_0^{V \times W}$. We can think of $V \times W \subset \mathbb{R}_0^{V \times W}$, and we will use the notation (v, w) for $f_{(v,w)}$.

Now consider the set of all elements of $\mathbb{R}_0^{V \times W}$ of the form

$$\begin{aligned}(a^1 v_1 + a^2 v_2, b^1 w_1 + b^2 w_2) &- a^1 b^1 (v_1, w_1) \\ &- a^1 b^2 (v_1, w_2) - a^2 b^1 (v_2, w_1) - a^2 b^2 (v_2, w_2)\end{aligned}$$

with $v_1, v_2 \in V$, $w_1, w_2 \in W$ and $a^1, a^2, b^1, b^2 \in \mathbb{R}$. They generate a subspace, N. Then we put

$$V \otimes W = \mathbb{R}_0^{V \times W} / N.$$

That is, $V \otimes W$ is a set of equivalence classes of functions from $V \times W$ to $\mathbb{R}$. The map

$$\otimes = \Pi\big|_{V \times W}$$

is the restriction to $V \times W$ of the projection $\Pi : \mathbb{R}_0^{V \times W} \to \mathbb{R}_0^{V \times W}/N$.

Theorem 3.6 *The pair $(V \otimes W, \otimes)$ defined above has the properties of Theorem 3.1.*

Proof (i) Since Π is linear and maps all of N to 0,

$$\begin{aligned}\Pi(a^1 v_1 + a^2 v_2, b^1 w_1 + b^2 w_2) = a^1 b^1 \Pi(v_1, w_1) &+ a^1 b^2 \Pi(v_1, w_2) \\ &+ a^2 b^1 \Pi(v_2, w_1) + a^2 b_2 \Pi(v_2, w_2)\end{aligned}$$

so $\otimes(a^1 v_1 + a^2 v_2, b^1 w_1 + b^2 w_2) = a^1 b^1 \otimes (v_1, w_2) + \cdots$ for all $v_1, v_2 \in V$, $w_1, w_2 \in W$, and $a^1, a^2, b^1, b^2 \in \mathbb{R}$. That is, $\otimes$ is bilinear. Also, since Π is linear and $V \times W$ spans $\mathbb{R}_0^{V \times W}$, $\otimes(V \times W) = \Pi(V \times W)$ spans $V \otimes W$.

(ii) Now suppose we have $b : V \times W \to Z$. We define a linear map $\Phi : \mathbb{R}_0^{V \times W} \to Z$ by $a^{ij}(v_i, w_j) \mapsto a^{ij} b(v_i, w_j)$. Then we define a map $\phi : V \otimes W \to Z$

by $\phi : V \otimes W \to Z$ by $\phi : [f] \mapsto \Phi(f)$ where $f \in \mathbb{R}_0^{V \times W}$ and $[f]$ is the equivalence class of f. It remains only to check that ϕ is well-defined, and that ϕ satisfies the four properties of part (ii) of Theorem 3.1; namely, uniqueness, linearity, surjectivity, and factoring of b. We leave these details for the problem at the end of this section. □

The first corollary of Theorem 3.1 is not meaningful in the infinite-dimensional case. The second corollary is valid in general. Theorems 3.2 and 3.3 - "associativity" and "commutativity" are valid in general.

Theorem 3.4 is not valid in general. From the second corollary of Theorem 3.1 with $Z = \mathbb{R}$ we get $(V \otimes W)^* \cong \mathfrak{L}(V, W; \mathbb{R})$, and, as we noted above, $\mathfrak{L}(V, W; \mathbb{R})$ can be larger than $V^* \otimes W^*$ in the infinite-dimensional case, so $(V \otimes W)^*$ can be larger than $V^* \otimes W^*$.

We can generalize Theorem 3.6 as follows.

Theorem 3.7 *Given n vector spaces $V_1, \ldots, V_n$, there exists an essentially unique vector space $V_1 \otimes \cdots \otimes V_n$ and mapping*

$$\otimes : V_1 \times \cdots \times V_n \to V_1 \otimes \cdots \otimes V_n$$

with the properties

(i) *$\otimes$ is n-linear, and $\otimes(V_1 \times \cdots \times V_n)$ spans $V_1 \otimes \cdots \otimes V_n$.*

(ii) *If Z is any vector space, and Ψ is any n-linear mapping, $\Psi : V_1 \times \cdots \times V_n \to Z$, then there exists a unique linear map $\phi : V_1 \otimes \cdots \otimes V_n \to Z$, onto $\langle \Psi(V_1 \times \cdots \times V_n) \rangle$, and such that $\Psi = \phi \circ \otimes$.*

Proof Problem 3.14. □

Thus far, as we have gone along, we have tried to focus some attention on infinite-dimensional vector spaces as well as finite-dimensional ones and point out some similarities and some differences. We did this because, while "classical" tensor analysis is restricted to finite dimensions, the infinite-dimensional spaces are important in many applications; for example, applications in which it is required to do tensor analysis on Banach or Frechet manifolds. (See, e.g., Marsden.) However, *in the following chapters we will assume, unless otherwise explicitly stated, that all our vector spaces are finite-dimensional.*

PROBLEM 3.10. Prove that the map ϕ in Theorem 3.6 is well-defined.

PROBLEM 3.11. Prove that the map ϕ in Theorem 3.6 has the four required properties.

PROBLEM 3.12. Denote the image of (v, w) under $\otimes$ by $v \otimes w$. With each element $v \otimes w$ of $V \otimes W$ we can associate a bilinear function $b_{v \otimes w}$ given by

$(\sigma, \tau) \mapsto (\sigma \cdot v)(\tau \cdot w)$. (i) Prove that this assignment defines an isomorphism of $V \otimes W$ with a subspace, Z, of $\mathfrak{L}(V^*, W^*; \mathbb{R})$. (ii) Prove that if V and W are finite-dimensional, then $Z = \mathfrak{L}(V^*, W^*; \mathbb{R})$.

PROBLEM 3.13. We can get a direct proof of $V^* \otimes W^* \subset (V \otimes W)^*$ as a nice application of Theorem 3.1. (Hint: Apply Theorem 3.1 for $V \otimes W \otimes V^* \otimes W^*$, choose b (not bilinear, but 4-linear) and show that for each $A \in V^* \otimes W^*$ there is a linear function $V \otimes W \to \mathbb{R}$).

PROBLEM 3.14. Prove Theorem 3.7.

4

TENSORS

We restrict our attention to tensor products of a single vector space, V, with itself and with products of V^*. We look, in these special cases, at the isomorphisms, and representations in terms of components we discussed in Chapters 2 and 3. Finally, we will describe several important mappings of these spaces.

4.1 Definitions and alternative interpretations

We can form a variety of tensor products all based on a single given space, V. Thus, we can form

$$V_0^r = \underset{r \text{ factors}}{V \otimes \cdots \otimes V} \quad \text{the contravariant tensor product spaces}$$

$$V_s^0 = \underset{s \text{ factors}}{V^* \otimes \cdots \otimes V^*} \quad \text{the covariant tensor product spaces}$$

and

$$V_s^r = \underbrace{V \otimes \cdots \otimes V}_{r \text{ factors}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s \text{ factors}} \quad \text{the mixed tensor product spaces.}$$

Recall that when V is finite-dimensional,

$$\underbrace{V \otimes \cdots \otimes V}_{r} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s} \cong \mathfrak{L}(\underbrace{V^*, \ldots, V^*}_{r}, \underbrace{V, \ldots, V}_{s}; \mathbb{R})$$

We will restrict ourselves to this case in the sequel (unless otherwise stated).

We could also form products containing both V's and V^*'s with V's and V^*'s in different orders from those in V_s^r. However, Theorems 3.2 and 3.3 give us some justification for ignoring these. In particular, we have $V_s^r \otimes V_q^p \cong V_{s+q}^{r+p}$. So we will now restrict ourselves to the tensor product spaces V_s^r, including the special cases V_0^r, V_s^0 and the space $V_0^0 \equiv \mathbb{R}$. Note that $\dim V_s^r = n^{r+s}$ if $\dim V = n$.

Definitions A *tensor of type* (r, s) is an element of V_s^r. It is *contravariant of degree* (*order*, or *rank*) r, and *covariant of degree* (*order*, or *rank*) s. If $r = 1$ and $s = 0$, and element of $V_0^1 (= \mathfrak{L}(V^*, \mathbb{R}) = V)$ is a *vector*. If $r = 0$ and $s = 1$ an element of $V_1^0 (= \mathfrak{L}(V, \mathbb{R}) = V^*)$ is a *covector* or *1-form* (or, *linear form, function,* or *functional*). An element of V_s^r of the form $v_1 \otimes v_2 \otimes \cdots \otimes v_r \otimes \sigma^1 \otimes \cdots \otimes \sigma^s$ is called *decomposable*.

In addition to restricting the possible types of tensor product spaces based on a single vector space, V, to the spaces V_s^r, the natural isomorphisms, Theorems 3.2 - 3.4, of the last chapter give us a certain "duality" in the set of spaces V_s^r when V is finite-dimensional. That is, in addition to the duality of V and V^*, and, from Theorem 3.4 and its Corollary, the duality of V_0^2 and V_2^0, we have

$$(\underbrace{V\otimes\cdots\otimes V}_{r} \;\otimes\; \underbrace{V^*\otimes\cdots\otimes V^*}_{s})^* \;\cong\; (\underbrace{V^*\otimes\cdots\otimes V^*}_{r}, \;\otimes\; \underbrace{V\otimes\cdots\otimes V}_{s}$$

$$\cong\; \underbrace{V\otimes\cdots\otimes V}_{s} \;\otimes\; \underbrace{V^*\otimes\cdots\otimes V^*}_{r}$$

so $(V_s^r)^* \cong V_r^s$, and V_s^r and V_r^s are dual spaces with respect to the natural pairing. That is, for $A \in V_s^r$ and $B^* \in V_r^s = (V_s^r)^*$ we have the natural pairing $(A, B^*) \mapsto \langle B^*, A\rangle$.

Recall that, in addition to the various natural isomorphisms between tensor product spaces which we observed in the last chapter, we had, in Section 2.2, Theorems 2.11 and 2.12, and Problem 2.14, natural isomorphisms between certain spaces of multilinear mappings. These latter give us alternative interpretations for the spaces V_s^r which are especially important for small values of r and s. We have the following explicit results for V_1^1, V_0^2, and V_2^0.

Theorem 4.1 (i) $V_0^2 \cong \mathfrak{L}(V^*, V)$
(ii) $V_1^1 \cong \mathfrak{L}(V, V) \cong \mathfrak{L}(V^*, V^*)$
(iii) $V_2^0 \cong \mathfrak{L}(V, V^*)$

Proof (i) In Theorem 2.11 put $V_1 = V_2 = V$ and $W = \mathbb{R}$ and by the corollary of Theorem 2.7 we can put $\mathfrak{L}(V^*, \mathbb{R}) = V$. (ii) and (iii): Problem 4.1. □

Corollary *A tensor of type* $(2,0), (1,1)$, *or* $(0,2)$ *is nondegenerate if and only if the corresponding linear mappings are isomorphisms* (cf., Corollary of Theorems 2.14 and 2.15).

For larger values of r and s the isomorphisms, and hence the alternative interpretations, proliferate rapidly. As one illustration, consider the space $V_2^1 = V\otimes V^*\otimes V^*$. We have the following examples.

(i) $V\otimes V^*\otimes V^* \cong \mathfrak{L}(V, V\otimes V^*)$; that is, according to Theorem 2.12, $\mathfrak{L}(V^*, V, V; \mathbb{R}) \cong \mathfrak{L}(V, \mathfrak{L}(V^*, V; \mathbb{R}))$, if $A \in V\otimes V^*\otimes V^*$, then the isomorphism, ϕ, is given by $\phi : A \mapsto \phi(A) \in \mathfrak{L}(V, \mathfrak{L}(V^*, V; \mathbb{R}))$ where $\phi(A)\cdot v \in \mathfrak{L}(V^*, V; \mathbb{R})$ is given by

$$(\phi(A)\cdot v)(\sigma, w) = A(\sigma, v, w) \tag{4.1}$$

(ii) $V\otimes V^*\otimes V^* \cong \mathfrak{L}(V, V; V)$; that is according to Problem 2.14 and Theorem 2.12, $\mathfrak{L}(V^*, V, V; \mathbb{R}) \cong \mathfrak{L}(V, V; \mathfrak{L}(V^*, \mathbb{R}))$. If $A \in V\otimes V^*\otimes V^*$ then the isomorphism, ϕ, is given by $\phi : A \mapsto \phi(A) \in \mathfrak{L}(V, V; \mathfrak{L}(V^*, \mathbb{R}))$ where $\phi(A)(v, w) \in \mathfrak{L}(V^*, \mathbb{R})$ is given by

$$(\phi(A)(v, w)) \cdot \sigma = A(\sigma, v, w) \tag{4.2}$$

In example (i) $\phi(A)$ is a (1,1) tensor valued, or linear operator-valued, 1-form. In example (ii) $\phi(A)$ is a vector-valued second-order covariant tensor, an important special case of which is a vector-valued "exterior 2-form" (Section 5.3). Important applications of both of these examples arise in geometry, Chapter 16. We will also have to deal with vector-valued 1-forms, and linear operator-valued exterior 2-forms.

PROBLEM 4.1 Prove parts (ii) and (iii) of Theorem 4.1. Exhibit the isomorphisms.

PROBLEM 4.2. Describe the linear transformations corresponding to $\langle -, - \rangle$ ($\langle -, - \rangle \in V_1^1$).

PROBLEM 4.3. Show that the elements of $\mathfrak{L}(V, V)$ and $\mathfrak{L}(V^*, V^*)$ corresponding to a given tensor of type $(1, 1)$ are dual linear transformations.

PROBLEM 4.4. List all the remaining possible interpretations of $V \otimes V^* \otimes V^*$ and describe the corresponding isomorphisms.

4.2 The components of tensors

In Section 3.2 we saw that if $\{e_i^1\}, \ldots, \{e_i^n\}$ are bases of $V_1, \ldots, V_n$, then $\{e_{i_1}^1 \otimes \cdots \otimes e_{i_n}^n\}$ is a basis of the tensor product space $V_1 \otimes \cdots \otimes V_n$, so that an element of $V_1 \otimes \cdots \otimes V_n$ can be "expanded" in terms of its *components* $A^{i_1 \cdots i_n}$, in this basis, as in (3.9).

As a special case of (3.9), any tensor $A \in V_s^r$ can be written

$$A = A^{i_1 \cdots i_r}_{j_1 \cdots j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_i} \otimes \cdots \otimes \varepsilon^{j_s} \tag{4.3}$$

where $\{e_i\}$ is a basis of V and $\{\varepsilon^k\}$ is its dual basis. As a special case of (3.10), the values of A are

$$A(\sigma^1, \ldots, \sigma^r, v_1, \ldots, v_s) = A^{i_1 \cdots i_r}_{j_1 \cdots j_s} \sigma^1_{i_1} \cdots \sigma^r_{i_r} v_1^{j_1} \cdots v_s^{j_s} \tag{4.4}$$

It is sometimes useful to describe the isomorphisms described in Section 4.1 in terms of the components of the tensors when a basis is chosen. Thus, for example, in the given illustration for tensors of type (1,2) we have for interpretation (i), $A = A^i_{jk} e_i \otimes \varepsilon^j \otimes \varepsilon^k$ and $(\phi \cdot A) \cdot e_m = \tilde{A}^i_{mk} e_i \otimes \varepsilon^k$ where $\{\varepsilon^i\}$ is the dual basis in V^*. If we evaluate A at $(\varepsilon^l, e_m, e_n)$ and evaluate $(\phi \cdot A) \cdot e_m$ at (ε^l, e_n) and then, according to (4.1), equate the results, we get $A^l_{mn} = \tilde{A}^l_{mn}$. That is, the components of A are the elements of the matrix of the linear transformation $\mathfrak{L}(V, V \otimes V^*)$. Finally, if $e_n^{lm} : e_p \mapsto \delta_p^l e_n \otimes \varepsilon^m$ is the basis of $\mathfrak{L}(V, V \otimes V^*)$ given by Theorem 2.3 we can write

$$\phi \cdot e_i \otimes \varepsilon^j \otimes \varepsilon^k = \phi_{ilm}^{njk} e_n^{lm}$$

and evaluating both sides on e_p we get

$$(\phi \cdot e_i \otimes \varepsilon^j \otimes \varepsilon^k) \cdot e_p = \phi_{ilm}^{njk} \delta_p^l e_n \otimes \varepsilon^m.$$

Evaluating both sides of this last equation, in turn, on (ε^s, e_t) we obtain by eq. (4.1) $e_i \otimes \varepsilon^j \otimes \varepsilon^k(\varepsilon^s, e_p, e_t) = \phi_{ilm}^{njk} \delta_p^l \delta_n^s \delta_t^m$ so $\phi_{ipt}^{sjk} = \delta_i^s \delta_p^j \delta_t^k$.

Once we choose a basis in V, and hence in V_s^r, we have exactly the same situation as in Section 1.2. That is, we have an isomorphism between V_s^r and the set of components $A_{j_1 \cdots j_s}^{i_1 \cdots i_r}$. Moreover, $A_{j_1 \cdots j_s}^{i_1 \cdots i_r} + B_{j_1 \cdots j_s}^{i_1 \cdots i_r}$ are the components of $A + B$ and for $a \in \mathbb{R}$, $aA_{j_1 \cdots j_s}^{i_1 \cdots i_r}$ are the components of aA. See Problem 1.16. This justifies the approach of "classical" tensor analysis where it is customary to work with the components of tensors rather with the tensors themselves.

It is important to know how the components of a tensor transform when we change bases in V. As in eq. (1.2), $\bar{e}_i = a_i^j e_j$, where (a_i^j) is the change-of-basis matrix, and according to Problem 2.3, $\bar{\varepsilon}^i = b_j^i \varepsilon^j$ where $a_j^i b_k^j = \delta_k^i$. We write $A \in V_s^r$ in terms of each basis, just as we did for any vector, v, in Section 1.1:

$$A = A_{j_1 \cdots j_s}^{i_1 \cdots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_s}$$

and

$$A = \bar{A}_{q_1 \cdots q_s}^{p_1 \cdots p_r} \bar{e}_{p_1} \otimes \cdots \otimes \bar{e}_{p_r} \otimes \bar{\varepsilon}^{q_1} \otimes \cdots \otimes \bar{\varepsilon}^{q_s}$$

and substitute into the first the expression for the e's and ε's in terms of the $\bar{e}$'s and $\bar{\varepsilon}$'s. Comparing the resulting expression for A with the second one above, we get our result:

$$\bar{A}_{q_1 \cdots q_s}^{p_1 \cdots p_r} = A_{j_1 \cdots j_s}^{i_1 \cdots i_r} a_{q_1}^{j_1} \cdots a_{q_s}^{j_s} b_{i_1}^{p_1} \cdots b_{i_r}^{p_r}. \tag{4.5}$$

Compare this with eq. (1.4).

Theorem 4.2 *Two sets of numbers, $A_{j_1 \cdots j_s}^{i_1 \cdots i_r}$, and $\bar{A}_{j_1 \cdots j_s}^{i_1 \cdots i_r}$, are components using different bases $\{e_i\}$ and $\{\bar{e}_i\}$ of the same tensor if and only if they are related by (4.5) where (a_j^i) is a nonsingular matrix and $a_j^i b_k^j = \delta_k^i$.*

Proof Problem 4.8. □

Theorem 4.2 characterizes a tensor in terms of its components. In "classical" treatments this characterization is taken as the definition. In specific applications one almost always works with components. Then we must however, be careful to be sure that our results do not depend on the particular choice of basis.

PROBLEM 4.5. Choose a basis for V, and write the specific form of (4.3) and (4.4) for the tensor $\langle -, - \rangle$ in V_1^1.

PROBLEM 4.6. Describe the nondegeneracy of tensors of type $(1,1), (2,0)$, and $(0,2)$ in terms of their components.

PROBLEM 4.7. Interpret the components of $A \in V \otimes V^* \otimes V^*$ as certain coefficients of elements of $\mathfrak{L}(V^*, V; V^*)$. See example below eq. (4.4).

PROBLEM 4.8. What are the matrices of the isomorphisms of Theorem 4.1(ii) and (iii)?

PROBLEM 4.9. Prove Theorem 4.2.

PROBLEM 4.10. Let $\{e_1, e_2, e_3\}$ be a basis for V, and let $A \in V_1^1$ be $A = A^i_j e_i \otimes \varepsilon^j$ where the components A^i_j arranged in a matrix, (A^i_j) with row index i, are

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

Let $\bar{e}_1 = e_1 + e_2$, $\bar{e}_2 = 2e_2$, $\bar{e}_3 = -e_2 + e_3$ be a new basis, and let $v = -e_1 + 2e_3$ and $\tau = 5\varepsilon^1 - 2\varepsilon^2 + \varepsilon^3$. (i) Evaluate A at (τ, v) and evaluate the two corresponding linear maps at τ and v, respectively. (ii) Find $\bar{\varepsilon}^i$ in terms of ε^i. (iii) Find the expression for v and τ in the new basis. (iv) Find the components of A in the new basis. (v) Show $\det(A^i_j) = \det(\bar{A}^i_j)$ and $\operatorname{tr}(A^i_j) = \operatorname{tr}(\bar{A}^i_j)$. (vi) Do the computation of part (i) in the new basis.

PROBLEM 4.11. (i) Suppose $A \in V_1^1$ has components δ^i_j for some basis of V. Show that then A has components δ^i_j for every basis of V, and $A = \langle -, - \rangle$.

(ii) If A is a tensor of type $(1,1)$ and A has the same components in every basis, show that A is a multiple of $\langle -, - \rangle$; i.e., the components of A are $A^i_j = a\delta^k_j$ for some $a \in \mathbb{R}$.

(Problem 4.11 gives the existence and "uniqueness" of (1,1) tensors having the same components in every basis.)

PROBLEM 4.12. If A is a tensor of type (r, s), and A has the same components in every basis, show that either $A = 0$, or $r = s$.

PROBLEM 4.13. If A is a tensor of type (2,2), and A has the same components in every basis, show that

$$A^{ij}_{pq} = a(\delta^i_p \delta^j_q + \delta^i_q \delta^j_p) + b(\delta^i_p \delta^j_q - \delta^i_q \delta^j_p) \tag{4.6}$$

See Problem 5.14. (There is the related topic of *isotropic tensors*, important in continuum mechanics, cf., Aris, 1962, p. 30, and in relativity, Section 24.2. There, only orthonormal bases are used.)

PROBLEM 4.14. If the following given set of numbers is the set of components of A for some basis of V, find the set of components of A for another basis of V.

(i) $\delta_{ij} = 1$ when $i = j = 0$ otherwise

(ii) $\delta^{i_1\cdots i_r}_{j_1\cdots j_r} = 1$ when $j_1 \cdots j_r$ are distinct integers between 1 and $p \geq r$, and $i_1 \cdots i_r$ is an even permutation of $j_1 \cdots j_r$

$= -1$ when $j_1 \cdots j_r$ are distinct and $i_1 \cdots i_r$ is an odd permutation of $j_1 \cdots j_r$

$= 0$ otherwise

(Hint: $\delta^{i_1\cdots i_r}_{j_1\cdots j_r} = \sum_\pi (\text{sgn } \pi)\ \delta^{i_{\pi(1)}}_{j_1} \cdots \delta^{i_{\pi(r)}}_{j_r}$ where π is a permutation of the integers $1, \ldots, r$.)

PROBLEM 4.15 Let $\varepsilon^{i_1\cdots i_r} = \delta^{i_1\cdots i_r}_{1\cdots r}$, which is defined as in Problem 4.14 (ii) except that $i_1, \ldots i_r$ can only take on the values $1, \ldots, r$. Show that $\bar{\varepsilon}^{j_1\cdots j_r} = \det(a^i_j)\varepsilon^{i_1\cdots i_r} b^{j_1}_{i_1} \cdots b^{j_r}_{i_r}$ have the same values as the $\varepsilon^{i_1\cdots i_r}$. ($(a^i_j)$ is the change-of-basis matrix and $a^i_j b^j_k = \delta^i_k$.) Two sets of numbers related this way are the components in their respective bases of *an* rth *order tensor density of weight 1.* $\varepsilon^{i_1\cdots i_r}$ (or $\varepsilon_{j_1\cdots j_s}$ defined the same way) is called *a permutation symbol.* (For a general discussion of relative tensors see Synge and Schild (1966). This concept is useful in integration theory in Chapter 12.)

4.3 Mappings of the spaces V_s^r

There are several types of mappings that can be defined for the tensor product spaces V_s^r. Some of the following ones can be generalized, but they are basic and most important in subsequent developments.

1. We saw in Section 2.4 that a given linear mapping $\Phi : V \to W$ "induces" another linear mapping, $\Phi^* : W^* \to V^*$, the dual, or transpose of Φ. Now that we have a whole hierarchy of spaces V_s^r, built on V, we can generalize this result.

Definitions If $\phi : V \to W$ is a linear mapping, and v_i are in V, then (i) *the* rth *power of* ϕ is the linear mapping $\phi_r : V_0^r \to W_0^r$ with the property

$$\phi_r(v_1 \otimes \cdots \otimes v_r) = \phi \cdot v_1 \otimes \cdots \otimes \phi \cdot v_r \tag{4.7}$$

and (ii) *the* sth *power of* ϕ^* is the linear mapping $\phi^s \cdot W_s^0 \to V_s^0$ with the property

$$\phi^s(\sigma^1 \otimes \cdots \otimes \sigma^s) = \phi^* \cdot \sigma^1 \otimes \cdots \otimes \phi^* \cdot \sigma^s \tag{4.8}$$

where σ^i are in W^*.

The existence of unique maps with the given properties comes from the generalization of Theorem 3.1.

We also have the following alternative description of these induced mappings.

Theorem 4.3 (i) $\phi_r : V_0^r \to W_0^r$ *has the property described by eq. (4.7) if and only if for* $A \in V_0^r$*, the values of* $\phi_r \cdot A$ *are given by*

$$(\phi_r \cdot A)(\sigma^1, \ldots, \sigma^r) = A(\phi^* \cdot \sigma^1, \ldots, \phi^* \cdot \sigma^r) \tag{4.9}$$

where σ^i *are in* W^*.

(ii) $\phi^s : W_s^0 \to V_s^0$ *has the property described by eq. (4.8) if and only if for* $A \in W_s^0$*, the values of* $\phi^s \cdot A$ *are given by*

$$(\phi^s \cdot A)(v_1, \ldots, v_s) = A(\phi \cdot v_1, \ldots, \phi \cdot v_s) \tag{4.10}$$

where v_i *are in* V.

Proof (i) (4.9) defines $\phi_r \cdot A$ as a multilinear function on $W^* \times \cdots \times W^*$ since ϕ^* is linear and A is multilinear; i.e., $\phi_r \cdot A \in W_0^r$. Now, if $\phi_r \cdot A$ satisfies (4.9) for all $A \in V_0^r$ then, in particular,

$$\begin{aligned}
&(\phi_r \cdot (v_1 \otimes \cdots \otimes v_r))(\sigma^1, \ldots, \sigma^r) \\
&\quad = (v_1 \otimes \cdots \otimes v_r)(\phi^* \cdot \sigma^1, \ldots, \phi^* \cdot \sigma^r) = \langle \phi^* \cdot \sigma^1, v_1 \rangle \cdots \langle \phi^* \cdot \sigma^r, v_r \rangle \\
&\quad = \langle \sigma^1, \phi \cdot v_1 \rangle \cdots \langle \sigma^r, \phi \cdot v_r \rangle \qquad \text{by Theorem 2.21} \\
&\quad = (\phi \cdot v_1 \otimes \cdots \otimes \phi \cdot v_r)(\sigma^1, \ldots, \sigma^r)
\end{aligned}$$

So $\phi_r \cdot (v_1 \otimes \cdots \otimes v_r) = \phi \cdot v_1 \otimes \cdots \otimes \phi \cdot v_r$. Thus, if ϕ_r satisfies (4.9) it satisfies (4.7).

For the converse, let $A = a^i(v_{i1} \otimes \cdots \otimes v_{ir})$. then

$$\phi_r \cdot A = a^i \phi_r \cdot (v_{i1} \otimes \cdots \otimes v_{ir}) = a^i(\phi \cdot v_{i1} \otimes \cdots \otimes \phi \cdot v_{ir})$$

by (4.7). Then

$$\begin{aligned}
\phi_r \cdot A(\sigma^1, \ldots, \sigma^r) &= a^i \langle \sigma^1, \phi \cdot v_{i1} \rangle \cdots \langle \sigma^r, \phi \cdot v_{ir} \rangle \\
&= a^i \langle \phi^* \cdot \sigma^1, v_{i1} \rangle \cdots \langle \phi^r \cdot \sigma^r, v_{ir} \rangle = A(\phi^* \cdot \sigma^1, \ldots, \phi^* \cdot \sigma^r).
\end{aligned}$$

Thus, if ϕ_r satisfies (4.7), it satisfies (4.9).

(ii) Problem 4.16. □

We can define generalizations of these mappings.

Definition If $\phi : V \to W$ and $\psi : X \to Y$ are linear, then *the tensor product of* ϕ *and* ψ is the linear mapping $\phi \otimes \psi : V \otimes X \to W \otimes Y$ with the property

$$\phi \otimes \psi(v \otimes x) = \phi \cdot v \otimes \psi \cdot x \tag{4.11}$$

In the special case $W = Y = \mathbb{R}$ we have a problem with notation, since in this case $\phi \otimes \psi$ has already been defined. In Section 3.1 $\phi \otimes \psi$ was defined as an element of $V^* \otimes X^*$. Here it is in $(V \otimes X)^*$. However, these two functions, with different domains, are corresponding elements in the isomorphism of Theorem 3.4 and have the same values, so we keep the same notation for both, relying on the context to make things clear. Thus, we have

$$\phi \otimes \psi(v, w) = \langle \phi \otimes \psi, v \otimes x \rangle \tag{4.12}$$

where on the left side $\phi \otimes \psi$ stands for a bilinear function belonging to $V^* \otimes X^*$ and on the right side $\phi \otimes \psi$ is the corresponding linear function belonging to $(V \otimes X)^*$.

More generally, we define a linear mapping, *the tensor product of* $\phi^1, \ldots, \phi^p$

$$\phi^1 \otimes \cdots \otimes \phi^p : V_1 \otimes \cdots \otimes V_p \to W_1 \otimes \cdots \otimes W_p$$

by

$$\phi^1 \otimes \cdots \otimes \phi^p(v_1 \otimes \cdots \otimes v_p) = \phi^1 \cdot v_1 \otimes \cdots \otimes \phi^p \cdot v_p \tag{4.13}$$

where $\phi^i : V_i \to W_i$, and in the special case $W_1 = \cdots = W_p = \mathbb{R}$ we have

$$\phi^1 \otimes \cdots \otimes \phi^p(v_1, \ldots, v_p) = \langle \phi^1 \otimes \cdots \otimes \phi^p, v_1 \otimes \cdots \otimes v_p \rangle \tag{4.14}$$

where on the left side $\phi^1 \otimes \cdots \otimes \phi^p$ stands for a multilinear function belonging to $V_1^* \otimes \cdots \otimes V_p^*$ and on the right side $\phi^1 \otimes \cdots \otimes \phi^p$ is the corresponding linear function belonging to $(V_1 \otimes \cdots \otimes V_p)^*$. Note that this result, from a generalization of Theorem 3.4, gives an explicit description of the pairings which, in the special cases discussed in Section 4.1, gives the dualities described there.

2. There are two other important general types of mappings; a multiplication $V_s^r \times V_q^p \to V_{s+q}^{r+p}$ and a contraction $V_q^p \to V_{q-1}^{p-1}$. These are frequently combined to yield other important bilinear mappings.

Theorem 4.4 *There exists one and only bilinear map $V_0^p \times V_0^q \to V_0^{p+q}$ such that for $v_1 \otimes \cdots \otimes v_p \in V_0^p$ and $w_1 \otimes \cdots \otimes w_q \in V_0^q$,*

$$(v_1 \otimes \cdots \otimes v_p, w_1 \otimes \cdots \otimes w_q) \mapsto v_1 \otimes \cdots \otimes v_p \otimes w_1 \otimes \cdots \otimes w_q.$$

Proof Let j be the natural isomorphism $j : V_0^p \otimes V_0^q \to V_0^{p+q}$ (see Theorem 3.2). In particular

$$j : (v_1 \otimes \cdots \otimes v_p) \otimes (w_1 \otimes \cdots \otimes w_q) \mapsto v_1 \otimes \cdots \otimes w_q.$$

So, $j \circ \otimes$ is a bilinear map with the required property. By the second corollary of Theorem 3.1 there is only one such map. $\square$

Definition $AB = j \circ \otimes(A, B) \in V_0^{p+q}$ is *the (outer) product* of the tensors $A \in V_0^p$ and $B \in V_0^q$ and the map $j \circ \otimes$ is called *tensor multiplication.*

Theorem 4.5 *If $A \in V_0^p$, and $B \in V_0^q$ then*
(i) *The values of AB are*

$$AB(\sigma^1, \ldots, \sigma^{p+q}) = A(\sigma^1, \ldots, \sigma^p)B(\sigma^{p+1}, \ldots, \sigma^{p+q}).$$

(ii) *The components of AB (relative to a basis of V_0^{p+q}) are $A^{i_1\cdots i_p}B^{j_1\cdots j_q}$.*

Proof (i) For $A = v_1 \otimes \cdots \otimes v_p$ and $B = w_1 \otimes \cdots \otimes w_q$,

$$\begin{aligned} &AB(\sigma^1, \cdots, \sigma^{p+q}) \\ &\quad = j \circ \otimes(v_1 \otimes \cdots \otimes v_p,\ w_1 \otimes \cdots \otimes w_q)(\sigma^1, \ldots, \sigma^{p+q}) \\ &\quad = (v_1 \otimes \cdots \otimes w_q)(\sigma^1, \ldots, \sigma^{p+q}) \\ &\quad = (v_1 \otimes \cdots \otimes v_p)(\sigma^1, \ldots, \sigma^p)(w_1 \otimes \cdots \otimes w_q)(\sigma^{p+1}, \ldots, \sigma^{p+q}) \\ &\quad = A(\sigma^1, \ldots, \sigma^p)B(\sigma^{p+1}, \ldots, \sigma^{p+q}). \end{aligned}$$

For linear combinations of elements of this form, the result follows from the bilinearity of $j \circ \otimes$.
(ii) Problem 4.19. □

Note that because of the natural isomorphism $j : V_0^p \otimes V_0^q \to V_0^{p+q}$, most writers do not bother to make a distinction between AB and $A \otimes B$. However, $A \otimes B$, a product of *vectors*, is in $V_0^p \otimes V_0^q$ and defined (in Chapter 3) on pairs $(\alpha, \beta) \in V_0^{p*} \times V_0^{q*}$, while AB, a product of *tensors*, is in V_0^{p+q} and defined on elements of $V^* \times \cdots \times V^*$. Thus, AB and $A \otimes B$ are different as mappings, since they have different domains, even though their values on "corresponding arguments" are the same – see Problem 4.21(ii).

The definition of the product of two tensors and Theorems 4.4 and 4.5 for V_0^p and V_0^q can be generalized to V_s^r and V_q^p with only slight complication of notation.

Theorem 4.6 *Multiplication of tensors is bilinear and associative, but not commutative.*

Proof Problem 4.20. □

Definition The *contraction*, C_μ^λ, is the unique linear mapping $V_q^p \to V_{q-1}^{p-1}$ with the property

$$\begin{aligned} &v_1 \otimes \cdots \otimes v_p \otimes \sigma^1 \otimes \cdots \otimes \sigma^q \mapsto \\ &\qquad \langle \sigma^\mu, v_\lambda \rangle v_1 \otimes \cdots \otimes \hat{v}_\lambda \otimes \cdots \otimes v_p \otimes \sigma^1 \otimes \cdots \otimes \hat{\sigma}^\mu \otimes \cdots \otimes \sigma^q \end{aligned} \tag{4.15}$$

where the "hats" above v_λ and σ^μ on the right mean that those factors are to be omitted.

It is not immediately obvious that there does indeed exist a unique linear mapping with this property. This comes from the generalization of Theorem 3.1 for V_s^r when we choose V_{q-1}^{p-1} for the vector space Z and choose

$$(v_1, \ldots, v_p, \sigma^1, \ldots, \sigma^q) \mapsto$$
$$\langle \sigma^\mu, v_\lambda \rangle v_1 \otimes \cdots \otimes \hat{v}_\lambda \otimes \cdots \otimes v_p \otimes \sigma^1 \otimes \cdots \otimes \hat{\sigma}^\mu \otimes \cdots \otimes \sigma^q \in V_{q-1}^{p-1}$$

for the multilinear map in part (ii) of that theorem.

As an example $C_1^2 : V_1^2 \to V$ takes $v \otimes w \otimes \sigma \mapsto \langle \sigma, w \rangle v$, and, in particular, takes $e_i \otimes e_j \otimes \varepsilon^k \mapsto \langle \varepsilon^k, e_j \rangle e_i = \delta_j^k e_i$. Hence

$$A_k^{ij} e_i \otimes e_j \otimes \varepsilon^k \mapsto A_k^{ij} \delta_j^k e_i = A_k^{ik} e_i.$$

This example illustrates the general result that if A has components $A_{j_1 \cdots j_s}^{i_1 \cdots i_r}$, then $C_\mu^\lambda \cdot A$ has components $A_{j_1 \cdots j_{\mu-1} k \cdots j_s}^{i_1 \cdots i_{\lambda-1} k \cdots i_r}$ (sum on k). Clearly, if r and s are large enough, one can perform several successive contractions. In particular, if $A \in V_1^1$, then $C_1^1 \cdot A = \text{trace } A$. See Problem 4.10(v). From this result for the components of $C_\mu^\lambda \cdot A$ we get

$$\begin{aligned} &C_\mu^\lambda \cdot A(\sigma^1, \ldots, \hat{\sigma}^\lambda, \ldots, \sigma^r, v_1, \ldots, \hat{v}_\mu, \ldots, v_s) \\ &= \sum_i A(\sigma^1, \ldots, \sigma^{\lambda-1}, \varepsilon^i, \ldots, \sigma^r, v_1, \ldots, v_{\mu-1}, e_i, \ldots, v_s) \end{aligned} \tag{4.16}$$

Finally, in certain cases, given two tensors A and B we can form the product and then contract. Thus, for example if, in a basis, $A = A_k^{ij} \varepsilon_i \otimes e_j \otimes \varepsilon^k \in V_1^2$, and $B = B_{lmn}^q e_q \otimes \varepsilon^l \otimes \varepsilon^m \otimes \varepsilon^n \in V_3^1$, then

$$C_3^2 \cdot AB = A_k^{ij} B_{ljn}^q e_i \otimes e_q \otimes \varepsilon^k \otimes \varepsilon^l \otimes \varepsilon^n \in V_3^2.$$

Important special cases of this operation will be described in Section 6.3.

In particular, the evaluation of a tensor, A, on the arguments $\sigma^1, \ldots, \sigma^r$, $v_1, \ldots, v_s$ can be performed by forming the product

$$A \otimes \sigma^1 \otimes \cdots \otimes \sigma^r \otimes v_1 \otimes \cdots \otimes v_s$$

and then contracting successively $r + s$ times.

In Theorem 4.2 we had a characterization of tensors in terms of components which is both of central importance in the classical theory and also very useful in doing actual calculations. Now, having introduced products and contractions we can describe other similar results known classically as *quotient rules.*

Theorem 4.7 *Two sets of numbers $A_{j_1 \cdots j_s}^{i_1 \cdots i_r}$ and $\bar{A}_{j_1 \cdots j_s}^{i_1 \cdots i_r}$ are components of the same tensor relative to bases $\{e_i\}$ and $\{\bar{e}_i\}$, respectively, if and only if for all tensors $B \in V_q^p$, $A_{j_1 \cdots j_s}^{i_1 \cdots k \cdots i_r} B_{j_{s+1} \cdots k \cdots j_{s+q}}^{i_{r+1} \cdots i_{r+p}}$ and $\bar{A}_{j_1 \cdots j_s}^{i_1 \cdots k \cdots i_r} \bar{B}_{j_{s+1} \cdots k \cdots j_{s+q}}^{i_{r+1} \cdots i_{r+p}}$ are components of the same tensor relative to these bases.*

Proof If: To keep the notation under control, we consider a special case. Suppose $A_k^{ij} B_{il}$ are components of a tensor of type (1,2) for all tensors B of type (0,2). Then

$$A_k^{ij} B_{il} e_j \otimes \varepsilon^k \otimes \varepsilon^l = \bar{A}_n^{pm} \bar{B}_{pq} \bar{e}_m \otimes \bar{\varepsilon}^n \otimes \bar{\varepsilon}^q \tag{4.17}$$

We put $\bar{e}_m = a_m^j e_j, \bar{\varepsilon}^n = b_k^m \varepsilon^k$, and $\bar{\varepsilon}^q = b_l^q \varepsilon^l$ where $a_j^i b_k^j = \delta_k^i$, and by Theorem 4.2, $\bar{B}_{pq} = B_{ir} a_p^i a_q^r$ into eq. (4.17). Then

$$A_k^{ij} B_{il} e_j \otimes \varepsilon^k \otimes \varepsilon^l = \bar{A}_n^{pm} B_{ir} a_p^i a_q^r a_m^j e_j \otimes b_k^n \varepsilon^k \otimes b_l^q \varepsilon^l$$

or

$$A_k^{ij} B_{il} = \bar{A}_n^{pm} B_{ir} a_p^i a_q^r a_m^j b_k^n b_l^q \quad \text{for all} \quad j, k, l$$

Since $a_q^r \cdot b_l^q = \delta_l^r$ and $B_{ir} \sigma_l^r = B_{il}$, and B_{il} can be chosen arbitrarily, we get

$$A_k^{ij} = \bar{A}_n^{pm} a_p^i a_m^j b_k^n$$

which is equivalent to the criterion Eq. (4.5) of Theorem 4.2.

Only if: Apply Theorem 4.2 to A and B, multiply, and contract. □

Note that instead of requiring all tensors $B \in V_1^p$ for the condition in Theorem 4.7 we need only a linearly independent set of pq elements of V_q^p. Also, it should be clear that any other contractions of AB or sequence of contractions could replace those indicated in Theorem 4.7.

PROBLEM 4.16. Prove part (ii) of Theorem 4.3.

PROBLEM 4.17. If $\phi : V \to W$ and $\psi : W \to X$, then $(\psi \circ \phi)_r = \psi_r \circ \phi_r$ and $(\psi \circ \phi)^s = \psi^s \circ \phi^s$.

PROBLEM 4.18. If $\phi : V \to W$ is an isomorphism, there are linear maps $(\phi^{-1})^s : V_s^0 \to W_s^0$ with $\sigma^1 \otimes \cdots \otimes \sigma^s \mapsto \phi^{-1*} \cdot \sigma^1 \otimes \cdots \otimes \phi^{-1*} \cdot \sigma^s$ and $(\phi^{-1})_r : W_0^r \to V_0^r$ with $w_1 \otimes \cdots \otimes w_r \mapsto \phi^{-1} \cdot w_1 \otimes \cdots \otimes \phi^{-1} \cdot w_r$. Define maps $\phi_{r,s} : V_s^r \to W_s^r$ and $\phi^{r,s} : W_s^r \to V_s^r$ and describe all four of these in terms of the values of their images as in eqs. (4.9) and (4.10).

PROBLEM 4.19. Prove part (ii) of Theorem 4.5.

PROBLEM 4.20. Prove Theorem 4.6.

PROBLEM 4.21. (i) Generalize eq. (4.14) to

$$B^*(v_1, \ldots, v_p) = \langle B^*, v_1 \otimes \cdots \otimes v_p \rangle$$

where B^* on the left is in $V_1^* \otimes \cdots \otimes V_p^*$ and B^* on the right is in $(V_1 \otimes \cdots \otimes V_p)^*$.

(ii) Show that if A is in V_p^0 and, equivalently, in V_0^{p*}, and if B is in V_q^0 and equivalently in V_0^{q*}, then

$$A \otimes B(v_1 \otimes \cdots \otimes v_p, v_{p+1} \otimes \cdots \otimes v_{p+q}) = AB(v_1, \ldots, v_{p+1})$$

PROBLEM 4.22. (i) Define linear maps $V_q^p \to V_{q-r}^{p-r}$ for $p, q \geq r$.

(ii) Define a nondegenerate bilinear function on $V_s^r \times V_r^s$ and thus conclude that V_s^r and V_r^s are dual spaces.

PROBLEM 4.23. For $A \in V_0^p$ and $B^* \in (V_0^p)^*, \langle B^*, A\rangle = C_p^p \cdots C_1^1 \cdot B^* A$.

5

SYMMETRIC AND SKEW-SYMMETRIC TENSORS

We will examine two classes of subspaces of the covariant tensor product spaces, V_s^0; the symmetric subspaces $\mathbf{S}^s(V^*)$, and the skew-symmetric (alternating) subspaces, $\bigwedge^s(V^*)$. In particular, we will describe bases of these spaces. Finally, we will look at special properties of $\mathbf{S}^2(V^*)$ and $\bigwedge^2(V^*)$. We simply mention here that obvious corresponding results are valid for contravariant tensors.

5.1 Symmetry and skew-symmetry

Recall that in Problem 3.8 we noted that in general the two functions $v \otimes w$ and $w \otimes v$ in $V \otimes V$ are not the same. That is, $\langle\sigma, v\rangle\langle\tau, w\rangle \neq \langle\sigma, w\rangle\langle\tau, v\rangle$ for all σ and τ. Or, equivalently, $v \otimes w(\sigma, \tau) \neq v \otimes w(\tau, \sigma)$ for all σ, τ.

Definition If the values of a tensor remain unchanged when two of its covariant arguments or two of its contravariant arguments are transposed, then the tensor is *symmetric in these two arguments.*

For example, if

$$A(\tau^1, \tau^2, \tau^3, \ldots, \tau^r, w_1, w_2, \ldots, w_s) = A(\tau^1, \tau^3, \tau^2, \ldots, \tau^r, w_1, w_2, \ldots, w_s)$$

for all values of all the τ's and w's then A is symmetric in its second and third covariant arguments. Notice that it makes no sense to transpose a covariant and contravariant argument.

Definition If the values of a tensor change sign when two of its covariant arguments or two of its contravariant arguments are transposed, then the tensor is *skew-symmetric* (or *antisymmetric,* or *alternating*) in these two arguments.

Theorem 5.1 *A is skew-symmetric in two (covariant, or contravariant) arguments if and only if whenever these two arguments have the same value, the value of A is zero.*

Proof We will do the proof for the first two covariant arguments of A to simplify the notation. (i) Suppose $A(\sigma^1, \sigma^2, \dots) = -A(\sigma^2, \sigma^1, \dots)$ for all σ^1 and σ^2. Then, when $\sigma^1 = \sigma^2$, $A(\sigma^2, \sigma^2, \dots) = -A(\sigma^2, \sigma^2, \dots)$ so that $A(\sigma^2, \sigma^2, \dots) = 0$. (ii) If $A(\sigma, \sigma, \dots) = 0$ for all σ, then $A(\sigma + \tau, \sigma + \tau, \dots) = 0$ for all σ and τ. Expanding the left side we get $A(\sigma, \tau, \dots) + A(\tau, \sigma, \dots) = 0$ for all σ, τ and so $A(\sigma, \tau, \dots) = -A(\tau, \sigma, \dots)$. □

Theorem 5.2 *A is symmetric in two (covariant, or contravariant) arguments if and only if the components $A^{ij\cdots}_{pq\cdots}$ are symmetric in their corresponding two (covariant, or contravariant) indices. A is skew-symmetric in two (covariant, or contravariant) arguments if and only if the components $A^{ij\cdots}_{pq\cdots}$ are skew-symmetric in their corresponding two (covariant, or contravariant) indices. Hence, in particular, the property of the components of a tensor being symmetric or skew-symmetric in two(covariant, or contravariant) indices is independent of the choice of a basis.*

Proof Problem 5.1. □

There is another way of describing the symmetry and skew-symmetry in two arguments of a tensor. We will discuss the symmetric case and leave the skew-symmetric case for a problem. Again, to simplify the notation we will work with the first two arguments.

If $v_1 \otimes v_2 \otimes \cdots \in V^r_s$, let $T : V^r_s \to V^r_s$ be such that $v_1 \otimes v_2 \otimes \cdots \mapsto v_2 \otimes v_1 \otimes \cdots$ where the dots on each side of the arrow are supposed to represent the same thing. By Theorem 3.1, or its generalization, T defines an automorphism of V^r_s.

Theorem 5.3 *If $A \in V^r_s$, and T is as given above, then $T \cdot A = A$ if and only if A is symmetric in its first two arguments.*

Proof The proof follows from the following lemma. □

Lemma $T \cdot A(\tau^1, \tau^2, \dots) = A(\tau^2, \tau^1, \dots)$.

Proof of lemma A is a linear combination of decomposable elements, $v_{i1} \otimes v_{i2} \otimes \cdots$. Write $A = a^i(v_{i1} \otimes v_{i2} \otimes \cdots)$. Then

$$T \cdot A = a^i T \cdot (v_{i1} \otimes v_{i2} \otimes \cdots) = a^i (v_{i2} \otimes v_{i1} \otimes \cdots).$$

So

$$\begin{aligned} T \cdot A(\tau^1, \tau^2, \dots) &= a^i (v_{i2} \otimes v_{i1} \otimes \cdots)(\tau^1, \tau^2, \dots) = a^i \langle \tau^1, v_{i2} \rangle \langle \tau^2, v_{i1} \rangle \cdots \\ &= a^i \langle \tau^2, v_{i1} \rangle \langle \tau^1, v_{i2} \rangle \cdots = a^i (v_{i1} \otimes v_{i2} \otimes \cdots)(\tau^2, \tau^1, \dots) \\ &= A(\tau^2, \tau^1, \dots). \end{aligned}$$

□

PROBLEM 5.1. Prove Theorem 5.2.

PROBLEM 5.2. (i) If a tensor of type (3,0) is symmetric in two arguments, and skew-symmetric in two arguments then it is the zero tensor. (ii) Generalize (i).

PROBLEM 5.3. (i) If the components of a tensor of type (1,1) are symmetric for every basis, then it is a scalar multiple of $\langle -, - \rangle$. (ii) If the components of a tensor of type (1,1) are skew-symmetric for every basis, then it is the zero tensor. These results indicate that the concepts of symmetry and skew-symmetry with respect to a covariant and a contravariant argument (index) are essentially vacuous.

PROBLEM 5.4. Define an automorphism, T, of V_s^r and prove that $T \cdot A = -A$ if and only if A is skew-symmetric in its first two arguments.

PROBLEM 5.5. Suppose $\dim V = 3$, and A is a tensor of type (0,3) whose components have the symmetries $A_{ijk} + A_{jki} + A_{kij} = 0$ and $A_{ijk} = -A_{ikj}$. How many components of A are independent? Choose an independent set, and express the remaining components in terms of them.

PROBLEM 5.6. Suppose A is a tensor of type (0,4) having the following symmetries:

(a) $A_{ijkl} = -A_{jikl}$

(b) $A_{ijkl} = -A_{ijlk}$

(c) $A_{ijkl} + A_{iklj} + A_{iljk} = 0$

Then

(i) $A_{ijkl} = A_{klij}$.

(ii) $A(v, w, v, w) = 0$ for all $v, w \in V$ implies $A = 0$. (Hence, if B and C are two tensors having the given symmetries and $B(v, w, v, w) = C(v, w, v, w)$ for all $v, w \in V$, then $B = C$.)

(iii) Let b be a symmetric tensor of type (0,2). Then $A_{ijkl} = b_{ik}b_{jl} - b_{il}b_{jk}$ has symmetries (a), (b), and (c).

PROBLEM 5.7. If $\dim V = n$, then a (0,4) tensor with the symmetries of Problem 5.6 has $n^2(n^2 - 1)/12$ distinct nonvanishing components. (Hint: Find the number of distinct nonvanishing components with 2, 3, and 4 distinct indices and add.)

5.2 The symmetric subspace of V_s^0

In the following discussion, we will be restricting our attention to the spaces V_s^0 of covariant tensors. It will be evident that a completely parallel development is valid for contravariant tensors.

Definition A tensor $A \in V_s^0$ is *symmetric* if it is symmetric in all pairs of its arguments. This is equivalent to the property $A(v_1, \dots, v_s) = A(v_{\pi(1)}, \dots, v_{\pi(s)})$ for all permutations, π, of $1, 2, \dots, s$.

Note that all zero and first order tensors are symmetric.

Theorem 5.4 *A is symmetric if and only if $A_{j_1 \cdots j_s} = A_{j_{\pi(1)} \cdots j_{\pi(s)}}$ for all permutations, π, of $1, 2, \dots, s$.*

We will let π also denote the automorphism of V_s^0 determined by $\sigma^1 \otimes \sigma^2 \otimes \cdots \otimes \sigma^s \mapsto \sigma^{\pi(1)} \otimes \sigma^{\pi(2)} \otimes \cdots \otimes \sigma^{\pi(s)}$. We get the following from Theorem 5.3.

Theorem 5.5 *A is symmetric if and only if $\pi \cdot A = A$ for all permutations, π, of $1, 2, \dots, s$.*

Proof Problem 5.8. □

Theorem 5.6 *The set, $\mathbf{S}^s(V^*)$, of all symmetric tensors of V_s^0 is a subspace of V_s^0.*

There are several other ways to describe the subspace of symmetric tensors of V_s^0.

Definitions The linear transformation

$$\mathfrak{S} : V_s^0 \to \mathbf{S}^s(V^*)$$

defined by

$$A \mapsto \frac{1}{s!} \sum_{\pi} \pi \cdot A \tag{5.1}$$

where $\sum_\pi$ is the sum over permutations of $1, \dots, s$ is called *the symmetrization operator on V_s^0*. $\mathfrak{S} \cdot A$ is called *the symmetric part of A*.

It is clear that $\mathfrak{S}$ given by (5.1) is linear, and it has values in $\mathbf{S}^s(V^*)$ since for any permutation, $\tilde{\pi}, \tilde{\pi} \cdot (\sum\limits_{\pi} \pi \cdot A) = \sum\limits_{\pi} \pi \cdot A$.

Theorem 5.7 $\mathfrak{S} \cdot A = \dfrac{1}{s!} \sum\limits_{\pi} \pi \cdot A$ *(i.e., (5.1)) if and only if*

$$\mathfrak{S} \cdot A(v_1, \dots, v_s) = \frac{1}{s!} \sum_{\pi} A(v_{\pi(1)}, \dots, v_{\pi(s)}) \tag{5.2}$$

Proof The proof is based on the following lemma. □

Lemma *For any permutation, π,*

$$\pi \cdot A(v_1, \dots, v_s) = A(v_{\pi^{-1}(1)}, \dots, v_{\pi^{-1}(s)})$$

Proof of lemma Let $A = a_i(\sigma^{i1} \otimes \cdots \otimes \sigma^{is})$. Then

$$\pi \cdot A = a_i(\sigma^{i\pi(1)} \otimes \cdots \otimes \sigma^{i\pi(s)}).$$

So

$$\begin{aligned}\pi \cdot A(v_1, \dots, v_s) &= a_i(\sigma^{i\pi(1)} \otimes \cdots \otimes \sigma^{i\pi(s)})(v_1, \dots, v_s) \\ &= a_i\langle \sigma^{i\pi(1)}, v_1\rangle\langle \sigma^{i\pi(2)}, v_2\rangle \cdots \langle \sigma^{i\pi(k)}, v_k\rangle \cdots \langle \sigma^{i\pi(s)}, v_s\rangle.\end{aligned}$$

Now, for some k, $\pi(k) = 1,\ k = \pi^{-1}(1)$, and $\langle \sigma^{i\pi(k)}, v_k\rangle = \langle \sigma^{i1}, v_{\pi^{-1}(1)}\rangle$. Another factor will be $\langle \sigma^{i2}, v_{\pi^{-1}(2)}\rangle$, and so on. So

$$\pi \cdot A(v_1, \dots, v_s) = a_i\langle \sigma^{i1}, v_{\pi^{-1}(1)}\rangle \cdots \langle \sigma^{is}, v_{\pi^{-1}(s)}\rangle = A(v_{\pi^{-1}(1)}, \dots, v_{\pi^{-1}(s)}).$$

□

Proof of theorem Now suppose $\mathfrak{S} \cdot A = \dfrac{1}{s!}\sum_\pi \pi \cdot A$. Then

$$\begin{aligned}\mathfrak{S} \cdot A(v_1, \dots, v_s) &= \left(\frac{1}{s!}\sum_\pi \pi \cdot A\right)(v_1, \dots, v_s) = \frac{1}{s!}\sum_\pi \pi \cdot A(v_1, \dots, v_s) \\ &= \frac{1}{s!}\sum_\pi A(v_{\pi^{-1}(1)}, \dots, v_{\pi^{-1}(s)}) \qquad \text{(by the lemma)} \\ &= \frac{1}{s!}\sum_\pi A(v_{\pi(1)}, \dots, v_{\pi(s)})\end{aligned}$$

All the steps of this argument are reversible, so the required equivalence is established. □

We saw that $\mathfrak{S}$ maps $V_s^0(V^*)$ into $\mathbf{S}^s(V^*)$. Actually, it is onto $\mathbf{S}^s(V^*)$, since if A is any element of $\mathbf{S}^s(V^*)$ then $\mathfrak{S}$ maps $A \in V_s^0$ into itself, so

$$\mathbf{S}^s(V^*) = \mathfrak{S}(V_s^0)$$

Moreover, $\mathfrak{S}^2 = \mathfrak{S}$; i.e., $\mathfrak{S}$ is a projection operator on V_s^0. So

$$V_s^0 = \ker \mathfrak{S} \oplus \mathbf{S}^s(V^*)$$

(see Problem 1.11).

There are alternative descriptions of ker $\mathfrak{S}$, and hence of $\mathbf{S}^s(V^*)$.

Theorem 5.8 *The linear closure of the set of all elements of V_s^0 which are skew-symmetric in two arguments is the same as the linear closure of the set of all elements of V_s^0 of the form*

$$\sigma^1 \otimes \cdots \otimes \sigma^i \otimes \cdots \otimes \sigma^j \otimes \cdots \otimes \sigma^s - \sigma^1 \otimes \cdots \otimes \sigma^j \otimes \cdots \otimes \sigma^i \otimes \cdots \otimes \sigma^s.$$

Proof Problem 5.9. □

Theorem 5.9 $N_s = \ker \mathfrak{S}$ *where N_s is the subspace of V_s^0 described in Theorem 5.8.*

Proof (i) Let A be an element of N_s of the form in Theorem 5.8. $\mathfrak{S} \cdot A$ is $1/s!$ times a sum of permutations of these elements. For each permutation, π, with $i \mapsto \pi(i)$ and $j \mapsto \pi(j)$, there is another permutation, $\tilde{\pi}$, which is the same as π except that $\tilde{\pi} : i \mapsto \pi(j)$ and $j \mapsto \pi(i)$. The terms in $\mathfrak{S} \cdot A$ corresponding to such pairs cancel each other, so for such elements $\mathfrak{S} \cdot A = 0$. Since every element of N_s is a linear combination of these elements, $\mathfrak{S} \cdot A = 0$ for all $A \in N_s$.

(ii) First we show that for $A \in V_s^0, \pi \cdot A - A \in N_s$. For $A = \sigma^1 \otimes \cdots \otimes \sigma^s$ and π a transposition this is clearly true. Assume it is true for $A = \sigma^1 \otimes \cdots \otimes \sigma^s$ and π a product of m transpositions, and proceed by induction. Since π is linear, it follows that $\pi \cdot A - A \in N_s$ for any $A \in V_s^0$. Now,

$$\mathfrak{S} \cdot A - A = \frac{1}{s!} \sum_{\pi} (\pi \cdot A - A) \in N_s.$$

So, if $A \in \ker \mathfrak{S}$, then $A \in N_s$. □

Corollary (i) $\mathbf{S}^s(V^*) \cong V_s^0 / N_s$; *and,* (ii) $V_s^0 = N_s \oplus \mathbf{S}^s(V^*)$.

To summarize, we have described a certain subspace of V_s^0 in three ways:
(i) the subset of V_s^0 stable (invariant) under symmetry automorphisms of V_s^0;
(ii) the image of the symmetrization operator; and
(iii) a quotient (factor) space.

We can express the components of $\mathfrak{S} \cdot A$ in a given basis of V_s^0 in terms of the components of A. We can write

$$A(v_{\pi(1)}, \ldots, v_{\pi(s)}) = A_{i_{\pi(1)} \cdots i_{\pi(s)}} v_{\pi(1)}^{i_{\pi(1)}} \cdots v_{\pi(s)}^{i_{\pi(s)}}$$

so from eq. (5.2),

$$\begin{aligned} \mathfrak{S} \cdot A(v_1, \ldots, v_s) &= \frac{1}{s!} \sum_{\pi} v_{\pi(1)}^{i_{\pi(1)}} \cdots v_{\pi(s)}^{i_{\pi(s)}} \, A_{i_{\pi(1)} \cdots i_{\pi(s)}} \\ &= \frac{1}{s!} v_1^{i_1} \cdots v_s^{i_s} \sum_{\pi} A_{i_{\pi(1)} \cdots i_{\pi(s)}} \\ &= \frac{1}{s!} \sum_{\pi} A_{i_{\pi(1)} \cdots i_{\pi(s)}} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_s}(v_1, \ldots, v_s) \end{aligned}$$

Thus,

$$\mathfrak{S} \cdot A = \frac{1}{s!} \sum_{\pi} A_{i_{\pi(1)} \cdots i_{\pi(s)}} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_s} \tag{5.3}$$

and so the components of $\mathfrak{S} \cdot A$ in the basis $\{\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_s}, i_1, \ldots, i_s = 1, \ldots, n\}$ of V_s^0 are $(1/s!) \sum_{\pi} A_{i_{\pi(1)} \cdots i_{\pi(s)}}$. Notice that there are n^s terms in (5.3) and the coefficients of the terms in which the values of $i_1, \ldots, i_s$ are simply permuted are all the same.

We can also write the symmetric part of A *in a basis of* $\mathbf{S}^s(V^*)$ with components in terms of the components of A in a basis of V_s^0. From (5.1), the definition of $\mathfrak{S}$, for any $A = A_{i_1 \cdots i_s} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_s} \in V_s^0$,

$$\mathfrak{S} \cdot A = \frac{1}{s!} A_{i_1 \cdots i_s} \sum_{\pi} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}} \tag{5.4}$$

for all $A \in V_s^0$. Note that the factors $\sum_{\pi} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}}$ are in $\mathbf{S}^s(V^*)$. In (5.4), $i_1, \ldots, i_s$ each take on values $1, \ldots, n$ so there will be n^s terms. We will now separate these into a certain number of bunches.

(i) Look at one term in which the values of $i_1, \ldots, i_s$ are distinct. There will be a total of $s!$ terms with $i_1, \ldots, i_s$ having these particular values, but permuted. For example, for $s = 3$, we might have $i_1 = 3$, $i_2 = 1$, and $i_3 = 4$. Then we also have $i_1 = 4$, $i_2 = 1$, $i_3 = 3$ and four more terms with indices $1, 3$, and 4. For all these $s!$ terms the second factor $\sum_{\pi} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}}$ in (5.4) is the same, so we can write the sum of these terms as

$$\frac{1}{s!}(A_{i_1 \cdots i_s} + A_{i_2 i_1 \cdots i_s} + \cdots)(\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_s} + \varepsilon^{i_2} \otimes \cdots \otimes \varepsilon^{i_s} + \cdots)$$

where now the values of $i_1, \ldots, i_s$ are fixed (no sum). Since the two factors of this expression are unchanged if we permute the $i_1, \ldots, i_s$, we can choose $i_1 < i_2 < \cdots < i_s$ and write this sum as

$$\sum_0 = \frac{1}{s!} \sum_{\pi} A_{i_{\pi(1)} \cdots i_{\pi(s)}} \sum_{\pi} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}}, \quad i_1 < i_2 < \cdots < i_s$$

(ii) If two of the values of the indices $i_1, \ldots, i_s$ are the same, there will be $s!/2$ terms of (5.4) with these values of the indices, and again each term will have the same factor $\sum_{\pi} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}}$. Now in this common factor, the same term appears twice, so we can write

$$\sum_{\pi} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}} = 2 \sum_{\pi *} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}}$$

where $\sum_{\pi *}$ means that we add up only half the terms - all the distinct ones. Further, in $\sum_{\pi} A_{i_{\pi(1)} \cdots i_{\pi(s)}}$ each of the $s!/2$ terms with the common factor

$\sum_{\pi *} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}}$ appears twice, so if we add up all the terms in which $i_1, \ldots, i_s$ have these values we get

$$\frac{1}{s!} \frac{\displaystyle\sum_{\pi} A_{i_{\pi(1)} \cdots i_{\pi(s)}}}{2} \cdot 2 \sum_{\pi *} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}}$$

$$= \frac{1}{s!} \sum_{\pi} A_{i_{\pi(1)} \cdots i_{\pi(s)}} \sum_{\pi *} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}}$$

Again, since the two factors of this expression are unchanged, if we permute the $i_1, \ldots, i_s$ we can choose $i_1 \leq i_2 \leq \cdots \leq i_s$, and write this sum as

$$\sum_1 = \frac{1}{s!} \sum_{\pi} A_{i_{\pi(1)} \cdots i_{\pi(s)}} \sum_{\pi *} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}} \quad i_1 \leq i_2 \leq \cdots \leq i_s$$

(iii) If the set $i_1, \ldots, i_s$ has a value repeated p times and another value repeated q times, there will be $s!/p!q!$ terms in (5.4) with these values of the indices, and each term will have the same factor

$$\sum_{\pi} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}} = p!q! \sum_{\pi *} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}}$$

since the $s!$ permutations of $i_1, \ldots, i_s$ yield only $s!/p!q!$ *distinct* terms. The sum of these terms will be

$$\sum_{pq} = \frac{1}{s!} \frac{\displaystyle\sum_{\pi} A_{i_{\pi(1)} \cdots i_{\pi(s)}}}{p!q!} \sum_{\pi} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}}$$

$$= \frac{1}{s!} \sum_{\pi} A_{i_{\pi(1)} \cdots i_{\pi(s)}} \sum_{\pi *} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}} \quad i_1 \leq i_2 \leq \cdots \leq i_s$$

Again $\sum_{\pi *}$ means we add up only distinct terms.

(iv) For any fixed set of values of $i_1, \ldots, i_s$ with no matter how many repetitions, we can draw out of (5.4) a set of terms whose sum is given by the same expression as in (i), (ii), and (iii). Thus, letting $i_1, \ldots, i_s$ range over all fixed sets of values, the sum of the terms of (5.4) will be the sum of sums of the form $\sum_0, \sum_1, \sum_{pq}, \ldots$. That is, we get

$$\mathfrak{S} \cdot A = \frac{1}{s!} \sum_{\pi} A_{i_{\pi(1)} \cdots i_{\pi(s)}} \sum_{\pi *} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}} \quad i_1 \leq i_2 \leq \cdots \leq i_s \tag{5.5}$$

Notice that the coefficients in (5.5) are exactly the same as those of (5.3). However, now they are not repeated as they were in (5.3).

Theorem 5.10 $\left\{\sum_{\pi *} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}}, \quad i_1 \leq i_2 \leq \cdots \leq i_s\right\}$ *forms a basis of* $\mathbf{S}^s(V^*)$, *and* (5.5) *is the representation of the symmetric part of* A *in that basis (in terms of the components of* A *in the* V_s^0 *basis).*

Proof Problem 5.11. □

Corollary *If* $\dim V = n$, *then* $\dim \mathbf{S}^s(V^*) = \binom{n+s-1}{s}$.

Proof The number of terms in (5.5) is the number of ways of choosing s integers between 1 and n in nondecreasing order (or, the number of ways of choosing n integers between 1 and s in nondecreasing order, if $n < s$). □

Finally, if A is symmetric, (5.5) becomes

$$A = A_{i_1 \cdots i_s} \sum_{\pi *} \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}} \quad i_1 \leq \cdots \leq i_s \tag{5.6}$$

Note that the components in this basis are simply a subset of the components of A in the V_s^0 basis.

PROBLEM 5.8. Prove Theorem 5.5.

PROBLEM 5.9. Prove Theorem 5.8.

PROBLEM 5.10. Prove $\mathfrak{S}$ commutes with the mapping ϕ_r of Section 4.3. In particular, symmetry is preserved under linear mappings.

PROBLEM 5.11. Prove Theorem 5.10.

5.3 The skew-symmetric (alternating) subspace of V_s^0

The following treatment closely parallels that of Section 5.2. Again we restrict our attention to the covariant tensors V_s^0. Obvious analogs exist for the spaces V_0^r.

Definitions A tensor $A \in V_s^0$ is *skew-symmetric* (or *alternating*) if it is skew-symmetric in all pairs of its arguments. This is equivalent to the property $A(v_1, \ldots, v_s) = (\operatorname{sgn} \pi) A(v_{\pi(1)}, \ldots, v_{\pi(s)})$ for all permutations π of $1, 2, \ldots, s$. A skew-symmetric tensor of type $(0, s)$ is called *an exterior s-form.* (A skew-symmetric tensor of type $(r, 0)$ is called *an r-vector.*)

Note that all zero and first order tensors are skew-symmetric. Note also, that by Theorem 5.1, in contrast to the symmetric case, the only skew-symmetric tensors of order greater than $\dim V$ are the zero tensors.

We will let π also denote the automorphism of V_s^0 determined by $\sigma^1 \otimes \sigma^2 \otimes \cdots \otimes \sigma^s \mapsto \sigma^{\pi(1)} \otimes \sigma^{\pi(2)} \otimes \cdots \otimes \sigma^{\pi(s)}$.

Theorem 5.11 *A is skew-symmetric if and only if $\pi \cdot A = (\text{sgn } \pi)A$ for all permutations, π, of $1, \ldots, s$.*

Proof Problem 5.12. □

Corollary *A is skew-symmetric if and only if $A_{i_1 \cdots i_s} = (\text{sgn } \pi) A_{i_{\pi(1)} \cdots i_{\pi(s)}}$ for all permutations, π, of $1, 2, \ldots, s$.*

Theorem 5.12 *The set, $\bigwedge^s(V^*)$, of all skew-symmetric tensors of V_s^0 is a subspace of V_s^0.*

There are several other ways to describe the subspace $\bigwedge^s(V^*)$ of skew-symmetric tensors of V_s^0.

Definitions The linear transformation

$$\mathfrak{A} : V_s^0 \to \textstyle\bigwedge^s(V^*)$$

defined by

$$A \mapsto \frac{1}{s!} \sum_{\pi} (\text{sgn } \pi) \pi \cdot A \tag{5.7}$$

is called *the alternating operator on V_s^0* and $\mathfrak{A} \cdot A$ is called *the skew-symmetric part* of A.

The following three theorems are analogs of Theorems 5.7 - 5.9.

Theorem 5.13 $\mathfrak{A} \cdot A = \frac{1}{s!} \sum_{\pi} (\text{sgn } \pi) \pi \cdot A$ *if and only if*

$$\mathfrak{A} \cdot A(v_1, \ldots, v_s) = \frac{1}{s!} \sum_{\pi} (sgn\ \pi) A(v_{\pi(1)}, \ldots, v_{\pi(s)}) \tag{5.8}$$

Corollary *If $A = \sigma^1 \otimes \cdots \otimes \sigma^s$, then $\mathfrak{A} \cdot A(v_1, \ldots, v_s) = \dfrac{1}{s!} \det(\sigma^i \cdot v_j)$.*

Theorem 5.14 *The linear closure of the set of all elements of V_s^0 which are symmetric in two arguments is the same as the linear closure of the set of all elements of V_s^0 of the form*

$$\sigma^1 \otimes \cdots \otimes \sigma^i \otimes \cdots \otimes \sigma^j \otimes \cdots \otimes \sigma^s + \sigma^1 \otimes \cdots \otimes \sigma^j \otimes \cdots \otimes \sigma^i \otimes \cdots \otimes \sigma^s$$

and the same as the linear closure of the set of decomposable elements with two or more factors the same.

Theorem 5.15 $N_A = \ker \mathfrak{A}$ *where N_a is the subspace of V_S^0 described in Theorem 5.14.*

Corollary (i) $\bigwedge^s(V^*) \cong V_s^0/N_A$ *and,*
(ii) $V_s^0 = N_A \oplus \bigwedge^s(V^*)$.

To summarize, we have described a certain subspace of V_s^0 in three ways: (i) the subset of V_s^0 stable (invariant) under skew-symmetry automorphisms of V_s^0; (ii) the image of the alternating operator; and (iii) a quotient (factor) space.

We have representations of $\mathfrak{A} \cdot A$ and A when A is skew-symmetric corresponding to those for the symmetric case. Thus, corresponding to (5.3), we can express the components of $\mathfrak{A} \cdot A$ in terms of those of A by

$$\mathfrak{A} \cdot A = \frac{1}{s!} \sum_{\pi} (\text{sgn } \pi) A_{i_{\pi(1)} \cdots i_{\pi(s)}} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_s} \tag{5.9}$$

As in the corresponding formula for the symmetric part of A, there are n^s terms in (5.9). However, now whenever two of the indices $i_1, \ldots, i_s$ have the same value, the coefficient vanishes.

Using

$$\mathfrak{A} \cdot A = \frac{1}{s!} A_{i_1 \cdots i_s} \sum_{\pi} (\text{sgn } \pi) \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}} \tag{5.10}$$

which we get from the definition, (5.7), of $\mathfrak{A}$, we can write the skew-symmetric tensor $\mathfrak{A} \cdot A$ in a basis of $\bigwedge^s(V^*)$ in terms of its components in a basis of V_s^0. Note that the factors $\sum_\pi (\text{sgn } \pi) \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}}$ in (5.10), which are in $\bigwedge^s(V^*)$, vanish whenever two of the indices $i_1, \cdots i_s$ have the same value. In particular, if $s > n$, then $\mathfrak{A} \cdot A = 0$. For every set of distinct values of $i_1, \ldots, i_s$, half the $s!$ factors $\sum_\pi (\text{sgn } \pi) \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}}$ with these values have the value

$$\sum_{\pi} (\text{sgn } \pi) \varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}},$$

and half have the value

$$-\sum_{\pi} (\text{sgn } \pi)\varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}}.$$

Hence

$$\mathfrak{A} \cdot A = \sum_{\pi_e} A_{i_{\pi(1)}\cdots i_{\pi(s)}} \sum_{\pi} (\text{sgn } \pi)\varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}}$$

$$+\sum_{\pi_o} A_{i_{\pi(1)}\cdots i_{\pi(s)}} \left(-\sum_{\pi} (\text{sgn } \pi)\varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}} \right) \tag{5.11}$$

where π_e in the coefficient of the first term is an even permutation of $1, \ldots, s$, and π_o in the coefficient of the second term is an odd permutation of $1, \ldots, s$. Moreover, for each set of distinct values of $i_1, \ldots, i_s$ we can choose the order $i_1 < i_2 < \cdots < i_s$. For either this order is an even permutation of the given order, in which case (5.11) is valid for $i_1 < \cdots < i_s$, or, if this order is an odd permutation of the given order, a minus sign must be introduced in the second factors of (5.11) and π_e and π_0 must be interchanged. The result is

$$\mathfrak{A} \cdot A = \frac{1}{s!} \sum_{\pi} (\text{sgn } \pi) A_{i_{\pi(1)}\cdots i_{\pi(s)}} \sum_{\pi} (\text{sgn } \pi)\varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}}$$
$$i_1 < i_2 < \cdots < i_s \tag{5.12}$$

(cf., eq. (5.5)).

Theorem 5.16 *The set* $\left\{ \sum_{\pi} (\text{sgn } \pi)\varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}};\ i_1 < \cdots < i_s \right\}$ *of tensors appearing in* (5.12) *form a basis of* $\bigwedge^s(V^*)$.

Corollary *If* $\dim V = n$, *then* $\dim \bigwedge^s(V^*) = \binom{n}{s}$. *In particular, if* $s = \dim V$, *then* $\bigwedge^s(V^*)$ *is 1-dimensional and spanned by* $\varepsilon^1 \otimes \cdots \otimes \varepsilon^s$. *If* $s > n$, *then* $\bigwedge^s(V^*)$ *has only the zero element.*

Finally, if A is skew-symmetric (5.12) reduces to

$$A = A_{i_1\cdots i_s} \sum_{\pi} (\text{sgn } \pi)\varepsilon^{i_{\pi(1)}} \otimes \cdots \otimes \varepsilon^{i_{\pi(s)}} \quad i_1 < i_2 < \cdots < i_s \tag{5.13}$$

We can simplify eqs. (5.12) and (5.13) if we introduce the following definition and notation.

Definition For $\sigma^i \in V^*$, $\mathfrak{A} \cdot \sigma^1 \otimes \cdots \otimes \sigma^s$ is called *the exterior* (or *wedge*, or *skew-symmetric*) *product* of $\sigma^1, \sigma^2, \ldots, \sigma^s$, and we write

$$\mathfrak{A} \cdot \sigma^1 \otimes \cdots \otimes \sigma^s = \sigma^1 \wedge \sigma^2 \wedge \cdots \wedge \sigma^s \tag{5.14}$$

Then

$$\sum_{\pi} (\operatorname{sgn} \pi) \sigma^{\pi(1)} \otimes \cdots \otimes \sigma^{\pi(s)} = s! \sigma^1 \wedge \cdots \wedge \sigma^s$$

and

$$\sum_{\pi} (\operatorname{sgn} \pi) \sigma^{i_{\pi(1)}} \otimes \cdots \otimes \sigma^{i_{\pi(s)}} = s! \sigma^{i_1} \wedge \cdots \wedge \sigma^{i_s}.$$

So, using the wedge product notation we can write (5.12) as

$$\mathfrak{A} \cdot A = \Big(\sum_{\pi} (\operatorname{sgn} \pi) A_{i_{\pi(1)} \cdots i_{\pi(s)}} \Big) \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_s} \quad i_1 < \cdots < i_s \tag{5.15}$$

and $\{\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_s} : i_1 < \cdots < i_s\}$ is a basis of $\bigwedge^s(V^*)$. If A is skew-symmetric, and we use this notation, then both (5.13) and (5.15) reduce to

$$A = s! A_{i_1 \cdots i_s} \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_s} \quad i_1 < \cdots < i_s \tag{5.16}$$

Comparing (5.15) with (5.12) and (5.16) with (5.13), we get that the components of a tensor in $\bigwedge^s(V^*)$ with respect to our wedge product basis are $s!$ times the components of Theorem 5.16.

(Note that the expression for $\mathfrak{A} \cdot A$ given by eq. (5.9) also simplifies if we use the wedge product notation. Thus, $\mathfrak{A} \cdot A = A_{i_1 \cdots i_s} \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_s}$. However, here $i_1, \ldots, i_s$ all run from 1 to n, and $\mathfrak{A} \cdot A$ is expressed in terms of a linearly dependent set and not a basis.)

It is important to note that our definition (5.14) of the exterior product is only one of two prevalent definitions in the literature which differ by a factor. According to the other definition,

$$\sigma^1 \wedge \cdots \wedge \sigma^s = s! \mathfrak{A} \cdot \sigma^1 \otimes \cdots \otimes \sigma^s.$$

With this definition the wedge product basis is the same as the basis of Theorem 5.16 and the components of $\mathfrak{A} \cdot A$ and A are just as they were in eqs. (5.12) and (5.13).

Finally, we point out that there are exactly corresponding definitions and results for the contravariant tensors, V_0^r. We have *the skew-symmetric* (*or, alternating*) *subspace*, $\bigwedge^r(V)$, the alternating operator, $\mathfrak{A}$, as in eq. (5.7), and three characterizations of $\bigwedge^r(V)$ corresponding to those for $\bigwedge^s(V^*)$. If we define

$$v_1 \wedge \cdots \wedge v_r = \mathfrak{A} \cdot v_1 \otimes \cdots \otimes v_r \tag{5.17}$$

we can write

$$\mathfrak{A} \cdot A = \sum_{\pi} (\operatorname{sgn} \pi) A^{i_{\pi(1)} \cdots i_{\pi(r)}} e_{i_1} \wedge \cdots \wedge e_{i_r} \quad i_1 < \cdots < i_r \tag{5.18}$$

corresponding to eq. (5.15), and if A is skew-symmetric its components in our wedge product basis are $s! A^{i_1 \cdots i_r}$.

PROBLEM 5.12. Prove Theorem 5.11.

PROBLEM 5.13. For $\bigwedge^2(V^*)$:

(i) If $\dim V = 3$, every element of $\bigwedge^2(V^*)$ is decomposable.

(ii) If $\dim V = 4$, not every element of $\bigwedge^2(V^*)$ is decomposable. (Decomposable elements of $\bigwedge^s(V^*)$ have the form $\sigma^1 \wedge \cdots \wedge \sigma^s$ (cf., Section 4.1).

PROBLEM 5.14. From the results of Chapters 2 and 3 there is natural isomorphism $\mathfrak{L}(V_2^0, V_2^0) \cong V_2^2$, so for $s = 2$, $\mathfrak{A} \in V_2^2$. If $\mathfrak{A}^{ij}_{pq}$ are components of $\mathfrak{A}$, show that in every basis $\mathfrak{A}^{ij}_{pq} = \delta^i_p \delta^j_q - \delta^i_q \delta^j_p$. (See Section 4.2 and Problem 4.13.)

PROBLEM 5.15. (i) If σ, τ, ω are in V^*, express the values and the components of $\sigma \wedge \tau \wedge \omega$ in terms of their values and components in a basis of V^*. (ii) Show that $\sigma^1 \wedge \cdots \wedge \sigma^s(v_1, \ldots, v_s) = (1/s!)\det(\langle \sigma^i, v_j \rangle)$.

PROBLEM 5.16. Write out the formulas (5.9), (5.12), and (5.13) for $\mathfrak{A} \cdot A$, and for a skew-symmetric tensor for the special case of covariant tensors of rank 3 with $\dim V = 4$.

PROBLEM 5.17. For the decomposable elements $\sigma^1 \otimes \cdots \otimes \sigma^s \in V_s^0$ we define *a symmetric product* by $\sigma^1 \circledast \cdots \circledast \sigma^s = \mathfrak{S} \cdot \sigma^1 \otimes \cdots \otimes \sigma^s$. Then a symmetric tensor can be written as a homogeneous polynomial in $\sigma^1, \ldots, \sigma^s$. Write the coefficients in terms of the $A_{i_1 \cdots i_s}$ of eq. (5.6).

5.4 Some special properties of $\mathbf{S}^2(V^*)$ and $\bigwedge^2(V^*)$

We saw, in Section 4.1, that the space of second-order covariant tensors, $V^* \otimes V^*$, is isomorphic to the space of linear mappings, $\mathfrak{L}(V, V^*)$, from V to V^*. Specifically, we have an isomorphism given by $\phi_1 : b \mapsto \mathcal{A}_1$ where $b \in V^* \otimes V^*$ and $\mathcal{A}_1 \in \mathfrak{L}(V, V^*)$ is given by $\mathcal{A}_1 : v \mapsto b(v, -)$, and we also have an isomorphism $\phi_2 : b \mapsto \mathcal{A}_2$ where $\mathcal{A}_2 : v \mapsto b(-, v)$ (cf., Theorem 2.11).

Also, since the elements of $V^* \otimes V^*$ are bilinear functions we have the concept of nondegeneracy and the result of Section 4.1 that a tensor of type $(0, 2)$ (i.e., an element of $V^* \otimes V^*$)) is nondegenerate iff $\mathcal{A}_1$ and $\mathcal{A}_2$ are nonsingular. Corresponding properties are, of course, valid for the symmetric and skew-symmetric subspaces, $\mathbf{S}^2(V^*)$ and $\bigwedge^2(V^*)$, of V_2^0. These two subspaces play prominent roles in geometry and mechanics, respectively, so we need to examine their properties a bit further.

(i) *Properties of* $\mathbf{S}^2(V^*)$. First of all we note that for $b \in \mathbf{S}^2(V^*)$, ϕ_1 and ϕ_2 are the same. For such a b, we write $\mathcal{A}_1 = \mathcal{A}_2 = b^\flat$ and the null space of the common linear mapping $b^\flat : V \to V^*$ is called *the null space of* b. It is the set $\{v \in V : b(v, w) = 0 \text{ for all } w \in V\}$ (cf., Theorem 2.14).

Next we look at the restriction of b to the diagonal elements of $V \times V$.

Definition Given b (symmetric), the map, Q, from V to $\mathbb{R}$ given by $Q : v \mapsto b(v,v)$ is the *quadratic function (form) associated with b.*

If $Q(v) = b(v,v)$, then $Q(v+w) = b(v+w, v+w)$, from which we get

$$b(v,w) = \frac{1}{2}[Q(v+w) - Q(v) - Q(w)] \quad \text{(the cosine law)} \tag{5.19}$$

which shows that the association $b \mapsto Q$ is $1-1$. Alternatively, using the expansion for $Q(v-w)$ in addition to that above for $Q(v+w)$ we get

$$b(v,w) = \frac{Q(v+w) - Q(v-w)}{4} \quad \text{(the polarization identity).}$$

The mapping $Q \mapsto b$ is called *the polarization of Q.*

Definition b is *positive definite* if $Q(v) > 0$ for all $v \neq 0$. b is *positive semidefinite* if $Q(v) \geq 0$ for all v. There are corresponding definitions with "positive" replaced by "negative". If b is not definite there can be *null vectors*, or *lightlike vectors* $v \neq 0$ for which $Q(v) = 0$. The *null cone of b* is the set of all null vectors of b. In the *indefinite* (neither positive semidefinite, nor negative semidefinite) case, vectors for which $Q(v) < 0$ are called *timelike*, and those for which $Q(v) > 0$ are called *spacelike.*

Theorem 5.17 (The Cauchy-Schwarz inequality) *If b is semidefinite, then it satisfies*

$$[b(v,w)]^2 \leq b(v,v)b(w,w)$$

for all $v, w \in V$.

Proof For all real a, $b(av+w, av+w) = a^2 b(v,v) + 2ab(v,w) + b(w,w)$. This expression is either always nonnegative, or always nonpositive. In either case as a quadratic, in a, its discriminant, $[b(v,w)]^2 - b(v,v)b(w,w)$, must be either negative or zero. (In contrast to the usual case, now $w = av$ is sufficient, but not necessary for equality.) □

Corollary *If b is semidefinite and not definite, then b is degenerate.*

Theorem 5.18 *b is definite if and only if b is semidefinite and nondegenerate.*

Proof Problem 5.19. □

Theorem 5.19 *b is definite if and only if it has no null vectors.*

Proof The only case not simply a matter of definition is when b is indefinite. Then, if $b(v,v) < 0$ and $b(w,w) > 0$, there is a number, $0 < a < 1$, such that for $z = av + (1-a)w$, $b(z,z) = 0$. □

For examples of the various possibilities for a symmetric tensor of type $(0,2)$, let $V = \mathbb{R}^n$ then with respect to the natural basis of $\mathbb{R}^n$, b is given by a set of numbers b_{ij} (Section 2.2). With these arranged in a matrix, (the matrix of $b^\flat$) consider the following examples.

(1) $(b_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ b is nondegenerate, positive definite

(2) $(b_{ij}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ b is nondegenerate, indefinite

(3) $(b_{ij}) = \begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix}$ b is nondegenerate, negative definite

(4) $(b_{ij}) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ b is degenerate, negative semidefinite

(5) $(b_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ b is degenerate, indefinite

We have been looking at the space $\mathbf{S}^2(V^*)$ and classifying its elements according to the possible values of $Q(v)$. Now we switch our point of view and pick one element out of $\mathbf{S}^2(V^*)$ and see what this does for V. First of all with a given $b \in \mathbf{S}^2(V^*)$ we can generalize the concepts of length and orthogonality of vectors in V.

Definitions A vector space, V, for which a nondegenerate element of $\mathbf{S}^2(V^*)$ is chosen is called *a scalar (inner) product vector space. The length (magnitude) of* v, $\|v\|$, is $|Q(v)|^{\frac{1}{2}}$ $(= |b(v,v)|^{\frac{1}{2}})$. *$v$ and w are orthogonal* if $b(v,w) = 0$. Since, in the semidefinite case, we have the Cauchy-Schwarz inequality, we can in that case define *the angle between v and w* by $\cos\vartheta = |b(v,w)|/(\|v\|\;\|w\|)$ when neither $\|v\|$ nor $\|w\|$ are zero.

With a given b, we can decompose V into a direct sum of subspaces: $V = V_+ \oplus V_- \oplus V_0$. V_0 is the null space of b. b is positive definite on V_+ and not on any larger subspace of V, and b is negative definite on V_- and not on any larger subspace of V. The choice of V_+ is not unique, though the dimension is. Similarly for V_-. See Greub (p. 265 ff).

Theorem 5.20 *Given b we can choose an orthonormal basis for V.*

Proof (i) The case $V = V_+$. (The Gram-Schmidt process.) Let $\{e_1, \ldots, e_n\}$ be a basis of V_+. We construct an orthogonal basis $\{\bar{e}_1, \ldots, \bar{e}_n\}$ as follows. Let $\bar{e}_1 = e_1$. Let $\bar{e}_2 = e_2$ – the projection of e_2 on $\bar{e}_1$,

$$= e_2 - \|e_2\| \frac{b(e_2, \bar{e}_1)}{\|e_2\| \ \|\bar{e}_1\|} \frac{\bar{e}_1}{\|\bar{e}_1\|} = e_2 - \frac{b(e_2, \bar{e}_1)}{b(\bar{e}_1, \bar{e}_1)} \bar{e}_1$$

Let $\bar{e}_3 = e_3$ – the projection of e_3 *on the plane of* $(\bar{e}_1, \bar{e}_2)$,

$$= e_3 - \frac{b(e_3, \bar{e}_2)}{b(\bar{e}_2, \bar{e}_2)} \bar{e}_2 - \frac{b(e_3, \bar{e}_1)}{b(\bar{e}_1, \bar{e}_1)} \bar{e}_1 \qquad \text{and so on}$$

These vectors will be orthogonal. Now divide each $\bar{e}_i$ by its magnitude.

(ii) For the general case, $V_+ \oplus V_- \oplus V_0$, we can construct a basis for V_+ and a basis for V_- as in (i), and any basis of V_0 will be orthonormal. Together these will form an orthonormal basis of $V_+ \oplus V_- \oplus V_0$ (loc. cit.). □

In terms of an orthonormal basis of V, b has the form

$$b = \varepsilon^1 \otimes \varepsilon^1 + \cdots + \varepsilon^r \otimes \varepsilon^r - \varepsilon^{r+1} \otimes \varepsilon^{r+1} - \cdots - \varepsilon^{r+s} \otimes \varepsilon^{r+s} \tag{5.20}$$

The matrix, in this basis, of the linear mapping $b^\flat : V \to V^*$ corresponding to b is

$$\begin{pmatrix} I_r & 0 & 0 \\ 0 & -I_s & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where I_r and I_s are respectively $r \times r$ and $s \times s$ identity matrices. In matrix language, Theorem 5.20 says that every real symmetric matrix is congruent to a diagonal matrix with 1's, - 1's, and 0's. See Problem 2.8.

To illustrate these results, we again consider the space $V = \mathbb{R}^n$, and suppose b is given by its components in the natural bases as in the examples above. The natural basis is not an orthonormal basis unless b has the form (5.20). In example (1), b has the form (5.20) so the natural basis is orthonormal. In example (2), there is a basis for which $b = \varepsilon^1 \otimes \varepsilon^1 - \varepsilon^2 \otimes \varepsilon^2$. In example (3), we can get $b = -\varepsilon^1 \otimes \varepsilon^1 - \varepsilon^2 \otimes \varepsilon^2$. In example (4) we can get $b = -\varepsilon^1 \otimes \varepsilon^1$. And in example

(5) the natural basis is again orthonormal. Going one step further for example (3), since

$$\begin{pmatrix} 0 & 1 \\ 2(\frac{1}{3})^{\frac{1}{2}} & (\frac{1}{3})^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} 0 & 2(\frac{1}{3})^{\frac{1}{2}} \\ 1 & (\frac{1}{3})^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

are congruent. Alternatively, according to (4.5),

$$\begin{pmatrix} 0 & 2(\frac{1}{3})^{\frac{1}{2}} \\ 1 & (\frac{1}{3})^{\frac{1}{2}} \end{pmatrix}$$

is the change of basis matrix, (a^i_j), so using (1.3) we find that for this b, $\{(0,1), (2(\frac{1}{3})^{\frac{1}{2}}, (\frac{1}{3})^{\frac{1}{2}})\}$ is an orthonormal basis for $\mathbb{R}^2$.

Definitions In a decomposition of V by b, the dimension of V_- is called *the index* of b. *A Euclidean vector space*, E^n_0, is a scalar (inner) product vector space with index 0. *A Lorentzian vector space*, E^n_1, is a scalar product vector space with index 1 (or $n-1$). An affine space (Problem 1.15) whose vector space is E^n_0 is called *a Euclidean affine space*, and is denoted by $\mathcal{E}^n_0$. An affine space whose vector space is E^n_1 is *a Lorentzian affine space*, $\mathcal{E}^n_1$.

In Lorentzian vector spaces, V can be decomposed as the direct sum of 1-dimensional timelike subspaces, and $n-1$-dimensional spacelike subspaces. Lorentzian vector spaces form the basis of Einstein's theory of relativity, whose "bizarre" geometry comes from the fact that Theorem 5.17 is no longer valid.

Theorem 5.21 (Backward Cauchy-Schwarz inequality) *If v and w are timelike vectors of a Lorentzian vector space, then*

$$[b(v,w)]^2 \geq b(v,v)b(w,w)$$

Proof We can write $w = av + z$ where $z \in V_+$. Then

$$b(w,w) = a^2 b(v,v) + b(z,z)$$

and $b(v,w) = ab(v,v)$. So

$$[b(v,w)]^2 = a^2[b(v,v)]^2 = b(v,v)[b(w,w) - b(z,z)]$$
$$= b(v,v)b(w,w) - b(v,v)b(z,z).$$

Since the last term is positive, we have our result. □

Two timelike vectors of a Lorentzian space cannot be orthogonal (Problem 5.23), so either $b(v, w) > 0$ or $b(v, w) < 0$. In the latter case we say that v and w are *forward-facing* or *future-pointing*. In that case we can define a function, ϑ, of v and w, by

$$\cosh \vartheta = \frac{-b(v, w)}{\|v\|\|w\|}$$

called *the hyperbolic angle between* v *and* w.

Theorem 5.22 *If* v *and* w *are timelike, then* (i) $b(v, w) < 0 \Rightarrow v + w$ *is timelike and forward-facing.*

(ii) $b(v, w) > 0 \Rightarrow v + w$ *is spacelike.*

Proof (i) $b(v + w, v + w) = b(v, v) + 2b(v, w) + b(w, w) < 0$ and $b(v + w, v) = b(v, v) + b(v, w) < 0$.

(ii) Apply the backward Cauchy-Schwarz inequality to the expansion in part (i). □

Finally, we will be interested in mappings of inner product spaces.

Definition If $\phi : V \to W$ is a linear mapping, b is a symmetric bilinear function in $\mathbf{S}^2(V^*)$, and B is a bilinear function in $\mathbf{S}^2(W^*)$ then ϕ is an *isometry* if

$$B(\phi \cdot v_1, \phi \cdot v_2) = b(v_1, v_2) \tag{5.21}$$

That is, these are maps of inner product spaces which preserve their structures.

Using the notation of eq. (4.10), we can write (5.21) as

$$\phi^2 \cdot B = b$$

and we say b is *the pull-back of* B *by* ϕ.

Frequently, the term isometry is reserved for the case where V and W have the same dimension. In particular, if ϕ is a linear transformation on a space V with inner product b, then ϕ is called *an orthogonal transformation.* If b is represented in an orthonormal basis by

$$\begin{pmatrix} I_r & \\ & -I_s \end{pmatrix}$$

then ϕ is an orthogonal transformation iff the matrix (ϕ^i_j) of ϕ in that basis satisfies

$$(\phi^i_j)^{\mathrm{tr}} \begin{pmatrix} I_r & \\ & -I_s \end{pmatrix} (\phi^i_j) = \begin{pmatrix} I_r & \\ & -I_s \end{pmatrix} \tag{5.22}$$

As an example, if $V = \mathbb{R}^2$ and b is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then the transformations represented by

$$\begin{pmatrix} \cosh\vartheta & \sinh\vartheta \\ \sinh\vartheta & \cosh\vartheta \end{pmatrix},$$

for all real ϑ are orthogonal.

If $b \in \mathbf{S}^2(V^*)$ and $B \in \mathbf{S}^2(W^*)$, another important class of mappings arise as a special case of eq. (2.4). If there, $V_1 = W_1 = V$, and $V_2 = W_2 = W$, then V and W are both self-dual, and a linear map $\phi : V \to W$ has a dual map, $\phi^* : W \to V$. In this case ϕ^* is called *the adjoint map* of ϕ. If $V = W$ and $\phi = \phi^*$, then eq. (2.4) becomes

$$b(v_1, \phi^* \cdot v_2) = B(\phi \cdot v_1, v_2)$$

and the linear operator ϕ is *self-adjoint*. The spectra of self-adjoint operators have special properties giving them a special importance in functional analysis (cf., Friedman).

(ii) *Properties of* $\bigwedge^2(V^*)$. First of all, for $b \in \bigwedge^2(V^*)$, $\phi_2 \cdot b = -\phi_1 \cdot b$, and we write $b^\flat = \phi_1 \cdot b$. Again the null space of b is the null space of $b^\flat$ and b is nondegenerate iff its null space is just zero.

Now it is clear that we cannot emulate for skew-symmetric tensors the classification of elements in terms of the values of $b(v,v)$ as we did for $\mathbf{S}^2(V^*)$, since $b(v,v) = 0$ now for all v. However, given an element, b, of $\bigwedge^2(V^*)$ we can get a result corresponding to Theorem 5.20; i.e., we can find a basis of V in terms of which b has a "canonical" representation.

Theorem 5.23 *Given* $b \in \bigwedge^2(V^*)$, *we can choose a basis* $\{\Xi^1, \ldots, \Xi^k\}$ *of the range of the linear map* $b^\flat$ *such that*

$$b = \Xi^1 \wedge \Xi^2 + \Xi^3 \wedge \Xi^4 + \cdots + \Xi^{2p-1} \wedge \Xi^{2p} \tag{5.23}$$

with $2p = k$.

Proof If $\{\varepsilon^i\}$ is a basis of V^*, then

$$b = a_{12}\varepsilon^1 \wedge \varepsilon^2 + a_{13}\varepsilon^1 \wedge \varepsilon^3 + \cdots + a_{1n}\varepsilon^1 \wedge \varepsilon^n$$
$$+ a_{23}\varepsilon^2 \wedge \varepsilon^3 + \cdots + a_{2n}\varepsilon^2 \wedge \varepsilon^n + \sum_{2<i<j} a_{ij}\varepsilon^i \wedge \varepsilon^j$$

$$= \varepsilon^1 \wedge \Xi^1 + \varepsilon^2 \wedge \Sigma^2 + \sum_{2<i<j} a_{ij}\varepsilon^i \wedge \varepsilon_j \tag{5.24}$$

where

$$\begin{aligned} \Xi^1 &= a_{12}\varepsilon^2 + \cdots + a_{1n}\varepsilon^n = a_{12}\varepsilon^2 + \Sigma^1 \\ \Sigma^2 &= a_{23}\varepsilon^3 + \cdots + a_{2n}\varepsilon^n \end{aligned} \tag{5.25}$$

We can assume $a_{12} \neq 0$, so from (5.25),

$$\varepsilon^2 = \frac{\Xi^1}{a_{12}} - \frac{\Sigma^1}{a_{12}} \tag{5.26}$$

Substituting (5.26) into (5.24) we get

$$\begin{aligned} b &= \varepsilon^1 \wedge \Xi^1 + \left(\frac{\Xi^1}{a_{12}} - \frac{\Sigma^1}{a_{12}}\right) \wedge \Sigma^2 + \sum_{2<i<j} a_{ij}\varepsilon^i \wedge \varepsilon^j \\ &= \Xi^1 \wedge \left(\frac{\Sigma^2}{a_{12}} - \varepsilon^1\right) - \frac{\Sigma^1 \wedge \Sigma^2}{a_{12}} + \sum_{2<i<j} a_{ij}\varepsilon^i \wedge \varepsilon^j \end{aligned}$$

so

$$b = \Xi^1 \wedge \Xi^2 + \sum_{2<i<j} \tilde{a}_{ij}\varepsilon^i \wedge \varepsilon^j \tag{5.27}$$

where

$$\Xi^2 = \frac{\Sigma^2}{a_{12}} - \varepsilon^1 = \frac{a_{23}\varepsilon^3 + \cdots + a_{2n}\varepsilon^n}{a_{12}} - \varepsilon^1$$

Let $\{e_i\}$ be the basis of V dual to $\{\varepsilon^i\}$ then,

$$\Xi^1(e_i) = a_{1k}\varepsilon^k(e_i) = a_{1i} \quad \text{and} \quad b(e_1, e_i) = \frac{a_{1i}}{2}$$

for $i \neq 1$ and $\Xi^1(e_1) = b(e_1, e_1) + 0$ so $\Xi^1(e_i) = b(2e_1, e_i)$ for all e_i, and hence, $\Xi^1 = b(2e_1, -)$. That is, Ξ^1 is the image of $2e_1$ under the linear map $b^\flat$, so Ξ^1 is in the range of $b^\flat$. Similarly, for Ξ^2. Clearly, Ξ^1 and Ξ^2 are linearly independent.

Now, either there is a nonzero term in the sum on the right side of (5.27), or all $\tilde{a}_{ij} = 0$. In the first case we can split $\sum_{2<i<j} \tilde{a}_{ij}\varepsilon^i \wedge \varepsilon^j$ into a term $\Xi^3 \wedge \Xi^4$ plus a sum containing two fewer ε's (by the same method we used on b). In any case we will eventually get $b = \Xi^1 \wedge \Xi^2 + \Xi^3 \wedge \Xi^4 + \cdots + \Xi^{2p-1} \wedge \Sigma^{2p}$ where the Ξ's are all linearly independent and are all in the image of $b^\flat$.

Now it only remains to show that $\{\Sigma^1, \ldots, \Xi^{2p}\}$ is a basis of the range of $b^\flat$. Extend $\{\Xi^1, \ldots, \Xi^{2p}\}$ to a basis of V^* and let $\{\bar{e}_1, \ldots, \bar{e}_n\}$ be the dual basis of V. If i is odd, and $\leq 2p-1$, then $b(\bar{e}_i, -) \cdot \bar{e}_j = b(\bar{e}_i, \bar{e}_j) = (\Xi^{i+1}/2)\bar{e}_j$ for all j, so $b^\flat(\bar{e}_i) = b(\bar{e}_i, -) = \Xi^{i+1}/2$ for i odd and $\leq 2p-1$. Similarly, $b^\flat(\bar{e}_i) = -\Xi^{i-1}/2$ for i even and $\leq 2p$, and $b^\flat(\bar{e}_i) = 0$ for $i > 2p$. Hence $b^\flat$ takes every $v \in V$ into the space $\langle \Xi^1, \ldots, \Xi^{2p} \rangle$. Since all Ξ^i are in the range of $b^\flat$ and $\{\Xi^1, \ldots, \Xi^{2p}\}$ is a linearly independent set, it is a basis of the range of $b^\flat$. $\square$

Corollary *The rank of* $b \in \bigwedge^2(V^*)$ *(defined to be the rank of* $b^\flat$*) is even, and* b *can be nondegenerate only if* $\dim V$ *is even.*

Definition A basis for V with respect to which b is written as eq. (5.23) is *a symplectic basis.* If b is nondegenerate, V is called *a symplectic vector space.*

Instead of (5.23), it is more common to write b in terms of the basis of Theorem 5.23 using a different indexing:

$$b = \varepsilon^1 \wedge \varepsilon^{1+p} + \varepsilon^2 \wedge \varepsilon^{2+p} + \cdots + \varepsilon^p \wedge \varepsilon^{2p} \tag{5.28}$$

The matrix in this basis of the linear mapping $b^\flat$, corresponding to b, is $\frac{1}{2}J$, where

$$J = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.29}$$

and I is the $p \times p$ identity matrix. Note that $J^2 : V \to V$ and $J^2 = -id$. (See Problem 1.14.) Theorem 5.23, in matrix terms, says that every skew-symmetric matrix is congruent to a matrix of type (5.29).

By means of $b \in \bigwedge^2(V^*)$ we can identify certain subspaces of V just as we did for elements of $\mathbf{S}^2(V^*)$. Thus, for $W \subset V$, let $W^\perp$ be the orthogonal complement of W with respect to b, then from $\dim b(W, -) = \dim V - \dim W^\perp$ and $\dim W = \dim(W \cap V^\perp) + \dim b(W, -)$ (Theorem 1.4) we get

$$\dim W + \dim W^\perp = \dim V + \dim(W \cap V^\perp) \tag{5.30}$$

We can use (5.30) to prove the following result.

Theorem 5.24 *If* $W \subset W^\perp$*, then* $W = W^\perp$ *iff* $\dim W = \frac{1}{2}(\dim V + \dim V^\perp)$.

Proof Problem 5.27. □

Definition A subspace W of V is (i) *Lagrangian* if $W = W^\perp$; (ii) *isotropic* if $W \subset W^\perp$; and (iii) *coisotropic* if $W^\perp \subset W$.

From eq. (5.30) we get that if W is isotropic (coisotropic) then $\dim W \leq (\geq) \frac{1}{2}(\dim V + \dim V^\perp)$.

It can be shown that every isotropic subspace can be enlarged to a Lagrangian subspace, and with every coisotropic subspace, W, we can associate a symplectic vector space, $W/W^\perp$.

We can illustrate these ideas in $\mathbb{R}^n$ (if the space is to be symplectic, n must be even). Thus, in $\mathbb{R}^2$, if W is the set of pairs $\{p = (a_1 t, a_2 t) : t \in \mathbb{R}\}$, then $W^\perp = \{q : b(q,p) = 0 \ \forall \ p \in W\}$ is W, so W is Lagrangian.

In addition to $\mathbb{R}^{2n}$, there are other important examples of symplectic vector spaces

1. Given V, the direct sum $V \oplus V^*$ with

$$b(v + \sigma, w + \tau) = \langle \sigma, v \rangle - \langle \tau, w \rangle$$

   is a symplectic space, and conversely, every symplectic space can be represented as a direct sum.
2. In Problem 1.14 we saw that for some $2n$-dimensional vector spaces (e.g., $\mathbb{R}^{2n}$) there is a complex structure, J. Suppose V also has a *Hermitian inner product*, that is, an inner product, b, such that $b(Jv, Jw) = b(v, w)$. Then with $B(v, w) = b(v, Jw)$, V is a symplectic vector space. Straightforward calculations show that B is skew-symmetric, nondegenerate and $B(Jv, Jw) = B(v, w)$.
3. Symplectic vector spaces abound in mathematical physics (cf., Guillemin and Sternberg). In particular, the "tangent spaces" of the "phase space" of a mechanical system have a natural symplectic structure. We will elucidate this remark in detail in Chapter 18 when we study mechanics. We have to introduce other ideas, such as manifolds, before we can explain further.

Finally, we have the concept of *a symplectic mapping*, analogous to the concept of isometry introduced at the end of subsection (i); that is, a linear mapping $\phi : V \to W$ which preserves the structures $b \in \bigwedge^2(V^*)$ and $B \in \bigwedge^2(W^*)$ according to eq. (5.21) just as before. The matrix form of the condition for a linear transformation on a symplectic vector space - *a symplectic transformation* - is the same as before (eq. (5.22)) with

$$\begin{pmatrix} I_r & \\ & -I_s \end{pmatrix}$$

replaced by

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

(This may be a good point at which to insert some parenthetical remarks of a more general, but less precise nature. We have already encountered, and will continue to encounter, situations which could be said to fit into the vague general context of having a set, which (i) has a certain "structure" defined on it, and (ii) "undergoes changes." A set with only one of these things does not define much. We have a defined concept if we have both (i) and (ii) and they are linked by "the changes preserve a given structure," or "the structures are preserved under a given change." These two opposite ways of looking at the same situation occur frequently. In the present case, when we defined symplectic transformations we focused on the transformations which preserve a given structure, and when we defined Hermitian inner products we focused on the structures which are preserved under a given transformation. The formal relationship, $b(Jv, Jw) = b(v, w)$ is the same for both problems, but the point of view is different.)

PROBLEM 5.18 (i) Show that Q satisfies the *parallelogram property*: $Q(v + w) + Q(v - w) = 2(Q(v) + Q(w))$ and $Q(v) = Q(-v)$. (ii) If $Q : V \to \mathbb{R}$ satisfies the parallelogram property, then b defined by (5.19) is bilinear and symmetric. (See Greub, 1981, p. 262.)

PROBLEM 5.19 Prove Theorem 5.18.

PROBLEM 5.20 Show there exist null vectors not in the null space iff b is indefinite.

PROBLEM 5.21 If b is semidefinite, we have the triangle (Minkowski) inequality:

$$(b(v + w, v + w))^{\frac{1}{2}} \leq (b(v, v))^{\frac{1}{2}} + (b(w, w))^{\frac{1}{2}}$$

PROBLEM 5.22 (i) We gave examples below Theorem 5.19 of elements of $\mathbf{S}^2(V^*)$ having all possible combinations of properties except one. Exemplify this last possibility. (ii) Find subspaces V_+, V_-, and V_0 for each example.

PROBLEM 5.23 Show that two timelike vectors of a Lorentzian vector space cannot be orthogonal.

PROBLEM 5.24 For forward-facing timelike vectors v and w of a Lorentzian vector space, we have the backward triangle inequality

$$[b(v + w, v + w)]^{\frac{1}{2}} \geq [b(v, v)]^{\frac{1}{2}} + [b(w, w)]^{\frac{1}{2}}$$

PROBLEM 5.25 Suppose P is a hyperplane of a Lorentzian vector space, E_1^n and n is a vector in E_1^n such that $b(n, t) = 0$ for all $t \in P$. Then

$$b(n, n) < 0 \ \Leftrightarrow P \quad \text{is a Euclidean vector space}$$

$$b(n, n) > 0 \ \Leftrightarrow P \quad \text{is a Lorentzian vector space}$$

$$b(n, n) = 0 \ \Leftrightarrow \ P = V_+ \otimes V_0 \ \text{ where } \ V_0 \ \text{ is 1-dimensional.}$$

(Hint: Choose a basis for E_1^n such that $n = n^0 e_0 + n^1 e_1$ where $e_0 \in V_-$ and $e_1 \in V_+$.)

PROBLEM 5.26 If $b \in \mathbf{S}^2(V^*)$ we can define a mapping

$$B : \textstyle\bigwedge^p(V^*) \times \bigwedge^p(V^*) \to \mathbb{R}$$

by $\langle \tau^1 \wedge \cdots \wedge \tau^p, \sigma^1 \wedge \cdots \wedge \sigma^p \rangle \mapsto \det(b(\tau^i, \sigma^j))$. Show that B is a nondegenerate element of $\mathbf{S}^2(\bigwedge^p(V^*))$; that is, B is a scalar (inner) product on $\bigwedge^p(V^*)$.

PROBLEM 5.27 Prove Theorem 5.24.

PROBLEM 5.28 Classify subspaces of $\mathbb{R}^4$ considered as a symplectic vector space.

PROBLEM 5.29 (i) The matrix of an orthogonal transformation satisfies

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\mathrm{tr}} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}^{-1}.$$

(ii) The matrix of a symplectic transformation satisfies

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\mathrm{tr}} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}^{-1}.$$

6

EXTERIOR (GRASSMANN) ALGEBRA

We will imbed collections of the spaces V_s^r into larger structures containing them. We will then focus primarily on the exterior, or Grassmann, algebras, $\bigwedge V$, and $\bigwedge V^*$.

6.1 Tensor algebras

Each of the spaces V_s^r, V_0^r, $\mathbf{S}^r(V)$, $\bigwedge^s(V^*)$, etc., is a vector space, and so we can add tensors within each of these spaces and multiply by scalars, and get elements within the same space. But we do not add tensors from two different spaces.

On the other hand, according to Section 4.3, we have a multiplication for tensors from any two tensor product spaces. As we saw there, this operation in the set $\bigcup_p V_0^p$ (or, more generally in $\bigcup_{p,q} V_q^p$) is bilinear and associative, but unlike addition and scalar multiplication, which form standard algebraic structures; i.e., vector spaces, the operation of tensor multiplication in $\bigcup_p V_0^p$ (or $\bigcup_{p,q} V_q^p$) does not. Moreover, it would be desirable to have addition and multiplication of tensors defined on the same set. We can accomplish this by essentially imbedding all our spaces in one large set which can be given the structure of an algebra.

There are several tensor algebras. To be specific we will look at *the contravariant tensor algebra.* That is, we look at the spaces $\mathbb{R}, V, V_0^2, \ldots, V_0^p, \ldots$. (Forget about the spaces $V_0^p \otimes V_0^q$, etc.). Now form the (weak) direct sum of these spaces; that is, the vector space consisting of all sequences $(A_0, A_1, A_2, \ldots, A_p, \ldots)$ where $A_0 \in \mathbb{R}$, $A_1 \in V, \ldots, A_p \in V_0^p, \ldots$ and all but a finite number of A's are zero. This is a vector space, $\oplus V_0^r$, where addition and scalar multiplication are defined "componentwise".

Finally, for $\boldsymbol{V} = (A_0, A_1, \ldots)$ and $\boldsymbol{W} = (B_0, B_1, \ldots)$ in $\oplus V_0^r$ we define multiplication in $\oplus V_0^r$ by

$$\mathbf{VW} = \left(A_0B_0, A_0B_1 + A_1B_0, \ldots, \sum_{i+j=p} A_iB_j, \ldots\right) \tag{6.1}$$

where A_iB_j is the product defined in Section 4.3.

Theorem 6.1 *With the operation of multiplication defined above, $\oplus V_0^r$ is an algebra; i.e., a vector space (or, more generally, an $\mathbb{A}$-module) with a bilinear multiplication. Moreover, this multiplication is associative, and $\oplus V_0^r$ has an identity element.*

Proof Problem 6.1. □

Note that for the special cases $\boldsymbol{V} = (0, \dots, A, \dots, 0, \dots)$ and $\boldsymbol{W} = (0, \dots, B, \dots, 0, \dots)$ where all components except the pth component of $\boldsymbol{V}$ and the qth components of $\boldsymbol{W}$ are zero, $\boldsymbol{VW} = (0, \dots, AB, \dots, 0, \dots)$ so AB is a special case of (6.1); i.e., $\oplus V_0^r$ is a "graded" algebra.

Evidently a tensor algebra, $\oplus V_s^0$, *the covariant tensor algebra*, can also be made out of the spaces $\mathbb{R}, V^*$, $V_2^0, V_3^0, \dots$, and one can be made out of all of the V_s^r. There are also *the symmetric algebras* $\mathbf{S}V$ and $\mathbf{S}V^*$, built out of the symmetric subspaces $\mathbf{S}^r(V) \subset V_0^r$ or out of the $\mathbf{S}^s(V^*) \subset V_s^0$. Finally, there are *the exterior algebras* $\bigwedge V$ and $\bigwedge V^*$ (or Grassmann algebras), formed from the spaces $\bigwedge^r(V) \subset V_0^r$ of r-vectors, or from the spaces $\bigwedge^s(V^*) \subset V_s^0$ of exterior s-forms. We will now focus our attention on the exterior algebras.

PROBLEM 6.1 (i) Prove Theorem 6.1.

(ii) A bilinear multiplication on the vector space $\oplus V_0^r$, which reduces in the special case as above, has the form (6.1).

6.2 Definition and properties of the exterior product

Since the exterior algebra $\bigwedge V^*$ is constructed from the direct sum of the spaces $\bigwedge^s(V^*)$, its elements are $n+1$-tuples $(A_0, A_1, \dots, A_n)$ where $A_0 \in \mathbb{R}$, $A_1 \in V^*, \dots, A_n \in \bigwedge^n(V^*)$. We have to define products so they have this form. We can make a definition like (6.1) if we first define a product in $\bigcup_s \bigwedge^s(V^*)$ where $0 \le s \le \dim V^*$.

Definition For $A \in \bigwedge^p(V^*)$ and $B \in \bigwedge^q(V^*)$ the *exterior* (or, *wedge*, or *skew-symmetric*) *product of* A *and* B is the element $A \wedge B \in \bigwedge^{p+q}(V^*)$ defined by

$$A \wedge B = \mathfrak{A} \cdot AB \tag{6.2}$$

Definition A p, q *shuffle* is a permutation π^*, of $1, 2, \dots, p+q$ such that

$$\pi^*(1) < \cdots < \pi^*(p)$$

and

$$\pi^*(p+1) < \cdots < \pi^*(p+q).$$

Theorem 6.2 (i) *The values of* $A \wedge B$ *are* $A \wedge B(v_1, \dots v_{p+q})$

$$= \frac{p!q!}{(p+q)!} \sum_* (\text{sgn } \pi) A(v_{\pi(1)}, \dots, v_{\pi(p)}) B(v_{\pi(p+1)}, \dots, v_{\pi(p+q)})$$

where $\sum_*$ *is the sum over* p, q *shuffles.*

(ii) *The components of $A \wedge B$ in a wedge product basis in $\bigwedge^{p+q}(V^*)$ are*

$$\sum_{*} (\text{sgn } \pi) A_{i_{\pi(1)} \cdots i_{\pi(p)}} B_{i_{\pi(p+1)} \cdots i_{\pi(p+q)}}$$

where $A_{i_{\pi(1)} \cdots i_{\pi(p)}}$ and $B_{i_{\pi(1)} \cdots i_{\pi(p)}}$ are the components of A and B, respectively, in wedge product bases. (Compare Theorem 4.5.)

Proof (i) $A \wedge B(v_1, \ldots, v_{p+q}) = (\mathfrak{A} \cdot AB)(v_1, \ldots, v_{p+q})$

$$= \frac{1}{(p+q)!} \sum_{\pi} (\text{sgn } \pi) AB(v_{\pi(1)}, \ldots, v_{\pi(p+q)})$$

$$= \frac{1}{(p+q)!} \sum_{\pi} (\text{sgn } \pi) A(v_{\pi(1)}, \ldots, v_{\pi(p)}) B(v_{\pi(p+1)}, \ldots, v_{\pi(p+q)})$$

$$\begin{aligned} = \frac{1}{(p+q)!} \Big[& p!q!\ A(v_1, \ldots, v_p) B(v_{p+1}, \ldots, v_{p+q}) \\ & + (-1)^p p!q!\ A(v_2, \ldots, v_{p+1}) B(v_1, v_{p+2}, \ldots, v_{p+q}) \\ & + \text{ terms in which at least one argument of } A \text{ is} \\ & \quad \text{interchanged with one argument of } B, \text{ and in} \\ & \quad \text{each term the indices are ordered as in a shuffle} \Big] \end{aligned}$$

(ii) From part (i), $(p+q)!\ A \wedge B(e_{i_1}, \ldots, e_{i_{p+q}})$

$$= \sum_{*} (\text{sgn } \pi) p!\ A(e_{i_{\pi(1)}}, \ldots, e_{i_{\pi(p)}}) q!\ B(e_{i_{\pi(p+1)}}, \ldots, e_{i_{\pi(p+q)}})$$

For $i_1 < \cdots < i_{p+q}$, the numbers on the left side are the coefficients of $\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_{p+1}}$ for $A \wedge B$ in eq. (5.16); that is they are the components of $A \wedge B$ in a wedge product basis. Since $i_{\pi(1)} < \cdots < i_{\pi(p)}$, $p!\ A(e_{i_{\pi(1)}}, \ldots, e_{i_{\pi(p)}})$ is the coefficient of $\varepsilon^{i_{\pi(1)}} \wedge \cdots \wedge \varepsilon^{i_{\pi(p)}}$ for A in eq. (5.16); that is, it is the component of A in a wedge product basis. Similarly for $q!\ B(e_{i_{\pi(p+1)}}, \ldots, e_{i_{\pi(p+q)}})$. □

To illustrate, suppose $A \in \bigwedge^1(V^*) = V^*$ and $B \in \bigwedge^2(V^*)$, then

(i) the values of $A \wedge B$ are

$$
\begin{aligned}
A \wedge B(v, w, x) &= (\mathfrak{A} \cdot AB)(v, w, x) \\
&= \frac{1}{3!}(AB(v, w, x) + AB(w, x, v) + AB(x, v, w) - AB(v, x, w) \\
&\qquad -AB(x, w, v) - AB(w, v, x)) \\
&= [(A(v)B(w, x) - A(v)B(x, w)) \\
&\qquad +(A(w)B(x, v) - A(w)B(v, x)) \\
&\qquad +(A(x)B(v, w) - A(x)B(w, v))] \\
&= \frac{1}{3}[A(v)B(w, x) - A(w)B(v, x) + A(x)B(v, w)]
\end{aligned}
$$

(ii) the components of $A \wedge B$ are

$$\sum_{*} (\text{sgn } \pi)\ A_{i_{\pi(1)}} B_{i_{\pi(2)} i_{\pi(3)}} = A_i B_{jk} - A_j B_{ik} + A_k B_{ij}$$

Recall that in Section 5.3 we have already introduced exterior products in V and in V^*. We will see shortly that the two definitions are consistent. Just as in Section 5.3, there is a commonly used alternative to the definition (6.2), namely

$$A \wedge B = \frac{(p+q)!}{p!q!}\ \mathfrak{A} \cdot AB$$

By introducing the factor $(p+q)!/p!q!$ in the definition we eliminate factors in some subsequent formulas such as in Theorem 6.2(i). (See, Warner, p. 59, ff.)

Theorem 6.3 *Exterior multiplication is bilinear.*

Proof Problem 6.3. □

Theorem 6.4 *Exterior multiplication is associative.*

Proof We wish to show that for $A \in V_p^0$, $B \in V_q^0$, and $C \in V_r^0$
(i) $\mathfrak{A} \cdot (\mathfrak{A} \cdot AB)C = \mathfrak{A} \cdot ABC$, and
(ii) $\mathfrak{A} \cdot A(\mathfrak{A} \cdot BC) = \mathfrak{A} \cdot ABC$.

The proof of (i) follows from the following lemma with $t = p + q$ and $u = p + q + r$, and from the linearity of $\mathfrak{A}$. □

Lemma *For any set* $\{\sigma^1, \ldots, \sigma^u\}$ *of* u *(not necessarily distinct) elements of* V^*,

$$\mathfrak{A} \cdot (\mathfrak{A} \cdot \sigma^1 \otimes \cdots \otimes \sigma^t)(\sigma^{t+1} \otimes \cdots \otimes \sigma^u) = \mathfrak{A} \cdot \sigma^1 \otimes \cdots \otimes \sigma^u \tag{6.3}$$

Proof The left-hand side is

$$\mathfrak{A} \cdot \left(\frac{1}{t!} \sum_\pi (\text{sgn } \pi) \sigma^{\pi(1)} \otimes \cdots \otimes \sigma^{\pi(t)} \right) (\sigma^{t+1} \otimes \cdots \otimes \sigma^u)$$

$$= \frac{1}{t!} \sum_\pi (\text{sgn } \pi) \mathfrak{A}(\sigma^{\pi(1)} \otimes \cdots \otimes \sigma^{\pi(t)} \otimes \sigma^{t+1} \otimes \cdots \otimes \sigma^u)$$

by the distributivity of multiplication in $\oplus V_p^0$, and the linearity of $\mathfrak{A}$ is

$$\frac{1}{t!} \sum_\pi (\text{sgn } \pi) \left(\frac{1}{u!} \sum_\lambda (\text{sgn } \lambda) \sigma^{\lambda\pi(1)} \otimes \cdots \otimes \sigma^{\lambda\pi(t)} \otimes \sigma^{\lambda(t+1)} \otimes \cdots \otimes \sigma^{\lambda(u)} \right)$$

where λ is a permutation of $1, \ldots, u$.

Now for each of the $t!/2$ even π's,

$$\sum_\lambda (\text{sgn } \lambda) \sigma^{\lambda\pi(1)} \otimes \cdots \otimes \sigma^{\lambda(u)} = \sum_\lambda (\text{sgn } \lambda) \sigma^{\lambda(1)} \otimes \cdots \otimes \sigma^{\lambda(u)}$$

so for each of the $t!/2$ even π's the term under $\sum_\pi$ is the same, namely,

$$\frac{1}{u!} \sum_\lambda (\text{sgn } \lambda) \sigma^{\lambda(1)} \otimes \cdots \otimes \sigma^{\lambda(u)}.$$

For each of the $t!/2$ odd π's,

$$\sum_\lambda (\text{sgn } \lambda) \sigma^{\lambda\pi(1)} \otimes \cdots \otimes \sigma^{\lambda(u)} = -\sum_\lambda (\text{sgn } \lambda) \sigma^{\lambda(1)} \otimes \cdots \otimes \sigma^{\lambda(u)}$$

and for each of these π's, sgn $\pi = -1$, so again, for each of these π's the term under $\sum_\pi$ is the same as for the even π's. Thus all $t!$ terms under $\sum_\pi$ are the same so the left side of (6.3) is $(1/u!) \sum_\lambda (\text{sgn } \lambda) \sigma^{\lambda(1)} \otimes \cdots \otimes \sigma^{\lambda(u)}$ which is precisely the right side of (6.3). □

Corollary $\sigma^1 \wedge \cdots \wedge \sigma^s = \mathfrak{A} \cdot \sigma^1 \otimes \cdots \otimes \sigma^s$.

Proof (i) For $s = 3$,

$$\begin{aligned}\mathfrak{A} \cdot (\sigma^1 \otimes \sigma^2 \otimes \sigma^3) &= \mathfrak{A} \cdot [(\mathfrak{A} \cdot (\sigma^1 \otimes \sigma^2))\sigma^3] = (\mathfrak{A} \cdot (\sigma^1 \otimes \sigma^2)) \wedge \sigma^3 \\ &= (\mathfrak{A} \cdot (\sigma^1 \sigma^2)) \wedge \sigma^3 = (\sigma^1 \wedge \sigma^2) \wedge \sigma^3 \\ &= \sigma^1 \wedge \sigma^2 \wedge \sigma^3\end{aligned}$$

$$\text{(ii)} \quad \begin{aligned}\mathfrak{A} \cdot (\sigma^1 \otimes \cdots \otimes \sigma^s \otimes \sigma^{s+1}) &= \mathfrak{A} \cdot [(\mathfrak{A} \cdot (\sigma^1 \otimes \cdots \otimes \sigma^s))\sigma^{s+1}] \\ &= \mathfrak{A} \cdot [(\sigma^1 \wedge \cdots \wedge \sigma^s)\sigma^{s+1}]\end{aligned}$$

by induction hypothesis, and this last expression is $\sigma^1 \wedge \cdots \wedge \sigma^{s+1}$. □

This corollary shows that definition (6.2) is consistent with definition (5.14).

Theorem 6.5 *If $A \in \bigwedge^p(V^*)$ and $B \in \bigwedge^q(V^*)$, then $A \wedge B = (-1)^{pq} B \wedge A$ (anticommutativity).*

Proof For elements of the form

$$\sigma^1 \wedge \cdots \wedge \sigma^p \in \textstyle\bigwedge^p(V^*)$$

and

$$\tau^1 \wedge \cdots \wedge \tau^q \in \textstyle\bigwedge^q(V^*)$$

we have

$$\begin{aligned}(\sigma^1 \wedge \cdots \wedge \sigma^p) \wedge (\tau^1 \wedge \cdots \wedge \tau^q) &= \sigma^1 \wedge \cdots \wedge \sigma^p \wedge \tau^1 \wedge \cdots \wedge \tau^q \\ &= (-1)^{pq} \tau^1 \wedge \cdots \wedge \tau^q \wedge \sigma^1 \wedge \cdots \wedge \sigma^p\end{aligned}$$

and A and B are linear combinations of such elements. □

Now that we have defined a product in $\bigcup_s \bigwedge^s(V^*)$ we can construct the exterior algebra of V^*. *The exterior algebra of* V^* is the vector space $\bigwedge V^* = \oplus \bigwedge^s(V^*) = \mathbb{R} \oplus \bigwedge^1(V^*) \oplus \cdots \oplus \bigwedge^n(V^*)$ (where $n = \dim V^*$) with a multiplication defined as in (6.1) with $A_i B_j$ replaced by $A_i \wedge B_j$. Note that $\dim \bigwedge V^* = \sum_s \dim \bigwedge^s(V^*) = 2^n$.

Analogously, we can define a wedge multiplication for r-vectors and construct *the exterior algebra, $\bigwedge V$, of V.*

PROBLEM 6.2 (i) Show there are $\binom{p+q}{q}$ terms in $\sum_*$, each one of which corresponds to a p, q shuffle. (ii) Write out the values and the components of $A \wedge B$ as in the illustration, if $A \in \bigwedge^2(V^*)$ and $B \in \bigwedge^3(V^*)$.

PROBLEM 6.3 Prove Theorem 6.3.

PROBLEM 6.4 Illustrate Theorems 6.3 - 6.5 with specific examples.

PROBLEM 6.5 In Corollary (i) of Theorem 5.15 we saw that $\bigwedge^s(V^*)$ can be described as a quotient space. For each s we have an alternation operator and a kernel. The direct sum of these kernels is an ideal in the algebra $\oplus V_s^0$. $\bigwedge V^*$ is isomorphic to the quotient of $\oplus V_s^0$ by this ideal (Greub, 1981, p. 102, ff).

6.3 Some more properties of the exterior product

Theorem 6.6 *The set $\{v_1, \ldots, v_p\}$ of elements of V is a linearly dependent set if and only if $v_1 \wedge \cdots \wedge v_p = 0$.*

Proof Problem 6.6. □

Recall (Problem 5.13) that elements of $\bigwedge^r(V)$ of the form $v_1 \wedge \cdots \wedge v_r$ are called decomposable.

Theorem 6.7 *There is a $1-1$ correspondence between r dimensional subspaces W of V and 1-dimensional subspaces, Z, of $\bigwedge^r(V)$ consisting of decomposable elements. Moreover, for $w \in W$ and $A \in Z, w \wedge A = 0$.*

Proof Given W, let $\{e_1, \ldots, e_r\}$ be a basis of W and let $Z = \bigwedge^r(W) = \{ae_1 \wedge \cdots \wedge e_r\}$. W and Z have the required properties. On the other hand, every 1-dimensional subspace of $\bigwedge^r(V)$ consisting of decomposable elements has the form $Z = \{av_1 \wedge \cdots \wedge v_r\}$. If $W = \langle\{v_1, \ldots, v_r\}\rangle$ then $Z = \bigwedge^r(W)$ and W and Z have the required property. Finally, $\overline{Z} = Z \Rightarrow e_1 \wedge \cdots \wedge e_r = a\bar{e}_1 \wedge \cdots \wedge \bar{e}_r \Rightarrow e_1 \wedge \cdots \wedge e_r \wedge \bar{e}_i = 0$ for all $\bar{e}_i \Rightarrow \bar{e}_i$ is a linear combination of $e_1, \ldots, e_r$ for all $\bar{e}_i \Rightarrow W = \overline{W}$. □

Theorem 6.8 *Suppose $v \in V$ and $A \in \bigwedge^r(V)$. Then there is a $B \in \bigwedge^{r-1}(V)$ such that $A = v \wedge B$ (i.e., A has v as a factor) if and only if $v \wedge A = 0$.*

Proof **Only if:** $A = v \wedge B$ implies $v \wedge A = 0$.
If: For $A \in \bigwedge^r(V)$ we can write

$$\begin{aligned} A &= C^{i_1 \cdots i_r} e_{i_1} \wedge \cdots \wedge e_{i_r} \quad i_1 < \cdots < i_r \\ &= C^{1 i_2 \cdots i_r} e_1 \wedge \cdots \wedge e_{i_r} + C^{i_1 \cdots i_r} e_{i_1} \wedge \cdots \wedge e_{i_r} \quad \text{with } i_1 > 1 \end{aligned}$$

Now choose a basis in V with e_1 the given vector v. Then $v \wedge A = 0$ implies $C^{i_1 \cdots i_r} = 0$ when $i_1 > 1$ and $A = e_1 \wedge C^{1 i_2 \cdots i_r} e_{i_2} \wedge \cdots \wedge e_{i_r} = v \wedge B$ with $B \in \bigwedge^{r-1}(V)$. □

Theorem 6.9 (Cartan's lemma) *Let $\{e_i\}$ be a basis of V, and suppose v_i, $i = 1, \ldots, p$ are elements of V such that $\sum_{i=1}^{p} e_i \wedge v_i = 0$. Then $v_i = A_i^j e_j$ where $A_i^j = A_j^i$ for $i, j = 1, \ldots, p$ and $A_i^j = 0$ for $j = p+1, \ldots, n$.*

Proof Problem 6.8. □

There are some important properties of exterior algebras corresponding to ones we have seen before for tensor algebras, in Section 4.3.

1. If $\phi : V \to W$ is a linear mapping, then there are induced linear maps in the tensor algebra corresponding to those we described in Section 4.3. That is, we have $\phi_r : \bigwedge^r(V) \to \bigwedge^r(W)$ given by

$$\phi_r : v_1 \wedge \cdots \wedge v_r \mapsto \phi \cdot v_1 \wedge \cdots \wedge \phi \cdot v_r \tag{6.4}$$

and $\phi^s : \bigwedge^s(W^*) \to \bigwedge^s(V^*)$ given by

$$\phi^s : \tau^1 \wedge \cdots \wedge \tau^s \mapsto \phi^* \cdot \tau^1 \wedge \cdots \wedge \phi^* \cdot \tau^s \tag{6.5}$$

In the special case when $W = V$, and $r = n = \dim V$, $\phi_n : \bigwedge^n(V) \to \bigwedge^n(V)$. If $\{e_1, \ldots, e_n\}$ is a basis for V, then $e_1 \wedge \cdots \wedge e_n$ is a basis for $\bigwedge^r(V)$ and

$$\begin{aligned}\phi_n : e_1 \wedge \cdots \wedge e_n &\mapsto \phi_1^{i_1} e_{i_1} \wedge \cdots \wedge \phi_n^{i_n} e_{i_n} \\ &= \sum_{\pi} (\mathrm{sgn}\ \pi) \phi_1^{\pi(1)} \cdots \phi_n^{\pi(n)} e_1 \wedge \cdots \wedge e_n \\ &= \det(\phi_j^i) e_1 \wedge \cdots \wedge e_n\end{aligned}$$

So for any $A \in \bigwedge^n(V)$

$$\phi_n \cdot A = \det(\phi_j^i) A \tag{6.6}$$

That is, ϕ_n is simply multiplication by $\det(\phi_j^i)$.

As interesting by-products, we get the results for determinants that (i) since ϕ_n is independent of the basis, $\det(\phi_j^i)$ is an invariant and we can talk about $\det\ \phi$, *the determinant of the linear transformation* ϕ, and (ii) from the result $(\phi \circ \psi)_r = \phi_r \circ \psi_r$ for compositions, we get that

$$\det(\phi_j^i)(\psi_k^j) = \det(\phi_j^i) \cdot \det(\psi_k^j).$$

(Compare this with the direct proof of this result.

2. Recall that in Section 4.3, from the definition of the tensor product of two linear mappings ϕ and ψ we obtained the result that (in the special case $W = Y = \mathbb{R}$) $\phi \otimes \psi$, as a tensor product of linear functions, was the linear function that corresponded to the bilinear function $\phi \otimes \psi$ in the isomorphism $(V \otimes X)^* \cong V^* \otimes X^*$ and that the pairing $\langle \phi \otimes \psi, v \otimes x \rangle$ induced by the duality is given by $\langle \phi \otimes \psi, v \otimes x \rangle = \phi \otimes \psi(v, x)$ (see eq. (4.12)).

We can get corresponding results in exterior algebra, but we cannot get these by following the procedure of Section 4.3. (In particular, we cannot define a general exterior product of linear mappings by the pattern in Section 4.3, since if $\phi \cdot v$ and $\psi \cdot x$ are in different spaces $\phi \cdot v \wedge \psi \cdot x$ is not defined.) Rather, we define, for σ^1 and σ^2 in V^*, a linear function, $\sigma^1 \wedge \sigma^2$, on $\bigwedge^2(V)$ by

$$\sigma^1 \wedge \sigma^2 = \mathfrak{A} \cdot \sigma^1 \otimes \sigma^2 = \frac{1}{2}(\sigma^1 \otimes \sigma^2 - \sigma^2 \otimes \sigma^1)$$

where $\sigma^1 \otimes \sigma^2$ and $\sigma^2 \otimes \sigma^1$ are the linear functions defined in eq. (4.11). Then using eq. (4.12).

$$\langle \sigma^1 \wedge \sigma^2, v_1 \wedge v_2 \rangle = \sigma^1 \wedge \sigma^2(v_1, v_2)$$

More generally, we define a linear function, $\sigma^1 \wedge \cdots \wedge \sigma^p$, on $\bigwedge^p(V)$ by

$$\sigma^1 \wedge \cdots \wedge \sigma^p = \mathfrak{A} \cdot \sigma^1 \otimes \cdots \otimes \sigma^p$$

where $\sigma^1 \otimes \cdots \otimes \sigma^p \in V^{p*}$. Then

$$\langle \sigma^1 \wedge \cdots \wedge \sigma^p, v_1 \wedge \cdots \wedge v_p \rangle = \sigma^1 \wedge \cdots \wedge \sigma^p(v_1, \ldots, v_p) \tag{6.7}$$

and by linearity

$$\langle A, v_1 \wedge \cdots \wedge v_p \rangle = A(v_1, \ldots, v_p) \tag{6.8}$$

where A on the left is in $\bigwedge^p(V)^*$ and A on the right is in $\bigwedge^p(V^*)$. Equation (6.8) gives the isomorphism between $\bigwedge^p(V)^*$ and $\bigwedge^p(V^*)$, and hence, the pairing of exterior p-vectors, $\bigwedge^p(V)$ and exterior p-forms, $\bigwedge^p(V^*)$.

Note that in the special case, eq. (6.7), from Problem 5.15 we have the formula

$$\langle \sigma^1 \wedge \cdots \wedge \sigma^p, v_1 \wedge \cdots \wedge v_p \rangle = \frac{1}{p!} \det(\langle \sigma^i, v_j \rangle) \tag{6.9}$$

Finally, there are two additional important mappings in exterior algebra.

1. The duality in exterior algebras described above enables us to define a pairing of exterior r-vectors and exterior s-forms, $s \geq r$. This is an important special case of the mapping described in Section 4.3, consisting of the composition of tensor multiplication followed by contractions.

Definition *The* (*left*) *interior product* is a bilinear map, $\rfloor$, from r-vectors and exterior s-forms to exterior $s-r$-forms for $s \geq r$

$$\rfloor : \bigwedge^r(V) \times \bigwedge^s(V^*) \to \bigwedge^{s-r}(V^*) \quad \text{with} \quad (A, B^*) \mapsto A\rfloor B^*$$

where $A\rfloor B^*$ is defined by

$$\langle A\rfloor B^*, C\rangle = \langle B^*, A \wedge C\rangle \tag{6.10}$$

for all $C \in \bigwedge^{s-r}(V)$. (Note that when $s = r$, $\bigwedge^{s-r}(V) = \mathbb{R}$, C is a real number, $A \in C = CA$, and (6.10) reduces to $A\rfloor B^* = \langle B^*, A\rangle$).

We can illustrate this definition schematically by arranging the spaces involved in a linear order. Thus, for $s \geq r$

$$\begin{array}{ccc}
A \wedge C \in \bigwedge^s(V) & & \\
C \in \bigwedge^{s-r}(V) & & \\
A \in \bigwedge^r(V) & & \\
V & & \\
 & \text{paired} & \text{paired} \\
V^* & & \\
\bigwedge^r(V^*) & & \\
A\rfloor B^* \in \bigwedge^{s-r}(V^*) & & \\
B^* \in \bigwedge^s(V^*) & &
\end{array}$$

The right side of (6.10) is a pairing of an s-vector and an exterior s-form, and the left side of (6.10) is a pairing of an $(s-r)$-vector and an exterior $(s-r)$-form.

With $\rfloor$ we have the two partial mappings $A\rfloor$ and $\rfloor B^*$.

Definition Given $A \in \bigwedge^r(V)$, we define a linear map $A\rfloor : \bigwedge^s(V^*) \to \bigwedge^{s-r}(V^*)$ for $s \geq r$ by $A\rfloor \cdot B^* = A\rfloor B^*$.

Comparing eq. (6.10) with eq. (2.4) we see that the mapping $A\rfloor$ is the transpose of the mapping $A\wedge$. The mapping $sA\rfloor : \bigwedge^s(V^*) \to \bigwedge^{s-1}(V^*)$ for the case $r = 1$ (i.e., $A \in V$) is denoted by i_A and called *interior multiplication by* A. This case will be examined further in Chapter 11, and will be important when we get to exterior and Lie derivatives.

Definition If we fix the second argument, B^*, in $\rfloor$ then we get a mapping

$$\rfloor B^* : \bigwedge^r(V) \to \bigwedge^{s-r}(V^*)$$

(from r-vectors to $s-r$-forms).

In the special case when $s = \dim V$, we get isomorphisms which map decomposable elements to decomposable elements, and the image of a decomposable element corresponding to a subspace W of V of dimension r (see Theorem 6.7) has as its corresponding subspace in V^* the orthogonal complement of W. Corresponding elements in such an isomorphism are called *dual tensors* in the classical literature (Synge and Schild, 1966, p. 247).

In the special case $r = 1$, $\ker \rfloor B^*$ is called *the associated subspace of* B^*. It is the orthogonal complement of the subspace of V^* generated by $A \rfloor B^*$ where $A \in \bigwedge^{s-1}(V)$. It will appear again in Section 14.3 when we discuss characteristics of forms.

2. We saw in Section 5.4, that given $b \in \mathbf{S}^2(V^*)$, a symmetric second-order covariant tensor, then V has an orthonormal basis with respect to b.

Definition Let $b \in \mathbf{S}^2(V^*)$ be nondegenerate and positive definite and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V. The set of $p, n-p$ shuffles defines a mapping of a basis of $\bigwedge^p(V)$ to $\bigwedge^{n-p}(V)$ by prescribing

$$* : e_{\pi(1)} \wedge \cdots \wedge e_{\pi(p)} \mapsto (\mathrm{sgn}\ \pi) e_{\pi(p+1)} \wedge \cdots \wedge e_{\pi(n)}$$

where π is a $p, n-p$ shuffle.

We can extend $*$ to all of $\bigwedge^p(V)$ by linearity so that $* : \bigwedge^p(V) \to \bigwedge^{n-p}(V)$. We define such a map for each p, so finally $* : \bigwedge V \to \bigwedge V$. $*$ is called *the* (*Hodge*) *star operator* on the exterior algebra $\bigwedge V$. With an analogous construction we also get the (Hodge) star operator on the exterior algebra $\bigwedge V^*$.

(In Problem 6.16 we will define a mapping $\bigwedge^p(V) \to \bigwedge^{n-p}(V)$. That definition will require a choice of $B^* \in \bigwedge^n(V^*)$ and a choice of a nondegenerate $b \in \mathbf{S}^2(V^*)$, but does not involve any choice of bases. That definition will generalize the one above which, in particular, shows that the definition of the star operator above does not actually depend on the particular choice of orthonormal basis of V.)

As an illustration, consider the case when $\dim V = 3$. Then, if $v = v^i e_i$ and $w = w^i e_i$ we get $v \wedge w = v^i w^j e_i \wedge e_j$ and $*(v \wedge w) = v^i w^j * (e_i \wedge e_j) = (v^1 w^2 - v^2 w^1) e_3 + (v^2 w^3 - v^3 w^2) e_1 + (v^3 w^1 - v^1 w^3) e_2$. Thus, when $\dim V = 3$, $*(v \wedge w) = v \times w$, the cross-product of vector analysis.

Theorem 6.10 $b(v, w) = *(v \wedge *w)$ *for* $v, w \in V$.

Proof Problem 6.14. □

(Note that in vector analysis we write $b(v, w) = v \cdot w$; that is, Theorem 6.10 gives a formula for the dot product.)

Theorem 6.11 $* \cdot * = (-1)^{p(n-p)} id$ *where id is the identity map on* $\bigwedge^p(V)$.

Proof Problem 6.15. □

PROBLEM 6.6 Prove theorem 6.6.

PROBLEM 6.7 Prove that for $v, w \in V$, $v \wedge w \neq 0$, and $A \in \bigwedge^r(V)$, $A = v \wedge w \wedge B$ for $B \in \bigwedge^{r-2}(V)$ if and only if $v \wedge A = 0$ and $w \wedge A = 0$ (cf. Theorem 6.8). Generalize.

PROBLEM 6.8 Prove Theorem 6.9.

PROBLEM 6.9 Show that the mappings given by (6.4) and (6.5) are the restrictions to $\bigwedge^r(V)$ and $\bigwedge^s(V^*)$, respectively, of the mappings ϕ_r and ϕ^s given by (4.7) and (4.8). In particular, they have the composition properties of Problem 4.17.

PROBLEM 6.10 Show (i) that there is a $1-1$ correspondence between alternating multilinear functions on the cartesian product of r factors of V and alternating linear functions on V_0^r; (ii) that there is a $1-1$ correspondence between alternating linear functions on V_0^r and linear functions on $\bigwedge^r(V)$; and (iii) that (i) and (ii) give the isomorphism given by eq. (6.8).

PROBLEM 6.11 If $A \in \bigwedge^r(V)$ and $B^* \in \bigwedge^r(V^*)$, then

$$A \rfloor B^* = C_r^r \cdots C_1^1 \cdot B^* A.$$

See Problem 4.23.

PROBLEM 6.12 If $\{e_i\}$ and $\{\varepsilon^i\}$ are dual bases of V and V^*, respectively, then

$$e_{i_1} \wedge \cdots \wedge e_{i_r} \rfloor \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_s}$$

$$= \begin{cases} \dfrac{(s-r)!}{s!}(\text{sgn } \pi)\varepsilon^{k_1} \wedge \cdots \wedge \varepsilon^{k_{s-r}} & \text{if } \{i_1, \ldots, i_r\} \subset \{j_1, \ldots, j_s\} \\ 0 & \text{otherwise} \end{cases}$$

Here the k's are the elements of $\{j_1, \ldots, j_s\}$ not in $\{i_1, \ldots, i_r\}$ and π is the permutation $j_1, \ldots, j_s \to i_1, \ldots, i_r, k_1, \ldots, k_{s-r}$.

PROBLEM 6.13

(i) $e_i \rfloor\ \varepsilon^{j_1} \wedge \cdots \wedge\ \varepsilon^{j_s} = \dfrac{1}{s} \displaystyle\sum_{k=1}^{s} (\text{sgn } \pi)\ \delta_i^{j_k} \varepsilon^{j_1} \wedge \cdots \wedge \hat{\varepsilon}^{j_k} \wedge \cdots \wedge \varepsilon^{j_s}$

(see Problem 6.12).

(ii) If $v \in V$ and $B^* \in \bigwedge^3(V^*)$, then $i_v B^* = 3 v^i B^*_{ijk} \varepsilon^j \wedge \varepsilon^k, j < k$.

PROBLEM 6.14 Prove Theorem 6.10.

PROBLEM 6.15 Prove Theorem 6.11.

PROBLEM 6.16 We noted above that once we choose B^* in $\bigwedge^n(V^*)$ (an "orientation"), then $\rfloor B^*$ defines an isomorphism from $\bigwedge^p(V)$ to $\bigwedge^{n-p}(V^*)$. Further, with a nondegenerate b we have the linear mapping, $b^\flat$, from V to V^*, (Section 5.4), which has an inverse, $b^\sharp = (b^\flat)^{-1}$, according to the corollary of Theorem 4.1, and this induces a linear map from $\bigwedge^{n-p}(V^*)$ to $\bigwedge^{n-p}(V)$ according to eq. (6.4). Form the composition

$$\bigwedge^p(V) \to \bigwedge^{n-p}(V^*) \to \bigwedge^{n-p}(V)$$

If $\{e_1, \ldots, e_n\}$ is an orthonormal basis of V, $B^* = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n$, and b is positive definite, then this composition reduces to the $*$ map defined above multiplied by $(n-p)!/n!$.

7

THE TANGENT MAP OF REAL CARTESIAN SPACES

Up to now we have been discussing certain abstract vector spaces, namely, tensor product spaces. These will be important in the context of our objective of modeling physical phenomena by various mathematical spaces with mappings between them. Tensor product spaces will be part of the structure of the image spaces of our mappings. For now we will focus on their domains.

For vector spaces, in general, and $\mathbb{R}^n$, in particular, we have linear mappings, and more generally, nonlinear mappings. For vector spaces we have an important device for dealing with the more general nonlinear mappings. We can approximate them by linear mappings – derivatives. We will recall some of the basic ideas of the differential calculus on $\mathbb{R}^n$ in the first section.

In our applications we will need to use for our domains spaces more general than $\mathbb{R}^n$, including spaces which are not vector spaces (surfaces in $\mathcal{E}_0^3$, for example). We will use spaces for which we can describe a device such as we have for vector spaces; a way of approximating mappings between such spaces (real-valued functions on surfaces in $\mathcal{E}_0^3$, for example) by linear ones – "derivatives". These spaces will be our differentiable manifolds.

The structures, tangent and cotangent spaces, that we need for the concept of a "derivative" of a mapping of differentiable manifolds can be described for the special cases of vector spaces, and, in particular, of cartesian spaces. It will turn out that we will be able to make a straightforward generalization of these concepts in the general setting of manifolds. Thus, we can temporarily avoid some of the abstract properties of manifolds, and see, in the special case of cartesian spaces, some of the important concepts that can be defined on them.

7.1 Maps of real cartesian spaces

Real cartesian n-space, $\mathbb{R}^n$, is the set of all ordered n-tuples of real numbers. A point in this space is an n-tuple $(a^1, a^2, \ldots, a^n)$, which we will frequently abbreviate by a. We will focus our attention on an arbitrary but fixed point $a \in \mathbb{R}^n$ and its neighborhoods, $\mathcal{U}_a$. These are defined by the "usual" topology of $\mathbb{R}^n$; that is, the topology given by the distance $d(a,b) = \left[\sum_{i=1}^n (a^i - b^i)^2\right]^{\frac{1}{2}}$.

We can, on the one hand, consider *functions* on $\mathcal{U}_a$, $f : \mathcal{U}_a \to \mathbb{R}$, from a neighborhood of a to the reals, and, on the other hand, we can consider (parametrized) *curves in* $\mathbb{R}^n$ *through* a, $\gamma_a : I \to \mathbb{R}^n$, from an open interval of $\mathbb{R}$ containing 0, and such that $\gamma_a(0) = a$.

There are very important examples of functions where $\mathcal{U}_a = \mathbb{R}^n$, namely, the n *natural coordinate functions*, (or *natural projections*), $\pi^i : \mathbb{R}^n \to \mathbb{R}$ given by $a \mapsto a^i$. There are n corresponding special curves through a, $\vartheta_{ak}:\mathbb{R} \to \mathbb{R}^n$ given by $u \mapsto (a^1, a^2, \ldots, a^k + u, \ldots, a^n)$. They are the n *natural coordinate curves through* a, (or, *natural injections*). Note that the π^i are linear, but the ϑ_{ak} are not.

With a given curve through a we can form its *component functions*, $\gamma_a^i = \pi^i \circ \gamma_a$. With a given function on $\mathcal{U}_a$ we can form the *partial functions* $f \circ \vartheta_{ak}$. The value of $f \circ \vartheta_{ak}$ at u is $f(a^1, a^2, \ldots, a^k + u, \ldots, a^n)$.

Since $\mathbb{R}$ and $\mathbb{R}^n$ are topological spaces, and we have the concept of continuity for mappings between any two topological spaces, we have the concept of continuity for functions and curves through a. (In particular, it is clear that π^i and ϑ_{ak} are continuous on their domains.)

Moreover, for any function, f, the $f \circ \vartheta_{ak}$ are functions from subsets of $\mathbb{R}$ to $\mathbb{R}$ so, for $k = 1, \ldots, n$, the ordinary derivatives, $D_0(f \circ \vartheta_{ak})$, of the partial functions of f are defined. We write these in the usual partial derivative notation; i.e., we define the notation $\partial f/\partial \pi^k \mid_a$ by

$$\left.\frac{\partial f}{\partial \pi^k}\right|_a = D_0(f \circ \vartheta_{ak}), \quad k = 1, \ldots, n$$

Similarly, since for any curve, γ_a, through a, the $\gamma_a^i = \pi^i \circ \gamma_a$ are functions from subsets of $\mathbb{R}$ to $\mathbb{R}$ we have the ordinary derivatives, $D_0 \pi^i \circ \gamma_a = D_0 \gamma_a^i$, $i = 1, \ldots, n$ of the component functions of γ_a.

Finally, if $\phi : \mathcal{U}_a \to \mathbb{R}^p$ (where $\mathcal{U}_a \subset \mathbb{R}^n$), and if $\bar{\pi}^i$, $i = 1, \ldots, p$ are the coordinate functions on $\mathbb{R}^p$, then we can form the compositions $\phi^i = \bar{\pi}^i \circ \phi$, $i = 1, \ldots, p$, *the component functions of the map* ϕ, and we can form the compositions $\phi \circ \vartheta_{ak}$, $k = 1, \ldots, n$, *the partial maps of* ϕ (Fig. 7.1). Composing these we have

$$\phi^i \circ \vartheta_{ak} = \bar{\pi}^i \circ (\phi \circ \vartheta_{ak}), \quad i = 1, \ldots, p \quad k = 1, \ldots, n$$

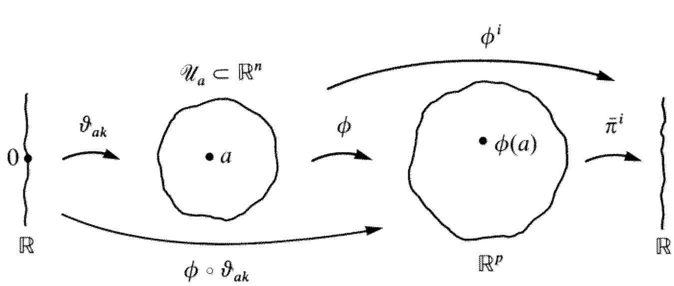

Figure 7.1

So, for each $i = 1, \dots, p$ we have n functions from neighborhoods of $0 \in \mathbb{R}$ to $\mathbb{R}$, and hence the ordinary derivatives at $0 \in \mathbb{R}$,

$$D_0(\phi^i \circ \vartheta_{ak}) = \left.\frac{\partial \phi^i}{\partial \pi^k}\right|_a \tag{7.1}$$

which, as indicated by the notation on the right, are the n partial derivatives of the p component functions, ϕ^i, of ϕ at a.

In summary, we point out that the ability to define partial derivatives depends on the existence in cartesian spaces of the natural projections, $\bar{\pi}^i$, and the coordinate curves ϑ_{ak}. Note that we have not yet invoked the natural vector space structure of Cartesian spaces.

Now, since cartesian spaces are normed vector spaces, we can go further and make the following definitions for maps between $\mathbb{R}^n$ and $\mathbb{R}^p$ (and, more generally, for maps between any two normed vector spaces V and W.)

Definitions Two maps ϕ and ψ (continuous at a) are *tangent* (*to each other*) *at* a if

$$\lim_{v \to a} \frac{\phi(v) - \psi(v)}{|v - a|} = 0$$

A continuous map $\phi : \mathcal{U}_a \to \mathbb{R}^p$ ($\mathcal{U}_a \subset \mathbb{R}^n$) is *differentiable at* a if it is tangent at a to an affine map, A, that is, a map $A : \mathbb{R}^n \to \mathbb{R}^p$ given by $A(v) = \phi(a) + (D_a\phi)(v - a)$ where $D_a\phi$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^p$. $D_a\phi$ is called *the derivative of* ϕ *at* a. If ϕ has a derivative at each $a \in \mathcal{U} \subset \mathbb{R}^n$ then $D\phi : \mathbb{R}^n \to \mathfrak{L}(\mathbb{R}^n, \mathbb{R}^p)$ is given by $D\phi : a \mapsto D_a\phi$; i.e., $D\phi(a) = D_a\phi$.

From the calculus on cartesian spaces we have the results: (1) If $\phi : \mathcal{U}_a \to \mathbb{R}^p$ is differentiable at a, then $\partial\phi^i/\partial\pi^k|_a$ exist, and (2) If ϕ is a C^1 map at a (that is, $\partial\phi^i/\partial\pi^k|_a$ exist in a neighborhood of a and $\partial\phi^i/\partial\pi^k$ are continuous at a), then ϕ is differentiable at a. In both cases, $D_a\phi$ is the Jacobian matrix, $(\partial\phi^i/\partial\pi^k|_a)$.

In particular, if ϕ is a curve, γ, taking a to $\gamma(a)$, and if $\gamma^i = \bar{\pi}^i \circ \gamma$ are the component functions of γ, then $D_a\gamma = (D_a\gamma^i)$, a column matrix, and if ϕ is a function, f on $\mathcal{U}_a \subset \mathbb{R}^n$, then $D_a f = (\partial f/\partial\pi^i|_a)$, a row matrix.

(More generally, if vector spaces V and W are not cartesian spaces we get partial derivatives after we choose bases, and then for each choice, the derivative is *represented by* a matrix.)

Now we recall several results from calculus in $\mathbb{R}^n$; namely, a couple of "chain rules" and a "product rule."

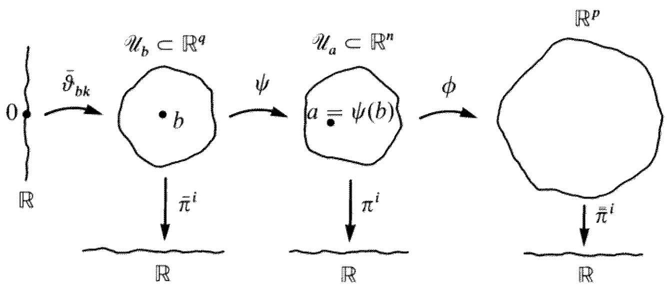

Figure 7.2

(i) If f is in C^1 at a, and γ_a is in C^1 at 0, then $f \circ \gamma_a$ is in C^1 at 0, and

$$D_0 f \circ \gamma_a = \left.\frac{\partial f}{\partial \pi^i}\right|_a D_0 \gamma_a^i \tag{7.2}$$

written as a sum of products of components, or

$$D_0 f \circ \gamma_a = \left(\left.\frac{\partial f}{\partial \pi^i}\right|_a \right) \circ (D_0 \gamma_a^i)$$

written as a composition of derivatives.

(ii) More generally, suppose $\mathcal{U}_b \subset \mathbb{R}^q$, $\mathcal{U}_a \subset \mathbb{R}^n$, $\psi : \mathcal{U}_b \to \mathbb{R}^n$, $\phi : \mathcal{U}_a \to \mathbb{R}^p$, and $\psi(b) = a$ (Fig. 7.2). If ϕ is C^1 at a, and ψ is C^1 at b, then the functions $\bar{\bar{\pi}}^i \circ \phi \circ \psi \circ \bar{\vartheta}_{bk} = \phi^i \circ \psi \circ \bar{\vartheta}_{bk}$ are C^1 at b, and

$$\left.\frac{\partial \phi^i \circ \psi}{\partial \bar{\pi}^k}\right|_b = \left.\frac{\partial \phi^i}{\partial \pi^j}\right|_a \left.\frac{\partial \psi^j}{\partial \bar{\pi}^k}\right|_b \qquad \begin{cases} i = 1, \ldots, p \\ j = 1, \ldots, n \\ k = 1, \ldots, q \end{cases} \tag{7.3}$$

or, equivalently, $\phi \circ \psi$ is C^1 at b, and

$$\left(\left.\frac{\partial \phi^i \circ \psi}{\partial \bar{\pi}^k}\right|_b \right) = \left(\left.\frac{\partial \phi^i}{\partial \pi^j}\right|_a \right) \circ \left(\left.\frac{\partial \psi^j}{\partial \bar{\pi}^k}\right|_b \right)$$

(iii) For functions f, g,

$$\left.\frac{\partial fg}{\partial \pi^i}\right|_a = D_0 fg \circ \vartheta_{ai} = D_0 (f \circ \vartheta_{ai})(g \circ \vartheta_{ai})$$

$$= f(a) \left.\frac{\partial g}{\partial \pi^i}\right|_a + g(a) \left.\frac{\partial f}{\partial \pi^i}\right|_a \tag{7.4}$$

As an example of the use of this notation, consider the function described in the usual notation by $f(x, y) = \sin(xy^2)$. We write $f(x^1, x^2) = \sin(x^1(x^2)^2)$, and using the projections we can also write $f = \sin \circ (\pi^1(\pi^2)^2)$. Then

$$\left.\frac{\partial f}{\partial \pi^1}\right|_a = D_0 \sin \circ (\pi^1(\pi^2)^2) \circ \vartheta_1 = (D_{a^1(a^2)^2} \sin)(D_0 \pi^1(\pi^2)^2 \circ \vartheta_1)$$

by the chain rule. Now the second factor

$$\begin{aligned} D_0\pi^1(\pi^2)^2 \circ \vartheta_1 &= D_0(\pi^1 \circ \vartheta_1)((\pi^2)^2 \circ \vartheta_1)) \\ &= \pi^1(a)\left.\frac{\partial(\pi^2)^2}{\partial \pi^1}\right|_a + (\pi^2(a))^2 \left.\frac{\partial \pi^1}{\partial \pi^1}\right|_a = (a^2)^2 \end{aligned}$$

by eq. (7.4). So $\partial f/\partial \pi^1|_a = (a^2)^2 \cos(a^1(a^2)^2)$.

There are two other results from the differential calculus in $\mathbb{R}^n$ that we will need later; the inverse and implicit function theorems, and an existence and uniqueness theorem for ordinary differential equations. We will recall these when the need for them arises in Sections 9.4 and 13.1, respectively.

PROBLEM 7.1 (a) Find $\partial f/\partial \pi^2|_a$ in the example above.

(b) If $f(x, y, z) = c_1x + c_2y + c_3z$ in the usual notation (c_i are constants), write f and compute $\partial f/\partial \pi^k|_a$ in our notation.

(c) If $f(x, y, z) = xyz$ in the usual notation, write f, and compute $(\partial f/\partial \pi^k)|_a$ in our notation.

PROBLEM 7.2 (a) If $f(x, y, z) = x^y$ write f in terms of the projections and compute $(\partial f/\partial \pi^i)|_a$.

(b) If $\phi(x, y, z) = (x^y, z)$ write an expression for ϕ and compute $(\partial \phi^i/\partial \pi^k)|_a$.

PROBLEM 7.3 Derive eq. (7.4) as a special case of eq. (7.3).

7.2 The tangent and cotangent spaces at a point of $\mathbb{R}^n$

In order to define a "derivative" for mappings between more general types of spaces, i.e., other than normed vector spaces, we introduce certain structures on them. Such structures can be introduced on the very special spaces $\mathbb{R}^n$, and when we eventually come to the description of our more general spaces - differentiable manifolds - it will be a straightforward matter to generalize these structures for our more general spaces.

For cartesian spaces we have available the concept of a C^1 map, i.e., a map whose domain is an open set on which it is continuously differentiable. So we can make the following definitions.

Since, for the time being, we will be working in neighborhoods of one fixed point, a, of $\mathbb{R}^n$, we will usually suppress the subscript a on the γ's, the partial derivatives, etc.

Definitions Two C^1 curves, γ_1 and γ_2, through a are *equivalent* (at a) if $D_0 f \circ \gamma_1 = D_0 f \circ \gamma_2$ for all C^1 functions, f, on neighborhoods of a. Two C^1 functions, f^1 and f^2, on neighborhoods of a are *equivalent* (at a) if $D_0 f^1 \circ \gamma = D_0 f^2 \circ \gamma$ for all C^1 curves, γ, through a. These definitions define equivalence relations. The equivalence class containing γ is called *the tangent of* γ (at a) and will be denoted by $[\gamma]$. That is, $[\gamma] = \{\alpha : D_0 f \circ \gamma = D_0 f \circ \alpha$ for all C^1 functions on neighborhoods of $a\}$. The equivalence class containing f is called *the differential of* f (at a) and will be denoted by $[f]$. That is, $[f] = \{g : D_0 f \circ \gamma = D_0 g \circ \gamma$ for all C^1 curves through $a\}$.

We wish to make several observations about these definitions.

(i) Note that though, for simplicity, we are restricting ourselves to $\mathbb{R}^n$, these definitions make sense for any normed vector space, since we have the concepts of differentiable curves and functions for such spaces. With a little more effort we could carry out the following development in this more general context. However, we choose to simply go from $\mathbb{R}^n$ to the full generalization of differentiable manifolds in Chapter 9 and note there that normed vector spaces are special cases.

(ii) It is important to note that for each curve, γ, through a we have defined a tangent *at a only*. We have not defined tangents at other points of γ. This will be done later, in Section 7.3. In the present development, eventually the curve will not be important, but the point will be. Thus, "the tangent of γ" will evolve into "the tangent at a."

(iii) The curve γ_1 given by $\gamma_1(u) = (a^1 + u^2, a^2 + \sin u)$ is equivalent to the coordinate curve ϑ_2. The curve γ_2 given by $\gamma_2(u) = (a^1, a^2 + 2u)$ is not equivalent to ϑ_2. This shows, in particular, that two curves can have the same set of values and have different tangents. Note that we define the tangent of γ so that we have a unique tangent vector at each point in contrast to the common picture of lots of tangent vectors at a point having different lengths.

(iv) From the chain rule (7.2) (taking $f = \pi^i$) we see that $\gamma_1 \sim \gamma_2 \Leftrightarrow D_0\gamma_1^i = D_0\gamma_2^i$, and (taking $\gamma = \vartheta_i$), $f^1 \sim f^2 \Leftrightarrow \partial f^1/\partial\pi^i = \partial f^2\ \partial\pi^i$. That is, we can say that two curves are equivalent if they have the same "tangent vector," in the usual calculus sense, and two functions are equivalent if their level surfaces have the same "normal vector" as in calculus.

Let T_a be the set of all tangents of curves at a, and let T^a be the set of all differentials of functions at a.

Theorem 7.1 *T^a is a real n-dimensional vector space.*

Proof With the usual definitions of addition of functions and multiplications of functions by real numbers, the corresponding operations $[f] + [g] = [(f+g)]$, and $c[f] = [cf]$ are well-defined. With these operations T^a is a vector space whose zero element is the equivalence class of functions constant on neighborhoods of a. Note that neither this theorem nor the definitions above depend on coordinate functions or coordinate curves. □

Theorem 7.2 *The set of differentials* $\{[\pi^i]\}$ *is a basis for* T^a *and for each* $[f] \in T^a$ *we have*

$$[f] = \frac{\partial f}{\partial \pi^i}[\pi^i] \tag{7.5}$$

Proof (1) Suppose $c_i[\pi^i] = 0$. (We are using the "summation convention"). Then $[c_i\pi^i] = 0$ by the definition of operations in T^1. But $0 \in T^a$ is the equivalence class of functions which are constant on $\mathcal{U}_a$. For such a function, g, $\partial g/\partial \pi^k = 0$ and since $c_i\pi^i$ and g are equivalent $\partial c_i\pi^i/\partial\pi^k = 0$, which implies that $c_i = 0$ for all i, so $\{[\pi^i]\}$ is a linearly independent set.

(2) Let $[f]$ be an arbitrary element of T^a. For any $f, f \sim (\partial f/\partial\pi^i)\pi^i$ since the partial derivatives of both sides are equal. So $[f] = [(\partial f/\partial\pi^i)\pi^i] = \partial f/\partial\pi^i[\pi^i]$ and hence the $[\pi^i]$ span T^a. □

Definitions T^a is called *the cotangent space at* a and $\{[\pi^i]\}$ is *the natural basis of* T^a.

Now we will also make T_a into a vector space. The procedure is not quite so direct as for T^a since $\gamma_1 + \gamma_2$ is not a curve through a. We first note that in each equivalence class there is precisely one differentiable curve, γ, of the form $\gamma^i(u) = a^i + r^i u$.*

We define $[\gamma_1] + [\gamma_2]$ to be the equivalence class containing the curve α given by $\alpha^i(u) = a^i + (r_1^i + r_2^i)u$ where γ_1 given by $\gamma_1^i(u) = a^i + r_1^i u$ is in $[\gamma_1]$ and γ_2 given by $\gamma_2^i(u) = a^i + r_2^i u$ is in $[\gamma_2]$. Similarly, $c[\gamma]$ is defined to be the equivalence class containing the curve given by $\beta^i(u) = a^i + cr^i u$, if $[\gamma]$ contains the curve given by $\gamma^i(u) = a^i + r^i u$.

Theorem 7.3 *With the given definitions of addition and scalar multiplication,* T_a *is a real vector space.*

Proof Straightforward verification of vector space properties. □

It should be noted that T_a as defined here cannot be constructed when we generalize beyond $\mathbb{R}^n$, as addition and scalar multiplication of vectors cannot

*This is the affine map in the class of curves tangent to each other at 0 (definition in Section 7.1). Thus, the class of curves equivalent at a is the same as the class of curves tangent to each other at 0.

be defined the way we did it. However, in Problem 7.5 we make a definition of T_a which gives the same space and can be generalized to spaces where we have derivatives.

Theorem 7.4 *The set* $\{[\vartheta_k]\}$ *is a basis for* T_a*, and for each* $[\gamma] \in T_a$ *we have*

$$[\gamma] = (D_0\gamma^i)[\vartheta_i] \tag{7.6}$$

Proof Problem 7.6. □

Definitions T_a is called *the tangent space at* a, and $\{[\vartheta_k]\}$ is *the natural basis of* T_a.

Theorem 7.5 T_a *and* T^a *are dual vector spaces and* $\{[\pi^i]\}$ *and* $\{[\vartheta_k]\}$ *are dual bases.*

Proof We have to show that the function from $T_a \times T^a$ to $\mathbb{R}$ given by $([\gamma], [f]) \mapsto D_0 f \circ \gamma$ is nondegenerate bilinear. To show nondegeneracy, choose successively $f = \pi^1, \pi^2, \ldots,$ and $\gamma = \vartheta_1, \vartheta_2, \ldots.$ To show linearity in the first argument we have to show $D_0 f \circ \gamma = D_0 f \circ \gamma_1 + D_0 f \circ \gamma_2$ for γ in the equivalence class $[\gamma_1] + [\gamma_2]$ and also that $D_0 f \circ \gamma = c D_0 f \circ \gamma_1$ for γ in the equivalence class $c[\gamma_1]$. These two equalities are verified easily when we use the special representatives α and β (described above Theorem 7.3) in each of the corresponding equivalence classes. □

Since T_a and T^a are finite-dimensional they can be considered *the* duals of one another; i.e., $T^a = T_a^*$ and $T_a = T^{a*}$, by Theorem 2.15. Thus, in particular, the differential, $[f]$, of f can be considered to be a linear function from T_a to $\mathbb{R}$, and from eqs. (7.5), (7.6), (7.2) we have

$$\langle [f], [\gamma] \rangle = D_0 f \circ \gamma \tag{7.7}$$

In the above development we defined the tangent of a curve at a point, in terms of the numbers $D_0 f \circ \gamma$, in a way which is strongly reminiscent of the ordinary geometrical definition; i.e., a tangent is a class of curves all of which have the same "tangent vector." Now we change our point of view somewhat and notice that the numbers $D_0 f \circ \gamma$ define certain mappings. Namely, for each γ through a we have a mapping, $\mathfrak{F}_a \to \mathbb{R}$, from the set of C^1 functions defined on neighborhoods of a to the real numbers, given by $f \mapsto D_0 f \circ \gamma$. Furthermore, all the mappings corresponding to the curves in an equivalence class are the same, so with each $[\gamma]$ we have a mapping

$$L_{[\gamma]} : \mathfrak{F}_a \to \mathbb{R} \tag{7.8}$$

given by $f \mapsto D_0 f \circ \gamma$.

The fact that the functions in $\mathfrak{F}_a$ do not all have the same domain causes problems in trying to give $\mathfrak{F}_a$ a simple standard algebraic structure, but we can define addition and multiplication in $\mathfrak{F}_a$ (in particular, scalar multiplication) on intersections of domains in the usual way.

Theorem 7.6 *The function $L_{[\gamma]}$ is a φ-derivation of $\mathfrak{F}_a$ to $\mathbb{R}$, i.e., $L_{[\gamma]}$ is linear, φ is the evaluation map, and*

$$L_{[\gamma]} \cdot fg = f(a) L_{[\gamma]} \cdot g + g(a) L_{[\gamma]} \cdot f \tag{7.9}$$

Proof Immediate from the properties of the derivative. □

Theorem 7.7 *The set $\{L_{[\gamma]} : [\gamma] \in T_a\}$, with the usual definition of addition and scalar multiplication of maps, is a vector space isomorphic to T_a.*

Proof To show that the map $[\gamma] \mapsto L_{[\gamma]}$ is linear we have to show $L_{[\gamma_1]} \cdot f + L_{[\gamma_2]} \cdot f = L_{[\gamma_1]+[\gamma_2]} \cdot f$ and $cL_{[\gamma]} \cdot f = L_{c[\gamma]} \cdot f$. That is, we must have, for all f, $D_0 f \circ \gamma_1 + D_0 f \circ \gamma_2 = D_0 f \circ \gamma$ for γ in the equivalence class of $[\gamma_1] + [\gamma_2]$ and $cD_0 f \circ \gamma = D_0 f \circ \gamma_1$ for γ_1 in the equivalence class of $c[\gamma]$. But these are the same two conditions we already verified in the proof of Theorem 7.5. □

On the basis of Theorem 7.7 we can think of tangents of curves as derivations, $L_{[\gamma]}$, of $\mathfrak{F}_a$ to $\mathbb{R}$ instead of equivalence classes, $[\gamma]$, of curves. The concept of a tangent being a function from $\mathfrak{F}_a$ to $\mathbb{R}$ also has a motivation from ordinary calculus, where the directional derivative gives a function from $\mathfrak{F}_a$ to $\mathbb{R}$ determined by the "vector" of coefficients.

Before proceeding we will pause to bring our notation into closer conformity with more common usage. We introduced the notations $[\gamma]$ and $[f]$ to emphasize the duality of the roles of curves and functions. However, $[f]$, which we called the differential of f, does have the properties and/or behavior we attribute (usually in a vague sort of way) to the ordinary calculus differential. In particular, if in eq. (7.5) we replace $[f]$ by df and $[\pi^i]$ by $d\pi^i$ we get

$$df = \frac{\partial f}{\partial \pi^i}\, d\pi^i \tag{7.10}$$

a familiar notation. Defining df as a vector or, more precisely, a covector, clarifies the usual calculus explanations of the relation between differentials and derivatives, the difference between differentials of dependent and independent variables, etc.

For $[\gamma]$, the situation is somewhat different. We do not have a good standard notation from calculus. We will keep $[\gamma]$ for the tangent vector of γ, except for the coordinate curves ϑ_k. Corresponding to $[\vartheta_k]$ we have $L_{[\vartheta_k]}$, and

$$L_{[\vartheta_k]} \cdot f = D_0 f \circ \vartheta_k = \frac{\partial f}{\partial \pi^k} \tag{7.11}$$

Equation (7.11) suggests replacing $L_{[\vartheta_k]}$ (and $[\vartheta_k]$) by $\partial/\partial\pi^k$, and writing (7.11) as

$$\frac{\partial}{\partial \pi^k} \cdot f = \frac{\partial f}{\partial \pi^k} \tag{7.12}$$

thus, defining $\dfrac{\partial}{\partial \pi^k}$ in terms of $\dfrac{\partial f}{\partial \pi^k}$.

Replacing $[\vartheta_k]$ by $\partial/\partial\pi^k$, eq. (7.6) becomes

$$[\gamma] = (D_0\gamma^i)\,\frac{\partial}{\partial \pi^i} \tag{7.13}$$

or, using $L_{[\gamma]} \cdot \pi^i = D_0\gamma^i$,

$$L_{[\gamma]} = (L_{[\gamma]} \cdot \pi^i)\,\frac{\partial}{\partial \pi^i} \tag{7.14}$$

In three dimensions, the tangent vector $\partial/\partial\pi^i$ of the coordinate curves are the $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ of calculus, and (7.13) is usually written something like

$$t_\gamma = \frac{dx}{du}\mathbf{i} + \frac{dy}{du}\mathbf{j} + \frac{dz}{du}\mathbf{k}$$

Finally, using (7.10) and (7.14), (7.7) can be written in the form

$$\langle df, L_{[\gamma]}\rangle = L_{[\gamma]} \cdot f \tag{7.15}$$

Now that we have discussed the concepts of tangent space and cotangent space at a point of $\mathbb{R}^n$ we would like to have these structures available on more general spaces. Our discussion involved at times (e.g., in the definitions of addition and scalar multiplication in T_a) certain specific properties of $\mathbb{R}^n$. Any generalization would have to be independent of such properties. There are several approaches, most of which require a prior description of the spaces (differentiable manifolds), on which these concepts are to be defined. However, while continuing to work in $\mathbb{R}^n$, a slight abstraction of our description of T_a as a certain set of derivations can be made which can be taken over to the more general spaces.

Instead of taking the specific functions $L_{[\gamma]}$ given by (7.8) for our tangents we could consider the functions from $\mathfrak{F}_a$ to $\mathbb{R}$ with one of the properties of $L_{[\gamma]}$, i.e., the set of all φ-derivations of $\mathfrak{F}_a$ to $\mathbb{R}$. This space is too large. We consider rather the set, $\{\boldsymbol{\Delta}\}$, of derivations, $\boldsymbol{\Delta}$, of $\mathfrak{F}_a$ to $\mathbb{R}$ with the additional property

$$\begin{array}{l}\text{there exists an } n\text{-tuple } (c^1, \ldots, c^n) \text{ of numbers such that} \\ \text{each derivation has the form } c^k \dfrac{\partial}{\partial \pi^k}\end{array} \tag{7.16}$$

We now have a set of derivations corresponding to n-tuples $(c^1, \ldots, c^n)$, instead of corresponding to curves, γ. There are no curves involved in this definition. We call such a function simply *a tangent vector at* a.

Notice that this definition explicitly involves the structure of $\mathbb{R}^n$, in particular, we can interpret (7.16) to say that the tangent space at a point of $\mathbb{R}^n$ is just $\mathbb{R}^n$.

Theorem 7.8 *The space of tangent vectors at a and the space of tangents of curves through a (given by (7.8)) are the same.*

Proof If $L_{[\gamma]}$ is a tangent of a curve, then

$$L_{[\gamma]} \cdot f = D_0 f \circ \gamma = D_0 \gamma^i \frac{\partial}{\partial \pi^i} \cdot f$$

by (7.14) so $L_{[\gamma]}$ has the property (7.16). On the other hand, if $\boldsymbol{\Delta}$ is a tangent vector, then $\boldsymbol{\Delta} = c^k \partial / \partial \pi^k$, which is a tangent to a curve, namely, the equivalence class containing the curve with component functions $\gamma^k : u \mapsto a^k + c^k u$ □

Now we finally identify T_a with the space of derivations of C^∞ functions on neighborhoods of a to $\mathbb{R}$.

Lemma *A derivation, v, has the property that if f is a constant function then* $v \cdot f = 0$.

Proof Problem 7.7. □

Theorem 7.9 *A derivation, v_a, on the set of C^∞ functions on neighborhoods of a satisfies property (7.16).*

Proof For any C^1 function f we have the first-order Taylor expansion

$$f(x) = f(a) + \sum_i (\pi^i(x) - \pi^i(a)) \int_0^1 \left.\frac{\partial f}{\partial \pi^i}\right|_{a+t(x-a)} dt$$

or

$$f = f(a) + \sum_i (\pi^i - \pi^i(a)) f^i \tag{7.17}$$

where the f^i are the functions (of x) defined by the integrals. If f is in C^2 we can take the partial derivative on both sides of (7.17) and apply the product rule on the terms on the right. We get $\partial f/\partial \pi^k|_a = f^k(a)$. If v_a is a derivation and we apply it to both sides of (7.17), and use the product rule on the terms on the right, we get, using the lemma above,

$$v_a \cdot f = \sum_i v_a \cdot \pi^i (f^i(a))$$

This step can be justified only if $f \in C^\infty$, since if $f \in C^k$, $f^i \in C^{k-1}$. Thus, using $f^i(a) = \partial f/\partial \pi^i|_a$ we get

$$v_a \cdot f = \sum_i v_a \cdot \pi^i \left.\frac{\partial f}{\partial \pi^i}\right|_a$$

which is of the form (7.16). □

Theorem 7.9 says $\{v : v \text{ is a derivation defined on } C^\infty \text{ functions}\} \subset \{c^k(\partial/\partial \pi^k)$ with domain C^∞ functions$\}$. The inclusion the other way is clear. The space of tangent vectors were derivations of the form $c^k(\partial/\partial \pi^k)$ with domain C^1 functions; i.e.,

$$\left\{ \mathbf{\Delta} = c^k \frac{\partial}{\partial \pi^k} : c^k \frac{\partial}{\partial \pi^k} \ \text{ has domain } \ C^1 \ \text{ functions} \right\}$$

Now there is a $1-1$ correspondence between maps $c^k(\partial/\partial \pi^k)$ with domain C^∞ functions and maps $c^k(\partial/\partial \pi^k)$ with domain C^1 functions, so we can conclude that $\{v\} \cong \{\mathbf{\Delta}\}$.

To summarize, we have described T_a, the tangent space at a point of $\mathbb{R}^n$ in four slightly different ways: a set of equivalence classes of C^1 curves, $[\gamma]$; a set of derivations, $L_{[\gamma]}$, on $\mathfrak{F}_a$; the set of derivations, $\mathbf{\Delta}$ on $\mathfrak{F}_a$ with the property (7.16); and the set of all derivations on C^∞ functions.

There are at least two other ways of describing T_a. With the exception of one "generally unimportant" situation, these descriptions can all be carried over to the spaces more general than $\mathbb{R}^n$ in which we will be interested, and they will all be equivalent.

In the future we will use the notation $v_a, w_a, \ldots$ for generic elements of T_a, and $\sigma_a, \tau_a, \ldots$ for generic elements of T^a. Since, for the time being, we will be working in neighborhoods of one fixed point, a, of $\mathbb{R}^n$, we will usually suppress the subscript a.

PROBLEM 7.4 Show that $f = (\pi^1)^2/2 + (\pi^2)2/2 + (1 - a^1)\pi^1 - a^2\pi^2$ is equivalent to π^1 at (a^1, a^2).

PROBLEM 7.5 (i) We can make T_a into a vector space without using the specific form of $\mathbb{R}^n$. Having made T^a into a vector space, we have its dual, T^{a*}.

Now define a map $\rho : T_a \to T^{a*}$ by $\rho[\gamma] \cdot [f] = D_0 f \circ \gamma$. Show that ρ is $1-1$ and define $[\gamma_1] + [\gamma_2] = \rho^{-1}(\rho[\gamma_1] + \rho[\gamma_2])$ and $c[\gamma] = \rho^{-1}(\rho c[\gamma])$.

(ii) If our space is $\mathbb{R}^n$, then this vector space is the same as the space T_a of Theorem 7.3.

PROBLEM 7.6 Prove Theorem 7.4.

PROBLEM 7.7 Prove the Lemma above Theorem 7.9.

PROBLEM 7.8 We abstracted certain properties of the functions $L_{[\gamma]}$, and called functions with these properties tangent vectors. Another set of properties of functions $\mathbf{v} : \mathfrak{F}_a \to \mathbb{R}$ giving the same set of functions is

(i) $\mathbf{v}$ is linear on $\mathfrak{F}_a$.

(ii) For f of the form $F \circ \psi$ where $\psi : \mathcal{U}_a \to \mathbb{R}^p$ and $F : \mathcal{U}_{\psi(a)} \subset \mathbb{R}^p \to \mathbb{R}$, and ψ and F are C^1,

$$\mathbf{v}(F \circ \psi) = \mathbf{v}(\psi^i(a)) \left. \frac{\partial F}{\partial \pi^i} \right|_{\psi(a)}$$

(cf., Willmore, p. 197 ff.)

PROBLEM 7.9 Show that the following functions are C^∞ and nonanalytic on $\mathbb{R}$ and sketch their graphs.

(a) $f(u) = \begin{cases} e^{-1/u} & u > 0 \\ 0 & u \leq 0 \end{cases}$

(b) $g(u) = \begin{cases} e^{1/u(u-1)} & 0 < u < 1 \\ 0 & u \leq 0 \text{ or } u \geq 1 \end{cases}$

(c) $h(u) = \dfrac{f(u)}{f(u) + f(1-u)}$ where $f(u)$ is defined in part (a).

7.3 The tangent map

Now we will study mappings from open sets of $\mathbb{R}^n$ to $\mathbb{R}^p$. For simplicity (to avoid having to deal simultaneously with different differentiability classes of functions) we will restrict ourselves to C^∞ mappings and C^∞ functions on $\mathbb{R}^n$ and $\mathbb{R}^p$. At each point of $\mathbb{R}^n$ we have a tangent space, and at each point of $\mathbb{R}^p$ there is a tangent space. In particular, if ϕ maps an open set containing a to $\mathbb{R}^p$, then we can look at the tangent space T_a at a and the tangent space $T_{\phi(a)}$ at $\phi(a)$.

If g is a C^∞ function in a neighborhood of $\phi(a)$, then $g \circ \phi$ is a C^∞ function in a neighborhood of a. Hence with each $v \in T_a$ we have a function w given by

$$w : g \mapsto v \cdot (g \circ \phi) \tag{7.18}$$

It is easy to check that this is a derivation so $w \in T_{\phi(a)}$ and we have a map from T_a to $T_{\phi(a)}$ given by $v \mapsto w$ (Fig. 7.3).

If γ is a curve through a, then $\phi \circ \gamma$ is a curve through $\phi(a)$. Hence with each $[\gamma]$ (or $L_{[\gamma]}$) in T_a we have a $[\phi \circ \gamma]$ (or $L_{[\phi\circ\gamma]}$) in $T_{\phi(a)}$. This mapping from T_a to $T_{\phi(a)}$ given by $L_{[\gamma]} \mapsto L_{[\phi\circ\gamma]}$ is the same as that given by $v \mapsto w$ above, since if $v = L_{[\gamma]}$, then

$$L_{[\phi\circ\gamma]} \cdot g = D_0 g \circ (\phi \circ \gamma) = D_0(g \circ \phi) \circ \gamma = L_{[\gamma]} \cdot g \circ \phi = v \cdot (g \circ \phi) = w \cdot g$$

so $L_{[\phi\circ\gamma]} = w$.

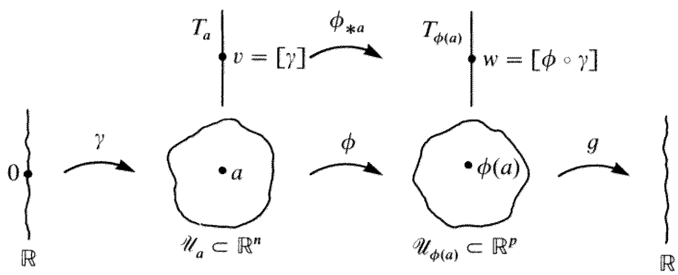

Figure 7.3

Definition The mapping $\phi_{*a} : T_a \to T_{\phi(a)}$ defined by $\phi_{*a}(v) = w$ where w is given by (7.18); i.e., $\phi_{*a}(v)g = v \cdot (g \circ \phi)$, or defined by $[\gamma] \mapsto [\phi \circ \gamma]$, is called *the tangent map* (or *the differential*, or *the derivative*) of ϕ at a.

Theorem 7.10 *ϕ_{*a} is linear.*

Proof Problem 7.10. □

Let us now evaluate ϕ_{*a} on the elements $\partial/\partial\pi^k\big|_a$, $k = 1, \ldots, n$, of the natural basis of T_a. If $\partial/\partial\bar{\pi}^j$, $j = 1, \ldots, p$, are the elements of the natural basis of $T_{\phi(a)}$, then, as in Eq. (1.6), for each k we should be able to write $\phi_{*a} \cdot \partial/\partial\pi^k\big|_a$ as a linear combination of the $\partial/\partial\bar{\pi}^j$. If g is a C^∞ function on $\mathcal{U}_{\phi(a)}$, then evaluating $\phi_* \cdot \partial/\partial\pi^k$ on g,

$$\begin{aligned}
\left(\phi_* \cdot \frac{\partial}{\partial\pi^k}\right) \cdot g &= \frac{\partial}{\partial\pi^k} \cdot g \circ \phi && \text{by (7.18)} \\
&= \frac{\partial g \circ \phi}{\partial\pi^k} && \text{by the definition of } \frac{\partial}{\partial\pi^k} \\
&= \frac{\partial g}{\partial\bar{\pi}^j}\,\frac{\partial\phi^j}{\partial\pi^k} && \text{by the chain rule, (7.3)} \\
&= \frac{\partial\phi^j}{\partial\pi^k}\,\frac{\partial}{\partial\bar{\pi}^j} \cdot g && \text{by the definition of } \frac{\partial}{\partial\bar{\pi}^j}
\end{aligned}$$

so

$$\phi_{*a} \cdot \left.\frac{\partial}{\partial \pi^k}\right|_a = \left.\frac{\partial \phi^j}{\partial \pi^k}\right|_a \left.\frac{\partial}{\partial \bar{\pi}^j}\right|_{\phi(a)} \tag{7.19}$$

and the coefficients of the linear transformation are $\partial\phi^j/\partial\pi^k$. Notice that if we arrange these as a $p \times n$ matrix we get the Jacobian matrix, or derivative, of ϕ. Finally, if v^k are the components of v in the basis $\partial/\partial\pi^k$ of T_a, and w^j are the components of $\phi_{*a} \cdot v$ in the basis $\partial/\partial\bar{\pi}^j$ of $T_{\phi(a)}$, then

$$w^j = \frac{\partial \phi^j}{\partial \pi^k}\, v^k \tag{7.20}$$

We consider two special cases: $n = 1$ and $p = 1$.

(i) If $n = 1$ then ϕ maps an open set $\mathcal{U}$ in $\mathbb{R}$ to $\mathbb{R}^p$, that is, ϕ is a curve, γ. Note that up to now we have been focusing mainly on curves *through a given point* in order to define the concept of tangent vector at a point. In the process we defined the tangent of γ, but not the tangents to arbitrary points of γ. Now γ is more general – its domain need not contain 0, and even our restricted concept of tangent of γ does not apply.

To define a general concept of tangents to a curve, we first make explicit that eq. (7.19) is valid for each point, a, in the domain of ϕ, and in this case, for $a = u \in \mathbb{R}$, it reduces to

$$\gamma_{*u} \cdot \left.\frac{d}{d\pi}\right|_u = \left.\frac{d\gamma^i}{d\pi}\right|_u \left.\frac{\partial}{\partial\bar{\pi}^i}\right|_{\gamma(u)}$$

Recall that $d/d\pi|_u = [\vartheta_u]$ is the tangent of ϑ_u, the coordinate curve through u, and $d\gamma^i/d\pi|_u = D_0(\gamma^i \circ \vartheta_u)$. So

$$\gamma_{*u} : [\vartheta_u] \mapsto D_0(\gamma^i \circ \vartheta_u) \left.\frac{\partial}{\partial\bar{\pi}^i}\right|_{\gamma(u)} = [\gamma \circ \vartheta_u]_{\gamma(u)}$$

by eq. (7.13). This result is also immediate from the definition of ϕ_{*u}.

Further, we have a map $d/d\pi : \mathbb{R} \to T^u$ defined by $u \mapsto \left.\frac{d}{d\pi}\right|_u = [\vartheta_u]$. Thus, finally, for each u in the domain of γ we have a vector in $T_{\gamma(u)}$ given by the composition

$$\dot{\gamma} = \gamma_{*u} \circ \frac{d}{d\pi} : u \mapsto [\vartheta_u] \mapsto [\gamma \circ \vartheta_u]_{\gamma(u)} \tag{7.21}$$

Definitions The map $\dot{\gamma}$ from the domain of γ to the set of tangent spaces of $\mathbb{R}^p$ is called *the canonical lift of* γ. $\dot{\gamma}(u) = [\gamma \circ \vartheta_u]_{\gamma(u)}$ is *the tangent* (or *velocity*) of γ at $\gamma(u)$.

Note that in the special case that γ is a curve through $\gamma(u)$, $\gamma(u) = \gamma(0)$, $u = 0$, and the tangent of γ at $\gamma(u)$ is the tangent of γ in our original sense; that is, $\dot{\gamma}(0) = [\gamma]$.

Since $D_0(\gamma^i \circ \vartheta_u) = D_u\gamma^i \circ D_0\vartheta_u = D_u\gamma^i$, we can write

$$[\gamma \circ \vartheta_u]_{\gamma(u)} = D_u\gamma^i \frac{\partial}{\partial \bar{\pi}^i}\bigg|_{\gamma(u)}.$$

Then

$$\dot{\gamma}(u) = D_u\gamma^i \frac{\partial}{\partial \bar{\pi}^i}\bigg|_{\gamma(u)} \tag{7.22}$$

so that $\dot{\gamma}(u)$ is a derivation, Δ, on C^1 functions. Then,

$$\dot{\gamma}(u)\cdot f = D_u\gamma^i \frac{\partial}{\partial \bar{\pi}^i}\bigg|_{\gamma(u)} \cdot f = D_u\gamma^i \frac{\partial}{\partial \bar{\pi}^i}\bigg|_{\gamma(u)} = D_u f \circ \gamma$$

The equation $\dot{\gamma}(u) \cdot f = D_u f \circ \gamma$ is sometimes used to define $\dot{\gamma}(u)$. With this definition $\dot{\gamma}(u)$ is clearly a derivation on C^∞ functions, and hence a tangent at $\gamma(u)$.

Finally, since ϕ_{*u} is a linear mapping on a 1-dimensional space when ϕ is a curve, it is sometimes identified with its image $\phi_{*u} \cdot d/d\pi|_u$ at $d/d\pi|_u$ according to Theorem 2.4; that is, if ϕ is a curve, the notation ϕ_{*u} is sometimes used for $\dot{\phi}(u)$.

(ii) If $p = 1$, then $\mathbb{R}^p = \mathbb{R}$, and the tangent spaces of $\mathbb{R}^p$ have a basis consisting of the single vector $d/d\bar{\pi}$. ϕ is a function $\mathcal{U}_a \subset \mathbb{R}^n \to \mathbb{R}$. So, $\phi_{*a} \cdot v = c(d/d\bar{\pi})$ and evaluating this at $\bar{\pi}$ gives $c = (\phi_{*a} \cdot v) \cdot \bar{\pi} = v \cdot (\bar{\pi} \cdot \phi) = v \cdot \phi$. Finally, putting $v = L_{[\gamma]}$ we get $c = \langle d\phi, v\rangle|_a$ by (7.15), so $\phi_{*a} \cdot v = \langle d\phi, v\rangle|_a(d/d\bar{\pi})$. This relationship accounts for the fact that ϕ_{*a} is often called the differential of ϕ at a, because in this case when ϕ is a function, we get $\phi_{*a} = d\phi$ if we equate $\phi_{*a} \cdot v$ and its coefficient, $\langle d\phi, v\rangle = d\phi \cdot v$.

Theorem 7.11 (The chain rule) *If $\phi : \mathcal{U}_a \to \mathbb{R}^p$, and $\psi : \mathcal{U}_{\phi(a)} \to \mathbb{R}^q$ where $\mathcal{U}_a \subset \mathbb{R}^n$ and $\mathcal{U}_{\phi(a)} \subset \mathbb{R}^p$, then $(\psi \circ \phi)_{*a} = \psi_{*\phi(a)} \circ \phi_{*a}$.*

Proof Problem 7.13. □

With ϕ_{*a} we have the transpose (or dual map) (see Section 2.4) $\phi^*_{\phi(a)} : T^*_{\phi(a)} \to T^*_a$ given by $\langle \phi^*_{\phi(a)} \cdot dg, v\rangle = \langle dg, \phi_{*a} \cdot v\rangle$ for all $v \in T_a$ and for all $dg \in T^*_{\phi(a)}$.

Theorem 7.12 *If g is a C^∞ function in a neighborhood of $\phi(a)$, then $\phi^*_{\phi(a)} \cdot dg = d(g \circ \phi)$.*

Proof

$$\begin{aligned}
\langle \phi^*_{\phi(a)} \cdot dg, v\rangle &= \langle dg, \phi_{*a} \cdot v\rangle && \text{by definition of transpose} \\
&= (\phi_{*a} \cdot v) \cdot g && \text{by (7.15)} \\
&= v \cdot (g \circ \phi) && \text{by (7.18)} \\
&= \langle d(g \circ \phi), v\rangle && \text{by (7.15)}
\end{aligned}$$

□

The coefficients of $\phi^*_{\phi(a)}$ in the natural bases of T^*_a and $T^*_{\phi(a)}$ are given by

$$\phi^*_{\phi(a)} \cdot d\bar{\pi}_k = \frac{\partial \phi^k}{\partial \pi^j}\, d\pi^j \tag{7.23}$$

If τ_k are the components of τ in the basis $d\bar{\pi}^k$ of $T^*_{\phi(a)}$, and σ_j are the components of $\phi^*_{\phi(a)} \cdot \tau$ in the basis $d\pi^j$ of T^*_a, then

$$\sigma_j = \frac{\partial \phi^k}{\partial \pi^j}\, \tau_k \tag{7.24}$$

One can proceed now to build the various tensor spaces and their algebras which we described in Chapters 4-6 on the spaces T_a, $T_{\phi(a)}$, T^*_a, and $T^*_{\phi(a)}$. Thus, in particular, we have the spaces $T_a \otimes \cdots \otimes T_a = (T_a)^r_0$ and $T^*_a \otimes \cdots \otimes T^*_a = (T_a)^0_s$, (Section 4.1), and their symmetric and skew-symmetric subspaces $\boldsymbol{S}^r(T_a)$, $\boldsymbol{S}^s(T^*_a), \bigwedge^r(T_a)$, and $\bigwedge^s(T^*_a)$, (Sections 5.2, 5.3).

Moreover, according to (4.7), and (4.8), the tangent map, ϕ_{*a} induces a linear map $\phi_{ra} : (T_a)^r_0 \to (T_{\phi(a)})^r_0$, and $\phi^*_{\phi(a)}$ induces a linear map $\phi^s_{\phi(a)} : (T_{\phi(a)})^0_s \to (T_a)^0_s$. In particular, we have the restrictions of these maps to $\bigwedge^r(T_a)$ and $\bigwedge^s(T^*_{\phi(a)})$ described in Section 6.3.

Now these maps can be extended in another direction. We have a space T_a for each $a \in \mathbb{R}^n$, so we also have the spaces $(T_a)^r_0$ for each $a \in \mathbb{R}^n$. Hence, if ϕ is defined on $\mathcal{U}$, we can define a map, ϕ_r, on $\bigcup_{a\in\mathcal{U}}(T_a)^r_0$ by its restrictions to the individual T_a's; i.e.,

$$\phi_r : \bigcup_{a\in\mathcal{U}} (T_a)^r_0 \to \bigcup_{b\in\mathbb{R}^p} (T_b)^r_0 \quad \text{by} \quad \phi_r\big|_{(T_a)^r_0} = \phi_{ra}.$$

Similarly, we have

$$\phi^s : \bigcup_{\phi(a)} (T_{\phi(a)})^0_s \to \bigcup_{a\in\mathbb{R}^n} (T_a)^0_s \quad \text{given by} \quad \phi^s\big|_{(T_{\phi(a)})^0_s} = \phi^s_{\phi(a)}$$

Notice, however, that while ϕ_r and ϕ^s are perfectly well-defined mappings we cannot say much about them until we know more about the properties of their domains and ranges. Thus, while ϕ itself is a C^∞ map, the statement that ϕ_r is a C^∞ map has no meaning until we have a meaning for differentiability of functions on $\bigcup_{a\in\mathcal{U}}(T_a)^r_0$. We will elucidate this point when we come to tensor fields in Chapter 11.

PROBLEM 7.10 Prove Theorem 7.10.

PROBLEM 7.11 Evaluate (ϕ_r) on a basis of $(T_a)^r_0$; i.e., generalize eq. (7.19).

PROBLEM 7.12 Illustrate eq. (7.22) with $\gamma : u \mapsto (1 + u^2, 1 + \sin u)$ and $a = \pi/2$.

PROBLEM 7.13 Prove Theorem 7.11.

PROBLEM 7.14 Derive eq. (7.3) from the result of Theorem 7.11. (The notation will be a little different.)

PROBLEM 7.15 Here is another way to describe the tangent map, ϕ_*. We define a mapping, ψ, from the set of C^∞ functions at $\phi(a)$ to the set of C^∞ functions at a by $\psi : g \mapsto g \circ \phi$. ψ is a linear map (strictly speaking, ψ should be a map between vector spaces of "germs" of functions; see comment below eq. (7.8)). The transpose, ψ^*, of ψ will map the space of linear functions on the C^∞ functions at a into the space of linear functions on the C^∞ functions at $\phi(a)$. The restriction of ψ^* to the derivations on the C^∞ functions at a will be the tangent map at a.

PROBLEM 7.16 Show that $\psi^* \cdot v$, where ψ^* is the map defined in Problem 7.15, depends only on dg (that is, it is the same for equivalent functions) so that the transpose of ψ^* restricted to derivations is ϕ^*. In particular then, we can write Theorem 7.12 as $\psi \cdot dg = d\psi \cdot g$.

8

TOPOLOGICAL SPACES

Chapter 1 was supposed to serve to recall some linear algebra needed for the following exposition of tensor algebra. This chapter is supposed to serve a similar purpose, to recall some topology which we will need to discuss differentiable manifolds.

8.1 Definitions, properties, and examples

Definitions If S is a set, and $\mathbf{T}$ is a set of subsets of S such that (i) every union of elements of $\mathbf{T}$ is an element of $\mathbf{T}$, (ii) the intersection of two elements of $\mathbf{T}$ is in $\mathbf{T}$, and finally, (iii) both S and the empty set are in $\mathbf{T}$, then $(S, \mathbf{T})$ is a *topological space* with *topology*, $\mathbf{T}$. The elements of $\mathbf{T}$ are *open sets*. A (open) *neighborhood of a point* is an open set containing that point. The complement (in S) of an open set is a *closed set*. If $A \subset S$ then $s \in S$ is *a limit point* of A if for all neighborhoods, $\mathcal{U}_s$ of s, $(\mathcal{U}_s - s) \cap A \neq \varnothing$. The *closure*, $\bar{A}$, of $A \subset S$ is the union of A and all its limit points. $(S, \mathbf{T})$ is a *Hausdorff space* if distinct elements of S are contained in disjoint elements of $\mathbf{T}$.

There are many simple consequences of the axioms, and many simple relations among the concepts defined. Thus, for example, finite unions of closed sets are closed, any intersection of closed sets is closed, A is closed if and only if $A = \bar{A}$, and so on.

A given set S can have many topologies defined on it, some of which may be quite "pathological." For example, every set has the *trivial* topology in which the only members of $\mathbf{T}$ are the empty set and S itself, and every set has the *discrete* topology in which the elements of $\mathbf{T}$ are all the subsets of S. If $\mathbf{T}_1 \subset \mathbf{T}_2$ then $\mathbf{T}_1$ is *weaker* than $\mathbf{T}_2$, or $\mathbf{T}_2$ is *finer* than $\mathbf{T}_1$. (See Problem 8.10.)

If $(S, \mathbf{T})$ is a topological space, and $A \subset S$, then $\mathbf{T}' = \{A \cap \mathcal{U} : \mathcal{U} \in \mathbf{T}\}$ is called the *relative* or *induced* topology of A, and $(A, \mathbf{T}')$ is a *subspace* of $(S, \mathbf{T})$. (See Problem 8.2.)

The most important examples of topological spaces are the real cartesian spaces, $\mathbb{R}^n$, with the topology defined by means of the "Euclidean" distance $d(a,b) = [\sum_{i=1}^n (a^i - b^i)^2]^{\frac{1}{2}}$, the open sets being the unions of open "balls," $\{b \in \mathbb{R}^n : d(a,b) < r, a \in \mathbb{R}^n, r \in \mathbb{R}^+\}$. These spaces, denoted by $(\mathbb{R}^n, d_2)$, are called *Euclidean metric spaces* (cf., Section 5.4(i)).

Euclidean metric spaces can be generalized in several directions.

1. If V^n is an n-dimensional vector space, then, once a basis is chosen, its elements are in $1-1$ correspondence with points of $\mathbb{R}^n$. A set in V^n will be open if its image in $\mathbb{R}^n$ is open. This definition is independent of the choice of basis since the components in the two bases are related by the change of basis matrix, a nonsingular continuous mapping of $\mathbb{R}^n$ onto $\mathbb{R}^n$ under which open sets are preserved (see Section 8.2). Thus, every finite-dimensional vector space can be considered to have a natural topology.

2. For any set on which we can define a distance function, or metric, we can define open sets just as we did for Euclidean metric spaces and get a topology.

Definitions A *metric* or *distance function* on a set S is a function $d : S \times S \to \mathbb{R}$ such that (i) $d(x, y) > 0$ if $x \neq y$, (ii) $d(x, y) = 0$ if $x = y$, (iii) $d(x, y) = d(y, x)$, and (iv) $d(x, z) \leq d(x, y) + d(y, z)$ for all x, y, z in S. A set on which a metric is defined is *a metric space.*

Other important examples of metric spaces are (i) $\mathbb{R}^n$ with the distance function $d(a, b) = \max_{1 \leq i \leq n} |a^i - b^i|$; (ii) the set of infinite sequences $\{(a^1, a^2, \ldots, a^n, \ldots) : a^i \in \mathbb{R}\}$ with $\sum_{i=1}^{\infty} (a^i)^2 < \infty$, so that we can define $d(a, b) = [\sum_{i=1}^{\infty} (a^i - b^i)^2]^{\frac{1}{2}}$; and (iii) the set of all continuous functions defined on the closed interval $[a, b]$, $(a, b \in \mathbb{R})$ with $d(f, g) = \max_{a \leq t \leq b} |f(t) - g(t)|$.

3. The Euclidean distance of the cartesian product $\mathbb{R}^n$ was defined in terms of the metric $d(a, b) = |a - b|$ of $\mathbb{R}$. If $X_1, \ldots, X_n$ are metric spaces, we can define a distance, and hence a topology, on the cartesian product $X_1 \times \cdots \times X_n$ by $d = (\sum (d_i)^2)^{\frac{1}{2}}$ where d_i is the metric on X_i. Example (i) above, in which the balls are cartesian products of open intervals, suggests a further generalization for when the X_i are not necessarily metric spaces. If $X_1, \ldots, X_n$ are topological spaces we define a topology on the cartesian product $X_1 \times \cdots \times X_n$ whose open sets are the unions of subsets of the form $\mathcal{U}_1 \times \cdots \times \mathcal{U}_n$ where $\mathcal{U}_i$ are open in X_i. With this *product topology*, $X_1 \times \cdots \times X_n$ is called a *product space.*

Theorem 8.1 *If (X, d_X) and (Y, d_Y) are metric spaces, the product topology on $X \times Y$ is the same as the metric topology induced by* $\max(d_X, d_Y)$.

Proof Problem 8.7. □

While vector spaces, metric spaces, and product spaces provide a vast generalization of real cartesian spaces, they do not include many topological spaces occurring in modern analysis, geometry, and applications. So we go back to general topological spaces.

Definitions Suppose $(S, \mathbf{T})$ is a given topological space, and suppose $\mathbf{B}$ is a set of open sets such that every open set is a union of elements of $\mathbf{B}$. Then $\mathbf{B}$ is *a basis for* $\mathbf{T}$. Alternatively, $\mathbf{B}$ is a set of open sets such that for every open set $\mathcal{U} \in \mathbf{T}$ and $s \in \mathcal{U}$ there is a $B \in \mathbf{B}$ contained in $\mathcal{U}$ containing s. Since $\mathbf{B}$ determines $\mathbf{T}$ we can say that $\mathbf{T}$ is the topology *generated* by $\mathbf{B}$.

Theorem 8.2 *The set of open balls with rational centers and rational radii is a basis of Euclidean space.*

Proof Given $\mathcal{U} \in \mathbf{T}$ and $s \in \mathcal{U}$, then s will be in some ball in $\mathcal{U}$. Inside of this ball there will be one with center s and rational radius, r. Finally, a ball, $\mathcal{O}$, with rational center a distance less than $r/3$ from s and radius $r/2$ will have the required property, $s \in \mathcal{O} \subset \mathcal{U}$. □

Corollary *Euclidean metric space has a countable basis.*

Not every metric space has a countable basis, see Problem 8.5. But, clearly, metric spaces are contained in the class characterized by the weaker property: each point has a countable family of neighborhoods at least one of which is contained in every neighborhood of the point (called *1st countability*).

In the definition of basis we started with a given topological space. If we start with just a set S, and pick a set of subsets of S, in general, this family will not constitute a basis for a topology in S. To be a basis it is necessary and sufficient that the family includes S itself (or covers S) and also includes a set $\mathcal{U}$ for each s in an intersection $\mathcal{U}_1 \cap \mathcal{U}_2$ such that $s \in \mathcal{U} \subset \mathcal{U}_1 \cap \mathcal{U}_2$. Two different bases can generate different topologies.

An array of concepts related to the idea of a covering will be important for us.

Definitions If $\{A_\alpha\}$ and $\{B_\beta\}$ are coverings of S, then $\{B_\beta\}$ is called a *refinement of* $\{A_\alpha\}$ if each B_β is contained in some A_α. If $\{A_\alpha\}$ is a covering of S such that each $s \in S$ has a neighborhood which intersects only a finite number of members of $\{A_\alpha\}$, then $\{A_\alpha\}$ is called *locally finite*. A subset, A, of S is *compact* if every open covering of A has a finite subcovering. S is *locally compact* if each $s \in S$ has a neighborhood whose closure is compact. S is *paracompact* if every open covering of S has an open locally finite refinement.

Lots of examples of these definitions can be found in $\mathbb{R}^n$. Thus, the open disks of radius 1, with centers (m, n) where m, and n are integers, cover $\mathbb{R}^2$. So do the open disks of radius $\frac{1}{2}$ with centers $(m/2, n/2)$. The latter is not a refinement of the former. The set of all intersections $\{A_\alpha \cap B_\beta\}$, where A_α belongs to the first covering and B_β belongs to the second, is a refinement of both. Also, a subcovering is a refinement of a covering. Each of the examples given is a locally finite covering. Each example also shows that $\mathbb{R}^2$ is not compact. An important theorem in analysis, the Heine-Borel theorem, says that the compact subsets of

$\mathbb{R}^n$ are the closed bounded sets. Thus, an open disk is not compact. We can cover it by concentric open disks of radius $r - 1/n$, for which there is no finite subcovering. Also, this is an example of a covering which is not locally finite.

$\mathbb{R}^n$ and open sets of $\mathbb{R}^n$ are locally compact, because each point is in a ball whose closure is compact. On the other hand, the set of rationals in $\mathbb{R}$ in the relative topology is not locally compact. The closure of every neighborhood of every point in this space can be covered by a set of open sets which has no finite subcovering.

Finally, a theorem we will need later, but whose proof is a little long, guarantees that a Hausdorff space which is locally compact and has a countable basis is paracompact, so, in particular, $\mathbb{R}^n$ and open sets in $\mathbb{R}^n$ are paracompact.

PROBLEM 8.1 Give an example of topological space which is not a Hausdorff space.

PROBLEM 8.2 (i) Show that $\mathbf{T}'$ is a topology on A (In the definition of relative topology). (ii) Show that if $\mathcal{U}_i$ is open in X_i, $i = 1, \ldots, n$, then the subsets of the form

$$\mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_n$$

are a topology on

$$X_1 \times X_2 \times \cdots \times X_n.$$

PROBLEM 8.3 Prove that a metric space is Hausdorff.

PROBLEM 8.4 Consider the functions $d_1(a,b) = \sum_{i=1}^n |a^i - b^i|$ and $d_\infty(a,b) = \max_{1\leq i\leq n}\{|a^i - b^i|\}$. Show they are both metrics on $\mathbb{R}^n$ and determine the same topology in $\mathbb{R}^n$ as the Euclidean distance.

PROBLEM 8.5 Show that for any set

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

is a metric. What is the topology determined by this metric?

PROBLEM 8.6 Give an example of a nonmetrizable topological space.

PROBLEM 8.7 Prove Theorem 8.1.

PROBLEM 8.8 Show that the two given descriptions of a basis for $\mathbf{T}$ both describe the same thing.

PROBLEM 8.9 Prove the statement below Theorem 8.2 giving necessary and sufficient conditions for a basis of a topology.

8.2 Continuous mappings

Definitions If X and Y are two topological spaces, a mapping $\phi : X \to Y$ is *continuous* if the inverse image of each open set in Y is open in X. ϕ is *continuous at* $s \in X$ if for every neighborhood $\mathcal{V}$ of $\phi(s)$ there is a neighborhood $\mathcal{U}$ of s such that $\phi(\mathcal{U}) \subset \mathcal{V}$. If ϕ is $1-1$, and onto Y and ϕ and ϕ^{-1} are both continuous, then ϕ is a *homeomorphism*. If there exists a homeomorphism between two spaces they are *homeomorphic*. If (X, d) and (Y, D) are metric spaces, and $D(\phi(x), \phi(y)) = d(x, y)$, then ϕ is *an isometry* (cf., eq. (5.21)).

Theorem 8.3 *ϕ is continuous if and only if ϕ is continuous at s for each $s \in X$.*

Proof Problem 8.12. □

Theorem 8.4 (1) *If $\phi : (X, d) \to (Y, D)$ is an isometry, then X and $\phi(X)$ are homeomorphic.* (2) *The identity map, $id : (X, d) \to (X, D)$ is continuous at x if there is a positive number, c, such that for all y in some d-neighborhood of x, $D(x, y) \leq cd(x, y)$.*

Proof Problem 8.13. □

We will need a variety of basic topological results in the following chapters. For the most part, we will deal with these situations as they arise. However, we insert here two important results of the sort we will need.

Theorem 8.5 *Suppose X and Y are topological spaces and $\phi : X \to Y$ is continuous, $1-1$ and onto. If, moreover, X is compact and Y is Hausdorff, then ϕ is homeomorphism.*

Proof We have to show that ϕ^{-1} is continuous. We will use the criterion that a map is continuous if (and only if) the inverse image of every closed set is closed. That is, we will show that if $F \subset X$ is closed then $(\phi^{-1})^{-1}(F) = \phi(F)$ is closed. Now, F is compact because a closed subset of a compact space is compact. Then $\phi(F)$ is compact because the image of a compact set under a continuous mapping is compact. Finally, $\phi(F)$ is closed because a compact subset of Hausdorff space is closed. □

Definitions supp f, *the support of f*, is the closure of the set on which $f \neq 0$. If X is a Hausdorff space, a set of nonnegative continuous functions $\{f_\alpha\}$ on X such that

(1) $\{\text{supp } f_\alpha\}$ is a locally finite covering of X.
(2) $\sum_\alpha f_\alpha(s) = 1$ for all $s \in X$

is called *a partition of unity on X*.

Theorem 8.6 *If X is a Hausdorff paracompact space and $\{A_\alpha\}$ is a covering of X, then there exists a partition of unity on X such that* $\{\text{supp } f_\alpha\}$ *is a refinement of $\{A_\alpha\}$.* (*A partition of unity* subordinate to *the given covering.*)

Proof Problem 8.16. □

PROBLEM 8.10 (i) If $Y \subset X$, then $(Y, \mathbf{T}_2)$ is a subspace of $(X, \mathbf{T}_1)$ iff the inclusion map $id\big|_Y : (Y, \mathbf{T}_2) \to (X, \mathbf{T}_1)$ is continuous. (ii) If $\mathbf{T}_1$ and $\mathbf{T}_2$ are two topologies on X, then the identity map $id : (X, \mathbf{T}_1) \to (X, \mathbf{T}_2)$ is continuous iff $\mathbf{T}_2 \subset \mathbf{T}_1$.

PROBLEM 8.11 With the natural topology on V^n, and the product topologies on $V^n \times V^n$ and $\mathbb{R} \times V^n$, show that the vector space operations on V^n are continuous and all linear functions on V^n are continuous.

PROBLEM 8.12 Prove Theorem 8.3.

PROBLEM 8.13 Prove Theorem 8.4.

PROBLEM 8.14 Our proof of Theorem 8.5 essentially consisted of stating four "lemmas" none of which we have proved. Prove any one or more of these.

PROBLEM 8.15 Suppose X and Y are sets and $\phi : X \to Y$.

(i) If Y is a topological space, then $\{\phi^{-1}(\mathcal{V}) : \mathcal{V} \text{ is open in } Y\}$ defines a topology on X.

(ii) If X is a topological space, then $\{\mathcal{V} \subset Y : \phi^{-1}(\mathcal{V}) \text{ is open in } X\}$ defines a topology on Y, *the identification topology.*

PROBLEM 8.16 Prove Theorem 8.6 (cf. Dugundji, p. 170).

9

DIFFERENTIABLE MANIFOLDS

At the beginning of Chapter 7 we mentioned that for the description of physical phenomena we have to generalize the concept of maps between cartesian spaces to those between more general domains and image spaces. We want to keep the domain as general as possible; i.e., with as little structure as possible. This is the perspective that motivates the study of sets on which local coordinates are defined. After general definitions, examples, and a brief discussion of mappings we impose the structures of Sections 7.2 and 7.3 on manifolds.

9.1 Definitions and examples

A differentiable manifold is a Hausdorff space, M, with certain additional structure on it.

Definitions An *n-dimensional chart* (or *a local coordinate system*) at $p \in M$ is a pair $(\mathcal{U}, \mu)$ where $\mathcal{U}$ is an open set of M containing p, and μ is a homeomorphism of $\mathcal{U}$ onto an open set of $\mathbb{R}^n$ (with the Euclidean topology). $\mathcal{U}$ is *a coordinate neighborhood*, and μ is *a coordinate map.* If π^i are the natural coordinate functions on $\mu(\mathcal{U})$ then $\mu^i = \pi^i \circ \mu$ are *local coordinate functions* (or *local coordinates*) on $\mathcal{U}$. A set of n-dimensional charts $\{(\mathcal{U}_\alpha, \mu_\alpha)\}$ such that $\{\mathcal{U}_\alpha\}$ covers M is an *atlas.* A Hausdorff space with an atlas is *a locally Euclidean space* (or *a topological manifold*).

Definitions A C^k *manifold* is a locally Euclidean space with a countable basis and with an atlas with the properties (1) if $(\mathcal{U}, \mu)$ and $(\mathcal{V}, \nu)$ are two charts such that $\mathcal{U} \cap \mathcal{V} \neq \varnothing$, then $\nu \circ \mu^{-1}$ is C^k on $\mu(\mathcal{U} \cap \mathcal{V})$ and $\mu \circ \nu^{-1}$ is C^k on $\nu(\mathcal{U} \cap \mathcal{V})$, and (2) if $(\mathcal{W}, \xi)$ has property (1) for every chart in the atlas then $(\mathcal{W}, \xi)$ is in the atlas. The mappings $\nu \circ \mu^{-1}$ and $\mu \circ \nu^{-1}$ are called *overlap mappings,* or *coordinate transformations*, and a pair of charts having property (1) are said to be C^k*-related*, or, C^k*-compatible.* An atlas with property (1) is called a C^k *atlas*, and a C^k atlas with property (2) is said to be *maximal.* A maximal C^k atlas is called *a* C^k *differentiable structure* (see Fig. 9.1).

We do not usually concern ourselves with property (2) in practice because we can get a C^k manifold by first putting a C^k atlas on a Hausdorff space with a countable basis and then throwing in all C^k-related (or, admissible) charts. An

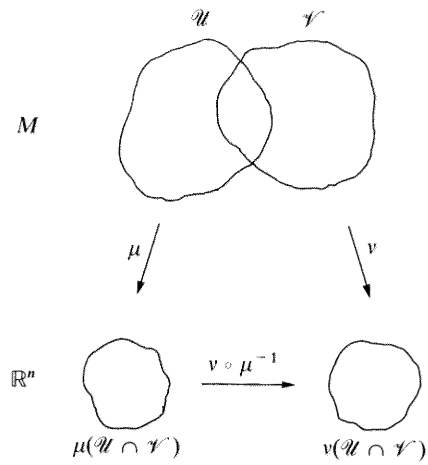

Figure 9.1

important result in differential topology says that when we do this (for $k > 0$), we can always find a subset of the set of these charts which cover M and are all C^∞-related; that is, M has a C^∞ atlas (cf., Munkres).

We note also that, according to the result mentioned in Section 8.2, since a manifold is locally compact, Hausdorff, and has a countable basis, it is paracompact, and hence, in particular, there exist partitions of unity on it.

Let us look at some examples of differentiable manifolds.

1. We can make $\mathbb{R}^n$ into a manifold by putting on it the atlas consisting of the single chart $(\mathbb{R}^n, id)$. The corresponding coordinate functions $\mu^i = \pi^i \circ id = \pi^i$ are called *the standard* (or *natural*) *coordinates* (see Section 7.1). If we add all possible C^ω-related charts we get a C^ω manifold. For example, on $\mathbb{R}^2$ we could add the chart $(\mathcal{U}, \mu)$ given by

$$\mu^1 : (a^1, a^2) \mapsto b_1 = \tan^{-1} \frac{a^2}{a^1}$$

$$\mu^2 : (a^1, a^2) \mapsto b_2 = \sqrt{(a^1)^2 + (a^2)^2}$$

defined on any open set, $\mathcal{U} \subset \mathbb{R}^2$ for which $a^1 > 0$. The manifold constructed on $\mathbb{R}^n$ in this way gives *the standard manifold structure* of $\mathbb{R}^n$. There are other manifold structures on $\mathbb{R}^n$. Thus, for $\mathbb{R}^1$, the single chart $(\mathbb{R}, \mu_0)$ where $\mu_0 : a \mapsto a^3$ is an atlas. This determines a C^ω manifold different from the standard one, since $(\mathbb{R}, id)$ and $(\mathbb{R}, \mu_0)$ are not C^ω-related. In fact they are not C^k-related for any k.

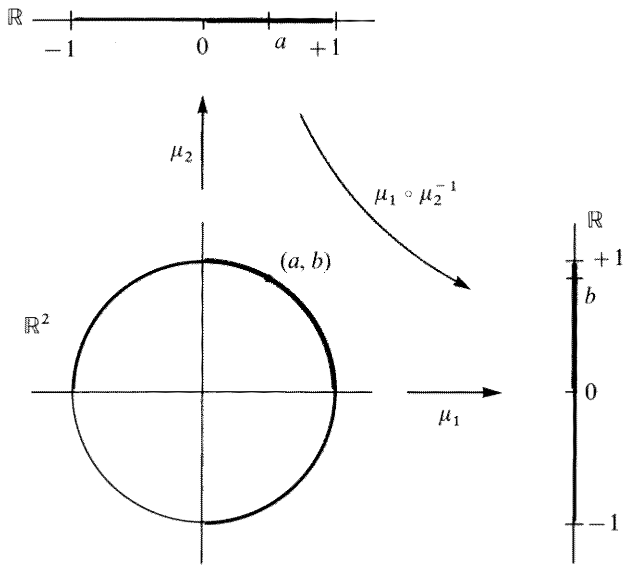

Figure 9.2

2. As a generalization of Example 1 we have that any n-dimensional vector space, V^n, with the standard, or natural, topology can be made into a manifold by choosing a basis, and then the single chart, (V^n, μ), will be an atlas, where $\mu{:}V^N \to \mathbb{R}^n$ maps v to its set of components. With another basis we get another chart (V^n, ν) and $\nu \circ \mu^{-1}$ is given by the change-of-basis matrix.

3. To a large extent the concept of a manifold is motivated by the desire to generalize and/or abstract the intuitive ideas of curves in $\mathbb{R}^2$ and $\mathbb{R}^3$, and surfaces in $\mathbb{R}^3$. For an example of this kind consider the unit circle $S^1 = \{(a,b){:}\, a^2 + b^2 = 1,\ a, b \in \mathbb{R}\}$ in $\mathbb{R}^2$ (see Fig. 9.2). S^1 is given the topology induced from $\mathbb{R}^2$. Let $\mathcal{U}_1 = \{(a,b) \in S^1 : a > 0\}$ and let $\mu_1 : \mathcal{U}_1 \to \mathbb{R}$ by $(a,b) \mapsto b$. Then $(\mathcal{U}_1, \mu_1)$ is a chart on S^1. Further, let $\mathcal{U}_2$, $\mathcal{U}_3$, $\mathcal{U}_4$ be the subsets of S^1 for which $b > 0$, $a < 0$, $b < 0$, respectively, and μ_2, μ_3, μ_4 take (a,b) to a, b, and a, respectively. Then the four charts $(\mathcal{U}_i, \mu_i)$, $i = 1,2,3,4$, form an atlas for S^1. On $\mathcal{U}_1 \cap \mathcal{U}_2$, since $\mu_1(a,b) = b$ and $\mu_2(a,b) = a$, $\mu_1 \circ \mu_2^{-1} : a \mapsto (a,b) \mapsto b = \sqrt{1 - a^2}$ for $0 < a < 1$, which is C^ω. We can check that all the other overlap maps are also C^ω so S^1 is a C^∞ manifold.

4. Consider the set, E (figure-eight) of points, (a,b) in $\mathbb{R}^2$ such that $(a,b) = (\sin 2u, \sin u)$, $0 < u < 2\pi$. These formulas determine a bijective map, μ, from E to $(0, 2\pi)$. (E, μ) will be a chart on E, and hence E will be a C^ω manifold if μ is a homeomorphism. $\mu(0,0) = \pi$, but no $\mathbb{R}^2$ neighborhood of $(0,0)$ maps into a neighborhood of π, so μ is not continuous if E has the induced, or relative, subset topology from $\mathbb{R}^2$, and (E, μ) is not a chart on E. Moreover, E, with the induced topology, cannot be made into a manifold since the only connected 1-dimensional manifolds are open intervals of $\mathbb{R}$ and S^1 (Bishop and Goldberg, p. 26). On the other hand, (E, μ) is a chart and E is a manifold if E is given the topology induced by μ from $\mathbb{R}$.

Examples 3 and 4 above illustrate two different ways of prescribing subsets of $\mathbb{R}^n$. Other examples appear in Problems 9.2, 9.3, 9.4, and 9.6. They prompt a generalization discussed in Section 9.4, and Chapter 10.

From the standpoint of the abstract definition of a manifold, Examples 3 and 4 (and their generalizations) are somewhat special in that we used an "ambient" space to construct them. Clearly, this is not always necessary, and when we do use an ambient space we have to be careful about thinking of the manifold as being "contained in" that space. This is suggested in Example 4, in which the topology of the manifold is not the subset topology of $\mathbb{R}^2$. Another example which illustrates this is the graph of $b = |a|$, $-1 < a < 1$. As a subset of $\mathbb{R}^2$ it has a corner, but as a 1-dimensional manifold it is "smooth". The concept of a manifold lying in a larger space will be elucidated in Chapter 10. It is important because (i) the basic familiar geometrical prototypes of manifolds, curves, and surfaces, are *in* Euclidean affine space, $\mathcal{E}_0^3$, and (ii) certain important theorems (cf., Auslander and MacKenzie, p. 116) show that in some sense every manifold is contained in $\mathbb{R}^n$ for large enough n. However, for many important manifolds this relation is not intuitively clear, and, in any event, it is instructive to study a manifold on its own (i.e., intrinsic properties) without reference to any $\mathbb{R}^n$ in which it may or may not be contained. This is illustrated by the following examples.

5. Consider the set of lines ℓ through a point of Euclidean affine space, $\mathcal{E}_0^3$. We make this set into a topological space by giving it the topology determined by that of a 2-sphere, S^2, with center at the given point, and with opposite points identified. This space is Hausdorff and has a countable basis. We represent $\mathcal{E}_0^3$ by $\mathbb{R}^3$ with $(0,0,0)$ corresponding to the given point.

Let $\mathcal{U}_1 = \{\ell : \ell \text{ is not in the plane } a^1 = 0\}$ and let $\mu_1 : \mathcal{U}_1 \to \mathbb{R}^2$ be given by

$$\mu_1^1 : \ell \mapsto \frac{a^2}{a^1} = \text{ slope of the projection of } \ell \text{ on the } a^1a^2 \text{ plane}$$

$$\mu_1^2 : \ell \mapsto \frac{a^3}{a^1} = \text{ slope of the projection of } \ell \text{ on the } a^1a^3 \text{ plane}$$

That is,

$$\mu_1(\ell) = \left(\frac{a^2}{a^1}, \frac{a^3}{a^1}\right) \quad \text{and } \mu_1 \text{ is } 1-1 \text{ onto } \mathbb{R}^2.$$

We define two more charts analogously. That is, $\mathcal{U}_2$ is the set of all lines except those in $a^2 = 0$, and $\mu_2(\ell) = (a^3/a^2, a^1/a^2)$, and μ_3 is the set of all lines except those in $a^3 = 0$, and $\mu_3(\ell) = (a^1/a^3, a^2/a^3)$. Clearly, $\{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3\}$ covers the space. We will rely on the assumption that it is intuitively evident that the

$\mathcal{U}$'s are open, and the μ's are homeomorphisms onto $\mathbb{R}^2$, and defer the proof of these facts to Problem 9.8. To complete the verification that we have a C^k manifold, we have to check the overlap maps of property (1).

Thus, consider, e.g.,

$$\mu_2 \circ \mu_1^{-1} : \mu_1(\mathcal{U}_1 \cap \mathcal{U}_2) \to \mu_2(\mathcal{U}_1 \cap \mathcal{U}_2).$$

$\mathcal{U}_1 \cap \mathcal{U}_2$ consists of all lines except those in the planes $a^1 = 0$ or $a^2 = 0$, and

$$\mu_2 \circ \mu_1^{-1} : \left(\frac{a^2}{a^1}, \frac{a^3}{a^1}\right) \mapsto \left(\frac{a^3}{a^2}, \frac{a^1}{a^2}\right) = \left(\frac{a^3/a^1}{a^2/a^1}, \frac{1}{a^2/a^1}\right)$$

or

$$\mu_2 \circ \mu_1^{-1} : (b^1, b^2) \mapsto \left(\frac{b^2}{b^1}, \frac{1}{b^1}\right) \quad \text{for all} \quad b^1 \neq 0$$

This mapping is C^ω and, similarly, checking the other overlap mappings we find they are all C^ω, so we have constructed a C^ω 2-manifold, *the analytic real projective plane*, $P^2(\mathbb{R})$.

6. Example 5, the projective plane, is a particular example of a class of manifolds called *Grassmann manifolds* of $\mathbb{R}^n$, or, more generally, of any vector space V^n.

Definitions We denote by $G(k, \mathbb{R}^n)$ the set of all *k-planes through the origin of* $\mathbb{R}^n$; that is, with $\mathbb{R}^n$ considered a vector space, the set of all k-dimensional subspaces of $\mathbb{R}^n$. We denote by $F(k, \mathbb{R}^n)$ the set of all *k-frames of* $\mathbb{R}^n$; i.e., the set of all linearly independent sets of k n-tuples.

$$\left\{ \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1k} \\ \vdots \\ a_{nk} \end{pmatrix} \right\}$$

Two k-frames are *equivalent* if they determine the same k-plane. Two k-frames will be equivalent iff their matrices A and B are related by $A = BC$ where C is a nonsingular $k \times k$ matrix.

Now corresponding to each k-plane, Π, we have an equivalence class of k-frames and hence an equivalence class of matrices. For each matrix some set of k rows must be linearly independent.

We can consider the set, $\mathcal{U}_1$, of all k-planes having a matrix representation with its first k rows linearly independent. Then each of these k-planes has a representation of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \\ a_{k+11} & & \cdots & a_{k+1k} \\ \vdots & & & \vdots \\ a_{n1} & & \cdots & a_{nk} \end{bmatrix}$$

Similarly, consider the set $\mathcal{U}_i$ of all k-planes whose matrices have some other set of k rows linearly independent. This set of matrices will have a representative with the identity $k \times k$ matrix in these k rows.

In this way all k-planes lie in one or more of the sets $\mathcal{U}_i$, $i = 1, \ldots, \binom{n}{k}$. The $\mathcal{U}_i$ will be coordinate neighborhoods in $G(k.\mathbb{R}^n)$ and with each Π in each $\mathcal{U}_i$ we can associate a unique element of $\mathbb{R}^{(n-k)k}$ given by stringing out the elements of the complement of the unit submatrix in the canonical representation of Π. If Π is in both $\mathcal{U}_i$ and $\mathcal{U}_j$, then the canonical representation in $\mathcal{U}_j$ is a multiple by a nonsingular $k \times k$ matrix of the canonical representation in $\mathcal{U}_i$, and this relation gives the overlap map.

To illustrate, for $G(2, \mathbb{R}^3)$ we have the atlas $\{(\mathcal{U}_i, \mu_i) : i = 1, 2, 3\}$ where $\mathcal{U}_i$ are those 2-planes whose canonical matrices are

$$\begin{pmatrix} a_{11} & a_{12} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ a_{21} & a_{22} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a_{31} & a_{32} \end{pmatrix}$$

respectively. Then $\mu_i : \Pi \mapsto (a_{i1}, a_{i2})$. Now if Π is in $\mathcal{U}_1$ and also in $\mathcal{U}_2$, and if Π is represented in $\mathcal{U}_1$ by

$$\begin{pmatrix} a_{11} & a_{12} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and in $\mathcal{U}_2$ by

$$\begin{pmatrix} 1 & 0 \\ a_{21} & a_{22} \\ 0 & 1 \end{pmatrix}$$

then

$$\begin{pmatrix} 1 & 0 \\ a_{21} & a_{22} \\ 0 & 1 \end{pmatrix} C = \begin{pmatrix} a_{11} & a_{12} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for some nonsingular matrix C. So

$$C = \begin{pmatrix} a_{11} & a_{12} \\ 0 & 1 \end{pmatrix}$$

where $a_{11} \neq 0$ and $a_{21} = 1/a_{11}$ and $a_{22} = -a_{12}/a_{11}$. Hence the overlap map $\mathbb{R}^2 - (0, r) \to \mathbb{R}^2$ is C^∞.

Now let us recall that, in this example, we started with a set $G(k, \mathbb{R}^n)$ of k-dimensional vector spaces. Out of this set we selected subsets, $\mathcal{U}_i$, which, together, cover it. With each $\mathcal{U}_i$ we have a map $\mu_i : \mathcal{U}_i \to \mathbb{R}^{(n-k)k}$ and hence we have an "atlas" on $G(k, \mathbb{R}^n)$. For $G(2, \mathbb{R}^3)$ we have shown that the overlap maps are C^ω. In the general case also, it can be shown we have a C^ω "atlas" on $G(k, \mathbb{R}^n)$. Thus, we have given $G(k, \mathbb{R}^n)$ a "differentiable structure", even though no topology has been defined on $G(k, \mathbb{R}^n)$. According to our definition, a manifold must be a topological space.

In this example, we can define a topology on the given set by recalling that its elements are in $1-1$ correspondence with equivalence classes of $n \times k$ matrices, so we can give $G(k.\mathbb{R}^n)$ the quotient topology; i.e., a set in $G(k, \mathbb{R}^n)$ is open if its inverse image in $F(k, \mathbb{R}^n)$ under the natural projection is open. We can then show that the $\mathcal{U}_i$ defined above are open, the μ_i are homeomorphisms and $G(k.\mathbb{R}^n)$ has a countable basis and is Hausdorff (Brickell and Clark, p. 41, p. 43), so $G(k, \mathbb{R}^n)$ has all the properties required in the definition of a C^ω manifold. In general, if one simply has a set with a "differentiable structure," as we had in this example, we can define an induced topology on the set for which the set of all coordinate domains forms a basis. (loc. cit. p. 22) Thus, the topology will have a countable basis if the set of coordinate domains is countable, otherwise it may not. Neither will it necessarily be Hausdorff.

7. Examples 5 and 6 are members of the general class of *quotient manifolds*. These are manifolds obtained by defining an equivalence relation on a given manifold. Clearly, the set of equivalence classes do not automatically form a differentiable manifold. We can give it a topology by means of the natural projection, π. If, moreover, π is a homeomorphism, then we can give it a differentiable structure. However, as noted at the end of Example 6, the Hausdorff condition may not always be satisfied. Quotient manifolds most frequently arise from (discontinuous) groups of transformations acting on M, the equivalence classes being the orbits in M of the group.

8. Finally, there are two important general methods of constructing new manifolds from old ones.

(i) One can take any open set of a manifold M and make it into a manifold of the same dimension as M by using the induced topology and restricting the coordinate maps. Such a manifold is called *an open submanifold of M*. For an important example, we note first that the set of all $n \times p$ real matrices can be given a natural manifold structure. For the $n \times n$ matrices we have the determinant function, which is continuous. So, since $\mathbb{R} - 0$ is open, the nonsingular $n \times n$ matrices form an open submanifold of the manifold of $n \times n$ matrices.

(ii) Starting with two manifolds M and N, and using the product topology in $M \times N$, we can make a manifold of dimension $\dim M + \dim N$. If $(\mathcal{U}, \mu)$ is a chart of M and $(\mathcal{V}, \nu)$ is a chart of N, then $(\mathcal{U} \times \mathcal{V}, \mu \times \nu)$ will be a chart of $M \times N$. Further, the set of all products $\mathcal{U} \times \mathcal{V}$ will cover $M \times N$, and on the intersection of $\mathcal{U}_1 \times \mathcal{V}_1$ and $\mathcal{U}_2 \times \mathcal{V}_2$, if $\mu_1 \times \nu_1(p,q) = (a,b)$ and $\mu_2 \times \nu_2(p,q) = (c,d)$, then the map $(a,b) \mapsto (\mu_1^{-1}(a), \nu_1^{-1}(b)) \mapsto (\mu_2 \circ \mu_1^{-1}(a), \nu_2 \circ \nu_1^{-1}(b))$ will be C^k if M and N are C^k manifolds. Thus, the set of charts so constructed is a C^k atlas and $M \times N$ is *the product manifold of M and N*. As an important example, we can construct the torus, $S^1 \times S^1$, from Example 3.

PROBLEM 9.1 (i) Check the remaining overlap maps of Example 3.

(ii) For S^1 as given in Example 3, let $\mathcal{U}_5 = S^1 - (0,1)$, $\mu_5 : \mathcal{U}_5 \to \mathbb{R}$ by $(a,b) \mapsto a/(1-b)$, $\mathcal{U}_6 = S^1 - (0,-1)$ and $\mu_6 : \mathcal{U}_6 \to \mathbb{R}$ by $(a,b) \mapsto a/(1+b)$. Then $\{(\mathcal{U}_5, \mu_5),\ (\mathcal{U}_6, \mu_6)\}$ is a C^ω atlas for S^1.

(iii) This atlas gives the same manifold as the one constructed in Example 3.

PROBLEM 9.2 For $S^n = \{(a^1, \ldots, a^{n+1}) : (a^1)^2 + \cdots + (a^n)^2 + (a^{n+1})^2 = 1,\ a_i \in \mathbb{R}\}$, we can choose $\mathcal{U}_1 = S^n - (0, \ldots, 1), \mu_1 : \mathcal{U}_1 \to \mathbb{R}^n$ by

$$\mu_1^i : (a^1, \ldots, a^{n+1}) \mapsto \frac{a^i}{1 - a^{n+1}},$$

$\mathcal{U}_2 = S^n - (0, \ldots, -1)$ and $\mu_2 : \mathcal{U}_2 \to \mathbb{R}^n$ by

$$\mu_2^i : (a^1, \ldots, a^{n+1}) \mapsto \frac{a^i}{1 + a^{n+1}}.$$

This is a C^∞ atlas for S^n (μ_1 and μ_2 are *stereographic coordinates*).

PROBLEM 9.3 Note that S^1 is also given by $S^1 = \{(a,b) \in \mathbb{R}^2 : a = \cos 2\pi u,\ b = \sin 2\pi u,\ u \in \mathbb{R}\}$. Make S^1 into a manifold with an atlas having two charts.

PROBLEM 9.4 S^{n-1} is the set of n-tuples, $(a^1, a^2, \ldots, a^n)$ of $\mathbb{R}^n$ given by

$$\begin{aligned}
a^1 &= \cos u^1 \\
a^2 &= \sin u^1 \cos u^2 \\
a^3 &= \sin u^1 \sin u^2 \cos u^3 \\
&\ \vdots \\
a^{n-1} &= \sin u^1 \ \cdots\ \sin u^{n-2} \cos u^{n-1} \\
a^n &= \sin u^1 \ \cdots\ \sin u^{n-2} \sin u^{n-1}, \qquad u^i \in \mathbb{R}
\end{aligned}$$

Make S^{n-1} into a manifold with an atlas having two charts.

PROBLEM 9.5 (i) Check the remaining overlap maps of Example 5. (ii) Show that $P^2(\mathbb{R}) = G(1, \mathbb{R}^3)$. (iii) Generalize to projective n space, $P^n(\mathbb{R})$.

PROBLEM 9.6 The torus in $\mathbb{R}^3$ can be given by either

$$\begin{aligned} x &= (a + b\sin v)\cos u \\ y &= (a + b\sin v)\sin u \\ z &= b\cos v \end{aligned}$$

or $(\sqrt{x^2+y^2} - a)^2 + z^2 = b^2$. Make it into a manifold with three charts and with six charts.

PROBLEM 9.7 Exhibit an atlas for $S^1 \times S^1$.

PROBLEM 9.8 As a set $P^2(\mathbb{R}) = \{[a] : a \in \mathbb{R}^3 - 0,$ and $a \sim b$ if $\exists\, t \neq 0 \ni a^i = tb^i\}$. Let $\pi =$ the natural projection of $\mathbb{R}^3 - 0 \to P^2(\mathbb{R})$ given by $a \mapsto [a]$. Give $P^2(\mathbb{R})$ the quotient topology. Let

$$\mathcal{U}_1 = \pi(\mathcal{V}_1) \quad \text{where} \quad \mathcal{V}_1 = \{a \in \mathbb{R}^3 - 0 : a^1 \neq 0\}$$

$$\mu_1 : \mathcal{U}_1 \to \mathbb{R}^2 \text{ given by } [a] \mapsto \left(\frac{a^2}{a^1}, \frac{a^3}{a^1}\right)$$

Prove that $\mathcal{U}_1$ is open, and μ_1 is a homeomorphism of $\mathcal{U}_1$ onto $\mathbb{R}^2$.

9.2 Mappings of differentiable manifolds

We said at the beginning of Chapter 7 that we want to introduce a concept of differentiability and a definition of a derivative for a class of spaces greater than just the class of normed vector spaces. In the last section, we defined a differentiable manifold as a topological space with a superimposed differentiable structure. On such a space we can say what we mean by maps between two manifolds being differentiable. In particular, we can say what we mean by a curve in M or a function on M being differentiable, as these are special cases of maps between manifolds. Having these two concepts available on M, we can, and will, in the next section, build the same structures on M that we did on $\mathbb{R}^n$ in Chapter 7. To define differentiability, the basic idea is to refer maps of manifolds back to maps of cartesian spaces by means of charts and then show that differentiability is independent of coordinates so that we have a well-defined concept.

Theorem 9.1 *If ϕ is a continuous mapping from M to N, and $\nu \circ \phi \circ \mu^{-1}$ is C^k for one chart $(\mathcal{U}, \mu)$ at p and one chart $(\mathcal{V}, \nu)$ at $\phi(p)$, then $\nu_1 \circ \phi \circ \mu_1^{-1}$ is C^k for any other charts $(\mathcal{U}_1, \mu_1)$ and $(\mathcal{V}_1, \nu_1)$ at p and $\phi(p)$.*

Proof $\nu_1 \circ \phi \circ \mu_1^{-1} = (\nu_1 \circ \nu^{-1}) \circ (\nu \circ \phi \circ \mu^{-1}) \circ (\mu \circ \mu_1^{-1})$ so, since $\nu \circ \phi \circ \mu^{-1}$ is C^k and $\nu_1 \circ \nu^{-1}$ and $\mu \circ \mu_1^{-1}$ are C^k by the definition of C^k manifold, then by the chain rule on cartesian spaces we have our result. □

Definitions If ϕ is a continuous mapping from a manifold M to a manifold N, and if $(\mathcal{U}, \mu)$ is a chart at p and $(\mathcal{V}, \nu)$ is a chart at $\phi(p)$, then $\hat{\phi} = \nu \circ \phi \circ \mu^{-1}$ is *a coordinate expression* (or *representation*) *of* ϕ *on* $\mathcal{U}$. ϕ *is* C^l *at* $p \in M$ if all coordinate expressions at p are C^l, or, by Theorem 9.1, if any one coordinate expression at p is C^l, $l \leq k$. If M and N are C^∞ manifolds and the coordinate expressions of ϕ are C^∞ then ϕ *is differentiable*. In particular, *a curve*, γ_p, *through* p *is differentiable* at p if for any chart $(\mathcal{U}, \mu)$, $\hat{\gamma} = \mu \circ \gamma$ is differentiable at 0, and *a function*, $f : \mathcal{U}_p \to \mathbb{R}$ *is differentiable* at p if $\hat{f} = f \circ \mu^{-1}$ is differentiable at $\mu^{-1}(p)$.

Theorem 9.2 *If* μ *is a coordinate map on a* C^k *manifold, then* μ *and* μ^{-1} *are* C^k.

Proof Problem 9.9. □

Theorem 9.3 *On* C^k *manifolds,* (i) *the injection* $\mathbf{i}_{\mathbf{q}_0} : \mathbf{M} \to \mathbf{M} \times \mathbf{N}$ *defined by* $p \mapsto (p, q_0)$ *where* q_0 *is fixed in* N *is* C^k, (ii) *the projection* $\boldsymbol{\pi}_1 : M \times N \to M$ *defined by* $(p, q) \mapsto p$ *for all* $q \in M$ *is* C^k.

Proof (i) For every chart $(\mathcal{U}, \mu)$ at p and $(\mathcal{V}, \nu)$ at (q_0) the coordinate expression $\hat{\mathbf{i}}_{q_0} : \mu(p) \mapsto a \mapsto (a, b_0) \mapsto (\mu(p), \nu(q_0))$, or $\hat{\mathbf{i}}_{q_0} : (\mu^1(p), \ldots, \mu^m(p)) \mapsto (\mu^1(p), \ldots, \mu^m(p),\ \nu^1(q_0), \ldots, \nu^n(q_0))$, or

$$\hat{\mathbf{i}}^j_{q_0}(\mu^1(p), \ldots, \mu^m(p)) = \mu^j(p) \qquad j = 1, \ldots, m$$

$$\hat{\mathbf{i}}^{m+j}_{q_0}(\mu^1(p), \ldots, \mu^m(p)) = \nu^j(q_0) \qquad j = 1, \ldots, n$$

So the component functions of $\hat{\mathbf{i}}_{q_0}$ are C^k functions.
(ii) Problem 9.10. □

Theorem 9.4 ϕ *is a* C^k *map at* p *if and only if for all* C^k *functions,* f, *on neighborhoods of* $\phi(p)$, *the functions* $f \circ \phi$ *are* C^k *at* p.

Proof Problem 9.11. □

From the point of view of the desire to define a concept of differentiability for functions and maps for spaces other than vector spaces, there is a more direct approach to the idea of a differentiable manifold. We impose directly the structure of (i.e., postulate the existence of) a class of differentiable functions on open sets of a topological space (see Chevalley; Chern; Sternberg). From there we can go in the opposite direction and construct a differentiable atlas (see Sternberg). In this approach, Theorem 9.4 is a definition and we can prove that a map ϕ is differentiable iff its coordinate expressions are – our definition.

Theorem 9.5 *On a C^∞ manifold, if f is differentiable on a neighborhood of p, then there exists a neighborhood $\mathcal{V}$ of p, and a differentiable function g on M such that $g(q) = f(q)$ for all $q \in \mathcal{V}$.*

Proof Choose a chart $(\mathcal{U}, \mu)$ at p such that $\mu(p) = (0, \ldots, 0)$, and choose r_2 such that $\{a : |a| < r_2\} \subset f(\mathcal{U})$. For $r_1 < r_2$ there is a C^∞ function, h, on $\mathbb{R}^n$ such that

$$h(a) = \begin{cases} 1 & \text{for } |a| \leq r_1 \\ 0 & \text{for } |a| \geq r_2 \end{cases}$$

Define g on $\mathcal{U}$ by $\hat{g}(a) = \hat{f}(h(a)a^1, \ldots, h(a)a^n)$ and $g(q) = g(p)$ for q outside of $\mathcal{U}$. □

When we generalize from cartesian spaces to manifolds it is useful to introduce an additional concept to help classify manifolds, or to describe when two manifolds are essentially the same.

Definition If M and N are C^k manifolds, then a $1-1$ map, ϕ, from M onto N such that both ϕ and ϕ^{-1} are C^k is *a C^k-diffeomorphism.*

Let us consider the following *examples.*

1. Let $\mathbb{R}^2$ have the standard structure, and let $D^2 = \{(a^1, a^2) \in \mathbb{R}^2 : (a^1)^2 + (a^2)^2 < 1\}$, the unit disk, have the structure given by the atlas $\{(D^2, id)\}$. Then $\phi : D^2 \to \mathbb{R}^2$ given by

$$\phi^1 : (a^1, a^2) \mapsto \frac{a^1}{1 - (a^1)^2 - (a^2)^2}$$

$$\phi^2 : (a^1, a^2) \mapsto \frac{a^2}{(1 - (a^1)^2 - (a^2)^2}$$

for $(a^1)^2 + (a^2)^2 < 1$ is a diffeomorphism.

2. In Section 9.1, Example 1, we made $\mathbb{R}$ into a manifold M with atlas $\{(\mathbb{R}, id)\}$ and into a manifold N with atlas $\{(\mathbb{R}, \mu_0)\}$. Let $\phi : N \to M$ be given by $a \mapsto a^{1/3}$. Then ϕ is a diffeomorphism. This illustrates the fact that two differentiable manifolds with different differentiable structures can be diffeomorphic. A major question in the classification of differentiable manifolds has been whether any two differentiable manifolds of the same dimension with the same topology but different differentiable structures must be diffeomorphic. This is true for $\dim M \leq 3$. In 1956, Milnor produced "exotic 7-spheres" not diffeomorphic with the standard S^7, and quite recently, the existence of "exotic" 4-dimensional manifolds was proved.

3. $S^2 = \{(a^1, a^2, a^3) : (a^1)^2 + (a^2)^2 + (a^3)^2 = 1\}$ with the induced topology from $\mathbb{R}^3$ is made into a manifold determined by six charts $(\mathcal{V}_i, \nu_i)$ where $\mathcal{V}_1 = \{(a^1, a^2, a^3) \in S^2 : a^1 > 0\}$, $\nu_1 : (a^1, a^2, a^3) \mapsto (a^2, a^3)$, etc. Now we can map S^2 onto the real projective plane, $P^2(\mathbb{R})$, as follows.

Let ϕ be a map from D^2 onto $\mathbb{R}^2$ given by

$$\phi^1 : (a^2, a^3) \mapsto \frac{a^2}{\sqrt{1 - (a^2)^2 - (a^3)^2}}$$

$$\phi^2 : (a^2, a^3) \mapsto \frac{a^3}{\sqrt{1 - (a^2)^2 - (a^3)^2}}$$

for $(a^2)^2 + (a^3)^2 < 1$. For a point on S^2 in $\mathcal{V}_1$, the values on the right are a^2/a^1 and a^3/a^1, which are the values of the map μ_1 of Example 5 in Section 9.1. Thus, we have $\mathcal{V}_1 \xrightarrow{\nu_1} D^2 \xrightarrow{\phi} \mathbb{R}^2 \xrightarrow{\mu_1^{-1}} \mathcal{U}_1$, and each map is a $1-1$ onto maps, so we have a $1-1$ map of a coordinate neighborhood of S^2 onto a coordinate neighborhood of $P^2(\mathbb{R})$. ν_1, ϕ, and μ_1^{-1} have differentiable inverses, so we have a diffeomorphism between $\mathcal{V}_1$ and $\mathcal{U}_1$. However, if we try to map *all* of S^2 onto $P^2(\mathbb{R})$ by this method we get a differentiable map, but not $1-1$, and hence not a diffeomorphism.

PROBLEM 9.9 Prove Theorem 9.2.

PROBLEM 9.10 Prove Theorem 9.3(ii).

PROBLEM 9.11 Prove Theorem 9.4.

9.3 The tangent and cotangent spaces at a point of M

Having the concepts of differentiability for curves and functions, we can now proceed exactly as in Section 7.2. Thus, for differentiable curves through p and differentiable functions on a neighborhood of p we can form $f \circ \gamma$ and define the tangent, $[\gamma]_p$, of a curve γ_p through p and the differential, $[f]_p$, of a function f as equivalence classes, exactly as in Section 7.2.

We can proceed to characterize the equivalence of curves and functions in terms of local coordinates, generalizing what we did in observation (iv) in Section 7.2. In order to do so we note that, just as for cartesian spaces, along with coordinate functions $\mu^i = \pi^i \circ \mu$ on neighborhoods of p, we also have *coordinate curves* ζ_{pi} *through* p given by

$$\zeta_{pi} = \mu^{-1} \circ \vartheta_{\mu(p)i} : u \mapsto \mu^{-1}(\mu^1(p), \ldots, \mu^i(p) + u, \ldots, \mu^n(p))$$

for some open interval containing 0 (Fig. 9.3). (Recall that the $\vartheta_{\mu(p)i}$ are the natural coordinate curves through $\mu(p)$ in $\mathbb{R}^n$.) Then along with the component functions, $\gamma^i = \mu^i \circ \gamma$, of γ, we have the partial functions, $f \circ \zeta_i$, of f, and we have the derivatives $D_0\gamma^i$, $D_0 f \circ \zeta_i = D_0 f \circ \mu^{-1} \circ \vartheta_i = D_0 \hat{f} \circ \vartheta_i = \partial \hat{f}/\partial \pi^i\big|_{\mu(p)}$,

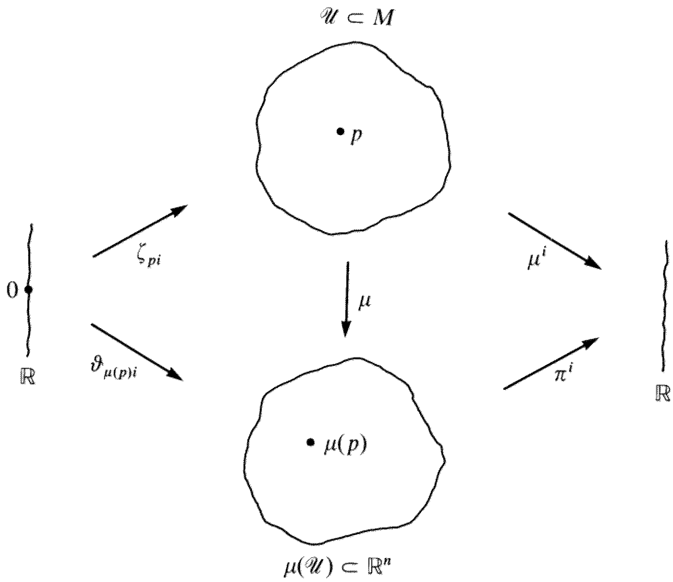

Figure 9.3

and $D_0 f \circ \gamma$. Introducing the notation

$$\left.\frac{\partial f}{\partial \mu^i}\right|_p = \left.\frac{\partial \hat{f}}{\partial \pi^i}\right|_{\mu(p)} \tag{9.1}$$

and using the chain rule, eq. (7.2), for cartesian spaces we get a chain rule

$$D_0 f \circ \gamma = \left.\frac{\partial f}{\partial \mu^i}\right|_p D_0 \gamma^i \tag{9.2}$$

relating these derivatives. Finally, from eq. (9.2), as in observation (iv) in Section 7.2,

$$\gamma_1 \sim \gamma_2 \;\Leftrightarrow\; D_0\gamma_1^i = D_0\gamma_2^i \quad \text{and} \quad f^1 \sim f^2 \Leftrightarrow \left.\frac{\partial f^1}{\partial \mu^i}\right|_p = \left.\frac{\partial f^2}{\partial \mu^i}\right|_p$$

The set of differentials at $p \in M$, T^p, is a vector space, *the cotangent space at* p. The differentials $[\mu^1], [\mu^2], \ldots, [\mu^n]$, of the coordinate functions constitute a *coordinate basis* of T^p. The proofs are just as in Section 7.2. With the notation $[\mu^i] = d\mu^i$ and $[f] = df$ we have, corresponding to (7.10), for all coordinate systems,

$$df\big|_p = \left.\frac{\partial f}{\partial \mu^i}\right|_p d\mu^i\big|_p \tag{9.3}$$

Using a local coordinate system, the constructions used in Section 7.2 to make the set of tangents of curves through $p \in M$ a vector space, *the tangent space at*

p, T_p, can be duplicated for general manifolds. An additional argument is now required to guarantee that our definitions are independent of coordinates. See eq. (9.7) and comments below eq. (9.8). Alternatively, we can use the method of Problem 7.5. The tangents $[\zeta_1], [\zeta_2], \ldots, [\zeta_n]$, of the coordinate curves constitute a *coordinate basis* of T_p. With the notation $[\zeta_i] = \partial/\partial\mu^i$ we have, corresponding to eq. (7.13), for all coordinate systems,

$$[\gamma] = D_0\gamma^i \left.\frac{\partial}{\partial\mu^i}\right|_p \tag{9.4}$$

As in Section 7.2, the elements, v_p, of the tangent space, T_p, at p may be described as ϕ-derivations on functions to $\mathbb{R}$. Thus, T_p is a set of derivations on $\mathfrak{F}_p$, (the set of C^1 functions defined on neighborhoods of p) and we have, corresponding respectively to eqs. (7.12) and (7.14),

$$\left.\frac{\partial}{\partial\mu^i}\right|_p \cdot f = \left.\frac{\partial f}{\partial\mu^i}\right|_p \tag{9.5}$$

and

$$v_p = (v_p \cdot \mu^i)\left.\frac{\partial}{\partial\mu^i}\right|_p \tag{9.6}$$

for the components, $v_p \cdot \mu^i$, of v_p.

If $v = [\gamma] \in T_p$ and $f \in \mathfrak{F}_p$, then $b\colon T_p \times T^p \to \mathbb{R}$ given by $(v, df) \mapsto v \cdot f = D_0 f \circ \gamma$ (from (9.4), (9.5), and (9.2)) is nondegenerate bilinear, so T_p and T^p are dual with respect to b, df can be considered to be a function from T_p to $\mathbb{R}$, and (since they are finite-dimensional) we can write $T^p = T_p^*$. Then

$$\langle df, v\rangle = v \cdot f = D_0 f \circ \gamma \tag{9.7}$$

In particular, $\langle d\mu^i, \partial/\partial\mu^j\rangle = \delta^i_j$; i.e., $\{\partial/\partial\mu^i\}$ and $\{d\mu^i\}$ are dual bases.

Finally, if $\sigma_p \in T^p$, then $\sigma_p = \sigma_i d\mu^i\big|_p$, and since $\sigma_i = \sigma_p \cdot \partial/\partial\mu^i\big|_p$,

$$\sigma_p = \left(\sigma_p \cdot \left.\frac{\partial}{\partial\mu^i}\right|_p\right) d\mu^i\big|_p \tag{9.8}$$

for the components, $\sigma_p \cdot \frac{\partial}{\partial\mu^i}$, of σ_p.

If M is a C^∞ manifold, T_p is the set of derivations on C^∞ functions at p. We get higher-order partial derivatives at p as follows. Having defined $\partial f/\partial\mu^j\big|_p$, eq. (9.1), we can define other functions on a neighborhood of p. Thus, if f is differentiable at each point of the coordinate neighborhood, $\mathcal{U}$, then we have the functions

$$\frac{\partial f}{\partial \mu^j} : p \mapsto \left.\frac{\partial f}{\partial \mu^j}\right|_p$$

on $\mathcal{U}$. Further, by definition, $\partial f/\partial \mu^j$ will be differentiable at p if $\partial f/\partial \mu^j \circ \mu^{-1}$ is differentiable at $\mu(p)$. But $\partial f/\partial \mu^j \circ \mu^{-1} = \partial \hat{f}/\partial \pi^j$. So if f is C^2 at p, $\partial f/\partial \mu^j$ will be C^1 functions at p, and we have

$$\left.\frac{\partial^2 f}{\partial \mu^i \partial \mu^j}\right|_p = \left.\frac{\partial}{\partial \mu^i}\left(\frac{\partial f}{\partial \mu^j}\right)\right|_p = \frac{\partial}{\partial \pi^i}\left(\frac{\partial f}{\partial \mu^j} \circ \mu^{-1}\right)_{\mu(p)} = \frac{\partial}{\partial \pi^i}\left(\frac{\partial \hat{f}}{\partial \pi^j}\right)_{\mu(p)}$$

Continuing, we get the result that if f is C^k at p, then the $(k-1)$th-order partial derivative functions will be C^1 functions at p, and we can take their partial derivatives.

We have generalized the concepts and formulas of Section 7.2 for $\mathbb{R}^n$ to concepts and formulas valid for a manifold, M. A new feature arises for manifolds which did not exist for $\mathbb{R}^n$ in Section 7.2 (but is relevant for $\mathbb{R}^n$ in Section 9.1 when we made it into a manifold). A formula may be given in terms of coordinates like eq. (9.6), or without the use of coordinates like eq. (9.7). In the former case, since the concepts described in the formula are independent of coordinates, the formula must transform properly on the intersection of coordinate domains. On $\mathcal{U} \cap \bar{\mathcal{U}}$, $\bar{v}^j = v \cdot \bar{\mu}^j = v \cdot d\bar{\mu}^j = v^i \frac{\partial}{\partial \mu^i} \cdot d\bar{\mu}^j = v^i \frac{\partial}{\partial \mu^i} \cdot \bar{\mu}^j$ by eqs. (9.6) and (9.7). So, by eq. (9.5)

$$\bar{v}^j = \frac{\partial \bar{\mu}^j}{\partial \mu^i}\, v^i \tag{9.9}$$

Similarly, putting $\sigma_i = \sigma \cdot \partial/\partial \mu^i$ and $\bar{\sigma}_i = \sigma \cdot \partial/\partial \bar{\mu}^i$ we get, from (9.8) and (9.7),

$$\bar{\sigma}_j = \frac{\partial \mu^i}{\partial \bar{\mu}^j}\, \sigma_i \tag{9.10}$$

Equations (9.9) and (9.10) for the transformations of the components of vectors and 1-forms can be generalized to formulas for tensors of any type. See eqs. (9.16) and (9.17). Also, compare these results with eq. (1.4), which relates the components v^i and $\bar{v}^i$ by the change-of-basis matrix. If a concept is defined or a relation is given in terms of components in a given basis, these formulas are used to show that the concept or relation is independent of coordinates.

Having now the concepts of tangent and cotangent spaces at a point $p \in M$ we can reproduce practically verbatim the discussion of Section 7.3 of the tangent map ϕ_{*p} at a point, and its transpose, or dual, $\phi^*_{\phi(p)}$.

In particular, if $\phi : M \to N$, $(\partial/\partial\mu^k)$ is a basis of T_p, $(\partial/\partial\nu^j)$ is a basis of $T_{\phi(p)}$, and $\phi^j = \nu^j \circ \phi$ are the component functions of ϕ, then eq. (7.19) becomes

$$\phi_{*p} \cdot \left.\frac{\partial}{\partial\mu^k}\right|_p = \left.\frac{\partial\phi^j}{\partial\mu^k}\right|_p \left.\frac{\partial}{\partial\nu^j}\right|_{\phi(p)} \tag{9.11}$$

and if $(d\mu^j)$ is a basis of T_p^* and $d\nu^k$ is a basis of $T_{\phi(p)}^*$, eq. (7.23) becomes

$$\left.\phi^*\right|_{\phi(p)} \cdot d\nu^k = \frac{\partial\phi^k}{\partial\mu^j}\, d\mu^j \tag{9.12}$$

Since $\partial\phi^j/\partial\mu^i = \partial\hat{\phi}^j/\partial\pi^i$, eq. (9.11) says that the (matrix) representation (with respect to the bases $\{\partial/\partial\mu^i\}$ and $\{\partial/\partial\nu^j\}$) of the tangent map of ϕ is the derivative of the representation $\hat{\phi} = \nu \circ \phi \circ \mu^{-1}$ (with respect to the coordinate maps μ and ν) of ϕ. Similarly, eq. (9.12) says that the (matrix) representation of the transpose (or dual) of the tangent map of ϕ is the transpose of the derivative of the representation of ϕ. Equations (7.20) and (7.24) become respectively,

$$w^j = \frac{\partial\phi^j}{\partial\mu^k}\, v^k \tag{9.13}$$

and

$$\sigma_j = \frac{\partial\phi^k}{\partial\mu^j}\, \tau_k \tag{9.14}$$

It is interesting to compare these equations for the transformation of components under a mapping, ϕ, with eqs. (9.9) and (9.10) for the transformation of components under a transformation of coordinates.

Just as for mappings, ϕ, between cartesian spaces in Section 7.3, we sort out two important cases.

(i) If $\phi = \gamma$, a curve defined on some open set in $\mathbb{R}$, we have the composition, for $a = u \in \mathbb{R}$,

$$\dot{\gamma}\colon u \mapsto [\vartheta_u] = \left.\frac{d}{d\pi}\right|_u \mapsto [\gamma \circ \vartheta_u]_{\gamma(u)} = D_u\gamma^i \left.\frac{\partial}{\partial\mu^i}\right|_{\gamma(u)}$$

the canonical lift of γ, and

$$\dot{\gamma}(u) = D_u\gamma^i \left.\frac{\partial}{\partial\mu^i}\right|_{\gamma(u)} \tag{9.15}$$

the tangent (or, *velocity*) of γ at $\gamma(u)$. Since $\dot{\gamma}(u) = \gamma_{*u} \cdot d/d\pi\big|_u$, the tangent of γ at u is sometimes denoted by γ_{*u}.

(ii) If ϕ is a function, f, since in this case, as before in Section 7.3, $\phi_{*p} \cdot v = \langle d\phi, v\rangle\, d/d\bar{\pi}$, the tangent map is sometimes called the differential of f.

With $T_p, T_{\phi(p)}$, T^p, and $T^{\phi(p)}$ we have all the various tensor product spaces and subspaces built on these vector spaces at p and $\phi(p)$, just as in Section 7.3. In a coordinate neighborhood, $\mathcal{U}$, we have coordinate representations such as $A^{i_1\cdots i_r}\partial/\partial\mu^{i_1}\otimes\cdots\otimes\partial/\partial\mu^{i_r}$ and $A_{j_1\cdots j_s}d\mu^{j_1}\otimes\cdots\otimes d\mu^{j_s}$ from which we get, on $\mathcal{U}\cap\mathcal{V}$, respectively,

$$\bar{A}^{j_1\cdots j_r} = A^{i_1\cdots i_r}\frac{\partial\bar{\mu}^{j_1}}{\partial\mu^{i_1}}\cdots\frac{\partial\bar{\mu}^{j_r}}{\partial\mu^{i_r}} \tag{9.16}$$

and

$$\bar{A}_{j_1\cdots j_s} = A_{i_1\cdots i_s}\frac{\partial\mu^{i_1}}{\partial\bar{\mu}^{j_1}}\cdots\frac{\partial\mu^{i_s}}{\partial\bar{\mu}^{j_s}} \tag{9.17}$$

(Compare with eq. (4.5).)

Moreover, the maps ϕ_{*p} and $\phi^*_{\phi(p)}$ are extended to the contravariant and covariant tensor algebras at p and $\phi(p)$, respectively, just as in Section 7.3. That is, we have $\phi_{rp}:(T_p)^r_0 \to (T_{\phi(p)})^r_0$ and $\phi^s_{\phi(p)}:(T_{\phi(p)})^0_s \to T_p)^0_s$.

Finally, note that we have been working all this time at a fixed point $p\in M$. As in Section 7.3, we extend ϕ_{rp} and $\phi^s_{\phi(p)}$ to maps

$$\phi_r\colon \bigcup_{p\in M} T_p)^r_0 \to \bigcup_{p\in N} T_p)^r_0 \quad\text{and}\quad \phi_s\colon \bigcup_{\phi(p)\in N} T_{\phi(p)})^0_s \to \bigcup_{p\in M} T_p)^0_s$$

In particular, we will write ϕ_* for ϕ_1, the extension of the tangent map at a point, and we will write ϕ^* for ϕ^1, the extension of the transpose (or dual) of the tangent map.

PROBLEM 9.12 Describe geometrically

(i) the set of values, $\zeta_i(u)$, of the coordinate curves
(ii) the hypersurfaces $\{p\colon\mu^i(p) = \text{constant}\}$

on the chart $(\mathcal{U},\mu)$ on

(1) $\mathbb{R}^3$ given by "cylindrical coordinates"
(2) S^2 given by "stereographic coordinates" (see Problem 9.2).

PROBLEM 9.13 Derive eq. (9.2).

PROBLEM 9.14 (i) Define $[\gamma_1]+[\gamma_2]$ and $c[\gamma]$ in T_p as in Section 7.2 and show that these definitions are independent of coordinates. (ii) If M is a finite-dimensional vector space, V^n, then there are natural isomorphisms $V^n \to T_p$ for each $p\in V^n$ given by $v\mapsto[\gamma_p(v)]$ where $\gamma_p(v) : t\mapsto p+tv$.

PROBLEM 9.15 Derive eq. (9.15).

PROBLEM 9.16 Derive the "transformation law" on intersections of coordinate domains for (r,s) tensors; i.e., generalize eqs. (9.9) and (9.10).

9.4 Some properties of mappings

The first thing we will show is that every differentiable mapping has a simple local representation; that is, in appropriate coordinates the equations of the mapping are simple. Then we will use these representations to describe certain classes of manifolds generalizing examples in Section 9.1.

Our main tool will be the "Inverse Function Theorem" for cartesian spaces.

Theorem 9.6 *Suppose* $\Phi : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^n$, Φ *is* C^k *on* $\mathcal{U}$, *and* $\det D_{a_0}\Phi \neq 0$ *for some* $a_0 \in \mathcal{U}$. *Then there exist open neighborhoods* $\mathcal{V}_{a_0}$ *and* $\bar{\mathcal{V}}_{\Phi(a_0)}$ *of* a_0 *and* $\Phi(a_0)$ *respectively such that*

(i) $\Phi\big|_{\mathcal{V}_{a_0}}$ *is* $1-1$ *and onto* $\bar{\mathcal{V}}_{\Phi(a_0)}$

(ii) Φ^{-1} *is* C^k *on* $\bar{\mathcal{V}}_{\Phi(a_0)}$

Proof Woll (p. 20). □

Suppose $\phi : M \to N$ is differentiable and $\dim M = m$ and $\dim N = n$. Since the overlap maps are all differentiable, the rank of the Jacobian matrix (the derivative) is the same for all representations of ϕ, and so we can talk about the *rank of* ϕ.

Theorem 9.7 (**The Rank Theorem**, Boothby, p. 70.) *A necessary and sufficient condition that* $\phi : M \to N$ *has rank* r *in a neighborhood of a point,* p, *is that there is a chart* $(\mathcal{U}, \mu)$ *at* p *and a chart* $(\mathcal{V}, \nu)$ *at* $\phi(p)$ *such that*

$$\nu^i \circ \phi = \mu^i \quad i = 1, \ldots, r$$

$$\nu^j \circ \phi = 0 \quad j = r+1, \ldots, n$$

on $\phi^{-1}(\mathcal{V}) \cap \mathcal{U}$.

Another way of stating this condition is that if $q \in \phi^{-1}(\mathcal{V}) \cap \mathcal{U}$, and $\mu(q) = (a^1, \ldots, a^m)$, then $\hat{\phi} = \nu \circ \phi \circ \mu^{-1} : (a^1, \ldots, a^m) \mapsto (a^1, \ldots, a^r, 0, \ldots, 0)$. Or, finally, if $\nu \circ \phi(q) = (b^1, \ldots, b^n)$ then the equations of the representation, $\hat{\phi}$, in these coordinates, of ϕ are $b^1 = a^1, \ldots, b^r = a^r$, $b^{r+1} = 0, \ldots, b^n = 0$.

Proof The proof is based on the following lemma. □

Lemma *If* ϕ *maps an open set* $\mathcal{U} \subset \mathbb{R}^m$ *to* $\mathbb{R}^n$ *and has* rank r *then for each* $a_0 \in \mathcal{U}$ *there is a diffeomorphism* $\Phi : \mathcal{V} \to \overline{\mathcal{V}}$ *with* $a_0 \in \mathcal{V}$, $\mathcal{V} \subset \mathcal{U}$, *and* $\overline{\mathcal{V}} \subset \mathbb{R}^m$, *and a diffeomorphism* $\Psi : \mathcal{W} \to \overline{\mathcal{W}}$ *with* $\phi(a_0) \in \mathcal{W}$, $\mathcal{W} \subset \phi(\mathcal{U})$, *and* $\overline{\mathcal{W}} \subset \mathbb{R}^n$ *such that*

$$\Psi \circ \phi \circ \Phi^{-1} : (\bar{a}^1, \ldots, \bar{a}^m) \mapsto (\bar{a}^1, \ldots, \bar{a}^r, 0, \ldots, 0)$$

where $(\bar{a}^1, \ldots, \bar{a}^m) \in \overline{\mathcal{V}}$ (see Fig. 9.4).

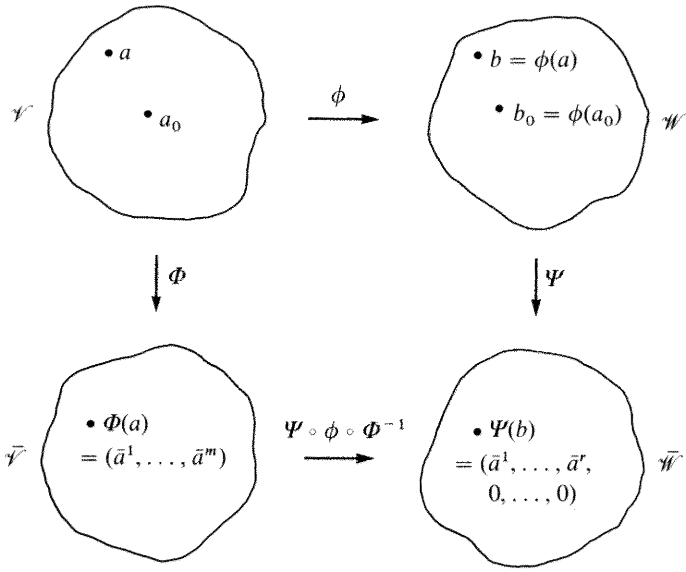

Figure 9.4

Proof (of lemma). We can assume, without loss of generality, that the upper left hand $r \times r$ submatrix of the Jacobian matrix of ϕ is nonsingular at a_0. Then we define Φ on $\mathcal{U}$ by

$$\begin{aligned} \Phi^i &= \phi^i \quad i = 1, \ldots, r \\ \Phi^j &= \pi^j \quad j = r+1, \ldots, m \end{aligned} \tag{9.18}$$

det $D_{a_0}\Phi \neq 0$ so by the inverse function theorem, we have $\overline{\mathcal{V}}$, and $\mathcal{V}$, with Φ a diffeomorphism between them. Writing out (9.18) we have

$$\begin{aligned} \bar{a}^1 &= \Phi^1(a) = \phi^1(a) = b^1 \\ &\vdots \\ \bar{a}^r &= \Phi^r(a) = \phi^r(a) = b^r \\ \bar{a}^{r+1} &= \Phi^{r+1}(a) = \pi^{r+1}(a) = a^{r+1} \\ &\vdots \\ \bar{a}^m &= \Phi^m(a) = \pi^m(a) = a^m \end{aligned}$$

Now $\phi \circ \Phi^{-1}$ is a map from $\overline{\mathcal{V}}$ to $\mathbb{R}^n$ given by

$$\begin{aligned}\phi \circ \Phi^{-1} : (a^1, \ldots, a^m) &\overset{\Phi^{-1}}{\mapsto} (a^1, \ldots, a^m)\\ &\overset{\phi}{\mapsto} (\phi^1(a), \ldots, \phi^n(a))\\ &= (\bar{a}^1, \ldots, \bar{a}^r, \phi^{r+1} \circ \Phi^{-1}(\bar{a}), \ldots, \phi^n \circ \Phi^{-1}(\bar{a}))\end{aligned}$$

$\phi \circ \Phi^{-1}$ has rank r on $\overline{\mathcal{V}}$ since ϕ has rank r on $\mathcal{V}$. This implies that for all $j = r+1, \ldots, m$ and $k = r+1, \ldots, n$, $\partial\phi^k \circ \Phi^{-1}/\partial\pi^j = 0$ on $\mathcal{V}$. That is, $\phi^k \circ \Phi^{-1}$ are functions of $\bar{a}^1, \ldots, \bar{a}^r$ only. $\phi^k \circ \Phi^{-1}$ are defined in a neighborhood of $\bar{a}^1_0 = b^1_0, \ldots, \bar{a}^r_0 = b^r_0$ so they are defined in a neighborhood of b_0.

Thus, we can define Ψ by

$$\begin{aligned}\Psi^i &= \pi^i & i &= 1, \ldots, r\\ \Psi^k &= \pi^k - \phi^k \circ \Phi^{-1} & k &= r+1, \ldots, n\end{aligned} \tag{9.19}$$

or, expressed alternatively,

$$\begin{aligned}\Psi : (b^1, \ldots, b^n) \mapsto (b^1, \ldots, b^r, b^{r+1} - \phi^{r+1} \circ \Phi^{-1}(b^1, \ldots, b^r), \ldots,\\ b^n - \phi^n \circ \Phi^{-1}(b^1, \ldots, b^r))\end{aligned}$$

det $D_{b_0}\Psi \neq 0$, so we have $\mathcal{W}$ and $\overline{\mathcal{W}}$, with Ψ a diffeomorphism between them.

Finally, putting $b^i = \bar{a}^i, i = 1, \ldots r$, and noting that $\phi^k \circ \Phi^{-1}(\bar{a}^1, \ldots, \bar{a}^r) = \phi^k(a) = b^k, k = r+1, \ldots, n$, the mapping (9.19) becomes

$$\Psi : (b^1, \ldots, b^n) \mapsto (\bar{a}^1, \ldots, \bar{a}^r, 0, \ldots, 0)$$

and

$$\Psi \circ \phi \circ \Phi^{-1} : (\bar{a}^1, \ldots, \bar{a}^m) \mapsto (\bar{a}^1, \ldots, \bar{a}^r, 0, \ldots, 0)$$

□

There are two special cases for $\phi : M \to N$ of maximum rank; immersions and submersions.

Definitions ϕ is *an immersion of* M *into* N if rank $\phi = \dim M$ at each point. ϕ is *a submersion of* M *into* N if rank $\phi = \dim N$ at each point. (Alternatively, an immersion is locally injective, or, ϕ_* is $1-1$, and a submersion is locally surjective, or, ϕ_* is onto.)

For these two special cases, the proof of the theorem shows:

1. A necessary and sufficient condition that ϕ is an immersion is that if $(\mathcal{V}, \nu)$ is any chart at $\phi(p)$, then there is a chart $(\mathcal{U}, \mu)$ at p with $\mu^i = \nu^i \circ \phi$, $i = 1, \ldots, m$. (To get $i = 1, \ldots, m$ it may be necessary to relabel the ν^i.

2. A necessary and sufficient condition that ϕ is a submersion is that if $(\mathcal{V},\nu)$ is any chart at $\phi(p)$ and $(\mathcal{W},\xi)$ is any chart at p, then there is a chart $(\mathcal{U},\mu)$ at p with $\mu^i = \nu^i \circ \phi$ for $i = 1,\ldots,n$, and $\mu^j = \xi^j$ for $j = n+1,\ldots,m$. (Again we may have to relabel the ν^i.)

We illustrate the theorem and the two special cases with the following examples.

(i) $m = 3, n = 3$, and $r = 2$. Suppose (a^1,a^2,a^3) are coordinate values in a neighborhood of p and (b^1,b^2,b^3) are coordinate values in a neighborhood of $\phi(p)$ and $\hat{\phi}:\mathbb{R}^3 \to \mathbb{R}^3$ according to

$$\hat{\phi} : \begin{cases} b^1 = a^1 + a^2 + a^3 \\ b^2 = a^1 + a^2 + a^3 \\ b^3 = a^1 + 3a^2 + a^3 \end{cases}$$

If we choose coordinates $(\bar{a}^1,\bar{a}^2,\bar{a}^3) = (a^1+a^2+a^3, a^1+3a^2+a^3, a^3)$ around p and $(\bar{b}^1,\bar{b}^2,\bar{b}^3) = (b^1, b^3, b^1 - b^2)$ around $\phi(p)$, then on the image of ϕ we have

$$\bar{b}^1 = \bar{a}^1, \qquad \bar{b}^2 = \bar{a}^2, \qquad \bar{b}^3 = 0$$

according to the Theorem.

(ii) $m = 2, n = 3$, and $r = 2$ (an immersion). Suppose, in a neighborhood of p, ϕ is represented by $\hat{\phi}:\mathbb{R}^2 \to \mathbb{R}^3$ according to

$$\hat{\phi} : \begin{cases} b^1 = a^1 + a^2 \\ b^2 = a^1 + a^2 \\ b^3 = a^1 + 3a^2 \end{cases}$$

If we choose coordinates $(\bar{a}^1,\bar{a}^2) = (a^1+a^2, a^1+3a^2)$ around p, then on the image of ϕ we have

$$b^1 = \bar{a}^1, \qquad b^3 = \bar{a}^2$$

and if we relabel b^3 we get the special case 1. Then, clearly, $\hat{\phi}$ has a differentiable inverse.

(iii) $m = 3, n = 2$, and $r = 2$ (a submersion). Suppose, in a neighborhood of p, ϕ is represented by $\hat{\phi}:\mathbb{R}^3 \to \mathbb{R}^2$ according to

$$\hat{\phi} : \begin{cases} b^1 = a^1 + a^2 + a^3 \\ b^2 = a^1 + 3a^2 + a^3 \end{cases}$$

If we choose coordinates $(\bar{a}^1,\bar{a}^2,\bar{a}^3) = (a^1+a^2+a^3, a^1+3a^2+a^3, a^3)$ around p, then on the image of ϕ we have

$$\bar{a}^1 = b^1, \quad \bar{a}^2 = b^2, \quad \bar{a}^3 = a^3$$

the special case 2, and, clearly, on $\mathcal{V}$ coordinates (b^1,b^2) do not determine a point in $\mathcal{U}$.

In Section 9.1 we had examples in which subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$, obtained as images of subsets of $\mathbb{R}$ and $\mathbb{R}^2$, respectively, were made into manifolds. We can generalize these cases.

Theorem 9.8 *The image of a* $1-1$ *immersion of* M *into* N *can be made into a manifold diffeomorphic with* M.

Proof Give $\phi(M)$ the topology induced by ϕ; i.e., $\mathcal{V} \subset \phi(M)$ is open if $\phi^{-1}(\mathcal{V})$ is open. Then $\phi^{-1}\big|_{\phi(M)}$ is a homeomorphism. If $(\mathcal{U}, \mu)$ is a chart at p then $(\phi(\mathcal{U}), \mu \circ \phi^{-1}\big|_{\phi(M)})$ will be a chart at $\phi(p)$. Any two charts at $\phi(p)$ will be C^k-related if the corresponding charts at p are C^k-related. □

(In the literature, a 1-1 immersion is sometimes called an imbedding. We will save this term for a stronger concept in Section 10.2.

Another characterization of immersion is given in terms of the relation between the classes of differentiable functions on M and N. Clearly, if $\phi: M \to N$ is differentiable, then corresponding to each differentiable function g on $N, g \circ \phi$ is a differentiable function on M. To go the other way we need a little more.

Theorem 9.9 $\phi: M \to N$ *is an immersion if and only if corresponding to each differentiable function,* f, *on* M *and each point* $p \in M$ *there is a differentiable function,* g, *on* N *such that* $g \circ \phi = f$ *on some neighborhood of* p. (Borrowing the terminology of the special case in Section 10.1 we can paraphrase this as every differentiable function on M is the "restriction" of some differentiable function on N.)

Proof If: Let $\mathcal{U}$ be a coordinate neighborhood of p and $f^1, \ldots, f^m$ a set of coordinate functions. Then on $\mathcal{U}$ there are functions g^i such that $g^i \circ \phi = f^i$, $i = 1, \ldots, m$. Now let v be a vector in T_p and form $\langle df^i, v\rangle$. Since $\langle df^i, v\rangle = \langle dg^i \circ \phi, v\rangle = \langle \phi^* \circ dg^i, v\rangle = \langle dg^i, \phi^* \circ v\rangle$ it follows that $\phi^* \circ v = 0$ implies $v = 0$ and so ϕ^* has rank m.

Only if: At each point $p \in M$, there is a neighborhood, $\mathcal{W}$, of $\phi(p)$ and a differentiable map, ψ, on $\mathcal{W}$ such that $\psi \circ \phi = id$ on $\tau(\mathcal{W})$ (Auslander and MacKenzie, p. 47). Then $f \circ \psi \circ \phi = f$ on $\psi(\mathcal{W})$. Let $f \circ \psi = g$. Finally, extend g to a function on N by Theorem 9.5. □

When a differentiable manifold is defined as indicated below Theorem 9.4 by prescribing a class of differentiable functions, then Theorem 9.9 can be used as a definition to transfer the differentiable structure of M to the image of M in N (loc. cit., p. 36).

In Section 9.1, we had examples in which subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$, obtained as inverse images of points of $\mathbb{R}$, were made into manifolds. These are examples coming from submersions in the special cases where $N = \mathbb{R}^n$.

If $N = \mathbb{R}^n$, then $\phi: M \to N$ has the form $\phi = (f^1, \ldots, f^n)$ and the second special case of the rank theorem says for a submersion there is a chart $(\mathcal{U}, \mu)$ at p with $\mu^i = \pi^i \circ \phi = f^i, i = 1, \ldots, n$. (Using the identity chart on $\mathbb{R}^n$.) That is, a set of functions, $f^1, \ldots, f^n$, with rank $(\partial f^i / \partial \xi^j) = n$ can be enlarged to a local coordinate system at p. Moreover, this chart has the property that if $b \in \phi(\mathcal{U})$, then on $\mathcal{U} \cap \phi^{-1}(b)$ the values of $f^1, \ldots, f^n$ are constant.

If we denote the level sets, or fibers, $\phi^{-1}(b)$, of a submersion $\phi : M \to \mathbb{R}^n$ by F_b, we can summarize these comments more formally as follows:

Theorem 9.10 *If $\phi: M \to \mathbb{R}^n$ is a submersion, then at each point of F_b there is a coordinate neighborhood $\mathcal{U}$ of M in which $q \in F_b \cap \mathcal{U}$ satisfy $f^1(q) = b^1, \ldots, f^n(q) = b^n$.*

Theorem 9.11 *The level sets of a submersion with the relative topology of M can be made into manifolds.*

Proof We will assume that $m = \dim M > n$. At each point of F_b there is a chart $(\mathcal{U}, \mu)$ with $\mu(q) = (b^1, \ldots, b^n, a^{n+1}, \ldots, a^m)$ with $(b^1, \ldots, b^n)$ fixed for all $q \in F_b \cap \mathcal{U}$. The projection $\boldsymbol{\pi}_a : (b^1, \ldots, b^n, a^{n+1}, \ldots, a^m) \mapsto (a^{n+1}, \ldots, a^m)$ is a diffeomorphism so $(F_b \cap \mathcal{U}, \boldsymbol{\pi}_a \circ \mu)$ is a chart on F_b. Since the coordinate maps $\boldsymbol{\pi}_a \circ \mu$ are diffeomorphisms, the overlap maps are differentiable. □

Definitions A level set of a submersion $\phi: M \to \mathbb{R}^n$ with the manifold structure of Theorem 9.11 will be called *a differentiable variety of ϕ*. In particular, if $\phi : M \to \mathbb{R}$ it is *a hypersurface of M*.

It is instructive to compare Theorems 9.11 and 9.8. In particular note that F_b has the topology of M in Theorem 9.11, but in Theorem 9.8 $\phi(M)$ need not have the topology of N. Also, F_b is closed in M, but $\phi(M)$ need not be closed in N.

PROBLEM 9.17 Prove Theorem 9.7 using the given lemma.

PROBLEM 9.18 Prove the results 1 and 2 for the special cases of Theorem 9.7.

PROBLEM 9.19 The subset S^{n-1} of $\mathbb{R}^n$ is a differentiable variety.

PROBLEM 9.20 Verify the properties of F_b and $\phi(M)$ as stated in the last paragraph above.

PROBLEM 9.21 At each point $p \in F_b$ the tangent space of F_b is a subspace of T_pM, and T_pF_b is the kernel of ϕ_{*p}.

10

SUBMANIFOLDS

The prototypical submanifold is a surface in ordinary space. There are various ways of describing surfaces in ordinary space. The two main methods are by parametric equations, $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, and the points (x, y, z) satisfying $F(x, y, z) =$ constant. We will extend these two descriptions to general differentiable manifolds. They correspond to, and derive, roughly, from the concepts of immersion and submersion, respectively.

10.1 Parametrized submanifolds

Let M be a given manifold. In analogy with parametrized curves, we focus on a mapping, ϕ, from another given manifold to M.

Definition *(P, ϕ) is an immersed submanifold of M* if P is a differentiable manifold and ϕ is a 1-1 immersion of P into M.

We will shorten "immersed submanifold" to simply "submanifold" in the following. A warning about terminology: in the literature sometimes the term "submanifold" is used as we are using it, and sometimes it stands for the stronger concept we have called an "imbedded submanifold" in Section 10.2. Compare the comment below Theorem 9.8.

It is usual (as in the case of curves or surfaces) to think of a submanifold of M as a subset of M. Hence, under the conditions of the above definition, we frequently say that $\phi(P)$ is a (immersed) submanifold of M. Now, according to Theorem 9.8, $\phi(P)$ can be made into a manifold diffeomorphic with P. Then, the inclusion map $id\big|_{\phi(P)} = \phi \circ \phi^{-1}\big|_{\phi(P)}$ is an immersion. Conversely, if S is a subset of M and if the inclusion $id\big|_S : S \to M$ is an immersion, (S having some differentiable structure), then $(S, id\big|_S)$ is a submanifold of M according to our definition. Thus, if we wish to think of a submanifold M as a subset of M, we have the criterion:

Theorem 10.1 *A subset, S, of M is a submanifold of M iff the inclusion map $id\big|_S : S \to M$ is an immersion.*

It is interesting to compare this criterion with that for a subspace of a topological space in Problem 8.10(i).

Just as in the case of parametrized curves and surfaces in ordinary Euclidean space, more than one submanifold can have the same image set in M. If (P, ϕ) is a submanifold of M, then the entire equivalence class $[(P, \phi)]$ containing (P, ϕ) given by $(Q, \psi) \sim (P, \phi)$ if there is a diffeomorphism $\alpha : P \to Q$ and $\psi \circ \alpha = \phi$ has the same image. By Theorem 10.1 this class contains $(\phi(P), id\big|_{\phi(P)})$.

Note that in Theorem 10.1 there remains some ambiguity about the differentiable structure of S. Theorem 10.2 will pin this down a bit. To motivate it, we consider the following example.

Let $P_1 = (0, 2\pi)$, $\phi_1(u) = (\sin 2u, \sin u)$, $P_2 = (-\pi, \pi)$, and $\phi_2(u) = (\sin 2u, \sin u)$. Then (P_1, ϕ_1) and (P_2, ϕ_2) are submanifolds of $\mathbb{R}^2$, and $\phi_1(P_1) = \phi_2(P_2) = E$, the figure eight. E with atlas (E, ϕ_1^{-1}) and E with atlas (E, ϕ_2^{-1}) are different, since $\phi_2^{-1} \circ \phi_1$ is not continuous at π. Also $[(P_1, \phi_1)] \neq [(P_2, \phi_2)]$ since there is no diffeomorphism α such that $\phi_2 \circ \alpha = \phi_1$ for the same reason. Thus, corresponding to two distinct differentiable structures on E we have two distinct equivalence classes of submanifolds of $\mathbb{R}^2$.

Theorem 10.2 *If $S \subset M$ has a given topology, then corresponding to each equivalence class of submanifolds with image S, there is at most one differentiable structure on S.*

Proof Problem 10.2. □

Consider the following examples, where $\mathbb{R}^n$ and open subsets of $\mathbb{R}^n$ have the standard structure.

1. P is the interval $(0, 1)$ in $\mathbb{R}$, and $\phi : (0, 1) \to \mathbb{R}^2$ according to $u \mapsto (\cos 2\pi u, \sin 2\pi u)$.
2. Same as 1 with $P = \mathbb{R}$.
3. $\phi : \mathbb{R} \to \mathbb{R}^2$ given by $u \mapsto (u^2, u^3)$.
4. $\phi : \mathbb{R} \to \mathbb{R}^2$ given by $u \mapsto (2u^2/1 + u^2,\ 2u^3/1 + u^2)$.
5. $\phi : \mathbb{R} \to \mathbb{R}^2$ given by $u \mapsto (0, \sin 2\pi u)$.
6. $\phi : \mathbb{R} \to S^1 \times S^1$ with the product structure on $S^1 \times S^1$, given by $u \mapsto (e^{2\pi i u}, e^{2\pi i \alpha u})$ where α is an irrational number.
7. P is an open submanifold of M, and $\phi = id\big|_P$ the inclusion map of P into M.
8. $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ given by $(u, v) \mapsto (u, v, \sqrt{u^2 + v^2})$.
9. $\phi : (0, \pi) \to \mathbb{R}^2$ given by $u \mapsto (1 + \cos u + \cos 2u,\ \sin u + \sin 2u)$.
10. P is the open parallelpiped $\mathcal{U} = \{(u^1, u^2, u^3) : (0, 0, 0) < (u^1, u^2, u^3) < (\pi, \pi, 2\pi)\}$ in $\mathbb{R}^3$, and $\phi : \mathcal{U} \to \mathbb{R}^4$ according to $(u^1, u^2, u^3) \mapsto (\cos u^1, \sin u^1 \cos u^2, \sin u^1 \sin u^2 \cos u^3, \sin u^1 \sin u^2 \sin u^3)$.
    (See Problem 9.4.)

Examples 1, 6, 7, 10 are submanifolds. The other are not. See Problem 10.1.

Along with the point of view that a submanifold is the image set of a $1-1$ immersion, we think of the tangent space, T_pP, of P at p, as a subspace of

$T_{\phi(p)}M$, the tangent space of M at $\phi(p)$. More precisely, we define the tangent space of the submanifold to be $\phi_*(T_pP)$ and then identify T_pP and $\phi_*(T_pP)$.

The results on immersions in Section 9.4 are valid, in particular, for submanifolds. Thus, if we represent a submanifold according to Theorem 10.1 by $(S, id\big|_S)$ where $S \subset M$ and $id\big|_S$ is the inclusion map, then Theorem 9.7 says that at each point of S there are charts with $\mu^i = \nu^i, i = 1, \ldots, p$ (p is the dimension of S), and $\nu^j = 0,\ j = p+1, \ldots, m$.

Theorem 10.3 *If P is compact, and (P, ϕ) is a submanifold of M, then $\phi(P)$ has the induced topology from M.*

Proof This is a corollary of Theorem 8.5. □

Finally, Theorem 9.9 says that if $S \subset N$ and S is a submanifold of N, then every differentiable function on S is the restriction of a differentiable function on N.

PROBLEM 10.1 Explain why each of the examples above is or is not a submanifold.

PROBLEM 10.2 Prove Theorem 10.2 (see Warner, p. 27).

10.2 Differentiable varieties as submanifolds

We saw in Section 10.1 that if (P, ϕ) is a submanifold of M, then each point of $\phi(P)$ has a $(\phi(P))$ neighborhood in a coordinate neighborhood of M whose points, q, satisfy $\nu^{p+1}(q) = 0, \ldots, \nu^m(q) = 0$ where ν^i are local coordinates on M. We saw also, in Theorem 9.10, that if $F_b = \phi^{-1}(b)$ is a differentiable variety in M, then at each point of F_b there is a coordinate neighborhood $\mathcal{U}$ of M in which all $q \in F_b \cap \mathcal{U}$ satisfy $f^1(q) = b^1,\ f^2(q) = b^2, \ldots, f^n(q) = b^n$ where f^i are local coordinate functions on M. This suggests that we draw our attention to subsets of coordinate neighborhoods of the types arising in these cases, and formulate our results in terms of them.

Definition Sets, U_k, in a coordinate neighborhood, $\mathcal{U}$, of M for which $m - k$ of the coordinate values are constant ($m = \dim M$) are called *k-dimensional coordinate slices of $\mathcal{U}$.*

Coordinate slices are, clearly, generalizations of coordinate curves, and, according to the paragraph above, they are special cases of differentiable varieties. Comparing the two situations described in that paragraph in the terminology of coordinate slices, we can say that for submanifolds at every point there is a chart, $(\mathcal{U}, \mu)$, of M having a coordinate slice, U_k such that $U_k \subset \phi(P) \cap \mathcal{U}$. That is, every point has a coordinate slice neighborhood. For differentiable varieties we have the stronger property that there is a chart at each point such that all points of F_b in $\mathcal{U}$ are in some coordinate slice of $\mathcal{U}$. That is, there is a coordinate slice, U_k of $\mathcal{U}$ such that $U_k = \phi(P) \cap \mathcal{U}$ (Fig. 10.1). We formalize this stronger property in the following definition.

Definitions A subset $S \subset M$ has the *k-submanifold property* if at each point of S, M has a chart, $(\mathcal{U}, \mu)$, such that $S \cap \mathcal{U}$ is a k-dimensional coordinate slice of $\mathcal{U}$. Such charts are *adapted to* S.

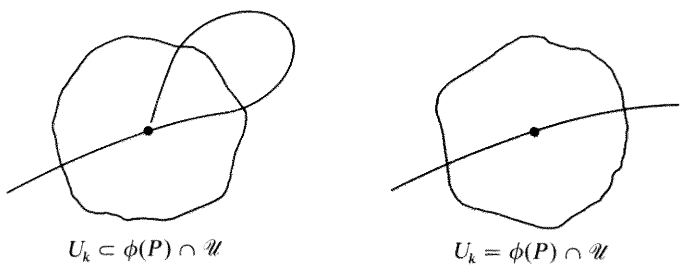

Figure 10.1

Theorem 10.4 *A differentiable variety has the k-submanifold property.*

Now, via the k-submanifold property, we will show that differentiable varieties are submanifolds of a special type; namely, with the relative topology of M, and closed in M.

Definitions $\phi : M \to N$ is *an embedding* if ϕ is a 1-1 immersion and, with $\phi(M)$ having the relative topology of N, ϕ is a homeomorphism of M onto $\phi(M)$. If ϕ is an imbedding, $\phi(M)$ is *an imbedded submanifold* of N (cf., comment below Theorem 9.8).

Theorem 10.5 *Locally, an immersion is an imbedding.*

Proof Problem 10.3. □

Theorem 10.6 *If $S \subset M$ has the k-submanifold property then it is an imbedded submanifold of M.*

Proof We will show that S is a manifold and the inclusion map is an imbedding. If $(\mathcal{U}, \mu)$ is a chart of M adapted to S, then using the relative topology for S, $\widetilde{\mathcal{U}} = \mathcal{U} \cap S$ is a coordinate neighborhood of S, and $\tilde{\mu} = \boldsymbol{\pi} \circ \mu|\widetilde{\mathcal{U}}$ (where $\boldsymbol{\pi}$ is the projection from $\mathbb{R}^m$ to $\mathbb{R}^k$) is a coordinate map, so $(\widetilde{\mathcal{U}}, \tilde{\mu})$ is a chart on S. Since $\tilde{\nu}^{-1} = \nu^{-1} \circ \mathbf{i}$, where $\mathbf{i}$ is an injection of $\mathbb{R}^k$ to $\mathbb{R}^m$, the overlap maps, $\tilde{\mu} \circ \tilde{\nu}^{-1}$, are differentiable, and S is a k-dimensional manifold. Finally, in the coordinate system $(\widetilde{\mathcal{U}}, \tilde{\mu})$ on S and the coordinate system $(\mathcal{U}, \mu)$ on M, the inclusion map is represented by $\mu \circ \tilde{\mu}^{-1} : (a^1, \ldots, a^k) \mapsto (a^1, \ldots, a^k, a_0^{k+1}, \ldots, a_0^m)$ where $a_0^{k+1}, \ldots, a_0^m$ are fixed. So this mapping from S to M is $1-1$ differentiable and a homeomorphism between S and its image. □

Combining Theorems 10.4 and 10.6 and the fact that a differentiable variety is closed, Problem 9.20, we have the following.

Theorem 10.7 *A differentiable variety in M is a closed imbedded submanifold of M.*

We complete the circle of implications establishing the equivalence of differentiable varieties, closed subsets with the k-submanifold property, and closed imbedded submanifolds with the following result.

Theorem 10.8 *A closed imbedded submanifold of M is a differentiable variety in M.*

Proof We noted in Section 10.1 that at each point, q, of $\phi(P)$ there is a chart $(\mathcal{V}, \nu)$ of M with $\nu^j = 0,\ j = p+1, \ldots, m$, on a $\phi(P)$ neighborhood of q. For any covering of M containing such charts we can use a partition of unity subordinate to the covering to construct a submersion, $\phi : M \to \mathbb{R}^{m-p}$, under which the submanifold goes to $(0, \ldots, 0)$. □

We also have the converse of Theorem 10.6. This result is part of another triad of equivalent concepts.

Theorem 10.9 *$S \subset M$ is an imbedded submanifold of M if and only if each point of S has an M-neighborhood, $\mathcal{U}$, such that $S \cap \mathcal{U}$ is the homeomorphic image, $\phi(\mathcal{V})$, under a 1-1 differentiable map, ϕ, of an open set, $\mathcal{V} \subset \mathbb{R}^p$, and ϕ has rank p at each point of V.*

Proof The "only if" part is immediately evident, and the "if" part may be based on the following lemma. □

Lemma *If S has the property described in Theorem 10.9, then S has the k-submanifold property.*

Proof Problem 10.5. □

Theorem 10.9 gives a local description of an imbedded submanifold. Thus, we have three equivalent ways of describing a set $S \subset M$; namely, S is the image of an imbedding, S has the property of Theorem 10.9, or S has the k-submanifold property.

PROBLEM 10.3 Prove Theorem 10.5.

PROBLEM 10.4 (i) Which of the examples in Section 10.1 are imbedded submanifolds? (ii) Which of the examples in Section 10.1 are differentiable varieties?

PROBLEM 10.5 Prove the lemma for Theorem 10.9.

PROBLEM 10.6 The dimension of a differentiable variety in M is $\dim M - n$.

11

VECTOR FIELDS, 1-FORMS, AND OTHER TENSOR FIELDS

Our discussion of vector fields will include their interpretation as derivations, construction of the tangent manifold, C^∞ vector fields, and the Lie bracket of two vector fields. Comments on 1-forms will be focused on analogies with vector fields. Finally, under mappings of tensor fields, we discuss interior multiplication of forms, the pull-back of $(0, s)$ tensor fields, and ϕ-related $(r, 0)$ tensor fields.

11.1 Vector fields

In Section 9.3 we defined, for each $p \in M$, a tangent space, T_p. If $S \subset M$, then we write $TS = \bigcup_{p \in S} T_p$, and $\pi : TS \to S$ for the mapping which projects each T_p to p.

Definition A *vector field on* S is a map $X : S \to TS$ such that for $p \in S$, $X(p) \in T_p$, or equivalently, $\pi \circ X = id$.

Theorem 11.1 *If* $\mathbb{R}^S$ *is the ring of functions on* S*, then with the definitions*

$$(X + Y)(p) = X(p) + Y(p)$$

$$(fX)(p) = f(p)X(p)$$

the set of vector fields on S *is a module over* $\mathbb{R}^S$.

Proof Problem 11.1. □

In Section 9.3 we described a tangent vector, $v_p \in T_p$ at a point $p \in M$ as a derivation on the C^1 functions, $\mathfrak{F}_p$, to the reals. So, in particular, given a vector field, X, we have $X(p) = v_p$ and

$$X(p) : \mathfrak{F}_p \to \mathbb{R}$$

according to

$$f \mapsto v_p \cdot f = X(p) \cdot f \tag{11.1}$$

Thus we can say

With a vector field X on an open set $\mathcal{U}$, if we pick a point, $p \in \mathcal{U}$, then with each function, $f \in \mathfrak{F}_{\mathcal{U}}$ (the set of C^1 functions on $\mathcal{U}$) we can form $X_p \cdot f = X(p) \cdot f$.

The fact that X depends on two arguments, $p \in \mathcal{U}$ and $f \in \mathfrak{F}_{\mathcal{U}}$, suggests that we may commute the choosing of p and f in this statement. Thus, we can also say

With a vector field X on an open set $\mathcal{U}$, if we first pick a function, $f \in \mathfrak{F}_{\mathcal{U}}$, we get a function given by

$$p \mapsto X_p \cdot f \tag{11.2}$$

Denoting this function by Xf we have

$$Xf(p) = X_p \cdot f \tag{11.3}$$

The map $f \mapsto Xf$ is frequently thought of as just another interpretation of a vector field. Denoting this interpretation of X by L_X, we have

$$L_X : \mathfrak{F}_{\mathcal{U}} \to \mathbb{R}^{\mathcal{U}}$$

according to

$$f \mapsto Xf$$

(cf., notation introduced in eq. (7.8)).

Theorem 11.2 *L_X is an R–linear derivation from $\mathfrak{F}_{\mathcal{U}}$ to $\mathcal{R}^{\mathcal{U}}$.*

Proof Problem 11.2. □

Theorem 11.3 *There is a 1-1 correspondence between vector fields on $\mathcal{U}$ and derivations in $\mathfrak{F}_{\mathcal{U}}$.*

Proof (i) If $\mathbf{\Delta}$ is a derivation on $\mathfrak{F}_{\mathcal{U}}$, define $X_{\mathbf{\Delta}}$ by

$$X_{\mathbf{\Delta}}(p) : f \mapsto (\mathbf{\Delta} \cdot f)(p)$$

$X_{\mathbf{\Delta}}$ is a vector field $\mathcal{U}$. Now $L_{X_{\mathbf{\Delta}}} \cdot f(p) = X_{\mathbf{\Delta}} f(p) = X_{\mathbf{\Delta}}(p) \cdot f = (\mathbf{\Delta} \cdot f)(p)$. So the mapping from vector fields to derivations is onto.

(ii) If $L_X = L_Y$, then $L_X \cdot f(p) = L_Y \cdot f(p)$ for all f and p. So $X(p) \cdot f = Y(p) \cdot f$ for all f and p. Hence $X = Y$ and the correspondence is 1-1. □

Finally, instead of fixing X and letting f vary as we have done, we can fix $f \in \mathfrak{F}_{\mathcal{U}}$ and consider the set of vector fields on $\mathcal{U}$. For each f we have a mapping from the module of vector fields on $\mathcal{U}$ to the module of functions on $\mathcal{U}$ given by

$$X \mapsto Xf \tag{11.4}$$

Theorem 11.4 *The map given by eq.* (11.4) *is a module homomorphism.*

Proof The proof follows immediately from the following Lemma. □

Lemma *Given $f \in \mathfrak{F}_{\mathcal{U}}$. For any vector fields X and Y on $\mathcal{U}$ and any $g \in \mathfrak{F}_{\mathcal{U}}$,*

$$(X+Y)f = Xf + Yf \quad \text{and} \quad (gX)f = g(Xf)$$

Proof Problem 11.3. □

In a local coordinate system $(\mathcal{U}, \mu)$ we define *the coordinate vector fields*

$$\frac{\partial}{\partial \mu^i} : p \mapsto \left.\frac{\partial}{\partial \mu^i}\right|_p \tag{11.5}$$

and have the functions

$$\frac{\partial}{\partial \mu^i} f : p \mapsto \left.\frac{\partial}{\partial \mu^i}\right|_p \cdot f \tag{11.6}$$

But in Section 9.3 we defined $\partial/\partial\mu^i\big|_p$ to be $[\zeta_i]_p$ so that $\partial/\partial\mu^i\big|_p \cdot f = D_0 f \circ \zeta_i = \partial f/\partial\mu^i\big|_p$, and we defined $\partial f/\partial\mu^i$ by $\partial f/\partial\mu^i : p \mapsto \partial f/\partial\mu^i\big|_p$, so

$$\frac{\partial f}{\partial \mu^i}(p) = \left.\frac{\partial}{\partial \mu^i}\right|_p \cdot f \tag{11.7}$$

From (11.6) and (11.7) we have $(\partial/\partial\mu^i)f(p) = (\partial f/\partial\mu^i)(p)$, or

$$\frac{\partial}{\partial \mu^i} f = \frac{\partial f}{\partial \mu^i} \tag{11.8}$$

In particular, since $L_{\partial/\partial\mu^i} : f \mapsto (\partial/\partial\mu^i)f$, we have $L_{\partial/\partial\mu^i} : f \mapsto \partial f/\partial\mu^i$; i.e., the derivations $L_{\partial/\partial\mu^i}$ map C^1 functions to their partial derivatives.

If X^i are functions on a coordinate neighborhood, $\mathcal{U}$, by Theorem 11.1 we have vector fields $X^i\partial/\partial\mu^i$. Conversely, given a vector field X on a neighborhood $\mathcal{U}$ we can define *component functions*, X^i, of X, with respect to a basis of coordinate vector fields by $X^i : p \mapsto X^i(p)$ where the $X^i(p)$ are the v_p^i given by $X(p) = v_p^i \, \partial/\partial\mu^i\big|_p$ according to eq. (9.6).

Theorem 11.5 $X^i = X\mu^i$.

Proof

$$\begin{aligned} X\mu^i(p) &= X(p) \cdot \mu^i && \text{by (11.1)} \\ &= v_p \cdot \mu^i = v_p^i \\ &= X^i(p) && \text{by the definition above} \end{aligned}$$

□

Theorem 11.6 *The set of vector fields on a coordinate domain, $\mathcal{U}$, form an n-dimensional $\mathfrak{F}_{\mathcal{U}}$-module with basis $\{\partial/\partial\mu^i : 1 = 1, \ldots, n\}$. That is, for every vector field X,*

$$X = X^i \frac{\partial}{\partial \mu^i} \tag{11.9}$$

or, in terms of functions

$$Xf = X^i \frac{\partial f}{\partial \mu^i} \quad \text{for all} \quad f \in \mathfrak{F}_{\mathcal{U}} \tag{11.10}$$

Proof These results follow immediately from the fact that $\partial/\partial\mu^i\big|_p$ form a basis at each point $p \in \mathcal{U}$. □

We have seen that, given a vector field, X, on $\mathcal{U}$, and a function $f \in \mathfrak{F}_{\mathcal{U}}$, we get a function Xf on $\mathcal{U}$. What can we say about Xf? The properties of Xf can be investigated by means of its coordinate representation,

$$\widehat{Xf} : b = \mu(p) \mapsto Xf(p) = X\mu^i(p)\frac{\partial f}{\partial \mu^i}(p) = \hat{X}^i(b)\frac{\partial \hat{f}}{\partial \pi^i}(b) \tag{11.11}$$

Alternatively, we have the representation (11.10) for Xf.

We will assume from now on (unless otherwise explicitly stated) that the manifold, M is C^∞, so we can talk about C^∞ functions. Recall, from Section 9.3 that if f is C^∞ then $\partial f/\partial\mu^i$ are C^∞. Then from either (11.10) or (11.11) we have the following result.

Theorem 11.7 *For a given vector field, X,*

$$Xf \in C^\infty \text{ for all } \ f \in C^\infty \ \Leftrightarrow \ X^i \in C^\infty$$

This give us information about Xf in terms of a local representation of X. We could use this to define $X \in C^\infty$. To give this an intrinsic definition (i.e., coordinate-free) we need some structure on the image set, $TM = \bigcup_{p \in M} T_p$, of X. Thus, we will make TM into a manifold.

For each chart $(\mathcal{U}, \mu)$ of M we have a bijective map, ν, of $\pi^{-1}(\mathcal{U})$ onto the open set $\mu(\mathcal{U}) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ given by $\nu : v \mapsto (\mu^1(\pi(v)), \ldots, \mu^n(\pi(v)), v \cdot \mu^1, \ldots, v \cdot \mu^n)$. If $\{\mathcal{U}_\alpha, \mu_\alpha)\}$ is an atlas of M, let $\mathcal{W}_\alpha = \mathcal{W} \cap (\mu_\alpha(\mathcal{U}_\alpha) \times \mathbb{R}^n)$ where $\mathcal{W}$ is an open set of $\mathbb{R}^{2n}$. We will say a set is open in TM if each point is contained in some element of $\{\mathcal{V}_\alpha^{-1}(\mathcal{W}_\alpha)\}$. With this topology TM is Hausdorff and $2nd$ countable since M is.

Theorem 11.8 *With the given topology, TM is a differentiable manifold,* the tangent manifold of M.

Proof We have to show (i) the set $\{(\pi^{-1}(\mathcal{U}_\alpha), \nu_\alpha)\}$ is an atlas, and (ii) $\nu_\alpha \circ \nu_\beta^{-1}$ is C^∞, so this atlas gives us a differentiable structure. We leave the details for Problem 11.5. □

Theorem 11.9 *X is C^∞ at $p \Leftrightarrow$ the component functions, X^i, are C^∞ at p.*

Proof We have to show that a representation $\nu \circ X \circ \mu^{-1}$ of X on pairs of charts is C^∞ iff the representations of X^i are C^∞. Now

$$\nu \circ X \circ \mu^{-1} : (\mu^1(p), \ldots, \mu^n(p)) \mapsto p \mapsto$$
$$X(p) \mapsto (\mu^1(p), \ldots, \mu^n(p), X^1(p), \ldots, X^n(p)).$$

The first n components are projections, and these are C^∞ by Theorem 9.2. The second n components are $\nu^{i+n} \circ X \circ \mu^{-1} = X^i \circ \mu^{-1}$, which are the representations of X^i and these are C^∞ iff the X^i are C^∞. □

Corollary (i) *$\partial/\partial\mu^i$ are C^∞ vector fields on a coordinate domain, $\mathcal{U}$.*
(ii) *$X \in C^\infty$ on an open set, $\mathcal{U} \Leftrightarrow Xf \in C^\infty$ for all $f \in \mathfrak{F}^\infty_{\mathcal{U}}$.*
(iii) *$X \in C^\infty$ on an open set, $\mathcal{U} \Leftrightarrow L_X : \mathfrak{F}^\infty_{\mathcal{U}} \to \mathfrak{F}^\infty_{\mathcal{U}}$.*

Theorem 11.10 *The set, $\mathfrak{X}M$, of C^∞ vector fields on a manifold, M, form an $\mathfrak{F}^\infty_M$-module, and on each chart $\{\partial/\partial\mu^i\}$ form a local basis.*

Proof We have to show that $X+Y$ and fX are C^∞ if X, Y, and f are, then we can use Theorem 11.6. By Corollary (ii) above, $X, Y \in C^\infty \Rightarrow Xf + Yf \in C^\infty$. Then by the Lemma for Theorem 11.4 $(X+Y)f \in C^\infty$ and so by Corollary (ii) again $X + Y \in C^\infty$. Similarly for fX. □

Notice that in Theorem 11.10 we referred to vector fields defined on an entire given manifold. While it does tell us something about a class of such vector fields, it leaves much to be desired. In particular, while, on the one hand, the module of vector fields described in Theorem 11.1 contains any vector field defined pointwise in an arbitrary manner on all of M, and on the other hand, we have, on a coordinate domain, the C^∞ vector fields $\partial/\partial\mu^i$, and hence any linear combination with C^∞ functions, we have not yet established the very existence of nontrivial C^∞ vector fields on a given manifold M. The way we get C^∞ vector fields on a manifold is by extending a locally given C^∞ vector field. This can be done in any way such that the component functions X^i and $\bar{X}^i$ in intersecting coordinate domains $\mathcal{U}$ and $\bar{\mathcal{U}}$ satisfy

$$\bar{X}^i = X^j \frac{\partial\bar{\mu}^i}{\partial\mu^j} \quad \text{on} \quad \bar{\mathcal{U}} \cap \mathcal{U} \tag{11.12}$$

cf., eq. (9.9).

For example, let $\{(\mathcal{U},\mu),(\bar{\mathcal{U}},\bar{\mu})\}$ be the stereographic atlas on S^2. That is,

$$\mathcal{U}=S^2-(0,0,1) \quad \mu:(a^1,a^2,a^3)\mapsto\left(\frac{a^1}{1-a^3},\ \frac{a^2}{1-a^3}\right)$$

$$\bar{\mathcal{U}}=S^2-(0,0,-1) \quad \text{and} \quad \bar{\mu}:(a^1,a^2,a^3)\mapsto\left(\frac{a^1}{1+a^3},\ \frac{a^2}{1+a^3}\right)$$

Then

$$\bar{\mu}^i=\frac{\mu^i}{(\mu^1)^2+(\mu^2)^2},\quad i=1,2,\quad \text{and} \quad (X^1,X^2)=(\mu^1-\mu^2,\mu^1+\mu^2)$$

and $(\bar{X}^1,\bar{X}^2)=(-\bar{\mu}^1-\bar{\mu}^2,\bar{\mu}^1-\bar{\mu}^2)$ satisfy (11.12), so the local vector fields

$$(\mu^1-\mu^2)\frac{\partial}{\partial\mu^1}+(\mu^1+\mu^2)\frac{\partial}{\partial\mu^2} \quad \text{on} \quad \mathcal{U}$$

$$(-\bar{\mu}^1-\bar{\mu}^2)\frac{\partial}{\partial\bar{\mu}^1}+(\bar{\mu}^1-\bar{\mu}^2)\frac{\partial}{\partial\bar{\mu}^2} \quad \text{on} \quad \bar{\mathcal{U}}$$

define a C^∞ vector field on S^2. (Note that at the points $(0,0,\pm1), X=0$, so this example also illustrates the fact that S^2 is not "parallelizable".)

Finally, we want to define a certain "multiplication" for vector fields. Given vector fields X,Y and a function f, we can form the functions Xf, Yf, $YXf=Y(Xf)$, and $XYf=X(Yf)$.

Theorem 11.11 *The mapping given by $f\mapsto XYf-YXf$ is a derivation.*

Proof Problem 11.8. □

We denote this derivation by $L_{[X,Y]}$. That is,

$$L_{[X,Y]}\cdot f=XYf-YXf \tag{11.13}$$

We saw, Theorem 11.3, that with each derivation, $\mathbf{\Delta}$, we have a corresponding vector field. In this case, $\mathbf{\Delta}=L_{[X,Y]}:f\mapsto XYf-YXf$ and the corresponding vector field is given by $Z_{\mathbf{\Delta}}(p):f\mapsto(XYf-YXf)(p)$.

Definition The vector field, $Z_{\mathbf{\Delta}}$, corresponding to the derivation, $L_{[X,Y]}$, of Theorem 11.11 is denoted by $[X,Y]$, and called *the Lie bracket of X and Y.*

That is,

$$\begin{aligned}[X,Y](p) : f \mapsto L_{[X,Y]} \cdot f(p) &= (XYf - YXf)(p) \\ &= X(p)Yf - Y(p)Xf\end{aligned}$$

or

$$[X,Y]f = XYf - YXf \tag{11.14}$$

In coordinates, using (11.10),

$$\begin{aligned}[X,Y]f &= X^i \frac{\partial}{\partial \mu^i} Yf - Y^i \frac{\partial}{\partial \mu^i} Xf \\ &= X^i \frac{\partial Y^j}{\partial \mu^i} \frac{\partial f}{\partial \mu^j} - Y^i \frac{\partial X^j}{\partial \mu^i} \frac{\partial f}{\partial \mu^j}\end{aligned}$$

and

$$[X,Y] = \left(X^i \frac{\partial Y^j}{\partial \mu^i} - Y^i \frac{\partial X^j}{\partial \mu^i} \right) \frac{\partial}{\partial \mu^j} \tag{11.15}$$

Theorem 11.12 *The map $\mathfrak{X}M \times \mathfrak{X}M \to \mathfrak{X}M$ given by $(X,Y) \mapsto [X,Y]$ has the following properties.*

(i) *It is $\mathbb{R}$-bilinear.*

(ii) *It is skew-symmetric.*

(iii) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ *(the Jacobi identity).*

(iv) $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$.

Proof Problem 11.9. □

By property (i), the set of C^∞ vector fields over $\mathbb{R}$ forms an algebra. Notice that the multiplication defined on $\mathfrak{X}M$ in Theorem 11.12 is not associative.

Definition A vector space with a multiplication with the properties (i), (ii) and (iii) of Theorem 11.12 is called a *Lie algebra*.

The algebra $\mathfrak{X}M$ is the prototypical Lie algebra.

PROBLEM 11.1 Prove Theorem 11.1.

PROBLEM 11.2 Prove Theorem 11.2.

PROBLEM 11.3 Prove the Lemma for Theorem 11.4.

PROBLEM 11.4 For points in the intersection, $\mathcal{U} \cap \bar{\mathcal{U}}$, of the two coordinate neighborhoods, $\partial/\partial\bar{\mu}^i = (\partial\mu^j/\partial\bar{\mu}^i)\partial/\partial\mu^j$ and $\bar{X}^i = X^j \partial\bar{\mu}^i/\partial\mu^j$ (eq. (11.12)).

PROBLEM 11.5 Complete the proof of Theorem 11.8.

PROBLEM 11.6 Using the atlas for $P^2(\mathbb{R})$ given in Section 9.1, show that the local vector fields

$$\mu_1^1 \frac{\partial}{\partial\mu_1^1} - \mu_1^2 \frac{\partial}{\partial\mu_1^2}$$

$$-2\mu_2^1 \frac{\partial}{\partial\mu_2^1} - \mu_2^2 \frac{\partial}{\partial\mu_2^2}$$

$$\mu_3^1 \frac{\partial}{\partial\mu_3^1} + 2\mu_3^2 \frac{\partial}{\partial\mu_3^2}$$

define a C^∞ vector field on $P^2(\mathbb{R})$.

PROBLEM 11.7 Show that $\pi^{-1}(p)$ is a closed imbedded submanifold of TM (you need a slight generalization of Theorem 10.7), whose tangent space $T_v\pi^{-1}(p) \subset TTM$ is the kernel of π_*.

PROBLEM 11.8 Prove Theorem 11.11.

PROBLEM 11.9 Prove Theorem 11.12.

PROBLEM 11.10 Prove that the vector space of all $n \times n$ matrices, $\mathcal{M}_n$, (Section 2.1) with the additional product defined by $[A, B] : (A, B) \mapsto AB - BA$ where AB and BA are the usual products, is a Lie algebra. $\mathcal{M}_n(\mathbb{R})$ with this additional structure is denoted by $g\ell(n, \mathbb{R})$.

11.2 1-Form fields

There are many analogies between vector fields and 1-form fields. In the following discussion, more or less "parallel" to that of vector fields, we will focus on these analogies.

Definition If $S \subset M$ and $T^*S = \bigcup_{p\in S} T_p^*$, then a 1-*form field on* S is a map $\sigma : S \to T^*S$ such that $\sigma(p) \in T_p^*$.

Theorem 11.13 *The set of 1-form fields on S is an $\mathbb{R}^S$-module.*

Recall that an element of $T_p^* = T^p$ has the form $df\big|_p$ for some $f \in \mathfrak{F}_p$, and we saw in Eq. (9.7) that $df\big|_p$ is a function $T_p \to \mathbb{R}$ given by $df\big|_p : v_p \mapsto \langle df\big|_p, v_p\rangle = v_p \cdot f$. So, in particular, if X is a given vector field, and σ is a given 1-form field, then $\sigma(p) = df\big|_p$ for some $f \in \mathfrak{F}_p$ and

$$\sigma(p) : T_p \to \mathbb{R}$$

according to

$$X(p) \mapsto \langle df\big|_p, X(p)\rangle = \langle \sigma(p), X(p)\rangle$$

Now, as in Section 11.1, let $S = \mathcal{U}$ be an open set in M. Given σ on $\mathcal{U}$, for each vector field, X, on $\mathcal{U}$ we have a function $\langle \sigma, X\rangle : \mathcal{U} \to \mathbb{R}$ given by $p \mapsto \langle \sigma(p), X(p)\rangle$. That is,

$$\langle \sigma, X\rangle(p) = \langle \sigma(p), X(p)\rangle \tag{11.16}$$

Equation (11.16) corresponds to eq. (11.3). Finally, for each σ we have, in analogy with the map L_X in Section 11.1, a map, $\rfloor\sigma$, from the vector fields on $\mathcal{U}$ to $\mathbb{R}^{\mathcal{U}}$ given by $X \mapsto \langle \sigma, X\rangle$. (See definition of $\rfloor B^*$ in Section 6.3 with $s = r = 1$.)

Theorem 11.14 *$\rfloor\sigma$ is a module homomorphism from the module of vector fields on $\mathcal{U}$ to $\mathbb{R}^{\mathcal{U}}$.*

The interpretation of a 1-form field, σ, as a module homomorphism, $\rfloor\sigma$, generalizes the definition of a 1-form as a linear function from a vector space to the reals.

Continuing the analogy with Section 1.1, instead of fixing σ and letting X vary, we can fix X and let σ vary so that we have a map

$$\sigma \mapsto \langle \sigma, X\rangle \tag{11.17}$$

Lemma *Given a vector field X, if σ and τ are 1-form fields, and $g \in \mathfrak{F}_{\mathcal{U}}$*

$$\langle \sigma + \tau, X\rangle = \langle \sigma, X\rangle + \langle \tau, X\rangle \quad \text{and} \quad \langle g\sigma, X\rangle = g\langle \sigma, X\rangle$$

Theorem 11.15 *The map given by* eq. (11.17) *is a module homomorphism.*

Definition If $f \in \mathfrak{F}_{\mathcal{U}}$, the 1-form field $df : \mathcal{U} \to T^*\mathcal{U}$ given by $df : p \mapsto df\big|_p \in T_p^*$ is *the differential of f*.

In a local coordinate system $(\mathcal{U}, \mu)$ we have *the coordinate 1-forms*

$$d\mu^i : p \mapsto d\mu^i\big|_p \tag{11.18}$$

Then, using eq. (9.7), $\langle d\mu^i, X\rangle : p \mapsto X\mu^i(p)$, and hence $\rfloor d\mu^i : X \mapsto X\mu^i$.

If σ_i are functions on $\mathcal{U}$, we can form $\sigma_i d\mu^i$, and conversely, given a 1-form σ on $\mathcal{U}$ we can define *component functions*, σ_i, of σ by $\sigma_i : p \mapsto \sigma_i(p)$ where the $\sigma_i(p)$ are the coefficients of σ_p according to eq. (9.8). With these component functions we have the following results corresponding to Theorems 11.5 and 11.6.

Theorem 11.16 $\sigma_i = \langle \sigma, \partial/\partial\mu^i\rangle$.

Theorem 11.17 *The set of 1-form fields on a coordinate domain form an $\mathfrak{F}_{\mathcal{U}}$-module with basis $\{d\mu^i : i, \ldots, n\}$. That is, for every 1-form σ*

$$\sigma = \sigma_i d\mu^i \tag{11.19}$$

or, in terms of functions,

$$\langle \sigma, X \rangle = \sigma_i X \mu^i \tag{11.20}$$

for all vector fields X on $\mathcal{U}$.

At this point we might note a certain breakdown in the "parallel" treatments of vector fields and 1-forms. On the one hand, we have a special class of 1-form fields for which we have no corresponding class of vector fields, namely, the 1-form fields df coming from functions $f \in \mathfrak{F}_{\mathcal{U}}$. On the other hand, while eqs. (9.3) and (9.8) are simply different notations for the same thing, we can extend the coefficients of (9.8) to all of $\mathcal{U}$ as we did in Theorems 11.16 and 11.17, but, in general we cannot extend (9.3) to the familiar equation $df = (\partial f/\partial \mu^i) d\mu^i$ by writing Theorem 11.16 in the simpler form $\sigma_i = \partial f/\partial \mu^i$. We get these results only if σ comes from a function, that is, $\sigma = df$. However, though to every $f \in \mathfrak{F}_{\mathcal{U}}$ we have a 1-form $\sigma = df$, we cannot say that every 1-form on $\mathcal{U}$ can be written df. In order for the latter to hold, the component functions σ_i of the given σ must satisfy "integrability conditions". A rough way of describing this distinction between vector fields and 1-form fields is that we can generally expect to be able to integrate vector fields to get curves, but more stringent conditions are required to get functions from 1-form fields. We will have more to say about this in Chapters 13 and 14.

Restricting ourselves to C^∞ manifolds again, we have from (11.20) that

$$\langle \sigma, X \rangle \in C^\infty \quad \text{for all} \quad X \in C^\infty \;\Leftrightarrow\; \sigma_i \in C^\infty \tag{11.21}$$

This corresponds to Theorem 11.7.

Corresponding to our introduction of C^∞ vector fields in Section 11.1, we now have C^∞ 1-form fields. Specifically, we can make $T^*M = \bigcup_{p \in M} T_p^*$ into a manifold, *the cotangent manifold of M*, in a manner analogous to what we did for TM, and then get results corresponding to Theorems 11.8-11.10. In particular, the set, $\mathfrak{X}^*M$, of C^∞ 1-form fields on M forms an $\mathfrak{F}_M^\infty$-module.

Having constructed the manifolds TM and T^*M we now have all the structures for them that we have for M. Thus, for example, for TM we have the tangent spaces, T_vTM, and cotangent spaces T_v^*TM, vector fields and 1-form fields on TM, and the tangent and cotangent manifolds, TTM and T^*TM.

Finally, the manifold T^*M has a special property other manifolds do not have, and which will be important when we study mechanics.

Theorem 11.18 *T^*M has a natural (canonical) 1-form field, θ_M.*

Proof If $\sigma \in T^*M$, $v \in T_\sigma T^*M$, and π is the natural projection, $\pi : T^*M \to M$ given by $\sigma \mapsto p$, then a 1-form $\theta_M : T^*M \to T^*T^*M$ is defined by $\langle \theta_M(\sigma), v\rangle = \langle \sigma, \pi_* \cdot v\rangle$. That is, on each vector space T_p^*M, θ_M is the transpose of π_* (see Fig. 11.1). □

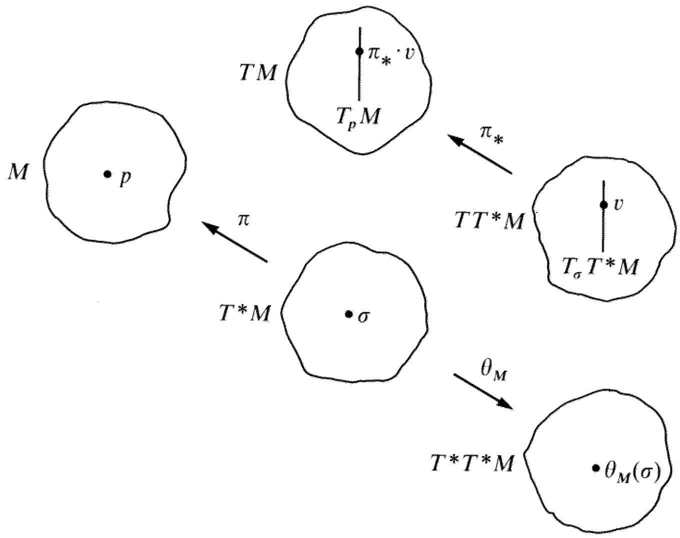

Figure 11.1

PROBLEM 11.11 Make T^*M into a manifold.

PROBLEM 11.12 State and prove any one of the results corresponding to Theorems 11.8-11.10 for T^*M and 1-form fields.

PROBLEM 11.13 For points in the intersection, $\mathcal{U} \cap \bar{\mathcal{U}}$, of two coordinate neighborhoods we have, for component functions and for coordinate 1-forms, respectively (i) $\bar{\sigma}_i = \sigma_j\ \partial\mu^j/\partial\bar{\mu}^i$; (ii) $d\bar{\mu}^i = (\partial\bar{\mu}^i/\partial\mu^j)d\mu^i$ (cf., eq. (11.12) and Problem 11.4). Moreover, (iii) $d\bar{\mu}^1 \wedge \cdots \wedge d\bar{\mu}^n = \det(\partial\bar{\mu}^i/\partial\mu^j)d\mu^1 \wedge \cdots \wedge d\mu^n$(cf., Section 6.3).

11.3 Tensor fields and differential forms

We will generalize the two previous sections (mainly introducing a lot more notation).

Recall that at the end of Section 9.3 we pointed out that having T_p and T_p^* at each point we can construct at each point the tensor product spaces $(T_p)^r_s$ as we did in Chapter 4. We also have the symmetric and skew-symmetric subspaces,

$$\left.\begin{matrix} \boldsymbol{S}^r(T_p) \\ \bigwedge^r(T_p) \end{matrix}\right\} \subset (T_p)^r_0 \quad \text{the vector spaces of contravariant tensors}$$

and

$$\left.\begin{array}{l} \boldsymbol{S}^s(T_p^*) \\ \\ \bigwedge^s(T_p^*) \end{array}\right\} \subset (T_p)_s^0 \quad \text{the vector spaces of covariant tensors.}$$

We can take the set, for all $p \in M$, of vector spaces of each type, and form differentiable manifolds, just as we did for TM and T^*M. Thus, for example, we have

$$T_s^r M = \bigcup_{p \in M} (T_p)_s^r, \quad \bigwedge^s(T^*M) = \bigcup_{p \in M} \bigwedge^s(T_p^*), \text{ etc.} \tag{11.22}$$

Definitions *An* (r, s) *tensor field* K, *on* M is a differentiable map $K : M \to T_s^r M$ given by $p \mapsto A \in (T_p)_s^r$. In particular, *an exterior* s*-form field on* M is a differentiable map $\omega : M \to \bigwedge^s(T^*M)$ from M into the manifold of exterior s-forms given by $p \mapsto \omega(p) \in \bigwedge^s(T_p^*)$, and an r*-vector field on* M is a differentiable map $F : M \to \bigwedge^r(TM)$ from M into the manifold of r-vectors given by $p \mapsto F(p) \in \bigwedge^r(T_p)$.

That is, the values of an exterior s-form field are skew-symmetric covariant tensors and the values of an r-vector field are skew-symmetric contravariant tensors. For us, of the two, the former will be much more important and will arise constantly in the sequel where we will call them simply *s-forms.*

Given an (r, s) tensor field, K, 1-forms $\sigma^1, \ldots, \sigma^r$, and vector fields $X_1, \ldots, X_s$ we can define a function $M \to \mathbb{R}$ by

$$K(\sigma^1, \ldots, \sigma^r, X_1, \ldots, X_s) : p \mapsto K(p)(\sigma^1(p), \ldots, X_s(p))$$

This generalizes the functions Xf and $\langle \sigma, X \rangle$ in Sections 11.1 and 11.2. Corresponding to Theorem 11.7 and Eq. (11.21) we have that $K(\sigma^1, \ldots, \sigma^r, X_1, \ldots, X_s)$ is a C^∞ function if $\sigma^1, \ldots, \sigma^r, X_1, \ldots, X_s$, and K are C^∞.

Finally, given an (r, s) tensor field, K, the mapping

$$\mathfrak{X}^*M \times \cdots \times \mathfrak{X}^*M \times \mathfrak{X}M \times \cdots \times \mathfrak{X}M \to \mathfrak{F}_M^\infty$$

from r 1-forms and s vector fields to $\mathfrak{F}_M^\infty$ given by

$$(\sigma^1, \ldots, \sigma^r, X_1, \ldots, X_s) \mapsto K(\sigma^1, \ldots, \sigma^r, X_1, \ldots, X_s) \tag{11.23}$$

generalizes the maps L_X and $\rfloor\sigma$ defined in Sections 11.1 and 11.2. In particular for s-forms we get $\omega(X_1, \ldots, X_s)$ and for r-vector fields we get $F(\sigma^1, \ldots, \sigma^r)$. (Sometimes an (r, s) tensor field is *defined* as a multilinear map of the form (11.23), and then a tensor field as we have defined it is derived; cf. O'Neill, pp. 35-37).

As before, we can describe differentiability of K, ω, F, etc., in terms of the functions $K(\sigma^1, \ldots, X_s), \omega(X_1, \ldots, X_s)$, etc., or in terms of *the component functions* of K, ω, F, etc., in a local coordinate system. For K we have, in a local coordinate system

$$K^{i_1 \cdots i_r}_{j_1 \cdots j_s} = K\left(d\mu^{i_1}, \ldots, d\mu^{i_r},\ \frac{\partial}{\partial \mu^{j_1}}, \ldots, \frac{\partial}{\partial \mu^{j_s}}\right) \tag{11.24}$$

and

$$K = K^{i_1 \cdots i_r}_{j_1 \cdots j_s} \frac{\partial}{\partial \mu^{i_1}} \otimes \cdots \otimes d\mu^{j_s} \tag{11.25}$$

For an s-form, ω, we have

$$\omega_{j_i \cdots j_s} = \omega\left(\frac{\partial}{\partial \mu^{j_1}}, \ldots, \frac{\partial}{\partial \mu^{j_s}}\right) \tag{11.26}$$

and

$$\omega = \omega_{j_1 \cdots j_s} d\mu^{j_1} \wedge \cdots \wedge d\mu^{j_s}, \qquad j_1 \ < \cdots < j_s \tag{11.27}$$

Of course, as before, for eqs. (11.25) and (11.27) to make sense, we have to make the set of (r, s) tensor fields and the set of s-forms into $\mathfrak{F}^{\infty}_M$-modules.

In addition to (r, s) tensor fields there is another generalization of Sections 11.1 and 11.2. To put it in context we recall that at the beginning of this section we started with various vector spaces built on T_p and T_p^* and made these into manifolds, eq. (11.22). Then we defined maps from M into these manifolds, in particular, (differential) s-forms. We can take another route, starting with the same vector spaces and constructing their tensor algebras as in Section 6.1. We can then make these into manifolds. The case that is important for us at present is the manifold $\bigwedge(T^*M)$ constructed from the exterior algebras, $\bigwedge T_p^*$ (Section 6.2); i.e.,

$$\bigwedge(T^*M) = \bigcup_{p \in M} \bigwedge T_p^* \tag{11.28}$$

A differentiable map $\omega : M \to \bigwedge(T^*M)$ with $p \mapsto \bigwedge T_p^*$ is *a differential form.* Finally, just as the set of s-forms "inherits" a $\mathfrak{F}^{\infty}_M$-module structure from its image space, we get an *algebra of differential forms* from the image set of exterior algebras. Later, Section 13.4, we will encounter *the algebra of contravariant tensor fields*, the differentiable maps of M into the manifold, $\oplus V_0^n(M)$ constructed from the contravariant tensor algebras $\oplus V_0^n$ built on the T_p's (Section 6.1).

PROBLEM 11.14 Make $\bigwedge^2(T^*M) = \bigcup_{p \in M} \bigwedge^2(T_p^*)$ into a manifold.

PROBLEM 11.15 State and prove any one of the results corresponding to Theorems 11.8-11.10 for $\mathbf{S}^2(T^*M)$ and symmetric $(0, 2)$ tensor fields.

11.4 Mappings of tensor fields and differential forms

We will now describe two important mappings of tensor fields and differential forms.

1. There is an operator on the algebra of differential forms on M which takes s-forms on M to $s-1$ forms on M.

Definition Given an r-vector field F we define the map $F\rfloor$, called *interior multiplication by F* on the algebra of differential forms taking s-forms to $s-r$ forms ($s \geq r$) by $F\rfloor : \omega \mapsto F\rfloor\omega$ where $(F\rfloor\omega)(p) = F(p)\rfloor\omega(p)$, the interior product defined in Section 6.3.

Theorem 11.19 *If ω is an s-form, $s > 1$, and $F = X$ is a vector field then* $X\rfloor\omega(X_1, \ldots, X_{s-1}) = \omega(X, X_1, \ldots, X_{s-1})$.

Proof By definition (see Section 6.3) $\langle X\rfloor\omega,\ Y\rangle = \langle\omega, X \wedge Y\rangle$. For $Y = X_1 \wedge \cdots \wedge X_{s-1}$, then

$$\langle X\rfloor\omega, X_1 \wedge \cdots \wedge X_{s-1}\rangle = \langle\omega, X \wedge X_1 \wedge \cdots \wedge X_{s-1}\rangle \tag{11.29}$$

But, for any s-form ω, and s vector fields $Y_1, \ldots, Y_s$, extending eq. (6.8) pointwise,

$$\langle\omega, Y_1 \wedge \cdots \wedge Y_s\rangle = \omega(Y_1, \ldots, Y_s) \tag{11.30}$$

Applying (11.30) to (11.29) we get the result. □

Definition If ω is an s-form, we write

$$\begin{aligned} i_X\omega &= sX\rfloor\omega \quad & s \geq 1 \\ i_X\omega &= 0 & s = 0 \end{aligned} \tag{11.31}$$

The operation i_X is also called *interior multiplication by X*.

Theorem 11.20 (i) *i_X is a skew-derivation in the algebra of differential forms on M; i.e., i_X is linear, and if σ is a p-form, and τ is a q-form, then for the $p+q$ form $\sigma \wedge \tau$*

$$i_X(\sigma \wedge \tau) = (i_X\sigma) \wedge \tau + (-1)^p\sigma \wedge (i_X\tau) \tag{11.32}$$

(ii) $i_{X+Y}\omega = i_X\omega + i_Y\omega$ *and* $i_{fX}\omega = fi_X\omega$.
(iii) $i_X df = L_X f = Xf$.

Proof We show these results are all valid at an arbitrary point M. They all come immediately from the definition of $X\rfloor\omega$, except eq. (11.32) which we prove by using linearity. Then (11.32) is verified for basis elements using Problem 6.13(i). □

Corollary (i) *If* $\omega = \sigma^1 \wedge \cdots \wedge \sigma^s$ *where the* σ^i *are 1-forms, then*

$$i_X \omega = \sum_{j=1}^{s} (-1)^{j+1} (\sigma^j \cdot X) \sigma^1 \wedge \cdots \wedge \hat{\sigma}^j \wedge \cdots \wedge \sigma^s \tag{11.33}$$

(ii) $i_X \circ i_X = 0$

2. Mappings of tensor fields may be induced by a differentiable map $\phi : M \to N$. Near the end of Section 9.3 we mentioned that with ϕ we have $\phi_{rp} : (T_p)^r_0 \to (T_{\phi(p)})^r_0$ for contravariant tensor spaces at each $p \in M$, and we have $\phi^s_{\phi(p)} : (T_{\phi(p)})^0_s \to (T_p)^0_s$ for covariant tensor spaces at each $p \in M$. Note that utilizing the duality in the set of tensor product spaces as expressed in Problem 4.21(i) it is often convenient to express $\phi^s_{\phi(p)}$ by

$$\langle \phi^s_{\phi(p)} \omega, A \rangle = \langle \omega, \phi_{sp} A \rangle \tag{11.34}$$

where $A \in (T_p)^s_0$ and $\omega \in (T_{\phi(p)})^0_s$.

Definitions If K is a covariant tensor field on N, the covariant tensor field on M, $\phi^* K : M \to T^0_s M$ given by $\phi^* K : p \mapsto \phi^s_{\phi(p)} \cdot K(\phi(p)) \in (T_p)^0_s$ is called *the pull-back of* K *by* ϕ. In particular, if $N = M$ *a covariant tensor field,* K*, on* M *is invariant under* ϕ if $\phi^* K(p) = K(p)$. (The reader will recall that the notation ϕ^* was introduced in Section 9.3 to denote the transpose of the tangent map and then extended to denote maps of cotangent manifolds. See comments on notation at the end of this section.)

Theorem 11.21 *If* $f \in \mathfrak{F}_M$*, then* $\phi^* df = d\phi^* f$.

Proof Extend Theorem 7.12 pointwise to 1-forms on manifolds. □

Theorem 11.22 *If* K *and* ϕ *are in* C^∞*, then* $\phi^* K \in C^\infty$.

Proof From eq. (11.25) $\phi^* K(p) = K_{j_1 \cdots j_s}|_{\phi(p)} \phi^s_{\phi(p)} \cdot (d\bar{\mu}^{j_1} \otimes \cdots \otimes d\bar{\mu}^{j_s})|_{\phi(p)}$ where $\{d\bar{\mu}^{j_1}, \ldots, d\bar{\mu}^{j_s}\}$ is a coordinate basis of $T^*_{\phi(p)}$. So,

$$\begin{aligned} \phi^* K(p) &= (K_{j_1 \cdots j_s} \circ \phi)(p) (\phi^* \cdot d\bar{\mu}^{j_1} \otimes \cdots \otimes \phi^* \cdot d\bar{\mu}^{j_s})_{\phi(p)} \\ &= (K_{j_1 \cdots j_s} \circ \phi)(p) \left(\frac{\partial \phi^{j_1}}{\partial \mu^{i_1}} d\mu^{i_1} \otimes \cdots \otimes \frac{\partial \phi^{j_s}}{\partial \mu^{i_n}} d\mu^{i_n} \right)(p) \end{aligned}$$

by eq. (9.12). Our result now follows from the generalizations of Theorem 11.9 and its corollaries. □

We have seen above that $\phi : M \to N$ induces a "backward" mapping of covariant tensor fields. Now suppose we start with a contravariant tensor field K on M and we want to define a contravariant vector field on N by means of the maps ϕ_{rp}. Then ϕ_{rp} maps $K(p)$ "forward" into an element of $(T_{\phi(p)})^r_0$. A problem arises now that did not occur in the covariant case, for if $p \neq q$, then, in general, $\phi_{rp}K(p) \neq \phi_{rq}K(q)$. So, if $\phi(p) = \phi(q)$, then K will not have a well-defined image.

Definitions If $\phi: M \to N$ and K_M and K_N are contravariant tensor fields on M and N respectively, then K_M *is ϕ-related to* K_N if $\phi_{rp} \cdot K_M(p) = K_N(\phi(p))$ for all $p \in M$. In particular, if $M = N$ and $K_M = K_N = K$, then K *is invariant under* ϕ; i.e., K is ϕ-related to itself.

Theorem 11.23 *A vector field, X, on M is ϕ-related to a vector field, Y, on N iff $X(g \circ \phi) = (Yg) \circ \phi$ for all $g \in \mathfrak{F}_N$.*

Proof $X(g\circ\phi)(p) = (Yg)\cdot\phi(p)$ iff $X(p)\cdot(g\circ\phi) = Y(\phi(p))\cdot g$ iff $(\phi_{*p}\cdot X(p))\cdot g = Y(\phi(p)\cdot g$ iff $\phi_{*p} \cdot X(p) = Y(\phi(p))$. □

Theorem 11.24 *If vector fields X_1 and X_2 are ϕ-related respectively to vector fields Y_1 and Y_2, then $[X_1, X_2]$ is ϕ-related to $[Y_1, Y_2]$.*

Proof Problem 11.18. □

Theorem 11.25 *If ϕ is regular on $M(\phi_{*p}$ is 1-1 for all $p \in M)$ and K is a contravariant tensor field on N, then there is a unique contravariant tensor field, ϕ^*K, on M, called* the pull-back of K by ϕ, *which is ϕ-related to K.*

Proof Define a linear map, $(\phi^{-1})_r$, from $\bigcup_{\phi(p)}(T_{\phi(p)})^r_0$ to $\bigcup_p (T_p)^r_0$ by

$$(\phi^{-1})_r\big|_{(T_{\phi(p)})^r_0} : w_1 \otimes \cdots \otimes w_r \mapsto \phi^{-1}_{*\phi(p)} \cdot w_1 \otimes \cdots \otimes \phi^{-1}_{*\phi(p)} \cdot w_r$$

where $w_i \in T_{\phi(p)}$ (see Problem 4.18). Then ϕ^*K defined by $\phi^*K(p) = (\phi^{-1})_r|_{(T_{\phi(p)})^r_0} \cdot K(\phi(p))$ will have the required property. □

Theorem 11.26 *If ω is an s-form on N and X is a vector field on N and ϕ is regular, then*

$$\phi^* i_X \omega = i_{\phi^* X} \phi^* \omega$$

That is, the operators i_X and ϕ^* commute.

Proof Problem 11.21. □

We must, finally, make another comment about notation. In order to construct pull-backs of covariant tensor fields we made use of the "backward" maps, ϕ^s. If ϕ is regular, $(\phi^{-1})_r$ give us pull-backs of contravariant tensor fields and the $\phi^{r,s}$ of Problem 4.18 gives us pull-backs of mixed tensor fields. The operator on tensor fields in all three cases will be denoted by ϕ^*. In order to simplify our notation we will frequently also write ϕ^* in place of ϕ^s, $(\phi^{-1})_r$, or $\phi^{r,s}$. Which of the two slightly distinct meanings is intended will, hopefully, be clear in context.

PROBLEM 11.16 The values of the pull-back by ϕ of a covariant tensor field, K, are $\phi^* K(X_1, \ldots, X_s) = K(\phi_* \cdot X_1, \ldots, \phi_* \cdot X_s)$.

PROBLEM 11.17 Let $M = \mathbb{R}^2$, $N = \mathbb{R}^3$, and $\phi : (a, b) \mapsto (\cos a, \sin a \cos b, \sin a \sin b)$. Then, if $\omega = x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy$ where x, y, z are the natural coordinate functions on $\mathbb{R}^3$, write $\phi^* \omega$ in terms of the natural coordinate functions u, v on $\mathbb{R}^2$.

PROBLEM 11.18 Prove Theorem 11.24.

PROBLEM 11.19 If K_M is ϕ-related to K_N, its values are $K_M(\sigma^1, \ldots, \sigma^r) = K_N(\phi^{*-1} \cdot \sigma^1, \ldots, \phi^{*-1} \cdot \sigma^r)$.

PROBLEM 11.20 If $\boldsymbol{\pi}_1 : M \times N \to M$ and $\boldsymbol{\pi}_2 : M \times N \to N$ are the projections and X is a vector field on M, then there is a unique vector field, $\widetilde{X}$, on $M \times N$ such that $\widetilde{X}$ is $\boldsymbol{\pi}_1$-related to X and $\widetilde{X}$ is $\boldsymbol{\pi}_2$-related to the zero vector field on N. (Similarly, for Y a vector field on N.)

PROBLEM 11.21 Prove Theorem 11.26.

PROBLEM 11.22 The natural 1-form, θ_M, on T^*M (Theorem 11.18) can be characterized by its behavior under pull-backs to M; θ_M is the natural 1-form on T^*M iff $\phi^* \theta_M = \theta_M$ for all maps $\phi : M \to T^*M$.

12

DIFFERENTIATION AND INTEGRATION OF DIFFERENTIAL FORMS

In Section 11.2 we mapped functions $f \in \mathfrak{F}_M$ into 1-forms, df. In Section 12.1 we will extend this mapping to an operation on the algebra of differential forms, and obtain some properties of this operation. In Section 12.2 we will describe a concept of integration on manifolds which reduces to the usual integrals on $\mathbb{R}^n$, and we prove a generalization of Stokes' theorem.

12.1 Exterior differentiation of differential forms

In Section 11.4, we defined, for each vector field, X, on M, an operator, i_X, on the algebra of differential forms on M which took s-forms on M into $s-1$-forms on M. We defined i_X by giving its values on ω, and it turned out that i_X was a skew-derivation, and $i_X df = Xf$ and $i_X f = 0$. Now, we define an operator, d, on the algebra of differential forms on M taking s-forms to $s+1$-forms. We define d, *exterior differentiation*, by listing properties it is to have. It will turn out that there is precisely one such operator.

Theorem 12.1 *There exists one and only one operator, d, on the algebra of differential forms with the following properties.*

(1) *d is a skew-derivation, i.e., d is $\mathbb{R}$-linear and if σ is a p-form and τ is a q-form then*

$$d(\sigma \wedge \tau) = (d\sigma) \wedge \tau + (-1)^p \sigma \wedge d\tau \tag{12.1}$$

(2) *For $f \in \mathfrak{F}_M$, $d\colon f \mapsto df$ (recall that df, the differential of f, was defined in Section 11.2).*

(3) *$d(df) = 0$.*

Proof (i) (*Local existence*) If

$$\omega = \omega_{i_1 \cdots i_s} d\mu^{i_1} \wedge \cdots \wedge d\mu^{i_s} \quad i_1 < \cdots < i_s,$$

on (U, μ) let

$$d\omega = d\omega_{i_1 \cdots i_s} \wedge d\mu^{i_1} \wedge \cdots \wedge d\mu^{i_s} \quad i_1 < \cdots < i_s \tag{12.2}$$

Property (2) is just a matter of definition and $\mathbb{R}$-linearity comes from $\mathbb{R}$-linearity of d on functions. For eq. (12.1) let $\sigma = f\ d\mu^{i_1} \wedge \cdots \wedge d\mu^{i_p}$ and $\tau = g\ d\mu^{j_i} \wedge \cdots \wedge d\mu^{j_q}$. Then

$$\begin{aligned} d(\sigma \wedge \tau) &= d(fg \wedge d\mu^{i_1} \wedge \cdots \wedge d\mu^{j_q}) \\ &= (df)g \wedge d\mu^{i_1} \wedge \cdots \wedge d\mu^{j_q} + f\ dg \wedge d\mu^{i_1} \wedge \cdots \wedge d\mu^{j_q} \\ &= (d\sigma) \wedge \tau + (-1)^p \sigma \wedge d\tau \end{aligned}$$

since d is a derivation on functions. We then get (12.1) for any σ and τ by the $\mathbb{R}$-linearity of d. Finally, $df = \partial f/\partial\mu^i\ d\mu^i$, so

$$d(df) = (d\ \partial f/\partial\mu^i) \wedge d\mu^i = (\partial^2 f/\partial\mu^i\ \partial\mu^j)d\mu^j \wedge d\mu^i = 0,$$

since the second partial derivatives are symmetric and the differentials are skew-symmetric.

(ii) (*Local uniqueness*) If $(\mathcal{U}, \mu)$ is a local coordinate system, then for a 1-form $\omega = \omega_i d\mu^i$ property (1) gives $d\omega = d\omega_i \wedge d\mu^i + \omega_i \wedge d\ d\mu^j$, which is determined by properties (2) and (3). Similarly, if ω is a higher-order form, writing $\omega = \omega_{i_1 \cdots i_s} d\mu^{i_1} \wedge \cdots \wedge d\mu^{i_s}$, then properties (1) and (3) give

$$d\omega = d\omega_{i_1 \cdots i_s} \wedge d\mu^{i_1} \wedge \cdots \wedge d\mu^{i_s},$$

which is determined by property (2).

(iii) Finally, if ω is defined on M, then by local existence and uniqueness we have a unique $d\omega$ on the coordinate neighborhoods $\mathcal{U}$ and $\mathcal{V}$ and hence on $\mathcal{U} \cap \mathcal{V}$, so $d\omega$ is unambiguously defined on all of M. □

Corollary *For any form,* ω, $d\ (d\omega) = 0$.

Example Let $M = \mathbb{R}^3$ with the standard structure. Denote the standard coordinate functions, π^i, by x, y, z. Then $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ form a basis of the module of vector fields, $\mathfrak{X}M$, and (dx, dy, dz) form a basis of the module 1-forms, $\mathfrak{X}^*M$.

(i) For a 0-form, f,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \tag{12.3}$$

(ii) For a 1-form, $\sigma = f\ dx + g\ dy + h\ dz$,

$$d\sigma = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz \wedge dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy \tag{12.4}$$

(iii) For a 2-form, $\omega = f\ dy \wedge dz + g\ dz \wedge dx + h\ dx \wedge dy$,

$$d\omega = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dx \wedge dy \wedge dz \tag{12.5}$$

(iv) For higher-order forms, ω, $d\omega = 0$.

Except for the fact that d operates on form fields instead of vector fields, it appears that the operator d unifies and generalizes the differential operators, grad, curl, and div of vector analysis.

To obtain the relation between d and grad we define a nondegenerate symmetric $(2,0)$ tensor field, $b\colon M \to \mathbf{S}^2(TM)$, on $\mathbb{R}^3$ by $b(dx, dx) = 1$, $b(dx, dy) = 0$, etc. Corresponding to b we have the linear mapping, $b^\sharp$, from $\mathfrak{X}^*$ to $\mathfrak{X}$ given by $b^\sharp : \sigma \mapsto X$ where $X : \tau \mapsto b(\tau, \sigma)$ (cf. Problem 6.16). Now we define grad f by grad $f = b^\sharp \cdot df$. By eq. (12.3)

$$\begin{aligned} b^\sharp \cdot df &= \frac{\partial f}{\partial x} b^\sharp \cdot dx + \frac{\partial f}{\partial y} b^\sharp \cdot dy + \frac{\partial f}{\partial z} b^\sharp \cdot dz \\ &= \frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial}{\partial z} \end{aligned}$$

so

$$\operatorname{grad} f = \frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial}{\partial z}$$

(In vector analysis notation $\partial/\partial x = \mathbf{i}$, $\partial/\partial y = \mathbf{j}$, and $\partial/\partial z = \mathbf{k}$.)

To obtain the relation between d and curl and the relation between d and div, we fix the order of the basis as (dx, dy, dz) and extend the Hodge star operator on the exterior algebra $\bigwedge V^*$ defined in Section 6.3 to an operator on the algebra of differential forms on $\mathbb{R}^3$. Then from (12.4) we get

$$*d\sigma = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dx + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dy + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dz$$

i.e., $*d$ maps 1-forms to 1-forms. From (12.5) we get

$$*d*\sigma = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

i.e., $*d*$ maps 1-forms to 0-forms. Finally, using $b^\sharp$ we obtain curl and div from $*d$ and $*d*$ respectively.

The linear operator $*d*$ on forms on $\mathbb{R}^3$ which we constructed in this example can be applied more generally to forms on $\mathbb{R}^n$ and takes s-forms to $s-1$ forms. We can go even further and apply it on certain manifolds.

Definitions On an orientable (Section 12.2) manifold with a metric structure (Section 15.1) we have *the codifferential operator* $\delta = (\pm 1) * d*$, on the algebra of differential forms, taking s-forms to $s-1$ forms. The coefficient, (± 1), of $*d*$ depends on the index of the metric tensor, the dimension of M, and the degree of the form. $\Delta = \delta d + d\delta$ is called *the Laplace-Beltrami operator* and takes s-forms to s-forms. A form, σ, for which $\Delta\sigma = 0$, is called *a harmonic form.* This notation and terminology generalizes that for functions on 3-dimensional Euclidean space (Section 15.1).

Recall that we defined $i_X\omega$ by giving its values on vector fields in terms of the values of ω (i.e., $i_X\omega(X_1, \ldots, X_{s-1}) = s\omega(X, X_1, \ldots, X_{s-1})$). We can describe the exterior differential of ω, $d\omega$, by its values on vector fields in terms of the values of ω on the vector fields and their Lie brackets.

Theorem 12.2 (i) *If ω is a 0-form,*

$$d\omega(X) = X\omega$$

(ii) *If ω is a 1-form,*

$$d\omega(X, Y) = \frac{1}{2}(X\omega \cdot Y - Y\omega \cdot X - \omega \cdot [X, Y])$$

(iii) *If ω is an s-form,*

$$d\omega(X_1, \ldots, X_{s+1}) = \frac{1}{s+1}\left(\sum_1^{s+1}(-1)^{i+1}X_i\omega(X_1, \ldots, \widehat{X}_i, \ldots, X_{s+1})\right.$$

$$\left. + \sum_{i<j}^{s+1}(-1)^{i+j}\omega([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{s+1})\right)$$

($\widehat{X}_i$ means that the argument is deleted.)

Proof (i) Directly from the definitions of df and Xf.

(ii) Using a coordinate system and the $\mathbb{R}$-linearity of d we can restrict ourselves to $\omega = f\ d\mu$. For these,

$$d\omega(X, Y) = (df \wedge d\mu)(X, Y) = \frac{1}{2}(df \otimes d\mu - d\mu \otimes df)(X, Y)$$

$$= \frac{1}{2}(df \cdot X\ d\mu \cdot Y - d\mu \cdot X\ df \cdot Y)$$

$$= \frac{1}{2}(Xf\ Y\mu - X\mu\ Yf) \tag{12.6}$$

Further,

$$X\langle\omega, Y\rangle = X\langle f\ d\mu, Y\rangle = X(fY\mu) = Xf\ Y\mu + fX\ Y\mu \tag{12.7}$$

Similarly,

$$-Y\langle\omega, X\rangle = -X\mu\ Yf - fY\ X\mu \tag{12.8}$$

Now form $2d\omega(X, Y) - X\langle\omega, Y\rangle + Y\langle\omega, X\rangle$. Replacing the terms in this expression by their expansions in (12.6), (12.7), and (12.8) we get $-fX\ Y\mu + fY\ X\mu = -f[X, Y]\mu = -f\ d\mu[X, Y] = -\omega \cdot [X, Y]$.

(iii) Problem 12.4. □

Definitions An s-form, θ, is (i) *closed* if $d\theta = 0$; (ii) *exact* if $\theta = d\omega$ for some ω; (iii) *locally exact* if $\theta = d\omega$ for some neighborhood of each point in the domain of θ.

The corollary of Theorem 12.1 says that if θ is exact then θ is closed. (This is sometimes called the Poincaré lemma, and sometimes its converse is so designated, see Theorem 12.3). In terms of the differential operator, d, θ is closed if $\theta \in \ker d$ and θ is exact if θ is in the image of d; cf., Problem 2.29. If the difference of two forms is exact, then we say they are related by *a gauge transformation* or are *cohomologous.*

In the example above, eq. (12.3) gives $d\ df = 0$ by direct computation. We can express this by saying that, if a form is the "gradient" of a function, then its "curl" vanishes. Similarly, from eq. (12.4), $d\ d\sigma = 0$ says that if a form is the "curl" of another form then its "divergence" vanishes. These are familiar results in vector analysis.

Theorem 12.3 *If θ is closed, then it is locally exact. More specifically, if θ is a closed s-form, then at each point there exists a cubical coordinate neighborhood, $\mathcal{U} = \{p : a^i < \mu^i(p) < b^i\}$ on which there exists an s-1 form ω such that $d\omega = \theta$.*

Outline of proof We exhibit a linear operator, H, on the algebra of differential forms on $\mathcal{U}$ taking s forms to s-1 forms and such that $H\ d\theta + d\ H\theta = \theta$. Such an operator is given by $H : \theta \mapsto H\theta$ where if $\theta : p \mapsto g(p) d\mu^1 \wedge \cdots \wedge d\mu^s$, then

$$H\theta : p \mapsto \left(\int_0^1 t^{s-1} (g \circ \Psi)(t,p) dt \right) i_X\ d\mu^1 \wedge \cdots \wedge d\mu^s$$

where $X = \mu^1 \frac{\partial}{\partial \mu^1} + \cdots + \mu^s \frac{\partial}{\partial \mu^s}$ and Ψ is the map from $[(0,1)] \times \mathcal{U}$ to $\mathcal{U}$ given by $\widehat{\Psi} : (t, \mu(p)) \mapsto t\mu(p) + (1-t)\mu(p_0)$. ($\Psi$ is a "homotopy" from the constant map $f_0(p) = \Psi(0,p) = p_0$ to the identity $f_1(p) = \Psi(1,p) = p$.) (See Singer and Thorpe, pp. 115 ff.) (Also note that this method is simply an abstraction and generalization of the classical method; see Kaplan, pp. 281-283.) □

In Section 11.4 we saw that a differentiable mapping $\phi : M \to N$ can induce mappings of tensor fields. In particular, for a covariant tensor field there is a pull-back, and for a differential form ω we have $\phi^*\omega$. Now we can generalize Theorem 11.21.

Theorem 12.4 *$\phi^* d\omega = d\phi^*\omega$. That is, exterior differentiation commutes with ϕ^*, or d is invariant under differentiable maps, ϕ.*

Proof Note that ϕ^* is linear, so we need only consider monomials for ω. Proceeding by induction we have

(i) for $\omega = f\ d\mu$,

$$\begin{aligned}\phi^* d\omega &= \phi^*(df \wedge d\mu) = \phi^* df \wedge \phi^* d\mu \\ &= d\phi^* f \wedge d\phi^* \mu = d(\phi^* f \wedge d\phi^* \mu) = d(\phi^* f \wedge \phi^* d\mu) \\ &= d\phi^*(f\ d\mu) = d\phi^* \omega\end{aligned}$$

(ii) Let $\omega = \omega_1 \wedge \omega_2$ where ω_1 is of order 1 and ω_2 is of order s. Then

$$\begin{aligned}\phi^* d\omega &= \phi^*(d\omega_1 \wedge \omega_2 - \omega_1 \wedge d\omega_2) \\ &= \phi^*(d\omega_1) \wedge \phi^* \omega_2 - \phi^* \omega_1 \wedge \phi^*(d\omega_2) \\ &= d\phi^* \omega_1 \wedge \phi^* \omega_2 - \phi^* \omega_1 \wedge d\phi^* \omega_2 = d(\phi^* \omega_1 \wedge \phi^* \omega_2) \\ &= d\phi^*(\omega_1 \wedge \omega_2) = d\phi^* \omega\end{aligned}$$

□

PROBLEM 12.1 Show that the components of $d\omega$ are

$$\frac{1}{s!}\sum_{\pi}(\text{sgn } \pi)\frac{\partial \omega_{i_{\pi(2)} \cdots i_{\pi(s+1)}}}{\partial \mu^{i_{\pi(1)}}}$$

and that they transform properly under coordinate transformations.

PROBLEM 12.2 Let f and g be functions on $\mathbb{R}^3$, and σ and τ be 1-forms on $\mathbb{R}^3$. Define dot and cross products by $\sigma \cdot \tau = b(\sigma, \tau)$ $(= *(\sigma \wedge *\tau) = *(\tau \wedge *\sigma))$, see Section 6.3) and $\sigma \times \tau = *(\sigma \wedge \tau)$ respectively, and write Grad, Curl, and Div for d, $*d$, and $*d*$ respectively. Prove the vector analysis formulas
(i) Grad $fg = g$ Grad $f + f$ Grad g
(ii) Curl $f\sigma =$ Grad $f \times \sigma + f$ Curl σ
(iii) Div $f\sigma =$ Grad $f \cdot \sigma + f$ Div σ
(iv) Div $\sigma \times \tau =$ Curl $\sigma \cdot \tau - \sigma \cdot$ Curl τ

PROBLEM 12.3 (i) If σ is a 1-form on $\mathbb{R}^3$, and ω is a 2-form on $\mathbb{R}^3$ and Δ is the Laplace-Beltrami operator, express σ and ω in terms of the natural basis of $\mathfrak{X}^*$ and compute $\Delta\sigma$ and $\Delta\omega$. (ii) Show that for a 1-form using the terminology in Problem 12.2, $\Delta\sigma =$ Grad Div $\sigma -$ Curl Curl σ.

PROBLEM 12.4 Prove part (iii) of Theorem 12.2.

12.2 Integration of differential forms

There are two apparently different types of integration situations that occur in calculus. In one case we want to integrate a given function over a given "3-dimensional" or "solid" subset of $\mathbb{R}^3$, and in the other case we form surface, or line integrals over surfaces, or curves, respectively, in $\mathbb{R}^3$.

While the first case generalizes "automatically" to $\mathbb{R}^n$, the generalization of the second case by the classical treatment is not easy (cf., Coburn, p. 196 ff). A presentation that simultaneously generalizes the second case to $\mathbb{R}^n$ and unifies the two situations would seem to be desirable. We can actually go beyond $\mathbb{R}^n$ to quite general differentiable manifolds. However, on manifolds it turns out that the second case, line and surface integrals, generalizes more naturally than the first, apparently simpler, case. Let us see what difficulties arise when we try to generalize the latter.

First, recall that in $\mathbb{R}^n$ for the latter case, given a certain class of subsets, S, (e.g., bounded sets whose boundary has zero content), and a certain class of functions, f (e.g., bounded continuous functions), we can define an integral as a function $\int : (S, f) \mapsto \int_S f$. This function has certain properties (e.g., linear in f, additive in S).

We want to define a concept on manifolds which reduces to this in the case $M = \mathbb{R}^n$. If $\mathbb{R}^n$ is to be a special case of M, we have to think of it as a manifold, and not just a set of n-tuples, and any concept defined on $\mathbb{R}^n$ as a manifold has to be independent of coordinates.

We have to define $\int_S f$ on $\mathbb{R}^n$ as a manifold, so that if $(\mathcal{U}, \mu)$ is the natural coordinate system, and hence any coordinate system, on $\mathbb{R}^n$, and $S \subset \mathcal{U}$, then $\int_S f = \int_{\mu(S)} f_\mu$, where f_μ is the representative of f on $\mu(S)$. That is, we require that $\int_{\mu(S)} f_\mu = \int_{\bar{\mu}(S)} f_{\bar{\mu}}$ for any two coordinate systems with $S \subset \mathcal{U} \cap \bar{\mathcal{U}}$. But by the "change of variables" theorem in the integral calculus on $\mathbb{R}^n$,

$$\int_{\bar{\mu}(S)} f_{\bar{\mu}} = \int_{\mu(S)} f_{\bar{\mu}} \circ (\bar{\mu} \circ \mu^{-1}) |\det J_{\bar{\mu} \circ \mu^{-1}}| \tag{12.9}$$

(cf., Bartle, p. 335) which, in general, is not $\int_{\mu(S)} f_\mu$. Clearly, the same situation occurs for any manifold, M, if we try to define $\int_S f$ in terms of representations via coordinate systems. So "the integral of a function" is not coordinate-invariant, and hence we cannot expect to be able to integrate *functions* on M.

Equation (12.9) suggests the proper way to proceed, if we write it in terms of iterated integrals, namely

$$\underset{\bar{\mu}(S)}{\int \cdots \int} f(y^1, \ldots, y^n)\, dy^1 \cdots dy^n$$

$$= \int_{\mu(S)} \cdots \int f(y^1(x^1 \cdots x^n), \ldots, y^n(x^1, \ldots, x^n)) \left| \det \left(\frac{\partial y^i}{\partial x^j} \right) \right| dx^1 \cdots dx^n \tag{12.10}$$

Recalling the transformation formula

$$d\bar{\mu}^1 \wedge \cdots \wedge d\bar{\mu}^n = \det \left(\frac{\partial \bar{\mu}^i}{\partial \mu^j} \right) d\mu^1 \wedge \cdots \wedge d\mu^n$$

(Problem 11.11) for n-forms; or slightly more generally, for a function, g,

$$g\, d\bar{\mu}^1 \wedge \cdots \wedge d\bar{\mu}^n = g \det\left(\frac{\partial \bar{\mu}^i}{\partial \mu^j}\right) d\mu^1 \wedge \cdots \wedge d\mu^n \tag{12.11}$$

we see that the integrands in eq. (12.10) transform something like n-forms. This suggests trying to define integrals of n-forms rather than of functions. This will lead, in particular, to a generalization of line and surface integrals of calculus, the second case considered above. The first case, the integration of *functions*, requires additional structure, and will be discussed in Section 15.1.

We first make a definition for n-forms on $\mathbb{R}^n$.

Definition If ω is an n-form on $\mathbb{R}^n$, then on any coordinate system on $\mathbb{R}^n$, it can be written $\omega = g dy^1 \wedge \cdots \wedge dy^n$. In particular, if $x^1, \ldots, x^n$ are the natural coordinate functions on $\mathbb{R}^n$, then $\omega = f dx^1 \wedge \cdots \wedge dx^n$. Now we define

$$\int_{\mathbb{R}^n} f dx^1 \wedge \cdots \wedge dx^n = \int_{\mathbb{R}^n} f$$

the right side being the usual integral defined on $\mathbb{R}^n$, if it exists. Then by (12.10) and (12.11), it makes sense to define

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f \tag{12.12}$$

Now we proceed to define the integral of an n-form, θ, over an n-dimensional manifold, M, in two steps: first, when supp θ is contained in a coordinate neighborhood, $\mathcal{U}$, and then for any θ of compact support (supp $\theta =$ supp f independent of coordinates). In the first step we "pull back" the integral to $\mathbb{R}^n$ by means of coordinates, and in the second step we define the integral as a sum of integrals over a covering of M by coordinate neighborhoods. In both steps we will need to prove that the definitions make sense.

So, to start, we suppose that supp $\theta \subset \mathcal{U}$, where $(\mathcal{U}, \mu)$ is a chart. Then we put

$$\int_M \theta = \int_{\mathbb{R}^n} \omega \tag{12.13}$$

where

$$\omega = \begin{cases} (\mu^{-1})^* \theta|_{\mathcal{U}} & \text{on } \ \mu(\mathcal{U}) \\ 0 & \text{outside of } \ \mu(\mathcal{U}) \end{cases}$$

so the right side of eq. (12.13) is defined by eq. (12.12).

Somewhat more explicitly, writing $\theta|_{\mathcal{U}} = f\ d\mu^1 \wedge \cdots \wedge d\mu^n$, then

$$(\mu^{-1})^*\theta|_{\mathcal{U}} = (\mu^{-1})^* f\ d\mu^1 \wedge \cdots \wedge d\mu^n = f_\mu (\mu^{-1})^* d\mu^1 \wedge \cdots \wedge d\mu^n$$
$$= f_\mu \frac{\partial(\mu^{-1})^1}{\partial x^{i_1}} dx^{i_1} \wedge \cdots \wedge \frac{\partial(\mu^{-1})^n}{\partial x^{i_n}} dx^{i_n}$$

by eq. (9.10), where x^i are the natural coordinate functions on $\mathbb{R}^n$. But $(\mu^{-1})^i = \mu^i \circ \mu^{-1} = x^i$, so $(\mu^{-1})^*\theta|_{\mathcal{U}} = f_\mu dx^1 \wedge \cdots \wedge dx^n$ and eq. (12.13) can be written

$$\int_M \theta = \int_{\mu(\mathcal{U})} f_\mu dx^1 \wedge \cdots \wedge dx^n \tag{12.14}$$

Since the definition, eq. (12.13), and the description, eq. (12.14), are given in terms of a coordinate system, it is necessary to show they are actually independent of coordinates.

Theorem 12.5 *If $(\mathcal{U}, \mu)$ and $(\bar{\mathcal{U}}, \bar{\mu})$ are two coordinate systems, and* supp $\theta \subset \mathcal{U} \cap \bar{\mathcal{U}}$, *then $\int_M \theta$ has the same value for both coordinate systems if and only if* det $(\partial\bar{\mu}^i/\partial\mu^j) > 0$.

Proof Using eq. (12.14), we have for the $(\bar{\mathcal{U}}, \bar{\mu})$ coordinates,

$$\int_M \theta = \int_{\bar{\mu}(\bar{\mathcal{U}})} g_{\bar{\mu}}\ dy^1 \cdots dy^n = \int_{\bar{\mu}(\mathcal{U}\cap\bar{\mathcal{U}})} g_{\bar{\mu}}\ dy^1 \cdots dy^n$$
$$= \int_{\mu(\mathcal{U}\cap\bar{\mathcal{U}})} g_{\bar{\mu}} \det\left(\frac{\partial y^i}{\partial x^j}\right) dx^1 \cdots dx^n$$

by the "change of variables" theorem, if and only if $\det(\partial y^i/\partial x^j) > 0$. But by eq. (12.11), $g_{\bar{\mu}} \det(\partial y^i/\partial x^j) = f_\mu$, so

$$\int_{\mu(\mathcal{U}\cap\bar{\mathcal{U}})} g_{\bar{\mu}} \det\left(\frac{\partial y^i}{\partial x^j}\right) dx^1 \cdots dx^n = \int_{\mu(\mathcal{U}\cap\bar{\mathcal{U}})} f_\mu\ dx^1 \cdots dx^n$$
$$= \int_{\mu(\mathcal{U})} f_\mu\ dx^1 \cdots dx^n$$

□

Since the condition $\det(\partial\bar{\mu}^i/\partial\mu^j) > 0$ cannot be satisfied for all coordinate systems whose coordinate neighborhoods contain supp θ, we need to impose further conditions in order to define the concept of an integral over M. It will turn out that we can define an "oriented integral" over a manifold which has an atlas with the property $\det(\partial\nu^i/\partial\mu^j) > 0$ for all coordinate systems in the atlas.

Definitions If there exists on M an atlas such that for all coordinate systems $\det(\partial\mu^i/\partial\nu^j) > 0$ then M is called *orientable* and the atlas is *an oriented atlas.* An n-form on M which never vanishes is called *a volume form.*

Theorem 12.6 *M has an oriented atlas if and only if M has a volume form.*

Proof Problem 12.5. □

Clearly, if M has a volume form it has others. Also, for every coordinate system, (μ^i), there is a coordinate system (ν^i) such that $\det(\partial\mu^i/\partial\nu^j) < 0$. Thus, if M is orientable we have two classes of atlases for which $\det(\partial\mu^i/\partial\nu^j) > 0$. If we choose one, we say that we *orient* M, and that M is then *an oriented manifold.*

Another way of describing this property of a manifold is that it is possible to order the bases of the tangent spaces the same way at all points. That is, for all points and for all charts, the bases, $\{\mu_*\partial/\partial\mu^1, \ldots, \mu_*\partial/\partial\mu^n\}$, of the tangent spaces of $\mathbb{R}^n$ all have the same order. That this property is equivalent to the definition is proved by using the relation $d\mu^1 \wedge \cdots \wedge d\mu^n = \det(\partial\mu^i/\partial\nu^j)d\nu^1 \wedge \cdots \wedge d\nu^n$. With this more "geometric" way of describing orientability, we can convince ourselves of the non-orientability of the standard example, the Möbius Strip.

Definition If M is an oriented n-dimensional manifold, $(\mathcal{U}, \mu)$ is any chart in the orientation of M, and θ is an n-form with supp $\theta \subset \mathcal{U}$, then $\int_M \theta$ is defined by eq. (12.13).

Now that we have a clearly defined concept of the integral over an oriented manifold, M, of an n-form whose support is contained in a coordinate neighborhood, we can extend the definition to encompass the more general case of n-forms with compact support.

Theorem 12.7 *If θ has compact support in M, then there exist functions $f_1, \ldots, f_k$ on M such that*

$$\theta = f_1\theta + \cdots + f_k\theta \tag{12.15}$$

where each $f_i\theta$ has support in a coordinate neighborhood of M.

Proof Since supp θ is compact, we can cover M by M-supp θ and coordinate neighborhoods $\mathcal{U}_1, \ldots, \mathcal{U}_k$. Since M is paracompact it has a partition of unity $f_1, \ldots, f_n$ subordinate to this cover. (The need to extend functions or forms defined locally to ones defined globally is the main reason for the assumption of paracompactness for manifolds.) □

Theorem 12.8 *The representation* (12.15) *of* θ *is not unique, but*

$$\sum_i \int_M f_i\theta = \sum_i \int_M g_i\theta$$

so we can define the integral of an n-form , θ, of compact support over an oriented n-dimensional manifold M by

$$\int_M \theta = \sum_i \int_M f_i\theta \tag{12.16}$$

for any covering and any partition of unity.

Proof If $\{g_i\}$ is a partition of unity with supp $g_j \subset \mathcal{V}_j$ for $j = 1, \ldots, l$ and $g_j = 0$ for $j > l$ then

$$\theta = g_1\theta + \cdots + g_l\theta$$

and

$$g_j\theta = f_1 g_j\theta + \cdots + f_k g_j\theta \tag{12.17}$$

for each j and supp $f_i g_j \subset \mathcal{V}_j$. Integrating (12.17), and summing on j we get

$$\sum_{j=1}^{l} \int g_j\theta = \sum_{j=1}^{l}\sum_{i=1}^{k} \int f_i g_j\theta = \sum_{i=1}^{k} \int f_i\theta$$

□

We must emphasize at this point that, from the point of view of generalizing integration on $\mathbb{R}^n$, our development so far has been very limited. Specifically, we usually want to integrate over given subsets of $\mathbb{R}^n$ in ordinary calculus - not all of $\mathbb{R}^n$ - while so far all we have defined is an integral over all of M, and not over subsets of M. Of course, for subsets of M that are manifolds, the above development applies.

Whether or not we can integrate over a given subset of $\mathbb{R}^n$ depends to a large extent, as we noted at the beginning of this section, on the nature of its boundary. As in ordinary calculus, we have to describe sufficient conditions for integrability on subsets of M.

When we generalize to manifolds we can take either of two essentially equivalent approaches (cf., Boothby, pp. 251-252); we can start with a given open submanifold of M with a smooth enough boundary or we can generalize the concept of a manifold to that of *a manifold with boundary* and deal with subsets of M intrinsically. We will take the former approach. The latter is indicated in Problem 12.7.

Definitions *A regular domain in M* is a set of points, D, in M such that its closure consists of (1) points, p, having a neighborhood of M in D, or (2) points, p, having a chart of M such that for $q \in \mathcal{U} \cap \bar{D}$, $\mu^n(q) \geq \mu^n(p)$. Points of type (1) are *interior points* and points of type (2) are called *boundary points*. Clearly the two sets are disjoint.

Theorem 12.9 *If D is a regular domain in M, then the boundary, ∂D, of D is an $n-1$ dimensional imbedded submanifold of M.*

Proof First of all, in the coordinate system $(\mathcal{U}, \mu)$ the set of boundary points of D is precisely the set of points q such that $\mu^n(q) = \text{const}$. That is, $\mathcal{U} \cap \partial D$ is an $n-1$ dimensional coordinate slice of $\mathcal{U}$. So ∂D has the $n-1$ dimensional submanifold property and we can apply Theorem 10.6. □

Theorem 12.10 *If M is orientable, and D is a regular domain in M, then ∂D is orientable and the orientation of ∂D can be described in terms of that of D.*

Proof On the intersection of the coordinate neighborhoods $\mathcal{U}$ and $\bar{\mathcal{U}}$ the Jacobian matrix of $\bar{\mu} \circ \mu^{-1}$ is $(\partial \bar{\mu}^i / \partial \mu^j)$. But $\bar{\mu}^n$ is constant along the boundary, given by $\mu^n = \text{constant}$, so along that boundary $\partial \bar{\mu}^n / \partial \mu^i = 0$ for $i = 1, \ldots, n-1$. Hence,

$$\left(\frac{\partial \bar{\mu}^i}{\partial \mu^j}\right) = \begin{bmatrix} \dfrac{\partial \bar{\mu}^1}{\partial \mu^1} & \dfrac{\partial \bar{\mu}^2}{\partial \mu^1} & \cdots & 0 \\[2ex] \dfrac{\partial \bar{\mu}^1}{\partial \mu^2} & \cdots & \cdots & 0 \\[2ex] \vdots & \cdots & \cdots & \vdots \\[2ex] \dfrac{\partial \bar{\mu}^1}{\partial \mu^n} & \cdots & \cdots & \dfrac{\partial \bar{\mu}^n}{\partial \mu^n} \end{bmatrix}$$

and using the coordinates $(\mathcal{V}, \nu)$ where $\mathcal{V} = \mathcal{U} \cap \partial D$ and $\nu = \mu \circ id\big|_{\partial D}$ on ∂D we get

$$\det\left(\frac{\partial \bar{\mu}^i}{\partial \mu^j}\right) = \det\left(\frac{\partial \bar{\nu}^i}{\partial \nu^j}\right) \frac{\partial \bar{\mu}^n}{\partial \mu^n}$$

Since the interior of $\mu(\mathcal{U} \cap \bar{D})$ maps into the interior of $\bar{\mu}(\mathcal{U} \cap \bar{D})$ and both nth coordinates increase as we go into the interior from the boundary, $\partial \bar{\mu}^n / \partial \mu^n > 0$, and so ∂D is orientable if M is and with the coordinates $(\mathcal{V}, \nu)$, ∂D has the induced orientation. □

Definition If M is oriented, θ has compact support on M, D is a regular domain in M, and c_D is the characteristic function of D, then

$$\int_D \theta = \int_M c_D \theta \tag{12.18}$$

is *the integral of* θ *over* D. (The integral on the right exists because on each chart $\mu(\mathcal{U} \cap \partial D)$ has content zero.)

Now that we have defined the concept of a regular domain, its boundary, and the integral over a regular domain of an n-form of compact support, we can prove basic integration theorems for manifolds.

The way in which $\int_D \theta$ transforms under a mapping of M is given by:

Theorem 12.11 *If ϕ is an orientation preserving diffeomorphism of M and D is a regular domain in M, then*

$$\int_D \phi^* \theta = \int_{\phi(D)} \theta \tag{12.19}$$

Proof We have to show $\int_M c_D \phi^* \theta = \int_M c_{\phi(D)} \theta$. If $\mathcal{U}_1, \ldots, \mathcal{U}_n$ are coordinate neighborhoods covering supp $c_D \phi^* \theta$, then $\mathcal{V}_1 = \phi(\mathcal{U}_1), \ldots, \mathcal{V}_n = \phi(\mathcal{U}_n)$ will cover supp $c_{\phi(D)} \theta$. If we use a partition of unity subordinate to each of these covers, the integrals on both sides can be written as sums of integrals of the forms $\int_M \phi^* \beta_i$ and $\int_M \beta_i$ where $\phi^* \beta_i$ are forms with support in $\mathcal{U}_i$ and β_i are forms with support in $\mathcal{V}_i$. But, for each i, we have

$$\int_M \phi^* \beta_i = \int_{\mu_i(\mathcal{U}_i)} \phi^* g_i dx^1 \wedge \cdots \wedge dx^n \tag{12.20}$$

and

$$\int_M \beta_i = \int_{\nu_i(\mathcal{V}_i)} g_i dx^1 \wedge \cdots \wedge dx^n \tag{12.21}$$

Now, if we think of the mapping $\hat{\phi}$ in $\mathbb{R}^n$ as a coordinate transformation, and the values of x^i on $\nu_i(\mathcal{V}_i)$ as the values of coordinates y^i on $\nu_i(\mathcal{V}_i)$, then on $\nu_i(\mathcal{V}_i)$, $g_i(x) dx^1 \wedge \cdots \wedge dx^n$ becomes $g_i(y) dy^1 \wedge \cdots \wedge dy^n$, $\int_{\mu_i(\mathcal{U}_i)} \phi^* g_i dx^1 \wedge \cdots \wedge dx^n$ becomes $\int_{\mu_i(\mathcal{U}_i)} g(y(x)) \det \left(\frac{\partial y^i}{\partial x^j} \right) dx^1 \cdots dx^n$ and $\int_{\nu_i(\mathcal{V}_i)} g_i dx^1 \ldots dx^n$ becomes $\int_{\nu_i(\mathcal{V}_i)} g_i(y) dy^1 \ldots dy^n$ so by the "change of variables" theorem, eq. (12.10), the integrals in (12.20) and (12.21) are the same. □

Evidently, Theorem 12.11 can also be thought of as a generalization to manifolds of the "change of variables" theorem in $\mathbb{R}^n$.

Theorem 12.12 (Stokes') *Suppose D is a regular domain of an n-dimensional, oriented manifold, M, and θ is an $n-1$ form on $\bar{D}$ with compact support. Then*

$$\int_D d\theta = (-1)^n \int_{\partial D} id\big|_{\partial D}^{*}\theta \tag{12.22}$$

In eq. (12.22) $id\big|_{\partial D}$ *is the inclusion map,* $id\big|_{\partial D} : \partial D \to \bar{D}$*, and the orientation of ∂D is the induced orientation.*

Proof (Sketch) Just as in the theorem on local exactness, Theorem 12.3, the proof is based on the classical proof generalized to n dimensions and extended to manifolds. By compactness, we can reduce the proof to the consideration of one of the forms $f_i\theta$ with compact support in a coordinate neighborhood in Theorem 12.7. Then we pull it back to $\mu(\mathcal{U})$ and put it in a box whose edges are coordinate hypersurfaces in $\mathbb{R}^n$. If the box does not intersect the boundary, then both sides of eq. (12.22) are zero. If the box does intersect the boundary, the $n-1$ form ω on $\mathbb{R}^n$ can be written

$$\omega = \sum_{j=1}^{n} (-1)^{j-1} \omega_j dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n$$

so that

$$d\omega = \sum_{j=1}^{n} \frac{\partial \omega_j}{\partial x^j} dx^1 \wedge \cdots \wedge dx^n$$

We integrate these forms using the definition given in Eq. (12.12). Now in the integrations over the box of $d\omega$ and over the boundary points in the box of ω, all terms except the one with coefficient ω_n evaluated on $x^n = 0$ drop out, and the terms remaining are the same except for the factor $(-1)^n$. □

We can eliminate the factor $(-1)^n$ in Stokes' theorem if ∂D has *the compatible orientation*, that is, the induced orientation if n is even and the opposite orientation if n is odd.

The subject of integration on manifolds can be pursued further in at least two different directions. We will only indicate these.

1. With Stokes' theorem we have integration by parts for s-forms and then with the Laplace-Beltrami operator we get generalizations of Green's identities which carry us finally into a generalization of classical potential theory (cf. Woll, pp. 111 ff).
2. We want to be able to integrate over a class of subsets of M larger than the set of regular domains, since, in particular, regular domains have smooth boundaries. This is done by constructing "p-chains" with their algebra (cf. Spivak; Bishop and Goldberg; Warner). Then we extend the class of subsets over which we can integrate to the class of "parametrizable oriented subsets" of M (Bishop and Goldberg, p. 195).

PROBLEM 12.5 Prove Theorem 12.6.

PROBLEM 12.6 Show that S^n and $P^2(\mathbb{R})$ are orientable.

PROBLEM 12.7 The 2-sphere, $x^2 + y^2 + z^2 = 1$, in $\mathbb{R}^3$, can be parametrized by the 2-cube $c\colon (u, v) \mapsto (\cos u, \sin u \cos v, \sin u \sin v)$ on $C = [0, \pi] \times [0, 2\pi]$ so $\int_{S^2} \theta = \int_C c^*\theta$. Evaluate $\int_{S^2} \theta$ if

$$\theta = (x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy)/r^3$$

and $r = \sqrt{x^2 + y^2 + z^2}$.

PROBLEM 12.8 A *manifold with boundary* is defined precisely the same as a manifold except that $\mathbb{R}^n$ is replaced by H^n where $H^n = \{a \in \mathbb{R}^n : a^n \geq 0\}$ with the relative topology of $\mathbb{R}^n$. Let $\partial H^n = \{a \in \mathbb{R}^n : a^n = 0\}$ and $\partial M = \{p \in M : \mu(p) \in \partial H^n\}$ (Note that this is independent of μ). Show that ∂M is an $n-1$-dimensional manifold imbedded in M.

PROBLEM 12.9 Fill in the details of the proof of Stokes' theorem.

13

THE FLOW AND THE LIE DERIVATIVE OF A VECTOR FIELD

The concept of a vector field on a manifold, M (or, on an open subset, $\mathcal{U}$, of M), has a simple intuitive definition, but also has broad and deep ramifications. In particular, we want to look at vector fields from two rather different and important points of view. Roughly, (1) a vector field can be thought of as determining, or being determined by, a family of curves, and (2) a vector field can be thought of as determining, or being determined by, a certain derivation on tensor fields on M (or $\mathcal{U}$). The first correspondence well known for cartesian spaces, $\mathbb{R}^n$, as the subject matter of ordinary differential equations will be examined in Sections 13.1-13.3. The second correspondence, introduced in restricted form in Section 11.1, will be described in Section 13.4.

13.1 Integral curves and the flow of a vector field

In Chapters 7 and 9, a major theme was the relation between tangents at $p \in M$ and curves through p. Now we wish, in a sense, to extend this relation to vector (i.e., tangent) fields and families of curves. Thus, suppose we start with a given field, X.

Definition A curve, γ, whose values lie in the domain of a vector field, X, and such that

$$\dot{\gamma}(u) = X(\gamma(u)) \tag{13.1}$$

for all u in the domain of γ is *an integral curve of* X. (Recall, from Section 9.3, that $\dot{\gamma}(u)$ is the tangent, or velocity, of γ at $\gamma(u)$.)

Alternatively, γ is an integral curve of X if $\gamma_* \circ d/d\pi = X \circ \gamma$, or if the following diagram commutes:

$$\begin{array}{ccc} TR & \xrightarrow{\gamma_*} & TM \\ \uparrow{\scriptstyle \frac{d}{d\pi}} & & \uparrow{\scriptstyle X} \\ \mathbb{R} & \xrightarrow{\gamma} & M \end{array}$$

or if $\frac{d}{d\pi}$ and X are γ-related. $\dot{\gamma} = \gamma_* \circ d/d\pi$ is the canonical lift of γ, and $X \circ \gamma$ is a vector field over γ.

This definition simply describes in a coordinate-free manner a condition on curves more commonly described by a system of differential equations. Thus, if we introduce a local coordinate system $(\mathcal{U}, \mu)$, at a point, $\gamma(u)$, in the domain of X, then

$$\dot{\gamma}(u) = D_u \gamma^i \left.\frac{\partial}{\partial \mu^i}\right|_{\gamma(u)} \quad \text{where} \quad \gamma^i = \mu^i \circ \gamma \text{ by Problem 9.15}$$

and

$$X(\gamma(u)) = \left(X^i \frac{\partial}{\partial \mu^i}\right)(\gamma(u)) = X^i(\gamma(u)) \left.\frac{\partial}{\partial \mu^i}\right|_{\gamma(u)}$$

So, in terms of local coordinates, (13.1) becomes

$$D_u \gamma^i = X^i(\gamma(u)) \quad i = 1, \ldots, n$$

Finally, in terms of the coordinate representations, $\hat{X}^i$, of the functions $X^i = \hat{X}^i \circ \mu$, this becomes the system

$$D_u \gamma^i = \hat{X}^i(\gamma^1(u), \ldots, \gamma^n(u)) \quad i = 1, \ldots, n \tag{13.2}$$

of ordinary differential equations. Thus, (13.2) is the local expression of the fact that γ is an integral curve of X.

Now having imposed condition (13.1) (or (13.2)) with a given vector field, X, we ask about the existence and properties of curves satisfying it.

Theorem 13.1 *Given $u_0 \in \mathbb{R}$, and $(a_0^1, \ldots, a_0^n)$ in the domains of all the $\hat{X}^i$. Then there exists an interval containing u_0, and a unique set of functions γ^i (on that interval) such that $\gamma^i(u_0) = a_0^i$ and γ^i satisfy* (13.2) *on that interval.**

Proof See any book on ordinary differential equations. □

One can get more on the nature of the γ^i in terms of whatever properties of the $\hat{X}^i$ may be available, and, of course we get much more in special cases such as in the case of linearity.

From the results of Theorem 13.1 for $\mathbb{R}^n$ we get corresponding results for curves on M.

We can also describe these results in a different way. First of all we notice that (13.2) is *an autonomous system*, i.e., the functions $\hat{X}^i$ do not have the argument u. A consequence of this is the following.

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*Note that we have omitted a hypothesis in the statement of Theorem 13.1; i.e., conditions on the $\hat{X}^i$. $\hat{X}^i \in C^1$ will suffice. In general, when such conditions are omitted it is to be understood that whatever differentiability is required is assumed.

Theorem 13.2 (Admissible change of parameter for integral curves.) *If γ_1 is an integral curve of X and γ_2 has the same range as γ_1, then γ_2 is an integral curve of X iff their parameters are related by a translation.*

Proof The proof is based on the following lemma. □

Lemma *Suppose $\gamma_2 = \gamma_1 \circ \rho$, $S \subset \mathbb{R}$ and $\rho(S) \subset$ domain of γ_1. Then $\dot{\gamma}_2 = \dot{\gamma}_1 \circ \rho \Leftrightarrow \rho = \chi$ where $\chi(t) = t + t_0$.*

Proof of lemma $\gamma_2 = \gamma_1 \circ \rho \Rightarrow \dot{\gamma}_2(t) = \dot{\gamma}_1(\rho(t))D_t\rho$. Then $\dot{\gamma}_2(t) = \dot{\gamma}_1(\rho(t)) \Leftrightarrow D_t\rho = 1 \Leftrightarrow \rho = \chi$. □

Proof of theorem **If:** If $\rho = \chi$, then $\gamma_1(u) = \gamma_2(t) \Rightarrow \dot{\gamma}_1(u) = \dot{\gamma}_2(t)$ by the lemma. Then $\dot{\gamma}_1(u) = X(\gamma_1(u)) \Rightarrow \dot{\gamma}_2(t) = X(\gamma_2(t))$.

Only if: If $\dot{\gamma}_1(u) = X(\gamma_1(u))$ and $\dot{\gamma}_2(t) = X(\gamma_2(t))$ and $\gamma_2(t) = \gamma_1(\rho(t))$, then $\dot{\gamma}_1(\rho(t)) = \dot{\gamma}_2(t) \Rightarrow \rho = \chi$ by the lemma. □

From this theorem we see that there is no important loss of generality if we always take $u_0 = 0$; that is, consider only curves, γ_p, through p (where $p \in M$ is in the domain, E, of X and corresponds to $a_0 \in \mathbb{R}^n$ in Theorem 13.1). Now if I_1 is one interval with γ^i given by Theorem 13.1 and I_2 is another interval with α^i given by Theorem 13.1, then on $I_1 \cap I_2$, $\alpha^i = \gamma^i$ and we can define unique functions on $I_1 \cup I_2$ which satisfy the conditions of Theorem 13.1.* Hence, there is a largest interval, I_p (containing 0), on which γ_p, the integral curve through p, is defined, and we have, for each $p \in E$,

$$\gamma_p : I_p \to E \tag{13.3}$$

with $u \mapsto \gamma_p(u)$.

Let E_u be the subset of E for each of whose points, $p, \gamma_p(u)$ is defined. That is, in E_u, the integral curves through p given by the existence theorem can be extended at least as far as u. Thus, for each $u \in \mathbb{R}$ we have a map

$$\Theta_u : E_u \to E \tag{13.4}$$

given by $p \mapsto \gamma_p(u)$.

Now we can rephrase the results of our existence theorem for integral curves in the form (1) a vector field X defined on E determines a family, $\{\gamma_p\}_{p \in E}$ of curves through points of E, each of which is an integral curve of X; or

*This is not as obvious as it may seem. In fact, this is one of the points where the Hausdorff property of M is required.

(2) a vector field X defined on E determines a family $\{\Theta_u\}_{u\in\mathbb{R}}$ of maps of subsets of E; or finally (3) a vector field X defined on E determines a map

$$\Theta\colon W \to E \tag{13.5}$$

given by $(u,p) \mapsto \Theta(u,p) = \Theta_u(p) = \gamma_p(u)$ where W is some subset of $\mathbb{R}\times E$, Θ_u is given by (13.4), and γ_p is the integral curve of X through p.

Definitions The map, Θ, defined in (13.5) is called *the flow generated by the given vector field,* X, and we say that the set $\{\Theta_u\}$ is *a local 1-parameter group* of local transformations generated by X.

EXAMPLE $M = \mathbb{R}^2$ with the standard atlas $\{(\mathcal{U},\mu)\} = \{(\mathbb{R}^2, id)\}$ with the given vector field $X = (\pi^1)^2\partial/\partial\pi^1$. The integral curves of X are represented by the solutions (γ^1,γ^2) of

$$\frac{d\gamma^1}{du} = (\gamma^1)^2 \quad \frac{d\gamma^2}{du} = 0$$

The solutions through (a_0, b_0) are given by

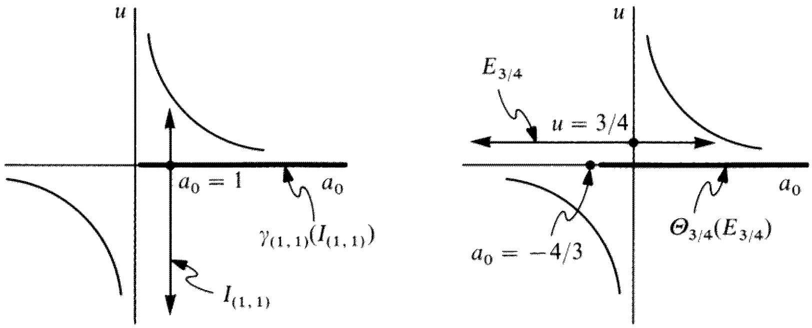
$$\gamma_{(a_0,b_0)}(u) = (\gamma^1_{(a_0,b_0)}(u), \gamma^2_{(a_0,b_0)}(u)) = \left(\frac{a_0}{1-ua_0}, b_0\right).$$

The maps Θ_u take (a_0, b_0) to $(a_0/(1-ua_0), b_0)$ and the flow is

$$\Theta\colon (u,(a_0,b_0)) \mapsto \left(\frac{a_0}{1-ua_0}, b_0\right)$$

The domain W of Θ is the set of pairs $(u,(a_0,b_0))$ where (u,a_0) is in the region of Fig. 13.1 between the two branches of the cylinder $ua_0 = 1$.

For example, if $(a_0,b_0) = (1,1)$, then the solution curve $\gamma_{(1,1)}$ is defined for $u \in (-\infty, 1)$. If $u = \frac{3}{4}$, then the domain, $E_{3/4}$, of $\Theta_{3/4}$ is the region of the (a_0b_0) plane with $a_0 < \frac{4}{3}$, and $\Theta_{3/4}$ takes $E_{3/4}$ into the piece of the plane in which $a_0 > -\frac{4}{3}$.



The section b_0 = constant

Figure 13.1

PROBLEM 13.1 Let $M = \mathbb{R}^2$ with the standard structure, and define X in terms of the natural coordinates $(\mathcal{U}, \mu) = (\mathbb{R}^2, id)$ by $X = \pi^1\, \partial/\partial\pi^1 + \pi^2\, \partial/\partial\pi^2$. Find the maximal integral curve through (a, b).

PROBLEM 13.2 Same as Problem 13.1 for $X = (\exp \circ \mu^1)^{-1}\, \partial/\partial\pi^1$.

PROBLEM 13.3 Find the representation in the given stereographic atlas of the maximal integral curve through a point of the vector field on S^2 given in the example in Section 11.1.

PROBLEM 13.4 If Θ is the flow of X, and if ϕ is a diffeomorphism of M, then the flow of $\phi_* X$ is given by the transformations $\phi \circ \Theta_u \circ \phi^{-1}$.

13.2 Flow boxes (local flows) and complete vector fields

The awkward situation in which the domain of Θ is not very well defined comes from the corresponding lack of information in Theorem 13.1 on the domain of the maximal integral curve through p. While we cannot do any better for individual integral curves, we can say a bit more when we consider families of curves.

Let S be any set in the common domain of the $\hat{X}^i$, and let $\mathcal{U} \subset S$ be the points of S which are centers of balls of radius r contained in S. Then a standard proof of Theorem 13.1 shows that if $\mathbf{K}$ is a bound for the $\hat{X}^i$ on their common domain, then for all $a_0 \in \mathcal{U}$ there is an integral curve through a_0 defined on the interval $(-r/\mathbf{K},\ r/\mathbf{K})$.

Definition If X is a vector field on $E \subset M$, a differentiable map $F : I \times \mathcal{V} \to M$ where $I = (-u_0, u_0) \subset \mathbb{R}$ and $\mathcal{V}$ is an open set in M is a *flow box* (or, *local flow*) of X if (1) for all $p \in \mathcal{V}$, $F_p : I \to M$ defined by $u \mapsto F(u,p)$ is an integral curve of X at p, and (2) for all $u \in I$, $F_u : \mathcal{V} \to M$ defined by $p \mapsto F(u,p)$ is a diffeomorphism of $\mathcal{V}$ onto $F_u(\mathcal{V})$. Briefly, a flow box is the restriction of a flow to a domain which is a product, and the maps $\Theta_u\big|_{\mathcal{V}}$ are diffeomorphisms.

For the example in Section 13.1 we have a flow box with $I = (-2,2)$, $\mathcal{V} = \{(a_0, b_0) : 0 < a_0 < \frac{1}{2},\ b_1 < b < b_2\}$ and $F : (u, (a_0, b_0)) \mapsto (a_0/(1 - ua_0), b_0)$.

Theorem 13.3 *For every p in the domain of X there is a flow box of X containing p.*

Proof Given p, choose a local coordinate system $(\mathcal{U}, \mu)$. The comments above the definition imply that there is a flow box, $\hat{F}$, of $\hat{X}$ on $\mu(\mathcal{U})$.* If $\hat{F} : I \times \hat{\mathcal{V}} \to \mu(\mathcal{U})$, then $F : I \times \mu^{-1}(\hat{\mathcal{V}}) \to \mathcal{U}$ given by $(u,p) \mapsto \mu^{-1}(\hat{F}(u, \mu(p)))$ will be a flow box of X containing p. □

Theorem 13.4 *If the values of the maximal integral curve, γ_p, through p of X are contained in a compact subset, C, of M, then γ_p is defined for all of $\mathbb{R}$.*

Proof Suppose the domain of γ_p, is (a,b). Let (b_n) be a sequence converging to b. Then since C is compact, there is a subsequence (b_{n_k}) such that $\gamma_p(b_{n_k}) \to q \in C$. Let $F : I \times \mathcal{V} \to M$ be a flow box at q with $I = (-c, c)$. All $\gamma_p(b_{n_k})$ will be in $\mathcal{V}$ for large enough k. We can find a point $x = \gamma_p(b_{n_k})$ on γ_p such that $|b_{n_k} - b| < c$. If F_x is the integral curve of the flow box at x, and $\alpha_x(t) \equiv \gamma_p(t + b_{n_k})$, then $F_x = \alpha_x$ where they are both defined. Since F_x is defined on a larger interval than α_x (F_x is defined on $(-c,c)$, and α_x is defined for $t + b_{n_k} < b$, or $t < b - b_{b_k} < c$), α_x is not the maximal integral curve through x, and hence γ_p is not the maximal integral curve through p. □

Definition A vector field, X, is *complete* if the domains of all its integral curves can be extended to all of $\mathbb{R}$.

The following is an immediate consequence of Theorem 13.4.

Corollary *A vector field on a compact manifold is complete.*

Note that if $M = \mathbb{R}^2$ (with the standard structure) and X is given by $X = X^1\ \partial/\partial\mu^1 + X^2\ \partial/\partial\mu^2$ where X^1 and X^2 are bounded functions on $\mathbb{R}^2$, then X is complete. However, if $M = \mathbb{R}^2 - (0,0)$ and $X = \partial/\partial\mu^1 + \partial/\partial\mu^2$, then X is not complete. Also, in the example of Problem 13.1 X is complete, and in the example of Problem 13.2 X is not complete.

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*From the comments above the definition, property 1 is clear, but property 2 is not obvious. We need continuity and differentiability of solutions of differential equations *with respect to initial conditions*. The hypothesis $X \in C^1$ will suffice. (See Abraham and Marsden, pp. 63 ff.)

Theorem 13.5 *X is complete iff the domain, W, of the flow, Θ, of X, is $\mathbb{R}\times E$.*

Proof Problem 13.7. □

Theorem 13.6 *If X is complete, then the flow, Θ, of X has the properties*

(1) $\Theta_0 = id$.

(2) $\Theta_u \circ \Theta_v = \Theta_{u+v}$ *for all* $u, v \in \mathbb{R}$.

(3) *Either:*

(i) $u \neq v \Rightarrow \Theta_u \neq \Theta_v$ *for all* $u, v \in \mathbb{R}$, *or*

(ii) *there exists a* $u_0 > 0$ *such that* $\Theta_{u_0} = \Theta_0 = id$, *and* $u \neq v \Rightarrow \Theta_u \neq \Theta_v$ *for all* $u, v \in [0, u_0)$.

Proof (1) Immediate from Theorem 13.1.

(2) If γ_p is the integral curve of X through p, then for each fixed v, by Theorem 13.2, $u \mapsto \gamma_p(u+v)$ is the integral curve γ_q of X through $q = \gamma_p(v)$. Then $\gamma_q(u) = \Theta_u(q) = \Theta_u(\theta_v(p))$ and $\gamma_q(u) = \gamma_p(u+v) = \Theta_{u+v}(p)$. So $\Theta_u \circ \Theta_v = \Theta_{u+v}$.

(3) We can find a point $p \in E$ and a neighborhood of p in which $X(q) \neq 0$ (assuming X is not identically zero). If $X^1(q) > 0$ in this neighborhood, and γ_p is an integral curve through p, then γ_p^1 is strictly increasing in some interval containing 0. Hence, if u, v are in such an interval, $u \neq v \Rightarrow \gamma_p^1(u) \neq \gamma_p^1(v) \Rightarrow \gamma_p(u) \neq \gamma_p(v) \Rightarrow \Theta_u(p) \neq \Theta_v(p) \Rightarrow \Theta_u \neq \Theta_v$. That is, there is some interval containing 0 for which $u \neq v \Rightarrow \Theta_u \neq \Theta_v$. Now suppose there exist $u, v > u$ such that $\Theta_u = \Theta_v$. Then $\Theta_{-u} \circ \Theta_u = \Theta_{v-u} \Rightarrow \Theta_{v-u} = \Theta_0$. So $w = v - u > 0$ is such that $\Theta_w = \Theta_0$. Let $u_0 > 0$ be the minimum w such that $\Theta_w = \Theta_0$. u_0 exists by the continuity of Θ with respect to u. Then $\Theta_u = \Theta_v$ for any $u, v > u$ in $[0, u_0)$ leads to a contradiction. □

The prototypical example of a flow of type 3(ii) is that of the vector field $X = -\mu_2\, \partial/\partial\mu^1 + \mu^1\, \partial/\partial\mu^2$ on $\mathbb{R}^2$.

Theorem 13.6 says that if we start with a complete vector field we can construct a mapping $\Theta : \mathbb{R} \times E \to E$ with certain properties. We can reverse our point of view. Instead of starting with a vector field, if we start with a mapping $\mathcal{A}:\mathbb{R}\times M \to M$ with certain properties, we can then construct a certain complete vector field.

Definitions A mapping $\mathcal{A}:\mathbb{R}\times M \to M$ with properties

(1) $\mathcal{A}(0, p) = p$ for all $p \in M$

(2) $\mathcal{A}(v, \mathcal{A}(u, p)) = \mathcal{A}(v + u, p) = \mathcal{A}(u, \mathcal{A}(v, p))$ for all $u, v \in \mathbb{R}$ and $p \in M$ is called *a 1-parameter group action*, and the set of partial maps $\{\mathcal{A}_u\}$ is *a 1-parameter group of transformations.*

The first two properties of Theorem 13.6 are precisely those of the definition of $\mathcal{A}$, so the flow of a complete vector field is a 1-parameter group action. (If we think of a 1-parameter group action as a special case of the action of a 1-dimensional Lie group, since the only connected 1-dimensional Lie groups are $\mathbb{R}$, and S^1, we get back property (3) of Theorem 13.6. Lie groups will be discussed in Chapter 25.)

Definition Given a 1-parameter group action, $\mathcal{A}$, for each $p \in M$ we have the curve $\mathcal{A}_p : \mathbb{R} \to M$ through p given by $\mathcal{A}_p : u \mapsto \mathcal{A}(u,p)$. *The infinitesimal generator of* $\mathcal{A}$ is the vector field, $X_{\mathcal{A}}$, defined by $X_{\mathcal{A}} : p \mapsto [\mathcal{A}_p]$ where $[\mathcal{A}_p]$ is the tangent of $\mathcal{A}_p$ at p (Section 9.3).

We can complete the circle for the concepts and relations described above by noting that since a flow is an action and since an infinitesimal generator is a vector field, we can start with a vector field X, construct its flow, Θ_X and construct the infinitesimal generator, Y_{Θ_X}, of Θ_X. Or, starting with an action, $\mathcal{A}$, we can construct its infinitesimal generator, $X_{\mathcal{A}}$, and then construct the flow, $\Theta_{X_{\mathcal{A}}}$ of $X_{\mathcal{A}}$.

Theorem 13.7 (i) *The flow $\Theta_{X_{\mathcal{A}}}$ of the infinitesimal generator, $X_{\mathcal{A}}$, of a 1-parameter group action, $\mathcal{A}$, is $\mathcal{A}$.* (ii) *The infinitesimal generator Y_{Θ_X} of the flow, Θ_X, of a complete vector field, X, is X.*

Proof (i) See Bishop and Goldberg (p. 127). (ii) See Boothby (p. 135). □

Corollary (i) *Every 1-parameter group action is the flow of a complete vector field.* (ii) *Every complete vector field is the infinitesimal generator of a 1-parameter group action.*

While the concept of a complete vector field is very neat, it is not adequate from the point of view of applications. Clearly, the terminology; i.e., the flow, comes from applications in fluid mechanics and in such applications "the flows" are rarely those of complete vector fields. In particular, the two general criteria we mentioned (bounded functions on $\mathbb{R}^n$ and compact manifolds) do not occur in those applications. In those cases "the flow" is only a *local* action; that is, a mapping defined only on a subset of $\mathbb{R} \times M$ and with (2) in the definition of $\mathcal{A}$ required to be valid only when the maps are defined (see Section 21.4). A local action has an infinitesimal generator just as before, but now it will not necessarily be a complete vector field. (See Bishop and Goldberg, pp. 126-127; Boothby, pp. 125-128.)

There is another important generalization – particularly for applications – which we have not discussed, and which we will need later. We noted in the beginning of the chapter that our vector fields were autonomous – independent of time. It is often necessary to consider vector fields, and more generally tensor fields which are defined on some interval of $\mathbb{R}$ as well as on $E \subset M$. In this case a time-dependent vector field has a flow, Θ, defined on some subset of $\mathbb{R} \times E$, where Θ_p is a curve such that $\dot{\Theta}_p(u) = X(\Theta_p(u), u)$ on some interval I_p containing u_0, and $\Theta_p(u_0) = p$.

PROBLEM 13.5 Verify the statement for the four examples given in the paragraph above Theorem 13.5.

PROBLEM 13.6 Prove that X is complete iff there is an $r > 0$ such that for every $p \in E$ the integral curve of X through p is defined on the interval $(-r, r)$.

PROBLEM 13.7 Prove Theorem 13.5.

PROBLEM 13.8 Show that if Θ is the flow of a complete vector field with property 3(ii) then Θ defines a map $\mathcal{A}: S^1 \times M \to M$. (Use $S^1 \cong \mathbb{R}/\langle u_0 \rangle$ where $\mathbb{R}/\langle u_0 \rangle$ is the quotient group of $\mathbb{R}$ mod u_0; i.e., $\mathbb{R}/\langle u_0 \rangle = \{[u] : u_1 \sim u_2$ if $u_1 - u_2$ is an integral multiple of $u_0\}$.)

PROBLEM 13.9 Show that $\mathcal{A} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by $(u, (a_0, b_0)) \mapsto (u + a_0, b_0 e^u)$ is a 1-parameter group action, and its infinitesimal generator is $1\ \partial/\partial\pi^1 + \pi^2\ \partial/\partial\pi^2$.

PROBLEM 13.10 Prove Theorem 13.7.

PROBLEM 13.11 The infinitesimal generator of a 1-parameter group action is invariant under this action; i.e., $\mathcal{A}_{u*}X(p) = X(\mathcal{A}_u(p))$ for all u (cf., Problem 13.4).

13.3 Coordinate vector fields

In a coordinate neighborhood we have n vector fields $\partial/\partial\mu^i$, $i = 1, \ldots, n$, *the coordinate vector fields* (Section 11.1). For the vector field $\partial/\partial\mu^1$, equation (13.2) is $D_0\gamma^1 = 1$ and $D_0\gamma^i = 0$, $i = 2, \ldots, n$, so the integral curve through p is given by $\gamma_p^1(u) = u + a^1$, $\gamma_p^i(u) = a^i$, $i = 2, \ldots, n$, where $\mu(p) = (a^1, \ldots, a^n)$. Hence, $\Theta_u : p \mapsto \Theta_u(p) = \gamma_p(u)$ takes p to a point $\Theta_u(p)$ with coordinates $(u + a^1, a^2, \ldots, a^n)$. If $\partial/\partial\mu^2$ is another coordinate vector field with flow H, then H_v takes $\Theta_u(p)$ to a point $H_v \circ \Theta_u(p)$ with coordinates $(u + a^1, v + a^2, a^3, \ldots, a^n)$. Clearly, we can reverse the order of these operations and get $\Theta_u \circ H_v = H_v \circ \Theta_u$. That is, the flows of two coordinate vector fields commute.

Suppose, on the other hand, we start with an arbitrary vector field.

Theorem 13.8 *Any vector field, X, can be chosen to be a coordinate vector field in a neighborhood of a point, p, of a manifold at which $X(p) \neq 0$.*

Proof Let $(\mathcal{U}, \mu)$ be a coordinate system at p such that $\mu(p) = (0, \ldots, 0)$ and $X(p) = \partial/\partial\mu^1|_p$. Define a map, ψ, from a neighborhood of $(0, \ldots, 0) \in \mathbb{R}^n$ to a neighborhood of p by $\psi\colon (u, a^2, \ldots, a^n) \mapsto \Theta_u \circ \mu^{-1}(0, a^2, \ldots, a^n)$ where Θ is the flow of X. The curves $u \mapsto \Theta_u \circ \mu^{-1}(0, a^2, \ldots, a^n)$ are integral curves of X through $\mu^{-1}(0, a^2, \ldots, a^n)$. Now, at $(0, \ldots, 0)$, $\psi_* \cdot \partial/\partial\pi^1 = X(p)$ by assumption, and $\psi_* \cdot \partial/\partial\pi^i|_{(0,0)} = \partial/\partial\mu^i|_p$, $i = 2, \ldots, n$, since on $u = 0$, $\psi = \mu^{-1}$. So ψ_* is nonsingular at $(0, \ldots, 0)$, and ψ has a local inverse. ψ^{-1} is a homeomorphism on $\mathcal{V} \subset \mathcal{U}$ so $(\mathcal{V}, \psi^{-1})$ is a local coordinate system with coordinate functions ν^i and coordinate curves ζ_i. The integral curves of X are now the coordinate curves ζ_1, so $X = \partial/\partial\nu^1$. □

Above Theorem 13.8 we saw that commutativity of their flows is a necessary condition for two vector fields to be coordinate vector fields. We also have the converse.

Theorem 13.9 *Two vector fields are coordinate vector fields of a local coordinate system iff their flows commute locally.*

Proof **Only if:** given above.

If: Suppose $\mu(p) = 0$, and map a neighborhood of $(0, \ldots, 0) \in \mathbb{R}^n$ back to a neighborhood of p by $(u, v, a^3, \ldots, a^n) \mapsto H_v \circ \Theta_u \circ \mu^{-1}(0, 0, a^3, \ldots, a^n)$ where Θ is the flow of X and H is the flow of Y. As in the proof of Theorem 13.8 we can show that this map has a local inverse which is a local coordinate map. The coordinate curves corresponding to $\{u = \text{const}, \ a^3 = \text{const}, \ldots, a^n = \text{const}\}$ are $v \mapsto H_v(\Theta_u(\mu^{-1}(0, 0, a^3, \ldots, a^n))) = H_v(q) = \gamma_q(v)$, which are integral curves of Y. Now since $H_v \circ \Theta_u = \Theta_u \circ H_v$ (locally) $(u, v, a^3, \ldots, a^n) \mapsto \Theta_u \circ H_v \circ \mu^{-1}(0, 0, a^3, \ldots, a^n)$ is the inverse of the same coordinate map and so the coordinate curves $u \mapsto \Theta_u(H_v(\mu^{-1}(0, 0, \ldots, a^n)))$ which correspond to $\{v = \text{const}, \ a^3 = \text{const}, \ldots, a^n = \text{const}\}$ are the integral curves of X. □

We make several observations in connection with Theorem 13.9.

1. Theorem 13.9 can be generalized to: A set of vector fields are coordinate vector fields of a coordinate system if and only if all their flows commute pairwise.
2. The commutativity of two flows is precisely equivalent to the vanishing of the Lie bracket of their vector fields, so Theorem 13.9 and its generalization can be expressed in terms of the vanishing of Lie brackets.
3. If we go from p to $\Theta_u(p)$, then to $H_v(\Theta_u(p))$, then to $\Theta_{-u}(H_v(\Theta_u(p)))$, and finally to $H_{-v}(\Theta_{-u}(H_v(\Theta_u(p))))$ we traverse a "parallelogram," and $uv[X, Y](p)$ is a second-order measure of the amount by which the "parallelogram" fails to close.

PROBLEM 13.12 Prove any one or more of the statements above (cf., Bishop and Goldberg, pp. 134-138).

13.4 The Lie derivative

At the beginning of this chapter we said that a vector field can be considered from two different points of view. Up to now we have taken one point of view. Now we take the other.

We saw in Section 11.1 that there is a 1-1 correspondence between vector fields on M and derivations on functions on M. In particular, given X we have an operation, L_X, on $\mathfrak{F}_M$. Now compare this with what we did in Section 12.1. There, by defining an operation d, on $\mathfrak{F}_M$ and on 1-forms we were able to define a unique extension to the entire algebra of differential forms. In our present case we can, in a similar manner, construct an extension of L_X to act on larger domains. Thus, with our definition of L_X on $\mathfrak{F}_M$, if we make the further definition

$$L_X : Y \mapsto [X, Y]$$

for $Y \in \mathfrak{X}M$, then a unique derivation is determined on the algebra of contravariant tensor fields, Section 11.3. Finally, we can go all the way to the algebra of all tensor fields on M if we postulate that our operation commutes with contractions. That is, for a vector field, Y, and a 1-form, ω,

$$L_X \langle \omega, Y \rangle = \langle \omega, L_X Y \rangle + \langle L_X \omega, Y \rangle$$

Thus, corresponding to each X we have, for each tensor field, K, $L_X K$, *the Lie derivative of K with respect to X.*

The proof of the existence and uniqueness of the Lie derivative proceeds by induction from tensor fields of one order to those of one order higher by means of the derivation property. Rather than formalize this description we will formalize a more "geometrical" description of the Lie derivative and show that the object defined has the properties described above, so that, by uniqueness, they are the same thing.

Suppose we are given a vector field, X, with a flow Θ, and a tensor field, K, of type (r, s). Pick $p \in M$. Then we have the integral curve γ_p of X through p defined on some interval, I, containing 0 (see Fig. 13.2). Fix $u \in I$. For each such u we get an element $K(\gamma_p(u)) \in (T_{\gamma_p(u)})^r_s$. Finally, recall (see Section 11.4) that Θ_u induces a linear map, Θ^*_u, from $(T_{\gamma_p(u)})^r_s$ back to $(T_p)^r_s$. (Θ^*_u exist for mixed tensor fields since Θ_u have inverses.) Thus, for each fixed p, we have a curve $K^\sharp_p : I \to (T_p)^r_s$ given by $u \mapsto \Theta^*_u \circ K(\gamma_p(u)) = (\Theta^*_u K)(p)$ where $\Theta^*_u K$ is the pull-back of K by Θ_u (Section 11.4).

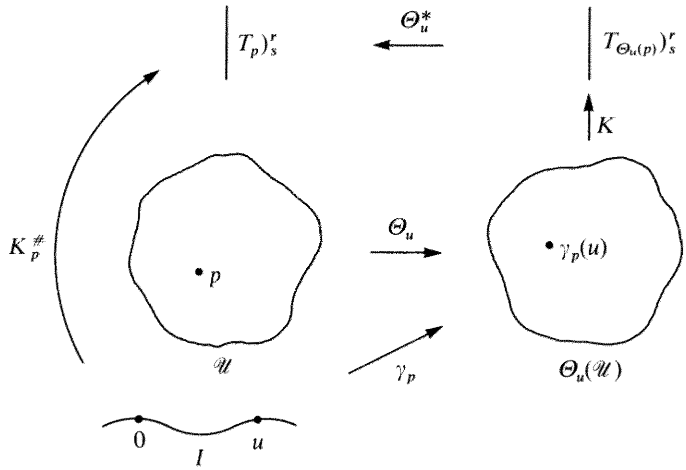

Figure 13.2

Definitions *The Lie derivative at* p *of* K *with respect to* X, denoted by $L_X K(p)$, is the derivative, $D_0 K_p^\sharp$, where $K_p^\sharp$ is the curve described above. *The Lie derivative of* K *with respect to* X is the map $L_X K : p \mapsto L_X K(p)$.

Note that since $K_p^\sharp$ is a map from one vector space to another, the concept of derivative is defined. (See Section 7.1.) Its existence can be inferred from that of the corresponding tangent maps which in turn comes from that of the maps which compose it. Note also that, as we have done on previous occasions, we identify the linear map $D_0 K_p^\sharp$ with its value at 1, which is in $(T_p)_s^r$ (see Theorem 2.4), so that the Lie derivative, L_X, maps (r, s) tensor fields to (r, s) tensor fields.

In order to obtain general properties of the Lie derivative, as well as to obtain useful descriptions for particular important tensor fields, K, it will be convenient to describe $L_X K$ in terms of its component functions in a basis of coordinate tensor fields. Thus, in a local coordinate system at p, we have

$$L_X K = (L_X K)^{i_1 \cdots i_r}_{j_1 \cdots j_s} \frac{\partial}{\partial \mu^{i_1}} \otimes \cdots \otimes d\mu^{j_s}$$

$L_X K$ will be described by expressing the component functions $(L_X K)^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ of $L_X K$ in terms of the component functions $K^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ of K.

If $(\mathcal{U}, \mu)$ is a local coordinate system at p, then under the map Θ_u, $\partial/\partial\mu^i|_p$ go to $\Theta_{u*} \cdot \partial/\partial\mu^i|_p$ and $d\mu^j|_p$ to $(\Theta_u^{-1})^* \cdot d\mu^j|_p$. The tensor products of the $\Theta_{u*} \cdot \partial/\partial\mu^i|_p$ and the $(\Theta_u^{-1})^* \cdot d\mu^j|_p$ form a basis of $(T_{\gamma_p(u)})_s^r$ so

$$K(\gamma_p(u)) = \bar{K}^{i_1 \cdots i_r}_{j_1 \cdots j_s}(\gamma_p(u)) \Theta_{u*} \cdot \frac{\partial}{\partial \mu^{i_1}}\bigg|_p \otimes \cdots \otimes (\Theta_u^{-1})^* \cdot d\mu^{j_s}\big|_p$$

and

$$\Theta_u^* \cdot K(\gamma_p(u)) = \bar{K}^{i_1 \cdots i_r}_{j_1 \cdots j_s}(\gamma_p(u)) \frac{\partial}{\partial \mu^{i_1}}\bigg|_p \otimes \cdots \otimes d\mu^{j_s}\big|_p \tag{13.6}$$

Equation (13.6) gives the curve $K_p^\sharp$ in terms of its component functions in the coordinate basis of $(T_p)^r_s$, so the representation of $D_0 K_p^\sharp$ in this basis is an ordered n^{r+s}-tuple of numbers formed from the $D_0 \bar{K}^{i_1 \cdots i_r}_{j_1 \cdots j_s} \circ \gamma_p$, or, as an element of $(T_p)^r_s$,

$$D_o K_p^\sharp = D_0 \bar{K}^{i_1 \cdots i_r}_{j_1 \cdots j_s} \circ \gamma_p \frac{\partial}{\partial \mu^{i_1}}\bigg|_p \otimes \cdots \otimes d\mu^{j_s}\big|_p \tag{13.7}$$

Thus,

$$(L_X K)^{i_1 \cdots i_r}_{j_1 \cdots j_s}(p) = D_0 \bar{K}^{i_1 \cdots i_r}_{j_1 \cdots j_s} \circ \gamma_p \tag{13.8}$$

Equation (13.8) gives the component description of the Lie derivative in terms of coordinate component functions of K.

Here are two examples of the calculation of the components of the Lie derivative by differentiating $\bar{K}^{i_1 \cdots i_r}_{j_1 \cdots j_s} \circ \gamma_p$ (according to eq. (13.8)). Let $M = \mathbb{R}^2$ with the standard structure given by the atlas $\{(\mathbb{R}^2, id)\}$. Let $X = \partial/\partial\mu^1 + 2\mu^1 \partial/\partial\mu^2 = \partial/\partial x + 2x\partial/\partial y$. The integral curves and the flow of X are obtained by integrating the system $(dx/du) = 1$, $(dy/du) = 2x$. We get $\Theta: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by $\Theta: (u, (x_0, y_0)) \mapsto (u + x_0, u^2 + 2ux_0 + y_0)$.

1. Let

$$K = Y = (\mu^1 + 1)^2 \frac{\partial}{\partial \mu^1} + \mu^2 \frac{\partial}{\partial \mu^2} = (x+1)^2 \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

To use eq. (13.8) we need $\bar{Y}^1(\gamma_p(u))$ and $\bar{Y}^2(\gamma_p(u))$. These are the components of Y with respect to the basis.

$$E_1 = \Theta_{u*} \cdot \frac{\partial}{\partial x}\bigg|_p \qquad E_2 = \Theta_{u*} \cdot \frac{\partial}{\partial y}\bigg|_p$$

of $T_{\gamma_p(u)}$.

Since $\Theta_u: (x_0, y_0) \mapsto (u + x_0, u^2 + 2ux_0 + y_0)$,

$$\begin{aligned} E_1 = \Theta_{u*} \cdot \frac{\partial}{\partial x}\bigg|_p &= \frac{\partial \Theta_u^1}{\partial x_0} \frac{\partial}{\partial x}\bigg|_{\gamma_p(u)} + \frac{\partial \Theta_u^2}{\partial x_0} \frac{\partial}{\partial y}\bigg|_{\gamma_p(u)} \\ &= 1 \frac{\partial}{\partial x}\bigg|_{\gamma_p(u)} + 2u \frac{\partial}{\partial y}\bigg|_{\gamma_p(u)} \end{aligned}$$

$$\begin{aligned} E_2 = \Theta_{u*} \cdot \frac{\partial}{\partial y}\bigg|_p &= \frac{\partial \Theta_u^1}{\partial y_0} \frac{\partial}{\partial x}\bigg|_{\gamma_p(u)} + \frac{\partial \Theta_u^2}{\partial y_0} \frac{\partial}{\partial y}\bigg|_{\gamma_p(u)} \\ &= 0 \frac{\partial}{\partial x}\bigg|_{\gamma_p(u)} + 1 \frac{\partial}{\partial y}\bigg|_{\gamma_p(u)} \end{aligned}$$

Now the components of Y are $(x+1)^2$ and y in the $\partial/\partial x,\ \partial/\partial y$ *basis*, so in the E_1, E_2 basis,

$$\begin{pmatrix}\bar{Y}^1\\ \bar{Y}^2\end{pmatrix} = \begin{pmatrix}1 & 0\\ -2u & 1\end{pmatrix}\begin{pmatrix}(x+1)^2\\ y\end{pmatrix}$$

and, hence, on the integral curve through (x_0, y_0),

$$\bar{Y}^1(\gamma_p(u)) = (u + x_0 + 1)^2$$

$$\bar{Y}^2(\gamma_p(u)) = -2u(u + x_0 + 1)^2 + (u^2 + 2ux_0 + y_0)$$

Finally,

$$D_0\bar{Y}^1 \circ \gamma_p = 2(x_0 + 1)$$

$$D_0\bar{Y}^2 \circ \gamma_p = -2(x_0^2 + x_0 + 1)$$

That is,

$$L_X Y = 2(x+1)\frac{\partial}{\partial x} - 2(x^2 + x + 1)\frac{\partial}{\partial y}$$

We can check this by noting the fact that $L_X Y$ is the Lie bracket $[X, Y]$ (Theorem 13.11(3)) whose components are $X^i(\partial Y^j/\partial\mu^i) - Y^i(\partial X^j/\partial\mu^i)$.

2. Let

$$K = \omega = (\mu^1 + 1)d\mu^1 + \mu^2\, d\mu^2 = (x+1)dx + y\, dy$$

To use eq. (13.8) we need $\bar{\omega}_1(\gamma_p(u))$ and $\bar{\omega}_2(\gamma_p(u))$. These are the components of ω with respect to the basis

$$\Xi^1 = (\Theta_u^{-1})^* \cdot dx\big|_p \quad \Xi^2 = (\Theta_u^{-1})^* \cdot dy\big|_p$$

of $T^*_{\gamma_p(u)}$.

Since Θ_u is given by

$$x = u + x_0 \quad y = u^2 + 2ux_0 + y_0$$

Θ_u^{-1} is given by

$$x_0 = x - u \quad y_0 = u^2 - 2ux + y$$

(note that $\Theta_u^{-1} = \Theta_{-u}$) and

$$\Xi^1 = (\Theta_u^{-1})^* \cdot dx\big|_p = \frac{\partial(\Theta_u^{-1})^1}{\partial x}dx\big|_{\gamma_p(u)} + \frac{\partial(\Theta_u^{-1})^1}{\partial y}dy\big|_{\gamma_p(u)}$$

$$= 1\ dx\big|_{\gamma_p(u)} + 0\ dy\big|_{\gamma_p(u)}$$

$$\Xi^2 = (\Theta_u^{-1})^* \cdot dy\big|_p = \frac{\partial(\Theta_u^{-1})^2}{\partial x}dx\big|_{\gamma_p(u)} + \frac{\partial(\Theta_u^{-1})^2}{\partial y}dy\big|_{\gamma_p(u)}$$

$$= -2u\ dx\big|_{\gamma_p(u)} + 1\ dy\big|_{\gamma_p(u)}$$

Now, the components of ω are $x+1$ and y *in the dx, dy basis*, so in the Ξ^1, Ξ^2 basis,

$$\begin{pmatrix} \bar{\omega}_1 \\ \bar{\omega}_2 \end{pmatrix} = \begin{pmatrix} 1 & 2u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x+1 \\ y \end{pmatrix}$$

and, hence, on the integral curve through (x_0, y_0),

$$\bar{\omega}_1(\gamma_p(u)) = u + x_0 + 1 + 2u(u^2 + 2ux_0 + y_0)$$

$$\bar{\omega}_2(\gamma_p(u)) = u^2 + 2ux_0 + y_0$$

Finally,

$$D_0\bar{\omega}_1 \circ \gamma_p = 1 + 2y_0 \quad D_0\bar{\omega}_2 \circ \gamma_p = 2x_0$$

That is, $L_X\omega = (1+2y)dx + 2x\,dy$. We can check this by deriving the components

$$X\omega_1 + \omega_1 \frac{\partial X^1}{\partial x} + \omega_2 \frac{\partial X^2}{\partial x} \quad X\omega_2 + \omega_1 \frac{\partial X^1}{\partial y} + \omega_2 \frac{\partial X^2}{\partial y}$$

of $L_X\omega$ from Theorem 13.11(2).

We saw in Theorem 13.8 that (where $X \neq 0$) we can always choose local coordinates so that X is a coordinate vector field. With such a choice eq. (13.8) simplifies.

Theorem 13.10 *In any coordinate system in which $X = \partial/\partial\mu^1$ the component functions of $L_X K$ are*

$$(L_X K)^{i_1 \cdots i_r}_{j_1 \cdots j_s} = \frac{\partial K^{i_1 \cdots i_r}_{j_1 \cdots j_s}}{\partial \mu^1}$$

Proof If $X = \partial/\partial\mu^1$, then the flow is just a translation, so $\bar{K}^{i_1 \cdots i_r}_{j_1 \cdots j_s} = K^{i_1 \cdots i_r}_{j_1 \cdots j_s}$. γ_p is the coordinate curve, α_1, through p with tangent $\partial/\partial\mu^1$, and in Section 9.3 we had, by definition of $(\partial f/\partial\mu^i)|_p$, for any function, f, $D_0 f \circ \alpha_i = (\partial f/\partial\mu^i)|_p$. So

$$D_0\bar{K}^{i_1 \cdots i_r}_{j_1 \cdots j_s} \circ \gamma_p = D_0 K^{i_1 \cdots i_r}_{j_1 \cdots j_s} \circ \gamma_p = \left.\frac{\partial K^{i_1 \cdots i_r}_{j_1 \cdots j_s}}{\partial \mu^1}\right|_p .$$

□

Using the components in Theorem 13.10, it is easy to confirm three basic properties of L_X, namely, (1) L_X is a derivation on the algebra of tensor fields on M, (2) $L_X K$ has the same symmetry and skew-symmetry properties as K does, and (3) L_X commutes with contractions.

Now we note explicitly coordinate-free descriptions of $L_X K$ in special cases.

Theorem 13.11 (1) *For a function, f, $L_X f = Xf$.*
(2) *For an exact 1-form, df, $L_X df = dXf$.*
(3) *For a vector field Y, $L_X Y = [X, Y]$.*

Proof (1) is immediate from Theorem 13.10. (Note that in this case our notation agrees with that already used in Section 11.1.) For (2) observe that in the coordinate system of Theorem 13.10 the components on the left side are $\partial/\partial\mu^i(\partial f/\partial\mu^1)$. For (3) in the special coordinates we have

$$L_X Y f = (L_X Y)^i \frac{\partial f}{\partial \mu^i} = \frac{\partial Y^i}{\partial \mu^1} \frac{\partial f}{\partial \mu^i}$$

and

$$\begin{aligned} XYf - YXf &= \frac{\partial}{\partial \mu^1}\left(Y^i \frac{\partial f}{\partial \mu^i}\right) - Y^i \frac{\partial X f}{\partial \mu^i} \\ &= \frac{\partial Y^i}{\partial \mu^1} \frac{\partial f}{\partial \mu^i} \end{aligned}$$

□

From (1) and (2) we have $L_X df = dL_X f$, i.e., in this case L_X and d commute. We will soon see that this generalizes.

Theorem 13.12 *For an (r, s) tensor field, K, we have the following formula for the values of $L_X K$:*

$$\begin{aligned} (L_X K)(\omega^1, \ldots, \omega^r, X_1, \ldots, X_s) &= L_X(K(\omega^1, \ldots, \omega^r, X_1, \ldots, X_s)) \\ &\quad - \sum_{i=1}^{r} K(\omega^1, \ldots, L_X \omega^i, \ldots, \omega^r, X_1, \ldots, X_s) \\ &\quad - \sum_{i=1}^{s} K(\omega^1, \ldots, \omega^r, X_1, \ldots, L_X X_i, \ldots, X_s) \end{aligned}$$

Proof Problem 13.16. □

Theorem 13.13 *In a local coordinate system $(\mathcal{U}, \mu)$ the component functions of $L_X K$ are*

$$\begin{aligned} (L_X K)^{i_1 \cdots i_r}_{j_1 \cdots j_s} &= X K^{i_1 \cdots i_r}_{j_1 \cdots j_s} - \sum_{\alpha} K^{i_1 \cdots i_{\alpha-1} h i_{\alpha+1} \cdots i_r}_{j_1 \cdots j_s} \frac{\partial X^{i_\alpha}}{\partial \mu^h} \\ &\quad + \sum_{\alpha} K^{i_1 \cdots i_r}_{j_1 \cdots j_{\alpha-1} h j_{\alpha+1} \cdots j_s} \frac{\partial X^h}{\partial \mu^{j_\alpha}} \end{aligned}$$

Proof In the formula in Theorem 13.12, replace ω^i by $d\mu^i$ and X_i by $\partial/\partial\mu^i$. □

Using the components in Theorem 13.13 we see immediately that the Lie derivative is $\mathbb{R}$-linear in its subscript argument, i.e., $L_{aX+bY}K = aL_X K + bL_Y K$.

Recall that in Section 11.4 we described the skew-derivation, i_X, interior multiplication by X, on the algebra of differential forms, and in Section 12.1 we introduced the operation, d, of exterior differentiation, also a skew-derivation on differential forms. We have the following important relation among operations on the algebra of differential forms.

Theorem 13.14 *On differential forms,*

$$L_X = i_X \circ d + d \circ i_X \tag{13.9}$$

Proof We will (1) show $i_X \circ d + d \circ i_X$ is a derivation; (2) confirm the formula for functions; (3) confirm the formula for 1-forms:

$$\begin{aligned}
(1)\ (i_X \circ d + d \circ i_X)\sigma \wedge \tau &= i_X(d\sigma \wedge \tau + (-1)^p \sigma \wedge d\tau) \\
&\quad + d(i_X\sigma \wedge \tau + (-1)^p \sigma \wedge i_X \tau) \\
&= i_X d\sigma \wedge \tau + (-1)^{p+1} d\sigma \wedge i_X \tau + (-1)^p i_X \sigma \wedge d\tau \\
&\quad + (-1)^{2p} \sigma \wedge i_X d\tau + d i_X \sigma \wedge \tau + (-1)^{p-1} i_X \sigma \wedge d\tau \\
&\quad + (-1)^p d\sigma \wedge d i_X \tau + (-1)^{2p} \sigma \wedge d i_X \tau \\
&= (i_X d\sigma + d i_X \sigma) \wedge \tau + \sigma \wedge (i_X d\tau + d i_X \tau)
\end{aligned}$$

(2) $L_X f = Xf$ and $(i_X \circ d + d \circ i_X)f = i_X df + d i_X f = Xf$
(3) $LXdf = dXf$ and $(i_X \circ d + d \circ i_X)df = 0 + d i_X df = dXf$ □

Corollary *On differential forms $d \circ L_X = L_X \circ d$.*

The following formula giving $D_u K_p^\sharp$ in terms of the Lie derivative will be useful.

Theorem 13.15 $D_u K_p^\sharp = \Theta_u^* \cdot L_X K(\Theta_u(p))$.

Proof From

$$K_p^\sharp : u \mapsto \Theta_u^* \cdot K(\gamma_p(u)) \quad \text{and} \quad K_{\Theta_{u(p)}}^\sharp : v \mapsto \Theta_v^* \cdot K(\gamma_{\Theta_{u(p)}}(v))$$

we have

$$K_p^\sharp : u + v \mapsto \Theta_u^* \cdot \Theta_v^* \cdot K(\gamma_p(u+v)) = \Theta_u^* \cdot K_{\Theta_{u(p)}}^\sharp(v)$$

So

$$D_u K_p^\sharp = \Theta_u^* \cdot D_0 K_{\Theta_{u(p)}}^\sharp = \Theta_u^* \cdot L_X K(\Theta_u(p))$$

from the definition of the Lie derivative. □

Theorem 13.16 *If D is a regular domain in M, α is an n-form and Θ is the flow of a vector field X, then*

$$\frac{d}{dt}\int_{\Theta_t(D)} \alpha = \int_{\Theta_t(D)} L_X \alpha$$

Proof This is a straightforward application of Theorems 13.15 and 12.11. □

Later, in Sections 18.2 and 21.4, we will need generalizations of Theorems 13.15 and 13.16.

PROBLEM 13.13 Using Theorem 13.11(1) we can write eq. (11.13) as $L_{[X,Y]}f = L_X L_Y f - L_Y L_X f = [L_X, L_Y]f$. Generalize by showing that this is valid if f is replaced by any tensor field K. (That is, $L_{[X,Y]}K$ is a measure of the noncommutativity of the Lie derivative of K.)

PROBLEM 13.14 Prove that for vector fields, X and Y, and a 1-form, ω,

$$L_X \langle \omega, Y \rangle = \langle \omega, L_X Y \rangle + \langle L_X \omega, Y \rangle$$

PROBLEM 13.15 (i) Write the formula of Theorem 13.12 for tensor fields of type $(1,0), (0,1), (1,1)$ and $(0,2)$. (ii) Write your formulas in (i) in terms of contractions.

PROBLEM 13.16 Prove Theorem 13.12. (Hint: Problem 13.13 shows that Theorem 13.12 says that Lie differentiation commutes with successive contractions.)

PROBLEM 13.17 Prove that

$$i_{[X,Y]} = L_X \circ i_Y - i_Y \circ L_X$$

on differential forms and, hence, in particular, $L_X \circ i_X = i_X \circ L_X$.

14

INTEGRABILITY CONDITIONS FOR DISTRIBUTIONS AND FOR PFAFFIAN SYSTEMS

In Section 13.1, we met the equation $\dot{\gamma} = X \circ \gamma$ and in Section 12.1 we met the equation $\theta = d\omega$. We saw that the former is an intrinsic (i.e., coordinate-free) description of a system of ordinary differential equations, and the latter is in intrinsic description of various systems of partial differential equations, depending on the degree of θ. We saw that differentiability conditions on X suffice to give local solutions of $\dot{\gamma} = X \circ \gamma$, but for $\theta = d\omega$ to have local solutions the additional condition $d\theta = 0$ is necessary. We will now generalize the situation $\theta = d\omega$ for θ a 1-form, to a system of equations given by a system of 1-forms.

14.1 Completely integrable distributions

A system of k linear homogeneous partial differential equations on an n-dimensional manifold, $M, k < n$, can be written in the form $X_i f = 0$, $i = 1, \dots, k$, where the X_i are local vector fields on M. We saw in Section 13.3 that $[X_1, X_2] = 0$ if and only if there is a coordinate system $(\mathcal{U}, \mu)$ in which $X_1 = \partial/\partial\mu^1$ and $X_2 = \partial/\partial\mu^2$. Thus, if $[X_1, X_2] = 0$, then $X_1\mu^3 = 0, \dots, X_1\mu^n = 0$ and $X_2\mu^3 = 0, \dots, X_2\mu^n = 0$. That is $\mu^3, \dots, \mu^n$ are solutions of the system $X_1 f = 0$, $X_2 f = 0$. Roughly, $\mu^3, \dots, \mu^n$ are functions on $\mathcal{U}$ which are constant when we move in the X_1 and X_2 directions. More precisely, if $[X_1, X_2] = 0$, then at each point $p \in \mathcal{U}$, $X_1 = \partial/\partial\mu^1$ is tangent to the coordinate curve $\bigcap\{q : \mu^i(q) = \mu^i(p)\}$, $i = 2, \dots, n$, and $X_2 = \partial/\partial\mu^2$ is tangent to the coordinate curve $\bigcap\{q : \mu^i(q) = \mu^i(p)\}$, $i = 1, 3, 4, \dots, n$, so X_1 and X_2 are both in the 2-dimensional coordinate slice consisting of the intersection of the hypersurfaces $\mu^i(q) = \mu^i(p)$, $i = 3, \dots, n$. That is, we have 2-dimensional submanifold on which μ_1 and μ_2 are coordinates, X_1 and X_2 are tangent vectors, and $\mu^3, \dots, \mu^n$ are constant.

For example, in $\mathbb{R}^4$, given X_1 and X_2, with $[X_1, X_2] = 0$ at each point in $\mathcal{U}$ the hypersurfaces $\mu^3 =$ constant and $\mu^4 =$ constant intersect in a 2-dimensional manifold whose tangent space contains X_1 and X_2.

We can generalize the discussion above by means of the observations following Theorem 13.9.

Theorem 14.1 *If* $[X_i, X_j] = 0$ $i, j = 1, \dots, k$ *on* $\mathcal{U}$, *then* (1) *there are* k*-dimensional submanifolds whose tangent spaces at each point contain*

$X_1, \ldots, X_k$, and, moreover, (2) there are functions $\mu^1, \ldots, \mu^n$ such that $\mu^1, \ldots, \mu^k$ are coordinates on the submanifolds, $\mu^{k+1}, \ldots, \mu^n$ are constant on the submanifolds, and $X_i \mu^j = 0,\ i = 1, \ldots, k,\ j = k+1, \ldots, n$.

Proof Problem 14.1. □

The example $[y\partial/\partial x, \partial/\partial y] = -\partial/\partial x$ in $\mathbb{R}^3$ shows that the hypothesis of Theorem 14.1 is not necessary for its conclusion. We will find a weaker condition which is both necessary and sufficient. We first introduce some standard terminology which involves the relationship between systems of vector fields and sections (see Chapter 26) of a manifold of Grassmann manifolds.

Definitions Let $G(k, T_p)$ be the Grassmann manifold of k-planes of the tangent space, T_p, of $p \in M$ (Section 9.1). A mapping $\mathcal{D}_k : M \to \bigcup_{p \in M} G(k, T_p)$ such that $\mathcal{D}_k(p) \in G(k, T_p)$ is called *a k-dimensional distribution on M*. A set of local vector fields $X_1, \ldots, X_k$, on $\mathcal{U} \subset M$ such that for all $p \in \mathcal{U},\ X_1(p), \ldots, X_k(p)$ is a basis of $\mathcal{D}_k(p)$ is *a local basis of $\mathcal{D}_k$*. Given a distribution, $\mathcal{D}_k$, a submanifold, N, of M is called *an integral submanifold of $\mathcal{D}_k$* if $T_p(N) \subset \mathcal{D}_k(p)$ for all $p \in N$. A given distribution, $\mathcal{D}_k$, is called *completely integrable* if it has a k-dimensional integral submanifold at each point.

In these terms, Theorem 14.1 gives sufficient conditions for a distribution to be completely integrable.

Theorem 14.2 *If the system $X_k f = 0,\ i = 1, \ldots, k$ has $n - k$ independent solutions, then the distribution, $\mathcal{D}_k$, of $\{X_i\}$ is completely integrable.*

Proof Let $\mu_{k+1}, \ldots, \mu_n$ be solutions of the system. Let $\mu_1, \ldots, \mu_k$ be functions such that $\{\mu_i\}$ is a coordinate system. Let N be the k-dimensional coordinate slice $\bigcap\{q : \mu^j(q) = \mu^j(p)\},\ j = k+1, \ldots, n$. Then $X(p) \in T_p(N)$ implies that $X(p)$ is in the tangent space of each of the hypersurfaces $\mu^j(q) = \mu^j(p),\ j = k+1, \ldots, n$. Now $\langle d\mu_j(p), X(p)\rangle = 0$ since, if $id\big|_N$ is the inclusion map $id\big|_N : N \to M$, then $id\big|_N^* d\mu_j = d\ id\big|_N^* \mu_j = 0$ which says that $d\mu_j$ vanishes on the tangent space of N. Since $\langle d\mu_j, X_i\rangle = 0$ for $i = 1, \ldots, k,\ j = k+1, \ldots, n,\ X(p) \in \mathcal{D}_k(p)$. □

Definition A distribution, $\mathcal{D}_k$, *is involutive* if for all X and Y in $\mathcal{D}_k$, $[X, Y]$ is in $\mathcal{D}_k$; i.e., $[X, Y] = f^i X_i$ where f^i are functions. (See Problem 14.3.)

Theorem 14.3 *If $\mathcal{D}_k$ is completely integrable, then $\mathcal{D}_k$ is involutive.*

Proof Let N be an integral submanifold of $\mathcal{D}_k$. Then at each point $p \in N$,

$$X(p), Y(p) \in \mathcal{D}_k(p) \Rightarrow X(p), Y(p) \in T_p(N) \Rightarrow [X(p), Y(p)] \in T_p(N)$$
$$\Rightarrow [X, Y](p) \in \mathcal{D}_k(p).$$

□

The converse of Theorem 14.3 follows from the following basic lemma.

Lemma *If $\mathcal{D}_k$ is involutive, then at each point, $p_0 \in M$, there is a coordinate system $(\mathcal{U}, \mu)$ such that $\partial/\partial\mu^1, \ldots, \partial/\partial\mu^k$ span $\mathcal{D}_k$ at each point of $\mathcal{U}$.*

Proof Proof by induction on k. For $k = 1$ the theorem is true by Theorem 13.8. Now suppose $X_1, \ldots, X_k$ is a local basis of $\mathcal{D}_k$. (The existence of such a basis is included in a careful statement of the hypothesis.). Let $(\mathcal{V}, \nu)$ be a coordinate system at p_0 with $\nu(p_0) = 0$, and $X_1 = \partial/\partial\nu^1$. Define $k-1$ vector fields by $Y_i = X_i - (X_i\nu^1)X_1,\ i = 1, \ldots, k$. These form a $k-1$-dimensional involutive distribution, $\mathcal{D}_{k-1}$, on $\mathcal{V}$. In particular, they form a $k-1$-dimensional involutive distribution on the coordinate slice $\nu^1(p) = 0$ (see Fig. 14.1). Now the induction hypothesis gives us a coordinate system $\xi^2, \ldots, \xi^n$ on $\nu^1(p) = 0$ such that $\partial/\partial\xi^2, \ldots, \partial/\partial\xi^k$ span $\mathcal{D}_{k-1}$ on a neighborhood, $\mathcal{W}$, of p_0 on $\nu^1(p) = 0$.

Let $\boldsymbol{\pi}$ be the projection $\boldsymbol{\pi}: \mathcal{V} \to \nu^1(p) = 0$ given by

$$q \mapsto \nu(q) \mapsto (0, \nu^2(q), \ldots, \nu^n(q)) \mapsto \nu^{-1}(0, \nu^2(q), \ldots, \nu^n(q)$$

Then on $\boldsymbol{\pi}^{-1}(\mathcal{W})$ we have the coordinate functions

$$\mu^1 = \nu^1,\ \mu^2 = \xi^2 \circ \boldsymbol{\pi}, \ldots, \mu^n = \xi^n \circ \boldsymbol{\pi}.$$

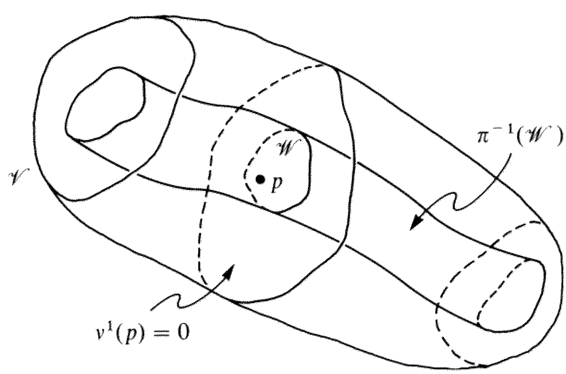

Figure 14.1

To show that $\partial/\partial\mu^1, \ldots, \partial/\partial\mu^k$ span $\mathcal{D}_k$ we will show that they span the same subspaces as $X_1, Y_2, \ldots, Y_k$. That is, putting $X_1 = Y_1$ we will show $Y_i\mu^j = 0$ for $i = 1, \ldots, k$, and $j = k+1, \ldots, n$.

Now, $Y_1, \ldots, Y_k$ is the basis of an involutive distribution, so $[Y_i, Y_j] = f^l_{ij} Y_l$. Hence,

$$Y_1(Y_i\mu^j) = [Y_1, Y_i]\mu^j = f^l_{1i} Y_l \mu^j \quad i = 2, \ldots, k \quad j = k+1, \ldots, n$$

That is, $Y_i\mu^j$ satisfy a system of linear homogeneous ordinary differential equations along μ^1 curves. On $\nu^1(p) = 0$, $\mu^i = \xi^i$ for $i \geq 2$ and $Y_i\xi^j = 0$ for $j > k$. So $Y_i\mu^j = 0$ for $i \geq 2$ and $j > k$ on $\nu^1(p) = 0$. Thus, by the uniqueness theorem for such systems of ordinary differential equations, $Y_i\mu^j \equiv 0$ for $i = 2, \ldots, k$ and $j > k$. Also $Y_1\mu^j = 0$ for $j > 1$, so $Y_i\mu^j = 0$ for $i = 1, \ldots, k$ and $j > k$. □

Theorem 14.4 *If $\mathcal{D}_k$ is involutive, then* (i) *$\mathcal{D}_k$ is completely integrable, and* (ii) *the system $X_i f = 0$, $i = 1, \ldots, k$ has $n - k$ solutions.*

Proof (i) On the coordinate slices

$$\{q \in \mathcal{U} : \mu^{k+1}(q) = c^{k+1}, \ldots, \mu^n(q) = c^n\}$$

of the coordinate system of the lemma, the tangent spaces are precisely those spanned by $\partial/\partial\mu^1, \ldots, \partial/\partial\mu^k$, so they are the k-planes, $\mathcal{D}_k(p)$, at each point.

(ii) Since $X_i = f^j_i \partial/\partial\mu^j$, $\mu^{k+1}, \ldots, \mu^n$ are solutions. □

Finally, combining Theorems 14.3 and 14.4 on the one hand and from Theorem 14.2 on the other hand we have the following.

Theorem 14.5 *$\mathcal{D}_k$ is completely integrable iff $X_i f = 0$, $i = 1, \ldots, k$ has $n - k$ independent solutions.*

At this point we should put the results above on distributions on manifolds into perspective. These abstract results have deep and extensive historical roots in important problems in differential geometry (cf. Sections 17.4 and 24.2) and continuum mechanics, where we are confronted with the problem of solving systems of nonlinear partial differential equations in $\mathbb{R}^n$ of the form

$$\frac{\partial y^j}{\partial x^i} = \psi^j_i(x^1, \ldots, x^k, y^{k+1}, \ldots, y^n) \; i = 1, \ldots, k \; j = k+1, \ldots, n \qquad (14.1)$$

A system of nonlinear partial differential equations of this special form has a couple of nice properties. (1) In matrix form the left side of (14.1) is the derivative of a map from $\mathbb{R}^k$ to $\mathbb{R}^{n-k}$ (Section 7.1), so (14.1) can be thought of as a generalization of an ordinary differential equation, and is sometimes called *a total differential equation* (Dieudonné, Vol. I, pp. 307 ff). (2) With eq. (14.1) we have *the associated linear system*

$$\frac{\partial f}{\partial x^i} + \psi_i^j \frac{\partial f}{\partial y^j} = 0 \quad i = 1, \ldots, k \quad j = k+1, \ldots, n \tag{14.2}$$

Clearly, this is a special case of the system with which we started this section. By the implicit function theorem, a set of $n - k$ solutions, f^j, of this system, with $f^j(a_0^1, \ldots, a_0^k, b_0^{k+1}, \ldots, b_0^n) = 0$ corresponds to a solution $(y^{k+1}, \ldots, y^n)$ of (14.1) with $y^j(a_0^1, \ldots, a_0^k) = b_0^j$ and

$$f^j(a^1, \ldots, a^k, y^{k+1}(a^1, \ldots, a^k), \ldots, y^n(a^1, \ldots, a^k)) = 0$$

for $(a^1, \ldots, a^k)$ in a neighborhood of $(a_0^1, \ldots, a_0^k)$. A solution of (14.2) is called *an integral of* (14.1).

Theorem 14.6 *The system* (14.1) *has a solution through* (a, b) *if and only if*

$$\frac{\partial \psi_i^j}{\partial x^l} + \psi_l^h \frac{\partial \psi_i^j}{\partial y^h} = \frac{\partial \psi_l^j}{\partial x^i} + \psi_i^h \frac{\partial \psi_l^j}{\partial y^h} \; i, l = 1, \ldots, k \quad j = k+1, \ldots, n \tag{14.3}$$

Proof Problem 14.5. □

PROBLEM 14.1 Prove Theorem 14.1.

PROBLEM 14.2 The vector fields

$$X_1 = \frac{\partial}{\partial \mu^1}, \quad X_2 = \frac{\partial}{\partial \mu^2} + c\frac{\partial}{\partial \mu^3} \quad \text{and} \quad X_3 = \frac{1}{(\mu^1)^2}\frac{\partial}{\partial \mu^2} + \frac{1}{c}\frac{\partial}{\partial \mu^3}$$

where c is a constant, determine three 1-dimensional, three 2-dimensional, and one 3-dimensional distributions in the chart $(\mathcal{U}, \mu)$ on a 3-dimensional manifold. Which are completely integrable and which are not? If $\mu^1 = r, \mu^2 = \vartheta$, and $\mu^3 = z$ are cylindrical coordinates on $\mathbb{R}^3$, interpret your results geometrically.

PROBLEM 14.3 Suppose $X_1, \ldots, X_k$ is a local basis of $\mathcal{D}_k$ and $[X_i, X_j]$ is in $\mathcal{D}_k$ for $i, j = 1, \ldots, k$. If $Y_i = f_i^k X_k$ where f_i^k are functions on $\mathcal{U} \subset M$ and $\det(f_i^k) \neq 0$, then $[Y_i, Y_j]$ is in $\mathcal{D}_k$.

PROBLEM 14.4 If a system of k first-order linear partial differential equations, $X_i f = 0$, is solved for k of the partial derivatives we can write it in the form (14.2). A system of the form (14.2) is called *a Jacobi system.* Prove that for a Jacobi system the integrability conditions reduce to $[X_i, X_j] = 0$.

PROBLEM 14.5 Prove Theorem 14.6. (See Problem 14.4.)

14.2 Completely integrable Pfaffian systems

We have referred to the fact that the problem of solving certain systems of partial differential equations is a venerable one appearing in many branches of mathematics. It has been treated by many mathematicians over many years and has taken many different forms. In particular, our results above can be stated in other equivalent ways. In recent years the name of Frobenius has been assigned to each of our several equivalent theorems by one or more authors.

Definitions A set of 1-forms $\{\sigma^i\}$ is called *a Pfaffian system* and equations $\sigma^i = 0$ are called *Pfaffian equations.* A Pfaffian system of $n - k$ linearly independent 1-forms defines *a codistribution* $\mathcal{D}^*_{n-k} : M \to \bigcup_{p \in M} G(k, T^*_p)$. *An integral submanifold of a Pfaffian system,* or of its *codistribution* $\mathcal{D}^*_{n-k}$ is a submanifold, N, such that at each point $\sigma^i(X) = 0$ for all i and for all X such that $X(p) \in T_p(N)$. A Pfaffian system or its codistribution, $\mathcal{D}^*_{n-k}$ is *completely integrable* if it has a k-dimensional integral submanifold at every point.

Note that with each codistribution $\mathcal{D}^*_{n-k}$ we have the associated distribution $\mathcal{D}_k$, given by $\langle \sigma^i, X_j \rangle = 0$. That is, at each point the corresponding subspaces of T_p and T^*_p are the orthogonal complements of one another (or annihilate one another) as described in Section 2.4.

We have the following characterizations of properties defined above.

Theorem 14.7 (1) *N is an integral submanifold of $\mathcal{D}^*_{n-k}$ iff N is an integral submanifold of its associated distribution.*

(2) (i) *A codistribution is completely integrable iff its associated distribution is completely integrable.* (ii) *A codistribution is completely integrable iff there exist $n - k$ functions f^j such that df^j span $\mathcal{D}^*_{n-k}$ at each point.*

Proof (1) **If:** $X(p) \in T_p(N) \Rightarrow X_p \in \mathcal{D}_k(p) \Rightarrow \sigma^i(p)(X(p)) = 0$.

Only if: $X(p) \in T_p(N) \Rightarrow \sigma^i(p)(X(p)) \Rightarrow X_p \in \mathcal{D}_k(p)$.

(2) (i) Immediate from (1). (ii) Let df^j span the associated distribution, $\mathcal{D}_k$. Then $\langle df^j, X_i \rangle = 0 \Leftrightarrow X_i f^j = 0 \Leftrightarrow \mathcal{D}_k$ is completely integrable (Theorem (14.5) $\Leftrightarrow \mathcal{D}^*_{n-k}$ is completely integrable (by part (2)(i)). □

We should note a couple of interesting things about the condition that the df^j span $\mathcal{D}^*_{n-k}$ in Theorem 14.7 (2)(ii). First of all we see that it is the dual of the property $\partial/\partial\mu^i$ span $\mathcal{D}_k$ in the basic lemma above, and secondly we can think of it as a generalization of the property of exactness for a single 1-form in Section 12.1.

Finally, we have a necessary and sufficient condition for a codistribution to be completely integrable in terms of the codistribution itself. This corresponds to the condition that $\mathcal{D}_k$ is involutive for a distribution.

Lemma *Let the vector fields* $X_1, \ldots, X_n$ *and the differential forms* $\sigma^1, \ldots, \sigma^n$ *be local bases of* TM *and* T^*M, *respectively, such that* $\langle \sigma^j, X_i \rangle = \delta_i^j$. *Let* $[X_i, X_j] = f_{ij}^l X_l$ *and* $d\sigma^i = g_{jl}^i \sigma^j \wedge \sigma^l$. *Then* $g_{jl}^i = -\frac{1}{2} f_{jl}^i$.

Proof Problem 14.6. □

Theorem 14.8 *The system* $\sigma^{k+1}, \ldots, \sigma^n$ *of* $n-k$ *1-forms is completely integrable if and only if* $d\sigma^i \wedge \sigma^{k+1} \wedge \cdots \wedge \sigma^n = 0$ *for* $i = k+1, \ldots, n$.

Proof From the lemma, if $f_{ij}^l = 0$ for $i, j = 1, \ldots, k$ and $l = k+1, \ldots, n$, then, for $i > k$, $d\sigma^i = g_{jl}^i \sigma^j \wedge \sigma^l$ with $g_{jl}^i = 0$ for $j, l \leq k$. □

From the proof of Theorem 14.8 we see that we can give the condition for complete integrability in another form.

Theorem 14.9 *The system* $\sigma^{k+1}, \ldots, \sigma^n$ *of* $n-k$ *1-forms is completely integrable if and only if* $d\sigma^j$, $j = k+1, \ldots, n$, *are in the ideal in the algebra of differential forms on* M *generated by* $\{\sigma^j\}$.

Proof Problem 14.8. □

PROBLEM 14.6 Prove the lemma above.

PROBLEM 14.7 For a single 1-form σ, the integrability condition is $d\sigma \wedge \sigma = 0$. Compare this with the integrability condition $d\sigma = 0$ in Section 12.1. Write these conditions in terms of the natural coordinates on $\mathbb{R}^3$, and then interpret them geometrically.

PROBLEM 14.8 Prove Theorem 14.9.

PROBLEM 14.9 What are the integrability conditions for the system of 1-forms $dy^j - \psi_i^j dx^i$, $j = k+1, \ldots, n$. How do they related to the integrability conditions for a Jacobi system (cf., Problem 14.4)?

14.3 The characteristic distribution of a differential system

Whether or not a given distribution is completely integrable, i.e., has k-dimensional integral submanifolds, it may have lower-dimensional integral submanifolds. We have seen that the complete integrability of a distribution corresponds to the existence of solutions of systems of linear homogeneous partial differential equations. Now the problem of finding solutions of a more general system of partial differential equations can be formulated as a problem of finding lower-dimensional integral submanifolds of a distribution.

Thus, the problem of solving a given partial differential equation of the general form $f(x^1, \ldots, x^n, \ \partial z/\partial x^1, \ldots, \partial z/\partial x^n, z) = 0$ means we are given a function, f, on a region in $\mathbb{R}^{2n+1}$ and we want to find an n-dimensional submanifold, (P, ϕ), of $\mathbb{R}^{2n+1}$ with ϕ given locally by

$$(x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^n, p_1(x^1, \ldots, x^n), \ldots, p_n(x^1, \ldots, x^n), z(x^1, \ldots, x^n))$$

such that on P

$$f(x^1, \ldots, x^n, p_1, \ldots, p_n, z) = 0$$

and

$$p_i = \frac{\partial z}{\partial x^i}$$

These conditions on P are respectively equivalent to the conditions that at each $p \in P$, and all $X_p \in T_pP$ for all p, the 1-forms

$$\sigma^1 = \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n + \frac{\partial f}{\partial p_1} dp_1 + \cdots + \frac{\partial f}{\partial p_n} dp_n + \frac{\partial f}{\partial z} dz$$

$$\sigma^2 = p_1 dx^1 + \cdots + p_n dx^n - dz$$

vanish. In particular, we are looking for an n-dimensional integral submanifold of a $2n-1$-dimensional distribution on $\mathbb{R}^{2n+1}$ not a $2n-1$-dimensional integral submanifold.

The way we proceed is by noting that the pair of 1-forms, σ^1, σ^2, determine a certain 1-dimensional distribution, the characteristic distribution of σ^1, σ^2 on P, and then with initial conditions we construct P.

We will first briefly discuss the general concept of the characteristic distribution of a set of forms. This class of distributions has interesting properties; in particular such distributions are completely integrable. We will then go back and obtain the explicit form of the characteristics for $\{\sigma^1, \sigma^2\}$.

Definitions If ω is an s-form, the space of vectors, X_p, satisfying

$$i_{X_p}\omega(p) = 0 \tag{14.4}$$

is *the associated space,* $A(\omega(p))$, *of* $\omega(p)$. The orthogonal complement, $A^*(\omega(p))$, of $A(\omega(p))$ in T_p^* is called *the enveloping space of* $\omega(p)$.

The condition (14.4) on X_p means that at $p \in M$ $i_X\omega(X_1, \ldots, X_{s-1}) = 0$, or $\omega(X, X_1, \ldots, X_{s-1}) = 0$ for all $X_i \in T_p$. By eq. (11.30) and the definition of $\rfloor$ in Eq. (6.10), this condition is equivalent to

$$\langle F \rfloor \omega, X \rangle = 0 \tag{14.5}$$

for all decomposable $(s-1)$-vectors, F, in $\bigwedge^{s-1}(T_p)$. The terminology "enveloping space of $\omega(p)$" comes from the fact (usually the definition) that this subspace of

T_p^* is the smallest one such that $\omega \in \bigwedge^s(A^*(\omega))$. This characterization of $A^*(\omega)$ comes from (14.5) using a basis of T_p^* containing one of $A^*(\omega)$.

Definitions If ω is in s-form, then a vector field which satisfies

$$i_X \omega = 0 \quad \text{and} \quad i_X d\omega = 0 \tag{14.6}$$

is called *a characteristic vector field of* ω. That is, at each point, p, *the characteristic vectors* are those in the intersection of $A(\omega(p))$ and $A(d\omega(p))$. The distribution of vector spaces spanned by the characteristic vectors, $X(p)$ where X is a characteristic vector field of ω is called *the characteristic distribution of* ω.

Theorem 14.10 $i_X\omega = 0$ *and* $i_X d\omega = 0 \Leftrightarrow i_X\omega = 0$ *and* $L_X\omega = 0$.

Proof Problem 14.10. □

Theorem 14.11 *The characteristic distribution of* ω *is completely integrable.*

Proof From the identities

$$i_{[X,Y]}\omega = L_X i_Y \omega - i_Y L_X \omega \quad \text{(Problem 13.17)}$$

$$L_{[X,Y]}\omega = L_X L_Y \omega - L_Y L_X \omega \quad \text{(Problem 13.13)}$$

and from Theorem 14.10 we see that $[X, Y]$ is in the distribution if X and Y are. □

EXAMPLE If ω is represented locally by $\omega = \mu^3 d\mu^1 \wedge d\mu^2$ on a 4-dimensional manifold, then the associated space of ω is the space spanned by $\partial/\partial\mu^3$ and $\partial/\partial\mu^4$, and $\partial/\partial\mu^4$ is a characteristic vector field of ω.

The conditions (14.6) describing a characteristic vector field relate a vector field, X, and an s-form, ω. We can reverse our point of view and start with a given vector field, X, instead of starting with a given s-form, ω.

Definition If X is a vector field, then an s-form, ω, which satisfies (14.6) is called *an absolute integral invariant of* X.

Clearly, by Theorem 14.10 an absolute integral invariant of X is an invariant of X. This concept, due to Poincaré and Cartan, is useful in mechanics, where we will see its significance in an important example.

We can extend our definition of characteristic vector field of a single form to the concept of a characteristic vector field of a set of forms. Again we have to make some preliminary definitions.

Definitions A set of forms on M is called *a differential system* (recall that a set of 1-forms is called a Pfaffian system). Let $\mathfrak{I}_p$ be the ideal in the algebra $\bigwedge T_p^*$ generated by set $\{\omega^i(p)\}$ of forms. That is, $\mathfrak{I}_p = \langle\{\omega^i(p)\}\rangle = \sum \theta^i(p) \wedge \omega^i(p)$ where $\theta^i(p) \in \bigwedge T_p^*$. *The associated space,* $A(\mathfrak{I}_p)$, of $\mathfrak{I}_p$ is the space of vectors X_p satisfying

$$i_{X_p}\omega(p) \in \mathfrak{I}_p \tag{14.7}$$

for all $\omega(p) \in \mathfrak{I}_p$ (or all $\omega^i(p)$). The orthogonal complement, $A^*(\mathfrak{I}_p)$, of $A(\mathfrak{I}_p)$ in T_p^* is called *the enveloping space of* $\mathfrak{I}_p$.

Again one can show that $A^*(\mathfrak{I}_p)$ is the smallest subspace of T_p^* such that $\mathfrak{I}_p \subset \bigwedge A^*(\mathfrak{I}_p)$ (cf., Choquet-Bruhat and DeWitt-Morette, p. 234).

EXAMPLE Let $\omega^1 = \varepsilon^1 + \varepsilon^2$ and $\omega^2 = \varepsilon^2 \wedge \varepsilon^3 + \varepsilon^1 \wedge \varepsilon^3 + \varepsilon^1 \wedge \varepsilon^2$ where $(\varepsilon^1, \varepsilon^2, \varepsilon^3)$ is a basis of T_p^*. Then $i_X\omega^1 \in \mathfrak{I} \Rightarrow i_X\omega^1 = 0$ and $i_X\omega^2 \in \mathfrak{I} \Rightarrow i_X\omega^2 = a\omega^1$ where $a \in \mathbb{R}$. If (e_1, e_2, e_3) is the dual basis in T_p, then the associated space of $\mathfrak{I}$ is spanned by e_3, and the enveloping space of $\mathfrak{I}$ is spanned by ε^1 and ε^2. Note that even though all three basis elements appear in the differential system, the associated space of $\mathfrak{I}$ is nontrivial.

Definitions If $\{\omega^i\}$ is the differential system, let $\mathfrak{I} = \langle\{\omega^i, d\omega^i\}\rangle$, the ideal generated by $\{\omega^i\}$ and $\{d\omega^i\}$. A vector field which satisfies

$$i_X\omega^i \in \mathfrak{I} \quad \text{and} \quad i_X d\omega^i \in \mathfrak{I} \tag{14.8}$$

is called *a characteristic vector field of the differential system,* $\{\omega^i\}$. The distribution of the vector spaces spanned by *the characteristic vectors,* $X(p)$, *where* X is a characteristic vector field of $\{\omega^i\}$, is called *the characteristic distribution of the differential system* $\{\omega^i\}$.

Theorem 14.12 *The characteristic distribution of a differential system is completely integrable.*

Proof Into $L_Z\theta = di_Z\theta + i_Z d\theta$ put, successively, $\theta = i_Y\omega$ and $Z = X$ and then $\theta = i_X\omega$ and $Z = Y$, and then substitute into

$$i_{[X,Y]}\omega = L_X i_Y\omega + L_Y i_X\omega$$

We get

$$i_{[X,Y]}\omega = di_X i_Y\omega + i_X di_Y\omega - di_Y i_X\omega - i_Y di_X\omega \tag{14.9}$$

Then, if X and Y satisfy (14.8), since $\omega \in \mathfrak{I} \Rightarrow d\omega \in \mathfrak{I}$, ($\{\omega^i, d\omega^i\}$ is a *closed* differential system), each term on the right side of (14.9) is in $\mathfrak{I}$ for all ω^i and $d\omega^i$ so $[X,Y]$ is in the distribution. □

Consider the special case where ω^i are all 1-forms; i.e., $\{\omega^i\}$ is a Pfaffian system and determines a codistribution, $\mathcal{D}^*$. $\mathfrak{I} = \sum \theta^i \wedge \omega^i$ has only the identically vanishing zero form, so the conditions on the characteristic vectors are

$$i_X \omega^i = 0 \quad \text{and} \quad i_X d\omega^i \in \mathcal{D}^*$$

In the case of the 1-forms σ^1 and σ^2 of the partial differential equation $f(x^1, \ldots, x^n, \partial z/\partial x^1, \ldots, \partial z/\partial x^n, z) = 0$ with which we started this section, the conditions on the characteristic vector fields are

$$\begin{aligned} &\sigma^1 \cdot X = 0 \\ &\sigma^2 \cdot X = 0 \\ &i_X d\sigma^1 = c_1^1 \sigma^1 + c_2^1 \sigma^2 \\ &i_X d\sigma^2 = c_1^2 \sigma^1 + c_2^2 \sigma^2 \end{aligned}$$

Putting $X = X^i(\partial/\partial x^i) + P^i(\partial/\partial p_i) + Z(\partial/\partial z)$, these conditions become

$$\frac{\partial f}{\partial x^i} X^i + \frac{\partial f}{\partial p_i} P^i + \frac{\partial f}{\partial z} Z = 0 \tag{14.10}$$

$$Z - p_i X^i = 0 \tag{14.11}$$

$$P^i = c_1^2 \frac{\partial f}{\partial x^i} + c_2^2 p_i \tag{14.12}$$

$$X^i = -c_1^2 \frac{\partial f}{\partial p_i} \tag{14.13}$$

$$0 = -c_1^2 \frac{\partial f}{\partial z} + c_2^2 \tag{14.14}$$

The last three equations come from equating coefficients in the fourth equation above. The third equation above is satisfied with $c_1^1 = c_2^1 = 0$, since σ^1 is closed. If we substitute, c_2^2 from (14.14) into (14.12) we get

$$P^i = c_1^2 \left(\frac{\partial f}{\partial x^i} + p_i \frac{\partial f}{\partial z} \right) \tag{14.15}$$

and substituting X^i from (14.13) into (14.11) we get

$$Z = -c_1^2 \, p_i \frac{\partial f}{\partial p_i} \tag{14.16}$$

Thus, dividing through by the common factor $-c_1^2$ in (14.13), (14.15), and (14.16), we get a characteristic vector field

$$\frac{\partial f}{\partial p_i}\frac{\partial}{\partial x^i} - \left(\frac{\partial f}{\partial x^i} + p_i\frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_i} + p_i\frac{\partial f}{\partial p_i}\frac{\partial}{\partial z}$$

The integral curves of this vector field are the solutions of the system of ordinary differential equations

$$\begin{aligned}
\frac{dx^i}{du} &= \frac{\partial f}{\partial p_i} \\
\frac{dp_i}{du} &= -\left(\frac{\partial f}{\partial x^i} + p_i\frac{\partial f}{\partial z}\right) \\
\frac{dz}{du} &= p_i\frac{\partial f}{\partial p_i}
\end{aligned} \tag{14.17}$$

The n-dimensional integral submanifold, (P, ϕ), is constructed from families of these curves through an initial manifold. In brief, choosing initial conditions on an $n-1$-dimensional initial manifold, we get an $n-1$-parameter family of solutions of (14.17),

$$\begin{aligned}
x^i &= x^i(u, t^\alpha) \quad \alpha = 1, \ldots, n-1 \\
p_i &= p_i(u, t^\alpha) \\
z &= z(u, t^\alpha)
\end{aligned} \tag{14.18}$$

If we eliminate u and t^α between the first and third equations, the resulting function, z, of the x^i will be a solution of the partial differential equation. The set (14.18) describes a family of "strips" on the submanifold. For details, see Courant and Hilbert (Vol. II, Chap. II).

PROBLEM 14.10 Prove Theorem 14.10.

PROBLEM 14.11 Apply the integrability conditions of Theorem 14.8 to the system $\{\sigma^1, \sigma^2\}$ at the beginning of this section for $n = 1$ and $n = 2$. Draw the appropriate conclusions.

15

PSEUDO-RIEMANNIAN GEOMETRY

In order to have "geometrical" concepts in a manifold, M, such as lengths of curves, surface area, or curvatures of curves or surfaces, we have to impose an additional "structure" on M. Here we assign a metric tensor, g, to M in terms of which we define the length of a curve and the distance between two points. We then examine the important special case when M has an atlas on each coordinate neighborhood of which the components of g are constant.

15.1 Pseudo-Riemannian manifolds

Definitions *A pseudo-Riemannian* (or *semi-Riemannian*) *manifold*, (M, g), is a manifold, M on which there is defined a nondegenerate, symmetric tensor field, g, of type $(0,2)$; i.e., $g: M \to \mathbf{S}^2(T^*M)$. g is called *a metric tensor field** on M. If g is positive definite (the index of g is 0) at each point of M, then (M, g) is *a Riemannian manifold.* If g has index 1, or index dim $M - 1$ at each point of M, then (M, g) is *a Lorentzian manifold* (cf., Section 5.4).

Note that if M is connected, then since g is nondegenerate everywhere, the index of g must be the same at each point.

Clearly, it is easy to devise local tensor fields with the required properties. Thus, for example, we can define g_{ii} to be either plus or minus one for each i, and $g_{ij} = 0$ for $i \neq j$ on a coordinate patch. It is another matter whether one can construct a differentiable tensor field on all of a given manifold, M. Since, by our definition, manifolds are paracompact, and hence have partitions of unity, we have the following.

Theorem 15.1 *Every differentiable manifold can be made into a Riemannian manifold.*

Proof Auslander and MacKenzie (p. 105). □

*Frequently, g is simply called a metric. Because the term "metric" occurs frequently with different meanings–we have already used it in connection with topological spaces, and see, for example, Laugwitz (pp. 176 ff)–one must be aware of the context in which it appears.

The non-Riemannian case depends on the topology of M. Thus, for example, a 2-sphere cannot be made into a Lorentzian manifold, but a 3-sphere can. When we are only interested in local properties, the question of existence does not arise. This is the case for the larger part of "classical" differential geometry.

We note, first of all, certain structures, induced by g, on M. Recall that because of the symmetry, the two linear maps from T_p to T_p^* corresponding to a point of $\mathbf{S}^2(T^*M)$ are the same. So g induces a linear mapping, $g^\flat$, from vector fields to 1-form fields; that is, $(g^\flat \cdot X) \cdot Y = g(X, Y)$, Section 5.4.

In local coordinates, if $\{\partial/\partial\mu^i\}$ is a local coordinate basis of vector fields, then

$$g^\flat : \frac{\partial}{\partial\mu^i} \mapsto g\left(\frac{\partial}{\partial\mu^i}, -\right) = g_{ik}\ d\mu^k$$

and if $X = X^i\ \partial/\partial\mu^i$ then

$$g^\flat\left(X^i \frac{\partial}{\partial\mu^i}\right) = X^i g_{ik}\ d\mu^k$$

The components, $X^i g_{ik}$, of the 1-form field $g^\flat \cdot X$ *geometrically equivalent* to the vector field X are written X_k. In particular, for Euclidean space ($M = \mathbb{R}^n$ with the standard differential structure and $g = \delta_{ij} d\pi^i \otimes d\pi^j$), $g^\flat : \partial/\partial\pi^i \mapsto d\pi^i$ and $X_i = X^i$.

Since g is nondegenerate, $g^\flat$ has an inverse, $g^\sharp$, taking 1-form fields to vector fields, If $g^\sharp \cdot d\mu^i = g^{ij}\partial/\partial\mu^j$, then

$$g^{ij} g_{jk} = \delta^i_k \tag{15.1}$$

Moreover, we have the following.

Theorem 15.2 *g^{ij} are the components of a nondegenerate symmetric tensor field of type* $(2, 0)$.

Proof Applying the general result described after Theorem 2.11 to this case, the components of the bilinear function $\mathfrak{X}^*M \times \mathfrak{X}^*M \to \mathbb{R}$ corresponding to $g^\sharp$ are g^{ij}. □

If ω_j are the components of a 1-form field ω then $g^{ij}\omega_j$ are the components of a vector field $g^\sharp \cdot \omega$ *geometrically equivalent* to the 1-form field ω, and are written ω^i. For higher orders there are many geometrically equivalent tensor fields, for example, $K^j_k = g^{jl} K_{kl}$ and $K^{ij} = g^{ik} g^{jl} K_{kl}$.

Definition If f is a function, grad $f = g^\sharp \cdot df$, a vector field geometrically equivalent to the 1-form df = Grad f (cf., Section 12.1). In local coordinates, grad $f = g^{ij}(\partial f/\partial\mu^i)\partial/\partial\mu^j$.

At each point, p, of (M, g), the tangent space, T_p, will contain spacelike, timelike, and null vectors if g is not definite (see Section 5.4). If (M, g) is non-Riemannian, it is convenient to sort out three classes of curves: *spacelike, timelike,* and *nonspacelike*, or *causal*, characterized, respectively, by their tangent vectors at all points.

We saw, in Section 12.2, that if M is orientable, then M has a volume form, i.e., an n-form field which never vanishes. Equivalently, M has an atlas such that for all coordinate systems $\det(\partial\mu^i/\partial\nu^j) > 0$.

Theorem 15.3 *On an orientable pseudo-Riemannian manifold a volume form is defined by* $\sqrt{|\det(g_{ij})|}\, d\mu^1 \wedge \cdots \wedge d\mu^n$.

Proof Choose an oriented atlas for M. We have to show that the given expression is the same in all coordinate systems. Under a change of coordinates, we have a transformation of basis 1-form fields given by $d\bar{\mu}^i = \partial\bar{\mu}^i/\partial\mu^j\, d\mu^j$. This induces a transformation of $\bigwedge^n(T^*M)$ (Problem 11.13) $d\bar{\mu}^1 \wedge \cdots \wedge d\bar{\mu}^n = \det(\partial\bar{\mu}^i/\partial\mu^j)\, d\mu^1 \wedge \cdots \wedge d\mu^n$. But, if $\bar{g}_{ij}$ are the components in $\bar{\mu}^i$ coordinates, and g_{ij} are the components in μ^i coordinates, then $g_{ij} = \partial\bar{\mu}^p/\partial\mu^i \partial\bar{\mu}^q/\partial\mu^j \bar{g}_{pq}$. Taking the determinant of both sides $|\det(g_{ij})| = (\det(\partial\bar{\mu}^i/\partial\mu^j))^2 |\det(\bar{g}_{ij})|$, and then taking the square root, we get $\sqrt{|\det(g_{ij})|} = \det(\partial\bar{\mu}^i/\partial\mu^j)\, \sqrt{|\det(\bar{g}_{ij})|}$ since M is orientable. □

Definitions $\Omega = \sqrt{|\det(g_{ij})|}\, d\mu^1 \wedge \cdots \wedge d\mu^n$ is called *the volume form of the orientable pseudo-Riemannian manifold,* (M, g). Expressing g_{ij} in terms of the lengths of the basis vectors $\frac{\partial}{\partial\mu^i}$, we can say that $\sqrt{|\det(g_{ij})|}$ is *the volume of the parallelopiped* spanned by these vectors.

If (M, g, Ω) is an oriented pseudo-Riemannian manifold with volume form Ω, and if D is a regular domain in M, f is a differentiable function of compact support on M, and X is a vector field on M, then *the volume of* D is $\int_D \Omega = \int_M c_D\Omega$, *the integral of* f *over* D is $\int_D f = \int_D f\Omega$, and *the divergence of* X, div X, is defined by $L_X\Omega = (\text{div } X)\Omega$. (Note that these three definitions involve Ω but not g, so they have obvious generalizations.)

Theorem 15.4 *In a coordinate neighborhood,*

$$\mathrm{divX} = \frac{1}{\sqrt{|\det(\mathrm{g_{ij}})|}}\, \frac{\partial}{\partial\mu^{\mathrm{k}}} \left(\sqrt{|\det(\mathrm{g_{ij}})|}\ \mathrm{X^k}\right)$$

Proof Problem 15.2. □

Now we consider differentiable mappings between two given pseudo-Riemannian manifolds (M, g) and (N, h), or, in particular, mappings of a pseudo-Riemannian manifold to itself.

Definitions If $\phi: (M, g) \to (N, h)$ and $\phi^* h = g$, then ϕ is *an isometry.* We say it "preserves," for "leaves invariant" the pseudo-Riemannian structures, g and h (Fig. 15.1). More generally, if c is a nonzero constant, and $\phi^* h = cg$, then ϕ is called *a homothety.*

To clarify the meaning of the condition $\phi^* h = g$, we write it in more explicit detail. Recall the definition of the pullback, $\phi^* h$, Section 11.4; if $p \in M$, then $\phi^* h(p) = \phi^* \cdot h(\phi(p))$. Now evaluating the right side on v_1 and v_2 in T_pM using the formula given by eq. (4.10) for $\phi^s \cdot A$, we have

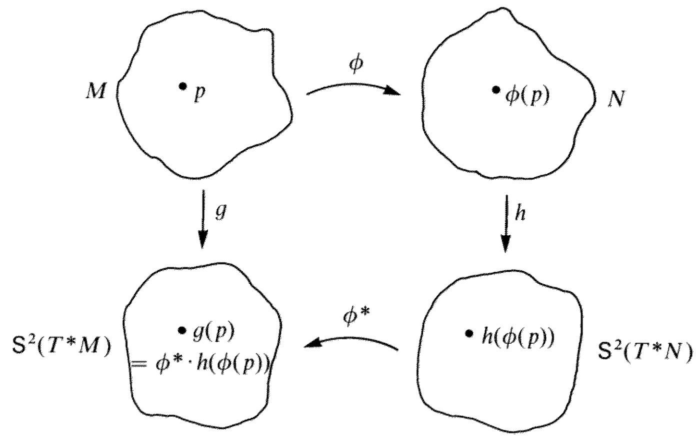

Figure 15.1

$\phi^* \cdot h(\phi(p))(v_1, v_2) = h(\phi(p))(\phi_* \cdot v_1, \phi_* \cdot v_2)$, so $\phi^* h = g$ can be written as

$$h(\phi(p))(\phi_* \cdot v_1, \phi_* \cdot v_2) = g(p)(v_1, v_2) \tag{15.2}$$

If (μ^i) are local coordinates at p and (ν^i) are local coordinates at $\phi(p)$, then in component notation (15.2) becomes

$$h_{ij}(\phi(p)) \, \frac{\partial \phi^i}{\partial \mu^k} \, \frac{\partial \phi^j}{\partial \mu^l} = g_{kl}(p) \tag{15.3}$$

Recall that in Section 5.4 we already used the term "isometry" for certain linear mappings of inner product vector spaces. Our present concept is a generalization of that earlier one. This is made evident by comparing eq. (15.2) with eq. (5.21) (and using the natural isomorphisms of Problem 9.14(ii)). In Section 8.2 we used the term "isometry" for maps of metric spaces. We will see, in the next section, that this concept can, in turn, be considered to be a generalization of our present one.

Finally, rather than studying maps between two given pseudo-Riemannian manifolds, we can alter our point of view, and start with a differentiable map, ϕ, between a manifold, P, and a pseudo-Riemannian manifold, (M, g). Then ϕ^*g is an induced structure on P. In particular, if (P, ϕ) is a submanifold of (M, g), and ϕ^*g is nondegenerate, then ϕ^*g is *the induced pseudo-Riemannian structure* on (P, ϕ). If, further, the submanifold is orientable, then it has an induced volume form, and each regular domain has a volume.

Theorem 15.5 $\int_D (\text{div } X)\Omega = \int_{\partial D} id\big|_{\partial D}^* i_X \Omega$ *using the compatible orientation on* ∂D. *(Here* $id\big|_{\partial D}^*$ *is the pullback of* $i_X\Omega$ *by the inclusion map* $id\big|_{\partial D} : \partial D \to \bar{D}$ *and* i_X *is interior multiplication by* X.*)*

Proof Problem 15.4. □

Theorem 15.5 is a generalization of the divergence theorem in ordinary vector analysis. We can make it look more like that theorem if on the left side we use the definition of Ω and Theorem 15.4. On the right side, by eq. (11.33), $i_X\Omega = \Sigma(-1)^{k+1}X^k\sqrt{|\det(g_{ij})|}\ d\mu^1 \wedge \cdots \wedge \widehat{d\mu}^k \wedge \cdots \wedge d\mu^n$. Then using the coordinates of Theorem 9.7 (with μ, ν interchanged) and eq. (15.3) (with g_{ij} and h_{ij} interchanged), $id\big|_{\partial D}^*\ i_X\Omega = \Sigma X^k(\sqrt{|\det(h_{ij})|}_k d\nu^1 \wedge_k \cdots \wedge d\nu^{n-1}_k$. Finally,

these terms can be combined using the unit normal vector (Section 17.4) of ∂D.

For a pseudo-Riemannian manifold (M, g) we can consider the maps, Θ_u, of a flow of a given vector field, X.

Definition If g is preserved under all the maps, Θ_u, of a given vector field, X, then X is a *Killing field* of g.

Theorem 15.6 X *is a Killing field of* g *iff* $L_X g = 0$.

Proof By Theorem 13.15, if $\{\Theta_u\}$ is the flow of X, and $g_p^\sharp : u \mapsto \Theta_u^* g(p)$ then $D_u g_p^\sharp = \Theta_u^* L_X g(p) = \Theta_u^* \cdot L_X g(\Theta_u(p)) = \Theta_u^* \cdot L_X g(\gamma_p(u))$. Now, $L_X g = 0$ in some neighborhood of $p \Rightarrow D_u g_p^\sharp = 0$ in some neighborhood of $u = 0 \Rightarrow \Theta_u^* g(p) = \Theta_0^* g(p) = g(p)$, so $\Theta_u^* g = g$. Conversely, for a p, if $\Theta_u^* g = g$ for u in some neighborhood of $u = 0$, then $D_0 g_p^\sharp = L_X g(p) = 0$. □

In a local coordinate system, by Theorem 13.13, we can write

$$(L_X g)_{jk} = X^i\ \frac{\partial g_{jk}}{\partial \mu^i} + g_{ji}\ \frac{\partial X^i}{\partial \mu^k} + g_{ki}\ \frac{\partial X^i}{\partial \mu^j}$$

and we have the following result.

Corollary *X is Killing field of g iff*

$$X^i \frac{\partial g_{jk}}{\partial \mu^i} + g_{ji} \frac{\partial X^i}{\partial \mu^k} + g_{ki} \frac{\partial X^i}{\partial \mu^j} = 0 \tag{15.4}$$

Equation (15.4) is called *the equation of Killing* (Eisenhart, 1949, p. 234).

We obtain important illustrations of the concepts introduced above if we start with the manifold $\mathbb{R}^n$ with the standard structure (Section 9.1). Using the standard coordinates (x^i) on $\mathbb{R}^n$ we put $g = dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n$. Then $(M, g) = (\mathbb{R}^n, g_\delta) = \mathcal{R}^n_0$ is called *n-dimensional Euclidean space.* Again, with the standard coordinates on $\mathbb{R}^n$ if we put instead $g = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + \cdots + dx^{n-1} \otimes dx^{n-1}$ then $(M, g) = (\mathbb{R}^n, g_\eta) = \mathcal{R}^n_1$ is called *n-dimensional Minkowski space.* (Compare with the terminology in Section 5.4.) The isometries of n-dimensional Euclidean space are *the Euclidean transformations*, and the Killing fields are

$$\left\langle \left\{ \frac{\partial}{\partial x^i},\ x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}; i, j = 1, \ldots, n \right\} \right\rangle$$

The isometries of Minkowski space are *the Lorentz transformations* and the Killing fields are

$$\left\langle \left\{ \frac{\partial}{\partial x^\alpha}, x^0 \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial x^0}, x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}; i, j = 1, \ldots, n-1, \alpha = 0, \ldots, n \right\} \right\rangle$$

Both of these are $n(n+1)/2$ dimensional vector spaces over $\mathbb{R}$.

The existence of isometries of a pseudo-Riemannian manifold depends on the existence of "symmetries" of the space, so, in general, a pseudo-Riemannian space has no isometries (cf., Problem 20.12).

PROBLEM 15.1 Show that the formula for the definition of div X is simply another form of the equation $dJ/dt = (\nabla \cdot \bar{v}), J = \det \Theta_{t*}$, which appears in classical fluid mechanics (cf., Coburn, p. 101).

PROBLEM 15.2 Prove Theorem 15.4.

PROBLEM 15.3 (i) One can define div X, in terms of the Hodge star operator, by div $X = *d*\omega = \text{Div}\ \omega$ where $\omega = g^\flat \cdot X$. Show that the two definitions give the same thing.

(ii) If $n = 3$, one can define $v \times w$ in terms of Ω by $\alpha \wedge \beta = i_{v \times w}\Omega$ where α and β are geometrically equivalent to v and w, respectively. Show that this definition is the same as the usual one in Section 6.3.

PROBLEM 15.4 Prove Theorem 15.5.

PROBLEM 15.5 If P is an open regular domain in $\mathbb{R}^2$ then the volume of $\phi(P) \subset M$ - the area of the surface $\phi(P)$ - is $\int_{\phi(P)} \left\| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right\| du \wedge dv$.

PROBLEM 15.6 S^3 is a submanifold of $\mathbb{R}^4$ (see Problem 9.19). Using the local coordinates of Example 10, Section 10.1, find the components, $(\phi^* g)_{\alpha\beta}$, of the induced structure.

PROBLEM 15.7 Show that the Killing fields of Euclidean space are as given above and find their integral curves.

PROBLEM 15.8 Same as Problem 15.7 for Minkowski space.

PROBLEM 15.9 Generalize Theorem 15.6; i.e., show that g can be replaced by any tensor field, K.

15.2 Length and distance

Using the concept of induced structure defined above, we can say, *the length of a curve, γ, between two points $\gamma(u_1)$ and $\gamma(u_2)$* in M is the induced volume of the image of the subinterval (u_1, u_2), and we write

$$\mathbf{L}(\gamma) = \int_{u_1}^{u_2} \sqrt{|\det h(u)|}\; du$$

where $h(u) = \gamma^* g(\gamma(u)) = g_{ij}(\gamma(u)) D_u \gamma^i D_u \gamma^j$ by (15.3) (with notation interchanged). This can also be described in familiar terms of the length of the velocity vector of the curve. That is, we can also write

$$\mathbf{L}(\gamma) = \int_{u_1}^{u_2} \|\dot{\gamma}(u)\|\; du$$

since

$$\begin{aligned}\sqrt{|\det h(u)|} &= \sqrt{|h(u)|} = \left| g_{ij}(\gamma(u)) D_u \gamma^i D_u \gamma^j \right|^{1/2} \\ &= \left| g(\dot{\gamma}(u), \dot{\gamma}(u)) \right|^{1/2} = \|\dot{\gamma}(u)\|\end{aligned}$$

Theorem 15.7 *The length of a curve is independent of its parametrization.*

Proof Just as in elementary calculus, on each segment with $\beta(\rho(u)) = \gamma(u)$ such that $d\rho/du \neq 0$,

$$\int_{u_1}^{u_2} \|\dot{\beta}(\rho(u))\|\; |d\rho/du|\; du = \int_{u_1}^{u_2} \|\dot{\gamma}(u)\|\; du$$

so that $\int_{t_1}^{t_2} \|\dot{\beta}(t)\|\; dt = \int_{u_1}^{u_2} \|\dot{\gamma}(u)\|\; du$ if $d\rho/du > 0$, and $\int_{t_2}^{t_1} \|\dot{\beta}(t)\|\; dt = \int_{u_1}^{u_2} \|\dot{\gamma}(u)\|\; du$ if $d\rho/du < 0$. $\square$

A particularly important parametrization, *arc-length*, is obtained when $\dot{\gamma}(u)$ never vanishes along γ. In that case, s, defined by $s(u) = \int_a^u \|\dot{\gamma}(t)\|\, dt$ has an inverse on the domain of γ, so γ can be reparametrized on the range of s.

Having the concept of length of a curve in (M, g), one can proceed to try to define a distance between two points of M. (Recall that a definition of distance has already been introduced in Section 8.1.) Let p and q be two points of M and consider the set of all curves joining them. This set is not empty if p and q are in the same component of M.

Definition *The distance, $d_g(p, q)$, between two points, p and q, on a Riemannian manifold, (M, g),* is the greatest lower bound, $\inf_\gamma \mathbf{L}(\gamma)$, where $\inf_\gamma$ is taken over all curves joining p and q.

Note that on a non-Riemannian manifold, according to this definition, the distance between any two points joined by a timelike curve would be zero, because it is possible to find timelike curves joining the points having arbitrarily small lengths. Hence, this concept must be altered for non-Riemannian manifolds.

Theorem 15.8 *On a Riemannian manifold, d_g is a distance function.*

Proof $d_g(p, p) = 0$ and symmetry are immediate from the definition. For any given positive number ε, there are curves γ_1 joining p and q and γ_2 joining q and r such that $\mathbf{L}(\gamma_1) < d_g(p, q) + \varepsilon$ and $\mathbf{L}(\gamma_2) < d_g(q, r) + \varepsilon$, respectively. So $d_g(p, r) \leq \mathbf{L}(\gamma_1 + \gamma_2) = \mathbf{L}(\gamma_1) + \mathbf{L}(\gamma_2) < d_g(p, q) + d_g(q, r) + 2\varepsilon$, from which we get the triangle inequality. The property if $p \neq q$ then $d_g(p, q) > 0$ comes out as a consequence of Theorem 15.9. Since the topology generated by d_g is the same as the original manifold topology, and the original manifold topology is Hausdorff, the topology generated by d_g is Hausdorff, from which the required property easily follows. □

Definition A curve joining p and q (on a Riemannian manifold) whose length is $d_g(p, q)$ is *a shortest curve joining p and q.*

For two given points p and q we need not have either the existence or uniqueness of shortest curves. Examples of the failure of existence and uniqueness are, respectively: (i) M is the Euclidean plane with the origin removed and with $p = (-1, 0)$ and $q = (1, 0)$; and (ii) M is a sphere in $\mathbb{R}^3$, and p and q are opposite poles.

We saw, in Section 8.1, that a distance function, d, on a set defines a Hausdorff topology. If we drop the property $d(p, q) > 0$ if $p \neq q$, then d, called a pseudodistance function, still defines a topology on M. Hence it is important to observe the following result.

Theorem 15.9 *The topology defined by d_g on a Riemannian manifold is the same as the original manifold topology.*

Proof Define a distance function $d(p,q) = [\Sigma(\mu^i(p) - \mu^i(q))^2]^{1/2}$ on a coordinate domain $\mathcal{U}$. Then there are positive numbers, c_1 and c_2, such that $d_g(p,q) \leq c_1 d(p,q)$ and $d(p,q) \leq c_2 d_g(p,q)$ for all $p,q \in \mathcal{U}$. (See Hicks, pp. 70-71.) Thus, by Theorem 8.4(ii) $(\mathcal{U}, d_g)$ and $(\mathcal{U}, d)$ are homeomorphic, so that each coordinate domain of M is a coordinate domain with the topology of d_g, and these coordinate domains generate the same topology as those with the original topology. □

Theorem 15.10 *For Riemannian manifolds ϕ is an isometry as defined above (i.e., ϕ preserves the Riemannian structures) iff ϕ is an isometry as defined in Section 8.2 (i.e., ϕ preserves distance functions).*

Proof (i) Since the set of curves joining $\phi(p)$ and $\phi(q)$ is precisely the set of images of the curves, γ, joining p and q we can write $d_g(\phi(p), \phi(q)) = d_g(p,q)$ as

$$\inf_{\gamma} \int_a^b [h(\phi_* \dot{\gamma}(t), \phi_* \dot{\gamma}(t))]^{1/2}\, dt = \inf_{\gamma} \int_1^b (g(\dot{\gamma}(t), \dot{\gamma}(t))]^{1/2}\, dt$$

Thus, clearly (15.2) implies that d_g is preserved.

(ii) The converse seems to require a technically rather involved proof; cf., Kobayashi and Nomizu, Vol. I, p. 169). □

As we have noted above, on a non-Riemannian manifold, where we sort out different classes of curves, a concept of distance between points p and q will have to depend on the kinds of curves that can join p and q. On a Lorentzian manifold (index $= 1$ or $n-1$), one can show that for certain pairs of points, p and q, that can be joined by a timelike curve, $\sup_\gamma \mathbf{L}(\gamma)$ where $\sup_\gamma$ is taken over all timelike curves joining p and q, has a finite value, and there is a curve (a geodesic) whose length has this value (Hawking and Ellis, pp. 213-217; O'Neill, p. 411). Such a curve is *a longest curve* joining the two points. These results, obviously more restrictive than for Riemannian manifolds since they do not apply to any arbitrary pair of points, are nevertheless important in the global study of spacetime. Pursuing this subject in the general case requires a careful examination of the causal structure of Lorentzian manifolds. We will look at certain special cases in the next section, and when we come to relativity.

Finally, there are slightly weaker concepts of "shortest curve" and "longest curve" which correspond to a relative rather than an absolute maximum or minimum in ordinary calculus, and which occur in more general form in the calculus of variations. Thus, given γ, consider all curves in a neighborhood of γ in the following sense. Let $I = [a,b]$, be in the domain of γ, and let

$[a,b] \times [-\varepsilon,\varepsilon] \to M$ be a map which restricted to $[(a,0),(b,0)]$ is γ. Given a curve γ, a map, Q_γ, with this property is called *a variation of* γ.

If $(u,t) \in [a,b] \times [-\varepsilon,\varepsilon]$ then for each fixed u we get a "vertical " curve γ_u, and for each fixed t we get a "horizontal" curve γ_t and $\gamma_0 = \gamma$ and a length $\mathbf{L}(\gamma_t)$. Let $J_{Q_\gamma}: t \mapsto \mathbf{L}(\gamma_t)$.

Definitions If for all variations, Q_γ, of γ, such that $Q_\gamma(a,t) = p$ and $Q_\gamma(b,t) = q$ for all $t \in [-\varepsilon,\varepsilon]$, we have $D_0 J_{Q_\gamma} = 0$, then γ is called *an extremal* (or, *critical point*, or *stationary point*) *of* $\mathbf{L}$. If for all such variations J_{Q_γ} has a maximum (minimum) at $t = 0$, then γ is *length-maximizing* (*minimizing*).

Theorem 15.11 *A shortest* (*longest*) *curve is length-minimizing* (*maximizing*).

Proof Problem 15.13. □

Sometimes, instead of using the length of a curve, it is more convenient to use *the energy of a curve* defined by $\mathbf{E}(\gamma) = \int_a^b g(\dot\gamma(t),\dot\gamma(t))dt$. $\mathbf{E}(\gamma)$ is simply related to $\mathbf{L}(\gamma)$. It has the advantage of doing away with the square root, but the disadvantage of being parameter-dependent.

Theorem 15.12 $\mathbf{L}(\gamma)^2 \leq (b-a)|\mathbf{E}(\gamma)|$, *and* $\mathbf{L}(\gamma)^2 = (b-a)|\mathbf{E}(\gamma)|$ *iff the parameter of* γ *is proportional to arc-length.*

Proof In the Cauchy-Schwarz inequality

$$\left(\int_a^b f(t)h(t)dt\right)^2 \leq \int_a^b (f(t))^2\,dt \int_a^b (h(t))^2\,dt$$

put $f = 1$ and $h = \|\dot\gamma\|$ and get the first result. For $f = 1$, Cauchy-Schwarz becomes an equality iff h is a constant. But by the definition of arc-length, s, $\|\dot\gamma(u)\| = $ constant iff the parameter of γ is proportional to s. □

PROBLEM 15.10 If $\beta = \gamma \circ s^{-1}$, then $\|\dot\beta(s)\| \equiv 1$.

PROBLEM 15.11 Fill in the details of the proof of Theorem 15.9.

PROBLEM 15.12 If $\gamma(u) = (u,u^2)$, construct a variation of γ, $Q_\gamma : [0,1] \times [-\varepsilon,\varepsilon] \to \mathbb{R}^2$ with $Q_\gamma(0,t) = (0,0)$, $Q_\gamma(1,t) = (1,1)$, and with vertical curves $\{(c,t) \in \mathbb{R}^2 : c = \text{const.}\}$. Is γ an extremal of $\mathbf{L}$?

PROBLEM 15.13 (i) Prove Theorem 15.11. (ii) Give an example of a length-minimizing curve which is not a shortest curve.

15.3 Flat spaces

We will consider a special class of pseudo-Riemannian manifolds, and for these, pursue some of the ideas of the previous section a bit further.

Definition A pseudo-Riemannian manifold, (M, g), which has a coordinate covering on each domain of which the components g_{ij} are constant is *a flat space.*

Clearly, this class of pseudo-Riemannian manifolds includes the Euclidean and Minkowski spaces mentioned in Section 15.1. Cylinders and cones in 3-dimensional Euclidean space are examples of flat spaces. (See Problem 15.14.)

If there are coordinates, (x^i), for which g_{ij} are constant, then there are coordinates (y^i) given by $x^i = a^i_j y^j$ for which $g_{ij} = \delta_{ij}$. That is, g_{ij} can be "diagonalized" by the coordinate transformation equations $g_{ij}\partial x^i/\partial y^p\ \partial x^j/\partial y^q = \delta_{pq}$, so there is no loss of generality to write g locally in the form $dx^1 \otimes dx^1 + \cdots + dx^r \otimes dx^r - dx^{r+1} \otimes dx^{r+1} - \cdots - dx^n \otimes dx^n$ (cf., eq. (5.20)). In short, a flat space is locally isometric to a pseudo-Euclidean manifold.

Definitions An *affine atlas* on a manifold is such that on the intersection of the coordinate neighborhoods of two overlapping charts the coordinates (x^i) and $(\bar{x}^i)$ are affinely related or, $\partial x^i/\partial \bar{x}^j$ are constants. *An affine manifold* is one which has an affine atlas.

Theorem 15.13 *A flat space is an affine manifold.*

Proof Let g_{ij} be the components of g on the chart with coordinates (x^i) and let $\bar{g}_{ij}$ be the components of g on the chart with coordinates $(\bar{x}^i)$. Then

$$\bar{g}_{ij} = \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j}\, g_{pq} \quad \text{and} \quad \frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j}\, g_{pq} + \frac{\partial x^p}{\partial \bar{x}^i}\frac{\partial^2 x^q}{\partial \bar{x}^k \partial \bar{x}^j}\, g_{pq} = 0$$

This says that

$$\frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^i}\frac{\partial x^q}{\partial \bar{x}^j}\, g_{pq} = g\left(\frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^i}\frac{\partial}{\partial x^p}, \frac{\partial}{\partial \bar{x}^j}\right)$$

is skew-symmetric in i and j, and since it is clearly symmetric in i and k it follows, from Problem 5.2, that

$$g\left(\frac{\partial^2 x^p}{\partial \bar{x}^k \partial \bar{x}^i}\frac{\partial}{\partial x^p}, \frac{\partial}{\partial \bar{x}^j}\right) = 0 \quad \text{for all} \quad i, j, k$$

which implies $\partial^2 x^p/\partial \bar{x}^k \partial \bar{x}^i = 0$ for all i, k, p. □

Definitions In a flat space coordinate neighborhood, a curve, α, given by $\alpha^i(u) = v^i u + a^i$ is *a straight line segment.* Two straight line segments, α and β with $\beta^i(t) = w^i t + b^i$ are *parallel* if $v^i = cw^i$ where c is not zero.

Theorem 15.14 (i) *A straight line segment joining p and q on a Riemannian flat space is a shortest curve joining p and q.* (ii) *A straight line segment on a Lorentzian flat space which joins two points p and q, which can be joined by a timelike curve, is a longest curve joining p and q.*

Proof (i) Let $\alpha^i(u) = v^i u + a^i$ be a straight line segment in a coordinate neighborhood, $\mathcal{U}$. Let γ be a curve in $\mathcal{U}$ joining p and q. At each point of γ we have a vector, v with components v^i. Then

$$\begin{aligned}\mathbf{L}(\gamma) &= \int_{t_1}^{t_2} \|\dot\gamma\| dt \geq \int_{t_1}^{t_2} \frac{g(v,\dot\gamma)}{\|v\|}\, dt \qquad \text{by the Cauchy-Schwarz inequality}\\ &= \int_{t_1}^{t_2} \frac{\sum_i v^i \frac{d\gamma^i}{dt}}{\|v\|} dt = \sum_i \frac{v^i}{\|v\|}(\gamma^i(t_2) - \gamma^i(t_1))\\ &= \sum \frac{v^i}{\|v\|}(\alpha^i(u_2) - \alpha^i(u_1)) = \mathbf{L}(\alpha)\end{aligned}$$

since $\alpha^i(u_2) - \alpha^i(u_1) = v^i(u_2 - u_1)$. A curve, γ, which does not lie entirely within $\mathcal{U}$ can be broken into pieces each of which lies in a flat coordinate neighborhood and can be compared with a straight line segment in that neighborhood.

(ii) Exactly the same as part (i), using instead the backward Cauchy-Schwarz inequality (see Theorem 5.21). $\square$

Theorem 15.15 *A shortest curve on a flat Riemannian manifold, and a longest curve on a flat non-Riemannian manifold for which $\|\dot\gamma(t)\|$ never vanishes is a straight line.*

Proof By Theorem 15.11 a shortest or longest curve is an extremal of $\mathbf{L}$; i.e., $D_0 J_{Q_\gamma} = 0$ for all variations, Q_γ. Further by Theorem 15.12 since $\|\dot\gamma(u)\| \neq 0$, such a curve can be reparametrized by arc-length, and hence, will be an extremal of $\mathbf{E}$. Now, consider the case where the endpoints, p and q, are contained in a coordinate neighborhood, and write the energy of the curves γ_t of a variation Q_γ in the neighborhood in component notation. Thus, for each γ_t, we have

$$\mathbf{E}(\gamma_t) = \int_p^q g_{ij} D_u \gamma_t^i D_u \gamma_t^j \; du \tag{15.5}$$

Now, if we let t vary, and write $\gamma_t^i(u) = \gamma^i(u,t)$, then (15.5) can be written as

$$f(t) = \int_p^q g_{ij} \frac{\partial \gamma^i}{\partial u}(u,t) \frac{\partial \gamma^j}{\partial u}(u,t) du$$

Finally, differentiating with respect to t, integrating by parts and setting $t = 0$ we get

$$f'(0) = 2g_{ij} \left[\frac{\partial \gamma^i}{\partial u}(q,0) \frac{\partial \gamma^i}{\partial t}(q,0) - \frac{\partial \gamma^i}{\partial u}(p,0) \frac{\partial \gamma^j}{\partial t}(p,0) \right]$$
$$-2 \int_p^q g_{ij} \frac{\partial^2 \gamma^i}{\partial u^2}(u,0) \frac{\partial \gamma^j}{\partial t}(u,0)\ du$$

Since

$$\frac{\partial \gamma^i}{\partial t}(q,0) = \frac{\partial \gamma^i}{\partial t}(p,0) = 0$$

$f'(0) = 0$ implies that

$$\int_p^q g_{ij} \frac{\partial^2 \gamma^i}{\partial u^2}(u,0) \frac{\partial \gamma^j}{\partial t}(u,0)\ du = 0 \tag{15.6}$$

Suppose $\partial^2 \gamma^i / \partial u^2 \neq 0$ at some point on γ_0. Then $g_{ij} \partial^2 \gamma^i / \partial u^2\ \partial \gamma^j / \partial t \neq 0$ at that point for some $\partial \gamma^i / \partial t$ since g is nondegenerate. But $\partial \gamma^i / \partial t$ can be chosen arbitrarily since Q_γ can be chosen arbitrarily, and then $\partial \gamma^i / \partial t$ can be extended along γ_0 in such a way to contradict (15.6). This gives $(\partial^2 \gamma^i / \partial u^2)(u,0) = 0$, so that

$$\gamma^i(u) = a^i u + b^i$$

i.e., γ is a straight line segment. □

PROBLEM 15.14 Take a nappe of a circular cone with angle ϑ in 3-dimensional Euclidean space, slice it along one of its generators and flatten it onto the $u - v$ plane perpendicular to its axis, the z-axis.

(i) Show that it has the equations

$$x = \sqrt{u^2 + v^2}\ \cos\left(\frac{\tan^{-1}(v/u)}{\sin \vartheta} \right) \sin \vartheta$$

$$y = \sqrt{u^2 + v^2}\ \sin\left(\frac{\tan^{-1}(v/u)}{\sin \vartheta} \right) \sin \vartheta$$

$$x = \sqrt{u^2 + v^2}\ \cos \vartheta$$

(ii) Show that the cone is flat.

PROBLEM 15.15 Prove the statement in Theorem 15.15 that for an arbitrary choice of $\partial\gamma^i/\partial t$ along γ_0 a variation Q_{γ_0} can be constructed which has those values of $\partial\gamma^i/\partial t$ on γ_0.

PROBLEM 15.16 Extend Theorem 15.15 to the general case in which p and q are not in the same coordinate neighborhood.

16

CONNECTION 1-FORMS

We will describe the Levi-Civita connection and its covariant derivative on a pseudo-Riemannian manifold. Then we abstract the idea of a connection to define a more general class of manifolds. After briefly describing their main features, we go back to pseudo-Riemannian manifolds and discuss their possible connections.

16.1 The Levi-Civita connection and its covariant derivative

We have now, as in Chapter 15, a symmetric, nondegenerate tensor field, g, of type $(0, 2)$ on M. We will construct a new structure on M by first working locally with g. Thus, we work in a chart $(\mathcal{U}, \mu)$ with coordinate neighborhood $\mathcal{U}$ and coordinate functions μ^i. In $\mathcal{U}$, g has components g_{ij} with respect to the coordinate basis $\{d\mu^i \otimes d\mu^j\}$.

Definitions $[ij, k] = \frac{1}{2}(\partial g_{jk}/\partial\mu^i + \partial g_{ki}/\partial\mu^j - \partial g_{ij}/\partial\mu^k)$ are called *the Christoffel symbols of the first kind.* $\left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\} = g^{lk}[ij, k]$, where g^{lk} are given by Eq. (15.1), are called *the Christoffel symbols of the second kind.*

We will now derive the relation between the Christoffel symbols in two coordinate systems (μ^i) and $(\bar{\mu}^i)$. From the basic relation

$$g_{ij} = \bar{g}_{rs}\,\frac{\partial\bar{\mu}^r}{\partial\mu^i}\,\frac{\partial\bar{\mu}^s}{\partial\mu^j}$$

for $p \in \mathcal{U} \cap \mathcal{V}$, putting $p_i^r = \partial\bar{\mu}^r/\partial\mu^i$ and differentiating with respect to μ^k, we get

$$\frac{\partial g_{ij}}{\partial\mu^k} = \frac{\partial\bar{g}_{rs}}{\partial\bar{\mu}^t}\,p_i^r p_j^s p_k^t + \bar{g}_{rs}\left(p_i^r\frac{\partial^2\bar{\mu}^s}{\partial\mu^i\partial\mu^k} + p_j^s\frac{\partial^2\bar{\mu}^r}{\partial\mu^i\partial\mu^k}\right)$$

By cyclically permuting the indices i, j, k and simultaneously permuting r, s, t, we get similar expansions for $\partial g_{jk}/\partial\mu^i$ and $\partial g_{ki}/\partial\mu^j$. Adding these last two and subtracting the one above, we get

$$\frac{\partial g_{jk}}{\partial\mu^i} + \frac{\partial g_{ki}}{\partial\mu^j} - \frac{\partial g_{ij}}{\partial\mu^k} = \left(\frac{\partial\bar{g}_{st}}{\partial\bar{\mu}^r} + \frac{\partial\bar{g}_{tr}}{\partial\bar{\mu}^s} - \frac{\partial\bar{g}_{rs}}{\partial\bar{\mu}^t}\right)p_i^r p_j^s p_k^t + 2\bar{g}_{rs}p_k^r\frac{\partial^2\bar{\mu}^s}{\partial\mu^i\partial\mu^j}$$

or

$$[ij,k] = [\,\overline{rs,t}\,]p_i^r p_j^s p_k^t + \bar{g}_{rs} p_k^r \frac{\partial^2 \bar{\mu}^s}{\partial \mu^i \partial \mu^j} \tag{16.1}$$

Multiplying (16.1) on the left by $g^{lk}(\partial\bar{\mu}^m/\partial\mu^l)$ and on the right by $\bar{g}^{mn}(\partial\mu^k/\partial\bar{\mu}^n)$ (these two things are equal) we get

$$\begin{Bmatrix} l \\ i\ j \end{Bmatrix} p_l^m = \overline{\begin{Bmatrix} m \\ r\ s \end{Bmatrix}} p_i^r p_j^s + \frac{\partial^2 \bar{\mu}^m}{\partial \mu^i \partial \mu^j} \tag{16.2}$$

Equations (16.1) and (16.2) are the transformation formulas for the Christoffel symbols in the intersection, $\mathcal{U}\cap\mathcal{V}$, of the coordinate neighborhoods $\mathcal{U}$ and $\mathcal{V}$. Note that because of the appearance of the second derivative terms, the Christoffel symbols are not tensors.

With the Christoffel symbols $\begin{Bmatrix} l \\ i\ j \end{Bmatrix}$ we can form a set of $(\dim M)^2$ 1-forms on each coordinate domain, namely,

$$\lambda_i^l = \begin{Bmatrix} l \\ i\ j \end{Bmatrix} d\mu^j$$

Then, putting $\bar{\lambda}_r^m = \overline{\begin{Bmatrix} m \\ r\ s \end{Bmatrix}} d\bar{\mu}^s$, the transformation formula (16.2) yields the transformation formula

$$p_l^m \lambda_i^l = p_i^r \bar{\lambda}_r^m + dp_i^m \tag{16.3}$$

for these 1-forms on the intersection $\mathcal{U}\cap\mathcal{V}$.

Definition The $(\dim\ M)^2$ 1-forms $\lambda_i^l = \begin{Bmatrix} l \\ i\ j \end{Bmatrix} d\mu^j$ defined on coordinate neighborhoods of M are called *the Levi-Civita connection 1-forms* of the pseudo-Riemannian manifold (M,g).

Now suppose $Y \in \mathfrak{X}M$ is a given vector field on M. In the intersecting coordinate neighborhoods $\mathcal{U}$ and $\mathcal{V}$, Y has coordinate components Y^i and $\bar{Y}^i$ related by $\bar{Y}^i = p_j^i Y^j$ on $\mathcal{U}\cap\mathcal{V}$, and

$$\begin{aligned} d\bar{Y}^i &= p_j^i dY^j + (dp_j^i)Y^j \\ &= p_j^i dY^j + (p_r^i \lambda_j^r - p_j^s \bar{\lambda}_s^i)Y^j \qquad \text{by (16.3)} \\ &= p_r^i(dY^r + Y^j\lambda_j^r) - \bar{Y}^s\bar{\lambda}_s^i \end{aligned}$$

Hence, on $\mathcal{U}\cap\mathcal{V}$,

$$d\bar{Y}^i + \bar{Y}^s\bar{\lambda}_s^i = p_r^i(dY^r + Y^j\lambda_j^r) \tag{16.4}$$

Equation (16.4) is an interesting result.

On the one hand, if we evaluate the n 1-forms on both sides of (16.4) on a vector field X at points in $\mathcal{U} \cap \mathcal{V}$ we get two sets of n functions which, at each point, are related like the components of a vector. Thus, for each Y we have a mapping $dY : \mathfrak{X}M \to \mathfrak{X}M$; i.e., *a vector-valued 1-form* given by

$$dY : X \mapsto \langle dY^i + Y^j \lambda^i_j, X \rangle \frac{\partial}{\partial \mu^i} \tag{16.5}$$

On the other hand, if we take the tensor product of both sides of eq. (16.4) with $\partial/\partial\bar{\mu}^i$ we get

$$\frac{\partial}{\partial \mu^i} \otimes (dY^i + Y^j \lambda^i_j) = \frac{\partial}{\partial \bar{\mu}^i} \otimes (d\bar{Y}^i + \bar{Y}^j \bar{\lambda}^i_j)$$

That is, the $(1,1)$ tensor field defined on $\mathcal{U}$ by $\dfrac{\partial}{\partial \mu^i} \otimes (dY^i + Y^j \lambda^i_j)$ and the $(1,1)$ tensor field defined on $\mathcal{V}$ by $\dfrac{\partial}{\partial \bar{\mu}^i} \otimes (d\bar{Y}^i + \bar{Y}^j \bar{\lambda}^i_j)$ are the same on $\mathcal{U} \cap \mathcal{V}$.

We can summarize our results as follows.

Theorem 16.1 *On a pseudo-Riemannian manifold* (M, g) *for each vector field,* Y*, there is a vector-valued 1-form,* dY*, given by* (16.5) *and there is a mapping* $\nabla : \mathfrak{X} \to$ *the module of* $(1,1)$ *tensor fields on* M *given by*

$$\nabla : Y \mapsto \frac{\partial}{\partial \mu^i} \otimes (dY^i + Y^j \lambda^i_j) \tag{16.6}$$

Definitions ∇ is called *the Levi-Civita connection* of (M, g), $d : Y \to dY$ is called *covariant differentiation*, dY is *the covariant differential* of the vector field Y, and the $(1-1)$ tensor field $\nabla Y = \partial/\partial\mu^i \otimes (dY^i + Y^j \lambda^i_j)$ is *the covariant derivative* of the vector field Y.

Note that at each point dY and ∇Y are corresponding elements in the isomorphism $\mathfrak{L}(T_p, T_p) \cong T_p \otimes T_p^*$ of Theorem 4.1(ii) since for all vector fields, X, and 1-form fields, θ,

$$(dY \cdot X) \cdot \theta = \nabla Y(\theta, X)$$

From this relation we can introduce the notation $dY \cdot X = \nabla_X Y$ and call this *the covariant derivative of* Y *in the* X *direction.*

Theorem 16.2 ∇ *is* $\mathbb{R}$*-linear on* $\mathfrak{X}M$*, and has the property, for a function,* f*, and a vector field,* Y*,*

$$\nabla f Y = Y \otimes df + f \nabla Y \tag{16.7}$$

Proof These properties come immediately from the definition (16.6). □

If $\{e_i\}$ is a basis of vector fields on an open set of M, and $\{\varepsilon^i\}$ is the dual basis, then $Y = Y^i e_i$ and from (16.7), $\nabla Y = e_i \otimes dY^i + Y^i \nabla e_i$, and the components of ∇Y are

$$Y^j_{;k} \equiv \nabla Y(\varepsilon^j, e_k) = dY^j(e_k) + Y^i \nabla e_i(\varepsilon^j, e_k) \tag{16.8}$$

In particular, if $e_i = \partial/\partial\mu^i$ are coordinate basis vectors, then

$$\nabla e_i(\varepsilon^j, e_k) = \left\{ \begin{matrix} j \\ i\ k \end{matrix} \right\}$$

(see Problem 16.3) and

$$Y^j_{;k} = \nabla Y\left(d\mu^j, \frac{\partial}{\partial\mu^k}\right) = \frac{\partial Y^j}{\partial\mu^k} + \left\{ \begin{matrix} j \\ i\ k \end{matrix} \right\} Y^i \tag{16.9}$$

Much of the preceding can be generalized, starting with a generalization of (16.4). Thus, if instead of Y, a tensor field is given, we get a generalization of (16.4). For example, if K is a given $(1,1)$ tensor field, then differentiating

$$\bar{K}^i_j = \frac{\partial\mu^q}{\partial\bar{\mu}^j}\frac{\partial\bar{\mu}^i}{\partial\mu_p}K^p_q$$

leads to

$$d\bar{K}^i_j + \bar{K}^r_j\bar{\lambda}^i_r - \bar{K}^i_s\bar{\lambda}^s_j = q^l_j p^i_k(dK^k_l + K^r_l\lambda^k_r - K^k_s\lambda^s_l) \tag{16.10}$$

where $q^l_j = \partial\mu^l/\partial\bar{\mu}^j$, which leads to the definition of a $(1,2)$ tensor field, ∇K, on (M,g), or a $(1,1)$ tensor-valued 1-form, dK (see Example (i), Section 4.1). Similarly, starting with an (r,s) tensor field, we get an $(r,s+1)$ tensor field. Thus, we can extend the domain of ∇ to the entire mixed tensor field algebra. If K is an (r,s) tensor field its covariant derivative ∇K is an $(r,s+1)$ tensor field.

PROBLEM 16.1 Let (g_{ij}) be a matrix of the linear mapping $g^\flat : T_p \to T^*_p$ corresponding to g. Then

$$\sum_i \left\{ \begin{matrix} i \\ i\ k \end{matrix} \right\} = \frac{\partial}{\partial\mu^k}\, ln\, \sqrt{\pm\det(g_{ij})}.$$

PROBLEM 16.2 The relation (16.3) between the λ's and the $\bar{\lambda}$'s in terms of $q^j_k = \partial\mu^j/\partial\bar{\mu}^k$ instead of the p's is

$$q^j_k\lambda^r_j = q^r_i\bar{\lambda}^i_k - dq^r_k \tag{16.11}$$

PROBLEM 16.3 Show that if $\{e_i\}$ are coordinate basis vectors and $\{\varepsilon^j\}$ is the dual basis then $\nabla e_i(\varepsilon^j, e_k) = \left\{ \begin{matrix} j \\ i\ k \end{matrix} \right\}$.

PROBLEM 16.4 If $\{e_i\}$ is a local basis of vector fields and $\{\varepsilon^i\}$ is the dual basis, then define $A^i_{jk} = \nabla e_j(\varepsilon^i, e_k)$, where ∇ has the properties of Theorem 16.2. If $(\bar{e}_i)$ is another local basis, and $e_i = p^j_i \bar{e}_j$, then

$$A^i_{jk} p^m_i = \bar{A}^m_{rs} p^r_j p^s_k + \langle dp^m_j, e_k \rangle \tag{16.12}$$

and multiplying by ε^k,

$$A^i_{jk} \varepsilon^k p^m_i = \bar{A}^m_{rs} \bar{\varepsilon}^s p^r_j + dp^m_j \tag{16.13}$$

so $A^i_{jk}\varepsilon^k$ satisfy a relation of the same form as Theorem 16.3.

PROBLEM 16.5 If θ is a 1-form on M, then $\nabla\theta = d\mu^i \otimes (d\theta_i - \theta_j \lambda^j_i)$ defines a $(0,2)$ tensor field on M, and the components of $\nabla\theta$ are

$$\theta_{i,j} = \frac{\partial \theta_i}{\partial \mu^j} - \left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\} \theta_k \tag{16.14}$$

PROBLEM 16.6 If K is a $(1,1)$ tensor field on M, then the components of ∇K are

$$K^i_{j,k} = \frac{\partial K^i_j}{\partial \mu^k} + \left\{ \begin{matrix} i \\ p\ k \end{matrix} \right\} K^p_j - \left\{ \begin{matrix} q \\ j\ k \end{matrix} \right\} K^i_q \tag{16.15}$$

PROBLEM 16.7 If A is an (r,s) tensor field, and B is a (p,q) tensor field, then $\nabla AB = A\nabla B + (\nabla A)B$.

PROBLEM 16.8 X is a Killing field, Section 15.1, iff $X_{i,j} + X_{j,i} = 0$.

16.2 Geodesics of the Levi-Civita connection

Definitions Suppose we are given a vector field, X and a curve γ. Then we can form ∇X, $\nabla X \circ \gamma$, a $(1,1)$ tensor field on γ, and $\dot{\gamma}$, a vector field on γ. Finally, the contraction, $C \cdot \dot{\gamma}(\nabla X \circ \gamma)$, of the product $\dot{\gamma}(\nabla X \circ \gamma)$ is a vector field on γ, called *the covariant derivative of X along γ*. If X is a vector field which satisfies $C \cdot \dot{\gamma}(\nabla X \circ \gamma) = 0$ on γ then X is *propagated* (*transported, translated*, or *displaced*) *parallelly along γ*.

The local expression for $C \cdot \dot{\gamma}(\nabla X \circ \gamma)$ is

$$D_u \gamma^k \left[\left(\frac{\partial X^j}{\partial \mu^k} + \left\{ \begin{matrix} j \\ i\ k \end{matrix} \right\} X^i \right) \circ \gamma \right] = D_u(X^j \circ \gamma) + \left(\left\{ \begin{matrix} j \\ i\ k \end{matrix} \right\} \circ \gamma \right) (D_u \gamma^k)(X^i \circ \gamma) \tag{16.16}$$

The right-hand side of (16.16) shows that we can extend the concept of a vector field being propagated parallelly along a curve, to that of a vector field *on a* *curve* being so propagated. That is, we ask for vector fields, $\widetilde{X}$, defined on γ, such that

$$D_u\widetilde{X}^i + \left(\left\{ {j \atop i\ k} \right\} \circ \gamma\right)(D_u\gamma_k)\widetilde{X}^i = 0 \tag{16.17}$$

on γ.

Equation (16.17) has two important properties. (i) The condition is independent of the parameter of the curve. (ii) Since the system is linear, there exists a unique solution with a given initial vector defined for all u. That is, a vector can be propagated arbitrarily far along a curve.

Definitions If the tangent vector, $\dot\gamma$, of γ satisfies (16.17), i.e., if

$$D_u^2\gamma^j + \left(\left\{ {j \atop i\ k} \right\} \circ \gamma\right) D_u\gamma^k D_u\gamma^i = 0 \tag{16.18}$$

then γ is *a geodesic*. For any curve, γ, the left side of (16.18) is called *the acceleration of* γ, or *the curvature vector of* γ.

Now we no longer have properties (i) and (ii) of eq. (16.17).

Theorem 16.3 *If γ^i satisfy* (16.18), *and $\bar\gamma^i(\bar u) = \gamma^i(u)$, then $\bar\gamma^i$ satisfy*

$$D_{\bar u}^2\bar\gamma^j + \left(\overline{\left\{ {j \atop i\ k} \right\}} \circ \bar\gamma\right) D_{\bar u}\bar\gamma^i D_{\bar u}\bar\gamma^k = 0$$

if and only if $\bar u = au + b$ and $aD_{\bar u}\bar\gamma^i = D_u\gamma^i$ where a and b are numbers (*cf.*, *Theorem 13.2*).

Proof Problem 16.10. □

Thus, we see that if we have two parametric curves with the same values, one could be a geodesic and the other not. However, if a parametric curve with parameter u is a geodesic, then $\|\dot\gamma(u)\|$ is constant (as we will see in Section 16.5 (Theorem 16.10)), which implies that $s = au + b$. That is, on a geodesic (on a pseudo-Riemannian manifold), as long as $\|\dot\gamma(u)\| \neq 0$, we can always use arc-length as a parameter.

Theorem 16.4 *For each $u_0 \in \mathbb{R}$, $p \in M$, and $v \in T_p$ there is an interval containing u_0 and a unique curve γ defined on that interval such that $\gamma(u_0) = p$, $\dot\gamma(u_0) = v$, and γ^i satisfy* (16.18).

Proof Follows immediately from Theorem 13.1 upon putting $D_u\gamma^i = v^i$. □

Finally, we have the following generalization of Theorem 15.15.

Theorem 16.5 *Shortest and longest curves in a pseudo-Riemannian space are geodesics.*

Proof The proof of Theorem 15.15 can be generalized. Thus, just as before a shortest or longest curve is an extremal of **E**, and we have eq. (15.5). Again we differentiate with respect to t, but now the g_{ij} are not constant. In the terms involving the derivatives of the g_{ij} we use

$$\frac{\partial g_{ij}}{\partial \mu^k} = g_{ih} \left\{ \begin{matrix} h \\ j\ k \end{matrix} \right\} + g_{jh} \left\{ \begin{matrix} h \\ i\ k \end{matrix} \right\} \tag{16.19}$$

(see eq. (16.35)) and in the other term we use

$$\frac{\partial}{\partial \mu}\left(g_{ij} \frac{\partial \gamma^i}{\partial t} \frac{\partial \gamma^j}{\partial u} \right) = g_{ij} \frac{\partial}{\partial u}\left(\frac{\partial \gamma^i}{\partial t} \right) \frac{\partial \gamma^j}{\partial u} + g_{ij} \frac{\partial \gamma^i}{\partial t} \frac{\partial^2 \gamma^j}{\partial u^2} + \frac{\partial g_{ij}}{\partial \mu^k} \frac{\partial \gamma^k}{\partial u} \frac{\partial \gamma^i}{\partial t} \frac{\partial \gamma^j}{\partial u}$$

The term on the left drops out after integration. The term on the right is expanded in terms of Christoffel symbols and some of these cancel with those above. We end up with

$$\int_p^q \left[g_{ij} \frac{\partial \gamma^i}{\partial t} \frac{\partial^2 \gamma^j}{\partial u^2} + g_{ih} \left\{ \begin{matrix} h \\ j\ k \end{matrix} \right\} \frac{\partial \gamma^i}{\partial t} \frac{\partial \gamma^j}{\partial u} \frac{\partial \gamma^k}{\partial u} \right] du = 0$$

The integrand is

$$g_{ij} \frac{\partial \gamma^i}{\partial t} \left(\frac{\partial^2 \gamma^j}{\partial u^2} + \left\{ \begin{matrix} j \\ h\ k \end{matrix} \right\} \frac{\partial \gamma^h}{\partial u} \frac{\partial \gamma^k}{\partial u} \right)$$

and now exactly as in the proof of Theorem 15.15 we conclude that the last factor must vanish. □

PROBLEM 16.9 In the following examples you are given a chart on a 2-dimensional manifold such that the representation of the g_{ij} are as shown. Find the representation of the Christoffel symbols and the representations of the geodesics (c is a real constant, and $(a, b) \in \mathbb{R}^2$).

	$\hat{g}_{11}(a,b)$	$\hat{g}_{12}(a,b)$	$\hat{g}_{22}(a,b)$
1	c^2	0	$c^2 \sin^2 a$
2	$1 + c^2$	0	a^2
3	$1 + 4a^2$	0	a^2

PROBLEM 16.10 Prove Theorem 16.3.

16.3 The torsion and curvature of a linear, or affine connection

In Section 16.1 we saw that having a tensor field, g, on M we were able to define a set of 1-forms, λ^i_j, on each coordinate patch in terms of which we subsequently constructed $(r, s+1)$ tensor fields on M. (With a given $(1,0)$ field we constructed a $(1,1)$ field, etc.)

We will now generalize somewhat as follows. Suppose we start with a manifold, M, not necessarily pseudo-Riemannian, and suppose there are given on each coordinate patch a set of 1-forms which satisfy eq. (16.3). That is, suppose we are given 1-forms, ω^l_i, such that

$$p^m_l \omega^l_i = p^r_i \bar{\omega}^m_r + dp^m_i \tag{16.20}$$

where $p^i_j = \partial\bar{\mu}^i/\partial\mu^j$ as before.

Definitions A manifold with a collection of local 1-forms, ω^l_i, satisfying condition (16.20) is *a manifold with a linear, or affine connection.* The 1-forms are called *connection 1-forms.*

Notice now that this structure is all that is needed to proceed to construct covariant derivatives. In particular, we have eqs. (16.4)-(16.8) and (16.10) and (16.11) with ω^l_i replacing λ^l_i. We can now define *the coefficients*, L^k_{ij}, *of a connection* (with respect to a coordinate basis) by

$$\omega^k_i = L^k_{ij} d\mu^j \tag{16.21}$$

Then eq. (16.2) and eqs. (16.9), (16.14), and (16.15) are valid with $\left\{ {k \atop i\ j} \right\}$ replaced by L^k_{ij}.

We can continue just as in Section 16.2 to define parallel propagation and geodesics with L^k_{ij} replacing $\left\{ {k \atop i\ j} \right\}$. The properties described in Section 16.2 of these concepts will still be valid, except that arc-length is no longer available since we do not have the length of a vector.

It is, however, important to notice that while $\left\{ {k \atop i\ j} \right\}$ are symmetric in i and j, this is not necessarily true of L^k_{ij}. From eq. (16.2) with $\left\{ {k \atop i\ j} \right\}$ replaced by L^k_{ij} we see that if L^k_{ij} is symmetric then $\bar{L}^k_{ij}$ is symmetric; i.e., the property of symmetry is independent of coordinates.

Theorem 16.6 *L^k_{ij} is symmetric (in its lower indices) on a chart $(\mathcal{U}, \mu)$ iff for each point $p \in \mathcal{U}$ there is a chart $(\bar{\mathcal{U}}, \bar{\mu})$ with $p \in \bar{\mathcal{U}}$ in which $\bar{L}^k_{ij}(p) = 0$.*

Proof If: If $\bar{L}^k_{ij}(p) = 0$ in $(\bar{\mathcal{U}}, \bar{\mu})$ then at p, eq. (16.2) becomes

$$L^l_{ij} p^m_l = \frac{\partial^2 \bar{\mu}^m}{\partial \mu^i \partial \mu^j}$$

so that at p, L^l_{ij} is symmetric, and we can get this for each $p \in \mathcal{U}$.

Only if: Pick $p \in \mathcal{U}$, and suppose $(\mathcal{U}, \mu)$ is a chart with $\mu(p) = 0$. Define new coordinates on $\mathcal{U}$ of the form

$$\bar{\mu}^i = \mu^i + \frac{1}{2} f^i_{jk} \mu^j \mu^k$$

Then at p, $p^m_l = \delta^m_l$ and $\partial^2 \bar{\mu}^m / \partial \mu^i \partial \mu^j = \frac{1}{2}(f^m_{ij} + f^m_{ji})$. So from eq. (16.2) the condition $\bar{L}^m_{rs}(p) = 0$ is the same as the condition $L^m_{ij}(p) = \frac{1}{2}[f^m_{ij}(p) + f^m_{ji}(p)]$. Since $L^m_{ij}(p)$ is symmetric, this condition can be satisfied by $f^m_{ij}(p) = L^m_{ij}(p)$. $\square$

Definition Coordinates at p for which $L^k_{ij}(p) = 0$ are called *geodesic coordinates.*

In the special case when M is pseudo-Riemannian and L^k_{ij} are the coefficients of the Levi-Civita connection, the vanishing of the L^k_{ij} at p is equivalent to the vanishing of all the partial derivatives of the components of g.

In view of eq. (16.17), one sometimes says that the symmetry of L^k_{ij} corresponds to the existence of coordinate systems at a point of which the components, X^r, of a vector are constant under *infinitesimal parallel displacement.*

There is another interesting geometrical interpretation of the symmetry conditions. (Cf., observation (3) Section 13.3.) We use the first-order approximation of the condition of parallel propagation, $\Delta X^r = -X^i L^r_{is} \Delta \mu^s$. If γ_1 and γ_2 intersect at p, let $\Delta_1 \mu^s$ and $\Delta_2 \mu^s$ be *small displacements* along γ_1 and γ_2, respectively (see Fig. 16.1). Now propagate $\Delta_2 \mu^s$ parallelly along γ_1, and propagate $\Delta_1 \mu^s$ parallelly along γ_2. We get, respectively, $\Delta_2 \mu^s + [-\Delta_2 \mu^i L^r_{is} \Delta_1 \mu^s]$ and $\Delta_1 \mu^s + [-\Delta_1 \mu^i L^r_{is} \Delta_2 \mu^s]$ so that the small vector between the end points of the new vectors is, to the second-order approximation,

$$(L^r_{is} - L^r_{si}) \Delta_1 \mu^s \Delta_2 \mu^i$$

Hence, in particular, if (and only if) the connection is symmetric we get an *infinitesimal parallelogram.*

We will now produce two tensor fields in terms of which we can describe the connection 1-forms.

(1) From eq. (16.2) with $\left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\}$ replaced by L^k_{ij} we see that the differences $L^i_{ji} - L^k_{ij}$ are the components of a $(1, 2)$ tensor field skew-symmetric in its covariant indices.

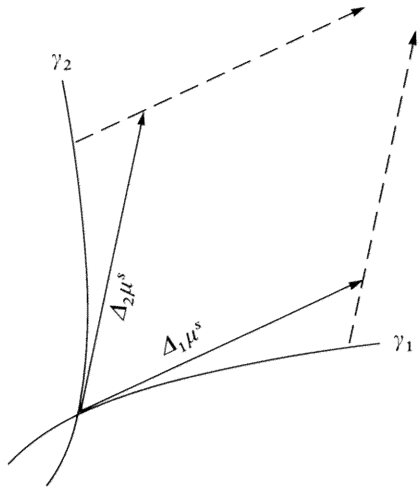

Figure 16.1

Definition $T^k_{ij} = L^k_{ji} - L^k_{ij}$ are the coordinate components of *the torsion tensor*, T, of the connection.

(2) For a tensor field of type (r, s) we can form $K^{i_1\cdots i_r}_{j_1\cdots j_s,k,l} - K^{i_1\cdots i_r}_{j_1\cdots j_s,l,k}$. There is an important set of identities, one for each (r, s), *the Ricci identities*, relating these differences to the torsion tensor and another important tensor field determined by the affine connection. We will derive the one for a vector field, X. Expanding, $X^i_{,j,k}$ using, successively, eqs. (16.15) and (16.9) we get

$$\begin{aligned}
X^i_{,j,k} &= \frac{\partial X^i_{,j}}{\partial \mu^k} + X^p_{,j} L^i_{pk} - X^i_{,q} L^q_{jk} \\
&= \frac{\partial^2 X^i}{\partial \mu^j \partial \mu^k} + \frac{\partial}{\partial \mu^k}(X^p L^i_{pj}) + X^p_{,j} L^i_{pk} - X^i_{,q} L^q_{jk} \\
&= \frac{\partial^2 X^i}{\partial \mu^j \partial \mu^k} + X^p \frac{\partial}{\partial \mu^k} L^i_{pj} + L^i_{pj}(X^p_{,k} - X^q L^p_{qk}) + L^i_{pk} X^p_{,j} - L^q_{jk} X^i_{,q}
\end{aligned}$$

Interchanging j and k and subtracting, we get

$$\begin{aligned}
X^i_{,j,k} - X^i_{,k,j} &= X^p \left(\frac{\partial}{\partial \mu^k} L^i_{pj} - \frac{\partial}{\partial \mu^j} L^i_{kp} \right) \\
&\quad - X^q (L^i_{rj} L^r_{qk} - L^i_{rk} L^r_{qj}) - X^i_{,q}(L^q_{jk} - L^q_{kj}) \\
&= X^p \left[\frac{\partial}{\partial \mu^k} L^i_{pj} - \frac{\partial}{\partial \mu^j} L^i_{pk} + (L^i_{rk} L^r_{pj} - L^i_{rj} L^r_{pk}) \right] \\
&\quad + X^i_{,q}(L^q_{kj} - L^q_{jk})
\end{aligned}$$

Denoting the coefficient of X^p in the first term on the right by $R^i_{\cdot pkj}$ we have the result

$$X^i_{,j,k} - X^i_{,k,j} = X^p R^i_{\cdot pkj} + X^i_{,q} T^q_{jk} \tag{16.22}$$

Definition

$$R^i_{\cdot pkj} = \frac{\partial}{\partial \mu^k} L^i_{pj} - \frac{\partial}{\partial \mu^j} L^i_{pk} + L^i_{rk} L^r_{pj} - L^i_{rj} L^r_{pk} \tag{16.23}$$

are the coordinate components of *the (Riemann) curvature tensor, R, of the connection.*

Note that $R^i_{\cdot pkj}$ is skew-symmetric in its last two covariant indices.

We noted in Section 16.1 that the covariant differential of a vector field corresponds to a vector-valued 1-form. Similarly, the torsion tensor, T, corresponds to a vector-valued 2-form, $\widetilde{T}$, defined by

$$\langle \omega, \widetilde{T}(X,Y) \rangle = T(\omega, X, Y)$$

We can write $\widetilde{T}(X,Y)$ in terms of a basis $\{e_i\}$ of local vector fields as

$$\widetilde{T}(X,Y) = T^k(X,Y) e_k \tag{16.24}$$

This defines the components $T^k(X,Y)$. $T^k : (X,Y) \mapsto T^k(X,Y)$ are k ordinary 2-forms. If we write $T = T^k_{ij} e_k \otimes \varepsilon^i \otimes \varepsilon^j$, we find

$$T^k = T^k_{ij} \varepsilon^i \otimes \varepsilon^j \tag{16.25}$$

Then

$$T^k(e_i, e_j) = T^k_{ij} \tag{16.26}$$

and, from (16.24),

$$\widetilde{T}(e_i, e_j) = T^k_{ij} e_k \tag{16.27}$$

Definitions The vector field $\widetilde{T}(X,Y)$ is called *the torsion translation* and the T^k are called *the torsion forms* of the connection.

Now $T^k(X,Y) = T^k_{ij} X^i Y^j = L^k_{ji} X^i Y^j - L^k_{ij} X^i Y^j$ when the T^k_{ij} are coordinate components of T. But

$$L^k_{ji} X^i Y^j - L^k_{ij} X^i Y^j = \omega^k_i(X) d\mu^i(Y) - \omega^k_i(Y) d\mu^i(X) = 2\omega^k_i \wedge d\mu^i(X,Y).$$

So

$$T^k = 2\omega^k_i \wedge d\mu^i \tag{16.28}$$

Equation (16.28) can be thought of as giving the torsion forms in terms of the connection 1-forms. This could be used to define the torsion tensor instead of the way we did it.

We gave the definition of the torsion tensor of a connection in terms of coordinate bases. We can now calculate the components of the torsion tensor in any basis of vector fields by means of the required transformation law. We would expect to then obtain a generalization of (16.28). We will get this by a different method in Section 17.3. The result is

$$T^k = 2(\omega_i^k \wedge \varepsilon^i + d\varepsilon^k) \tag{16.29}$$

Clearly (16.29) reduces to (16.28) for coordinate vector fields.

Again, as we did for the covariant derivative and the torsion tensor we can describe the curvature tensor, which we defined as a $(1,3)$ tensor, in a sometimes convenient alternative fashion. From the isomorphisms

$$\mathfrak{L}(T_p^*, T_p, T_p, T_p; \mathbb{R}) \cong \mathfrak{L}(T_p, T_p; \mathfrak{L}(T_p^*, T_p; \mathbb{R}))$$

(Theorem 2.12) and $\mathfrak{L}(T_p^*, T_p; \mathbb{R}) \cong \mathfrak{L}(T_p, \mathfrak{L}(T_p^*, \mathbb{R})) \cong \mathfrak{L}(T_p, T_p)$, we see that a $(1,3)$ tensor corresponds to a linear operator-valued $(0,2)$ tensor. Since $R^i_{\cdot pkj}$ is clearly skew-symmetric in its last two covariant indices, R, the curvature tensor, corresponds to a linear operator-valued 2-form, $\widetilde{R}$, defined by

$$\langle \omega, \widetilde{R}(X,Y) \cdot Z \rangle = R(\omega, Z, X, Y)$$

We can write $\widetilde{R}(X,Y)$ in terms of a basis $\{e_i\}$ of local vector fields as

$$\widetilde{R}(X,Y) \cdot e_j = R_j^i(X,Y) e_i \tag{16.30}$$

This defines the components $R_j^i(X,Y)$. $R_j^i : (X,Y) \mapsto R_j^i(X,Y)$ are ordinary 2-forms. If we write $R = R^i_{\cdot jkl} e_i \otimes \varepsilon^j \otimes \varepsilon^k \otimes \varepsilon^l = e_i \otimes \varepsilon^j \otimes R^i_{\cdot jkl} \varepsilon^k \otimes \varepsilon^l$ we find

$$R_j^i = R^i_{\cdot jkl} \varepsilon^k \otimes \varepsilon^l \tag{16.31}$$

Then

$$R_j^i(e_p, e_q) = R^i_{\cdot jpq} \tag{16.32}$$

and, from (16.30),

$$\widetilde{R}(e_i, e_l) \cdot e_j = R^i_{\cdot jkl} e_i \tag{16.33}$$

Definitions The linear operator $\widetilde{R}(X,Y)$ on $\mathfrak{X}M$ is called *the curvature transformation of the connection*, and the R_j^i are called *the curvature forms of the connection.*

A computation analogous to the one for the torsion tensor leads to

$$R^i_j = 2(\omega^i_k \wedge \omega^k_j + d\omega^i_j) \tag{16.34}$$

which gives the curvature forms in terms of the connection forms. Equations (16.29) and (16.34) are called *the (Cartan) structural equations of the linear,* or *affine connection.* Instead of thinking of these equations as giving the torsion and curvature forms in terms of the ε^k and ω^i_j, one can also view them the other way around, as imposing integrability conditions on the 1-forms ε^k and ω^i_j given the torsion and curvature.

PROBLEM 16.11 Derive the Ricci identities (generalization of eq. (16.22):

$$K^{i_1\cdots i_r}_{j_1\cdots j_s,k,l} - K^{i_1\cdots i_r}_{j_1\cdots j_s,l,k} = \sum_p^r K^{i_1\cdots \hat{i}\cdots i_r}_{j_1\cdots\cdots\cdots j_s} R^{i_p}_{\cdot ilk}$$

$$- \sum_q^s K^{i_1\cdots\cdots\cdots i_r}_{j_1\cdots \hat{j}\cdots j_s} R^{j}_{\cdot j_q lk} + K^{i_1\cdots i_r}_{j_1\cdots j_s,r} T^r_{kl}$$

PROBLEM 16.12 For a manifold with a symmetric (torsionless) connection prove

(i) the 1st Bianchi identity,

$$R^i_{\cdot pkj} + R^i_{\cdot kjp} + R^i_{\cdot jpk} = 0$$

or,

$$\widetilde{R}(X,Y)\cdot Z + \widetilde{R}(Y,Z)\cdot X + \widetilde{R}(Z,X)\cdot Y = 0$$

(ii) the 2nd Bianchi identity,

$$R^i_{\cdot jkl,m} + R^i_{\cdot jlm,k} + R^i_{\cdot jmk,l} = 0$$

or

$$dR^i_j = 0$$

PROBLEM 16.13 For a pseudo-Riemannian manifold (M,g) define $R_{ijkl} = g_{ip}R^p_{\cdot jkl}$. Then (i) $R_{ijkl} = -R_{jikl}$; (ii) $R_{ijkl} = R_{klij}$; and (iii) R_{ijkl} has $(n^2(n^2-1))/12$ distinct nonvanishing components. (See Problems 5.6 and 5.7.)

PROBLEM 16.14 For the examples of Problem 16.9 calculate $\widehat{R}_{ijkl}$ and $\widehat{R}_{ijkl}/\det(\hat{g}_{ij})$.

16.4 The exponential map and normal coordinates

From the existence theorem, Theorem 16.4, we see that through each fixed point, p, of M there are lots of geodesics. This enables us to construct a diffeomorphism between a neighborhood of 0 of T_p and a neighborhood of p, and, as a consequence, a certain useful coordinate system at p.

Recall that Theorem 13.3 in Section 13.2 affirmed the existence of flow boxes for $\dot{\gamma}(u) = X(\gamma(u))$. There is a corresponding result for eq. (16.18); namely, for each $p \in M$ and $v \in T_p$, there exists a neighborhood $\mathcal{W}$ of (p, v) in TM and a $u_0 \in \mathbb{R}$, such that all solutions of (16.18) with initial values in $\mathcal{W}$ are defined for $0 \leq u < u_0$. Hence, in particular, for a fixed point, p, there is a neighborhood $\mathcal{U}$ of 0 in T_p for which all geodesics through p are defined for $0 \leq u < u_0$. Denote the geodesic corresponding to v by γ_v. If $u_0 > 1$ we have a differentiable mapping $\mathcal{U} \to M$ given by $v \mapsto \gamma_v(1)$. If $u_0 \leq 1$ we can make a change of parameter $\bar{u} = au$ with $a > 1/u_0$ and get geodesics with shorter initial vectors, according to Theorem 16.3, and with parameter values going to 1. Thus, in any event, we have a mapping $\exp_p$ from some neighborhood of $0 \in T_p$ to M given by $\exp_p : v \mapsto \gamma_v(1)$. $\exp_p$ is called *the exponential map at p.*

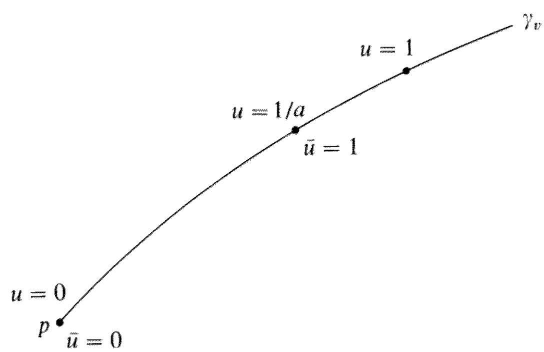

Figure 16.2

Theorem 16.7 *For a given $p \in M$ and $v \in T_p$, $\exp_p$ has the property $\exp_p(tv) = \gamma_v(t)$ for all $t \leq 1$.*

Proof Suppose γ_v has a parameter u. Reparametrize γ_v with $\bar{u} = au$ so that $\bar{u} = 1$ between $u = 0$ and $u = 1$. That is, choose $a > 1$ (Fig. 16.2). Then we get the geodesic $\bar{\gamma}_{v/a}$ with initial vector v/a, by Theorem 16.3, and $\exp_p : v/a \mapsto \bar{\gamma}_{v/a}(1) = \gamma_v(1/a)$. For each $a > 1$ we have such a geodesic, so we have a map $(1/a)v \mapsto \gamma_v(1/a)$ for all $a \geq 1$. $\square$

Now let $\{e_i\}$ be a basis of T_p (with $e_i \in \mathcal{U}$). Then for $v \in \mathcal{U}, v = v^i e_i$. So we have a map, ϕ, of a neighborhood of $0 \in \mathbb{R}^n$ to a neighborhood of p. In particular, $(u, 0, \ldots, 0) \mapsto ue_1 \mapsto \gamma_{ue_1}(1) = \gamma_{e_1}(u)$, etc. The tangent map of ϕ at $0 \in \mathbb{R}^n$ is nonsingular since the tangents to the coordinate curves at $0 \in \mathbb{R}^n$ map into the linearly independent set $\{e_i\}$. Hence by the inverse function theorem there is a diffeomorphism between a neighborhood of $0 \in \mathbb{R}^n$ and a neighborhood, $\mathcal{V}$, of $p \in M$, and $(\mathcal{V}, \phi^{-1}|_{\mathcal{V}})$ is a local coordinate system at p. In terms of these coordinates, the equations of a geodesic are $\gamma_v^i(u) = v^i u$, so $D_u^2 \gamma_v^k = 0$, and geodesics satisfy $(L_{ij}^k \circ \gamma) D_u \gamma^i D_u \gamma^j = 0$ (cf., eq. (16.18)). We can summarize these results as follows.

Theorem 16.8 *At each point, p, on a manifold with a connection there is* (i) *a diffeomorphism between a neighborhood $\mathcal{V}$ of p and a neighborhood* 0 *of T_p. In particular, for each $q \in \mathcal{V}$ there is a geodesic segment starting at p and ending at q;* (ii) *a local coordinate system, called* a normal (*or* Riemannian, *or* geodesic) *coordinate system in which the geodesics through p are given by $\gamma_v^i(u) = v^i u$ and if L_{ij}^k is symmetric, $L_{ij}^k(p) = 0$.*

Theorem 16.9 *On a pseudo-Riemannian manifold the length of the geodesic segment joining the points p and $q = \exp_p(v) = \gamma_v(1)$ is the length of the initial tangent vector, v.*

Proof Problem 16.15. □

PROBLEM 16.15 Prove Theorem 16.9.

16.5 Connections on pseudo-Riemannian manifolds

In Section 16.3 we described a certain structure which we imposed on an arbitrary manifold called a connection. The question arises whether one or more such structures can be imposed on a given manifold, i.e., questions of existence and uniqueness. Since every manifold can be given a Riemannian structure, every manifold can be given the connection of Section 16.1. The question of uniqueness is also easily disposed of. From eq. (16.2) it is evident that if L_{ij}^k are the coefficients of a connection, and K_{ij}^k are the components of a tensor, then $L_{ij}^k + K_{ij}^k$ are the coefficients of a connection, and that is all there are.

We see, in particular, that for a pseudo-Riemannian manifold there is a whole class of connections in addition to the Levi-Civita connection defined in terms of the given metric tensor field in Section 16.1. However, it is possible to sort out this particular connection by means of its properties.

Theorem 16.10 *The Levi-Civita connection has the properties*

(1) *Its torsion is zero.*

(2) *If X and Y are both parallel along a curve, γ, then $D_u g(X,Y) \circ \gamma = 0$ (the connection "preserves" or "is compatible with" g).*

Proof (1) is immediate from the definition of T and the symmetry of the Christoffel symbols.

(2) From the definition of $[ij,k]$ and the symmetry of g we get

$$\frac{\partial g_{ik}}{\partial \mu^j} = [ij,k] + [kj,i] = g_{kl} \left\{ {l \atop i\ j} \right\} + g_{il} \left\{ {l \atop k\ j} \right\} \tag{16.35}$$

Multiplying both sides of (16.35) by $D_u \gamma^j X^i Y^k$ we get

$$\begin{aligned} X^i Y^k \frac{\partial g_{ik}}{\partial \mu^j} D_u \gamma^j &= Y^k X^i D_u \gamma^j \left\{ {l \atop i\ j} \right\} g_{kl} + X^i g_{il} Y^k D_u \gamma^i \left\{ {l \atop k\ j} \right\} \\ &= -Y^k g_{kl} D_u X^l - X^i g_{il} D_u Y^l \end{aligned}$$

using the parallelism of X and Y along γ. Thus, $D_u(g_{ik} X^i Y^k) = 0$ on γ. □

Theorem 16.11 (The Ricci Lemma) *On any pseudo-Riemannian manifold with a connection, property (2) of Theorem 16.10 is equivalent to $g_{ik,j} = 0$.*

Proof Notice first that we have already, for the Levi-Civita connection, from (16.35) that $g_{ik,j} = 0$. Now

$$\begin{aligned} D_u(g_{ik} X^i Y^k) &= X^i Y^k \frac{\partial g_{ik}}{\partial \mu^j} D_u \gamma^j + Y^k g_{kl} D_u X^l + X^i g_{il} D_u Y^l \\ &= X^i Y^k \frac{\partial g_{ik}}{\partial \mu^j} D_u \gamma^j - Y^k X^i D_u \gamma^j L^l_{ij} g_{kl} - X^i g_{il} Y^k D_u \gamma^j L^l_{kj} \end{aligned}$$

if X and Y are parallel along γ. So

$$D_u(g_{ik} X^i Y^k) = X^i Y^k D_u \gamma^j \left[\frac{\partial g_{ik}}{\partial \mu^j} - L^l_{ij} g_{kl} - L^l_{kj} g_{il} \right]$$

The conclusion follows from the fact that at any given point X^i and Y^k can be chosen arbitrarily. □

Now we can show that the Levi-Civita connection is the only connection on a pseudo-Riemannian manifold with properties (1) and (2) of Theorem 16.10; i.e., properties (1) and (2) of Theorem 16.10 actually characterize the Levi-Civita connection among the connections on a pseudo-Riemannian manifold. This is frequently called *the fundamental theorem of Riemannian geometry.*

Theorem 16.12 *If L^i_{jk} is a connection on a pseudo-Riemannian manifold and L^i_{jk} has properties* (1) *and* (2) *of Theorem 16.10, then* $L^i_{jk} = \left\{ \begin{matrix} i \\ j\ k \end{matrix} \right\}$.

Proof From the Ricci lemma, $\partial g_{ik}/\partial \mu^j = L^l_{ij} g_{kl} + L^l_{kj} g_{il}$. Permute the indices and form $\left\{ \begin{matrix} i \\ j\ k \end{matrix} \right\}$ on the left side. The right side will reduce to L^i_{jk}. □

Definition A connection on a pseudo-Riemannian manifold with property (2) of Theorem 16.10 (or, equivalently with $g_{ik,j} = 0$) is called *a metric connection.*

If we drop the condition that the connection of (M, g) preserves g (property (2) of Theorem 16.10), we can construct the following class of "semimetric" connections.

Theorem 16.13 *If (M, g) is a pseudo-Riemannian manifold and ω is a 1-form, then* $W^i_{jk} = \left\{ \begin{matrix} i \\ j\ k \end{matrix} \right\} + \frac{1}{2}(\delta^i_j \omega_k + \delta^i_k \omega_j - g_{jk} g^{il} \omega_l)$ *are the coefficients of a symmetric connection on (M, g). Such a connection is called* a Weyl (*or* conformal) connection.

Proof Problem 16.17. □

Theorem 16.14 *The Weyl connections are the symmetric connections characterized by the property $g_{mn,r} + g_{mn}\omega_r = 0$.*

Proof Problem 16.19. □

Clearly, the Levi-Civita connection is a special case of a Weyl connection.

By Theorem 16.13 we see that with each pair (g, ω) we can associate a connection. This correspondence is not 1-1, for if $g'_{mn} = f g_{mn}$ and

$$\omega'_r = \omega_r - \frac{\partial \ln |f|}{\partial \mu^r}$$

where f is a function on M, then the pair (g', ω') has the same connection as (g, ω). The map $\omega_r \mapsto \omega_r - \partial \ln |f| / \partial \mu^r$; i.e., adding an exact differential, or gradient to ω_r is *a gauge transformation*, (cf., Section 12.1) and $g_{mn} \mapsto f g_{mn}$ is *a conformal transformation.* (See Section 15.1.)

Weyl spaces arose out of an historically important attempt to create a unified field theory of electromagnetic and gravitational forces, using the fact that here ω transforms like an electromagnetic potential (Section 21.3) and g transforms with a factor depending on position, i.e., the scale or "gauge" can vary from point to point (Weyl, (1922), Section 35, Bergmann, Chap. XVI). *Projective spaces* are another class of manifolds with a symmetric connection (Schouten, Chap. VI).

PROBLEM 16.16 Using eq. (16.35) and an orthonormal basis show that on a pseudo-Riemannian manifold the Levi-Civita connection 1-forms satisfy $\lambda_j^i + \lambda_i^j = 0$.

PROBLEM 16.17 Prove Theorem 16.13.

PROBLEM 16.18 On a pseudo-Riemannian manifold the coefficients of a connection are

$$L_{ij}^l = \left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\} + \frac{1}{2}\widetilde{T}_{ij}^l - \frac{1}{2}\tilde{g}_{ij}^l$$

where

$$\widetilde{T}_{ij}^l = T_{ij}^l + g^{lk}g_{sj}T_{ik}^s + g^{lk}g_{is}T_{jk}^s$$

and

$$\tilde{g}_{ij}^l = -g^{lk}g_{ij,k} + g^{lk}g_{jk,i} + g^{lk}g_{ki,j}$$

PROBLEM 16.19 Prove Theorem 16.14. (See Problem 16.18.)

PROBLEM 16.20 For a Weyl connection, instead of property (2) of Theorem 16.10, we have that, if X and Y are parallel along a curve, then the ratio of their lengths and the angle between them are preserved.

17

CONNECTIONS ON MANIFOLDS

In Section 16.3 we defined what we mean by a manifold with a linear (or affine) connection. We made the definition in terms of local 1-form fields. There are several other ways of describing connections. We will first present two of these, and then describe the most important structures arising out of a connection on a manifold.

17.1 Connections between tangent spaces

In Sections 16.2 and 16.3, we noted that with a connection we can introduce the idea of parallel propagation of vectors along curves. Now we will show that we can go the other way. Hence our definition of a connection and parallel propagation along curves are equivalent. Indeed, the idea of parallel propagation perhaps motivates the term "connection" better than its definition. For on a manifold in general there is no connection between tangent spaces at two different points, but with the introduction of parallel propagation of vectors along a curve we have a mapping $T_{p_0} \to T_p$ between tangent spaces at p_0 and p and thus quite literally a connection between the tangent spaces at different points of M.

Let us go along a given curve, γ, joining p_0 to $\gamma(u) = p$ in a coordinate neighborhood, $\mathcal{U}$, of p_0. Now suppose we have functions, ϕ^i_j, on $\mathcal{U}$ such that

$$X^i_p = \phi^i_j(\gamma(u))X^j_{p_0} \tag{17.1}$$

for all vectors X_{p_0} in T_{p_0}. Then, assuming adequate differentiability, we get $D_u X^i_p = (D_u \phi^i_j) X^j_{p_0}$ along γ, or

$$D_u X^i_p + f^i_j X^j_p = 0 \tag{17.2}$$

where $f^i_j = -(D_u \phi^i_j)(\phi^i_j)^{-1}$ (matrix multiplication).

Now we want the linear mappings of the tangent spaces to be independent of the choice of parameter of γ, and also of the choice of coordinates. So eq. (17.2) must have these properties. The first condition is satisfied since the f^i_j are linear functions of $D_u \gamma^k$; that is,

$$f^i_j = \ell^i_{jk} D_u \gamma^k \tag{17.3}$$

The second condition requires a certain transformation law for the f^i_j. Namely, from $\bar{X}^i = p^i_j X^j$,

$$D_u \bar{X}^i = \frac{\partial^2 \bar{\mu}^i}{\partial \mu^j \partial \mu^k}(D_u \gamma^k) X^j + p^i_j (D_u X^j)$$

and from (17.2),

$$-\bar{f}^i_j \bar{X}^j = \frac{\partial^2 \bar{\mu}^i}{\partial \mu^j \partial \mu^k}(D_u \gamma^k) X^j - p^i_j f^j_k X^k$$

or

$$\left(-\bar{f}^i_j p^j_k + p^i_j f^j_k - \frac{\partial^2 \bar{\mu}^i}{\partial \mu^j \partial \mu^k} D_u \gamma^j\right) X^k = 0$$

or, since this is to be independent of X^k, and using (17.3), we get

$$\left(\ell^j_{kt} p^i_j - \bar{\ell}^i_{js} p^s_t p^j_k - \frac{\partial^2 \bar{\mu}^i}{\partial \mu^k \partial \mu^t}\right) D_u \gamma^t = 0 \tag{17.4}$$

Finally, requiring (17.1) to be valid for all curves, γ, implies that (17.4) must be independent of $D_u \gamma^t$, so

$$\ell^j_{kt} p^i_j = \bar{\ell}^i_{js} p^j_k p^s_t + \frac{\partial^2 \bar{\mu}^i}{\partial \mu^i \partial \mu^t} \tag{17.5}$$

Thus, the ℓ^i_{kt} satisfy the same equations, (eq. (16.2)), as the Christoffel symbols, $\left\{ \begin{matrix} i \\ k\ t \end{matrix} \right\}$, and the same equations as the coefficients, L^i_{kt}, of an affine connection, and as the functions A^i_{kt} defined in Problem 16.4.

17.2 Coordinate-free description of a connection

Recall that in Section 16.1 we defined the Levi-Civita connection of a pseudo-Riemannian manifold. We now generalize.

Definition *A connection at a point*, $p \in M$, is a linear map, $\mathbf{D}_p$, from T_p to the vector space of derivations from $\mathfrak{X}M$ to T_p. That is,

$$\mathbf{D}_p : v_p \mapsto \mathbf{D}_p(v_p)$$

where

$$\mathbf{D}_p(v_p) : Y \in \mathfrak{X}M \mapsto \mathbf{D}_p(v_p)Y \in T_p$$

with the following properties, where we write $\mathbf{D}_{v_p}$ instead of $\mathbf{D}_p(v_p)$, replacing the redundant subscript, p, on $\mathbf{D}$ with the argument v_p.

1. $\mathbf{D}_{av_p+bw_p} = a\mathbf{D}_{v_p} + b\mathbf{D}_{w_p}$ for $a, b \in \mathbb{R}$.
2. (i) $\mathbf{D}_{v_p}$ is $\mathbb{R}$-linear.
   (ii) $\mathbf{D}_{v_p} fY = (v_p f)Y_p + f(p)\mathbf{D}_{v_p} Y$ for $f \in \mathfrak{F}_p$.

Now, suppose we have a connection at each point $p \in M$. Since each $v_p = X(p)$ for some vector field $X \in \mathfrak{X}M$, we can write $\mathbf{D}_{v_p} = \mathbf{D}_{X(p)}$ for some $X \in \mathfrak{X}M$, and we have

(1) a vector field

$$\mathbf{D}_X Y : M \to TM$$

given by $\mathbf{D}_X Y : p \mapsto \mathbf{D}_{X(p)} Y$, and

(2) a map

$$\mathbf{D} : \mathfrak{X}M \times \mathfrak{X}M \to \mathfrak{X}M$$

given by $\mathbf{D} : (X, Y) \mapsto \mathbf{D}_X Y$.

Clearly, $\mathbf{D}_X$ satisfies the same conditions as $\mathbf{D}_{v_p}$ with a and b replaced by f and g. $\mathbf{D}$ is frequently called *a Koszul connection* on M (Spivak 1979, Vol. II, Chap. 6).

Definitions The bilinear map

$$\bar{\nabla} Y : \mathfrak{X}^* M \times \mathfrak{X}M \to \mathfrak{F}_M$$

given by $\bar{\nabla} Y : (\theta, X) \mapsto \mathbf{D}_X Y(\theta)$ is called *the covariant derivative of* Y of *the connection* $\bar{\nabla}$, a map from vector fields to $(1,1)$ tensor fields given by $Y \mapsto \bar{\nabla} Y$.

Theorem 17.1 *$\bar{\nabla}$ satisfies Theorem 16.2, so it is a generalization of the Levi-Civita connection, ∇.*

Proof Problem 17.1. □

Notation: To avoid excessive notation we will drop the bar on ∇, and it will be made clear in a particular discussion whether we are talking about the general or the particular case. Also, we will replace $\mathbf{D}_X Y$ by $\nabla_X Y$.

Now with a local basis of vector fields we can proceed as we did in Section 16.1. In particular, if $Y = Y^i e_i$, according to Theorem 17.1 we have

$$\nabla Y = e_i \otimes dY^i + Y^i \nabla e_i \tag{17.6}$$

We define *the coefficients, Γ^i_{jk}, of the connection, ∇, with respect to the basis* $\{e_i\}$, by

$$\nabla e_j = \Gamma^i_{jk} e_i \otimes \varepsilon^k \tag{17.7}$$

or $\nabla_{e_k} e_j = \Gamma^l_{jk} e_l$ or $\Gamma^i_{jk} = \nabla e_j(\varepsilon^i, e_k) = \nabla_{e_k} e_j(\varepsilon^i)$. Then from (17.6) and (17.7),

$$\nabla Y = (\langle dY^i, e_j \rangle + Y^k \Gamma^i_{kj}) e_i \otimes \varepsilon^j \tag{17.8}$$

so, contracting,

$$C \cdot (X\nabla Y) = e_i\langle dY^i, X\rangle + Y^k\Gamma^i_{kj}e_i\langle \varepsilon^j, X\rangle$$

and (Problem 17.6)

$$\nabla_X Y = (XY^i + Y^k\Gamma^i_{kj}X^j)e_i \tag{17.9}$$

Clearly, in a coordinate system, the coordinate components of ∇Y in (17.8) are precisely the functions in (16.9), with $\left\{ {j \atop i\ k} \right\}$ replaced by Γ^j_{ik}, and the coordinate components of $\nabla_X Y$ in (17.9) are $X^j Y^i_{,j}$ where $Y^i_{,j}$ are the components of ∇Y.

With the Γ^i_{jk} defined here precisely as the A^i_{jk} were defined in Problem 16.4, the transformation law for the Γ^i_{jk} is the same as that for A^i_{lk} (and for $\left\{ {i \atop j\ k} \right\}$, and for Γ^i_{jk}, and for ℓ^i_{jk}). Finally, if we define 1-forms ε^i_j by $\varepsilon^i_j = \Gamma^i_{jk}\varepsilon^k$, they will transform just as the λ^i_j of Section 16.1, and the connection 1-forms, ω^i_j, of Section 16.3.

Recall that in Section 16.1, after constructing the covariant derivative, ∇Y, of a vector field, we indicated that the construction can be generalized to obtain the covariant derivative, ∇K, of any tensor field. (We exhibited two specific results in Problem 16.5, and Problem 16.6.) We make a similar generalization now. Note, first that, while starting with the vector field $\nabla_X Y$, we defined the $(1,1)$ tensor field ∇Y, we can just as well, starting with $\nabla_X Y$, make the following definition.

Definition The map $\nabla_X : \mathfrak{X}M \to \mathfrak{X}M$ given by $Y \mapsto \nabla_X Y$ is called *covariant differentiation in the X direction.*

Theorem 17.2 *∇_X is $\mathbb{R}$-linear on $\mathfrak{X}M$ and has the property, for a function, f, and a vector field, Y,*

$$\nabla_X fY = XfY + f\nabla_X Y$$

Proof Problem 17.3. □

We will generalize ∇_X instead of ∇Y. We use the same idea that we used for the Lie derivative.

Theorem 17.3 *Given a vector field, X, there exists a unique derivation, ∇_X, on the algebra of tensor fields on M with the following properties.*

(i) $\nabla_X f = Xf$ *for* $f \in \mathfrak{F}_M$.

(ii) $\nabla_X Y$ *is the covariant derivative of* Y *in the* X *direction (with respect to the given connection).*

(iii) $\nabla_X\langle\theta, Y\rangle = \langle\theta, \nabla_X Y\rangle + \langle\nabla_X\theta, Y\rangle$ *for* $Y \in \mathfrak{X}$ *and* $\theta \in \mathfrak{X}^*$ *("∇_X commutes with contractions," see Problem 13.12).*

Proof ∇_X is defined uniquely on f, X, and θ by (i), (ii), and (iii). For contravariant tensor fields define $\nabla_X K$ locally by

$$\begin{aligned}\nabla_X K^{i_1\cdots i_r} e_{i_1}\otimes\cdots\otimes e_{i_r} &= (\nabla_X K^{i_1\cdots i_r} e_{i_1}\otimes\cdots\otimes e_{i_{r-1}})\otimes e_{i_r}\\ &+ (K^{i_1\cdots i_r} e_{i_1}\otimes\cdots\otimes e_{i_{r-1}})\otimes\nabla_X e_{i_r}\end{aligned}$$

and for mixed tensor fields define $\nabla_X K$ locally by

$$\begin{aligned}\nabla_X K^{i_1\cdots i_r}_{j_1\cdots j_s} e_{i_1}\otimes\cdots\otimes\varepsilon^{j_s} &= (\nabla_X K^{i_1\cdots i_r}_{j_1\cdots j_s} e_{i_1}\otimes\cdots\otimes\varepsilon^{j_{s-1}})\otimes\varepsilon^{j_s}\\ &+ (K^{i_1\cdots i_r}_{j_1\cdots j_s} e_{i_1}\otimes\cdots\otimes\varepsilon^{j_{s-1}})\otimes\nabla_X\varepsilon^{j_s}\end{aligned}$$

To show $\nabla_X JK = J\nabla_X K + (\nabla_X J)K$, write the local expression for both sides and use the given definitions to get an identity. For any derivation, specification of the operation on the lower-order tensors determines it (uniquely) on the higher-order ones. Finally, existence and uniqueness on each coordinate neighborhood gives existence and uniqueness on the entire manifold (cf., Theorem 12.1). □

In terms of a given basis of vector fields, we have, in particular, for a 1-form, θ,

$$\nabla_X\theta = \nabla_X\theta_i\varepsilon^i = (\nabla_X\theta_i)\varepsilon^i + \theta_i\nabla_X\varepsilon^i = (X\theta_i)\varepsilon^i + \theta_i\nabla_X\varepsilon^i$$

by Theorem 17.3(i). By Theorem 17.3(iii), $\langle\nabla_X\varepsilon^i, e_j\rangle = -\langle\varepsilon^i, \nabla_X e_j\rangle$, so $\nabla_X\varepsilon^i\cdot e_j = -X^k\langle\varepsilon^i, \nabla_{e_k}e_j\rangle = -X^k\Gamma^i_{jk}$ by the definition of Γ^i_{jk} in eq. (17.7). So, finally,

$$\nabla_X\theta = (X\theta_i - \theta_j X^k\Gamma^j_{ik})\varepsilon^i \tag{17.10}$$

Now we define ∇K for any tensor field, K, in terms of $\nabla_X K$ by

$$\nabla K(\theta^1,\ldots,\theta^r, X, X_1,\ldots,X_s) = \nabla_X K(\theta^1,\ldots,\theta^r, X_1,\ldots,X_s)$$

In particular, $\nabla f(X) = \nabla_X f$, $\nabla Y(\theta, X) = \nabla_X Y(\theta)$, and $\nabla\theta(X, X_1) = \nabla_X\theta(X_1)$. From eqs. (17.9) and (17.10), respectively, we get the coordinate components of ∇Y and $\nabla\theta$, already obtained in eqs. (16.9) and (16.14).

Finally, we want to describe parallel propagation of vectors along curves and geodesics (Section 16.2) in terms of the present notation.

If X is a vector field over a curve, γ, then with our definition of $\nabla_X Y$, a vector field Y is propagated parallelly along γ if $\nabla_{\dot\gamma}Y = 0$ on γ.

If Y is a vector field over a (immersed) curve, γ, we can extend it locally to a vector field, Z, on a neighborhood of γ. Then $Y = Z\circ\gamma$ and we define

1. $\nabla_{\dot\gamma}Y = (\nabla_{\dot\gamma}Z)\circ\gamma$, the covariant derivative of Y along γ (cf., Section 16.2).
2. Y is propagated parallelly along γ if $\nabla_{\dot\gamma}Y = 0$.

3. γ is a geodesic if $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$.

PROBLEM 17.1 Prove Theorem 17.1.

PROBLEM 17.2 Using eq. (17.9) and the formula of Theorem 12.2(ii) for the differential of 1-forms, show that

$$\nabla_X Y - \nabla_Y X - [X, Y] = 2(d\varepsilon^i + \varepsilon^i_j \wedge \varepsilon^j)(X, Y)e_i \tag{17.11}$$

PROBLEM 17.3 Prove Theorem 17.2.

PROBLEM 17.4 (i) Suppose X_i, $i = 1, \ldots, n$, are independent vector fields on an n-dimensional manifold, M, (i.e., $\{X_i\}$ is a parallelization of M). For any vector fields X and $Y = Y^i X_i$, define $\nabla_X Y$ by $\nabla_X X_i = 0$ and $\nabla_X Y = XY^i X_i$. Then ∇_X satisfies Theorem 17.2. Each X_i is said to be *a parallel vector field* (with respect to this connection).

(ii) We will say a manifold, M, has *a flat connection,* ∇, if it has a coordinate covering on each domain of which the coefficients of ∇ are zero. Show that an affine manifold (see Section 15.3) has a flat connection. (Note that this is a global property of M.)

PROBLEM 17.5 Prove that for any three vector fields, X, Y, Z, and any second-order covariant tensor field, g,

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) + (\nabla_Z g)(X, Y) \tag{17.12}$$

PROBLEM 17.6 Using contractions we can write the relations between ∇Y and $\nabla_X Y$ as $\nabla_X Y = C\cdot(X\nabla Y)$ and we can write Eq. (17.12) as $\nabla_Z C^1_1 \cdot C^2_2 \cdot (XYg) = C^1_1 \cdot C^2_2 \cdot \nabla_Z (XYg)$.

PROBLEM 17.7 Show that covariant differentiation in the X direction satisfies the same formula as Lie differentiation in Theorem 13.12. (Hint: Generalize the second result of Problem 17.6.)

PROBLEM 17.8 Show that the definition of $\nabla_{\dot{\gamma}} Y$ is independent of the vector field Z.

17.3 The torsion and curvature of a connection

In Section 16.3, we introduced and discussed the concepts of the torsion and the curvature of a connection. These tensors were defined in terms of their components in a coordinate basis. We now give an alternative approach which starts with coordinate-free definitions.

Definition Given a connection, we can define a map

$$\widetilde{T}: \mathfrak{X}M \times \mathfrak{X}M \to \mathfrak{X}M$$

by

$$(X, Y) \mapsto \nabla_X Y - \nabla_Y X - [X, Y]$$

That is,

$$\widetilde{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{17.13}$$

Clearly, $\widetilde{T}$ is a vector-valued 2-form.

By Problem 17.2, the components of the vector $\widetilde{T}(X, Y)$ with respect to an arbitrary basis are $2(d\varepsilon^i + \varepsilon^i_k \wedge \varepsilon^k)(X, Y)$. In particular, with respect to a coordinate basis its components are $(2\varepsilon^i_k \wedge d\mu^k)(X, Y)$. But these are the components, in a coordinate basis, of the vector $\widetilde{T}(X, Y)$ defined in Section 16.3 - as we showed in the paragraph just above eq. (16.28). So these vectors are the same. It follows, in particular, that eq. (16.29) is the general form of eq. (16.28). The 2-forms T^k are called *the torsion forms of the connection*, and the $(1, 2)$ tensor, T, defined by $T(\omega, X, Y) = \langle \omega, \widetilde{T}(X, Y) \rangle$ is *the torsion tensor of the connection.*

Definition Given a connection, we can define a map

$$\widetilde{R}: \mathfrak{X}M \otimes \mathfrak{X}M \to \mathfrak{L}(\mathfrak{X}M, \mathfrak{X}M)$$

by

$$(X, Y) \mapsto \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

That is,

$$\widetilde{R}(X, Y) \cdot Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{17.14}$$

Clearly, $\widetilde{R}(X, Y)$, *the curvature transformation of the connection*, is a linear operator on $\mathfrak{X}M$, and $\widetilde{R}$ is a linear operator-valued 2-form. If the components of $\widetilde{R}(X, Y)$ in a basis are $R^i_j(X, Y)$ and a $(1, 3)$ tensor, R, is defined by $R(\omega, Z, X, Y) = \langle \omega, \widetilde{R}(X, Y) \cdot Z \rangle$, then $R^i_j = R^i_{.jkl} \varepsilon^k \otimes \varepsilon^l$. R is *the (Riemann) curvature tensor of the connection*, and the 2-forms R^i_j are *the curvature forms of the connection.*

We have to show that these $R^i_{.jkl}$ are the $R^i_{.jkl}$ in Section 16.3 when a coordinate basis is used. We write the left side of (17.14) in any basis,

$$\widetilde{R}(X, Y) \cdot Z = R^i_{.pkj} X^k Y^j Z^p e_i \tag{17.15}$$

and, in view of our observations just below eq. (17.9), the components of the right side of (17.14) are

$$X^k(Y^j Z^k_{,j})_{,k} - Y^j(X^k Z^i_{,k})_{,j} - [X,Y]^r Z^i_{,r} \tag{17.16}$$

In a coordinate basis, $[X,Y]^r = XY^r - YX^r$, so expanding (17.16) we get

$$X^k Y^j [Z^i_{,j,k} - Z^i_{,k,j} + (L^q_{jk} - L^q_{kj}) Z^i_{,q}] \tag{17.17}$$

Equating (17.17) and the coefficient of the right side of (17.15), we see that $R^i_{.pkj}$ satisfy the condition (16.22) by which these coefficients were originally defined.

We now have a rather neat formal description of the curvature tensor of a connection in eq. (17.14). We have some analytical meaning for this concept from the way it was introduced in Section 16.3, namely, as a measure of the noncommutativity of second-order covariant derivatives. This description can also be compared with the noncommutativity relation $L_{[X,Y]}Z = L_X L_Y Z - L_Y L_X Z$ for Lie derivatives. See Problem 13.11. We have, finally, to consider its geometrical significance. This is illustrated by two of its properties.

(1) Let X be a vector field, and let Y be a vector field such that $[X,Y] = 0$; for example, the tangent field of the horizontal curves of a variation of some curve (Section 15.2) with the given X along it. Then putting $Z = X$ in eq. (17.14) and substituting $\nabla_Y X$ into it from eq. (17.13), we get

$$\widetilde{R}(X,Y) \cdot X = \nabla_X(\nabla_X Y - \widetilde{T}(X,Y)) - \nabla_Y \nabla_X X$$

If, further, the integral curves of X are geodesics, the last term on the right vanishes, and we have the result that, if X is a vector field whose integral curves are geodesics and Y is a vector field such that $[X,Y] = 0$, then X and Y satisfy

$$\nabla_X \nabla_X Y = \widetilde{R}(X,Y) \cdot X + \nabla_X \widetilde{T}(X,Y)$$

In particular, if we pick out a geodesic, γ, then on γ, the restriction, Y_γ, of Y satisfies

$$\nabla_{\dot\gamma} \nabla_{\dot\gamma} Y_\gamma = \widetilde{R}(\dot\gamma, Y_\gamma) \cdot \dot\gamma + \nabla_{\dot\gamma} \widetilde{T}(\dot\gamma. Y_\gamma) \tag{17.18}$$

Equation (17.18) is called *the Jacobi equation*, or *the equation of geodesic deviation.* It is second-order ordinary differential equation. A solution is called *a Jacobi vector field.*

We can interpret (17.18) as a relation between the curvature and the deviation of the geodesics if we consider the case of a Riemannian manifold with the Levi-Civita connection (so $\widetilde{T} = 0$) and the geodesics above all passing through a point p on γ. (p cannot be in any coordinate neighborhood in which X and Y are coordinate vector fields, but it can be a limit point of such a neighborhood.)

Let s be the arc-length along γ measured from p, and introduce *the sectional (Riemannian) curvature*

$$K_{xy} = g_{ij} R^i_{.klm} x^j y^k x^l y^m$$

determined by the unit vectors $x = \dot{\gamma}$ and $y = Y_\gamma / \|Y_\gamma\|$. (See Problem 17.10.) Then taking the scalar product of (17.18) with y we get

$$\frac{d^2\|Y_\gamma\|}{ds^2} + \|Y_\gamma\| g(y, \nabla_x \nabla_x \, y) + \|Y_\gamma\| K_{xy} = 0$$

One can show that in the limit as $s \to 0$, the middle term drops out (Problem 17.14) and we get

$$\lim_{s\to 0} \frac{1}{\|Y_\gamma\|} \frac{d^2\|Y_\gamma\|}{ds^2} = -K_{xy} \tag{17.19}$$

Let s^* be the length parameter along the integral curves of Y. Then $s^* = \|Y_\gamma\| v$ + terms of order v^2, where v is a parameter for Y. So

$$\frac{d^2\|Y_\gamma\|}{ds^2} = \lim_{v\to 0} \frac{\partial^2}{\partial s^2} \left(\frac{\partial s^*}{\partial v} \right)$$

and hence along γ near $s = 0$, if $K_{xy} > 0$ the rate of separation of the geodesics decreases, and if $K_{xy} < 0$ the rate of separation of the geodesics increases.

We will see eq. (17.18) (with $\widetilde{T} = 0$) again in Section 22.3. There it will be related to an equation of "relative" motion, in which Y_γ is the "relative position vector" of points on neighboring geodesics, subject to a "tidal force."

(2) We have another illustration of the geometrical meaning of the curvature tensor in the special case $R = 0$.

Definition A vector field X which satisfies $\nabla X = 0$ is called *a parallel vector field.* (Cf., Problem 17.4.)

Theorem 17.4 *On a manifold with a connection, ∇, there exists a neighborhood of p in which $R = 0$ iff there exist n linearly independent parallel vector fields in some neighborhood of p.*

Proof **If:** At each point, $R(\omega, Z, X, Y) = \langle \omega, \widetilde{R}(X, Y) \cdot Z \rangle = 0$ for all ω, Z, X, Y, since by Eq. (17.14) $\widetilde{R}(X_i, X_j) \cdot X_k = 0$ for the parallel vector fields X_i, $i = 1, \ldots, n$.

Only if: We will give a proof for the case when ∇ is the Levi-Civita connection of a pseudo-Riemannian manifold. For the general case see Bishop and Goldberg (pp. 236-237).

Consider two points p and q, and two curves γ_1 and γ_2 joining p and q (Fig. 17.1). Given an initial vector X_p at p, move it parallelly along γ_1 and γ_2 to get X_{q_1} and X_{q_2}, respectively, at q. We will show $R = 0 \Rightarrow X_{q_1} = X_{q_2}$.

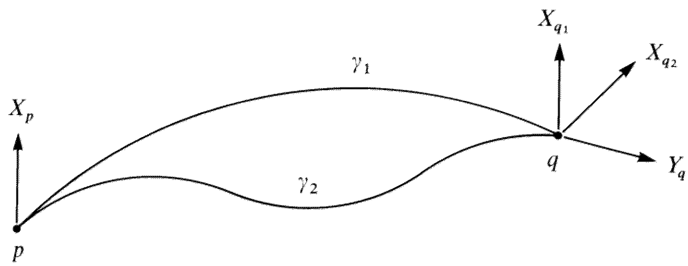

Figure 17.1

Parametrize both γ_1 and γ_2 by u going between u_1 and u_2. Imbed γ_1 and γ_2 in a family of curves. Thus, for example, if in coordinates $\gamma_1 : u \mapsto \gamma_1^i(u)$ and $\gamma_2 : u \mapsto \gamma_2^i(u)$, we can construct the family $f^i(u,t) = \gamma_1^i(u) + t(\gamma_2^i(u) - \gamma_1^i(u))$ on $I = \{(u,t) \in \mathbb{R}^2 : u_1 \le u \le u_2 \text{ and } 0 = t_1 \le t \le t_2 = 1\}$. Now we can also propagate X_p along all the curves of the family and get a vector field, X, on the image of $I - q$. Finally, choose a vector Y_q at q and propagate it back to p along all the curves and get another vector field.

Let v_u be the tangent vector of the $u =$ const. curves, and let v_t be the tangent vector of the $t =$ const. curves. Then

$$\begin{aligned}
&\nabla_{v_u}(g(X,Y)) - g(X, \nabla_{v_u} Y) \\
&\quad = g(\nabla_{v_u} X, Y) && \text{by eq. (17.12) and Theorem 16.11} \\
&\quad = \int_{u_1}^{u} \frac{\partial}{\partial u} g(\nabla_{v_u} X, Y) du && \text{since } \nabla_{v_u} X = 0 \text{ at } u = u_1 \\
&\quad = \int_{u_1}^{u} \nabla_{v_t} g(\nabla_{v_u} X, Y) du \\
&\quad = \int_{u_1}^{u} g(\nabla_{v_t} \nabla_{v_u} X, Y) du && \text{by eq. (17.12), and since } \nabla_{v_t} Y = 0 \\
&\quad = \int_{u_1}^{u} g((\nabla_{v_u} \nabla_{v_t} X + \widetilde{R}(v_t, v_u) \cdot X), Y) du && \text{by eq. (17.14)} \\
&\quad = \int_{u_1}^{u} g(\widetilde{R}(v_t, v_u) \cdot X, Y) du && \text{since } \nabla_{v_t} X = 0
\end{aligned}$$

In particular, at $u = u_2$ we get

$$\nabla_{v_u}(g(X,Y)) = \int_{u_1}^{u_2} g(\widetilde{R}(v_t, v_u) \cdot X, Y)\, du$$

Finally, integrating from t_1 to t_2 we get

$$g(X_{q_2}, Y_q) - g(X_{q_1}, Y_q) = \int_{t_1}^{t_2} \int_{u_1}^{u_2} g(\widetilde{R}(v_t, v_u) \cdot X, Y)\, du\, dt$$

and hence, since Y_q is arbitrary, if $R = 0$, then $X_{q_2} = X_{q_1}$. Corresponding to n linearly independent vectors X_p at p we thus get n linearly independent parallel vector fields. □

Theorem 17.5 *If in Theorem 17.4, ∇ is the Levi-Civita connection of a pseudo-Riemannian manifold, then there is a local coordinate system at p for which*

$$g_{ij} = \begin{cases} \pm 1 & i = j \\ 0 & i \neq j \end{cases}$$

Proof If X^i are components of a vector field, X then we get a 1-form ω with components $\omega_i = g_{ij}X^j$ and $\omega_{i,k} = g_{ij}X^j_{,k}$. If X is a parallel vector field ($\nabla X = 0$) then $\omega_{i,k} = 0$ and hence $d\omega = 0$. We get then, locally, a function f such that $df = \omega$, or $\partial f / \partial \mu^i = \omega_i$.

Let ω^α, and f^α be, respectively, n 1-forms and n functions corresponding to n linearly independent parallel vector fields, X_α. Choose new coordinates by

$$\bar{\mu}^\alpha = f^\alpha(\mu^j)$$

Then

$$\bar{\omega}^\alpha_j = \frac{\partial \mu^k}{\partial \bar{\mu}^j}\omega^\alpha_k = \frac{\partial \mu^k}{\partial \bar{\mu}^j}\,\frac{\partial f^\alpha}{\partial \mu^k} = \delta^\alpha_j$$

Now for two parallel vector fields, X_α and X_β we have $\nabla_g(X_\alpha, X_\beta) = 0$ (by eq. (17.12) and the Ricci lemma), or $\partial/\partial\mu^k\ (g_{rs}X^r_\alpha X^s_\beta) = 0$. But $g_{rs}X^r_\alpha X^s_\beta = g^{rs}\omega^\beta_r\omega^\alpha_s = \bar{g}^{rs}\delta^\beta_r\delta^\alpha_s = \bar{g}^{\beta\alpha}$. Thus, the $\bar{g}^{\beta\alpha}$ are constants. By a linear transformation we get metric coefficients of the kind required. □

Theorem 17.6 *$R = 0$ and $T = 0$ locally $\Leftrightarrow$ there exist n parallel coordinate vector fields $\Leftrightarrow$ there is a coordinate system for which the coefficients of the connection vanish; i.e., the connection is* locally flat (Chern 1959, p. 82).

Proof The first equivalence comes from eq. (17.13) and the fact (see Section 13.3) that X_i are coordinate vector fields if and only if $[X_i, X_j] = 0$. The second equivalence comes from the definition of the Γ's in eq. (17.7). □

PROBLEM 17.9 Show that (i) the properties of the Levi-Civita connection in Theorem 16.10 can be written, respectively,

(1) $[X, Y] = \nabla_X Y - \nabla_Y X$ (torsion-free).

(2) $Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$ (metric).

(ii) (1) and (2) together are equivalent to

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y])$$

(the Koszul formula).

PROBLEM 17.10 (i) Show that in a pseudo-Riemannian manifold with the Levi-Civita connection,

$$K_{XY} = \frac{g(\widetilde{R}(X, Y) \cdot Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2}$$

depends only on the plane (in T_pM) determined by X and Y at each point. K_{XY} is called *the sectional (Riemannian) curvature of the plane determined by X and Y*.

(ii) Show that R is determined if K_{XY} is known for all pairs X, Y. (See Problem 5.6.)

PROBLEM 17.11 For S^3 with the coordinates and the Riemannian structure of Problem 15.6, compute K_{XY} for pairs of coordinate vectors.

PROBLEM 17.12 The sectional, or Riemannian, curvature, K_{XY}, at a point, p, is the Gaussian curvature of the surface formed by the geodesics through p in the directions of the subspace $\langle\{X, Y\}\rangle$. (Cf., Laugwitz, pp. 125 ff, or Synge and Schild, pp. 94 ff.)

PROBLEM 17.13 K_{XY} is independent of X and Y at a point iff

$$R_{ijkl} = \varkappa(g_{ik}g_{jl} - g_{il}g_{jk})$$

or

$$\widetilde{R}(X, Y) \cdot Z = \varkappa(g(Z, Y)X - g(Z, X)Y)$$

and $\varkappa$ is their common value. (See Problem 5.6.)

PROBLEM 17.14 Show $\lim_{s \to 0} \|Y_\gamma\| g(y, \nabla_x \nabla_x y) = 0$ in the derivation of eq. (17.19) (Synge and Schild, p. 97).

PROBLEM 17.15 Prove the last statement in the proof of Theorem 17.4.

17.4 Some geometry of submanifolds

If (P, ϕ) is a submanifold of a pseudo-Riemannian manifold (M, h), then, in general, (P, ϕ) has a natural induced pseudo-Riemannian structure, $\phi^* h$. See Section 15.1. In coordinate components the relation between h and $g = \phi^* h$ is given by eq. (15.3). $\phi^* h$ has the required properties except that in the non-Riemannian case $\phi^* h$ could be degenerate, for example, on the null curves of a Lorentzian manifold.

We have a criterion for the nature of a submanifold in the following special case

Theorem 17.7 *Suppose (P, ϕ) is a hypersurface of a Lorentzian manifold, (M, h).* (i) *If $\phi^* h$ is nondegenerate, then there exists* a normal vector, n, *at each point such that $h(n, n) < 0$ ($h(n, n) > 0$) iff (P, ϕ) is Riemannian (Lorentzian).* (ii) *$\phi^* h$ is degenerate iff there exists a normal vector, n, at each point such that $h(n, n) = 0$.*

Proof These results are valid at each point according to Problem 5.25. □

A submanifold, (P, ϕ), if $\phi^* h$ is nondegenerate, also has a natural induced connection. We can think of P as contained in M, and ϕ as the inclusion map (Section 10.1). Then at each $p \in P$ the tangent space $T_p P$ is a subspace of $T_p M$. For (P, g) pseudo-Riemannian and, hence, g nondegenerate, we can write $T_p M = T_p P \oplus (T_p P)^\perp$ where $(T_p P)^\perp$ is the orthogonal complement of $T_p P$ and we have the projections $\boldsymbol{\pi} \cdot T_p M = T_p P$ and $\boldsymbol{\pi}^\perp \cdot T_p M = (T_p P)^\perp$.

If X and Y are vector fields on P, we can extend them to M, and if ∇' is a connection on (M, h), then at points of P we have

$$\nabla'_X Y = \boldsymbol{\pi} \cdot \nabla'_X Y + \boldsymbol{\pi}^\perp \cdot \nabla'_X Y \tag{17.20}$$

Theorem 17.8 *The map given by $(X, Y) \mapsto \boldsymbol{\pi} \cdot \nabla'_X Y$ is $\mathbb{R}$-bilinear and $\boldsymbol{\pi} \cdot \nabla'_X fY = (Xf)Y + f\boldsymbol{\pi} \cdot \nabla'_X Y$.*

Proof Problem 17.16. □

Definition Theorem 17.8 says that $\nabla : \mathfrak{X}P \times \mathfrak{X}P \to \mathfrak{X}P$ given by $(X, Y) \mapsto \boldsymbol{\pi} \cdot \nabla'_X Y$ is a connection, *the induced connection*, and we write $\boldsymbol{\pi} \cdot \nabla'_X Y = \nabla_X Y$.

Theorem 17.9 *If ∇' is the Levi-Civita connection of a pseudo-Riemannian manifold (M, h), then ∇ is the Levi-Civita connection of $(P, \phi^* h)$.*

Proof (i) Extend the vector fields X, Y on P to vector fields on M. Then on P,

$$\nabla'_X Y - \nabla'_Y X - [X,Y] = \nabla_X Y + \boldsymbol{\pi}^\perp \cdot \nabla'_X Y - (\nabla_Y X + \boldsymbol{\pi}^\perp \cdot \nabla'_Y X) - [X,Y]$$

so the vanishing of the left side entails the vanishing of

$$\nabla_X Y - \nabla_Y X - [X,Y] \text{ (the tangential field)}$$

and

$$\boldsymbol{\pi}^\perp \cdot \nabla'_X Y - \boldsymbol{\pi}^\perp \cdot \nabla'_Y X \text{ (the normal field)}$$

Thus, we have, in addition to the vanishing of the torsion, the extra result that the map given by $(X,Y) \mapsto \boldsymbol{\pi}^\perp \cdot \nabla'_X Y$ is symmetric.

(ii) The condition (2) on (M,h) of Problem 17.9 reduces on P to

$$Xg(Y,Z) = g(\nabla'_X Y, Z) + g(Y, \nabla'_X Z)$$

on P, where X,Y,Z are vector fields on P and $g = \phi^* h$. But $g(\nabla'_X Y, Z) = g(\nabla_X Y + \boldsymbol{\pi}^\perp \cdot \nabla'_X Y, Z) = g(\nabla_X Y, Z)$, since $g(\boldsymbol{\pi}^\perp \cdot \nabla'_X Y, Z) = 0$. Similarly, for $g(Y, \nabla'_X Z)$, which gives us condition (2) of Problem 17.9 on (P,g). □

Theorem 17.10 *For (M,h) with the Levi-Civita connection, the map $II : \mathfrak{X}P \times \mathfrak{X}P \to \mathfrak{X}P^\perp$ given by $(X,Y) \mapsto \boldsymbol{\pi}^\perp \cdot \nabla'_X Y$ is symmetric and bilinear over $\mathfrak{F}_p$.*

Proof Problem 17.7. □

Definition The map, II, of Theorem 17.10 is called *the second fundamental form* or *shape factor*, of P. (Since II is symmetric, the word "form" is a misnomer.)

In terms of the notations ∇ and II, eq. (17.20), for (P,h) with the Levi-Civita connection, is

$$\nabla'_X Y = \nabla_X Y + II(X,Y) \tag{17.21}$$

Equation (17.20) or eq. (17.21) is called *the Gauss formula* or *Gauss decomposition.*

Now we consider another decomposition. If $X \in \mathfrak{X}P$ and $Z(p) \in (T_pP)^\perp$ for $p \in P$, we can extend these vector fields to M and write

$$\nabla'_X Z = \boldsymbol{\pi} \cdot \nabla'_X Z + \boldsymbol{\pi}^\perp \cdot \nabla'_X Z \tag{17.22}$$

as we did before for $X, Y \in \mathfrak{X}P$.

Theorem 17.11 *The mapping* $\widetilde{\mathfrak{S}}:\mathfrak{X}P \times \mathfrak{X}P^\perp \to \mathfrak{X}P$ *given by* $(X, Z) \to \boldsymbol{\pi} \cdot \nabla'_X Z$ *is* $\mathfrak{F}_p$ *bilinear.*

Proof Write $\nabla'_X fZ = f\nabla'_X Z + (Xf)Z = f(\boldsymbol{\pi} \cdot \nabla'_X Z + \boldsymbol{\pi}^\perp \nabla'_X Z) + (Xf)Z$ and $\nabla'_X fZ = \boldsymbol{\pi} \cdot \nabla'_X fZ + \boldsymbol{\pi}^\perp \cdot \nabla'_X fZ$ and compare tangential components. Similarly, for $\nabla'_{fX} Z$. □

Theorem 17.12 *The map* $\nabla^\perp:\mathfrak{X}P\times\mathfrak{X}P^\perp \to \mathfrak{X}P^\perp$ *given by* $(X, Z) \mapsto \boldsymbol{\pi}^\perp \cdot \nabla_X Z$*, called* the normal connection*, has the properties of a connection.*

Proof Problem 17.18. □

In terms of the notations $\widetilde{\mathfrak{S}}$ and $\nabla^\perp$, eq. (17.22) is

$$\nabla'_X Z = \widetilde{\mathfrak{S}}(X, Z) + \nabla^\perp_X Z \tag{17.23}$$

Equation (17.22) or eq. (17.23) is called *the Weingarten formula* or *Weingarten decomposition.*

Theorem 17.13 *For* (M, h) *with the Levi-Civita connection,* $h(\widetilde{\mathfrak{S}}(X, Z), Y) = -h(II(X, Y), Z)$ *for* $X, Y \in \mathfrak{X}P$ *and* $Z \in \mathfrak{X}P^\perp$.

Proof Differentiate $h(Y, Z) = 0$ covariantly with respect to X. Then $h(\nabla'_X Y, Z) + h(Y, \nabla'_X Z) = 0$. Substitute for $\nabla'_X Y$ and $\nabla'_X Z$ from (17.21) and (17.23), respectively, and eliminate two terms by orthogonality. □

We will illustrate these concepts in the geometry of submanifolds in the special cases where P is a hypersurface in the pseudo-Riemannian manifold (M, h) with the Levi-Civita connection.

1. If Z is a unit normal vector field, then differentiating $h(Z, Z) = 1$ we get $h(\nabla'_X Z, Z) = 0$, so $h(\nabla^\perp_X Z, Z) = 0$, by (17.23). Since, M is a hypersurface, $\nabla^\perp_X Z$ and Z are proportional, so $\nabla^\perp_X Z = 0$. Thus, for Z a unit normal vector field, eq. (17.23) reduces to

$$\nabla'_X Z = \widetilde{\mathfrak{S}}(X, Z) \tag{17.24}$$

For a fixed $Z \in \mathfrak{X}P^\perp$, the map given by $X \mapsto \widetilde{\mathfrak{S}}(X, Z)$ is a linear operator on $\mathfrak{X}P$, so in this case the map $\mathfrak{S}: X \mapsto \boldsymbol{\pi} \cdot \nabla'_X Z$ is a linear operator on $\mathfrak{X}P$. $\mathfrak{S}$ is called *the shape operator* (O'Neill) or the *Weingarten map* (Hicks; Spivak). In this case we can write $II(X,Y) = l(X,Y)Z$, and the formula in Theorem 17.13 becomes

$$g(\mathfrak{S}(X), Y) = -\, l(X,Y) \tag{17.25}$$

where $g = \phi^* h$. From this we see that $\mathfrak{S}$ is a self-adjoint operator. Moreover, in a basis of T_pP at a point, p, $\mathfrak{s}_{\alpha\beta} = -\, l_{\alpha\beta}$, $\alpha, \beta = 1, \ldots, n-1$, where $(\mathfrak{s}_{\alpha\beta})$ is the matrix of $\mathfrak{S}$ in that basis. The algebraic invariants of $\mathfrak{S}$ in the special case when P is a surface in the Euclidean space $\mathcal{R}_0^3$ are the Gaussian and mean curvatures of P and the eigenvalues of $\mathfrak{S}$ are the principal curvatures.

2. If we specialize further to the case where P is a hypersurface in $\mathcal{R}_0^n$ the classical results of that subject emerge.

(i) Let Z be a unit normal vector field. Then Z has components Z^i, $i = 1, \ldots, n$, with the property $\sum_{i=1}^n Z^i Z^i = 1$. Define the map $\phi\colon P \to S^{n-1} \subset \mathcal{R}_0^n$ by $p \mapsto (Z^1(p), \ldots, Z^n(p))$. ϕ is called *the spherical map of Gauss.* For $X \in \mathfrak{X}P$, the components of $\nabla'_X Z$ are $X^j(\partial Z^i/\partial \pi^j)$. Let γ be a curve on P with $[\gamma] = X$. Then the tangent map of ϕ takes $X = [\gamma]$ to $[\phi \circ \gamma]$ and $[\phi \circ \gamma]$ has components

$$D_0(Z^i \circ \gamma) = \frac{\partial Z^i}{\partial \pi^j} D_0 \gamma^j = \frac{\partial Z^i}{\partial \pi^j} X^j$$

which are the components of $\nabla'_X Z$. Thus, $\mathfrak{S}$ is the tangent map of the Gauss spherical map.

(ii) To find the components, $l_{\alpha\beta}$, of l in a coordinate system (μ^α) on (P, ϕ) we evaluate $\nabla'_{\partial/\partial\mu^\alpha}(\partial/\partial\mu^\beta)$ and take its normal component. Thus, since

$$\nabla'_{\partial/\partial\mu^\alpha}\left(\frac{\partial}{\partial\mu^\beta}\right) = \nabla'_{\partial/\partial\mu^\alpha}\left(\frac{\partial\phi^k}{\partial\mu^\beta}\,\frac{\partial}{\partial\pi^k}\right) = \frac{\partial^2\phi^k}{\partial\mu^\alpha\partial\mu^\beta}\,\frac{\partial}{\partial\pi^k}$$

we have

$$l_{\alpha\beta} = h\left(II\left(\frac{\partial}{\partial\mu^\alpha}, \frac{\partial}{\partial\mu^\beta}\right), Z\right) = h\left(\frac{\partial^2\phi^k}{\partial\mu^\alpha\partial\mu^\beta}\,\frac{\partial}{\partial\pi^k}, Z^i\frac{\partial}{\partial\pi^i}\right) = \sum_{k=1}^n Z^k \frac{\partial^2\phi^k}{\partial\mu^\alpha\partial\mu^\beta}$$

This is the usual classical expression for the components of the second fundamental form (cf., Eisenhart, 1947, p. 215).

(iii) In this case, using coordinate vector fields $\partial/\partial\mu^\alpha$, $\partial/\partial\mu^\beta$ for X, Y, the Gauss formula (17.21) reduces to

$$\frac{\partial^2\phi^i}{\partial\mu^\alpha\partial\mu^\beta} = \Gamma^\gamma_{\alpha\beta}\,\frac{\partial\phi^i}{\partial\mu^\gamma} + l_{\alpha\beta}Z^i \tag{17.26}$$

where Z^i are the components of a unit normal vector. Similarly, using Theorem 17.13, the Weingarten formula (17.24) becomes

$$\frac{\partial Z^i}{\partial\mu^\alpha} = -\, l_{\alpha\gamma}\, g^{\gamma\beta}\,\frac{\partial\phi^i}{\partial\mu^\beta} \tag{17.27}$$

Equations (17.26) and (17.27) are, respectively, the classical equations of Gauss and Weingarten (cf., Eisenhart, 1947, pp. 216-217).

Another important result in the geometry of submanifolds is the relationship between the Riemann curvature tensors of P and M. For M with X, Y, Z tangent to P, we have from (17.14),

$$\widetilde{R}'(X,Y)\cdot Z = \nabla'_X\nabla'_Y Z - \nabla'_Y\nabla'_X Z - \nabla'_{[X,Y]}Z$$

Using the Gauss and Weingarten formulas (17.21) and (17.23) on the right side, we get

$$\begin{aligned}\widetilde{R}'(X,Y)\cdot Z = \widetilde{R}(X,Y)\cdot Z &+ \widetilde{\mathfrak{S}}(X, II(Y,Z)) - \widetilde{\mathfrak{S}}(Y, II(X,Z)) \\ &+ II(X,\nabla_Y Z) - II(Y,\nabla_Y Z) - II([X,Y],Z) \\ &+ \nabla^{\perp}_X II(Y,Z) - \nabla^{\perp}_Y II(X,Z) \qquad (17.28)\end{aligned}$$

Note that the first three terms on the right are tangent to P, and the last five terms are normal to P. If W is another tangent vector field, then from (17.28), using Theorem 17.13,

$$\begin{aligned}h(\widetilde{R}'(X,Y)\cdot Z, W) = h(\widetilde{R}(X,Y)\cdot Z, W) &+ h(II(Y,W), II(X,Z)) \\ &- h(II(X,W), II(Y,Z)) \qquad (17.29)\end{aligned}$$

Equation (17.29) is called *the Gauss curvature equation.* The fact that the normal part of $\widetilde{R}'(X,Y)\cdot Z$ equals the last five terms of (17.28) is called *the Codazzi-Mainardi equation.*

In the special case where (P,ϕ) is a hypersurface in $\mathcal{R}^n_0$, the left side of (17.28) vanishes, and the tangential and normal parts on the right are separately zero, so

$$\widetilde{R}(X,Y)\cdot Z = l(X,Z)\mathfrak{S}\cdot Y - l(Y,Z)\mathfrak{S}\cdot X \qquad (17.30)$$

and

$$II(X,\nabla_Y Z) - II(Y,\nabla_X Z) - II([X,Y],Z) + \nabla^{\perp}_X II(Y,Z) - \nabla^{\perp}_Y II(X,Z) = 0 \qquad (17.31)$$

For X, Y, Z coordinate fields $\partial/\partial\mu^\alpha$, $\partial/\partial\mu^\beta$, $\partial/\partial\mu^\gamma$, and using eq. (17.15), $\mathfrak{S}\cdot X = \mathfrak{s}^\lambda_\mu\,\partial/\partial\mu^\lambda$, etc., and (17.25) on (17.30), and using (17.7), and $\nabla^\perp_X II(Y,Z) = Xl(Y,Z)N$, etc., where N is a unit normal vector field on (17.31), eqs. (17.30) and (17.31) become, respectively,

$$R^{\delta}_{\cdot\gamma\alpha\beta} = l^\delta_\alpha l_{\beta\gamma} - l^\delta_\beta l_{\alpha\gamma} \qquad (17.32)$$

$$\frac{\partial}{\partial\mu^\alpha} l_{\beta\gamma} + l_{\alpha\delta}\Gamma^\delta_{\alpha\beta} = \frac{\partial}{\partial\mu^\beta} l_{\alpha\gamma} + l_{\beta\delta}\Gamma^\delta_{\gamma\alpha} \qquad (17.33)$$

Equations (17.32) and (17.33) are, respectively, the classical equations of Gauss (*another* "Gauss equation") and Codazzi-Mainardi (cf., Eisenhart, 1947, p. 219).

We conclude this section with a few important observations, without going into details.

(1) Recall that the left side of (17.32), as defined in eq. (16.23), is just a certain complicated combination of the components of g and their first and second derivatives, so the combination of l's on the right side, in terms of which Gauss' definition of the curvature of a surface in $\mathcal{R}_0^3$ can be expressed, depends only on g. Hence Gauss' curvature is an intrinsic property of the surface - Gauss' celebrated *Theorema Egregium* (1827).

(2) The Gauss-Weingarten equations (17.26) and (17.27) can be considered to be an analog of the Frenet-Serret equations of curve theory in $\mathcal{R}_0^3$ (or in $\mathcal{R}_0^n$). They give the derivatives of the tangent and normal vectors in terms of the vectors themselves. For curves in $\mathcal{R}_0^3$, the coefficients, curvature and torsion determine the curve uniquely up to a rigid motion. That is the Fundamental Theorem of Curve Theory. For hypersurfaces in $\mathcal{R}_0^n$, there is an analog, the Fundamental Theorem for Hypersurfaces. That theorem says that the coefficients, the g's and l's, in eqs. (17.26) and (17.27) determine a unique surface in $\mathcal{R}_0^n$ up to a rigid motion. In the case of hypersurfaces, however, since the Gauss-Weingarten equations are partial differential equations (the corresponding Frenet-Serret equations are ordinary differential equations), the coefficients have to satisfy integrability conditions. We saw that (17.32) and (17.33) were necessary conditions for a hypersurface of $\mathcal{R}_0^n$. It turns out that they are also sufficient; i.e., they are the integrability conditions.

PROBLEM 17.16 Prove Theorem 17.8.

PROBLEM 17.17 Prove Theorem 17.10.

PROBLEM 17.18 Prove Theorem 17.12.

PROBLEM 17.19 Derive eqs. (17.26) and (17.27).

PROBLEM 17.20 Find $l_{\alpha\beta}$ for S^3 in the local coordinates of Problem 15.6.

PROBLEM 17.21 Verify eqs. (17.32) and (17.33) for S^3 in the local coordinates of Problem 17.20.

PROBLEM 17.22 Show that if X and Y determine a plane in T_pP then the sectional curvatures for P and M are related by

$$K'_{XY} = K_{XY} + \frac{g(II(X,X), II(Y,Y)) - g(II(X,Y), II(X,Y))}{g(X,X)g(Y,Y) - g(X,Y)^2}$$

PROBLEM 17.23 If Ω is the volume form of an oriented pseudo-Riemannian manifold, and N is the positive unit normal vector field on a hypersurface, then $i_N\Omega$ is the volume form of the hypersurface.

18

MECHANICS

Our exposition of geometry in Chapters 15, 16, and 17 consisted of starting with a manifold and superimposing a symmetric tensor field of type $(0, 2)$, or prescribing a covariant derivative for vector fields, and then studying the properties of such abstract structures. A motivation for each generalization and abstraction is that it includes as a special case the classical well-established geometry with which we are familiar.

Starting with a manifold we can superimpose other kinds of tensor fields, or k-dimensional distributions, and then study the resulting structures. The study of manifolds with skew-symmetric tensor fields of type $(0, 2)$ and of manifolds with $n-1$-dimensional distributions is motivated by the fact that classical mechanics can be described in terms of these structures as a special case.

In classical mechanics, the "motion" of "systems of particles" and "rigid bodies" is described by describing each "position" or "location" by a set of numbers called generalized coordinates. The set of all positions is called *the configuration space*. To get a coordinate-free description we will, in each specific case, model configuration space by a manifold. Then the tangent and cotangent manifolds of the configuration space will turn out to be what are classically called *the state space* and *the phase space*. We will find, in particular, that there is a natural (or "canonical") skew-symmetric tensor field on the cotangent manifold in terms of which we can write the laws of classical mechanics. Again, rather than giving a detailed exposition of this model of classical mechanics at this time, we will generalize, which should enhance both efficiency and insight, and we will point out the relationships to mechanics as we go along.

18.1 Symplectic forms, symplectic mappings, Hamiltonian vector fields, and Poisson brackets

Recall that in Section 11.3 we defined an s-form to be a differentiable map from a manifold, M, to the manifold $\bigwedge^s(T^*M)$ of exterior s-forms. If $s = 2$, that is, if ω is a 2-form, we say it is nondegenerate if $\omega(p)$ is nondegenerate for all $p \in M$.

Definitions A nondegenerate closed 2-form, ω, on M is called *a symplectic* (or *Hamiltonian*) *form*, and the pair (M, ω) is *a symplectic* (or *Hamiltonian*) *manifold.*

Recall that in Section 5.4(ii) we looked at $\bigwedge^2(V^*)$ and introduced symplectic terminology for vector spaces. It is clear from the Corollary of Theorem 5.23 that, if M is finite-dimensional, and (M, ω) is symplectic then M must be even-dimensional. Locally we have

$$\omega = \omega_{ij} d\mu^i \wedge d\mu^j$$

The requirement of differentiability on M, of course, raises the question of existence of a symplectic form on a given M.

Just as in Chapter 15, when we had the symmetric tensor, g, with the symplectic form, ω, we have a linear map $\omega^\flat : \mathfrak{X} \to \mathfrak{X}^*$ given by $X \mapsto \omega^\flat \cdot X$ where $(\omega^\flat \cdot X) \cdot Y = \omega(X, Y)$. Since ω is nondegenerate we also have the inverse $\omega^\sharp : \mathfrak{X}^* \to \mathfrak{X}$. In contrast to the symmetric case, where, if M is orientable, with g we can define two volume forms, ω determines a unique $2n$-form.

$$\omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n}$$

on M; that is, a symplectic manifold is oriented.

Now we want to consider mappings between two given symplectic manifolds (M, ω) and (N, θ), or, in particular, mappings from a symplectic manifold to itself.

Definition A differentiable mapping $\phi : M \to N$ is a *symplectic map* if it "preserves," or "leaves invariant" the symplectic structures.

This descriptive definition is suggestive and convenient, and corresponds to the definition of isometry for pseudo-Riemannian manifolds in Section 15.1. In symbols, $\phi^*\theta = \omega$, or more explicitly, recalling the definition of pull-back, $\phi^*\theta(p) = \phi^* \cdot \theta(\phi(p))$, and the definition of $\phi^* \cdot \theta(\phi(p))$ by eq. (4.10), ϕ has the property

$$\theta(\phi(p))(\phi_* \cdot v_1, \phi_* \cdot v_2) = \omega(p)(v_1, v_2) \tag{18.1}$$

where $v_i \in T_p(M)$. Compare eq. (15.2).

Theorem 18.1 *If* $\dim M = \dim N$, *then a symplectic map,* ϕ, *is volume-preserving and* ϕ *is a local diffeomorphism.*

Proof (i) Since $\phi^*\theta = \omega$, $\phi^*\theta \wedge \cdots \wedge \theta = \omega \wedge \cdots \wedge \omega$.

(ii) From (18.1) we have

$$\theta_{ij}(\phi(p)) \frac{\partial \phi^i}{\partial \mu^k} \frac{\partial \phi^j}{\partial \mu^1} = \omega_{kl}(p)$$

and taking det on both sides we get $\det(\partial\phi^i/\partial\mu^i) \neq 0$ since ω and θ are nondegenerate. □

Definition X is a Hamiltonian vector field on (M, ω) if X preserves ω; i.e., if all the maps, $\{\Theta_u\}$, of the flow, Θ, of X preserve the symplectic structure of M.

Theorem 18.2 *X is Hamiltonian* $\Leftrightarrow L_X\omega = 0$.

Proof This is just a special case of Problem 15.9. □

Theorem 18.3 *X is Hamiltonian $\Leftrightarrow i_X\omega$ is closed $\Leftrightarrow \omega^\flat \cdot X$ is closed.*

Proof Problem 18.1. □

Theorem 18.3 says that if X is a Hamiltonian vector field, then there is, locally, a function, H, *a Hamiltonian function* such that $i_X\omega = dH$. Clearly, H is not unique. On the other hand, since ω is nondegenerate, and $\omega^\flat \cdot \omega^\sharp = id$, any closed 1-form σ yields a Hamiltonian vector field $\omega^\sharp \cdot \sigma$. In particular, starting with a function, H, we get a Hamiltonian vector field $\omega^\sharp \cdot dH$.

Definitions If we are given a function H on M, the vector field $X_H = \frac{1}{2}\omega^\sharp \cdot dH$ determined by H is called *the Hamiltonian vector field of H*. Then

$$i_{X_H}\omega = 2\omega^\flat \cdot X_H = dH \tag{18.2}$$

and (M, ω, H) is called *a Hamiltonian system.* (In many of the standard references $i_X\omega = \omega^\flat \cdot X$. The fact that in our work $i_X\omega = 2\omega^\flat \cdot X$, which arises from the fact that our $(\omega^\flat \cdot X) \cdot Y = \omega(X, Y)$ is half of theirs, will lead to some equations which look slightly different from those found in those references.)

Definition Suppose f is a function on M and X is a vector field on M with flow Θ (or, Θ is a 1-parameter group action with infinitesimal generator X). See Section 13.3. If f is constant on integral curves, γ_p, of X, or equivalently, on translations, Θ_u, of Θ, then f is *an integral of X*, or f is *an invariant* of Θ.

It is interesting to compare Hamiltonian vector fields with vector fields defined in Riemannian geometry in Section 15.1. By Theorem 18.2, Hamiltonian vector fields correspond to Killing fields in geometry (but they are much more prevalent). Coming from functions, they are precisely analogous to gradients. (However, gradients are not, in general, Killing fields.)

Theorem 18.4 *If (M, ω, H) is a Hamiltonian system, then H is an integral of X_H.*

Proof At each point $D_u(H \circ \gamma) = D_0(H \circ (\gamma \circ \theta_u)) = \langle dH(\gamma(u)), \dot{\gamma}(u) \rangle$ by eq. (9.7). Putting $\dot{\gamma}(u) = X_H(\gamma(u))$ we get

$$D_u(H \circ \gamma) = \langle dH, X_H \rangle(\gamma(u)) = \langle 2\omega^\flat \cdot X_H, X_H \rangle(\gamma(u))$$

by eq. (18.2). But $\langle \omega^\flat \cdot X_H, X_H \rangle = 0$ since ω is skew symmetric. □

This result is interpreted mechanically as a statement of conservation of energy. We will examine this more closely in Section 19.2.

As an alternative to imposing a symplectic structure, ω, on M, one can impose a Poisson bracket structure on the differentiable functions, $\mathfrak{F}_M$. Thus, on the one hand, starting with a given ω one can form the following product in $\mathfrak{F}_M$.

Definition For $f, g \in \mathfrak{F}_M$, *the Poisson bracket of* f *and* g *is* $\{f, g\} = 2\omega(X_f, X_g)$.

Theorem 18.5 *Useful alternative descriptions of* $\{f, g\}$ *are given by*

$$\{f, g\} = L_{X_g} f = X_g f$$

Proof $L_{X_g} f = X_g f = \langle df, X_g \rangle = 2\omega^\flat \cdot X_f(X_g) = 2\omega(X_f, X_g)$. □

Theorem 18.6 *The Poisson bracket has the following three properties.*

(i) *The product* $(f, g) \mapsto \{f, g\}$ *makes* $\mathfrak{F}_M$ *a Lie algebra.* (See Section 11.1.)

(ii) *Each partial map is a derivation.*

(iii) $\{f, g\} = 0 \Leftrightarrow f$ *is constant on integral curves of* X_g

$\Leftrightarrow g$ *is constant on integral curves of* X_f

(*i.e.,* f *is an integral of* X_g *and* g *is an integral of* X_f).

Proof (i) Both the bilinearity and skew-symmetry of the product come almost immediately from the definitions. We get the Jacobi identity from

$$\{f, \{g, h\}\} = L_{X_f} L_{X_g} h = X_f X_g h,$$

$$\{g, \{h, f\}\} = -L_{X_g} L_{X_f} h = -X_g X_f h,$$

and

$$\{h, \{f, g\}\} = L_{X_{\{f,g\}}} h = X_{\{f,g\}} h$$

But $X_{\{f,g\}} = -[X_f, X_g]$ (see the Corollary of Theorem 18.7). So $\{h\{f, g\}\} = -[X_f, X_g]h$ and adding, we get zero.

(ii) and (iii). Problem 18.3. □

Now, we can also go in the opposite direction. If on M we have a pairing $\mathfrak{F}_M \times \mathfrak{F}_M \to \mathfrak{F}_M$ with the three properties of Theorem 18.6, then $F:(df, dg) \mapsto \{f, g\}$ defines a 2-vector field on M (recall, Section 11.3, that a 2-vector field is a differentiable map from M to $\bigwedge^2(TM) = \bigcup_{p\in M} \bigwedge^2(T_p))$. F is nondegenerate and we have the maps $F^\sharp : \mathfrak{X}^* \to \mathfrak{X}$ and $F^\flat : \mathfrak{X} \to \mathfrak{X}^*$. The corresponding 2-form field, $(v, w) \mapsto (F^\flat \cdot v, F^\flat \cdot w) \mapsto F(F^\flat \cdot v, F^\flat \cdot w)$ will be symplectic.

We should notice that we have not defined a variety of related skew-symmetric structures on M which we can compare according to the following listing.

(i) ω, the symplectic structure, gives a pairing of vector fields on M to functions on M

(ii) $\{\ \}$, the Poisson bracket, gives a pairing of functions on M to functions on M

(iii) $[\ ]$, the Lie bracket, gives a pairing of vector fields on M to vector fields on M.

(iv) F, defined in the paragraph above, gives a pairing of differential 1-forms on M to functions on M.

One more structure will complete this list.

Definition For $\sigma, \tau \in \mathfrak{X}^*M$, *the Poisson bracket of* σ *and* τ is $\{\sigma, \tau\} = -\frac{1}{2}\omega^\flat \cdot [\omega^\sharp \cdot \sigma, \omega^\sharp \cdot \tau]$.

Theorem 18.7 *If τ is closed, then* $\{\sigma, \tau\} = \frac{1}{2}L_{\omega^\sharp \cdot \tau}\sigma$.

Proof Since $X\rfloor\omega = \omega^\flat \cdot X$, $\omega^\sharp \cdot \sigma\rfloor\omega = \sigma$. Then $L_{\omega^\sharp \cdot \tau}\sigma = L_{\omega^\sharp \cdot \tau}\omega^\sharp\sigma\rfloor\omega = (L_{\omega^\sharp \cdot \tau}\omega^\sharp \cdot \sigma)\rfloor\omega + \omega^\sharp \cdot \sigma\rfloor L_{\omega^\sharp \cdot \tau}\omega$ by Problems 6.11 and 13.15. But $L_{\omega^\sharp \cdot \tau}\omega = di_{\omega^\sharp \cdot \tau}\omega + i_{\omega^\sharp \cdot \tau}d\omega = d2\omega^\flat \cdot \omega^\sharp \cdot \tau + i_{\omega^\sharp \cdot \tau}d\omega = 0$ since $d\tau = d\omega = 0$. So, $L_{\omega^\sharp \cdot \tau}\sigma = [\omega^\sharp \cdot \tau, \omega^\sharp \cdot \sigma]\rfloor\omega = \omega^\flat \cdot [\omega^\sharp \cdot \tau, \omega^\sharp \cdot \sigma] = -\omega^\flat \cdot [\omega^\sharp \cdot \sigma, \omega^\sharp \cdot \tau] = 2\{\sigma, \tau\}$. □

Corollary. (i) *The relation between the Poisson bracket of functions and the Poisson bracket of 1-forms is given by*

$$d\{f, g\} = \{df, dg\} \tag{18.3a}$$

i.e., Poisson brackets commute with exterior differentiation.

(ii)

$$X_{\{f,g\}} = -[X_f, X_g] \tag{18.3b}$$

and the Hamiltonian vector fields form a Lie algebra.

PROBLEM 18.1 Prove Theorem 18.3.

PROBLEM 18.2 Prove properties (ii) and (iii) of Theorem 18.6.

PROBLEM 18.3 $\{f, g\} = 0$ iff f is constant on integral curves of X_g iff g is constant on integral curves of X_f.

PROBLEM 18.4 Show that F as specified above defines a nondegenerate 2-vector field, and that the corresponding 2-form field is symplectic.

PROBLEM 18.5 Prove the corollary of Theorem 18.7.

18.2 The Darboux theorem, and the natural symplectic structure of T^*M

There are two key results that pin down to specific concrete cases the abstract structures we have been describing and, in particular, form the bridge to classical mechanics. One is that every symplectic manifold has certain "natural" (or "canonical") local coordinates: "Darboux's theorem for symplectic manifolds." The second is that the cotangent manifold of any manifold has a "natural" (or "canonical") symplectic structure. An important related fact is that the "natural" coordinates on T^*M induced by a coordinate system on M (Section 11.2) are "natural" with respect to its symplectic structure.

Theorem 18.8 (Darboux) *Suppose M has dimension $2n$ and ω is a nondegenerate 2-form on M. Then $d\omega = 0$ on M if and only if at each point of M there is a coordinate system with coordinate functions $x^1, \ldots, x^n, y^1, \ldots, y^n$ such that*

$$\omega = dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n \tag{18.4}$$

Proof The one way is obvious. To go the other way, we note, first of all, that on $\mathbb{R}^{2n}$ there is the standard 2-form

$$\hat{\omega}_1 = d\pi^1 \wedge d\pi^{n+1} + \cdots + d\pi^n \wedge d\pi^{2n}$$

where π^i are the standard coordinate functions on $\mathbb{R}^{2n}$. Now, given ω, select a chart $(\mathcal{U}, \mu)$ at p_0 such that the representation $\hat{\omega} = \omega \circ \mu^{-1}$ of ω on $\mu(\mathcal{U})$ has the form of $\hat{\omega}_1$ at $a_0 = \mu(p_0)$. Such a chart exists by Theorem 5.23. The idea of the proof is to find a map, Θ_1, from a neighborhood $\mathcal{V} \subset \mu(\mathcal{U})$ of $a_0 = \mu(p_0)$ in $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$ (see Fig. 18.1) such that $(\mu^{-1}(\mathcal{V}), \Theta_1 \circ \mu)$ is the required kind of coordinate system; i.e., a coordinate system in which ω has the form (18.4).

Define a "time-dependent" 2-form, Ω, on a neighborhood of $\mu(p_0)$ by

$$\Omega(a, t) = \hat{\omega}(a) + t(\hat{\omega}_1(a) - \hat{\omega}(a))$$

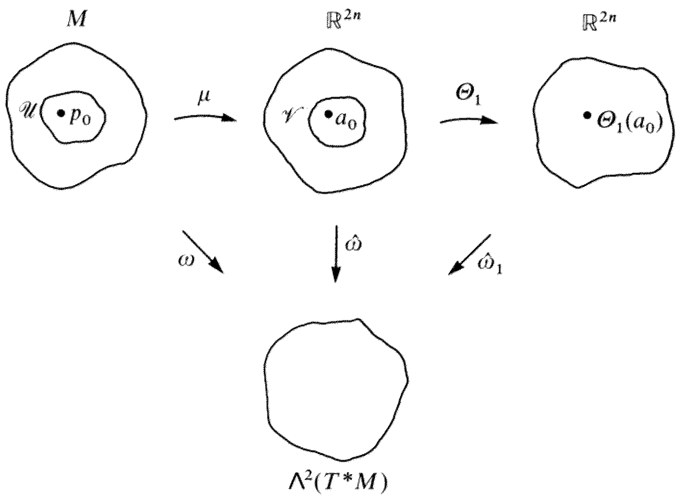

Figure 18.1

with $a \in \mu(\mathcal{U})$ and $t \in \mathbb{R}$. Now there is a neighborhood, $\mathcal{V}$, of a_0 for which

(i) $\hat{\omega} - \hat{\omega}_1 = d\sigma$ and $\sigma(a_0) = 0$ since $\hat{\omega}$ and $\hat{\omega}_1$ are closed,

(ii) Ω is nondegenerate for $0 \leq t \leq 1$ since $\Omega(a_0, t) = \hat{\omega}(a_0)$ is nondegenerate,

(iii) A "time-dependent" vector field, Z, with $Z(a_0, t) = 0$ is defined by $i_Z\Omega = \sigma$ for $0 \leq t \leq 1$ since Ω is nondegenerate for $0 \leq t \leq 1$.

(iv) A local flow, Θ, of Z is defined for $0 \leq u \leq 1$, by a generalization of Theorem 13.3.

In order to proceed with the proof we make use of a generalization of the formula of Theorem 13.15. If K is a time-dependent tensor field, and Θ is the flow of a time-dependent vector field Z, then we have the mapping

$$K_p^\sharp : (u, t) \in I_1 \times I_2 \mapsto \Theta_u^* K(p, t) \in (T_p)_s^r$$

and when $t = u$, by the chain rule and Theorem 13.15

$$D_u K_p^\sharp = \Theta_u^* \cdot L_Z K(\Theta_u(p), u) + \Theta_u^* \cdot \frac{\partial K}{\partial u}(\Theta_u(p), u) \tag{18.5}$$

Finally, applying (18.5) to our time-dependent 2-form, Ω, we get

$$D_u \Omega_a^\sharp = \Theta_u^* \cdot L_Z \Omega(\Theta_u(a), u) + \Theta_u^* \cdot \frac{\partial \Omega}{\partial t}(\Theta_u(a), t)$$

for $a \in V$. But, since $d\Omega = 0$, $L_Z\Omega = di_Z\Omega = d\sigma = \hat{\omega} - \hat{\omega}_1$, and $\partial\Omega/\partial t = \hat{\omega}_1 - \hat{\omega}$, so $D_u\Omega_a^\sharp = 0$. That is, $\Theta_u^* \cdot \Omega(\Theta_a(u), u)$ is constant. Putting $u = 0$ and $u = 1$, we get

$$\hat{\omega}(a) = \Theta_1^* \cdot \hat{\omega}_1(\Theta_a(1)) = \Theta_1^* \cdot \hat{\omega}_1(\Theta_1(a))$$

for $a \in \mathcal{V}$. That is, $\hat{\omega}$ and $\hat{\omega}_1$ are representations of the same form on $\mathcal{V}$ and $\Theta_1(\mathcal{V})$ respectively, so $\Theta_1 \circ \mu$ is the required coordinate map, and $x^i = \pi^i \circ \Theta_1 \circ \mu$ and $y^i = \pi^{n+1} \circ \Theta_1 \circ \mu$ $(i = 1, \ldots, n)$. $\square$

It is instructive to note that (i) Theorem 18.8 uses the canonical form for an *exterior*-form at a *point*, Theorem 5.23, to get a canonical form for a 2-form on a *coordinate neighborhood*, and (ii) Theorem 18.8 says that there is only one symplectic form on a manifold. Contrast these two statements with the geometric situation when M has a pseudo-Riemannian structure.

There is more general Darboux theorem which gives a local canonical representation for 1-forms. The result above comes easily form this general theorem as does also a parallel result giving a local canonical representation for contact forms. The proof of the general theorem is in Sternberg (pp. 137 ff). The proof above is given by Abraham and Marsden (p. 175).

Now we go to the other key result mentioned above.

Theorem 18.9 *For any manifold M, there exists a symplectic structure on T^*M.*

Proof We saw in Theorem 11.18 that T^*M has a natural 1-form, θ_M. We will show that θ_M is differentiable and $d\theta_M$ is symplectic. This will be done by introducing coordinates.

If $(\mathcal{U}, \mu)$ is a chart on M, and π is the projection $T^*M \to M$, then the induced chart, $(\mathcal{V}, \nu)$ on T^*M is obtained by putting $\mathcal{V} = \bigcup_{p \in \mathcal{U}} T^*_p$, $\nu^i = q^i = \mu^i \circ \pi$, $i = 1, \dots, n$, and $\nu^{n+j} = p_j = \sigma_j \circ \pi$, $j = 1, \dots, n$, where σ_j is given by

$$\sigma(p) = \sigma_i(p) d\mu^i \big|_p \tag{18.6}$$

for $p \in M$ and $\sigma \in \pi^{-1}(p)$. See Problem 11.11.

With the coordinates q^i, p_j the coordinate vector fields $\partial/\partial q^i$, $\partial/\partial p_j$ form a basis of vector fields on $\mathcal{V}$. Thus

$$X = X^i \frac{\partial}{\partial q^i} + Y_j \frac{\partial}{\partial p_j} \tag{18.7}$$

for some functions X^i and Y_j. But according to eq. (9.11)

$$\pi_* \cdot \frac{\partial}{\partial q^i} = \frac{\partial \pi^k}{\partial q^i} \frac{\partial}{\partial \mu^k} \quad \text{and} \quad \pi_* \cdot \frac{\partial}{\partial p_i} = \frac{\partial \pi^k}{\partial p_i} \frac{\partial}{\partial \mu^k}$$

where π^k are the component functions of π. So $\pi_* \cdot \partial/\partial q^i = \partial/\partial \mu^i$ and $\pi_* \cdot \partial/\partial p_i = 0$, and from eq. (18.7),

$$\pi_* \cdot X = X^i \frac{\partial}{\partial \mu^i}. \tag{18.8}$$

By the definition of θ_M in Theorem 11.18

$$\langle \theta_M(\sigma), X \rangle = \langle \sigma, \pi_* \cdot X \rangle. \tag{18.9}$$

Substituting from (18.8) and (18.6) into the right side of (18.9), we get

$$\langle \theta_M(\sigma), X(\sigma)\rangle = \sigma_i(p)X^i(\sigma) = p_i(\sigma)X^i(\sigma) \tag{18.10}$$

for $p = \pi(\sigma)$. But $\theta_M(\sigma)$ has the form

$$\theta_M(\sigma) = \xi_i(\sigma)dq^i + \eta^j(\sigma)dp_j \tag{18.11}$$

so from (18.7),

$$\langle \theta_M(\sigma), X(\sigma)\rangle = \xi_i(\sigma)X^i(\sigma) + \eta^j(\sigma)Y_j(\sigma). \tag{18.12}$$

Comparing (18.10) and (18.12), noting that they are valid for arbitrary choices of X^i and Y_j, we find we must have $\xi_i(\sigma) = p_i(\sigma)$ and $\eta^j(\sigma) = 0$. Hence, from (18.11),

$$\theta_M(\sigma) = p_i(\sigma)dq^i. \tag{18.13}$$

Since p_i are coordinate functions, $\theta_M : \sigma \mapsto \theta_M(\sigma)$ is differentiable and $d\theta = dp_i \wedge dq^i$ is clearly nondegenerate and closed. □

Corollary *The natural coordinate system induced on T^*M by a given coordinate system on M, as described in Section 11.2, is a natural (or canonical) coordinate system on the symplectic manifold T^*M in the sense that it is one of the kind guaranteed by Darboux's theorem.*

Theorem 18.10 *If ϕ is a diffeomorphism from M to N, then $\phi^* : T^*N \to T^*M$ takes the natural 1-form, θ_M on T^*M to the natural 1-form, θ_N, on T^*N; i.e., $(\phi^*)^*\theta_M = \theta_N$.*

Proof The proof is by chasing around pairings with the help of Fig. 18.2. If $\sigma \in T_p^*N$, $\mathbf{w} \in T_\sigma T^*N$, π_M is the projection of T^*M on M and π_N is the projection of T^*N on N, then

$$\begin{aligned}
&\langle (\phi^*)^*\theta_M(\sigma), \mathbf{w}\rangle \\
&\quad = \langle \theta_M(\phi^*(\sigma)), (\phi^*)_*(\mathbf{w})\rangle && \text{the property of the transpose of } (\phi^*)_* \\
&\quad = \langle \phi^*(\sigma), (\pi_{M*} \cdot (\phi^*)_*(\mathbf{w}))\rangle && \text{the definition of } \phi \\
&\quad = \langle \phi^*(\sigma), ((\pi_M \circ \phi^*)_*(\mathbf{w}))\rangle \\
&\quad = \langle \sigma, (\phi_*((\pi_M \circ \phi^*)_*(\mathbf{w}))\rangle && \text{the property of the transpose of } \phi_* \\
&\quad = \langle \sigma, ((\phi \circ \pi_M \circ \phi^*)_*(\mathbf{w}))\rangle \\
&\quad = \langle \sigma, \pi_{N*}(\mathbf{w})\rangle \\
&\quad = \langle \theta_N(\sigma), \mathbf{w}\rangle && \text{the definition of } \theta_N.
\end{aligned}$$

So $(\phi^*)^*\theta_M = \theta_N$. □

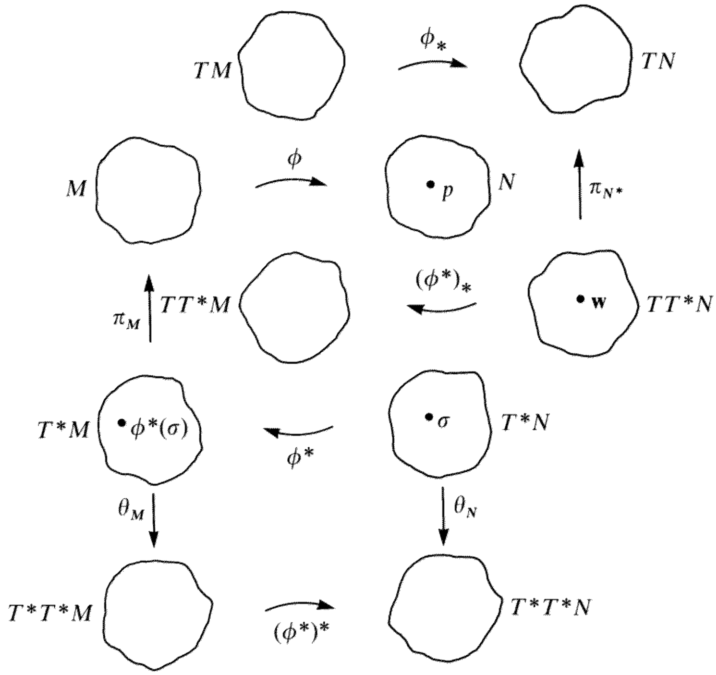

Figure 18.2

Corollary *If ϕ is any diffeomorphism from M to N, then $\phi^*: T^*N \to T^*M$ is a symplectic map.*

Proof Problem 18.8. □

We can think of a coordinate transformation between overlapping coordinate systems on a manifold as a special case of a mapping of manifolds. A symplectic form on the manifold will have a local representation in each coordinate system.

Definition In a symplectic manifold a coordinate transformation is *a canonical coordinate transformation* if the overlap map is symplectic.

Theorem 18.11 *If a symplectic form has the Darboux representation in one coordinate system, then it has the Darboux representation in a second coordinate system iff the coordinate transformation is canonical.*

Proof Problem 18.9. □

Theorem 18.12 *Two sets of natural (Darboux) coordinates (q^i, p_i) and (Q^i, P_i) satisfy the following relations at each point of the overlap of their domains.*

$$\frac{\partial p_i}{\partial P_j} = \frac{\partial Q^j}{\partial q^i} \quad \frac{\partial p_i}{\partial Q^j} = -\frac{\partial P_j}{\partial q^i}$$
$$\frac{\partial q^i}{\partial P_j} = -\frac{\partial Q^j}{\partial p_i} \quad \frac{\partial q^i}{\partial Q^j} = \frac{\partial P_j}{\partial p_i}.$$

Proof Problem 18.10. □

PROBLEM 18.6 Show that in natural (Darboux) coordinates,
(i) if ω is a symplectic form,

$$i_X\omega = \sum X^i dy^i - Y^i dx^i$$

where X^i, Y^i are the component functions of X.
(ii) the Poisson bracket of f and g is

$$\{f, g\} = \sum \left(\frac{\partial f}{\partial y^i} \frac{\partial g}{\partial x^i} - \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial y^i} \right).$$

PROBLEM 18.7 Prove Theorem 18.10 by choosing coordinates and showing $P_i dQ^i = (\phi^*)^* \cdot p_i dq^i$.

PROBLEM 18.8 Prove the corollary of Theorem 18.10.

PROBLEM 18.9 Prove Theorem 18.11.

PROBLEM 18.10 Prove Theorem 18.12. (Hint: use eq. (5.22) with $\begin{pmatrix} I_r & \\ & -I_s \end{pmatrix}$ replaced by J.)

18.3 Hamilton's equations. Examples of mechanical systems

Theorem 18.13 *In a natural coordinate system, the Hamiltonian vector field, X_H, of a Hamiltonian system (M, ω, H) has the component functions $\left(\frac{\partial H}{\partial y^i}, \frac{\partial H}{\partial x^i}\right)$.*

Proof Write both sides of $i_{X_H}\omega = dH$ in terms of the basic fields dx^i, dy^i. The left side is given by Problem 18.6(i) and

$$dH = \frac{\partial H}{\partial x^i} dx^i + \frac{\partial H}{\partial y^i} dy^i$$

so, equating coefficients, $X_H^i = \partial H/\partial y^i$ and $Y_H^i = -\partial H/\partial x^i$. □

In applications to mechanics, in the Hamiltonian system (M, ω, H) M is the cotangent manifold of the configuration space, so we write q^i, p_i instead of x^i, y^i for coordinate functions. In these coordinates $\omega = dq^i \wedge dp_i$. The famous *Hamilton canonical equations* are simply the equations in natural coordinates, of the integral curves, γ, of a given Hamiltonian vector field, X_H. Thus, by Theorem 18.13, the Hamilton canonical equations are the equations

$$\frac{dq^i \circ \gamma}{dt} = \frac{\partial H}{\partial p_i} \quad \frac{dp_i \circ \gamma}{dt} = -\frac{\partial H}{\partial q^i} \tag{18.14}$$

for the curves, γ.

Let us see how Hamilton's equations describe the behavior of a mechanical system. In mechanics we are concerned with the "motion" of "systems (finite sets) of particles" and/or "rigid bodies."

For the case of systems of particles we assume that, at each instant, we can represent the "position" of the system as a set of n points in $\mathcal{E}_0^3$, a 3-dimensional Euclidean affine space. Recall that by definition, an n-dimensional Euclidean affine space, $\mathcal{E}_0^n$, is an affine space whose vector space, V, is a Euclidean vector space (see Section 5.4(i)). If we choose a point $p_0 \in \mathcal{E}_0^n$, we get a 1-1 correspondence between points of $\mathcal{E}_0^n$ and points of V. Then choosing an orthonormal basis, $\{e_i\}$, in V we get (rectangular cartesian) coordinates on $\mathcal{E}_0^n$, and, finally, $\mathcal{E}_0^n$ acquires the structure of a Riemannian manifold with rectangular cartesian coordinates by means of the isomorphisms, $e_i \leftrightarrow \partial/\partial\mu^i\big|_p$, between V and $T_p\mathcal{E}_0^n$ for each $p \in \mathcal{E}_0^n$ (Problem 9.14(ii)). Clearly, $\mathcal{E}_0^n$ with this structure is isometric with $\mathcal{R}_0^n$.

We assume that the "motion" of the systems of particles will be determined by "forces" and "constraints," the former describable by a function, a potential, on $\mathcal{E}_0^3$, and the latter determining a submanifold of $\mathbb{R}^{3n}$. The submanifold is called *the configuration space* of the system, and its cotangent manifold is called *the (momentum) phase space* of the system.

For the case of rigid bodies, we can represent a position of the body as the image of an isometry of $\mathcal{E}_0^3$. In this case *the configuration space* is the set of isometries, and again the cotangent manifold is *the phase space.*

In both cases, the precise way in which forces and constraints determine the motion is assumed to be represented, in local coordinates, by the solutions of Hamilton's equations. The major significance of this model of a mechanical system, in contrast to, for example, the Newtonian model is that, since the induced coordinates in the phase space are always natural, eqs. (18.14) have the same form for *any* choice of coordinates in configuration space.

Consider the following *examples* of simple mechanical systems. In all these examples H will have the form

$$H = \boldsymbol{\phi} \circ \pi + \mathbf{T}^\flat$$

where π is the natural projection of the phase space on the configuration space, and $\mathsf{T}^\flat$ is a function on the phase space which comes from T, a function on the tangent (state) space, by the Legendre transformation to be described in the next section (see Problem 18.13). The choice of H of this form is justified in Section 19.1.

1. THE SIMPLE PLANE PENDULUM. We are given two positive numbers, a length, ℓ, and a mass, m. The position of the ball of the pendulum is represented by a point in $\mathcal{E}_0^3$. Introducing rectangular cartesian coordinates (a, b, c), the configuration space is the circle in $\mathbb{R}^3$,

$$S^1 = \{(a, b, c) : a^2 + b^2 = \ell^2,\ c = 0\}$$

with local coordinate, ϑ, given by $a = \ell \sin \vartheta$, $b = -\ell \cos \vartheta$. Then local coordinates (q, p) of a point, σ, in the cotangent manifold, the phase space, are given by $q = \vartheta \circ \pi$ and $\sigma = p\, d\vartheta$.

The force of gravity is described in a potential function in $\mathcal{E}_0^3$ given by $\boldsymbol{\phi}(a, b, c) = mgb (g = \text{constant})$. Then on S^1

$$\boldsymbol{\phi}(\vartheta) = -mg\ell \cos \vartheta \tag{18.15}$$

The kinetic energy is a function on the tangent manifold TS^1 given by

$$\mathsf{T}(\vartheta, v) = \frac{1}{2} m\ell^2 v^2 \tag{18.16}$$

where v is the component function of a vector with respect to the coordinate basis $\partial/\partial\vartheta$. Choose $H = \boldsymbol{\phi} \circ \pi + \mathsf{T}^\flat$ where $\mathsf{T}^\flat$ is the function on T^*S^1 obtained by putting $v = p/m\ell^2$ in (18.16). Then we get, from (18.15) and (18.16),

$$H(q, p) = -mg\ell \cos q + \frac{1}{2m} \frac{p^2}{\ell^2} \tag{18.17}$$

(Strictly speaking, $\boldsymbol{\phi}(\vartheta)$, $\mathsf{T}(\vartheta, v)$ and $H(q, p)$ should be $\hat{\boldsymbol{\phi}} \circ \vartheta$, $\hat{\mathsf{T}} \circ (\vartheta, v)$, and $\hat{H} \circ (p, q)$, respectively. We will follow the usual practice of suppressing the extra notation.)

Differentiating (18.17), $\partial H/\partial q = mg\ell \sin q$ and $\partial H/\partial p = p/m\ell^2$ and so Hamilton's equations for the simple plane pendulum are

$$\frac{dq}{dt} = \frac{p}{m\ell^2}, \quad \frac{dp}{dt} = -mg\ell \sin q$$

2. THE SIMPLE HARMONIC OSCILLATOR. We consider the motion of a particle subject to a central force, of which this example is a special case. Since such a motion occurs in a plane, the position of the particle is represented by a point in $\mathcal{E}_0^3$ where we now use cylindrical coordinates (r, ϑ, z). The central force is described by a potential function

$$\phi(r, \vartheta, z) = f(r) \tag{18.18}$$

The configuration space is $\mathbb{R}^2$, the set of pairs (r, ϑ). Local coordinates (q^1, q^2, p_1, p_2) of a point, σ, in the cotangent manifold, the phase space, are given by $q^1 = r$, $q^2 = \vartheta$, and $\sigma = p_1 dr + p_2 d\vartheta$. Again $H = \phi \circ \pi + \mathsf{T}^\flat$, and now the kinetic energy, T, is given by

$$\mathsf{T}(v^1, v^2) = \frac{1}{2}m[(v^1)^2 + r^2(v^2)^2] \tag{18.19}$$

where $v^1 = dr/dt$ and $v^2 = d\vartheta/dt$. (Note that T is really a function of *four* coordinate functions on the tangent manifold.) Putting $v^1 = p_1/m$ and $v^2 = p_2/mr^2 = p_2/m(q^1)^2$ into (18.19) we get from (18.18) and (18.19),

$$H(q^1, q^2, p_1, p_2) = f(q^1) + \frac{1}{2m}\left(p_1^2 + \frac{p_2^2}{(q^1)^2}\right). \tag{18.20}$$

Note that the right side of (18.20) does not actually contain q^2. (Consequently q^2 is called a cyclic coordinate.) Differentiating (18.20) we get Hamilton's equations for this case:

$$\frac{dq^1}{dt} = \frac{p_1}{m}, \quad \frac{dq^2}{dt} = \frac{p_2}{m(q^1)^2}$$

$$\frac{dp_1}{dt} = -f'(q^1) + \frac{p_2^2}{m(q^1)^3}, \quad \frac{dp_2}{dt} = 0.$$

In particular, we get $p_2 = mr^2 d\vartheta/dt =$ constant, a well-known result. (Kepler's equal areas in equal times.) For the special case of a simple harmonic oscillator, q^2 is constant, $p_2 = 0$, and $f(r) = \frac{1}{2}kr^2$ where k is a constant, and Hamiltonian's equations reduce to

$$\frac{dq^1}{dt} = \frac{p_1}{m}, \quad \frac{dp_1}{dt} = -kq^1.$$

3. THE DOUBLE PLANE PENDULUM. We are given two lengths ℓ_1 and ℓ_2 and two masses m_1 and m_2. The position of the system is represented by two points in $\mathcal{E}_0^3$. Because of the constraints the position will be determined by two angles ϑ_1 and ϑ_2. The pair $(\vartheta_1, \vartheta_2)$ can be thought of as local coordinates on a certain submanifold of $\mathbb{R}^6$, namely, the 2-dimensional torus. In terms of the natural coordinates x, y, z on $\mathbb{R}^3$,

$$x = (\ell_1 + \ell_2 \sin\vartheta_2)\cos\vartheta_1$$

$$y = (\ell_1 + \ell_2 \sin\vartheta_2)\sin\vartheta_1$$

$$z = \ell_2 \cos\vartheta_2.$$

Hence for the double plane pendulum the configuration space is the 2-dimensional torus. Local coordinates (q^1, q^2, p_1, p_2) of a point, σ, in the cotangent bundle, the phase space, are given by $q^1 = \vartheta_1$, $q^2 = \vartheta_2$, and $\sigma = p_1\, d\vartheta_1 + p_2\, d\vartheta_2$. The force is again gravity with a potential $\boldsymbol{\phi}$ defined in $\mathcal{E}_0^3$. In terms of ϑ_1 and ϑ_2, and hence q^1 and q^2, we have

$$\boldsymbol{\phi}(q^1, q^2) = -m_1 g \ell_1 \cos q^1 - m_2 g(\ell_1 \cos q^1 + \ell_2 \cos q^2) \tag{18.21}$$

and

$$\begin{aligned} \mathsf{T}(v^1, v^2) = &\frac{1}{2} m_1 \ell_1^2 (v^1)^2 \\ &+\frac{1}{2} m_2 [(\ell_1 \sin\vartheta_1\; v^1 + \ell_2 \sin\vartheta_2\; v^2)^2 \\ &+(\ell_1 \cos\vartheta_1\; v^1 + \ell_2 \cos\vartheta_2\; v^2)^2] \end{aligned} \tag{18.22}$$

where $v^1 = d\vartheta_1/dt$ and $v^2 = d\vartheta_2/dt$. (Again, note that T is really a function of *four* coordinates.) Again writing the v's in (18.22) in terms of q's and p's and adding the result to the expression for $\boldsymbol{\phi}$ in (18.21) we get the Hamiltonian function.

4. LAGRANGE'S TOP. In the previous examples we had systems of particles. Now we consider the motion of a rigid body, in the special case in which one point of the body is fixed. For any rigid body moving with a fixed point (the body need not have any symmetry) the set of positions has the structure of a 3-dimensional submanifold of $\mathbb{R}^9$; specifically, the rotations, $SO(3)$, the orthogonal matrices with determinant equal to 1 in the set of 3 by 3 matrices. That is, the configuration space of a rigid body moving with a fixed point is $SO(3)$. We can choose the Euler angles, χ, ϑ, φ, as local coordinates (χ and ϑ determine the direction of the axis of rotation, and φ gives the rotation about that axis). The force acting on this system (as in Examples 1 and 3) is gravity, so

$$\boldsymbol{\phi}(\chi, \vartheta, \varphi) = mg\ell \cos\vartheta \tag{18.23}$$

(We have, among the given properties of this system, a density distribution on $\mathcal{E}_0^3$ which determines m and ℓ.) There are standard expressions for T in terms of angular velocity; cf., MacMillan (pp. 180-181), or Arnold (p. 137). Similarly, for angular velocity in terms of Euler angles (op. cit. pp. 184-185, p. 150, respectively). The result is

$$\mathsf{T}(v^1, v^2, v^3) = \frac{1}{2}[A(v^1)^2 + A(v^3)^2 \sin^2\vartheta + C(v^2 + v^3 \cos\vartheta)^2] \tag{18.24}$$

where $v^1 = d\vartheta/dt$, $v^2 = d\varphi/dt$, $v^3 = d\chi/dt$ and A and C are (constant) components of the moments of inertia. Finally, as before, we write the v's in terms of q's and p's to get the Hamiltonian function, in this case we have

$$v^1 = \frac{p_1}{A}$$

$$v^2 + v^3 \cos q^1 = \frac{p_2}{C}$$

$$v^3 \sin^2 q^1 = \frac{p_3 - p_2 \cos q^1}{A}$$

(op. cit. pp. 378, 151, respectively). Note that, like in Example 2, not all q's will appear explicitly in the expression for $H(q^1, \ldots, p_3)$; p_2, and p_3 are integrals of Hamilton's equations and the system reduces to two equations in two unknowns.

In all the examples above, one step involved expressing the v's in terms of the q's and p's. Neither the fact that this can be done nor the specific form of the relations was justified. The reason for this gap is that, while the concrete physical examples above were introduced at this time to illustrate the considerable previous abstract structure, additional abstract structure, introduced in the next section, is required to fully explain even these simple examples. See Problem 19.3.

PROBLEM 18.11 Show that if H has k cyclic coordinates, then Hamilton's equations reduce to a system of $2(n-k)$ equations in $2(n-k)$ unknowns.

PROBLEM 18.12 Derive Hamilton's equations for the spherical pendulum.

18.4 The Legendre transformation and Lagrangian vector fields

There is another standard model of classical mechanics in terms of the so-called Lagrange equations. These may be derived either directly from Hamilton's equations, or independently starting with *Hamilton's Variational Principle.* If we follow the latter method, we expect, of course, that we can then proceed to derive Hamilton's equations from Lagrange's. To point up this reciprocity, or duality, between Hamilton's and Lagrange's equations we note further that while Hamilton's equations are on the cotangent manifold or (momentum) phase space of the given configuration space, the Lagrange equations are on the tangent manifold, now called *the state space* or *the velocity phase space* of the configuration space. In Hamilton's equations we had a given function, H, *the Hamiltonian* on the cotangent manifold. We proceed now by describing certain structures induced by a given function, L, called *a Lagrangian*, on the tangent manifold.

Let L be a given function on the tangent manifold of a manifold, M. We will fix $p \in M$ and look at only the restriction of L to the vector space T_p. To avoid excessive indexing we will again denote this restriction by L. Let $v \in T_p$, and note that the derivative at v of L, $D_vL : T_p \to \mathbb{R}$, a linear function on T_p, is in T_p^*. Let $DL : T_p \to T_p^*$ according to $v \mapsto D_vL$ and suppose DL has an inverse. Then given a function L on T_p we can define a function H on T_p^* by

$$H : \sigma \mapsto \langle \sigma, DL^{-1}(\sigma) \rangle - L \circ DL^{-1}(\sigma) \tag{18.25}$$

for $\sigma \in T_p^*$. Thus, we have a mapping, $\mathfrak{L}$, from functions on T_p to functions on T_p^*.

Following the standard conventions, we denote the coordinate functions on a coordinate neighborhood of TM by $(q^i, \dot{q}^i)$. Notice that we use the same notation for the first n coordinate functions on TM as on T^*M (see Theorem 18.9), and the notation $\dot{q}^i$ is suggested by the fact that the component functions v^i of v can always be thought of as the derivatives at 0 of the component functions of some curve. In these coordinates

$$DL : (\dot{q}^i(v), \ldots, \dot{q}^n(v)) \mapsto \left(\frac{\partial L}{\partial \dot{q}^1}(v), \ldots, \frac{\partial L}{\partial \dot{q}^n}(v) \right) = (p_1(DL \cdot v), \ldots, p_n(DL \cdot v))$$

where (q^i, p_i) are coordinates on T^*M, and (18.25) can be written

$$H(p_1, \ldots, p_n) = p_i \dot{q}^i - L(\dot{q}^i, \ldots, \dot{q}^n) \tag{18.26}$$

where the $\dot{q}^i$ on the right are the solutions of $\partial L / \partial \dot{q}^i = p_i$. (See comment on notation below eq. (18.17).) Moreover, differentiating (18.26) (using $\partial L / \partial \dot{q}^i = p_i$) we have

$$dH = v^i dp_i \tag{18.27}$$

Theorem 18.14 *If DL has an inverse, then $\mathfrak{L}$ is 1-1 and onto.*

Proof Immediate from (18.26). □

Definitions L is a function on TM, then for each $p \in M$ we have a DL, so we have a map, denoted again by DL, from TM to T^*M called *the Legendre transformation of L*, or *the fiber derivative of L*. For sets $\mathcal{U} \subset M$ at which DL has an inverse, $\mathfrak{L} : \mathfrak{F}_{T\mathcal{U}} \to \mathfrak{F}_{T^*\mathcal{U}}$ is called *the Legendre transformation.*

We have been assuming that the given function, L, is such that DL has an inverse. In this case L is called *hyperregular.* We will continue to assume L is hyperregular, though for many of the subsequent results we need only that DL is locally invertible (i.e., L is *regular*).

An important example is given by

$$L(v) = \frac{1}{2} g_p(v, v)$$

which we can construct in the case when M has a given pseudo-Riemannian structure, g. Then $D_v L \cdot w = g_p(v, w)$. Thus, at each $p \in M$, $DL : v \to D_v L$ is the linear map $g^\flat$ from T_p to T_p^* induced by g. Further, we can write (18.25) as $H(DL(v)) = \langle DL(v), v \rangle - L(v)$, so, in this case $H(DL(v)) = g_p(v, v) - \frac{1}{2} g_p(v, v)$, or $H \circ DL = L$, or $H = L \circ DL^{-1}$. As indicated above, we can make a model of a mechanical system based on L (instead of H). This particular form of L will correspond to a mechanical system in which no forces are acting.

Now, with the derivative, DL, induced by a given Lagrangian function on TM, we can proceed to build additional structures. Specifically, we have the pull-backs of the natural 1-form, θ, and the natural symplectic structure, $d\theta$, on T^*M. (We are discarding the subscript M on θ.) We write

$$\theta_{L} = DL^*\theta \quad \text{and} \quad \omega_{L} = -DL^* d\theta.$$

Since $d\theta$ is nondegenerate and L is hyperregular, ω_{L} is nondegenerate. By Theorem 12.4 $\omega_{L} = -d\theta_{L}$ so ω_{L} is closed, and hence ω_{L} is a symplectic form on TM, and (TM, ω_{L}) is a symplectic manifold.

Theorem 18.15 *θ_{L} can also be given by the formula*

$$\langle \theta_{L}(v), \mathbf{a} \rangle = \langle DL(v), \pi_* \cdot \mathbf{a} \rangle$$

where $v \in TM$, $\mathbf{a} \in T_vTM$, and π is the natural projection of TM on M. (Cf., Theorem 11.18.)

Proof $\theta_{L}(v) = DL^*\theta(v) = DL^* \cdot \theta(DL(v))$ by the definition of the pull-back. So

$$\begin{aligned}
\langle \theta_{L}(v), \mathbf{a} \rangle &= \langle DL^* \cdot \theta(DL(v)), \mathbf{a} \rangle \\
&= \langle \theta(DL(v)), DL_* \cdot \mathbf{a} \rangle \\
&= \langle DL(v), \bar{\pi}_* \cdot DL_* \cdot \mathbf{a} \rangle \qquad \text{by the definition of } \theta \\
&\qquad\qquad\qquad\qquad\qquad (\bar{\pi} \text{ is the projection of } T^*M \text{ on } M) \\
&= \langle DL(v), (\bar{\pi} \circ DL)_* \cdot \mathbf{a} \rangle \qquad \text{by the chain rule} \\
&= \langle DL(v), \pi_* \cdot \mathbf{a} \rangle
\end{aligned}$$

□

We saw, eq. (18.13), that in coordinates (q^i, p_i) on T^*M the natural 1-form, θ, has the form $\theta = p_i dq^i$. Therefore, in coordinates $(q^i, \dot{q}^i)$ on TM we have

$$\theta_{L} = \frac{\partial L}{\partial \dot{q}^i} dq^i \tag{18.28}$$

and

$$\omega_{L} = \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} d\dot{q}^i \wedge dq^j \tag{18.29}$$

Definitions *The action of* L is a map $A_L : TM \to \mathbb{R}$ given by $v \mapsto \langle DL(v), v\rangle$. *The energy of* L is $E_L = A_L - L$. *The Lagrangian vector field of* L is the vector field, X_L, on TM such that

$$i_{X_L}\omega_L = dE_L \tag{18.30}$$

Note that since ω_L is nondegenerate, there is one and only one X_L. In coordinates

$$A_L(v) = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} \tag{18.31}$$

(Another comment on terminology. The "action" of L defined above clearly has no relation to the "action" of a 1-parameter group action which was defined in Section 13.2 and which will appear again in Section 19.2. On the other hand, the term "action" as used in mechanics more commonly refers to an integral of what we, following Abraham and Marsden, have called action and is the celebrated quantity in the venerable Maupertius Principle of Least Action.)

Now we will write eq. (18.30), the definition of Lagrangian vector field, in coordinate form. We write

$$X_L = Y^k \frac{\partial}{\partial q^k} + Z^l \frac{\partial}{\partial \dot{q}^l}$$

and substitute this and the expression for ω_L in (18.29) into the left side of (18.30). Using the linearity of $i_{X_L}\omega_L$ with respect to X_L and ω_L (and denoting partial derivatives of L by subscripts) we get

$$\begin{aligned} i_{X_L}\omega_L = {} & (Y^k L_{\dot{q}^i q^j}) i_{\partial/\partial q^k} dq^i \wedge dq^j + (Y^k L_{\dot{q}^i \dot{q}^j}) i_{\partial/\partial q^k} d\dot{q}^i \wedge dq^j \\ & + (Z^l L_{\dot{q}^i q^j}) i_{\partial/\partial \dot{q}^l} dq^i \wedge dq^j + (Z^l L_{\dot{q}^i \dot{q}^j}) i_{\partial/\partial \dot{q}^l} d\dot{q}^i \wedge dq^j \end{aligned}$$

By Theorem 11.20, $i_{\partial/\partial q^k} dq^i \wedge dq^j = \delta^i_k dq^j - \delta^j_k dq^i$, etc. So

$$i_{X_L}\omega_L = -Y^j L_{\dot{q}^i q^j} dq^i + Y^i L_{\dot{q}^i q^j} dq^j + Y^i L_{\dot{q}^i \dot{q}^j} d\dot{q}^j - Z^j L_{\dot{q}^i \dot{q}^j} dq^i \tag{18.32}$$

The right side of (18.30) is

$$dA_L - dL = L_{\dot{q}^i} d\dot{q}^i + \dot{q}^i L_{q^j \dot{q}^i} dq^j + \dot{q}^i L_{\dot{q}^j \dot{q}^i} d\dot{q}^j - (L_{q^i} dq^i + L_{\dot{q}^i} d\dot{q}^i)$$

Simplifying, we get

$$dE_L = -L_{q^i} dq^i + \dot{q}^i L_{q^j \dot{q}^i} dq^j + \dot{q}^i L_{g^j \dot{q}^i} d\dot{q}^j \tag{18.33}$$

Putting (18.32) and (18.33) into (18.30) and equating coefficients of $d\dot{q}^i$, we get

$$Y^i = \dot{q}^i \tag{18.34}$$

and equating coefficients of dq^i and using (18.34), we get

$$Y^j L_{\dot{q}^i q^j} + Z^j L_{\dot{q}^i \dot{q}^j} = L_{q^i}. \tag{18.35}$$

Equations (18.34) and (18.35) together are the coordinate form of eq. (18.30). That is, X_L satisfies (18.30) iff the component functions (Y^i, Z^i) of X_L satisfy (18.34) and (18.35).

Finally, if γ is an integral curve of X_L, then the component functions $q^i \circ \gamma$ and $\dot{q}^i \circ \gamma$ satisfy

$$\frac{dq^i \circ \gamma}{dt} = \hat{Y}^i(q^j \circ \gamma, \dot{q}^k \circ \gamma), \qquad \frac{d\dot{q}^i \circ \gamma}{dt} = \hat{Z}^i(q^j \circ \gamma, \dot{q}^k \circ \gamma)$$

Thus, slightly abusing the notation, on an integral curve (18.34) becomes

$$\frac{dq^i}{dt} = \dot{q}^i \tag{18.36}$$

and (18.35) becomes

$$\frac{dq^j}{dt} L_{\dot{q}^i q^j}(q^k, \dot{q}^l) + \frac{d\dot{q}^j}{dt} L_{\dot{q}^i \dot{q}^j}(q^k, \dot{q}^l) = L_{q^i}(q^k, \dot{q}^l)$$

or

$$\frac{d}{dt} L_{\dot{q}^i}(q^k, \dot{q}^l) = L_{q^i}(q^k, \dot{q}^l). \tag{18.37}$$

(See comment on notation below eq. (18.17).) Equation (18.37) with $\dot{q}^i$ given by eq. (18.36) is the Euler-Lagrange equation.

To reiterate, in our approach the Euler-Lagrange equation arose as simply the coordinate form of the definition of (the integral curves) of the Lagrangian vector field of L. The significance of this vector field (and its integral curves) appears in the approach in which the Euler-Lagrange equation arises from a *variational principal* for the integral of L.

PROBLEM 18.13 With the Lagrangian function, L, in the form $L = \mathsf{T} - \boldsymbol{\phi} \circ \pi$ obtain the v's in terms of the q's and p's in the four examples in Section 18.3.

PROBLEM 18.14 Make a figure for Theorem 18.15 corresponding to the one accompanying Theorem 18.10.

19

ADDITIONAL TOPICS IN MECHANICS

We will continue to compare the Hamiltonian and Lagrangian formulations of mechanics, and get particular results when we think of configuration space as a pseudo-Riemannian manifold. We will then touch on the important ideas of momentum mappings and conservation laws, and finally on the Hamilton-Jacobi theory for solving the Hamilton canonical equations.

19.1 The configuration space as a pseudo-Riemannian manifold

In Section 18.3, we explained how the motion of a mechanical system can be obtained by means of the Hamilton canonical equations from a given Hamiltonian function on the cotangent space of the configuration space, and in Section 18.4 by means of the Legendre transformation we transferred structures on T^*M to analogs on TM. The correspondence between structures on T^*M and TM is completed by the following relations.

Theorem 19.1 *With the notation of Chapter 18,*
(i) $H = E_L \circ DL^{-1}$
(ii) $X_H = DL_* X_L$
(iii) $A_L = \langle \theta, X_H \rangle \circ DL$

Proof (i) For $\sigma \in T_p^*$,

$$\begin{aligned} E_L \circ DL^{-1}(\sigma) &= (A_L - L) \circ DL^{-1}(\sigma) = A_L \circ DL^{-1}(\sigma) - L \circ DL^{-1}(\sigma) \\ &= A_L(v) - L \circ DL^{-1}(\sigma) = \langle DL(v), v \rangle - L \circ DL^{-1}(\sigma) \\ &= \langle \sigma, DL^{-1}(\sigma) \rangle - L \circ DL^{-1}(\sigma) = H(\sigma) \end{aligned}$$

(ii) For $v \in TM$ and $\mathbf{a} \in T_v TM$,

$$\begin{aligned} \omega(DL_* X_L(v), DL_* \mathbf{a}) &= \omega_L(X_L(v), \mathbf{a}) \quad \text{since } \omega_L \text{ is the pull-back of } \omega \\ &= \frac{1}{2} dE_L(v)(\mathbf{a}) \quad \text{by (18.30)} \\ &= \frac{1}{2} dH(DL(v))(DL_* \mathbf{a}) \quad \text{since } dE_L \text{ is the pull-back of } dH \\ &= \omega(X_H(DL(v)), DL_* \mathbf{a}) \quad \text{by (18.2)} \end{aligned}$$

So, since $\mathbf{a}$ is arbitrary and ω is nondegenerate,

$$DL_* X_L(v) = X_H DL(v)$$

(iii) Problem 19.1. □

From Theorem 19.1(ii) we see that the integral curves of X_H and X_L go into one another and, hence, the projection of these curves on M are the same. The conclusion for mechanics is that the solutions of the Euler-Lagrange equations give motions of mechanical systems. That is, rather than starting with H on T^*M and solving the Hamilton canonical equations (18.14), we can start with L on TM and solve the Euler-Lagrange equations (18.36) and (18.37).

The standard form of *the Lagrangian*, L, in applications to mechanics is

$$L = \mathbf{T} - \boldsymbol{\phi} \circ \pi \tag{19.1}$$

where $\mathbf{T}$ is *the kinetic energy* and $\boldsymbol{\phi}$ is *the potential energy* of the system or rigid body, and $\pi : TM \to M$. $\mathbf{T}$ is a function on TM quadratic on each tangent space of M, and $\boldsymbol{\phi}$ is a function on the configuration space, M. The motivation for this form of L comes from the fact that in special cases in terms of rectangular cartesian coordinates on M the Euler-Lagrange equations reduce to Newton's equations, as we will see below.

Thus, from a slightly more abstract point of view, we can always think of the configuration space of a mechanical system (in which forces come from a potential) as a pseudo-Riemannian manifold on which a function, $\boldsymbol{\phi}$, is prescribed, the pseudo-Riemannian structure coming from the polarization of the quadratic function $\mathbf{T}$ (Section 5.4).

In a local coordinate system $(q_i, \dot{q}^i)$ on TM on (M, g) we have $g(v,w)(p) = g_{ij}(q^k)\dot{q}^i(v)\dot{q}^j(w)$, and $g(v,v)(p) = g_{ij}(q^k)\dot{q}^i\dot{q}^j$, so

$$D_v g = \left(\frac{\partial g}{\partial \dot{q}^1}(v), \ldots, \frac{\partial g}{\partial \dot{q}^n}(v)\right) = 2(g_{ij}\dot{q}^j, \ldots, g_{nj}\dot{q}^j) = 2g^\flat(v) \tag{19.2}$$

at each point $p \in M$.

Theorem 19.2 *On a pseudo-Riemannian manifold* (M, g) *let* $\mathbf{T}(v) = \frac{1}{2}g(v,v)$. *Then, with* L *given by* (19.1) *we get*

(i) $DL = g^\flat$

(ii) $\langle DL(v), w\rangle = g(v,w)$

(iii) $A_L = 2\mathbf{T}$

(iv) $E_L = \mathbf{T} + \boldsymbol{\phi} \circ \pi$

Proof Problem 19.2. □

Theorem 19.3 *On a pseudo-Riemannian manifold (M, g) let $\theta_g = (g^\flat)^*\theta$ and $\omega_g = -(g^\flat)^* d\theta$ where θ is the natural 1-form on T^*M. Then in local coordinates,*

$$\theta_g = g_{ij}\dot{q}^i dq^j \tag{19.3}$$

$$\omega_g = g_{ij} dq^i \wedge d\dot{q}^j + \frac{\partial g_{ij}}{\partial q^k}\dot{q}^i dq^j \wedge dq^k \tag{19.4}$$

Proof $(g^\flat)^* p_j dq^j = p_i(g^\flat(v))dq^i = g_{ij}\dot{q}^i dq^j$, and ω_g is obtained by exterior differentiation. □

Corollary *If (M, g) has a function L on TM as in* Theorem 19.2 *the coordinate expressions for θ_L and ω_L are given by* eqs. (19.3) *and* (19.4).

Theorem 19.4 *If $\phi: M \to N$ is an isometry, then $(\phi_*)^*\theta g_N = \theta g_M$. That is, the 1-form, θ_g, is preserved, and $\phi_*: TM \to TN$ is symplectic.*

Proof Since $\phi_* = g_N^\sharp \circ (\phi^*)^{-1} \circ g_M^\flat$,

$$\begin{aligned}
(\phi_*)^*\theta_{gN} &= (g_N^\sharp \circ (\phi^*)^{-1} \circ g_M^\flat)\theta_{gN} = (g_M^\flat)^*((\phi^*)^{-1})^*(g_N^\sharp)^*\theta_{gN} \\
&= (g_M^\flat)^*((\phi^*)^{-1})^*\theta_N \quad \text{by definition of } \theta_{gN} \\
&= (g_M^\flat)^*\theta_M \quad \text{by Theorem 18.10} \\
&= \theta_{gM} \quad \text{by definition of } \theta_{gM}
\end{aligned}$$

□

Theorem 19.5 *With L of the form* (19.1), *the projections $\gamma_0 = \pi \circ \gamma$ of the integral curves of X_L (i.e., the solutions of the Euler-Lagrange equations) satisfy*

$$\nabla_{\dot{\gamma}_0}\dot{\gamma}_0 = -\text{grad}\,\phi(\gamma_0) \tag{19.5}$$

Proof With L in the form (19.1), eq. (18.37) is

$$\frac{d}{dt}(g_{ij}\dot{q}^j) = \frac{1}{2}\,\frac{\partial g_{jk}}{\partial q^i}\dot{q}^j\dot{q}^k - \frac{\partial \phi}{\partial q^i}$$

Expanding the left side, multiplying both sides by g^{ik} and putting $\dot{q}^i = dq^i/dt$ according to eq. (18.36) we get

$$\begin{aligned}
\ddot{g}^k &= g^{lk}\left(\frac{1}{2}\,\frac{\partial g_{ij}}{\partial q_l} - \frac{\partial g_{li}}{\partial q^j}\right)\dot{q}^i\dot{q}^j - g^{ik}\frac{\partial \phi}{\partial q^i} \\
&= \left\{ {k \atop i\ \ j} \right\}\dot{q}^i\dot{q}^j - g^{ik}\frac{\partial \phi}{\partial q^i}
\end{aligned} \tag{19.6}$$

and this is the coordinate form of (19.5). □

Corollaries (i) *If there are no forces acting so that, in particular, $\boldsymbol{\phi} = 0$, then the projections, γ_0, of the integral curves of X_L are precisely the geodesics of the Levi-Civita connection of (M, g).*

(ii) *If M is pseudo-Euclidean space and q^i are rectangular cartesian coordinates, then* (19.6) *are Newton's equations.*

PROBLEM 19.1 Prove Theorem 19.1(iii).

PROBLEM 19.2 Prove Theorem 19.2.

PROBLEM 19.3 Use (19.1) to derive the relations mentioned at the end of Section 18.3.

19.2 The momentum mapping and Noether's theorem

In Section 18.1, after introducing the concept of a Hamiltonian vector field, X on a symplectic manifold, (M, ω), we focused on the Hamiltonian vector field, X_H, obtained from a given function, H, on (M, ω). Now we switch our point of view, and think of X as coming from a 1-parameter group action. That is, we start with a 1-parameter group action, $\mathcal{A}_M$, on M that preserves the symplectic structure, *a symplectic action*, and we get a vector field, X_M, on M the infinitesimal generator of the action (see Section 13.2).

Definition A function, $\boldsymbol{J}$, on a symplectic manifold, M is called *a momentum mapping* if

$$i_{X_M}\omega = d\boldsymbol{J} \tag{19.7}$$

where X_M is the infinitesimal generator of a symplectic action. (Note: this concept is a special case of a more general one which requires more information about Lie groups than we have available at this point.)

A class of momentum mappings comes from the following result.

Theorem 19.6 *If a 1-form, θ, on (M, ω) such that $d\theta = -\omega$ is invariant under a symplectic action on M, then $i_{X_M}\theta$ is a momentum mapping.*

Proof This comes immediately from the identity $di_{X_M}\theta + i_{X_M}d\theta = L_{X_M}\theta$. □

The next three theorems give conditions under which the hypothesis of Theorem 19.6 is satisfied, and give formulas for the associated momentum mappings.

Theorem 19.7 *If the manifold of Theorem 19.6 is a cotangent manifold, T^*M, and $\mathcal{A}_M$ is any action on M, then*

(i) *θ, the natural 1-form on T^*M, is preserved under $\mathcal{A}_{T^*M}$, and $i_{X_{T^*M}}\theta$ is a momentum mapping on T^*M.*

(ii)
$$i_{X_{T^*M}}\theta(\sigma_p) = \langle \sigma_p, X_M(p)\rangle \tag{19.8}$$

for $p \in M$ and $\sigma_p \in T_p^*M$.

Proof (i) The induced action, $\mathcal{A}_{T^*M}$ is symplectic and θ is preserved by Theorem 18.10 and its corollary. Hence, by Theorem 19.6, $i_{X_{T^*M}}\theta$ is a momentum mapping on T^*M.

(ii)

$$\begin{aligned} i_{X_{T^*M}}\theta(\sigma_p) &= \langle \theta(\sigma_p), X_{T^*M}(\sigma_p)\rangle = \langle \sigma_p, \pi_* \circ X_{T^*M}(\sigma_p)\rangle \quad \text{by definition of } \theta \\ &= \langle \sigma_p, X_M \circ \pi(\sigma_p)\rangle = \langle \sigma_p, X_M(p)\rangle \end{aligned}$$

□

Example Let $M = \mathbb{R}^n$ and $\mathcal{A}_M : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by $(u, (a^1, \ldots, a^n)) \mapsto (a^1 + ur^1, \ldots, a^n + ur^n)$ with $r^i \in \mathbb{R}$. Then $X_M : (a^1, \ldots, a^n) \mapsto (r^1, \ldots, r^n)$ and $\langle \sigma, X_M\rangle(a) = p_i r^i$ where $p_i = \sigma_i \circ \pi$ and σ_i are the components of σ. Thus, in this case, the momentum mapping on T^*M gives the X_M component of the momentum.

Theorem 19.8 *If the manifold of Theorem 19.6 is the tangent manifold of a pseudo-Riemannian manifold (M, g), and $\mathcal{A}_M$ is an action on M that preserves g, then*

(i) *θ_g, the pull-back of θ by $g^\flat$, is preserved under $\mathcal{A}_{TM}$, and $i_{X_{TM}}\theta_g$ is a momentum mapping on TM.*

(ii)
$$i_{X_{TM}}\theta_g(v_p) = g(v_p, X_M(p)) \tag{19.9}$$

for $p \in M$ and $v_p \in T_pM$.

Proof (i) The induced action $\mathcal{A}_{TM}$ is symplectic and θ_g is preserved by Theorem 19.4. Hence, by Theorem 19.6 $i_{X_{TM}}\theta_g$ is a momentum mapping on TM.

(ii) $i_{X_{TM}}\theta_g(v_p)$

$$
\begin{aligned}
&= \langle \theta_g(v_p), X_{TM}(v_p)\rangle \\
&= \langle (g^\flat)^*\theta(v_p), X_{TM}(v_p)\rangle && \text{by the definition of } \theta_g \\
&= \langle \theta(g^\flat(v_p)), g^\flat_* X_{TM}(v_p)\rangle && \text{by the property of transpose} \\
&= \langle g^\flat(v_p), \tilde{\pi}_* \cdot g^\flat_* X_{TM}(v_p)\rangle && \text{by the definition of } \theta \\
& && (\tilde{\pi} \text{ is the projection } T^*M \to M) \\
&= \langle g^\flat(v_p), \pi_* X_{TM}(v_p)\rangle && (\pi \text{ is the projection } TM \to M) \\
&= \langle g^\flat(v_p), X_M(p)\rangle = g(v_p, X_M(p))
\end{aligned}
$$

□

EXAMPLE Let $M = \mathbb{R}^3$ and $\mathcal{A}_M$ be an action on $\mathbb{R}^3$ such that for each $u \in \mathbb{R}$, the map $\mathbb{R}^3 \to \mathbb{R}^3$ is a rotation. The infinitesimal generator $X_M(a)$ at $a \in M$ is the velocity of a, and this can be written in the form $r_{ij}a^i$ where (r_{ij}) is a skew-symmetric matrix (Synge and Schild, section 5.3). If we write

$$
(r_{ij}) = \begin{pmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{pmatrix}
$$

then $X_M(a) = (-r_3a^2 + r_2a^3, r_3a^1 - r_1a^3, -r_2a^1 + r_1a^2) = r \times a$, the ordinary vector product. Hence, $g(v(a), X_M(a)) = g(v(a), r \times a) = g(r \times a, v(a)) = r_i(a \times mv(a))^i$, so the momentum mapping on TM gives the X_M component of the angular momentum.

Theorem 19.9 *If the manifold of Theorem 19.6 is a tangent manifold, TM, and $\mathcal{A}_M$ is an action on M such that L, a Lagrangian on TM, is preserved under $\mathcal{A}_{TM}$, then*

(i) *θ_L, the pull-back of θ by DL, is preserved under $\mathcal{A}_{TM}$, and $i_{X_{TM}}\theta_L$, is a momentum mapping on TM.*

(ii)
$$
i_{X_{TM}}\theta_L(v_p) = \langle DL(v_p), X_M(p)\rangle \tag{19.10}
$$

for $p \in M$ and $v_p \in T_pM$.

Proof Problem 19.4. □

In the previous chapter we focused on vector fields coming from Hamiltonian or Lagrangian functions, and so far in this section we have focused on a structure, the momentum mapping, arising from the point of view of a vector field as the infinitesimal generator of an action. From the first point of view, a vector field, through the Hamilton canonical equations or the Euler-Lagrange equations, *determines* a motion of a mechanical system, so that an integral of the vector field (i.e., of the differential equations) gives some information about the motion (cf., Theorem 18.4). From the second point of view, a vector field *arises* from the motion of a mechanical system, and an integral of the vector field is something conserved during the motion. The relation between these two points of view is expressed by *conservation laws.* They are contained in the following formulation.

Theorem 19.10 *If X_H is a Hamiltonian vector field on M, and X_M is the infinitesimal generator of a symplectic action with moment map, J, on M, then H is an integral of X_M iff J is an integral of X_H.*

Proof From (19.7) we can write $X_M = X_J$, and the result follows from Theorem 18.6(iii). □

We have the following important consequence of this general result.

Theorem 19.11 (E. Noether, Nachr. Ges. Göttingen, 1918) *Suppose a 1-parameter group acts on a manifold M (e.g., the configuration space of a mechanical system). If a given Lagrangian, L, is invariant under $\mathcal{A}_{TM}$, then $i_{X_{TM}}\theta_L$ is an integral of X_L, the Lagrangian vector field of L on TM, and is given by* eq. (19.10).

Proof (i) The fact that L is invariant under $\mathcal{A}_{TM}$ implies that

$$\langle D_v L, w\rangle = \langle D_{\gamma_v(u)}L, \gamma_{w(u)}\rangle$$

for $v, w \in TM$ and γ_v, γ_w are integral curves of X_{TM}. Putting $w = v$ we get that the action of L is invariant under $\mathcal{A}_{TM}$ and hence the energy, E_L, of L is invariant under $\mathcal{A}_{TM}$, or E_L is an integral of X_{TM}.

(ii) Let $J = i_{X_{TM}}\ \theta_L$. Then $dJ = i_{X_{TM}}\omega_L$ since θ_L is invariant according to Theorem 19.9. So X_{TM} has momentum map, J. Also X_L is the Hamiltonian vector field of E_L. So by part (i) and Theorem 19.10, J is an integral of X_L. □

Under the actions in the examples above L is invariant, and $DL(v_p) = p_i$, so $\langle DL(v_p),\ X_M(p)\rangle$ becomes, respectively, $p_i r^i$ and $p_i(r \times a)^i = r^i(a \times DL(v))$, and these are constants of the respective motions.

PROBLEM 19.4 Prove Theorem 19.9.

19.3 Hamilton-Jacobi theory

Having formulated a mechanical problem in terms of the Hamilton canonical equations, the solution is obtained by solving these equations. There is an interesting and somewhat effective classical method for this.

We will first put this method into the general context of "the method of characteristics" which we touched on in Section 14.3. We noted there that solutions of certain types of partial differential equations can be obtained from the solution of the system (14.17) of the characteristic vector field of the differential system of the partial differential equation. We can go in the opposite direction.

Theorem 19.12 *If a function, S, of $x^1, \dots, x^n$ is a solution of*

$$F\left(x^i, \frac{\partial S}{\partial x^i}, S\right) = 0 \tag{19.11}$$

and x^i are solutions of

$$\frac{dx^i}{du} = \frac{\partial F}{\partial p_i}(x^i, p_i, z) \tag{19.12}$$

where $z = S \circ x$ and $p_i = \dfrac{\partial S}{\partial x^i} \circ x$, then z and p_i satisfy

$$\frac{dp_i}{du} = -\left(\frac{\partial F}{\partial x^i}(x^i, p_i, z) + p_i \frac{\partial F}{\partial z}(x^i, p_i, z)\right) \tag{19.13}$$

$$\frac{dz}{du} = p_i \frac{\partial F}{\partial p_i}(x^i, p_i, z) \tag{19.14}$$

Proof Since S satisfies (19.11), S satisfies

$$\frac{\partial F}{\partial x^i} + \frac{\partial F}{\partial z}\frac{\partial S}{\partial x^i} + \frac{\partial F}{\partial p_j}\frac{\partial^2 S}{\partial x^i \partial x^j} = 0$$

or

$$\frac{\partial F}{\partial p_j}\frac{\partial^2 S}{\partial x^i \partial x^j} = -\left(\frac{\partial F}{\partial x^i} + \frac{\partial F}{\partial z}\frac{\partial S}{\partial x^i}\right) \tag{19.15}$$

Now, if x^i is a solution of (19.12), then from the definition of p_i,

$$\frac{dp_i}{du} = \frac{\partial^2 S}{\partial x^i \partial x^j}\frac{dx^j}{du} = \frac{\partial^2 S}{\partial x^i \partial x^j}\frac{\partial F}{\partial p_j}$$

which is precisely the left side of (19.15), and

$$\frac{dz}{du} = \frac{\partial S}{\partial x^i}\frac{dx^i}{du} = \frac{\partial S}{\partial x^i}\frac{\partial F}{\partial p_i}$$

which is eq. (19.14). □

Corollary *If F is independent of z, S is a solution of (19.11), and x^i are solutions of*

$$\frac{dx^i}{du} = \frac{\partial F}{\partial p_i}(x^i, p_i) \tag{19.16}$$

then

$$\frac{dp_i}{du} = -\frac{\partial F}{\partial x^i}(x^i, p_i) \tag{19.17}$$

Clearly, in different notation, eqs. (19.16) and (19.17) are the Hamilton canonical equations, so the corollary gives us solutions of the Hamilton canonical equations with Hamiltonian function, H, from solutions of the partial differential equation $H\left(q^i, \dfrac{\partial S}{\partial x^j}\right) = 0$.

Example We saw, in Section 18.3, Example 2, that the Hamiltonian of a particle subject to a central force is

$$H(q^1, q^2, p_1, p_2) = f(q^1) + \frac{1}{2m}\left(p_1^2 + \frac{p_2^2}{(q^1)^2}\right) \tag{18.20}$$

so we want to solve

$$f(q^1) + \frac{1}{2m}\left(\left(\frac{\partial S}{\partial q^1}\right)^2 + \frac{1}{(q^1)^2}\left(\frac{\partial S}{\partial q^2}\right)^2\right) = 0$$

Slightly simplifying the notation by putting $q^1 = r$ and $q^2 = \vartheta$ and looking for solutions S in the form $S(r, \vartheta) = \mathbf{R}(r) + \mathbf{\Theta}(\vartheta)$ we get $\mathbf{dR}/dr)^2 = -2mf(r) - A^2/r^2$ and $(d\mathbf{\Theta}/d\vartheta)^2 = A^2$ where A^2 is an arbitrary (nonnegative) constant. Then

$$\frac{dr}{du} = \frac{\partial H}{\partial p_1} = \frac{1}{m}\,p_1 = \frac{1}{m}\frac{\partial S}{\partial r} = \pm\frac{1}{m}\sqrt{-2mf(r) - \frac{A^2}{r^2}}$$

$$\frac{d\vartheta}{du} = \frac{\partial H}{\partial p} = \frac{1}{mr^2}\,p_2 = \frac{1}{mr^2}\frac{\partial S}{\partial \vartheta} = \pm\frac{A}{mr^2}$$

$$\frac{dp_1}{du} = -\frac{\partial H}{\partial q^1} = -f'(r) + \frac{1}{mr^3}\,p_2^2 = -f'(r) + \frac{A^2}{mr^3}$$

$$\frac{dp_2}{du} = -\frac{\partial H}{\partial q^2} = 0$$

While in this particular example the problem can be "solved by quadratures," in general, the problem of solving $H\left(q^i, \dfrac{\partial S}{\partial x^j}\right) = 0$ could be as formidable as solving the original Hamilton canonical equations. Thus, as an effective method for solving the Hamilton canonical equations, this straightforward application of the general theory for first-order partial differential equations has its limitations.

A variation of this general method, attributed to C.G.J. Jacobi, circumvents the need to solve a system of ordinary equations. We first have to reformulate the partial differential equation (19.11) in the case where the dependent variable is missing.

If F is a function of x^α and p_α, $\alpha = 0, 1, \ldots, n$, we assume we can separate out p_0 in $F(x^\alpha, p_\alpha) = 0$, writing it in the form $p_0 + H(x^0, x^i, p_i) = 0$, $i = 1, \ldots, n$. Then we look for functions, S, of x^α such that

$$\frac{\partial S}{\partial x^0} + H\left(x^0, x^i, \frac{\partial S}{\partial x^i}\right) = 0, \tag{19.18}$$

the Hamilton-Jacobi equation.

Definition A function $S : (x^0, x^i, Q^i) \mapsto S(x^0, x^i, Q^i)$ in some neighborhood, $\mathcal{U}$, of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ in which $\partial S/\partial x^i$ satisfy (19.18) and $\det(\partial^2 S/\partial x^i \partial Q^j) \neq 0$ is called *a complete integral of* (19.18).

Theorem 19.13 *Suppose we have a complete integral, S, of* eq. (19.18), *and arbitrary constants, P_i. Let x^i be a solution of*

$$\frac{\partial S}{\partial Q^i} = -P_i \tag{19.19}$$

and let

$$p_i = \frac{\partial S}{\partial x^i} \tag{19.20}$$

where on the right side of (19.20), the arguments, x^i, are the solutions of $\partial S/\partial Q^i = -P_i$. (Hence, x^i and p_i are functions of x^0 and Q^i). Then x^i and p_i are solutions of

$$\frac{dx^i}{dx^0} = \frac{\partial H}{\partial p_i} \tag{19.21}$$

$$\frac{dp_i}{dx^0} = -\frac{\partial H}{\partial x^i} \tag{19.22}$$

(Note: (1) Equations (19.21) and (19.22) are part of the set of characteristic equations of the Hamilton-Jacobi equation and, (2) eqs. (19.21) and (19.22) are the Hamilton canonical equations in slightly different notation.)

Proof With S a function of x^0, x^i, Q^i we differentiate (19.18) with respect to Q^i and (19.19) with respect to x^0 and we get, respectively,

$$\frac{\partial^2 S}{\partial Q^i \partial x^0} + \frac{\partial H}{\partial p_j} \frac{\partial^2 S}{\partial x^j \partial Q^i} = 0$$

$$\frac{\partial^2 S}{\partial Q^i \partial x^0} + \frac{\partial^2 S}{\partial Q^i \partial x^j} \frac{dx^j}{dx^0} = 0$$

which, since $\det(\partial^2 S/\partial Q^i \partial x^j) \neq 0$, yields (19.21). Similarly, differentiating (19.18) with respect to x^i and (19.20) with respect to x^0, we get, respectively,

$$\frac{\partial^2 S}{\partial x^0 \partial x^i} + \frac{\partial H}{\partial x^i} + \frac{\partial H}{\partial p_j} \frac{\partial^2 S}{\partial x^i \partial x^j} = 0$$

$$\frac{\partial^2 S}{\partial x^0 \partial x^i} + \frac{\partial^2 S}{\partial x^i \partial x^j} \frac{dx^i}{dx^0} = \frac{dp_i}{dx^0}$$

Combining these, using (19.21) we get (19.22). □

EXAMPLE We will apply the Hamilton-Jacobi method to the problem of the previous example; the motion of a particle with the Hamiltonian (18.20). Now we want to solve

$$\frac{\partial S}{\partial u} + f(r) + \frac{1}{2m} \left[\left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \vartheta} \right)^2 \right] = 0 \tag{19.23}$$

with $S(u, r, \vartheta) = \mathbf{U}(u) + \mathbf{R}(r) + \mathbf{\Theta}(\vartheta)$ we get $d\mathbf{U}/du = -Q^1$, $(d\mathbf{R}/dr)^2 = -2m(f(r) - Q^1) - (Q^2)^2/r$, and $(d\mathbf{\Theta}/d\vartheta)^2 = (Q^2)^2$, so

$$S(u, r, \vartheta, Q^1, Q^2) = -Q^1 u + \int \sqrt{-2m(f(r) - Q^1) - \frac{(Q^2)^2}{r^2}}\, dr + Q^2 \vartheta \tag{19.24}$$

is a complete integral of (19.23). Differentiating S with respect to Q^1 and Q^2, and using (19.19), we get, respectively,

$$-u + m \int \frac{1}{I_1}\, dr = -P_1$$

$$\vartheta - Q^2 \int \frac{1}{r^2 I_1}\, dr = -P^2$$

where I_1 is the integrand in (19.24). The solutions of these equations for r and ϑ include, when $Q^1 = 0$, the solutions obtained previously using the method of Theorem 19.12.

Finally, the rather formal analytical result of Theorem 19.13 can be given some geometrical content by utilizing the fact that eqs. (18.14) are valid for every canonical (natural) coordinate system ("the Hamilton canonical equations are invariant under canonical coordinate transformations") and introducing the concept of a generating function of a free canonical coordinate transformation.

It follows from Theorem 18.10 that if a canonical coordinate transformation $(q^i, p_i) \mapsto (Q^i, P_i)$ on T^*M comes from a coordinate transformation $(q^i) \mapsto (Q^i)$ on M, then $P_i dQ^i = p_i dq^i$. Now we want to consider the alternative situation. Specifically, on the graph of $(q^i, p_i) \mapsto (Q^i, P_i)$, i.e., at points in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ at which (with slight abuse of notation)

$$Q^i - Q^i(q^1, \dots, q^n, p_1, \dots, p_n) = 0$$

$$P_i - P_i(q^1, \dots, q^n, p_1, \dots, p_n) = 0 \quad i = 1, \dots, n$$

we want to be able to use q^i and Q^i as coordinates. According to the implicit function theorem, the condition we now impose is $\det(\partial Q^i/\partial p_j) \neq 0$.

If (q^i, p_i) and (Q^i, P_i) are two sets of canonical coordinates on T^*M, then on the intersection of their domains, or, equivalently, on the graph in $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ of the coordinate transformation, $d(p_1 dq^i) = d(P_i dQ^i)$, or $d(p_i dq^i - P_i dQ^i) = 0$, so, locally, $p_i dq^i - P_i dQ^i = dS(q^i, p_i)$. If now we can take q^i and Q^i as coordinates on the graph, then

$$dS(q^i, Q^i) = p_i dq^i - P_i dQ^i \tag{19.25}$$

and

$$\frac{\partial S}{\partial q^i} = p_i, \quad \frac{\partial S}{\partial Q^i} = -P_i \tag{19.26}$$

Definitions A coordinate transformation with the property $\det(\partial Q^i/\partial p_j) \neq 0$ is called *free*, and a function, S, of q^i and Q^i obtained as above, from a free coordinate transformation is called a *generating function of the transformation.*

We have seen that a free canonical coordinate transformation produces functions of q^i and Q^i. We can go in the opposite direction.

Theorem 19.14 *Every function S, on a $2n$-dimensional submanifold with coordinates q^i and Q^i of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ on which $\det(\partial^2 S/\partial Q^i \partial q^j) \neq 0$ is a generating function of a free canonical coordinate transformation.*

Proof Define p_i by $p_i = \partial S/\partial q^i$ (the first equation in (19.26)). Then using our hypothesis we can solve this equation for Q^i as functions of q^i and p_i. Define $P_i(q^i, p_i) = -(\partial S/\partial Q^i)(q^i, Q^i(q^j, p_j))$. Then $p_i dq^i - P_i dQ^i = (\partial S/\partial q^i) dq^i - (\partial S/\partial Q^i) dQ^i = dS(q^i, Q^i)$, so the transformation is canonical and has generating function, S. From the definition of p_i, $(\partial p_i/\partial Q^j) = (\partial^2 S/\partial q^i \partial Q^j)$, so $\det(\partial^2 S/\partial q^i \partial Q^j) \neq 0 \Rightarrow \det(\partial Q^i/\partial p_j) \neq 0$ and hence the transformation is free.

□

Theorem 19.15 *Suppose S is a function of q^i and Q^i with* $\det(\partial^2 S/\partial q^i \partial Q^j) \neq 0$. *Then by Theorem 19.14 we get a free canonical coordinate transformation, Ψ, given by (19.26). Suppose S also satisfies*

$$H(q^i, p_j) = K(Q^i) \tag{19.27}$$

where K is a given function of Q^i. (That is, S is a complete integral of (19.27)). Then the free canonical coordinate transformation, Ψ, maps $H(q^i, p_j)$ to $K(Q^i)$.

Proof Let $\bar{H}$ be the image of H under Ψ; i.e., $\bar{H} \circ \Psi = H$. Then

$$\begin{aligned} \bar{H}(Q^i(q^i, p_j), P_j(q^i, p_j)) &= H(q^i, p_j) \\ &= H\left(q^i, \frac{\partial S}{\partial q^j}\right) \quad \text{by the first equation of (19.26)} \\ &= K(Q^i) \quad \text{by our hypothesis} \end{aligned}$$

□

Now the process of solving the Hamilton canonical equations for q^i and p_j consists of first solving the Hamilton canonical equations for Q^i and P_j in the new coordinates, which is easy since

$$\frac{dQ^i}{du} = \frac{\partial K}{\partial P_i} = 0, \quad \frac{dP_j}{du} = -\frac{\partial K}{\partial Q^j} = \text{ constants} \tag{19.28}$$

and then finding the equations of transformation, which we get from (19.26) once we have a complete integral of (19.27).

Example We will apply this method to the same problem as in the two previous examples. Now we want to solve

$$f(r) + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \vartheta}\right)^2\right] = K(Q^1, Q^2) \tag{19.29}$$

With $S(r, \vartheta) = \mathbf{R}(r) + \mathbf{\Theta}(\vartheta)$ we get

$$\left(\frac{d\mathbf{R}}{dr}\right)^2 = -2mf(r) - \frac{(Q^2)^2}{r^2} + 2mK(Q^1, Q^2) \quad \text{and} \quad \left(\frac{d\mathbf{\Theta}}{d\vartheta}\right)^2 = (Q^2)^2$$

so

$$S(r, \vartheta, Q^1, Q^2) = \int \sqrt{-2mf(r) - \frac{(Q^2)^2}{r^2} + 2mK(Q^1, Q^2)}\, dr + Q^2\vartheta \tag{19.30}$$

is a complete integral of (19.29). Differentiating S with respect to Q^1 and Q^2, and using the second equation of (19.26), we get, respectively,

$$m\frac{\partial K}{\partial Q^1}\int \frac{1}{I_2}\, dr = -P_1$$

$$-Q^2\int \frac{1}{r^2 I_2}\, dr + m\frac{\partial K}{\partial Q^2}\int \frac{1}{I_2}\, dr + \vartheta = -P_2$$

where I_2 is the integrand in (19.30). Finally, from (19.28),

$$m\frac{\partial K}{\partial Q^1}\int \frac{1}{I_2}\, dr = -P_1 + u\frac{\partial K}{\partial Q^1}$$

$$-Q^2\int \frac{1}{r^2 I_2}\, dr + m\frac{\partial K}{\partial Q^2}\int \frac{1}{I_2}\, dr + \vartheta = -P_2 + u\frac{\partial K}{\partial Q^2}$$

where P_1 and P_2 are constants, which, when $K(Q^1, Q^2) = Q^1$, give precisely the solutions obtained in the previous example. Alternatively, if $\partial K/\partial Q^1 \neq 0$,

$$m\int \frac{1}{I_2}\, dr - u = \frac{p_1}{\partial K/\partial Q^1}$$

$$-Q^2\int \frac{1}{r^2 I_2}\, dr + \vartheta = -P_2 + P_1\frac{\partial K/\partial Q^2}{\partial K/\partial Q^1}$$

which give the same solutions as the previous example except for the designation of the constants.

PROBLEM 19.5 Apply the methods in this section to the other examples in Section 18.3.

20

A SPACETIME

In Chapters 18 and 19, we described models of "the motion of systems of particles and rigid bodies" the subject of classical analytical mechanics. In this chapter and the next we describe a certain model of "spacetime," called Einstein's theory of relativity.* This is usually more specifically called the special theory of relativity, or the theory of special relativity to distinguish it from Einstein's theory of gravitation which we will look at in Chapters 22 - 24.

We could start by simply taking four-dimensional Minkowski space (Section 15.1) as our model of spacetime, and then describe the consequences of the Lorentz transformations, the isometries of Minkowski space (Section 15.1), for mechanics, electrodynamics, etc. However, to obtain a deeper understanding of some of these apparently not so intuitively obvious consequences, we will take the longer route and develop our model from Galileo's and Newton's notions of space and time.

20.1 Newton's mechanics and Maxwell's electromagnetic theory

Newton's model of "physical space" was a 3-dimensional Euclidean affine space, $\mathcal{E}_0^3$, a 3-dimensional affine space whose vector space has a positive definite symmetric bilinear form, $b \in \mathbf{S}^2(V^*)$ (cf., Section 5.4(i)). Hence, in this model we do not have an origin, but we do have straight lines, planes, parallelism, and the distance between two points as the length of the straight line segment joining them.

In Newton's model we have "matter," i.e., "particles" and "bodies." These can be described by associating positive real functions, *mass*, or *mass density*, with certain possible "positions" or "locations" in $\mathcal{E}_0^3$ represented by certain subsets of $\mathbb{R}^n$ (cf., Section 18.3). "Motion" is manifested by the changing of distances between particles. Hence we think of our distances as a function of a single real variable, t. The existence of repetitive changes gives us time units. ("Time is defined so that motion looks simple," Misner et al. 1973.) In short, we define *a particle in* $\mathcal{E}_0^3$ as a pair, (m, γ), where m is a number and γ is a curve in $\mathcal{E}_0^3$ parametrized by t.

Newton could use $\mathcal{E}_0^3$ as a model of physical space because he used the "fixed stars" in terms of which he could locate a point. In $\mathcal{E}_0^3$ we can "pick a point,"

*And called the relativity theory of Poincaré and Lorentz by Whittaker, 1960.

and define the velocity of a particle. However, in physical space there is no way to "pick a point," or talk about the velocity of a particle without reference to "particles" which may be "moving." In order to avoid using fixed stars we introduce frames of reference.

A frame of reference is usually pictured as a moving set of coordinate axes. The essential ingredients of this picture are not the coordinates, but the motion, a concept we associated with matter (particles and bodies), and a concept of rigidity. Thus, a frame of reference is sometimes described as a body in rigid motion (or a rigid body in motion). Most simply, we will define *a frame of reference in* $\mathcal{E}_0^3$ to be a (differentiable) 1-parameter family, $\{\phi_t\}$, of rigid motions, or isometries, of $\mathcal{E}_0^3$. *A point,* $\mathcal{P}$, *of a frame of reference* is an orbit of $\{\phi_t\}$. If μ is any rectangular cartesian coordinate system on $\mathcal{E}_0^3$ (Section 18.3), then $\mu \circ \phi_t^{-1}$ are *rectangular cartesian coordinates of the frame of reference,* $\{\phi_t\}$. In other words, the coordinates of p with respect to $\{\phi_t\}$ are the coordinates of $p_0 = \phi_t^{-1}(p)$. If ν is another rectangular cartesian coordinate system, then $\nu \circ \phi_t^{-1} = (\nu \circ \mu^{-1}) \circ \mu \circ \phi_t^{-1}$ and the two sets of coordinate functions $x^i = \mu^i \circ \phi_t^{-1}$ and $y^i = \nu^i \circ \phi_t^{-1}$ of the frame of reference $\{\phi_t\}$ are related by

$$y^i(p) = a^i_j x^j(p) + b^i \tag{20.1}$$

where a^i_j are the components of an orthogonal matrix (Section 5.4).

Finally, our particles in $\mathcal{E}_0^3$ will be associated with frames of reference; i.e., *a particle in a frame of reference,* $\{\phi_t\}$, is a pair $(m, \phi_t^{-1} \circ \gamma)$, and *the velocity in* (or, *with respect to*) $\{\phi_t\}$ is the velocity of the curve $\phi_t^{-1} \circ \gamma$ (Section 9.3). The component functions (with respect to the frame $\{\phi_t\}$) are $\gamma^i = x^i \circ \gamma = \mu^i \circ \phi_t^{-1} \circ \gamma$ so the velocity of the curve $\phi_t^{-1} \circ \gamma$ at t is $(\phi_t^{-1} \circ \gamma)^{\cdot}(t) = D_t \gamma^i \; \partial/\partial \mu^i$.

If $\{\psi_t\}$ is another frame of reference and ν is another rectangular cartesian coordinate system, then for a given point, p, the relation between the coordinates $\mu \circ \phi_t^{-1}$ of p and the coordinates $\nu \circ \psi_t^{-1}$ of p will depend on t. That is, the coordinate functions $x^i = \mu^i \circ \phi_t^{-1}$ and $y^i = \nu^i \circ \psi_t^{-1}$ will satisfy relations of the form

$$y^i(p) = f^i(x^j(p), t) \tag{20.2}$$

for all p and t.

By introducing frames of reference we effectively get rid of $\mathcal{E}_0^3$, replacing it by the set of all frames of reference.

Having modeled the basic ingredients in our version of Newtonian mechanics, in order to proceed we have to relate these concepts by "physical laws." That is, we have to throw in some physics, specifically, *Newton's First Law,* or, *The Principle of Inertia.* The effect of this law is essentially to sort out a particular class of frames of reference, so-called *inertial frames.* Newton said (*Principia,* 1687) that there exists a frame of reference (the fixed stars) in $\mathcal{E}_0^3$ with respect to which all "free particles" move in a straight line with constant speed. We translate "free particle" to mean that there are no particles or bodies close enough to it to have any noticeable effect on it. The last phrase in Newton's law is described in our terms by the first of the following definitions.

Definitions A particle in $\mathcal{E}_0^3$, (m, γ), *moves in a straight line with constant speed with respect to a frame of reference*, $\{\phi_t\}$, if there is a coordinate system, μ, such that $v^i = D_t\gamma^i$ are constant, where $\gamma^i = x^i \circ \gamma = \mu^i \circ \phi_t^{-1} \circ \gamma$. In particular, (m, γ) is *at rest in* $\{\phi_t\}$ if there is a coordinate system such that $v^i = 0$. By eq. (20.1), these definitions are independent of the coordinate system of $\{\phi_t\}$.

Definition If the functions f^i in eq. (20.2) are given by

$$f^i(x^j, t) = a^i_j x^j - v^i t + b^i \tag{20.3}$$

where a^i_j are the elements of an orthogonal matrix, and v^i and b^i are constants, then (20.2) becomes

$$y^i(p) = a^i_j x^j(p) - v^i t + b^i \tag{20.4}$$

and (20.4) are called *the general Galilean transformations.*

Theorem 20.1 The Newtonian (Galilean) Principle of Relativity. *Suppose* $\{\phi_t\}$ *and* $\{\psi_t\}$ *are two frames of reference and* $\{\phi_t\}$ *is inertial. Then* $\{\psi_t\}$ *is inertial iff* $\{\phi_t\}$ *and* $\{\psi_t\}$ *have coordinates* $x^i = \mu^i \circ \phi_t^{-1}$ *and* $y^i = \nu^i \circ \psi_t^{-1}$ *which are related by a general Galilean transformation.*

Proof Problem 20.1. □

Another common way of describing (at least in part) the result above is to say the form of the equation $d^2x^i/dt^2 = 0$ is invariant (or the equation is covariant) under Galilean transformations. One can show that the usual form of Newton's Second Law, in the case where the force comes from a potential, is also invariant under Galilean transformations. See Problem 20.2. The important thing about these forms is that they are independent of the transformations themselves; i.e., they do not involve the coefficients of (20.4). (Cf. last paragraph of observation (1), at the end of this section.)

We now turn to a brief account of Maxwell's electromagnetic theory.

Coulomb found that there is an *electric force field*, E, produced by stationary charges. Coulomb's law has the same form as Newton's law of gravitation which implies that

$$\text{curl } E = 0, \quad \text{div } E = \frac{\eta}{\varepsilon} \tag{20.5}$$

where η is the charge density, and ε, the permutivity, is a constant whose value depends on the units used.

Ampere found that there is a force on a closed current-carrying wire due to a second closed current-carrying wire. This force is expressed in terms of a vector field, B, *the magnetic induction.* B is proportional to the magnitude of the second current times a certain integral around that wire (the law of Biot and Savart). As a consequence we find

$$\text{div } B = 0, \quad \text{curl } B = \mu J \tag{20.6}$$

where J is the current density in the second wire, and μ, the permeability, is another constant whose value depends on the units used.

Faraday discovered that there is a current in a wire in the presence of a varying magnetic field, so then a term $-\partial B/\partial t$ must be added to the right side of Coulomb's curl equation in (20.5). Maxwell added $\varepsilon\mu\ \partial E/\partial t$ to the right side of Ampere's curl equation in (20.6) in order to provide for conservation of charge. With these additions the system of equations governing E and B becomes

$$\begin{gathered} \text{curl } E = -\frac{\partial B}{\partial t}, \quad \text{div } E = \frac{\eta}{\varepsilon} \\ \text{div } B = 0, \quad \text{curl } B = \mu J + \varepsilon\mu\ \frac{\partial E}{\partial t} \end{gathered} \tag{20.7}$$

The constants ε and μ appearing in the last two equations of (20.7) come in as a result of the proportionality factors in the Coulomb and Ampere laws. We might expect to be able to eliminate them by choosing proper units, but there are not enough units to be chosen. We can think of Coulomb's law as determining units for charge, and Ampere's law (or Biot-Savart's) as determining units for current, but charge density and current density are related by $J = \eta V$ where V is the velocity of the charged particles of the current. Thus, we cannot get rid of both ε and μ, and no matter how units are chosen there remains one empirically determined constant in the set of equations.

Now we make two observations about equations (20.7) which evince an important distinction between Newton's mechanics and Maxwell's electromagnetic theory.

(1) In contrast to Newton's mechanics, in Maxwell's equations there is no explicit motion and no involvement of frames of reference. The functions and vector fields in Maxwell's equations are defined on some "medium" and can vary with time. Thus, if we choose for our medium $\mathcal{E}_0^3$, then equations (20.7) are defined on $\mathcal{E}_0^3 \times \mathbb{R}$.

Hertz supposed the medium could be any frame of reference moving uniformly with respect to $\mathcal{E}_0^3$ - "the aether contained within ponderable bodies moves with them" (Whittaker, Vol. I, p. 328) - and he generalized eq. (20.7) to apply to moving media (or moving bodies). If this motion is described by a given vector field, V, the generalized equations have an additional term $-\text{curl } V \times B$ on the left side of the curl E equation, and an additional term, $\varepsilon\mu[\text{curl } V \times E - (\sigma/\varepsilon)V]$ on the left side of the curl B equation. ($V \times B$ and $V \times E$ are the vector products of vector analysis.) See Problem 20.3. Thus, for example, we can imagine a copper wire moving through a static magnetic field in such a way that at points of the wire the variation of B is exactly the same as it was for a stationary wire in a varying magnetic field. Then the field, E, in the wire will be produced by the term curl $V \times B$ in the first equation of the generalized (20.7) instead of $\partial B/\partial t$.

Clearly, by construction, the *form* of the generalized equations is invariant under the transformations (20.4) between two frames of references. However, these equations contain the coefficients of the transformation, specifically, V (cf., comment below Theorem 20.1). V is a velocity with respect to one particular frame of reference, and thus the equations identify a unique frame of reference which should be physically distinguishable.

(2) Maxwell's equations (20.7) have a special property which the generalized system does not have. Take the curl of the last equation of (20.7) and use the first equation to eliminate E. Then each of the components, in rectangular cartesian coordinates, of B satisfies

$$\sum_{j=1}^{3} \frac{\partial^2 B^i}{\partial x^{j^2}} - \varepsilon\mu \frac{\partial^2 B^i}{\partial t^2} = \text{first-order terms} \tag{20.8}$$

The components of E satisfy equations of the same kind.

The solutions of equations of the from (20.8) are waves. Thus, the components of E and B are propagated as waves with speed $1/\sqrt{\varepsilon\mu}$ with respect to the frame $\mathcal{E}_0^3$. The significance of this fact is that having gotten rid of $\mathcal{E}_0^3$ in mechanics by introducing frames of reference, we find it popping up again in electromagnetic theory and having a certain unique importance. While we have been able to do mechanics without worrying about having to do it in some unique frame of reference, electromagnetic theory tells us we should look for a certain unique frame of reference. Still more remarkable is the result that the velocity of propagation $1/\sqrt{\varepsilon\mu}$ turns out to be precisely the empirically measured speed of light!

PROBLEM 20.1 Prove Theorem 20.1. Hint: For the "only if" part show that, for any particle, $(m, \phi_t^{-1} \circ \gamma)$, the assumption that whenever there is a coordinate system, μ, for $\{\phi_t\}$ such that $d\gamma^i/dt =$ constant, then there is a coordinate system, ν, for $\{\psi_t\}$ such that $d\gamma'^i/dt =$ constant implies that the functions f^i in eq. (20.2) satisfy $\partial^2 f^i/\partial x^j \partial x^k = 0$. Cf., also, the derivation of the Lorentz transformation in Section 20.3.

PROBLEM 20.2 Prove that Newton's Second Law, in the case where the force comes from a potential, is covariant under Galilean transformations.

PROBLEM 20.3 Derive the form of the Maxwell equations for a moving medium. (See Robertson and Noonan, Section 2.5.)

20.2 Frames of reference generalized

The fact that $1/\sqrt{\varepsilon\mu}$ is the speed of light leads to the conclusion that light is simply a particular type of electromagnetic radiation, and so various experiments with light were made to clarify the status of Maxwell's equations.

In Section 20.1, we conceived of Maxwell's equations as being defined on $\mathcal{E}_0^3 \times \mathbb{R}$. We could try to generalize by claiming these equations are only valid *locally* - on pieces of $\mathcal{E}_0^3 \times \mathbb{R}$. This permits bodies in relative motion to be at rest with respect to the medium - i.e., bodies drag the medium along with them (there is an ether drag). This possibility was excluded by the phenomenon of aberration of light rays from the stars observed by Bradley in 1728. The only other possibility is that bodies, in particular the earth, move with respect to the medium (there is an ether drift). The Michelson-Morley experiment, 1887, effectively squelched that idea. These experiments (and several others) created a logical impasse. The way in which Einstein resolved this problem amounts, in our terms, to generalizing our concept of frame of reference. In order to motivate this, we first examine the implications of eq. (20.8).

For our purposes it will suffice to examine the special case of eq. (20.8) when the right side is zero. This will occur when $\eta = J = 0$. We then have precisely the same equation for the components of E. Thus, we are led to consider the basic wave equation

$$\sum_{i=1}^{3} \frac{\partial^2 f}{\partial x^{i^2}} - \frac{1}{c}\frac{\partial^2 f}{\partial t^2} = 0 \tag{20.9}$$

Solutions of eq. (20.9) are determined by initial conditions which are propagated on *wave fronts*. A wave front can be thought of as a moving surface in $\mathcal{E}_0^3$, and, in coordinates, as a set of solutions of

$$h(t, x^1, x^2, x^3) = \text{ constant} \tag{20.10}$$

for some function h. We will now obtain a necessary condition for h if (20.10) represents a wave front.

A property of a wave front is that it separates regions in which f is constant from those in which a derivative of f is not zero. That is, a wave front consists of points at which the solution of (20.10) may not be analytic. On the other hand, with certain exceptions, the characteristic hypersurfaces, if f and $\partial f/\partial t$ are analytic on a hypersurface they will be propagated as analytic functions in a neighborhood of the hypersurface. Thus, the wave fronts must be characteristic hypersurfaces. The necessary and sufficient conditions that a hypersurface be characteristic is given by the Cauchy-Kowalewski construction. To obtain it for our case we first simplify our notation.

We write eq. (20.9) in the form

$$\sum_{\alpha=0}^{3} \underset{\alpha}{e} \frac{\partial^2 f}{\partial x^{\alpha^2}} = 0 \tag{20.11}$$

where $x^0 = ct$, $\underset{0}{e} = -1$, and $\underset{1}{e} = \underset{2}{e} = \underset{3}{e} = 1$. Now, the hypersurface (20.11) can be embedded in a family $\{h(x^\lambda) = a\colon a \in \mathbb{R}\}$, and these in turn can be used to construct coordinate transformations

$$y^\lambda = y^\lambda(x^\mu), \quad \lambda,\ \mu = 0, 1, 2, 3$$

with $y^0 = h$. By the chain rule,

$$\frac{\partial^2 f}{\partial x^{\alpha^2}} = \frac{\partial^2 f}{\partial y^\lambda \partial y^\mu}\, \frac{\partial y^\lambda}{\partial x^\alpha}\, \frac{\partial y^\mu}{\partial x^\alpha} + \frac{\partial f}{\partial y^\lambda}\, \frac{\partial^2 y^\lambda}{\partial x^{\alpha^2}}$$

and substituting this into (20.11) we get

$$\sum_{\alpha=0}^{3} \underset{\alpha}{e}\, \frac{\partial y^i}{\partial x^\alpha}\, \frac{\partial y^j}{\partial x^\alpha}\, \frac{\partial^2 f}{\partial y^i \partial y^j} + 2\sum_{\alpha=0}^{3} \underset{\alpha}{e}\, \frac{\partial y^i}{\partial x^\alpha}\, \frac{\partial y^0}{\partial x^\alpha}\, \frac{\partial^2 f}{\partial y^i \partial y^0}$$

$$+ \sum_{\alpha=0}^{3} \underset{\alpha}{e} \left(\frac{\partial y^0}{\partial x^\alpha}\right)^2 \frac{\partial^2 f}{\partial y^{0^2}} + \text{ first-order terms } = 0, \quad i, j = 1, 2, 3 \tag{20.12}$$

Now notice that if we are given f and $\partial f/\partial y^0$ on any hypersurface $y^0(x^\lambda) =$ constant, then all first and second partial derivatives of f can be calculated on this hypersurface except $\partial^2 f/\partial y^{0^2}$, and if $\sum_{\alpha=0}^{3} \underset{\alpha}{e}(\partial y^0/\partial x^\alpha)^2 \neq 0$ we get $\partial^2 f/\partial y^{0^2}$ from eq. (20.12). We can proceed to calculate all the higher-order derivatives of f by differentiating (20.12) and get a power series expansion for f in a neighborhood of $y^0(x^\lambda) =$ constant. Thus, a necessary condition for the function h in eq. (20.10), if eq. (20.10) represents a wave front, is that h satisfies

$$\sum_{\alpha=0}^{3} \underset{\alpha}{e} \left(\frac{\partial h}{\partial x^\alpha}\right)^2 = 0 \tag{20.13}$$

A particular solution of (20.13) is given by

$$h(t, x^1, x^2, x^3) = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} - ct$$

This is a spherical wave front, due to a flash of light at $x^i = t = 0$. At time t this flash will be observed on a sphere of radius ct. These spherical wave solutions suggest a way of interpreting the seemingly contradictory empirical evidence mentioned at the beginning of this section: light is propagated in the same way in any two frames of reference, and yet the "medium" in which Maxwell's equations are valid cannot coincide with both frames.

Suppose points $\mathcal{P}$ and $\mathcal{Q}$ of two relatively moving frames of reference coincide at some instant at which time and place a flash of light is emitted. Then, according to $\mathcal{P}$, t seconds later the light will have reached p and q. But according to $\mathcal{Q}$, after he/she counts t seconds, the light will be at p' and q' (see Fig. 20.1). Thus, according to $\mathcal{Q}$, the light arrived at p in less than t seconds, and arrived at q in more than t seconds. This suggests that in relatively moving frames, time has to be measured differently. In particular, two simultaneous occurrences according to $\mathcal{P}$ are not simultaneous according to $\mathcal{Q}$.

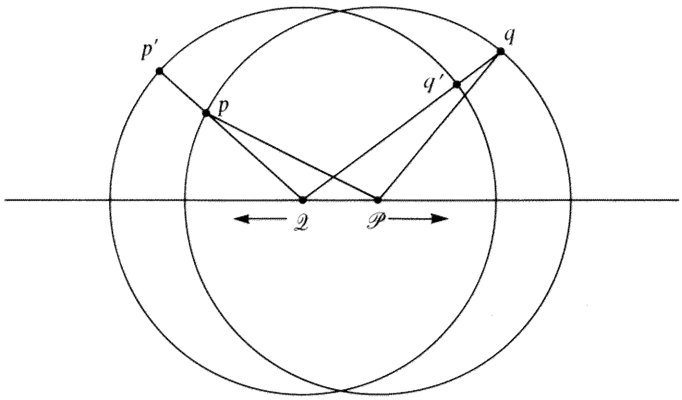

Figure 20.1

Our original definition of a frame of reference in $\mathcal{E}_0^3$ involved a parameter, t. We used the same parameter for all frames of reference, so, in particular, in coordinate transformations such as (20.2) and (20.4) only one "time" appears. We now allow different frames of reference to be parametrized by different times. Now we consider the set of all such *generalized frames*, and in analogy with our statement of Newton's Principle of Inertia we express *Maxwell's law* by the statement: there exists one of these frames of reference with respect to which, in some rectangular cartesian coordinate system, Maxwell's equations are valid. *Einstein's Postulate*, on which his theory of relativity is based, is that there exists a generalized frame of reference for which both Newton's and Maxwell's laws are valid.

The Galilean Principle of Relativity, Theorem 20.1, gave us an entire class of frames of reference, once we had Newton's law. Now, having both Newton's law and Maxwell's law we would like to obtain a class of our generalized frames of reference. In Section 20.1 the class of frames was described by eq. (20.4). Our procedure now will be to try to find transformation laws, corresponding to eq. (20.4), which describe the class of generalized frames of reference in which *both* Newton's law and Maxwell's law are valid.

20.3 The Lorentz transformations

Suppose (x^1, x^2, x^3) are rectangular cartesian coordinates and t is the parameter for a frame of reference $\{\phi_t\}$. Suppose $\{\psi_{t'}\}$ is a second frame of reference with rectangular cartesian coordinates (y^1, y^2, y^3) and parameter t' such that (1) if Newton's law is valid in $\{\phi_t\}$ then Newton's law is valid in $\{\psi_{t'}\}$, and (2) if Maxwell's law is valid in $\{\phi_t\}$ then Maxwell's law is valid in $\{\psi_{t'}\}$. These two conditions will impose a relation between the coordinates and time on the first frame and the coordinates and time on the second frame which we will now obtain. More specifically, we will obtain functions f^λ for which

$$y^\lambda = f^\lambda(x^\alpha) \tag{20.14}$$

where $\alpha,\ \lambda = 0, 1, 2, 3$ and $x^0 = ct$ and $y^0 = ct'$.

The first condition, the "invariance" of Newton's Law, reads in coordinates,

I. If (t, x^1, x^2, x^3) are related by

$$x^i - x^i_* = v^i(t - t_*) \tag{20.15}$$

in the first frame, then (t', y^1, y^2, y^3) are related by

$$y^i - y^i_* = {}'v^i(t' - t'_*) \tag{20.16}$$

in the second frame (v^i and $'v^i$ are constants).

Equations (20.15) and (20.16) give the trajectory of a "free particle." v^i and $'v^i$ are its velocity in the respective frames, and the star subscripts denote initial values. (For simplicity we are writing x^i and y^i instead of the more accurate $x^i \circ \gamma(t)$ and $y^i \circ \gamma(t)$.)

The second condition, the "invariance" of Maxwell's Law, implies the "invariance" of wave fronts; i.e.,

II. If h satisfies eq. (20.13) in the first frame, then h', the composition of h and the inverse of (20.14), satisfies

$$\sum_{\alpha=0}^{3} \underset{\alpha}{e} \left(\frac{\partial h'}{\partial y^\alpha}\right)^2 = 0 \tag{20.17}$$

in the second frame.

We can write this condition in an equivalent, more useful fashion.

Recall (Section 14.3), that corresponding to any given first-order partial differential equation there is an equivalent system of characteristic equations (14.17). That is, corresponding to a solution of the partial differential equation there is a family of solutions of (14.17), and conversely. Applying (14.17) to our present case where the first-order partial differential equation is (20.13), f is given by

$$f\left(x^\alpha, \frac{\partial h}{\partial x^\alpha}, h\right) = \sum_{\alpha=0}^{3} \underset{\alpha}{e} \left(\frac{\partial h}{\partial x^\alpha}\right)^2$$

and the first two sets of equations in (14.17) are, respectively,

$$\frac{dx^\alpha}{du} = 2 \underset{\alpha}{e} \frac{\partial h}{\partial x^\alpha}, \quad \text{no sum on } \alpha$$

and

$$\frac{d}{du}\left(\frac{\partial h}{\partial x^\alpha}\right) = 0$$

So, $\partial h/\partial x^\alpha$ are constant along the integral curves of the system, and integrating the first set, choosing $u = (t - t_*)/2$ we get

$$c = -\frac{1}{c}\frac{\partial h}{\partial t} \tag{20.18}$$

and

$$x^i - x^i_* = \frac{\partial h}{\partial x^i}\,(t - t_*) \tag{20.19}$$

By using (20.13) we can replace (20.18) with

$$\sum_{i=1}^{3}\left(\frac{\partial h}{\partial x^i}\right)^2 = c^2 \tag{20.20}$$

If in eq. (20.19) we fix t, and let x^i_* vary over some initial surface, then eqs. (20.19) are the equations of a wave front moving with velocity $\partial h/\partial x^i$. On the other hand, if we think of the x^i_* as fixed and t varying, then eqs. (20.19) are the equations of *the trajectory of a photon*, or *a ray of light*.

The desired formulation of our second condition, the "invariance" of Maxwell's law, can now be stated as follows.

II′. If (t, x^1, x^2, x^3) are related by eq. (20.19), in which the constant coefficients $\partial h/\partial x^i$ are subject to the condition (20.20), on the first frame, then (t', y^1, y^2, y^3) are related by

$$y^i - y^i_* = \frac{\partial h'}{\partial y^i}(t' - t'_*) \tag{20.21}$$

in which the constant coefficients $\partial h'/\partial y^i$ are subject to

$$\sum_{i=1}^{3}\left(\frac{\partial h'}{\partial y^i}\right)^2 = c^2 \tag{20.22}$$

on the second frame

The form of the transformation (20.14) can now be obtained by a rather formal application of eqs. (20.15), (20.16), (20.19) - (20.22). Notice first that there is a certain redundancy in these equations. The first condition says that (20.15) implies (20.16) for arbitrary constant coefficients v^i and $'v^i$. The second condition says that a relation (20.19) of the same form as (20.15) implies a relation (20.21) of the same form as (20.16) *with the additional condition on the coefficients imposed by (20.20) and (20.22).* Thus, the set of conditions reduces to (20.15) implies (20.16), and if the coefficients, v^i, of (20.15) satisfy (20.20) then the coefficients, $'v^i$, of (20.16) satisfy (20.22).

To simplify the notation, we introduce $x^0 = ct$ as before, and k^0, a fixed positive number. Further, put $\tau = (x^0 - x^0_*)/k^0$ in place of t, and define constants, k^i, by $v^i/c = k^i/k^0/(i = 1, 2, 3)$. Similarly for the y's and the primed symbols. Then (20.15) becomes

$$x^\alpha = x^\alpha_* + k^\alpha \tau \tag{20.23}$$

(20.16) becomes

$$y^\alpha = y^\alpha_* + {'k^\alpha} \tau' \tag{20.24}$$

(20.20) becomes

$$\sum_{\alpha=0}^{3} \underset{\alpha}{e}(k^\alpha)^2 = 0 \tag{20.25}$$

and (20.22) becomes

$$\sum_{\alpha=0}^{3} \underset{\alpha}{e}('k^\alpha)^2 = 0 \tag{20.26}$$

Substituting (20.23) into (20.14) and differentiating with respect to τ, we get $dy^\lambda/d\tau = (\partial f^\lambda/\partial x^\alpha)k^\alpha$. From (20.24) (eliminating τ') $dy^i/dy^0 = {'k^i}/{'k^0}$. So

$$\frac{\partial f^i}{\partial x^\alpha} k^\alpha = \frac{dy^i}{d\tau} = \frac{dy^i}{dy^0}\ \frac{dy^0}{d\tau} = \frac{'k^i}{'k^0}\ \frac{\partial f^0}{\partial x^\alpha} k^\alpha \tag{20.27}$$

Differentiate both sides of (20.27) again with respect to τ. Since we want (20.14) to be invertible we assume $\det(\partial f^\lambda/\partial x^\alpha) \neq 0$. Then $(\partial f^0/\partial x^\alpha)k^\alpha \neq 0$ for nonvanishing "vectors" k^α and we get

$$\frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta} k^\alpha k^\beta = \frac{\partial f^i}{\partial x^\alpha}\ k^\alpha \ \frac{\dfrac{\partial^2 f^0}{\partial x^\alpha \partial x^\beta} k^\alpha k^\beta}{\dfrac{\partial f^0}{\partial x^\alpha} k^\alpha} \tag{20.28}$$

Note that eqs. (20.28) are identities in x^α and k^α. Now fix x^α. Since for "vectors" k^α for which $(\partial f^i/\partial x^\alpha)k^\alpha = 0$ we must also have $(\partial^2 f^i/\partial x^\alpha \partial x^\beta)k^\alpha k^\beta = 0$, it follows that $(\partial f^\lambda/\partial x^\alpha)k^\alpha$ divide $(\partial^2 f^\lambda/\partial x^\alpha \partial x^\beta)k^\alpha k^\beta$. In other words, there exist functions g_β, of x^α such that

$$\frac{\partial^2 f^\lambda}{\partial x^\alpha \partial x^\beta} k^\alpha k^\beta = \left(\frac{\partial f^\lambda}{\partial x^\alpha} k^\alpha\right) (2 g_\beta k^\beta)$$

or, dropping the k's,

$$\frac{\partial^2 f^\lambda}{\partial x^\alpha \partial x^\beta} = \frac{\partial f^\lambda}{\partial x^\alpha} g_\beta + \frac{\partial f^\lambda}{\partial x^\beta} g_\alpha \tag{20.29}$$

Now, using (20.27) in (20.26), the latter becomes

$$\left(\sum_{\lambda=0}^{3} \underset{\lambda}{e} \frac{\partial f^\lambda}{\partial x^\alpha} \frac{\partial f^\lambda}{\partial x^\beta}\right) k^\alpha k^\beta = 0 \tag{20.30}$$

But since (20.26) is to follow from (20.25), (20.30) is also to be a consequence of (20.25) and, hence, the coefficients of (20.30) and (20.25) must be proportional; i.e.,

$$\sum_{\lambda=0}^{3} \underset{\lambda}{e} \frac{\partial f^\lambda}{\partial x^\alpha} \frac{\partial f^\lambda}{\partial x^\beta} = \Gamma \underset{\alpha}{e} \delta_{\alpha\beta} \tag{20.31}$$

where Γ is a function of the x^α's. Finally, if we differentiate (20.31) and apply (20.29) and (20.31) to the result, we get $g_\alpha = 0$ and $\Gamma =$ constant. Thus, from (20.29), the f^λ of Eq. (20.14) are affine functions. That is, (20.14) is

$$y^\lambda = \sum_{\alpha=0}^{3} \underset{\alpha}{e}\, a^\lambda_\alpha\, x^\alpha + b^\lambda \tag{20.32}$$

Using (20.32) for f^λ, (20.31) becomes

$$\sum_{\lambda=0}^{3} \underset{\lambda}{e}\, a^\lambda_\alpha\, a^\lambda_\beta = \Gamma \underset{\alpha}{e} \delta_{\alpha\beta} \tag{20.33}$$

Now, (20.33) is supposed to be valid for any two sets of rectangular cartesian coordinates for any two frames. In particular, for two sets of rectangular cartesian coordinates with the same origin in the same frame, $t' = t$, and $a^0_i = 0$ so Eq. (20.33) reduces to

$$\sum_{i=1}^{3} a^i_j\, a^i_k = \Gamma\, \delta_{jk} \tag{20.34}$$

But two such rectangular cartesian coordinate systems are related by an orthogonal transformation, i.e., the matrix (a^i_j) is orthogonal, so from (20.34) $\Gamma = 1$ and (20.33) becomes

$$\sum_{\lambda=0}^{3} \underset{\lambda}{e}\, a_\alpha^\lambda\, a_\beta^\lambda = \underset{\alpha}{e}\, \delta_{\alpha\beta} \tag{20.35}$$

(Note that the matrix (a_α^λ), has the property, described in Section 5.4(i), of the matrix of an orthogonal transformation of a 4-dimensional vector space with an inner product, b, of index 1.)

We have now accomplished the objective stated at the beginning of this section.

Theorem 20.2 *Equation* (20.32) *with the orthogonality condition* (20.35) *on the matrix* (a_α^λ) *are the relations imposed by Newton's and Maxwell's laws on the rectangular cartesian coordinates and time in two frames of reference.*

Definitions Equation (20.32) with the condition (20.35) on the coefficients a_α^λ are called *the Lorentz transformations.* (Cf., Section 15.1 and Problem 20.12.) The frames related by the Lorentz transformations are called *the Lorentzian frames.*

We note in passing that the Lorentz transformations are usually considered the essence of special relativity. Special relativity is frequently described as the theory of the Lorentz transformations. There are shorter "derivations," the short-cuts being achieved (as Einstein did) by invoking homogeneity and/or isotropy of $\mathcal{E}_0^3$ and/or time. Ours, following Fock, is a simplification of a derivation based on geometrical interpretation of the two required conditions (Weyl, 1923). See also, Zeeman, for a derivation depending only on causality.

PROBLEM 20.4 Justify the proportionality argument used in deriving (20.31).

PROBLEM 20.5 Carry out the details for obtaining the Lorentz transformations from eqs. (20.29) and (20.31).

PROBLEM 20.6 The result of this section can be described as the "only if" part of a theorem corresponding to Theorem 20.1. Prove the converse; namely, if the relations between rectangular cartesian coordinates and time in two generalized frames of reference are given by the Lorentz transformations, and Newton's and Maxwell's laws are valid in one frame, then they are valid in the other frame.

20.4 Some properties and forms of the Lorentz transformations

The orthogonality property (20.35) of the coefficient matrices (a_α^λ) of the Lorentz transformations (20.32) implies that $\det(a_\alpha^\lambda) = \pm 1$. Note that we can also write (20.35) in the form

$$\sum_{\alpha=0}^{3} \underset{\alpha}{e}\, a_{\alpha}^{\lambda} a_{\alpha}^{\mu} = \overset{\lambda}{e}\, \delta^{\lambda\mu} \quad (\overset{\lambda}{e} = \underset{\lambda}{e}) \tag{20.36}$$

Finally, we can invert (20.32) to get

$$x^{\beta} = \sum_{\lambda=0}^{3} \underset{\lambda}{e}\, \tilde{a}_{\lambda}^{\beta} y^{\lambda} - c^{\beta} \tag{20.37}$$

where $\tilde{a}_{\lambda}^{\beta} = a_{\beta}^{\lambda}$, and $c^{\beta} = \sum_{\lambda=0}^{3} \underset{\lambda}{e}\, \tilde{a}_{\lambda}^{\beta} b^{\lambda}$.

We will now show how a Lorentz transformation relating two generalized frames of reference can be described in terms of a "relative velocity" between these two frames.

Consider a particle at rest in the frame $\{\psi_{t'}\}$; i.e., $'v^i = d(y^i \circ \gamma)/dt' = 0$. In the frame $\{\phi_t\}$ the velocity of this particle is $v^i = d(x^i \circ \gamma)/dt$, thinking of γ as a function of t. Differentiating the Lorentz transformations (20.37) with respect to t (thinking of y^{λ} as functions of t) we get

$$c = -a_0^0 \frac{dy^0}{dt} \quad \text{and} \quad v^i = -a_i^0 \frac{dy^0}{dt}$$

So

$$\frac{v^i}{c} = \frac{a_i^0}{a_0^0} \tag{20.38}$$

Thus, all particles at rest in the frame $\{\psi_{t'}\}$ have the same velocity, v^i, in the frame $\{\phi_t\}$, which fact we can rephrase by saying that $\{\psi_{t'}\}$ has (*relative*) *velocity* v^i *with respect to* $\{\phi_t\}$.

Because of the symmetry of the relation between two frames, we can, by differentiating the Lorentz transformation (20.32) with respect to t', derive

$$\frac{'v^i}{c} = \frac{a_0^i}{a_0^0} \tag{20.39}$$

where $'v^i$ are the components with respect to $\{\psi_{t'}\}$ of the velocity of a particle fixed in the $\{\phi_t\}$ frame, and can say that $\{\phi_t\}$ has velocity $'v^i$ with respect to $\{\psi_{t'}\}$.

Having introduced the relative velocity, v^i, we can now write the Lorentz transformations (20.32) in terms of this velocity and an orthogonal matrix, (A_j^i). First, from (20.36) putting $\lambda = \mu = 0$ we have

$$(a_0^0)^2 - \sum_{j=1}^{3} (a_j^0)^2 = 1 \tag{20.40}$$

From (20.40) and (20.38) we get

$$a_0^0 = \pm \frac{1}{\sqrt{1-(v^2/c^2)}} \tag{20.41}$$

where $v^2 = \sum(v^i)^2$; from (20.38) and (20.41) we get

$$a_i^0 = \pm \frac{v^i}{c} \; \frac{1}{\sqrt{1-(v^2/c^2)}} \tag{20.42}$$

and from (20.39) and (20.41) we get

$$a_0^i = \pm \frac{{}'v^i}{c} \; \frac{1}{\sqrt{1-(v^2/c^2)}} \tag{20.43}$$

Next, we put $\lambda = 0$, and $\mu = i$ in (20.36), and we put $\alpha = 0$ and $\beta = i$ in (20.35). We get, respectively (using (20.42) and (20.43)),

$$a_0^0 \, {}'v^i = \sum_j v^j a_j^i \tag{20.44}$$

and

$$a_0^0 v^i = \sum_j a_i^j \, {}'v^j \tag{20.45}$$

Finally, we define a set of numbers, A_j^i by

$$a_j^i = A_j^i - \frac{a_0^i a_j^0}{1 - a_0^0} \tag{20.46}$$

Then from (20.36) with $\lambda = i$, and $\mu = j$ and (20.44) and (20.45) we conclude that

$$\sum_j A_j^i A_j^k = \delta^{ik} \tag{20.47}$$

and

$${}'v^i = -\sum_j A_j^i v^j \tag{20.48}$$

Equations (20.47) and (20.48) are important properties of the A_j^i's.

Now, by substituting the expressions for the a_α^λ's from (20.41), (20.42), (20.43), and (20.46) into the Lorentz transformations, (20.32), we can express them in terms of the relative velocity, v^i, and the orthogonal transformation, A^i_j. We first split (20.32) into

$$ct' = -a^0_0 ct + \sum_j a^0_j x^j + b^0$$

$$y^i = -a^i_0 ct + \sum_j a^i_j x^j + b^i \quad (i = 1, 2, 3)$$

When $b^0 = v^i = 0$ the first equation (noting eq. (20.42)) reduces to $t' = -a^0_0 t$. If we require that $t = t'$ in this case, then we have to use the lower signs in eqs. (20.41) - (20.43) and so $a^0_0 \leq -1$. Now, with the substitutions for a^λ_α, and restricting ourselves to the homogeneous transformations, $b^\lambda = 0$, without much loss of generality, we get

$$ct' = \frac{1}{\sqrt{1-(v^2/c^2)}}\, ct - \sum_i \frac{v^i}{c\sqrt{1-(v^2/c^2)}}\, x^i \tag{20.49}$$

$$y^i = \sum_j A^i_j \left[-\frac{v^j}{\sqrt{1-(v^2/c^2)}}\, t + \sum_{k=1}^{3} \left(\delta^{jk} + \left(\frac{1}{\sqrt{1-(v^2/c^2)}} - 1 \right) \frac{v^j v^k}{v^2} \right) x^k \right] \tag{20.50}$$

Finally, with a suitable rotation of coordinates in the frame $\{\psi_{t'}\}$, Eq. (20.50) reduces to

$$y^i = -\frac{v^i}{\sqrt{1-(v^2/c^2)}}\, t + \sum_{k=1}^{3} \left(\delta^{jk} + \left(\frac{1}{\sqrt{1-(v^2/c^2)}} - 1 \right) \frac{v^i v^k}{v^2} \right) x^k \tag{20.51}$$

We must make several important observations about the equations we have just obtained.

(1) The determinant of the linear system (20.49), (20.51) is 1. This system has three parameters. These linear transformations are sometimes called *boosts.*

(2) The determinant of the linear system (20.49), (20.50) is ± 1 depending on $\det(A^i_j)$. This has six parameters, and is called *the Lorentz group.* The system (20.49), (20.50) with determinant $+1$ is called *the proper Lorentz group.*

(3) If we add arbitrary constants to the system (20.49), (20.50), then these equations will give all possible relations between a set of rectangular cartesian coordinates and time in one frame of reference and a set of rectangular cartesian coordinates and time in another frame of reference. This set of equations will have 10 parameters and is called *the Poincaré group.* (Cf., the Lorentz transformations, Section 15.1, and Problem 20.12.)

(4) If we make the same rotation of coordinates in $\{\psi_{t'}\}$ that reduced (20.50) to (20.51), then (20.48) reduces to $'v^i = -v^i$. Thus we can say that if $\{\psi_{t'}\}$ has velocity v^i with respect to $\{\phi_t\}$, then $\{\phi_t\}$ has velocity $-v^i$ with respect to $\{\psi_{t'}\}$.

(5) If we replace v^i/c by 0, then the Poincaré group reduces to the general Galilean group, (20.4).

(6) If in (20.49) and (20.51) we choose coordinates in $\{\phi_t\}$ such that the velocity of $\{\psi_{t'}\}$ is in the direction of the $x = x^1$ axis, i.e., $v^2 = v^3 = 0$, then these equations reduce to

$$\begin{aligned} ct' &= \frac{1}{\sqrt{1-(v^2/c^2)}}\, ct - \frac{v}{c\sqrt{1-(v^2/c^2)}}\, x \\ y &= -\frac{v}{c\sqrt{1-(v^2/c^2)}}\, ct + \frac{1}{\sqrt{1-(v^2/c^2)}}\, x \end{aligned} \tag{20.52}$$

If we put $\cosh\vartheta = 1/\sqrt{1-(v^2/c^2)}$, then (20.52) take the form

$$\begin{aligned} ct' &= (\cosh\vartheta)ct - (\sinh\vartheta)x \\ y &= -(\sinh\vartheta)ct + \cosh\vartheta)x \end{aligned} \tag{20.53}$$

We will see in Section 21.1 that ϑ as defined here is precisely the hyperbolic angle described in Section 5.4.

PROBLEM 20.7 Carry out the details of the derivations of eqs. (20.47) and (20.48).

PROBLEM 20.8 Same as Problem 20.7 for eqs. (20.49) and (20.50).

PROBLEM 20.9 Confirm the observations made at the end of the section.

PROBLEM 20.10 Derive the general systems (20.49) and (20.51) starting with the spacial case (20.52) by writing $x^i = x^i_{\|} + x^i_{\perp}$ and $y^i = y^i_{\|} + y^i_{\perp}$ where $x^i_{\|}$ and $y^i_{\|}$ are projections on a line parallel to v^i, and then use (20.52) on $x^i_{\|}$ and $y^i_{\|}$ (cf., Pauli, p. 10).

20.5 Minkowski spacetime

According to the result of Section 20.3, Theorem 20.2, the coordinates and the parameter values of generalized frames for which both Newton's principle of inertia and Maxwell's equations hold are related by (20.32), in which the coefficients satisfy (20.35). These equations express Einstein's relativity principle, which accommodates both mechanics and electromagnetism (thus generalizing the Newton-Galileo principle which applied only to mechanical phenomena).

Not the least important feature of this result is that the quantities and relations appearing can be interpreted as, or identified with, simple structures and relations in a mathematical model conceptually simpler than our model involving frames of reference.

Thus, let (M, g) be a 4-dimensional pseudo-Riemannian manifold with a global coordinate system, μ, on which $g = g_\eta$ has components

$$g_{00} = g\left(\frac{\partial}{\partial\mu^0}, \frac{\partial}{\partial\mu^0}\right) = -1,$$

$$g_{ii} = g\left(\frac{\partial}{\partial\mu^i}, \frac{\partial}{\partial\mu^i}\right) = 1, \quad i = 1, 2, 3,$$

and

$$g_{\alpha\beta} = g\left(\frac{\partial}{\partial\mu^\alpha}, \frac{\partial}{\partial\mu^\beta}\right) = 0, \quad \alpha \neq \beta \ (\alpha, \beta = 0, 1, 2, 3)$$

That is, (M, g_η) is a Lorentzian manifold (Section 15.1) isometric with Minkowski space, $\mathcal{R}_1^4$. Then, from the relations

$$\bar{g}_{\lambda\nu} = \frac{\partial\mu^\alpha}{\partial\bar{\mu}^\lambda} \frac{\partial\mu^\beta}{\partial\bar{\mu}^\nu} \, g_{\alpha\beta}$$

relating the components of g_η in two coordinate systems, the coordinate transformation

$$\mu^\lambda = \sum_{\alpha=0}^{3} \underset{\alpha}{e} \, a_\alpha^\lambda \, \bar{\mu}^\alpha + b^\lambda \tag{20.54}$$

with $(\underset{\lambda}{e} \, a_\alpha^\lambda)$ given by (20.35), gives another coordinate system in which g_η has the same values. Thus, the Lorentz transformations can be interpreted as special coordinate transformations on (M, g_η). Coordinate functions on (M, g_η) related by (20.54) are called *Lorentzian coordinates.* They correspond to the coordinates, x^i, and the parameter values, t, of Lorentzian frames.

Thus, in the presence of a Lorentzian frame (or a Lorentzian coordinate system), we have a 1-1 correspondence between points $p \in (M, g_\eta)$ and pairs $(t, \bar{p})$, where t is a parameter value of the frame and $\bar{p} \in \mathcal{E}_0^3$, and

$$x^i(p) = x^i(\bar{p}),$$

$$x^0(p) = ct \tag{20.55}$$

where on the left the x^α, $\alpha = 0, 1, 2, 3$ are coordinate functions on (M, g_η), and on the right the x^i, $i = 1, 2, 3$ are coordinate functions of the frame.

The tangent space at each point, p, of (M, g_η) is a Lorentzian vector space, and by means of the exponential maps, Section 16.4, tangent vectors go into straight line segments. Hence, properties of vectors in the tangent spaces of (M, g_η) are carried over to properties of lines and line segments in (M, g_η).

Definitions Choose a timelike vector field, X, on (M, g_η). Then at each point, p, X_p is in one of the nappes, *the future nappe*, of the null cone at T_pM. With this additional structure, (M, g_η) is *time-oriented* and (M, g_η) is *a Minkowski spacetime*. A point in Minkowski spacetime is called *an event*; a timelike curve, λ, is *a world line*; and a pair, (m, λ), where λ is a future-pointing world line (Section 5.4), is *a particle in* (M, g_η).

If p is a point on a timelike curve, λ, then in a Lorentz frame having (20.55) we get $x^0(\lambda(u)) = ct$, so t is a function of u. Inverting this function we get

$$\bar{\lambda}(t) = \lambda(u(t)) \tag{20.56}$$

Again from (20.55) $x^i(p) = \mu^i \circ \phi_t^{-1}(\bar{p})$, so that if p is replaced by $\bar{\lambda}(t)$, $\bar{p}$ will be a function, γ, of t. Thus, each timelike curve, λ, in (M, g_η) has a "projection," $\phi_t^{-1} \circ \gamma$, a curve in the frame of reference $\{\phi_t\}$ in $\mathcal{E}_0^3$.

A curve in (M, g_η), in particular a straight line, is either spacelike, timelike or null depending on the corresponding property of its tangent vectors, and, for Lorentzian coordinates, the coordinate curves are mutually orthogonal straight lines. By Theorem 16.9 the length of a line segment joining two points (events) p and q with parameter values 0 and 1 in (M, g_η) is the length of the initial vector in T_pM, and by definition (of distance, Section 15.2) this is the distance, $d(p, q)$, or "separation" between p and q.

Theorem 20.3 *In Lorentzian coordinates, x^α,*

$$d(p, q) = \left[\sum_{\alpha=0}^{3} \underset{\alpha}{e}\, (x^\alpha(p) - x^\alpha(q))^2\right]^{\frac{1}{2}} \tag{20.57}$$

Proof Problem 20.11. □

The null lines through p form a (hyper-) cone, given, in Lorentzian coordinates, by

$$\sum_{\alpha=0}^{3} \underset{\alpha}{e}(x^\alpha(p) - x^\alpha(q))^2 = 0 \tag{20.58}$$

Lines through p with

$$\sum_{\alpha=0}^{3} \underset{\alpha}{e}(x^\alpha(p) - x^\alpha(q))^2 < 0 \tag{20.59}$$

correspond, under the exponential map to vectors, v, in T_pM with $g_\eta(v, v) < 0$ and hence are timelike. They can be either future-pointing or past-pointing. Lines through p with

$$\sum_{\alpha=0}^{3} \underset{\alpha}{e}(x^\alpha(p) - x^\alpha(q))^2 > 0 \tag{20.60}$$

correspond to vectors, v, with $g_\eta(v, v) > 0$, and hence are spacelike.

Clearly, the definition (20.57) and the conditions (20.58), (20.59), and (20.60) are independent of the choice of Lorentzian coordinates.

The changing of coordinates in Minkowski spacetime is illustrated by *Minkowski diagrams,* Fig. 20.2. (They will be useful in Chapter 21.) For a given set of coordinates, look at the points of (M, g_η) for which $x^2 = x_*^2$ and $x^3 = x_*^3$. This is a plane. Through each point, p, with coordinates $x_*^\alpha(p)$, we have orthogonal x^0 and x^1 coordinate curves in this plane. We represent these by perpendicular lines on paper in the usual fashion (with the x^0 axis vartical). Points whose coordinates satisfy $-(x^0 - x_*^0)^2 +_(x^1 - x_*^1)^2 = 0$, generators of the null cone, are drawn as lines making 45° angles with the coordinate lines. These points are zero distance from p. Points whose coordinates satisfy $-(x^0 - x_*^0)^2 + (x^1 - x_*^1)^2 =$ constant are drawn as hyperbolas. These points are all equidistant from p.

If we change coordinates by a Lorentz transformation, we get y^0 and y^1

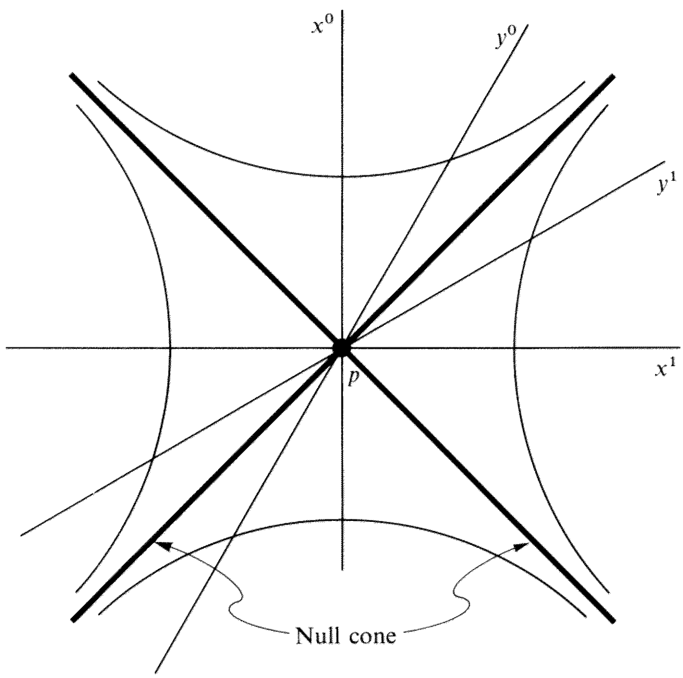

Figure 20.2

coordinate curves which are orthogonal. The null cone and the hyperbolas will look the same. However, from eq. (20.52)

$$x^0 = \frac{v}{c}\, x^1 + \sqrt{1-(v^2/c^2)}\; y^0 \quad \text{and} \quad x^0 = \frac{c}{v}\, x^1 - \frac{c}{v}\, \sqrt{1-(v^2/c^2)}\; y^1$$

so the y^α coordinate lines through p have x^α coordinate equations $x^0 = (c/v)x^1 + a^1$ and $x^0 = (v/c)x^1 + a^2$ where a^1 and a^2 are constants. Thus, the new coordinate lines no longer look perpendicular, but they do have the property that they make equal Euclidean angles with the generators of the null cone. We note from this that, in particular, the original representation of the x^0 and x^1 coordinate curves by perpendicular lines on the paper is merely a matter of convenience and has no special significance for Minkowski spacetime.

Going back to some physical interpretation we put $x^0 = ct$ where t is the time coordinate function. Then from (20.19) and (20.20), the null lines on (20.58) correspond to the light rays or the (trajectories of the) photons through p. Putting $\Delta = (\sum_{\alpha=1}^{3}(x^\alpha(p) - x^\alpha(q))^2)^{\frac{1}{2}}$, $t_1 = t(p)$, and $t_2 = t(q)$, eq. (20.59) gives

$$t_2 - t_1 > \frac{\Delta}{c}$$

or

$$t_1 - t_2 > \frac{\Delta}{c}$$

and eq. (20.60) gives

$$-\frac{\Delta}{c} < t_2 - t_1 < \frac{\Delta}{c}$$

In the first case we say the events p and q are *time-ordered* and either *q occurs after p* or *q occurs before p*. In the second case the sign of $t_2 - t_1$ is not independent of coordinates, so there is no way to say which events occurs first. In this case we say the events are *quasisimultaneous.*

Actually, these alternatives reflect a certain symmetry which does not occur in Newtonian mechanics. There, for two different events which do not occur at the same time, it is always possible to find a frame of reference in which they occur at the same place, but for two different events which do not occur at the same place, it is not always possible to find a frame in which they occur at the same time. Now we have the following.

Theorem 20.4 (i) *If p and q are time-ordered, then there exists a frame* (*i.e., coordinates in* (M, g_η)) *in which p and q occur at "the same place."* (ii) *If p and q are quasisimultaneous, then there is a frame in which p and q are "simultaneous."*

Proof Write a Lorentz transformation for each event and subtract. If we choose $v^i = (x_2^i - x_1^i)/(t_2 - t_1)$ then $y_2^i = y_1^i$, and if we choose $v^i = \frac{1}{2}c^2(t_2 - t_1)(x_2^i - x_1^i)$ then $t_2' = t_1'$. Note that by our hypotheses, for both choices $v^2 < c^2$, so they can both be made. □

If p and q are time-ordered, we can think of p and q as lying on a coordinate curve $y^i = 0$, and $t' =$ constant as consisting of spacelike hyperplanes of simultaneous events, and if p and q are quasi-simultaneous, then we can think of p and q as lying in a coordinate hyperplane $t' = 0$, and $y^i =$ constants as consisting of events occurring at the same place.

To get a better understanding of Minkowski spacetime we will look at its trigonometry. First of all, the hyperbolic hypersurfaces

$$\sum_{\alpha=0}^{3} \underset{\alpha}{e} \, (x^\alpha(p) - x^\alpha(q))^2 = \text{ constant}$$

as the set of points whose distance from p is constant are the analogs of spheres.

Spacelike triangles lie in Euclidean submanifolds and hence have the usual properties. We want to consider two other cases.

Theorem 20.5 (A triangle inequality) *If ℓ_1 and ℓ_2 are future-pointing timelike lines intersecting at q, and p on l_1 occurs before q, and q occurs before r on l_2, then the line, ℓ_3, through p and r is future-pointing timelike and $d(p,r) \geq d(p,q) + d(q,r)$.*

Proof The future-pointing line, ℓ_4, through p parallel to ℓ_2 has the same tangent vector as ℓ_2. The tangent vectors of ℓ_4, ℓ_3, and ℓ_2 are in T_pM and their lengths satisfy the corresponding inequality (Problem 5.24). □

Theorem 20.6 *Suppose a timelike line, ℓ_1, and a spacelike line, ℓ_2 intersect at q. With p on ℓ_1 and r on ℓ_2 form the triangle pqr. Let ℓ_3 be the line through the points p and r* (see Fig. 20.3).

(i) *If ℓ_3 is spacelike and ℓ_1 and ℓ_2 are orthogonal then*

$$d^2(q,r) = d^2(p,q) + d^2(p,r)$$

(ii) *If ℓ_3 is a null line, then*

$$d(p,q) = d(q,r) \Leftrightarrow \ell_1 \text{ and } \ell_2$$

are orthogonal.

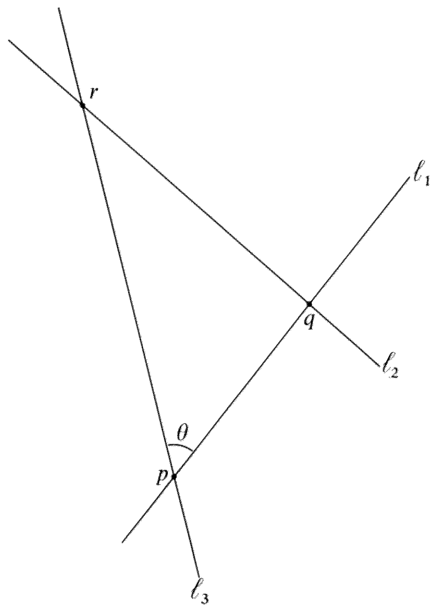

Figure 20.3

(iii) *If ℓ_3 is timelike and ℓ_1 and ℓ_3 are both future-pointing, then*

$$d^2(q,r) = 2d(p,q)d(p,r)\cosh\vartheta - d^2(p,q) - d^2(p,r)$$

where ϑ is the hyperbolic angle (Section 5.4) between ℓ_1 and ℓ_3. If further, ℓ_2 and ℓ_1 are orthogonal, then

$$d^2(q,r) = d^2(p,q) - d^2(p,r)$$

and

$$d(p,q) = d(p,r)\cosh\vartheta$$

Proof The line ℓ_4 through p and parallel to ℓ_2 has the same tangent vector as ℓ_2. Let $\exp_p(v_1) = q$, $\exp_p(v_3) = r$ and $\exp_p(v_4) = r'$ where $d(q,r) = d(p,r')$. Then v_1, v_3, and v_4 satisfy $v_3 = v_1 + v_4$. The results of (i) and (ii) follow by taking the inner product on both sides of this equation. For (iii) we use the relation $\cosh\vartheta = -[g_\eta(v_1,v_3)/\|v_1\|\|v_3\|]$, and when ℓ_1 and ℓ_2 are orthogonal, $g_\eta(v_1,v_3) = -d^2(p,q)$. □

Finally, we have three "nonorthogonality properties" for intersecting timelike and null lines.

Theorem 20.7

(i) *Two lightlike lines are orthogonal iff they are the same.*
(ii) *Two timelike lines are never orthogonal.*
(iii) *A timelike line and a lightlike line are never orthogonal.*

Proof (i) If ℓ_1 and ℓ_2 are lightlike lines with tangennt vectors v_1 and v_2, then $v_1 = a_1 z + w_1$ and $v_2 = a_2 z + w_2$ where z is a unit timelike vector and w_1 and w_2 are spacelike vectors orthogonal to z (cf., Theorem 5.20). Taking scalar products we get $0 = -a_1 a_2 + g_\eta(w_1, w_2)$, $a_1^2 = g_\eta(w_1, w_1)$ and $a_2^2 = g_\eta(w_2, w_2)$, from which we get $[g_\eta(w_1, w_2)]^2 = g_\eta(w_1, w_1) g_\eta(w_2, w_2)$, which implies that $w_1 = a w_2$ and $a_1 = a a_2$ so $v_1 = a v_2$.

(ii) and (iii) Problem 20.15. □

PROBLEM 20.11 Prove Theorem 20.3.

PROBLEM 20.12 Show that

$$\sum_{\alpha=0}^{3} \underset{\alpha}{e}(x^\alpha(p) - x^\alpha(q))^2 = \sum_{\alpha=0}^{3} \underset{\alpha}{e}(y^\alpha(p) - y^\alpha(q))^2$$

for two frames and for $p, q \in (M, g_\eta)$ if and only if the coordinates x^α and y^α are related by a Lorentz transformation. (Thus, if we think of the Lorentz transformations as mappings instead of coordinate transformations, then they are precisely the ones that preserve distance, eq. (20.57), the isometries. One can show that the isometries of a pseudo-Riemannian manifold form a group with at most $[n(n+1)/2]$ parameters, and we see, in particular, that the Lorentz transformations defined in Section 15.1 and the Poincaré group defined in Section 20.4 are just different manifestations of this maximal symmetry.)

PROBLEM 20.13 Show that for a triangle containing two spacelike sides any two of the following implies the third:

(i) The triangle is spacelike.
(ii) The two given sides are perpendicular.
(iii) The Pythagorean theorem is satisfied.

PROBLEM 20.14 The distance between two timelike lines is constant iff they are parallel (cf., Synge, pp. 44 ff).

PROBLEM 20.15 Prove parts (ii) and (iii) of Theorem 20.7.

21

SOME PHYSICS ON MINKOWSKI SPACETIME

We will sketch some "kinematic" consequences of the Lorentz transformations - kinematic from the viewpoint of $\mathcal{E}_0^3$, though from the viewpoint of spacetime they are just transformations of coordinate intervals. Then we will present formulations of particle dynamics, electromagnetism, and fluid mechanics on (M, g_η).

21.1 Time dilation and the Lorentz-Fitzgerald contraction

In this section, we go back again to the consideration of a particle in a frame of reference, $(m, \phi_t^{-1} \circ \gamma)$, moving with constant velocity. In this case we can describe the situation in terms of two (generalized) frames of reference; the frame $\{\phi_t\}$, and a frame $\{\psi_{t'}\}$ in which the particle is at rest; *a proper frame*, or *rest frame* of the particle. A particle at rest in $\{\phi_t\}$ is called an *observer*, and $\{\phi_t\}$ is the frame of the observer.

Suppose the particle is at the point $\mathcal{P}_j$ in $\{\phi_j\}$ when $t = t_j$, $j = 1, 2$. $(\mathcal{P}_j, t_j)$ determine unique points $\bar{p}_j \in \mathcal{E}_0^3$. Then choosing coordinates in $\{\phi_t\}$ according to observation (6) in Section 20.4, the Lorentz transformations give

$$t_j = \frac{1}{\sqrt{1 - v^2/c^2}}\, t_j' + \frac{v}{c^2\sqrt{1 - v^2/c^2}}\, y(\bar{p}_j) \tag{21.1}$$

(inverting (20.52)).

Now, since $(m, \phi_t^{-1} \circ \gamma)$ is at rest in $\{\psi_{t'}\}$, $\bar{p}_1$ and $\bar{p}_2$ are two $\mathcal{E}_0^3$ points of the same point, $\mathcal{Q}$, of $\{\psi_{t'}\}$ so $y(\bar{p}_1) = y(\bar{p}_2)$. Taking the difference in (21.1) for $\bar{p}_1$ and $\bar{p}_2$ we get

$$t_2 - t_1 = \frac{1}{\sqrt{1 - v^2/c^2}}\, (t_2' - t_1') \tag{21.2}$$

Since $\bar{p}_1$ and $\bar{p}_2$ correspond to $\mathcal{Q}$, $\mathcal{Q}$ has the corresponding parameter values t_1' and t_2'. Hence $t_2' - t_1'$ on the right side of (21.2) is the time interval measured on a clock located at $\mathcal{Q}$.

Finally, for the left side of (21.2), the parameter value t_1 is a parameter value for all points $\mathcal{P}$ of $\{\phi_t\}$. Similarly, for t_2. Hence $t_2 - t_1$ is the time difference for *any* point in $\{\phi_t\}$. In particular, it is the time interval measured either on a clock at $\mathcal{P}_1$ or on a clock at $\mathcal{P}_2$. Thus, we have obtained the relation between time measurements in the proper frame of a particle and those in the frame of an observer.

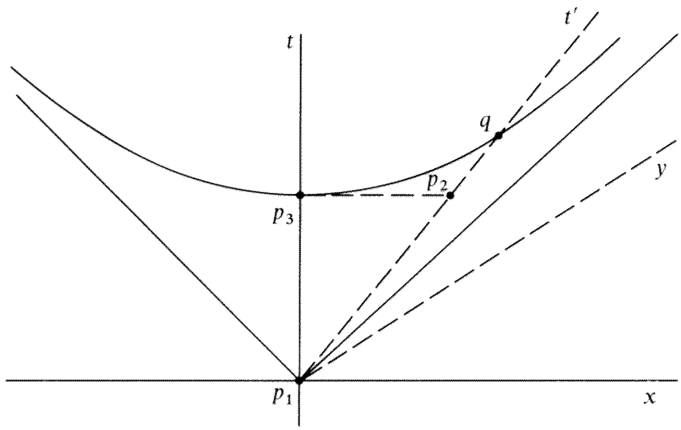

Figure 21.1

The mathematical steps in the above derivation are quite simple; basically, just the algebra of going from the Lorentz transformations to (21.2). It is the physical interpretation which is not so simple.

Another derivation of the phenomenon of time dilation makes use of properties of Minkowski spacetime. Let an observer be represented by the t axis in a Minkowski diagram, Fig. 21.1. (For simplicity take $c = 1$, or $x^0 = t$.) The particle, at rest in $\{\psi_{t'}\}$, can be represented by the t' axis in another set of Lorentzian coordinates. Since the length of the segment p_1p_3 equals the length of the segment p_1q, for any point p_2 on p_1q, $t(p_3) - t(p_1) > t'(p_2) - t'(p_1)$. If p_2 is on the line parallel to the x axis, then $t(p_2) = t(p_3)$, so $t(p_2) - t(p_1) > t'(p_2) - t'(p_1)$, which gives the time dilation.

We can go further. Since the t axis and the line through p_3 and p_2 are orthogonal,

$$t(p_2) - t(p_1) = [t'(p_2) - t'(p_1)] \cosh \vartheta \tag{21.3}$$

by Theorem 20.6(iii). Comparing (21.2) and (21.3) we get the result anticipated at the end of Section 20.4:

Theorem 21.1 *The "angle" introduced in eqs. (20.53) is the hyperbolic angle between the t and t' axes.*

Proof Immediate. □

We can summarize the phenomenon of time dilation by saying that *proper time* intervals, time intervals measured by a particle in its own frame (its "proper" frame), are always smaller than those measured by any observer with respect to whom the particle is moving.

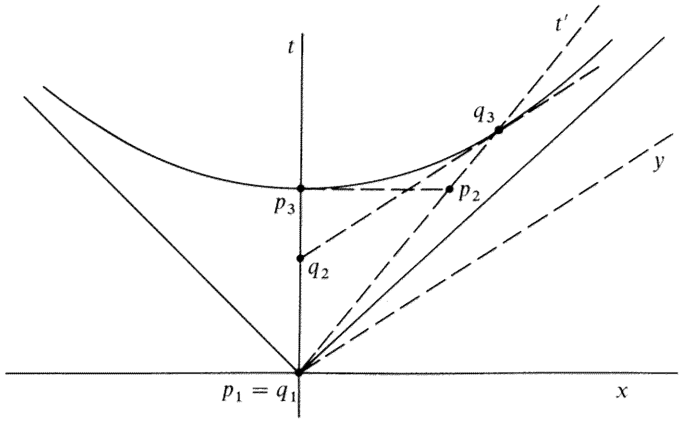

Figure 21.2

In spite of "thought experiments" involving trains and rockets, a completely satisfactory physical feeling for this phenomenon seems somewhat elusive, and it has been marked by apparent "paradoxes." Thus, for example, the principle of relativity expressed by eqs. (20.32) and (20.35) essentially proclaims the equivalence of two frames of a certain class. If the frame of the particle is moving with respect to that of the observer, then the observer frame is moving with respect to that of the particle, so that by the above equivalence, there should be a reciprocity in the measurement of time intervals in the two frames, which, on the surface, our result seems to contradict. A more careful analysis resolves this "paradox" (cf., Fock, pp. 40-42). While, again, the physical description in terms of moving particles and points in reference frames is quite involved, a simple description of the situation is obtained by means of a Minkowski diagram.

In Fig. 21.2, p_3p_2 is parallel to the x axis and q_2q_3 is parallel to the y axis. If the t axis is the observer and the t' axis is the particle, then $t'(p_2) - t'(p_1) < t(p_2) - t(p_1)$ (as before), but if the t' axis is the observer and the t axis is the particle, then $t(q_2) - t(q_1) < t'(q_2) - t'(q_1)$.

A physical description of the companion consequence of the Lorentz transformations, the Lorentz-Fitzgerald contraction, will be left to the interested reader. Here we will simply note that we can write the Lorentz transformation for any two events p_1 and p_2 as

$$y(p_i) = -\frac{v}{\sqrt{1 - v^2/c^2}}\, t(p_i) + \frac{1}{\sqrt{1 - v^2/c^2}}\, x(p_i)$$

$i = 1, 2$. Then putting $t(p_1) = t(p_2)$ and subtracting we get

$$y(p_2) - y(p_1) = \frac{1}{\sqrt{1 - v^2/c^2}}\, [x(p_2) - x(p_1)] \tag{21.4}$$

Now, if p_1 and p_2 correspond to the ends of a rod moving in the direction of its length with respect to an observer, (t, x) are the observer's coordinates, and

(t', y) are the rod's coordinates, then eq. (21.4) says that the length of the rod as measured in its frame, its *proper* or *rest length*, is always greater than that measured by an observer.

Again, this phenomenon can be described by a Minkowski diagram. In Fig. 21.3 the length of p_1p_3 is greater than the length of p_1p_2 which are respectively the proper and observer lengths of the rod.

The two consequences of the Lorentz transformations which we have been discussing, time dilation and the Lorentz-Fitzgerald contraction, are frequently summarized by the statement that "motion slows down clocks and shortens rods," or, as is commonly stated, clocks *appear* to be slowed down and rods *appear* to be shortened. This description seems to suggest that time and space behave the same, whereas they should more properly be described as behaving oppositely: the time interval measured by the clock in its own frame of reference is smaller than that measured by the observer in his frame, and the length measured by the rod in its own frame of reference is larger than that measured by the observer.

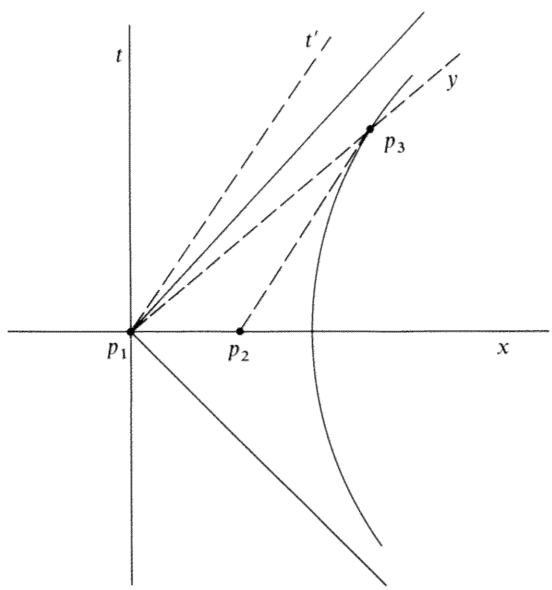

Figure 21.3

Another consequence, concerns relative velocities, or the addition of velocities. Recall that we have the concepts of the velocity of a particle with respect to a frame (Section 20.1), and the relative velocity of one frame with respect to another (Section 20.4).

Definition *The relative velocity of a particle, γ_2, with respect to a particle, γ_1,* is the velocity of γ_2 in a frame in which γ_1 is at rest.

Suppose that in $\{\phi_t\}$ $\phi_t^{-1} \circ \gamma_1$ has velocity v^i and $\phi_t^{-1} \circ \gamma_2$ has velocity w^i. Let $\{\psi_{t'}\}$ be a frame in which γ_1 is at rest. Then the velocity of $\{\psi_{t'}\}$ with respect to $\{\phi_t\}$ is v^i, and if y^i are rectangular cartesian coordinates in $\{\psi_{t'}\}$ then the relative velocity of γ_2 with respect to γ_1 is dy^i/dt'.

Differentiating the Lorentz transformations (20.51) we get

$$\frac{dy^i}{dt'} = \left[\frac{-v^i}{\sqrt{1-v^2/c^2}} + \sum_{k=1}^{3}\left(\delta^{ik} + \left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)\frac{v^i v^k}{v^2}\right) w^k\right]\frac{dt}{dt'}$$

since $dx^k/dt = w^k$ is the velocity of $\phi_t^{-1} \circ \gamma_2$ in $\{\phi_t\}$. Now using the Lorentz transformations (20.49) to get dt/dt' we have the general result

$$\frac{dy^i}{dt'} = \frac{\sqrt{1-v^2/c^2}}{1-\sum\frac{v^j w^j}{c^2}}\left(w^i - v^i + \left(1 - \frac{1}{\sqrt{1-v^2/c^2}}\right)\left(1 - \frac{\sum v^j w^j}{v^2}\right)v^i\right) \tag{21.5}$$

Note that here the v^i are constants, but w^i and dy^i/dt' need not be.

In particular,

(i) If v^i and w^i are parallel, (21.5) becomes

$$\frac{dy^i}{dt'} = \frac{w^i - v^i}{1 - \frac{\sum v^j w^j}{c^2}} \tag{21.6}$$

(ii) If v^i and w^i are perpendicular, (21.5) becomes

$$\frac{dy^i}{dt'} = \sqrt{1-v^2/c^2}\; w^i - v^i \tag{21.7}$$

We get a somewhat different interpretation of these formulas if we suppose that $\phi_t^{-1} \circ \gamma_1$ has velocity $-v^i$ instead of v^i and again $\{\psi_{t'}\}$ is the frame in which γ_1 is at rest. Then we get (21.5) with v^i replaced by $-v^i$. If $\{\chi_{t''}\}$ is a frame in which γ_2 is at rest then (21.5) with v^i replaced by $-v^i$ gives the velocity of $\{\chi_{t''}\}$ with respect to $\{\psi_{t'}\}$ in terms of the velocity, v^i, of $\{\phi_t\}$ with respect to $\{\psi_{t'}\}$ and the velocity, w^i, of $\{\chi_{t''}\}$ with respect to $\{\phi_t\}$. The special case

$$\frac{dy^i}{dt'} = \frac{w^i + v^i}{1 + \frac{\sum v^j w^j}{c^2}} \tag{21.8}$$

when v^i and w^i are parallel is Einstein's formula for the addition of velocities.

A third interpretation of the formulas (21.5)-(21.7) is that they give the relation between the components w^i and dy^i/dt' of the velocity of γ_2 in two different Lorentzian frames. Note that when v^i and w^i are small compared with c these formulas all reduce to the usual simple relation.

In the cases which we have been considering, where particles move with constant velocity with respect to a frame, $\{\phi_t\}$, there is a frame in which the particle is at rest. Times and lengths measured in this frame were called proper. Now, if a particle is subject to force, its velocity is not constant in any Lorentzian frame. By generalizing the concept of proper time to include circumstances in which forces are present, we obtain an important invariant.

If λ is a timelike curve in (M, g_η), then its length (Section 15.2) is

$$\int_{u_1}^{u_2} \|\dot{\lambda}(u)\| du = \int_{u_1}^{u_2} [-g(\dot{\lambda}(u), \dot{\lambda}(u))]^{\frac{1}{2}} du = \int_{u_1}^{u_2} \left[\left(\frac{d\lambda^0}{du}\right)^2 - \sum_{i=1}^{3} \left(\frac{d\lambda^i}{du}\right)^2\right]^{\frac{1}{2}} du$$

in Lorentzian coordinates. If λ happens to be a straight line, it will be a coordinate curve in a Lorentzian coordinate system in which the particle is at rest, and using x^0 as the parameter we get

$$x_2^0 - x_1^0 = c(t_2 - t_1) = \int_{u_1}^{u_2} \|\dot{\lambda}(u)\| \, du$$

where $t_2 - t_1$ is the proper time interval. This special case obviously fits into the following general definition.

Definition $\tau(u) = (1/c) \int_{u_1}^{u} \|\dot{\lambda}(u)\| du$ is called *the proper time* of the particle λ (measured from $\lambda(u_1)$).

Theorem 21.2 *If (m, λ) is a particle in (M, g_η), then in a Lorentzian coordinate system with associated Lorentzian frame $\{\phi_t\}$, λ has a "projection" $\phi_t^{-1} \circ \gamma$ (Section 20.5). Then*

$$\frac{d\tau}{dt} = \sqrt{1 - v^2/c^2} \tag{21.9}$$

where $v = \|(\phi_t^{-1} \circ \gamma)^{\cdot}\|$.

Proof

$$\frac{d\tau}{dt} = \frac{d\tau}{du}\frac{du}{dt} = \frac{1}{c}\left[\left(\frac{d\lambda^0}{du}\right)^2 - \sum \left(\frac{d\lambda^i}{du}\right)^2\right]^{\frac{1}{2}} \frac{du}{dt}$$

$$= \frac{1}{c}\left[\left(\frac{d\bar{\lambda}^0}{dt}\right)^2 - \sum \left(\frac{d\bar{\lambda}^i}{dt}\right)^2\right]^{\frac{1}{2}} \qquad \text{by eq. (20.56)}$$

$$= \frac{1}{c}\left[c^2 - \sum \left(\frac{d\gamma^i}{dt}\right)^2\right]^{\frac{1}{2}} = [1 - v^2/c^2]^{\frac{1}{2}}$$

□

Definition The tangent vector, or velocity, $\dot{\lambda}$, of a particle, λ, parametrized by its proper time is frequently called *the 4-velocity* or, the *relativistic velocity* of the particle.

Theorem 21.3 *The components of the velocity, $\dot{\lambda}$, of a particle in terms of the velocity of the associated $\mathcal{E}_0^3$ curve in a Lorentz frame are*

$$\left(\frac{c}{\sqrt{1-v^2/c^2}}, \frac{v^i}{\sqrt{1-v^2/c^2}}\right) \tag{21.10}$$

Proof Problem 21.6. □

Note that, in contrast to previous velocities we have considered, $(\phi_t^{-1} \circ \gamma)^{\cdot}$ and $\dot{\lambda}$ are not necessarily constant. However, now the magnitude of $\dot{\lambda}$ is constant, i.e., $g(\dot{\lambda}, \dot{\lambda}) = -c^2$, and $\|\dot{\lambda}\| = c$.

Finally, we make one more definition which we will need in the next section.

Definition $\mathbf{P} = m\dot{\lambda}$ is *the 4-momentum* or *energy-momentum* of a particle $(m, \dot{\lambda})$.

PROBLEM 21.1 The explanation, using Fig. 21.2, of the "reciprocity" between two frames of time dilation depended on the fact that p_2 and q_2 both lie beneath the hyperbola. Prove that that is indeed the case.

PROBLEM 21.2 The formula (21.4), for the Lorentz-Fitzgerald contraction is for the case in which the rod moves in the direction of its length. If, more generally, φ is the angle between the x axis and the rod, then

$$l_r = \frac{\sqrt{1-(v^2/c^2)\sin^2\varphi}}{\sqrt{1-v^2/c^2}}\, l_0$$

where l_r is the rest length and l_0 is the length measured by an observer.

PROBLEM 21.3 Construct a figure to explain the "reciprocity" between two frames of the Lorentz-Fitzgerald contraction.

PROBLEM 21.4 Derive eqs. (21.6) and (21.7).

PROBLEM 21.5 Choose coordinates in $\{\phi_t\}$ and $\{\psi_{t'}\}$ so that $v^2 = v^3 = w^2 = w^3 = 0$ where v^i is the velocity of $\{\phi_t\}$ with respect to $\{\psi_{t'}\}$, and w^i is the velocity of $\{\chi_{t''}\}$ with respect to $\{\phi_t\}$. Then (i) $\tanh\vartheta_1 = v_1/c$ and $\tanh\vartheta_2 = w_1/c$ where ϑ_1 is the hyperbolic angle between the t and t' axes and ϑ_2 is the hyperbolic angle between the t and t'' axes, and (ii) the velocity of $\{\chi_{t''}\}$ with respect to $\{\psi_{t'}\}$ is $c\tanh(\vartheta_1 + \vartheta_2)$.

PROBLEM 21.6 Show that Einstein's addition formula respects the limitation of the velocity of one frame with respect to another.

PROBLEM 21.7 Prove Theorem 21.3.

21.2 Particle dynamics on Minkowski spacetime

In classical Newtonian dynamics in $\mathcal{E}_0^3$ we introduce the concepts of momentum, energy and force. These concepts should be extendible to relativity. Also in classical dynamics we have conservation laws which we want to still be valid.

For particles, the only kinds of interactions which can be meaningfully formulated in the context of the theory of relativity are collisions and disintegrations since "action at a distance" is excluded. (We will consider interaction between particles and fields in the next section.) In the case of two particles $(m_1, \phi_t^{-1} \circ \gamma_1)$ and $(m_2, \phi_t^{-1} \circ \gamma_2)$ colliding elastically, the classical law of conservation of momentum is

$$m_1(\phi_t^{-1} \circ \gamma_1)^{\cdot} + m_2(\phi_t^{-1} \circ \gamma_2)^{\cdot} = \bar{m}_1(\phi_t^{-1} \circ \bar{\gamma}_1)^{\cdot} + \bar{m}_2(\phi_t^{-1} \circ \bar{\gamma}_2)^{\cdot} \qquad (21.11)$$

(where now, γ_1 and γ_2, and $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are the trajectories before and after collision, respectively, and $\bar{m}_i = m_i,\ i = 1, 2$).

A simple example (Lewis and Tolman) shows that if formula (21.11) is valid for the frame of reference $\{\phi_t\}$ it will no longer hold if we simply replace ϕ_t by $\psi_{t'}$ in a second frame of reference, $\{\psi_{t'}\}$. In that example, the velocity components satisfy

$$\frac{m_1 v_1^i}{\sqrt{1 - v_1^2/c^2}} + \frac{m_2 v_2^i}{\sqrt{1 - v_2^2/c^2}} = \frac{\bar{m}_1 \bar{v}_1^i}{\sqrt{1 - \bar{v}_1^2/c^2}} + \frac{\bar{m}_2 \bar{v}_2^i}{\sqrt{1 - \bar{v}_2^2/c^2}} \qquad (21.12)$$

in both frames. (Here the v's in the denominators are the magnitudes of the corresponding velocities, and, again in this example, $\bar{m}_i = m_i$.)

On the basis of (1) this particular example, and perhaps others (cf., Taylor and Wheeler), (2) the fact that (21.12) reduces to the classical law for small velocities, and, most importantly, (3) the fact that it has been confirmed experimentally in particle accelerators, eq. (21.12) is *accepted as a physical law.* Along with eq. (21.12) it is sometimes conventient to introduce the following terminology for a particle: $mv^i/\sqrt{1 - v^2/c^2}$ is its *relativistic 3-momentum,* $m/\sqrt{1 - v^2/c^2}$ is its *relativistic mass,* and m is its *rest* or *proper mass.*

The elastic collision of two particles which we have been discussing can be viewed from the standpoint of Minkowski spacetime as the intersection at some point (event), p, in Minkowski spacetime of four particles; $(m_1, \lambda_1), (m_2, \lambda_2)$, $(\bar{m}_1, \bar{\lambda}_1)$, and $(\bar{m}_2, \bar{\lambda}_2)$. They have 4-momenta $m_1\dot{\lambda}_1$, $m_2\dot{\lambda}_2$, $\bar{m}_1\dot{\bar{\lambda}}_1$, and $\bar{m}_2\dot{\bar{\lambda}}_2$ at p. Form the vector $m_1\dot{\lambda}_1 + m_2\dot{\lambda}_2 - \bar{m}_1\dot{\bar{\lambda}}_1 - \bar{m}_2\dot{\bar{\lambda}}_2$. Now, if any three of the components of a vector at p are zero in all (Lorentzian) coordinate systems, then so is the fourth component. Hence, in view of (21.10), we get, as a consequence of (21.12),

$$\frac{m_1 c}{\sqrt{1 - v_1^2/c^2}} + \frac{m_2 c}{\sqrt{1 - v_2^2/c^2}} = \frac{\bar{m}_1 c}{\sqrt{1 - \bar{v}_1^2/c^2}} + \frac{\bar{m}_2 c}{\sqrt{1 - \bar{v}_2^2/c^2}} \tag{21.13}$$

an additional conservation law for the collision process we have been discussing. This is the source of one of the dramatic results of the theory of relativity.

In order to summarize and generalize in a theorem what we have found above, we first make the following definitions, which we will subsequently have to justify.

Definitions For a particle, $(m, \phi_t^{-1} \circ \gamma)$, $\mathbf{E} = mc^2/\sqrt{1 - v^2/c^2}$ is called its *total* or *relative energy*, and $\mathbf{E}_0 = mc^2$ is called its *proper* or *rest energy*.

Theorem 21.4 *In any interaction (elastic or inelastic collision, or disintegration) of any number of particles, conservation of relativistic 3-momentum implies both conservation of total energy and 4-momentum.*

Proof The proof is based as before on the simple observation concerning the vanishing of the components of a vector, and introducing a lot more notation to provide for arbitrary numbers of particles before and after the interaction. □

To justify our definition of total energy and rest energy we first make another definition.

Definition *The (relativistic) kinetic energy* of a particle is

$$\mathbf{T} = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 \tag{21.14}$$

This definition is acceptable because:

(1) $$\frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = mc^2\, \frac{1 - \sqrt{1 - v^2/c^2}}{\sqrt{1 - v^2/c^2}}$$

$$= mc^2\, \frac{(1 - \sqrt{1 - v^2/c^2})(1 + \sqrt{1 - v^2/c^2})}{\sqrt{1 - v^2/c^2}(1 + \sqrt{1 - v^2/c^2})}$$

$$= \frac{mv^2}{\sqrt{1 - v^2/c^2}(1 + \sqrt{1 - v^2/c^2})}$$

which becomes the classical Newtonian kinetic energy in the limit as $v/c \to 0$.

(2) In his 1905 paper Einstein found this expression for the kinetic energy of charged particle accelerating in an electromagnetic field. See Problem 21.14 (next section).

Now, in Newtonian mechanics, by conservation of energy, the total kinetic energy of our systems of colliding particles would be constant, since no forces are acting. But because of eq. (21.13)

$$\mathbf{T}_1 + \mathbf{T}_2 + m_1c^2 + m_2c^2 = \frac{m_1c^2}{\sqrt{1 - v_1^2/c^2}} + \frac{m_2c^2}{\sqrt{1 - v_2^2/c^2}}$$

is conserved. To avoid giving up the idea of conservation of energy we think of the quantity conserved as the energy of the system. Then $m_1c^2 + m_2c^2$ will be some kind of energy also. Thus, we call $\mathbf{E} = \sum(m_ic^2/\sqrt{1 - v_i^2/c^2})$ *the total energy of the system* and $\mathbf{E}_0 = \sum m_ic^2$ *the rest energy of the system.*

The fact that, in a given process $\mathbf{E} = \mathbf{E}_0 + \mathbf{T}$ is conserved (constant) suggests that changes of $\mathbf{T}$ must be compensated for by changes of the rest mass of the system, and vice versa. Thus, for example if two particles of equal mass, m, collide head-on inelastically, producing a single stationary particle of mass, $\bar{m}$, then the total energy before collision will be $2mc^2/\sqrt{1 - v^2/c^2}$ and after collision it will be $\bar{m}c^2$. So $\bar{m} = 2m/\sqrt{1 - v^2/c^2}$, which is more than just the sum of the two masses.

The idea of the equivalence of mass and energy was suggested by experiments in electromagnetism as far back as 1881 (Whittaker, p. 51) and it was attractive, esthetically and philosophically, to postulate this equivalence for all kinds of energy in other processes. This has of course been established experimentally, and dramatically utilized. For a derivation for energy other than kinetic energy, see Pauli (pp. 121 ff).

Having disposed of the concepts of momentum and energy, there remains only the concept of force to be considered.

Definition For a particle, (m, λ), in Minkowski spacetimee with tangent vector $\dot{\lambda}$, the *relativistic* (*four*, or *Minkowski*) *force*, $\mathbf{F}$, acting on the particle is $\nabla_{\dot{\lambda}}\mathbf{P}$ (cf., Section 17.2).

In a Lorentzian coordinate system with $u^\alpha = d\lambda^\alpha/d\tau$,

$$\mathbf{F} = \frac{d}{d\tau}(mu^\alpha)\frac{\partial}{\partial x^\alpha} = \frac{d}{d\tau}\left(\frac{mc}{\sqrt{1 - v^2/c^2}}\right)\frac{\partial}{\partial x^0} + \frac{d}{d\tau}\left(\frac{mv^i}{\sqrt{1 - v^2/c^2}}\right)\frac{\partial}{\partial x^i} \tag{21.15}$$

Since $u_\alpha u^\alpha = -c^2$ we have $u_\alpha du^\alpha/d\tau = 0$ and $u_\alpha dmu^\alpha/d\tau = -c^2 dm/d\tau$. So

$$u_\alpha \frac{d}{d\tau}(mu^\alpha) = -c^2\frac{d}{d\tau}m \tag{21.16}$$

In particular, we have, from (21.16), that the relativistic force and the relativistic velocity are orthogonal iff the (rest) mass, m, of the particle is constant. Expanding (21.16), and using (21.9) we get

$$\frac{-c}{\sqrt{1-v^2/c^2}}\frac{d}{d\tau}\frac{mc}{\sqrt{1-v^2/c^2}}+\frac{v_i}{1-v^2/c^2}\frac{d}{dt}\left(\frac{mv^i}{\sqrt{1-v^2/c^2}}\right)=-c^2\frac{d}{d\tau}m \quad (21.17)$$

Using (21.17) in (21.15) in the case $m=$ constant we get

$$\mathbf{F}=\frac{1}{\sqrt{1-v^2/c^2}}\left(\frac{v_i}{c}\mathbf{F}^i_N\frac{\partial}{\partial x^0}+\mathbf{F}^i_N\frac{\partial}{\partial x^i}\right) \quad (21.18)$$

where

$$\mathbf{F}^i_N=\frac{d}{dt}\left(\frac{mv^i}{\sqrt{1-v^2/c^2}}\right) \quad (21.19)$$

the components of *the relativistic 3-force* (which approximates the usual Newtonian force when v/c is small.

Finally, we get a relation between the relativistic kinetic energy of a particle and the relativistic 3-force acting on it which generalizes the classical relation. Thus, from $d\mathbf{T}/d\tau = d\mathbf{E}/d\tau - d\mathbf{E}_0/d\tau$ we have

$$\frac{1}{\sqrt{1-v^2/c^2}}\frac{d\mathbf{T}}{d\tau}=\frac{1}{\sqrt{1-v^2/c^2}}\frac{d}{d\tau}\left(\frac{mc^2}{\sqrt{1-v^2/c^2}}\right)-\frac{c^2}{\sqrt{1-v^2/c^2}}\frac{dm}{d\tau} \quad (21.20)$$

From (21.17) and (21.20),

$$\frac{1}{\sqrt{1-v^2/c^2}}\frac{d\mathbf{T}}{d\tau}=\frac{v_i}{1-v^2/c^2}\mathbf{F}^i_N+c^2\left(1-\frac{1}{\sqrt{1-v^2/c^2}}\right)\frac{dm}{d\tau}$$

or

$$\frac{d\mathbf{T}}{d\tau}=v_i\mathbf{F}^i_N-\frac{v^2\sqrt{1-v^2/c^2}}{\sqrt{1-v^2/c^2}+1}\frac{dm}{d\tau} \quad (21.21)$$

Equation (21.21) reduces to the Newtonian relation when $v/c\to 0$ and m is constant. It is sometimes taken to define the relativistic kinetic energy.

PROBLEM 21.8 Describe the motion of a particle if the force acting on it is constant. Show, in particular, that $v<c$.

PROBLEM 21.9 Show that

$$c^2 \frac{d}{dt} \frac{m}{\sqrt{1 - v^2/c^2}} = \mathbf{F}^i_N v_i$$

and if $a^i = dv^i/dt$ is *the 3-acceleration,*

$$\mathbf{F}^i_N = \frac{m}{\sqrt{1 - v^2/c^2}}\, a^i + \frac{\mathbf{F}^j_N v_j}{c^2}\, v^i$$

21.3 Electromagnetism on Minkowski spacetime

Recall that in Chapter 20, the requirement to build a relativity theory encompassing both Newton's mechanical laws and electromagnetic theory led to a Minkowski spacetime formulation. In this formulation we saw, in the last section, that Newton's laws had to be slightly modified. It turns out that Maxwell's equations can be formulated in Minkowski spacetime without modification of their classical form. This formulation is in terms of an electromagnetic tensor. There are various motivations for introducing the electromagnetic tensor. We will introduce it by its relation to an electromagnetic force. (For an approach starting with an eletromagnetic potential 4-vector see, for example, Rindler, pp. 120 ff.)

Theorem 21.5 *If a particle, λ, with "charge", e, is subject to a relativistic force, $\mathbf{F}$, whose components in a Lorentzian frame are given by*

$$\mathbf{F}^i_N = e(E^i + (v \times B)^i) \tag{21.22}$$

in eq. (21.18), where v is the velocity in the frame, of the curve γ associated with λ, E and B are vector fields in the frame, and $v \times B$ is the cross-product of vector analysis (Section 6.3), then a 2-form, $\mathcal{F}$, defined by

$$(g_\eta)^\flat \cdot \mathbf{F} = -e i_{\dot\lambda} \mathcal{F} \tag{21.23}$$

on λ has the form

$$\mathcal{F} = -\frac{E_i}{c} dx^0 \wedge dx^i + B_3 dx^1 \wedge dx^2 - B_2 dx^1 \wedge dx^3 + B_1 dx^2 \wedge dx^3 \tag{21.24}$$

Proof If we substitute (21.22) into (21.18) and map the result to a 1-form by $(g_\eta)^\flat$, we get

$$(g_\eta)^\flat \cdot \mathbf{F} = \frac{e}{\sqrt{1 - v^2/c^2}} \Big(- E_i \frac{v^i}{c} dx^0 + (E_i + (v \times B)_i) dx^i \Big)$$

$$= \frac{e}{\sqrt{1 - v^2/c^2}} \Big(- E_i \frac{v^i}{c} dx^0 + E_i dx^i + (v_2 B_3 - v_3 B_2) dx^1$$

$$+ (v_3 B_1 - v_1 B_3) dx^2 + (v_1 B_2 - v_2 B_1) dx^3 \Big)$$

and computing $i_{\dot\lambda}\mathcal{F}$ from (21.24) we get (21.23). □

We assume (21.22) is valid in every Lorentzian frame. Consequently, $\mathcal{F}$ has the form (21.24) in every frame. The validity of (21.22) is just a matter of the definition of the E^i and The B^i. That is, whatever the force may be, we write it in the form (21.22). (In a different approach, in which E^i and B^i have some prior physical meaning, the 3-force $\mathbf{F}^i_N$ of eq. (21.22) is called *the Lorentz force.*)

Definition $\mathcal{F}$ is called *the electromagnetic field tensor.*

It is common practice to arrange the components $\mathcal{F}_{\alpha\beta}$ of $\mathcal{F}$ into a matrix:

$$\mathcal{F}_{\alpha\beta} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{pmatrix}$$

The components of $\mathcal{F}$ in two Lorentzian coordinate systems are, of course, related by the Lorentz transformations. In particular, for the Lorentz transformations (20.52) we get

$$\begin{aligned} E_1' &= E_1 & B_1' &= B_1 \\ E_2' &= \frac{1}{\sqrt{1-v^2/c^2}}(E_2 - vB_3) & B_2' &= \frac{1}{\sqrt{1-v^2/c^2}}(B_2 + vE_3) \\ E_3' &= \frac{1}{\sqrt{1-v^2/c^2}}(E_3 + vB_2) & B_3' &= \frac{1}{\sqrt{1-v^2/c^2}}(B_3 - vE_2) \end{aligned} \tag{21.25}$$

in which the "primed" frame is moving with velocity v with respect to the "unprimed" frame. It is interesting to note that the first set of these relations also come directly from (21.22). For if the particle is at rest in the "primed" frame then $\mathbf{F}^{i'}_N = eE^{i'}$. But it can be shown (Rindler, pp. 98, 130) that the relativistic 3-force transforms, in this case, by

$$\mathbf{F}^{1'}_N = \mathbf{F}^1_N \text{ and } \mathbf{F}^{i'}_N = \frac{1}{\sqrt{1-v^2/c^2}}\,\mathbf{F}^i_N, \quad i = 2, 3$$

Note that in the above discussion we did not use any properties of E or B. Classical electromagnetism requires that E and B satisfy the Maxwell equations (20.7). We can write conditions on $\mathcal{F}$ in Minkowski spacetime which contain the Maxwell equations in Lorentzian coordinates. These conditions are

$$d\mathcal{F} = 0 \tag{21.26}$$

$$d * \mathcal{F} = \mathcal{S} \tag{21.27}$$

where $\mathcal{S} = i_Z \Omega$ in which the charge-current density

$$Z = c\mu\eta \, \frac{\partial}{\partial x^0} + \mu J^i \, \frac{\partial}{\partial x^i}$$

where μ, η, and $J = J^i \partial/\partial x^i$ are defined in Section 20.1, and

$$\Omega = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

the volume form of Minkowski spacetime.

If we write the component expressions for the Maxwell equations, (20.7), in a Lorentzian coordinate system, and put $B^1 = B_1 = \mathcal{F}_{23}$, $B^2 = B_2 = \mathcal{F}_{31}$, etc., then div B is the coefficient of $dx^1 \wedge dx^2 \wedge dx^3$ in $d\mathcal{F}$. With $E^1/c = E_1/c = \mathcal{F}_{01}$, etc., the components of curl $E + \partial B/\partial t$ are the coefficients of the remaining terms of $d\mathcal{F}$. The other two Maxwell equations give $d * \mathcal{F} = \mathcal{S}$. This is again obtained quite automatically if we write the E's and B's as components of $\mathcal{F}$, once we write explicit forms of $*\mathcal{F}$ and $\mathcal{S}$. These are

$$*\mathcal{F} = B_i dx^0 \wedge dx^i + \frac{E_1}{c} dx^2 \wedge dx^3 - \frac{E^2}{c} dx^1 \wedge dx^3 + \frac{E_3}{c} dx^1 \wedge dx^2 \tag{21.28}$$

$$\begin{aligned} \mathcal{S} = &\frac{\eta}{c\varepsilon} dx^1 \wedge dx^2 \wedge dx^3 \\ &-\mu \, dx^0 \wedge (J_1 \, dx^2 \wedge dx^3 - J_2 \, dx^1 \wedge dx^3 + J_3 \, dx^1 \wedge dx^2) \end{aligned} \tag{21.29}$$

(Note that the $*$ operator is not exactly the same as in Section 6.3 where it was defined for a positive definite inner product-the E_i/c terms have opposite sign. Cf., Flanders, pp. 15-16.) Thus, we obtain eqs. (21.26) and (21.27) as a spacetime formulation of Maxwell's equations.

In terms of components of $\mathcal{F}$, eqs. (21.26) and (21.27) can be written, respectively, as

$$\frac{\partial}{\partial x^\alpha} \, \varepsilon^{\alpha\beta\gamma\delta} \, \mathcal{F}_{\beta\delta} = 0 \tag{21.30}$$

where $\varepsilon^{\alpha\beta\gamma\delta}$ is the permutation symbol, Problem 4.15, and

$$\frac{\partial \mathcal{F}^{\alpha\beta}}{\partial x^\beta} = Z^\alpha \tag{21.31}$$

Just as we saw in mechanics, where the relativistic formulation produced an extra invariant in spacetime, eq. (21.13), the relativistic formulation, eqs. (21.26) and (21.27), gives us information in classical electromagnetism which cannot be derived in the classical formulation. Specifically, two of the Maxwell equations can be derived from the other two.

Theorem 21.6 *If* div $B = 0$ *in all Lorentzian systems, then* $d\mathcal{F} = 0$*, and consequently* curl $E + \partial B/\partial t = 0$.

Proof From (21.24),

$$\begin{aligned} d\mathcal{F} &= \frac{\partial E_i/c}{\partial x^\alpha} dx^\alpha \wedge dx^i \wedge dx^0 + \frac{\partial B_1}{\partial x^\alpha} dx^\alpha \wedge dx^2 \wedge dx^3 \\ &\quad + \frac{\partial B_2}{\partial x^\alpha} dx^\alpha \wedge dx^3 \wedge dx^1 + \frac{\partial B_3}{\partial x^\alpha} dx^\alpha \wedge dx^1 \wedge dx^2 \\ &= \frac{\partial E_i/c}{\partial x^\alpha} dx^\alpha \wedge dx^i \wedge dx^0 + \frac{\partial B_1}{\partial x^0} dx^0 \wedge dx^2 \wedge dx^3 \\ &\quad + \frac{\partial B_2}{\partial x^0} dx^0 \wedge dx^3 \wedge dx^1 + \frac{\partial B_3}{\partial x^0} dx^0 \wedge dx^1 \wedge dx^2 \\ &\quad + \frac{\partial B_1}{\partial x^i} dx^i \wedge dx^2 \wedge dx^3 + \frac{\partial B_2}{\partial x^i} dx^i \wedge dx^3 \wedge dx^1 \\ &\quad + \frac{\partial B_3}{\partial x^i} dx^i \wedge dx^1 \wedge dx^2 \end{aligned}$$

By the hypothesis, the sum of the last three terms is zero, so

$$\begin{aligned} d\mathcal{F} = \bigg(&\frac{\partial E_i/c}{\partial x^\alpha} dx^\alpha \wedge dx^i + \frac{\partial B_1}{\partial x^0} dx^2 \wedge dx^3 \\ &+ \frac{\partial B_2}{\partial x^0} dx^3 \wedge dx^1 + \frac{\partial B_3}{\partial x^0} dx^1 \wedge dx^2 \bigg) \wedge dx^0 \end{aligned} \tag{21.32}$$

and, consequently, $d\mathcal{F}(\partial/\partial x^i, \partial/\partial x^j, \partial/\partial x^k) = 0$. By our hypothesis this will be valid for all Lorentzian systems, so, since any timelike vector can be written as a sum of spacelike vectors, $d\mathcal{F}$ vanishes on all triples in each tangent space. □

Theorem 21.7 *If the relation* div $E = \eta/\varepsilon$ *holds in all Lorentzian systems then* $d * \mathcal{F} = \mathcal{S}$*, and, consequently,* curl $B - \partial E/\partial t = \mu J$.

Proof From (21.28),

$$\begin{aligned} d * \mathcal{F} = &-\frac{\partial B_i}{\partial x^\alpha} dx^\alpha \wedge dx^i \wedge dx^0 + \frac{\partial E_1/c}{\partial x^0} dx^0 \wedge dx^2 \wedge dx^3 \\ &+ \frac{\partial E_2/c}{\partial x^0} dx^0 \wedge dx^3 \wedge dx^1 + \frac{\partial E_3/c}{\partial x^0} dx^0 \wedge dx^1 \wedge dx^2 \\ &+ \sum \frac{\partial E_i/c}{\partial x^i} dx^1 \wedge dx^2 \wedge dx^3 \end{aligned}$$

and using (21.29),

$$
\begin{aligned}
d * \mathcal{F} - \mathcal{S} = &-\frac{\partial B_i}{\partial x^\alpha} dx^\alpha \wedge dx^i \wedge dx^0 \\
&+\left(\frac{\partial E_1/c}{\partial x^0} dx^2 \wedge dx^3 + \frac{\partial E_2/c}{\partial x^0} dx^3 \wedge dx^1 \right. \\
&\qquad \left. + \frac{\partial E_3/c}{\partial x^0} dx^1 \wedge dx^2 \right) \wedge dx^0 \\
&+ \sum \frac{\partial E_i/c}{\partial x^i} dx^1 \wedge dx^2 \wedge dx^3 - \frac{\sigma}{c\varepsilon} dx^1 \wedge dx^2 \wedge dx^3 \\
&+ \mu(J_1\ dx^2 \wedge dx^3 + J_2\ dx^3 \wedge dx^1 + J_3\ dx^1 \wedge dx^2) \wedge dx^0
\end{aligned}
$$

By the hypothesis, all terms which do not have a factor of dx^0 drop out, and we can finish the argument just as in Theorem 21.6. □

Finally, there are several immediate, but important, consequences of eqs. (21.26) and (21.27).

(1) $d\mathcal{F} = 0$ implies that, locally, $\mathcal{F}$ is the exterior differential of a 1-form, A, an electromagnetic potential. In coordinates, $A = f\ dt + \omega_i\ dx^i$. Then from (21.24)

$$g^\flat B = \text{Curl}\ \omega, \quad g^\flat E = \text{Grad}\ f - \frac{\partial \omega}{\partial t}$$

(2) $\mathcal{F} = dA$ implies that $\mathcal{F} = dA'$ also, where $A' = A + d\chi$. That is, the electromagnetic potential is not unique; we can go from one to another by a "gauge transformation" (see Sections (12.1) and (16.5)).

(3) $d * \mathcal{F} = \mathcal{S}$ implies conservation of charge. That is, we have $0 = d\mathcal{S} = di_Z\Omega = L_Z\Omega = (\text{div}\ Z)\Omega$ so $\text{div}\ Z = 0$.

PROBLEM 21.10 Derive eq. (21.25).

PROBLEM 21.11 Derive eqs. (21.30) and (21.31).

PROBLEM 21.12 If $\mathcal{F}$ satisfies Maxwell's equations, $\sigma = 0$, and curl $J = 0$, then $\mathcal{F}$ is a harmonic form (Section 12.1).

PROBLEM 21.13 Use eqs. (21.19) and (21.22) to describe the motion of a particle of mass m and charge e, moving in a plane and subject to a uniform magnetic field, B, acting perpendicular to the plane.

PROBLEM 21.14 Following Einstein (1923a, §10), use eqs. (21.21) and (21.25), and the Lorentz transformations to derive the formula (21.14) for the relativistic kinetic energy of a particle of charge e accelerated from rest.

21.4 Perfect fluids on Minkowski spacetime

We build on the Newton-Euler-Cauchy laws. In classical continuum mechanics in $\mathcal{E}_0^3$ we have a mass density, ρ, and a flow Θ (i.e., a local action, see Section 13.2), which satisfy

$$\text{(i)} \qquad \frac{d}{dt}\int_{\Theta(D,t)} \rho\Omega = 0 \quad \text{(conservation of mass)} \tag{21.33}$$

and

$$\text{(ii)} \qquad \frac{d}{dt}\int_{\Theta(D,t)} \rho v\Omega = \int_{\Theta(D,t)} \rho f\Omega + \int_{\partial\Theta(D,t)} \sigma i_\nu \Omega \quad \text{(equation of motion)} \tag{21.34}$$

for D a bounded regular domain of $\mathbb{R}^3$ (Section 12.2). ν is the positive unit normal vector on $\partial\Theta(D,t)$, and $i_\nu\Omega$ is the volume form on $\partial\Theta(D,t)$ (see Problem 17.23). v is the velocity of Θ, ρf is the force acting on a unit volume of the fluid by gravity, or an electromagnetic field (if the fluid is charged) or any other external agent, and σ is the internal stress due to pressure, viscosity, etc. The stress vector σ depends on a direction, n, at each point as well as on the point itself. Using (21.34) one can show, following Cauchy, that $\sigma^i(x^j,t,n_j) = \sigma^{ij}(x^k,t)n_j$, i.e., σ is linear in n. Finally, *the Cauchy stress tensor*, σ^{ij}, is symmetric. This property is equivalent to a second equation of motion which involves moments of the terms in (21.34).

Now we write conditions (21.33) and (21.34) as differential equations. We use a "transport theorem" (see Problem (21.15)),

$$\frac{d}{dt}\int_{\Theta(D,t)} f\Omega = \int_{\Theta(D,t)} \left(\frac{\partial f}{\partial t} + \operatorname{div} fv\right)\Omega \tag{21.35}$$

Applying (21.35) to (21.33) we get, in rectangular cartesian coordinates,

$$\frac{\partial\rho}{\partial t} + \frac{\partial \rho v^i}{\partial x_i} = 0 \tag{21.36}$$

Applying (21.35) to (21.34), using $\sigma^i = \sigma^{ij}n_j$, the divergence theorem (Theorem 15.5) and rectangular cartesian coordinates again, we get

$$\int_{\Theta(D,t)} \left(\frac{\partial \rho v^i}{\partial t} + \frac{\partial \rho v^i v^j}{\partial x^j}\right)\Omega = \int_{\Theta(D,t)} \rho f^i\Omega + \int_{\Theta(D,t)} \frac{\partial\sigma^{ij}}{\partial x^j}\Omega$$

and with (21.36),

$$\rho\left(\frac{\partial v^i}{\partial t} + v^j\,\frac{\partial v^i}{\partial x^j}\right) = \rho f^i + \frac{\partial \sigma^{ij}}{\partial x^j} \tag{21.37}$$

or

$$\frac{\partial}{\partial t}(\rho v^i) + \frac{\partial}{\partial x^j}(\rho v^i v^j - \sigma^{ij}) = \rho f^i \tag{21.38}$$

A fluid is a continuous medium in which the stress, σ^{ij}, is a function of the rate of strain,

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i}\right)$$

such that when the latter vanishes an isotropic part remains. We sort out of the large class of possible types of fluids the mathematically simplest.

Definitions A *Newtonian fluid* is one for which

$$\sigma_{ij} = -p\ \delta_{ij} - \frac{2}{3}\mu tr(d_{ij})\delta_{ij} + 2\mu\ d_{ij}$$

where $\sigma_{ij} = \sigma^{ij}$, p is a function called the (*mean*) *pressure*, and μ is a *coefficient of viscosity*. A *perfect fluid* is a Newtonian fluid in which $\mu = 0$ (nonviscous). Then

$$\sigma_{ij} = -p\ \delta_{ij}$$

and eq. (21.38) becomes

$$\frac{\partial}{\partial t}(\rho v^i) + \frac{\partial}{\partial x^j}(\rho v^i v^j + p\ \delta^{ij}) = \rho f^i \tag{21.39}$$

Now look at (21.36) and (21.39). It is clear that when $f^i = 0$ they can be combined into a single equation $\partial T^{\alpha\beta}/\partial x^\alpha = 0$, α, $\beta = 0, 1, 2, 3$ if we make proper definitions. This suggests that we might be able to write a related spacetime equation for a related spacetime invariant which reduces to this in some limiting case.

We go to Minkowski spacetime, and consider a family of particles each parametrized by proper time. Let U be the vector field which at each point has the value of the 4-velocity $\dot{\lambda}$ of the particle. Form the tensor field $\rho\ U \otimes U/c^2$ with components

$$M^{\alpha\beta} = \rho\ \frac{U^\alpha U^\beta}{c^2} \tag{21.40}$$

where ρ is *the proper density* (=proper mass/proper volume). $U^0 = c/\sqrt{1 - v^2/c^2}$, $U^i = v^i/\sqrt{1 - v^2/c^2}$ in Lorentzian coordinates, by eq. (21.10), and the density in the given coordinate system will be $\bar{\rho} = \rho/(1 - v^2/c^2)$ (with notation of Section 20.5) since the mass is $m/\sqrt{1 - v^2/c^2}$ and the volume is $\sqrt{1 - v^2/c^2}$ times the proper volume. Hence, in terms of v^i and $\bar{\rho}$, (21.40) becomes

$$M^{ij} = \bar{\rho}\,\frac{v^i v^j}{c^2} \tag{21.41}$$

$$M^{i0} = \bar{\rho}\,\frac{v^i}{c} \tag{21.42}$$

$$M^{00} = \bar{\rho} \tag{21.43}$$

From (21.41) and (21.42) for $i = 1, 2, 3$,

$$\frac{\partial M^{i\alpha}}{\partial x^\alpha} = \frac{\partial}{\partial x^j}\,\frac{\bar{\rho} v^i v^j}{c^2} + \frac{\partial}{\partial x^0}\,\frac{\bar{\rho} v^i}{c} = \frac{1}{c^2}\,\left[\frac{\partial}{\partial t}(\bar{\rho} v^i) + \frac{\partial}{\partial x^j}(\bar{\rho} v^i v^j)\right] \tag{21.44}$$

and from (21.42) and (21.43),

$$\frac{\partial M^{0\alpha}}{\partial x^\alpha} = \frac{\partial}{\partial x^i}\,\frac{\bar{\rho} v^i}{c} + \frac{\partial}{\partial x^0}\bar{\rho} = \frac{1}{c}\left(\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x^i}\bar{\rho} v^i\right) \tag{21.45}$$

Thus, comparing (21.44) and (21.39) and comparing (21.45) and (21.36), and identifying ρ and $\bar{\rho}$ in corresponding equations, we have that

$$\frac{\partial M^{\alpha\beta}}{\partial x^\beta} = 0 \tag{21.46}$$

for *a particle flow* or *an incoherent fluid*; i.e., a "fluid" in which there are no internal forces.

To get the proper description in Minkowski spacetime of a fluid with internal stresses we have to add a stress tensor $S^{\alpha\beta}$. For a perfect fluid, $\sigma_{ij} = -p\,\delta_{ij}$, we define

$$S^{\alpha\beta} = \frac{p}{c^2}\left(\frac{U^\alpha U^\beta}{c^2} + g^{\alpha\beta}\right) \tag{21.47}$$

Note that $S^{\alpha\beta} U_\beta = 0$, and in Lorentzian coordinates,

$$S^{ij} = \frac{p}{c^2}\left(\frac{v^i v^j}{c^2(1 - v^2/c^2)} + \delta^{ij}\right) \sim \frac{p}{c^2}\delta^{ij} \quad \text{for small } p \text{ and } v^i$$

$$S^{i0} = \frac{p}{c^2}\,\frac{v^i}{c^2(1 - v^2/c^2)} \qquad \sim 0 \quad \text{for small } p \text{ and } v^i$$

$$S^{00} = \frac{p}{c^2}\left(\frac{1}{1 - v^2/c^2} - 1\right) \qquad \sim 0 \quad \text{for small } p \text{ and } v^i$$

In this approximation

$$\frac{\partial}{\partial x^\beta}(M^{\alpha\beta} + S^{\alpha\beta}) = 0 \tag{21.48}$$

reduce to the classical equations (21.36), conservation of mass and (21.39), the equation of motion, for a perfect fluid with no external forces. $M^{\alpha\beta} + S^{\alpha\beta}$ is called *the stress-energy tensor*, or *the energy-momentum tensor of a perfect fluid with no external forces.*

A case in which there are external forces can occur if the fluid is charged. We saw in the last section that a force on a charged particle can be ascribed to an electromagnetic field tensor $\mathcal{F}$ related to the force by eq. (21.23). If instead of a single charged particle we have a "cloud" of charged particles, we change eq. (21.23) to $(g\eta)^\flat \cdot \mathbf{F} = -\sigma i_U \mathcal{F}$, with coordinates $-\sigma U^\alpha \mathcal{F}_{\alpha}^{\cdot\beta}$ where η is the charge density of the fluid and $\mathbf{F}$ is the force per unit volume. We replace ρf^i in eq. (21.39) by the components of the 3-force per unit volume. Then by eq. (21.18) $\sqrt{1 - v^2/c^2}\, \mathbf{F}^i = \rho f^i$. To conform to the pattern developed in the previous cases, we would like the effect of ρf^i to be described by a second-order tensor such that the conservation of mass and the equation of motion are expressed as a vanishing "divergence."

Theorem 21.8 *If we define* $E^{\alpha\beta}$ *by*

$$E^{\alpha\beta} = \frac{1}{\mu}(\mathcal{F}^{\alpha}_{\cdot\nu}\mathcal{F}^{\nu\beta} - \frac{1}{4}g^{\alpha\beta}\mathcal{F}^{\lambda\nu}\mathcal{F}_{\lambda\nu}) \tag{21.49}$$

then

$$\frac{\partial}{\partial x^\beta}E^{\alpha\beta} = \frac{1}{\mu}Z^\nu \mathcal{F}_{\nu}^{\cdot\beta}$$

Proof Problem 21.17. □

For small electromagnetic fields and velocities

$$\frac{\partial}{\partial x^\beta}(M^{\alpha\beta} + S^{\alpha\beta} + E^{\alpha\beta}) = 0 \tag{21.50}$$

reduces to eqs. (21.36) and (21.39) for a charged perfect fluid, and $M^{\alpha\beta} + S^{\alpha\beta} + E^{\alpha\beta}$ is the stress-energy tensor in this case.

PROBLEM 21.15 Derive the "transport theorem," eq. (21.35). (Use Theorem 13.16, eq. (18.5) and the fact that for a volume form $L_X f\Omega = L_{fX}\Omega$).

PROBLEM 21.16 The form $h^\alpha_\gamma = g_{\beta\gamma}((U^\alpha U^\beta/c^2) + g^{\alpha\beta})$ of the tensor appearing in eq. (21.47) is a projection operator; it has the property $h^\alpha_\gamma h^\gamma_\delta = h^\alpha_\delta$, and it maps vectors in each tangent space to their projections in the subspace orthogonal to U.

PROBLEM 21.17 Use Maxwell's equations (21.26) or (21.30) and (21.31), $Z^\alpha = \mu\eta\sqrt{1 - v^2/c^2}\, U^\alpha$, and eq. (21.49) to prove Theorem 21.8.

22

EINSTEIN SPACETIMES

In this chapter, we will construct Einstein's cosmological models as represented by his field equations, eqs. (22.17) and (22.27). We will include some motivation and justification for these models and one interesting general consequence.

22.1 Gravity, acceleration, and geodesics

To introduce Einstein's cosmological models we go back to the Minkowski spacetime model, and note a couple of ways in which it seems incomplete.

(1) We saw in Problem 21.13 that in Minkowski spacetime we could calculate the trajectory of a charged particle subject to an electromagnetic force field. On the other hand, the existence of a family of Lorentz frames (coordinates), and hence (M, g_η), was predicated on Newton's Principle of Inertia, affirming the existence of a frame in which free particles move on straight lines with constant speed. But in the presence of gravity there really are not any free particles, since every (massive) particle is subject to gravity, so this model excludes gravity. The fact that Einstein's special theory can deal with one of the two macroscopic force fields of physics, but not the other, gives it a certain unsatisfactory incompleteness.

(2) Einstein's principle of relativity, Theorem 20.2, while it extends the Newtonian (Galilean) principle, Theorem 20.1, to electromagnetism, has the same basic limitation as the latter; that is, it is a restricted type of relativity in the sense that physics is the same in every reference frame in a certain restricted class - reference frames moving relative to one another with constant relative velocity. Complete relativity would not require such a restriction. It would also include frames accelerated with respect to one another.

The principle of equivalence is the concept that relates these two deficiencies, and thereby simultaneously corrects both. The result, from the standpoint of the first, is called the theory of gravitation, and from the standpoint of the second is called the theory of general relativity (or, the general theory of relativity).

The principle of equivalence has many, more or less equivalent, formulations. One, roughly, is that gravity is simply a manifestation of using the wrong frame of reference.

Einstein's elevator "thought experiment" is the classical pellucid argument for the principle of equivalence. Thus, suppose a person is enclosed in a (small) elevator in a tall building. If the elevator is falling freely there will be nothing holding him/her to the floor - no gravity. Incidentally, this could not happen if gravitational mass were not the same as inertial mass - otherwise his/her wallet might move relative to him/her. Alternatively, if the elevator is not moving, he/she could interpret the force exerted on him/her by the floor to be due to an upward acceleration.

Another way to look at the principle of equivalence is to consider what happens to equations of motion in different frames of reference. For example, suppose a particle (m, γ) moves according to

$$\frac{d^2\gamma^i}{dt^2} = 0 \tag{22.1}$$

in some frame of reference. In a rotating frame of reference given by

$$x = r\cos(\vartheta + \omega t), \quad y = r\sin(\vartheta + \omega t), \quad z = z$$

where ω is a constant, the equations of motion of the particle are

$$\begin{aligned} \frac{d^2r}{dt^2} - r\left(\frac{d\vartheta}{dt}\right)^2 &= \omega^2 r + 2\omega r\frac{d\vartheta}{dt} \\ r\frac{d^2\vartheta}{dt^2} + 2\frac{dr}{dt}\frac{d\vartheta}{dt} &= -2\omega\frac{dr}{dt} \\ \frac{d^2z}{dt^2} &= 0 \end{aligned} \tag{22.2}$$

(where we have written dr/dt for $dr \circ \gamma/dt$, etc.) The left sides are the usual cylindrical coordinate components of acceleration, and on the right sides there are the centrifugal and Coriolis forces per unit mass arising from the rotation. Then, if we go backwards, we can eliminate the centrifugal and Coriolis forces by using the right frame of reference - just like when we let the elevator drop.

If we go to a general frame of reference, then eq. (22.1) will go to a more general form of (22.2). Since a Lorentz frame in $\mathcal{E}_0^3$ corresponds to Lorentz coordinates in (M, g_η), a general frame in $\mathcal{E}_0^3$ will correspond to general coordinates in (M, g_η) so in (M, g_η) going from (22.1) to the generalization of (22.2) corresponds to going from the equation of motion of a particle, (m, λ),

$$\frac{d^2\lambda^\alpha}{d\tau^2} = 0 \tag{22.3}$$

to the equation of motion

$$\frac{d^2\lambda^\alpha}{d\tau^2} = -\left\{ \begin{matrix} \alpha \\ \beta\ \delta \end{matrix} \right\} \frac{d\lambda^\beta}{d\tau}\frac{d\lambda^\delta}{d\tau} \tag{22.4}$$

of λ in general coordinates in (M, g_η). Again we can think of the right side as forces per unit mass. They could include gravity, since the acceleration of gravity is independent of mass. However, if gravity could be included on the right side of eq. (22.4) as suggested by the principle of equivalence as stated above, then we would have a model of gravity in (M, g_η) which cannot be, as we have seen.

A crucial refinement in our discussion of the elevator example is that although in the falling elevator the effects of gravity are small, they are there, and we can only eliminate gravity *completely at one point*. In terms of eq. (22.4), this says that there are no coordinates in which contributions to the right side due to gravity completely vanish (gravity is not a fictitious force), though in some coordinate system the right side will vanish at a point and be "small" near the point. This statement has the following two implications.

1. If eq. (22.4) is to be valid, and the right side cannot be made to vanish, then we can not think of it simply as a coordinate transformation of eq. (22.3) in (M, g_η). But we can think of it as the equation of a geodesic in arbitrary coordinates in a 4-dimensional manifold, so the presence of gravity corresponds to the presence of some nonzero curvature: gravity is a manifestation of the curvature of a 4-dimensional manifold (or conversely).

2. In a normal coordinate system, physics near the center of coordinates looks approximately like physics without gravity in (M, g_η). In other words, in a neighborhood of each point, physics can be approximated, by means of the exponential map, by physics in the Minkowski spacetime tangent space at the point. In particular, the 4-dimensional manifold of (1) must be Lorentzian (Section 15.1).

22.2 Gravity is a manifestation of curvature

Under the assumption that our 4-dimensional manifold is Lorentzian - the second point above - a simple calculation makes the first point above accessible to experimental verification.

Recall, first (Section 15.1), that in a Lorentzian manifold we have spacelike, timelike, and causal curves. For a future-pointing timelike curve, λ, we define *proper time*, τ just as before in Section 21.1. Now, let us see what the terms on the right side of (22.4) look like when we go from (M, g_η) to "slightly curved" Lorentzian space and with these meanings of λ and τ.

We assume our manifold has coordinates (x^α) in terms of which $g = (\eta_{\alpha\beta} + \varepsilon h_{\alpha\beta})dx^\alpha \otimes dx^\beta$ and ε is small. Let $t = x^0/c$. Writing out the components $\alpha = i = 1, 2, 3$ of (22.4) in terms of t and putting x^i for $\lambda^i = x^i \circ \lambda$ we get

$$\frac{d^2x^i}{dt^2} = -\frac{dx^i}{dt}\frac{d^2t}{d\tau^2}\left(\frac{d\tau}{dt}\right)^2 - \left(\left\{ {i \atop 0\,0} \right\}\right.$$

$$\left. + \left\{ {i \atop 0\,1} \right\} fracdx^1dt \Big/ c + \cdots + \left\{ {i \atop 3\,3} \right\} \left(\frac{dx^3}{dt}\right)^2 \Big/ c^2 \right) c^2 \tag{22.5}$$

Theorem 22.1 *If $h_{i\alpha}$ are independent of t, then to first order in ε and $(dx^i/dt)/c$, eq. (22.5) reduces to*

$$\frac{d^2x^i}{dt^2} = \frac{\partial}{\partial x^i}\left(\frac{c^2}{2}\varepsilon h_{00}\right) \tag{22.6}$$

Proof

$$\begin{aligned}
\left\{ {i \atop 0\,0} \right\} &= \frac{1}{2}g^{i\alpha}\left(\frac{\partial g_{0\alpha}}{\partial x^0} + \frac{\partial g_{\alpha 0}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\alpha}\right) \\
&= \frac{1}{2}(\eta^{i\alpha} + \varepsilon\tilde{h}^{i\alpha})\left(2\frac{\partial}{\partial x^0}(\eta_{i\alpha} + \varepsilon h_{i\alpha}) - \frac{\partial}{\partial x^\alpha}(\eta_{00} + \varepsilon h_{00})\right) \\
&= \frac{1}{2}\eta^{i\alpha}\left(2\frac{\partial}{\partial x^0}\varepsilon h_{i\alpha} - \frac{\partial}{\partial x^\alpha}\varepsilon h_{00}\right) \quad \text{to first order in } \varepsilon \\
&= -\frac{1}{2}\varepsilon\frac{\partial}{\partial x^i}h_{00} \quad \text{if } h_{i\alpha} \text{ are independent of } t
\end{aligned} \tag{22.7}$$

Similarly, it is seen that all the $\left\{ {i \atop \alpha\,\beta} \right\}$ have a factor of ε, and in all terms of the term on the right in (22.5), except $\left\{ {i \atop 0\,0} \right\}$, there is at least one factor of $(dx^i/dt)/c$ so the last term on the right reduces to $-\left\{ {i \atop 0\,0} \right\}c^2$. Finally, $(dx^i/dt)(d^2t/d\tau^2)$ is of second order in $(dx^i/dt)/c$ and

$$\begin{aligned}
\left(\frac{d\tau}{dt}\right)^2 &= -\frac{1}{c^2}(\eta_{\alpha\beta} + \varepsilon h_{\alpha\beta})\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} \\
&= 1 - \varepsilon h_{00} \quad \text{to first order} \\
&= -g_{00}
\end{aligned} \tag{22.8}$$

so the first term on the right of (22.5) vanishes to first order and eq. (22.5), to that order is

$$\frac{d^2x^i}{dt^2} = -\left\{ {i \atop 0\,0} \right\}c^2 \tag{22.9}$$

Substituting (22.7) into (22.9) we get (22.6). □

Note that in eq. (22.6) the "force" on the right side has a potential, which is a property of gravity. Writing

$$\mathbf{\Phi} = -\frac{c^2}{2}\varepsilon h_{00} \tag{22.10}$$

we get the simple relation

$$-g_{00} = 1 + \frac{2\mathbf{\Phi}}{c^2} \tag{22.11}$$

which accounts for the designation, in the literature, of the metric coefficients as "gravitational potentials."

We have not proved that $\mathbf{\Phi}$ actually is the potential of gravity. However, it has been experimentally confirmed that it has a property in common with gravity - the "redshift" expressed in terms of $\mathbf{\Phi}$ is what is observed when the values of the gravitational potential are used for $\mathbf{\Phi}$. A derivation of the relation between redshift and $\mathbf{\Phi}$ follows. There are others.

Suppose we have two particles, e, (earth) and, s, (sun) and suppose that s emits photons during a time interval Δt, and these photons arrive at e. The time interval during which these photons arrive at e will be the same, Δt. The proper time in s's frame will be $\Delta\tau_s$, and in e's frame it will be $\Delta\tau_e$. The number of waves emitted by s will be the same as the number of waves received by e. Hence

$$\nu_s \Delta\tau_s = \nu_e \Delta\tau_e \tag{22.12}$$

where ν_s and ν_e are the proper frequencies of the radiation observed by s and e, respectively. Now by (22.8), (to first order)

$$\Delta\tau_s = \sqrt{-g_{00}(s)}\ \Delta t, \quad \Delta\tau_e = \sqrt{-g_{00}(e)}\ \Delta t$$

From (22.11) and (22.12)

$$\nu_s\sqrt{1 + \frac{2\mathbf{\Phi}_s}{c^2}} = \nu_e\sqrt{1 + \frac{2\mathbf{\Phi}_e}{c^2}}$$

or

$$\nu_s = \nu_e\left(1 + \frac{2(\mathbf{\Phi}_e - \mathbf{\Phi}_s)/c^2}{1 + 2(\mathbf{\Phi}_s/c^2)}\right)^{\frac{1}{2}}$$

or, to first order,

$$\frac{\nu_s - \nu_e}{\nu_e} = \frac{\mathbf{\Phi}_e - \mathbf{\Phi}_s}{c^2} \tag{22.13}$$

Note that $\mathbf{\Phi}_e > \mathbf{\Phi}_s$ since force $= -\text{grad}\ \mathbf{\Phi}$ points toward s, so $\nu_s > \nu_e$, a shift toward the red. Equation (22.13) is the redshift formula. The value of the left side of (22.13) has been determined by a variety of experiments. If the values of the gravitational potential are used for $\mathbf{\Phi}_e$ and $\mathbf{\Phi}_s$ on the right, (22.13) is satisfied.

PROBLEM 22.1 Discuss the redshift formula. Give a derivation using conservation of energy and/or give some numerical values for the quantities in (22.13). (See Weinberg, pp. 79-86.)

22.3 The field equation in empty space

We would like to pin down the manifold which according to our assumptions in Section 22.1 describes gravity. First of all, according to the second point at the end of Section 22.1, the manifold must be Lorentzian. More specifically, we will restrict ourselves to the following class.

Definitions *A spacetime* is a connected, 4-dimensional, oriented, time-oriented Lorentzian manifold with the Levi-Civita connection. Thus, at each point, p, the tangent space, T_p, is a Lorentzian vector space, and each future-pointing timelike vector $v \in T_p$ has *a local rest space*, the subspace of T_p orthogonal to v.

Spacetimes have been studied extensively in recent years (cf., Hawking and Ellis; Beem and Ehrlich). Causality concepts have been defined in terms of which one examines the behavior of nonspacelike curves, and the occurrence of singularities. We will refer to the general results of this theory only in the context of the specific spacetimes we will be describing.

The result of Section 22.2 suggests that with any spacetime we can associate a "force" which has at least some of the properties of gravity. Different spacetimes will give different forces, so, since gravity should be determined uniquely by the distribution of mass in the universe, we can not expect more than one of these manifolds to actually correspond to the force of gravity. Thus, we need to impose some conditions on our manifolds. We can get a clue by looking at the conditions that actually *characterize* the gravitational field (not merely noting properties possessed by the gravitational field, as we have been doing). The conditions that characterize, i.e., determine, Newtonian gravity are that for points in *empty space* it has a potential that satisfies Laplace's equation $\Delta\Phi = 0$ and boundary conditions.

Since the curvature, R, of its connection is one of the main invariants of a pseudo-Riemannian manifold, it would seem reasonable to try to impose some condition in R. Clearly $R = 0$ will not do, for that characterizes flat space, which, on our space, gives only Minkowski spacetime. A weaker condition can be obtained by setting a contraction of R equal to zero. Note that there is no problem of choice of a particular contraction since $C_1^1 \cdot R = 0$ and $C_3^1 \cdot R = -C_2^1 \cdot R$.

Definition The contraction

$$\mathrm{Ric} = C_2^1 \cdot R$$

of R is a $(0, 2)$ tensor field called *the Ricci curvature tensor.*

In terms of a basis

$$\mathrm{Ric} = R^{\alpha}_{\cdot\beta\alpha\delta}\varepsilon^{\beta} \otimes \varepsilon^{\delta} \tag{22.14}$$

Its values, in terms of the values of R, are

$$\mathrm{Ric}(X, Y) = \sum_{\alpha} R(\varepsilon^{\alpha}, X, e_{\alpha}, Y) \tag{22.15}$$

(see eq. (4.16)) and its components are

$$\mathrm{Ric}_{\beta\delta} = R^{\alpha}_{\cdot\beta\alpha\delta} \tag{22.16}$$

Hence, in these terms, the suggestion above for a condition on R amounts to

$$\mathrm{Ric} = 0 \tag{22.17}$$

Theorem 22.2 Ric *has the following properties:*
(i) Ric *is symmetric*
(ii) $\mathrm{Ric}(X, Y) = \sum_{\alpha} g_{\alpha\alpha} g(\tilde{R}(e_{\alpha}, X) \cdot Y, e_{\alpha})$ *where* $\{e_{\alpha}\}$ *is any orthonormal basis and* $g_{\alpha\alpha} = \pm 1$.
(iii) *If* u *is a unit vector,* $\mathrm{Ric}(u, u)$ *is* $\pm$ *the sum of the sectional curvatures of the planes containing* u.

Proof Problem 22.5. □

Now we will show that on our spacetime with the proposed condition $\mathrm{Ric} = 0$, we can identify a function $\mathbf{\Phi}$ with the property $\Delta\mathbf{\Phi} = 0$ - the quantity we associated with gravity in regions of empty space.

In our spacetime let X be the tangent field of a family of forward-facing timelike geodesics. We consider the special (but important - see Section 24.2) case in which X has orthogonal hypersurfaces.

Suppose F is a vector field on spacetime lying on these hypersurfaces. Two nearby geodesics, γ, γ', intersect a hypersurface at points $p, q = p + Y$ so that

$$F^{i}(q) - F^{i}(p) = F^{i}(p + Y) - F^{i}(p) = \frac{\partial F^{i}}{\partial \mu^{j}}\, Y^{j}_{\gamma}$$

approximately, where Y_{γ} is the restriction to γ of a vector field on spacetime which is tangent to Y at p.

If we think of Y as the position of a particle moving on γ' relative to that of an observer on γ with the relative acceleration $\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}Y_{\gamma}$, and $F(q) - F(p)$ as a relative force, then, with F in terms of a potential, $\mathbf{\Phi}$, $F^{i} = g^{ij}\frac{\partial \mathbf{\Phi}}{\partial \mu^{j}}$ and substituting this into the right side of the equation above we expect Y_{γ} to satisfy the equation of motion

$$(\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}Y_{\gamma})^{i} = g^{ik}\mathbf{\Phi}_{,j,k}Y^{i}_{\gamma} \tag{22.18}$$

Theorem 22.3 *If $\mathbf{\Phi}$ satisfies (22.18), then*

$$\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) = g^{ij}\mathbf{\Phi}_{,i,j}, \tag{22.19}$$

Proof The $(1,1)$ tensor $g^{ik}\mathbf{\Phi}_{,j,k}$ is a linear operator, $\mathcal{T}$, on the local rest space of γ at a point of γ, so by (22.18) $\mathcal{T}: Y_\gamma \mapsto \nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}Y_\gamma$. But Y_γ also satisfies the Jacobi equation (17.18) (with $\widetilde{T} = 0$) so $\mathcal{T}: Y_\gamma \mapsto \widetilde{R}(\dot{\gamma}, Y_\gamma)\cdot\dot{\gamma}$. The trace of $\mathcal{T}$ is $\mathrm{tr}\,\mathcal{T} = \sum_{i=1}^{3}\mathcal{T}(e_i, e_i)$ where e_i is any orthonormal basis of the local rest space. By the original representation of $\mathcal{T}$, $\mathrm{tr}\,\mathcal{T} = g^{ij}\mathbf{\Phi}_{,i,j}$, and by the last representation $\mathrm{tr}\,\mathcal{T} = \sum_{i=1}^{3} g(\widetilde{R}(\dot{\gamma}, e_i)\cdot\dot{\gamma}, e_i) = \mathrm{Ric}(\dot{\gamma}, \dot{\gamma})$ by Theorem 22.2(ii). □

From Theorem 22.3 it is clear that Ric $= 0$ implies that $\Delta\mathbf{\Phi} = 0$ independently of the choice of X. Conversely, if $\Delta\mathbf{\Phi} = 0$ for all choices of X, then so is $\mathrm{Ric}(\dot{\gamma}, \dot{\gamma})$, which implies that Ric $= 0$ (Problem 22.6).

We can write out Ric in terms of the Christoffel symbols (cf., eq. (16.23)) and thence in terms of the metric coefficients. So eq. (22.17), Ric $= 0$, is a set of partial differential equations for the g's. Specifically, they are 10 second-order quasilinear partial differential equations for the 10 functions $g_{\alpha\beta}$. Thus, from a certain point of view we have reduced our problem of finding the relation between gravity and the geometry of spacetime in empty space to solving a system of partial differential equations. While the theory of partial differential equations gives us, in this case, certain important qualitative information, and some local existence and uniqueness theorems (see Adler et al., Chap. 7) we get physically interesting solutions by trying to model certain simple physical situations. We will do this in the next chapter.

Once we have a solution of Ric $= 0$, and hence, an empty space solution for a spacetime, the motion of a particle is governed by eq. (22.4), the "equation of motion" with which we started. We have to be a little careful, though. In the approximation in Section 22.2 the potential, $\mathbf{\Phi}$, introduced into the right side and appearing in the redshift formula was supposed to correspond to the gravity of mass other than that of the particle itself. A massive particle could alter the geometry in such a way that it is no longer a geodesic. Hence, we interpret eq. (22.4) as the equation of motion of *a test particle* - a particle of "vanishingly small" mass.

From Ric we can form two more important invariants, which we will need in the next section and in Chapter 24.

Definitions The *scalar curvature* is $S = g^{\beta\delta}\mathrm{Ric}_{\beta\delta}$. The *Einstein tensor* is $G = \mathrm{Ric} - \frac{1}{2}Sg$, or in components $G_{\alpha\beta} = \mathrm{Ric}_{\alpha\beta} - \frac{1}{2}Sg_{\alpha\beta}$.

Clearly G is symmetric. It has two other important properties.

Theorem 22.4 *g satisfies $G = 0$ iff g satisfies* $\mathrm{Ric} = 0$.

Proof If g satisfies $\mathrm{Ric} = 0$, then g satisfies $S = 0$, so g satisfies $G = 0$. Conversely, if g satisfies $G = 0$, then g satisfies $g^{\alpha\beta}G_{\alpha\beta} = 0$, but $g^{\alpha\beta}G_{\alpha\beta} = S - \frac{1}{2}g^{\alpha\beta}g_{\alpha\beta}S = -S$, so g satisfies $\mathrm{Ric} = 0$. □

Theorem 22.5 *The divergence of G vanishes.*

Proof From the second Bianchi identity (Problem 16.12),

$$R^{\alpha\eta}_{\cdot\cdot\beta\gamma,\delta} + R^{\alpha\eta}_{\cdot\cdot\gamma\delta,\beta} + R^{\alpha\eta}_{\cdot\cdot\delta\beta,\gamma} = 0$$

Put $\beta = \alpha$ and $\gamma = \eta$. Then $S_{,\delta} + R^{\alpha\eta}_{\cdot\cdot\eta\delta,\alpha} + R^{\alpha\eta}_{\cdot\cdot\delta\alpha,\eta} = 0$, or

$$S_{,\delta} - 2\ \mathrm{Ric}^{\alpha}_{\cdot\delta,\alpha} = 0$$

Then

$$g^{\delta\beta}S_{,\delta} - 2\ \mathrm{Ric}^{\alpha\beta}_{,\alpha} = 0 \quad \text{or} \quad (g^{\alpha\beta}S - 2\ \mathrm{Ric}^{\alpha\beta})_{,\alpha} = 0$$

□

PROBLEM 22.2 Show that a spacetime has a timelike Killing field, X, iff it has coordinates such that $\partial g_{\alpha\beta}/\partial\mu^0 = 0$. Such a spacetime is called *stationary.*

PROBLEM 22.3 Show that a timelike vector field, U, on a spacetime has orthogonal hypersurfaces, *rest spaces*, iff $\mathrm{curl}\, U(X, Y) = 0$ for all X, Y orthogonal to U. Such a vector field is called *irrotational.* (By definition $\mathrm{curl}\, U = d(g^\flat \cdot U)$, and note that for vector fields X, Y, Z, $\mathrm{curl}\, X(Y, Z) = g(\nabla_Y X, Z) - g(Y, \nabla_Z X)$).

PROBLEM 22.4 Show that $C^1_1 \cdot R = 0$ and $C^1_3 \cdot R = -C^1_2 \cdot R$.

PROBLEM 22.5 Prove Theorem 22.2. (Cf., Eisenhart, 1949, p. 113.)

PROBLEM 22.6 Prove that if $\mathrm{Ric}(\dot\gamma, \dot\gamma) = 0$ for all timelike geodesic vector fields, then $\mathrm{Ric} = 0$ (cf., Sachs and Wu, p. 114).

22.4 Einstein's field equation (Sitz, der Preuss Acad. Wissen., 1917)

In Section 21.4, we obtained several symmetric second-order tensors, *stress-energy*, or *energy-momentum* tensors, each of which is supposed to describe some type of matter/energy content of Minkowski spacetime. In the previous section, in setting up the field equation for empty space, no mention was made of any matter/energy distribution of spacetime. Recall that there we were led to $\mathrm{Ric} = 0$ by the argument that in a certain limiting sense this condition reduces to $\Delta\mathbf{\Phi} = 0$, which is satisfied by gravity at points of empty space. In Newtonian theory, in regions filled with matter, we have to satisfy $\Delta\mathbf{\Phi} = 4\pi\mathbf{g}\rho$, Poisson's equation with gravitational constant $\mathbf{g}$. The obvious corresponding generalization of Ric $= 0$ is to replace the right side of Ric $= 0$ with something representing the matter/energy distribution of the region. It would have to be a second-order symmetric tensor, and the energy-momentum tensors we defined in Section 21.4 have these properties. *We now assume that these tensors represent the mass/energy content of an arbitrary spacetime.* This assumption is sometimes called *the principal of minimal coupling*, or *the strong principal of equivalence.* The fact that these tensors are divergenceless suggests using G on the left rather than Ric, and assuming $G = aT$ where T stands for any stress-energy tensor including those defined in Section 21.4.

We could try to generalize. Thus, for example,

$$G = aT + bg \tag{22.20}$$

where a and b are constants, has all the properties we have required. Some justification for stopping with this much generality is given by the theorem that if T is a second-order symmetric divergenceless tensor whose components are linear in the second-order partial derivatives of the components of g with coefficients which are functions of, at most, the components of g and their first derivatives, then G must have the form $T = c_1 G + c_2 g$ where c_1 and c_2 are constants.* Thus we can, alternatively, either postulate the relation or equivalently postulate that every energy-momentum tensor has the properties of this theorem. Finally, although there has been some recent interest in models of spacetime having a nonzero *cosmological constant*, $-b = \Lambda$, for the classical models, there are problems with physical interpretations of solutions when $T = 0$. We will, by and large, restrict ourselves to

$$G = aT \tag{22.21}$$

Now, in lieu of empirical verification of this postulate, perhaps the main argument for its validity is to show, as we have done in previous situations, that in the "Newtonian limit" we get the standard classical results - in this case the Poisson equation. In the process we will evaluate the constant a.

First note that from

$$\mathrm{Ric}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} S = aT_{\alpha\beta}$$

we get

---

*This result was proved in a long paper by E. Cartan in 1922 and in 4 dimensions, more recently, by Lovelock (1972). Other results from which this can be easily derived are in Weyl, 1922, Appendix II and Weinberg, p. 113.

$$\mathrm{Ric}_{\alpha\beta} = a(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T^{\lambda}_{\lambda}) \tag{22.22}$$

expressing Ric in terms of the stress-energy tensor. We will use (22.22) in our subsequent calculations.

Now let $g_{\alpha\beta} = \eta_{\alpha\beta} + \varepsilon h_{\alpha\beta}$ as we did in Section 22.2 and find the expansion of $\mathrm{Ric}_{\alpha\beta}$ to first order in ε. $\mathrm{Ric}_{\alpha\beta}$ is written in terms of the Christoffel symbols using eq. (16.23). To first order,

$$\mathrm{Ric}_{\alpha\beta} = \frac{\partial}{\partial x^{\gamma}}\left\{ {\gamma \atop \alpha\ \beta} \right\} - \frac{\partial^2}{\partial x^{\alpha}\partial x^{\beta}} \ln \sqrt{|\det(g_{\alpha\beta})|} \tag{22.23}$$

since products of the Christoffel symbols are all of second order. To first order,

$$|\det(g_{\alpha\beta})| = 1 + \varepsilon \sum_{\alpha} h_{\alpha\alpha}$$

and

$$\ln|\det(g_{\alpha\beta})| = \varepsilon \sum_{\alpha} h_{\alpha\alpha} \tag{22.24}$$

We can make the same calculations for $\left\{ {i \atop \alpha\ \beta} \right\}$ that we did in Section 22.2 for $\left\{ {i \atop 0\ 0} \right\}$, so

$$\left\{ {i \atop \alpha\ \beta} \right\} = \frac{\varepsilon}{2}\left(\frac{\partial h_{\beta i}}{\partial x^{\alpha}} + \frac{\partial h_{i\alpha}}{\partial x^{\beta}} - \frac{\partial h_{\alpha\beta}}{\partial x^{i}} \right) \tag{22.25}$$

to first order in ε.

Now from (22.23), (22.24), and (22.25) and assuming, as before, that the g's are independent of x^0,

$$\mathrm{Ric}_{00} = -\frac{\varepsilon}{2}\sum_{i} \frac{\partial^2 h_{00}}{(\partial x^i)^2} = -\frac{1}{2}\sum_{i} \frac{\partial^2 g_{00}}{(\partial x^i)^2}$$

Now consider the case of an incoherent fluid, eq. (21.40). Then

$$T_{00} = g_{0\alpha}g_{0\beta}T^{\alpha\beta} = T^{00} = \rho$$

to first order in ε, v^i/c and ρ. Similarly $T^{\lambda}_{\lambda} = g_{\alpha\beta}T^{\alpha\beta} = -T^{00}$ to the same approximation. From (22.22)

$$\mathrm{Ric}_{00} = a(T_{00} - \frac{1}{2}g_{00}T^{\lambda}_{\lambda})$$

so

$$-\frac{1}{2}\sum_i \frac{\partial^2 g_{00}}{(\partial x^i)^2} = a\frac{1}{2}T^{00}$$

or

$$\sum_i \frac{\partial^2 g_{00}}{(\partial x^i)^2} = -aT^{00} \tag{22.26}$$

Putting $T^{00} = \rho$, putting $g_{00} + 1 = -2\mathbf{\Phi}/c^2$ (eq. (22.11)), and using $\sum_i \partial^2\mathbf{\Phi}/(\partial x^i)^2 = 4\pi\mathbf{g}\rho$, Poisson's equation with gravitational constant $\mathbf{g}$, we get from (22.26)

$$a = \frac{8\pi\mathbf{g}}{c^2}$$

Further, with $h_{\alpha\beta} = 0$, $\alpha \neq \beta$ and $h_{ii} = h_{00}$, all the other equations of (22.22) are either satisfied identically or with (22.26). (Problem 22.7.) The *Einstein field equation* is

$$G = \frac{8\pi\mathbf{g}}{c^2}T \tag{22.27}$$

In the derivation of Einstein's field equation we assumed that T is divergenceless. We could, alternatively, have assumed Einstein's equation from which it follows that T is divergenceless. We insert here an interesting consequence of this property of T.

Combining all the cases we considered in Section 21.4, we have

$$T^{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right)\frac{U^\alpha U^\beta}{c^2} + \frac{p}{c^2}g^{\alpha\beta} + \frac{1}{\mu}[\mathcal{F}^{\alpha}_{\cdot\nu}\mathcal{F}^{\nu\beta} - \frac{1}{4}g^{\alpha\beta}\mathcal{F}^{\lambda\mu}\mathcal{F}_{\lambda\mu}]$$

Then, taking the divergence, we get, using Theorem 21.8,

$$U^\alpha\left[\left(\rho + \frac{p}{c^2}\right)U^\beta\right]_{,\beta} + \left[\left(\rho + \frac{p}{c^2}\right)U^\beta\right]U^\alpha_{,\beta} + g^{\alpha\beta}p_{,\beta} + \frac{1}{\mu}Z^\beta\mathcal{F}_{\beta}^{\cdot\alpha} = 0 \tag{22.28}$$

Multiplying by U_α we get

$$-c^2\left[\left(\rho + \frac{p}{c^2}\right)U^\beta\right]_{,\beta} + U^\beta p_{,\beta} + \frac{1}{\mu}Z^\beta U_\alpha\mathcal{F}_{\beta}^{\cdot\alpha} = 0$$

Then substituting this into (22.28), and using $Z^\beta = \mu\sigma_0 U^\beta$, (Problem 21.17)

$$U^\alpha\left[\frac{1}{c^2}(U^\beta p_{,\beta} + \sigma_0 U^\beta U_\gamma\mathcal{F}_{\beta}^{\cdot\gamma})\right] + \left(\rho + \frac{p}{c^2}\right)U^\beta U^\alpha_{,\beta} + g^{\alpha\beta}p_{,\beta} + \sigma_0 U^\beta\mathcal{F}_{\beta}^{\cdot\alpha} = 0$$

or

$$\left(\rho + \frac{p}{c^2}\right)U^\beta U^\alpha_{,\beta} = \left(\frac{U^\alpha U^\beta}{c^2} + g^{\alpha\beta}\right)\left(-p_{,\beta} - \sigma_0\mathcal{F}_{\gamma\beta}U^\gamma\right) \tag{22.29}$$

From Problem 21.16, eq. (22.29) says that the acceleration of a curve, γ, with tangent vector U is in the local rest space of U, and depends only on the components of pressure and electromagnetic forces in that space. In particular, γ is a geodesic iff the sum of these forces vanishes.

Equation (22.29) is frequently called an equation of motion, and in the sense indicated above it is a consequence of the field equation. However, it seems that there is less here than meets the eye. Like eq. (22.4), eq. (22.29) can be interpreted as the equation of motion of a "test particle" - one of "vanishingly small" mass. The motion of a massive particle is indeed determined by the field equation, but that is a much more complicated matter, cf., Misner, Thorne, and Wheeler, pp. 471-480 and Bergmann, Chap. XV.

PROBLEM 22.7 Show that for an incoherent fluid, Einstein's equation, when linearized as we did, is satisfied with $h_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $h_{ii} = h_{00}$.

23

SPACETIMES NEAR AN ISOLATED STAR

A solution of the field equations gives the geometry, i.e., the metric tensor, of the spacetime model - at least locally. The geometry is supposed to be "due to," or "determined by" a distribution of mass (energy). Thus, a solution will depend on an assumed mass distribution in the spacetime model.

We get simple models for the geometry near a massive star far from other stars if we assume that we can neglect all other mass in the universe. This is the case of a model being given by solutions of $G = 0$, the metric tensor at points not occupied by mass due to a mass distribution elsewhere.

In a model of spacetime of this kind, one would expect spacetime to have certain *general* geometrical properties. Hence, we *assume* these general properties. This has the effect of limiting to a certain class our search for a solution, and simplifies our problem.

23.1 Schwarzschild's exterior solution

We look for a spacetime which is a solution of $G = 0$ with the properties:

1. It is *static*: it has a timelike Killing vector field, X, orthogonal to a family of spacelike hypersurfaces.
2. It is spherically symmetric about one of the integral curves, λ_0, of X; λ_0 is the world line of a particle, (m, λ_0), which represents the star.
3. It approaches Minkowski spacetime far enough away from λ_0.

Theorem 23.1 *Property* (1) *implies that there exists a coordinate system with a coordinate function t such that the $g_{\alpha\beta}$ are independent of t and $t =$ constant are the orthogonal hypersurfaces, so $g_{0i} = 0,\ i = 1, 2, 3$.*

Proof Problem 23.1. □

Now suppose that at each point where λ_0 intersects a $t =$ constant hypersurface we construct the geodesics on that hypersurface and go out a distance s on each. The assumption (2) of spherical symmetry says that the set of points $\{t = \text{constant},\ s = \text{constant}\}$ forms an ordinary Euclidean 2-sphere and hence it has a metric tensor of the form $r^2(d\vartheta^2 + \sin^2\vartheta\ d\varphi^2)$. (Here and in the following formulas we write expressions like $d\vartheta \otimes d\vartheta$, $d\varphi \otimes d\varphi$, etc., in the usual way as $d\vartheta^2$, $d\varphi^2$, etc.) Moreover, this 2-sphere is normal to the geodesics, which can be chosen to be coordinate curves. If r' is the coordinate along the geodesics then with these coordinate our spacetime has a metric tensor of the form $-A'c^2dt^2 + B'dr'^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$ and by spherical symmetry A', B', and r^2 are functions of only r'. Finally, in terms of r we get a metric tensor, g, of the form

$$-A(r)c^2dt^2 + B(r)dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) \tag{23.1}$$

To summarize, we have inferred, from the given mass distribution in spacetime - namely, a single mass point - that spacetime should have certain "symmetries", properties 1-3 above. Using the first two of these we have inferred that such a spacetime must have local coordinates in which g is represented by eq. (23.1). Notice that we are left with only two functions of r available to satisfy the 10 equations $G_{\alpha\beta} = 0$. Thus, that a solution *exists* is remarkable - though uniqueness (which we will get by using the third property) is not sursprising.

(There is another result on existence and uniqueness, Birkhoff's theorem, (Hawking and Ellis, Appendix B) which turns things around. It turns out that by assuming only the property of spherical symmetry we get *uniqueness*.)

Now with the form of g given by (23.1) in coordinates $(t, r, \vartheta, \varphi)$ we will show we can satisfy $G_{\alpha\beta} = 0$ and we will find the form of A and B. Since $G_{\beta\delta} = 0$ is equivalent to $\mathrm{Ric}_{\beta\delta} = 0$, we want to show we can satisfy the latter equation when we write the components

$$\mathrm{Ric}_{\beta\delta} = \frac{\partial}{\partial x^\alpha}\left\{ {\alpha \atop \beta\ \delta} \right\} - \frac{\partial}{\partial x^\delta}\left\{ {\alpha \atop \beta\ \alpha} \right\} + \left\{ {\alpha \atop \varepsilon\ \alpha} \right\}\left\{ {\varepsilon \atop \beta\ \delta} \right\} - \left\{ {\alpha \atop \varepsilon\ \delta} \right\}\left\{ {\varepsilon \atop \beta\ \alpha} \right\} \tag{23.2}$$

in terms of A and B. Thus, we first need to express the Christoffel symbols in terms of A and B. A straightforward procedure makes use of the definition of the Christoffel symbols in Section 16.1.

Since we will also be wanting the equations of the geodesics, an interesting indirect approach will "kill two birds with one stone" as well as considerably shorten the calculations. (This method is given in Adler, Bazin, and Schiffer. Other, more geometrical methods are given by Frankel and by O'Neill).

To get the equations of the geodesics, we first note that they are the extremals of the energy function defined in Section 15.2. This fact was essentially demonstrated in Theorem 16.5. Now we employ the useful result from the calculus of variations that the extremals of a function of the form $\int_{u_1}^{u_2} f(x^\alpha, d\lambda^\alpha/du)du$ can be obtained directly from f. Specifically, a curve, λ, with parameter u is an extremal of $\int_{u_1}^{u_2} f(x^\alpha, y^\alpha)du$ iff it satisfies

$$\frac{d}{du}\left(\frac{\partial f}{\partial y^\alpha}\right) = \frac{\partial f}{\partial x^\alpha} \tag{23.3}$$

when $x^\alpha = \lambda^\alpha$ and $y^\alpha = \dot{\lambda}^\alpha$. These are the Euler-Lagrange equations. (To be precise, certain minimum differentiability properties have to be assumed for f).

In our case, for the energy function, **E**, $f(x^\alpha, y^\alpha) = g(\dot\lambda, \dot\lambda)$. For nonnull curves we can use the arc length parameter, s, and with g given by eq. (23.1) with $A = e^\xi$ and $B = e^\eta$, and with $x^0 = ct,\ x^1 = r,\ x^2 = \vartheta,\ x^3 = \varphi$,

$$f\left(x^\alpha, \frac{d\lambda^\alpha}{ds}\right) = -e^\xi c^2 \left(\frac{dt}{ds}\right)^2 + e^\eta \left(\frac{dr}{ds}\right)^2 + r^2 \left[\left(\frac{d\vartheta}{ds}\right)^2 + \sin^2\vartheta \left(\frac{\delta\varphi}{ds}\right)^2\right] \quad (23.4)$$

Substituting (23.4) into (23.3) we get for $\alpha = 0, 1, 2, 3$, respectively, with $\eta' = d\eta/dr$ and $\xi' = d\xi/dr$,

$$\frac{d}{ds}\left(e^\xi c^2 \frac{dt}{ds}\right) = 0 \quad (23.5)$$

$$\frac{d}{ds}\left(2e^\eta \frac{dr}{ds}\right) = -e^\xi \xi' c^2 \left(\frac{dt}{ds}\right)^2 + e^\eta \eta' \left(\frac{dr}{ds}\right)^2 + 2r\left[\left(\frac{d\vartheta}{ds}\right)^2 + \sin^2\vartheta \left(\frac{d\varphi}{ds}\right)^2\right] \quad (23.6)$$

$$\frac{d}{ds}\left(r^2 \frac{d\vartheta}{ds}\right) = r^2 \sin\vartheta \cos\vartheta \left(\frac{d\varphi}{ds}\right)^2 \quad (23.7)$$

$$\frac{d}{ds}\left(r^2 \sin^2\vartheta \frac{d\varphi}{ds}\right) = 0 \quad (23.8)$$

Equations (23.5) - (23.8) are the equations of the (nonnull) geodesics of a space-time with a metric of the form (23.1).

Now compare eqs. (23.5) - (23.8) with the components of

$$\frac{d^2\lambda^\alpha}{ds^2} + \left\{ {\alpha \atop \beta\ \gamma} \right\} \frac{d\lambda^\beta}{ds}\frac{d\lambda^\gamma}{ds} = 0$$

We get

$$\left\{ {0 \atop 1\ 0} \right\} = \left\{ {0 \atop 0\ 1} \right\} = \frac{1}{2}\xi' \quad \text{from (23.5)}$$

$$\left.\begin{array}{ll} \left\{ {1 \atop 0\ 0} \right\} = \tfrac{1}{2}\xi' e^{\xi-\eta} & \left\{ {1 \atop 1\ 1} \right\} = \tfrac{1}{2}\eta' \\ \left\{ {1 \atop 2\ 2} \right\} = -re^{-\eta} & \left\{ {1 \atop 3\ 3} \right\} = -r\sin^2\vartheta\ e^{-\eta} \end{array}\right\} \quad \text{from (23.6)}$$

$$\left\{ {2 \atop 1\ 2} \right\} = \left\{ {2 \atop 2\ 1} \right\} = \frac{1}{r} \qquad \left\{ {2 \atop 3\ 3} \right\} = -\sin\vartheta\cos\vartheta \quad \text{from (23.7)}$$

and

$$\left\{ {3 \atop 1\ 3} \right\} = \left\{ {3 \atop 3\ 1} \right\} = \frac{1}{r} \quad \left\{ {3 \atop 2\ 3} \right\} = \left\{ {3 \atop 3\ 2} \right\} = \cot\vartheta \quad \text{from (23.8)}$$

and all other Christoffel symbols are zero.

Now we substitute these Christoffel symbols into (23.2) using, to simplify the second and third terms of (23.2),

$$\left\{ \begin{matrix} \alpha \\ \beta \quad \alpha \end{matrix} \right\} = \frac{\partial}{\partial x^\beta} \ln \sqrt{|\det(g_{\alpha\beta})|}$$

and similarly for $\left\{ \begin{matrix} \alpha \\ \varepsilon \quad \alpha \end{matrix} \right\}$, (Problem 16.1), and we get the components, $\mathrm{Ric}_{\beta\delta}$, of the Ricci tensor for a spacetime with the metric (23.1), namely,

$$\frac{2\,\mathrm{Ric}_{00}}{e^{\xi-\eta}} = \xi'' + \frac{1}{2}(\xi')^2 - \frac{1}{2}\eta'\xi' + \frac{2\xi'}{r} \tag{23.9}$$

$$2\,\mathrm{Ric}_{11} = \xi'' + \frac{1}{2}(\xi')^2 - \frac{1}{2}\eta'\xi' - \frac{2\eta'}{r} \tag{23.10}$$

$$-\mathrm{Ric}_{22} = (e^{-\eta}r)' - 1 + e^{-\eta}r\frac{\eta' + \xi'}{2} \tag{23.11}$$

$$\mathrm{Ric}_{33} = \mathrm{Ric}_{22}\ \sin^2\vartheta \tag{23.12}$$

and the remaining six components are identically zero.

Setting Ric $= 0$, from (23.9) and (23.10) we get $\eta' + \xi' = 0$. Using the condition (3) with which we started, that this spacetime $\to$ Minkowski spacetime for $r \to \infty$, we must have $\eta = -\xi$. Then from (23.11),

$$e^\xi = 1 - \frac{2m}{r} \quad \text{and} \quad e^\eta = \frac{1}{1 - (2m/r)}$$

where $2m$ is a constant of integration. Clearly, we must have $r > 2m$. Putting $A = e^\xi$ and $B = e^\eta$ into (23.1) we get "*the Schwarzschild line element*" for $r > 2m$,

$$-\left(1 - \frac{2m}{r}\right) c^2 dt^2 + \frac{dr^2}{1 - (2m/r)} + r^2(d\vartheta^2 + \sin^2\vartheta\ d\varphi^2) \tag{23.13}$$

Finally, we can express the constant of integration, $2m$, in terms of the mass M of the star, or particle, of the model. We suppose that the quantity $\mathbf{\Phi} = -(c^2/2)\varepsilon h_{00} = -(c^2/2)(g_{00} + 1)$ found in the small curvature and velocity case in Section 22.2 (eqs. (22.10) and (22.11)) *does* correspond to the Newtonian gravitational potential of M. Then if $\mathbf{g}$ is the gravitational constant, $\mathbf{\Phi} = -(\mathbf{g}M/r)$ and

$$\frac{c^2}{2}(g_{00} + 1) = \frac{\mathbf{g}M}{r}$$

But according to (23.13), $g_{00} = -1 + (2m/r)$, so $(c^2/2)(2m/r) = (\mathbf{g}M/r)$ or

$$2m = \frac{2\mathbf{g}M}{c^2} \tag{23.14}$$

which is called *the Schwarzschild radius.*

It was noted above that the formula (23.13) for the Schwarzschild line element was derived under the condition that $r > 2m$. We get an idea of the magnitude of the Schwarzschild radius, $2m$, by putting $(2\mathbf{g}/c^2) = 1.48 \times 10^{-28}$ cm/gm. Then $2m = 1.48 \times 10^{-28} M$. For the sun, $M = 2 \times 10^{33}$ gm, so in this case the radius is approximately 3×10^5 cm. Thus, for an "ordinary" star, the Schwarzschild radius is well inside the star, so $r > 2m$ is, for practical purposes, no restriction since we have already restricted ourselves to the exterior of the star. (We will make a model of the interior of certain types of stars in the next chapter.)

The trajectory of a test particle in Schwarzschild spacetime must satisfy the geodesic equations (23.5) - (23.8) with $e^{\xi} = e^{-\eta} = 1 - (2m/r)$. Also, from eq. (23.4), for a timelike geodesic,

$$-\left(1 - \frac{2m}{r}\right) c^2 \left(\frac{dt}{ds}\right)^2 + \frac{1}{1 - \frac{2m}{r}} \left(\frac{dr}{ds}\right)^2 + r^2 \left[\left(\frac{d\vartheta}{ds}\right)^2 + \sin^2\vartheta \left(\frac{d\varphi}{ds}\right)^2\right] = -1 \tag{23.15}$$

In Newtonian mechanics, a particle in a central force field moves in a plane. This suggests that we try a solution with $\vartheta = \pi/2$, the equatorial plane. This will satisfy eq. (23.7). Then eq. (23.5) integrates, with proper time $\tau = s/c$, to

$$\left(1 - \frac{2m}{r}\right) \frac{dt}{d\tau} = l = \text{ constant} \tag{23.16}$$

and eq. (23.8) integrates to

$$r^2 \frac{d\varphi}{d\tau} = h = \text{ constant} \tag{23.17}$$

and, with (23.16) and (23.17), eq. (23.15) reduces to

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{mc^2}{r} + \frac{h^2}{2r^2} - \frac{mh^2}{r^3} = \frac{(l^2 - 1)c^2}{2} \tag{23.18}$$

Comparing these results with the Newtonian case of planetary motion we see that eq. (23.17) is precisely the area integral, Kepler's second law, with h the angular momentum of the planet, and, with $mc^2/r = \mathbf{g}M/r$, eq. (23.18) is precisely the energy integral with the extra term mh^2/r^3 on the left. There are five qualitatively different classes of (trajectories of) particles in the exterior of an isolated star depending on the relative magnitudes of m, h, and l (Problem 23.4).

A photon is supposed to be a null geodesic, i.e., a solution of

$$\frac{d^2\lambda^\alpha}{du^2} + \left\{ \begin{matrix} \alpha \\ \beta \quad \gamma \end{matrix} \right\} \frac{d\lambda^\beta}{du}\frac{d\lambda^\gamma}{du} = 0 \tag{23.19}$$

with

$$g_{\alpha\beta}\frac{d\lambda^{\alpha}}{du}\frac{d\lambda^{\beta}}{du} = 0 \tag{23.20}$$

With f the same now as it was for a particle, except that the parameter is now u instead of s, eqs. (23.5) - (23.8) with s replaced by u describe the trajectory of a photon. In addition, in this case (23.4) is

$$-\left(1-\frac{2m}{r}\right)c^2\left(\frac{dt}{du}\right)^2 + \frac{1}{1-(2m/r)}\left(\frac{dr}{du}\right)^2 + r^2\left[\left(\frac{d\vartheta}{du}\right)^2 + \sin^2\vartheta\left(\frac{d\varphi}{du}\right)^2\right] = 0 \tag{23.21}$$

Again we put $\vartheta = \pi/2$. From (23.5) we get (23.16) with τ replaced by u, and from (23.8) we get (23.17) with τ replaced by u, and putting these in (23.21) we get

$$c^2 l^2 - \left(\frac{dr}{du}\right)^2 = \left(1-\frac{2m}{r}\right)\frac{h^2}{r^2} \tag{23.22}$$

or, using (23.17) with τ replaced by u,

$$\frac{c^2 l^2}{h^2} - \frac{1}{r^2}\left(\frac{dr}{d\varphi}\right)^2 = \frac{1}{r^2}\left(1-\frac{2m}{r}\right) \tag{23.23}$$

so that for photons, in contrast to the case for particles, the family of trajectories can be written in terms of a single parameter, c^2l^2/h^2.

PROBLEM 23.1 Prove Theorem 23.1. (Hint: The local flow $\Theta : I \times \mathcal{U} \to M$ of X induces a diffeomorphism of a neighborhood $\mathcal{V} \subset \mathcal{U}$ of a point of an orthogonal hypersurface, S, with $I \times S$, and Θ_t are local isometries of M. (See O'Neill, p. 361.)

PROBLEM 23.2 Show that if we allow A and B to be functions of t, as well as of r, in our initial form of the metric tensor, eq. (23.1), the vanishing of Ric forces us back to their independence of t.

PROBLEM 23.3 Show that $\mathrm{Ric}_{\alpha\beta} = 0$ for $\alpha \neq \beta$, for the metric tensor (23.1).

PROBLEM 23.4 Examine the possible cases of the trajectories of particles and photons in the exterior of a star in Schwarzschild's solution (cf., Robertson and Noonan, p. 242 ff, or O'Neill, p. 374 ff).

23.2 Two applications of Schwarzschild's solution

Einstein proposed three tests of general relativity. One was to predict the redshift of light from the sun or other star, which we discussed in Section 22.2. The other two are applications of Schwarzschild's solution.

(i) The precession of the orbit of Mercury

In terms of the parameter φ instead of τ, eq. (23.18) becomes

$$\frac{h^2}{2r^4}\left(\frac{dr}{d\varphi}\right)^2 - \frac{mc^2}{r} + \frac{h^2}{2r^2} - \frac{mh^2}{r^3} = \frac{(l^2-1)c^2}{2}$$

Then, putting $r = 1/\omega$ and differentiating with respect to φ, we get either

$$\frac{d\omega}{d\varphi} = 0 \tag{23.24}$$

or

$$\frac{d^2\omega}{d\varphi^2} + \omega = \frac{mc^2}{h^2} + 3m\omega^2 \tag{23.25}$$

The trajectories of (23.24) are r = constant. To find the trajectories described by (23.25), we solve (23.25) by a perturbation method, since $3m\omega^2$ is small compared with mc^2/h^2. Hence we write (23.25) as

$$\frac{d^2\omega}{d\varphi^2} + \omega = C + \frac{\varepsilon}{C}\omega^2$$

where $C = mc^2/h^2$ and $\varepsilon = 3m^2c^2/h^2$ and we look for a solution in the form $\omega(\varphi, \varepsilon) = \omega_0 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \cdots$.

Then

$$\frac{d^2\omega_0}{d\varphi^2} + \omega_0 = C \tag{23.26}$$

$$\frac{d^2\omega_1}{d\varphi^2} + \omega_1 = \frac{\omega_0^2}{C} \tag{23.27}$$

From (23.26), $\omega_0 = C + D\cos(\varphi + \delta)$, where D and δ are arbitrary constants, and putting this into (23.27) with $\delta = 0$ and solving we get a particular solution,

$$\omega_1 = C + \frac{D^2}{2C} + D\varphi\sin\varphi - \frac{D^2}{6C}\cos 2\varphi$$

To first order

$$\frac{1}{r} = \omega = \omega_0 + \varepsilon\omega_1 = C + \varepsilon\left(C + \frac{D^2}{2C}\right) + D\cos\varphi + \varepsilon\left(D\varphi\sin\varphi - \frac{D^2}{6C}\cos 2\varphi\right)$$

or

$$\frac{1}{r} = C + D\cos(\varphi - \varepsilon\varphi) + \varepsilon\left(C + \frac{D^2}{2C} - \frac{D^2}{6C}\cos 2\varphi\right)$$

Now if ε were zero, the trajectory would be a conic. The effect of the last term is to perturb the radial distance periodically. The ε in the middle term destroys the periodicity of the orbit. The maximum value of this term (giving a minimum for $r = 1/\omega$) occurs now not for φ a multiple of 2π, but for $\varphi - \varepsilon\varphi$ a multiple of 2π. Thus, two successive minima of r occur at $\varphi_1 - \varepsilon\varphi_1 = 2\pi n$ and $\varphi_2 - \varepsilon\varphi_2 = 2\pi n + 2\pi$ or $(\varphi_2 - \varphi_1)(1 - \varepsilon) = 2\pi$, or the change in angle is approximately $\Delta\varphi = 2\pi(1 + \varepsilon)$.

Finally, putting $\varepsilon = 3m^2c^2/h^2$, and using the Newtonian approximation $2m = 2\mathbf{g}M/c^2$, we get an advance in the perihelion of the orbit of $6\pi(\mathbf{g}^2M^2/h^2c^2)$ radians. All these quantities can be determined for the case of Mercury moving in the sun's gravitational field - we get an advance of 42 - 43 s. of arc/century (cf., Problem 23.6).

(ii) The bending of a light ray passing the sun

Proceeding as before in (i) starting with (23.23) instead of (23.18),

$$\frac{d^2\omega}{d\varphi^2} + \omega = 3m\omega^2 \tag{23.28}$$

instead of (23.25). $3m\omega^2$ is evidently small compared with ω, so putting $\varepsilon = 3m$, and $\omega(\varphi, \varepsilon) = \omega_0 + \omega_1\varepsilon + \cdots$, we get

$$\frac{d^2\omega_0}{d\varphi^2} + \omega_0 = 0 \tag{23.29}$$

$$\frac{d^2\omega_1}{d\varphi^2} + \omega_1 = \omega_0^2 \tag{23.30}$$

Equation (23.29) gives

$$\omega_0 = D\cos\varphi \tag{23.31}$$

and putting this into (23.30) we get

$$\omega_1 = \frac{2}{3}D^2 - \frac{1}{3}B^2\cos^2\varphi$$

So, to first order, the trajectory is

$$\frac{1}{r} = D\cos\varphi + \frac{\varepsilon D^2}{3}(2 - \cos^2\varphi).$$

(Fig. 23.1). Note that if $\varepsilon = 0$, then the trajectory is a straight line. With the extra term we get a curve symmetric about $\varphi = 0$ and as r goes to infinity, φ goes to a solution of

$$0 = D\cos\varphi + \frac{\varepsilon D^2}{3}(2 - \cos^2\varphi)$$

from which, to first order in ε,

$$\cos\varphi = -\frac{2\varepsilon}{3}D = -2mD$$

so $\varphi = \pm(\pi/2 + 2mD)$ and the angle between the asymptotes is $4mD = (4\mathrm{g}M/c^2)D$. From (23.31). $1/D$ is the minimum value of r in the solution of (23.29). For a light ray which just grazes the sun this gives about 2 s. of arc.

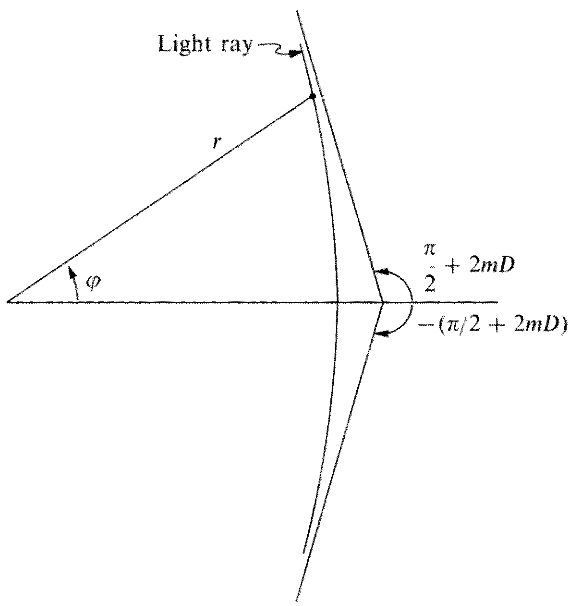

Figure 23.1

PROBLEM 23.5 Solve eq. (23.25) without the second-degree term on the right to get the Newtonian orbits

$$r = \frac{h^2/mc^2}{1 + (Ah^2/mc^2)\cos\varphi}$$

where $A \geq 0$ is a constant of integration. Thus, the orbits are ellipses with eccentricity Ah^2/mc^2 (Kepler's first law), and from this equation h^2 is obtained from the geometry (eccentricity and principal axes) of the orbit.

PROBLEM 23.6 Look up astronomical data needed in the formulas given for the perihelion advance and the bending of light rays and confirm the values given in our applications.

23.3 The Kruskal extension of Schwarzschild spacetime

In Section 23.1 we obtained a Ricci flat spacetime for $r > 2m$, where m is a small positive number. The metric tensor (23.13) gives a Ricci flat spacetime for $0 < r < 2m$, also. We can show this most easily by putting $A = -e^{\xi}$ and $B = -e^{\eta}$ instead of the substitution above eq. (23.4). Then $\mathrm{Ric}_{22} = 0$ becomes $(e^{-\eta}r)' + 1 = 0$, so $e^{-\eta} = -1 + 2m/r = e^{\xi}$, and we get the same form as before.

Equations (23.5), (23.7), and (23.8) are the same as before, and eq. (23.15) is still valid for a timelike geodesic, so a test particle still has eqs. (23.16) - (23.18), and a photon has eqs. (23.16) and (23.17) with u in place of τ and eq. (23.22).

Thus, we have a spacetime on $0 < r < 2m$ and a spacetime on $r > 2m$. Alternatively, we can say we have a spacetime on $r > 0$ with a singularity at $r = 2m$. As we have seen in Section 23.1, we have a physical interpretation only for r considerably greater than $2m$. Nevertheless, the occurrence of a singularity provoked considerable interest in its significance. With that in mind, and with a certain prescience, investigators asked what would happen if there *were* an isolated body small and massive enough so that it lay inside the Schwarzschild radius. That there actually *are* such bodies, the end states of gravitational collapse, or "black holes," has been generally recognized only very recently. We will see that while $r = 2m$ is not a singularity (of a slightly modified spacetime), it *does* have an important physical significance.

To see what happens near $r = 2m$ we can restrict ourselves to examining the behavior of particles and photons moving radially; that is, with ϑ and φ constant and, hence, $h = 0$. The equations governing the radial motion of photons are, from eq. (23.16), $(1 - 2m/r)dt/du = l$, and from eq. (23.22), $(dr/du)^2 = c^2l^2$, and from these two we get

$$\pm\, c\frac{dt}{dr} = \frac{1}{1-(2m/r)} \tag{23.32}$$

so

$$\pm\, ct = r + 2m\ln|r-2m| \;+\; \text{constant} \tag{23.33}$$

From Fig. 23.2 it is evident that for $r > 2m$ no timelike curve (in particular, no particle) can approach $r = 2m$ in finite time, t. In physical terms, for an observer at rest with respect to the small massive isolated body and watching a particle fall toward the body, the particle will never pass $r = 2m$. (Also, if in the discussion of the redshift at the end of Section 22.2, s and e are, respectively, the particle and the observer, the relation

$$\frac{\nu_s - \nu_e}{\nu_e} = \frac{\sqrt{-g_{00}(s)} - \sqrt{-g_{00}(e)}}{\sqrt{-g_{00}(s)}}$$

shows that the observer will see a dramatic red shift of the light coming from the particle as it approaches $r = 2m$.) This result can also be gotten analytically from the differential equation for r as a function of t obtained from (23.16) and (23.18). From that equation we get for timelike curves that also for $r < 2m$ as $r \to 2m$, $t \to \pm\infty$. Since for $r < 2m$ the r coordinate curves are timelike and the t coordinate curves are spacelike, in this region a particle travels between a spacelike coordinate value, t, of plus or minus infinity and a finite value during a coordinate time interval $0 < r < 2m$.

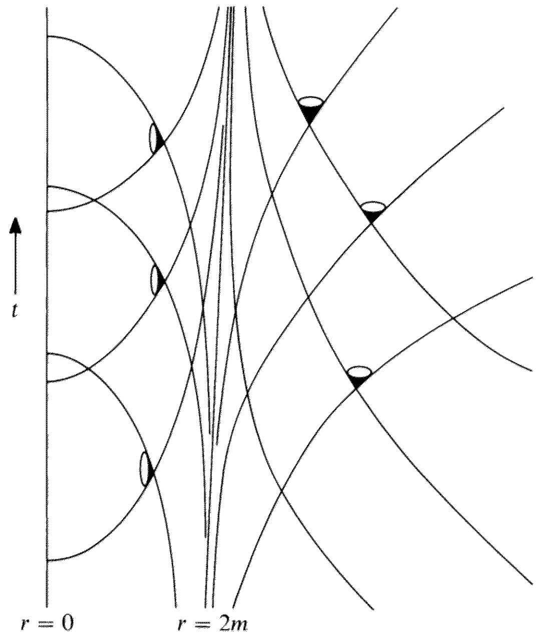

Figure 23.2

Equation (23.16) gives us a clue that the fact that for an observer at rest with respect to the body a particle never crosses $r = 2m$ could be a consequence of what happens to the observer's time more than of what happens to the particle. Hence, instead of examining r as a function of t as we did above, we go to eq. (23.18) and examine r as a function of τ. A straightforward integration for $h = 0$ shows that for all l, there are solutions of (23.18) for which r decreases continuously with increasing τ and which reach $r = 0$ at a finite τ. In particular, there is no problem at $r = 2m$.

Moreover, if we compute $\det(g_{\alpha\beta})$, the sectional curvatures, and other invariants of g and R, we find no problem at $r = 2m$, all of which suggests trying a coordinate change.

Using the minus sign in eq. (23.33) we get the incoming null curves, and clearly these will be straightened out if we put

$$c\bar{t} = ct + r + 2m \ln |r - 2m|, \quad \bar{r} = r \tag{23.34}$$

(Fig. 23.3). With these coordinates, (23.13) becomes

$$-\left(1-\frac{2m}{\bar{r}}\right)c^2\ d\bar{t}^2+2\ d\bar{t}\ d\bar{r}+\bar{r}^2(d\vartheta^2+\sin^2\ \vartheta\ d\varphi^2) \qquad (23.35)$$

The singularity is gone. The null curves are $\bar{t}$-constant and the solutions of

$$\frac{d\bar{t}}{d\bar{r}}=\frac{2}{c[1-(2m/\bar{r})]} \qquad (23.36)$$

(cf., eq. (23.32)).

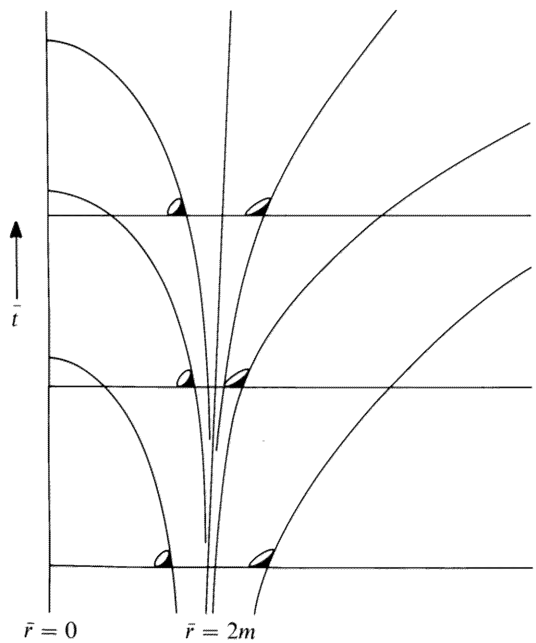

Figure 23.3

Equation (23.35) can be considered to be a coordinate representation of the metric tensor of a spacetime into which we have isometrically imbedded the two pieces of Schwarzschild spacetime. Similarly, by choosing the plus sign in Eq. (23.33) instead of the minus sign, the outgoing null curves can be straightened out and we get a spacetime without a singularity and containing the two Schwarzschild pieces. These two spacetimes are called *Eddington-Finkelstein spacetimes*. For more details, including their motivation for a further extension, *the Kruskal* (or *Kruskal-Szekeres*) *spacetime*, see either Hawking and Ellis (pp. 150-156), or Misner et al. (pp. 828-832).

The Eddington-Finkelstein spacetimes have the undesirable feature that $r = 2m$ is a null curve. We can avoid that if we can find coordinates (or an isometry) for which the metric tensor has the form

$$F(\bar{t},\bar{r})(-d\bar{t}^2 + d\bar{r}^2) + G(\bar{t},\bar{r})(d\vartheta^2 + \sin^2\ \vartheta\ d\varphi^2) \tag{23.37}$$

Kruskal found a transformation $\psi : (\bar{t},\bar{r}) \mapsto (\psi_1(\bar{t},\bar{r}), \psi_2(\bar{t},\bar{r})) = (t,r)$ from a region of $\mathbb{R}^2$ symmetric about $(0,0)$ with $\psi(\bar{t},\bar{r}) = \psi(-\bar{t},-\bar{r})$ onto $\mathbb{R} \times \mathbb{R}^+$, and such that $F(\bar{t},\bar{r}) = f^2(\psi_2(\bar{t},\bar{r})) = f^2(r)$ and $G(\bar{t},\bar{r}) = \psi_2^2(\bar{t},\bar{r}) = r^2$. A fairly straightforward approach is to solve the partial differential equations

$$g_{\alpha\beta} = \frac{\partial\bar{\mu}_\gamma}{\partial\mu^\alpha}\ \frac{\partial\bar{\mu}_\delta}{\partial\mu^\beta}\ \bar{g}_{\gamma\delta}$$

for ψ^{-1} (locally). (See Adler et al., pp. 226-230.) The result is

$$\bar{t} = \bar{t}_I(t,r) = \sqrt{\frac{r}{2m} - 1}\ e^{r/4m}\ \sinh\ \frac{ct}{4m}$$

$$\bar{r} = \bar{r}_I(t,r) = \sqrt{\frac{r}{2m} - 1}\ e^{r/4m}\ \cosh\ \frac{ct}{4m}$$

for $r \geq 2m$

$$\bar{t} = \bar{t}_{II}(t,r) = \sqrt{1 - \frac{r}{2m}}\ e^{r/4m}\ \cosh\ \frac{ct}{4m}$$

$$\bar{r} = \bar{r}_{II}(t,r) = \sqrt{1 - \frac{r}{2m}}\ e^{r/4m}\ \sinh\ \frac{ct}{4m}$$

for $r < 2m$, and

$$f^2(r) = \frac{32m^3}{r}\ e^{-r/2m}$$

for $r > 0$. Note that the region $-\infty < t < \infty,\ r > 0$ maps onto points of the $\bar{t},\bar{r}$ plane with $\bar{t}+\bar{r} > 0$, and $\bar{t}+\bar{r} = 0$ corresponds to $t = -\infty$, and $\bar{t} = \bar{r}$ corresponds to $t = +\infty$.

By the symmetry with respect to the origin,

$$\bar{t} = \bar{t}_{I'}(t,r) = -\bar{t}_I(t,r)$$

$$\bar{r} = \bar{r}_{I'}(t,r) = -\bar{r}_I(t,r)$$

for $r \geq 2m$, and

$$\bar{t} = \bar{t}_{II'}(t,r) = -\bar{t}_{II}(t,r)$$

$$\bar{r} = \bar{r}_{II'}(t,r) = -\bar{r}_{II}(t,r)$$

for $r < 2m$ with the same f, is also a solution.

The entire Kruskal extension is shown in Fig. 23.4. The $r =$ constant curves are hyperbolas and the $t =$ constant curves are straight lines in the $(\bar{t},\bar{r})$ plane.

From (23.27) we see that the null lines in a ϑ = constant, φ = constant section have slope ± 1. A ray of light or a particle starting at p in Region I, "the Normal Schwarzschild region," heading for $r = 0$ has to pass $t = \infty$ as we have already seen. We make the additional observation that every ray of light, or particle, starting at a point q inside the Schwarzschild radius has to end up in either Region I or I', but a ray of light or particle going into Region I from point q will never be seen by an observer in Region I, since it has to go through $t = -\infty$.

PROBLEM 23.7 Show that the sectional curvatures of Schwarzschild spacetime are all proportional to $1/r^3$. (But they are not all the same.)

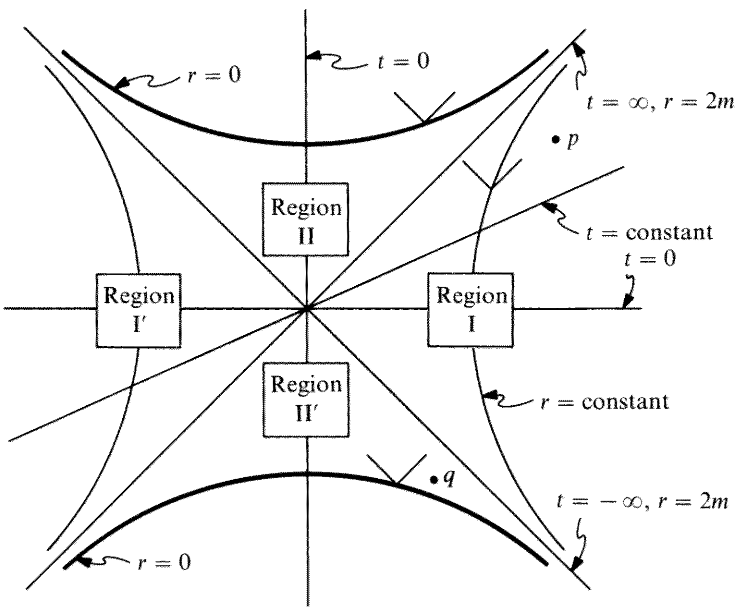

Figure 23.4

23.4 The field of a rotating star

In recent years, generalization of the Schwarzschild–Kruskal solution have been found. These give fields due to charged massive bodies, rotating massive bodies, and charged rotating massive bodies (Kerr-Newman geometry; Misner, Thorne, and Wheeler, p. 877 ff).

The Kerr solution for a rotating massive body has a coordinate representation for the metric tensor (in Boyer and Lindquist coordinates),

$$\begin{aligned} -&\left(1 - \frac{2mr}{r^2 + a^2\cos^2\vartheta}\right) c^2\, dt^2 + \frac{r^2 + a^2\cos^2\vartheta}{r^2 - 2mr + a^2} dr^2 + (r^2 + a^2\cos^2\vartheta) d\vartheta^2 \\ &-2\,\frac{2mar\sin^2\vartheta}{r^2 + a^2\cos^2\vartheta} c\, dt\, d\varphi + \left[r^2 + a^2 + \frac{2ma^2 r\sin^2\vartheta}{r^2 + a^2\cos^2\vartheta}\right] \sin^2\vartheta\, d\varphi^2 \qquad (23.38) \end{aligned}$$

One can verify that this satisfies Ric $= 0$. We can note also that (1) when $a = 0$, (23.38) reduces to the Schwarzschild solution (23.13); (2) (23.38) is time-independent and axially symmetric; (3) if the signs of a and φ are simultaneously changed, (23.38) does not change, which suggests that a corresponds to a rotation; and (4) the cross-product term $dt\, d\varphi$ which appears corresponds to what we get with a rotating coordinate system in flat space.

We will skip (1) a derivation of (23.38) and (2) further justification for identifying the parameter a with a rotation. Both involve considerable calculations, the latter involving a comparison with an approximate solution (see Adler et al., Chap. 7). We will content ourselves with observing a sort of generalization of the Schwarzschild radius.

For $a^2 > m^2$ the coefficients in (23.38) have no singularities, so this spacetime could model a rotating star. For $a^2 < m^2$ a singularity occurs at

$$r^2 - 2mr + a^2 = 0 \qquad (23.39)$$

This represents two hyperspheres

$$r_+ = m + \sqrt{m^2 - a^2} \quad \text{and} \quad r_- = m - \sqrt{m^2 - a^2}$$

both with $r < 2m$, so (23.38) could still model a rotating star. But, as in the Schwarzschild case, it can also model a black hole. The Kerr solution can be extended to a spacetime which contains it without the singularity, but, like in the Schwarzschild case, the hypersurfaces where the singularity occurred, the two hyperspheres, still have physical significance.

Note, first of all, that the hypersurface

$$r^2 - 2mr + a^2\cos^2\vartheta = 0 \qquad (23.40)$$

where the coefficient of dt^2 vanishes is now, in contrast to the Schwarzschild metric, (23.13), different from (23.39). At the Schwarzschild radius several things happened:

(i) The t coordinate curves changed from timelike to spacelike.

(ii) The redshift became infinite.

(iii) Photons and particles could only travel inward.

Now, clearly the first two phenomena occur on (23.40). We will show that now the third phenomenon occurs on (23.39) and another interesting phenomenon occurs on (23.40).

Theorem 23.2 *At each point of the hypersurface $r^2 - 2mr + a^2 = 0$ the tangent space of the hypersurface has a 1-dimensional null subspace and all the other vectors are spacelike, i.e., the tangent space of $r^2 - 2mr + a^2 = 0$ is tangent to the null cone.*

Proof Let $f(r) = r^2 - 2mr + a^2$. Then grad f is a normal vector to $f(r) = 0$. In coordinates

$$g(\text{grad } f, \text{grad } f) = g^{ij}\frac{\partial f}{\partial \mu^i}\frac{\partial f}{\partial \mu^j}$$

so in our case

$$g(\text{grad } f, \text{grad } f) = \frac{r^2 - 2mr + a^2}{r^2 + a^2\cos^2\vartheta}(2r - 2m)^2$$

and on $f(r) = 0$ this vanishes. Thus, we have a normal vector which satisfies the criterion of Problem 5.25 for our conclusion. □

Corollary *A particle or photon can cross (23.29) in only one direction.*

Now consider the hypersurface (23.40). It has two parts,

$$r = m \pm \sqrt{m^2 - a^2\cos^2\vartheta}$$

We will denote the outer part by r_∞.

Definitions r_∞ is called *the stationary* (or, *static*) *limit surface.* r_+ is called *an event horizon*, and the region between them is called *the ergosphere* (Fig. 23.5).

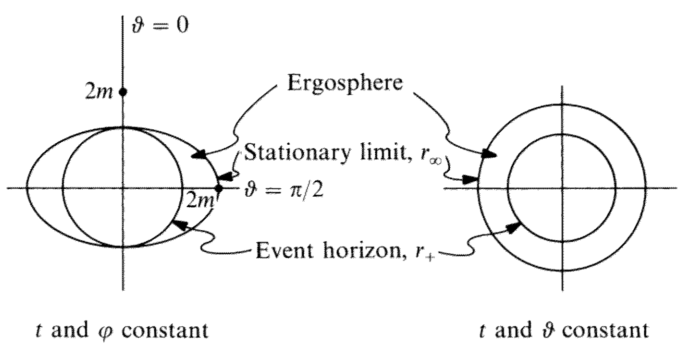

Figure 23.5

Consider curves in this spacetime parametrized by t in the coordinate slice r and ϑ constant. They will have tangent vectors $\partial/\partial t + (d\varphi/dt)\ \partial/\partial\varphi$. They can be visualized as coming out of the paper from the t and ϑ constant slice and lying on circular cylinders centered at the origin.

Theorem 23.3 *At each point of the ergosphere there are timelike and null curves parametrized by t in the coordinate slice r and ϑ constant, and for all such curves $d\varphi/dt > 0$.*

Proof The condition the tangent vectors, v, of our curves have to satisfy is $g(v, v) < 0$ for timelike and $g(v, v) = 0$ for null curves. In our case this condition is

$$g\left(\frac{\partial}{\partial t} + \frac{d\varphi}{dt}\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial t} + \frac{d\varphi}{dt}\frac{\partial}{\partial\varphi}\right) = g_{00} + 2g_{03}\frac{d\varphi}{dt} + g_{33}\left(\frac{d\varphi}{dt}\right)^2 \tag{23.41}$$

is negative or zero, respectively. The discriminant, $g_{03}^2 - g_{00}g_{33}$, of the quadratic (23.41) is positive for $r > r_+$, Problem 23.8. Since $g_{33} > 0$, the parabola $y = g_{00} + 2g_{03}x + g_{33}x^2$ opens upward, so the required condition is satisfied. For $r > r_\infty$, $g_{00} < 0$ so there are both negative and positive possible values for $d\varphi/dt$. In particular, we can have $d\varphi/dt = 0$. However, for $r < r_\infty$, since $g_{00} > 0$, all possible values of $d\varphi/dt$ must be positive. In particular, $d\varphi/dt$ cannot be zero. □

In physical terms, the fact that in the ergosphere $d\varphi/dt$ cannot be zero is interpreted to mean that particles in the ergosphere cannot be at rest with respect to the rotating body, but are dragged along with its rotation. There is a further argument that this phenomenon could account, at least in part, for the source of the large energy detected in some stellar observations.

PROBLEM 23.8 Show that the discriminant of (23.41) is positive for $r > r_+$.

24

NONEMPTY SPACETIMES

We will obtain two types of solutions of Einstein's equation, (22.27); a model of the interior of a star, and a class of models of the universe as a whole, a class of cosmologies.

24.1 Schwarzschild's interior solution

To make a model of the interior of a star, we start with Einstein's equation in the form (22.22),

$$\mathrm{Ric}_{\alpha\beta} = a(T_{\alpha\beta} - \frac{1}{2}T^{\lambda}_{\lambda}g_{\alpha\beta}) \tag{24.1}$$

with $a = 8\pi\mathbf{g}/c^2$.

We assume our medium is a perfect fluid with no external forces. Then from eqs. (21.40) and (21.47),

$$T_{\alpha\beta} = \rho\frac{U_\alpha U_\beta}{c^2} + \frac{p}{c^2}\left(\frac{U_\alpha U_\beta}{c^2} + g_{\alpha\beta}\right) \tag{24.2}$$

Recall that the velocity field, U, by definition, Section 21.4, is the field of a particle flow, so, since for a particle, λ, $g(\dot{\lambda}, \dot{\lambda}) = -c^2$, we get $g_{\alpha\beta}U^\alpha U^\beta = -c^2$, and from (24.2), $T_{\alpha\beta}U^\beta = -\rho U_\alpha$.

Now we can take U as a coordinate vector field, $\partial/\partial x^0$. Then $U^i = 0$, $i = 1, 2, 3$, the fluid particles are at rest with respect to such a coordinate system, or the coordinates are "comoving". Further, in this case we have $g_{00}(U^0)^2 = -c^2$ and $U_0^2 = -g_{00}c^2$, so

$$T_{00} = -\rho g_{00}, \quad T^0_0 = -\rho, \quad T^1_1 = T^2_2 = T^3_3 = \frac{p}{c^2}$$

and

$$T^i_\alpha = 0, \quad \alpha \neq i, \quad \alpha = 0, 1, 2, 3, \quad i = 1, 2, 3$$

For the model of a star we assume our stress-energy tensor, T, is spherically symmetric, so the resulting spacetime would be expected to be spherically symmetric also, as in Schwarzschild's exterior solution.

We assume further that the velocity field, U, is irrotational so that, by Problem 22.3 it has orthogonal hypersurfaces, rest spaces.

Finally, as we noted in Problem 23.2 we do not need the additional assumption that U is a Killing field, i.e., $g_{\alpha\beta}$ are independent of t, but just as before it will turn out that U will have to be one, so we start with a metric tensor of the form (23.1), as for Schwarzschild's exterior solution:

$$g = -e^{\xi}c^2dt^2 + e^{\eta}dr^2 + r^2(d\vartheta^2 + \sin^2\ \vartheta\ d\varphi^2) \tag{24.3}$$

where ξ and η are functions of only r.

The components of Ric will be given by eqs. (23.9) - (23.12) just as before. By the orthogonality, $U_i = 0$, and by the results below eq. (24.2),

$$T_{00} - \frac{1}{2}g_{00}T^{\lambda}_{\lambda} = \frac{e^{\xi}}{2}\left(\rho + \frac{3p}{c^2}\right) \tag{24.4}$$

$$T_{11} - \frac{1}{2}g_{11}T^{\lambda}_{\lambda} = \frac{e^{\eta}}{2}\left(\rho - \frac{p}{c^2}\right) \tag{24.5}$$

$$T_{22} - \frac{1}{2}g_{22}T^{\lambda}_{\lambda} = \frac{1}{2}\left(\rho - \frac{p}{c^2}\right)r^2 \tag{24.6}$$

With eqs. (24.4) - (24.6) and the components of Ric given by eqs. (23.9) - (23.11), the $(0,0), (1,1)$, and $(2,2)$ components of the field equation (24.1) are, respectively,

$$\frac{e^{\xi-\eta}}{2}\left[\xi'' + \frac{1}{2}(\xi')^2 - \frac{1}{2}\xi'\eta' + \frac{2\xi'}{r}\right] = a\frac{e^{\xi}}{2}\left(\rho + \frac{3p}{c^2}\right) \tag{24.7}$$

$$-\frac{1}{2}\left[\xi'' + \frac{1}{2}(\xi')^2 - \frac{1}{2}\xi'\eta' - \frac{2\eta'}{r}\right] = a\frac{e^{\eta}}{2}\left(\rho - \frac{p}{c^2}\right) \tag{24.8}$$

$$1 - (re^{-\eta})' - re^{-\eta}\ \frac{\xi' + \eta'}{r} = \frac{a}{2}\left(\rho - \frac{p}{c^2}\right)r^2 \tag{24.9}$$

We can combine these conditions in various ways. Thus, adding $e^{\xi-\eta}$ times the $(1,1)$ component and the $(0,0)$ component, we get

$$a\left(\rho + \frac{p}{c^2}\right) = e^{-\eta}\frac{\eta' + \xi'}{r} \tag{24.10}$$

If we solve (24.10) and the $(2,2)$ components for ρ and p, the density and pressure, we get

$$a\rho = \frac{1}{r^2} - e^{-\eta}\left(\frac{1}{r^2} - \frac{\eta'}{r}\right) \tag{24.11}$$

and

$$a\frac{p}{c^2} = -\frac{1}{r^2} + e^{-\eta}\left(\frac{1}{r^2} + \frac{\xi'}{r}\right) \tag{24.12}$$

and we see, in particular, that ρ, and p are functions of only r. To eqs. (24.10) - (24.12) we can adjoin the condition $T^{\alpha\beta}{}_{,\beta} = 0$. From eq. (22.29) we get

$$\frac{p'}{c^2} = -\frac{\xi'}{2}\left(\rho + \frac{p}{c^2}\right) \tag{24.13}$$

At this point, in order to proceed, we realize that just as in ordinary fluid mechanics we need another piece of physical information, namely, *an equation of state* for the medium. If we know ρ as a function of p, then we can integrate (24.13) and get

$$-\frac{\xi}{2} = \int \frac{dp}{c^2\left(\rho + \dfrac{p}{c^2}\right)} + \text{ const.} \tag{24.14}$$

With ρ as a function of p and ξ as a function of p, we can, in principle, find p as a function of r from (24.13) and then find ρ, ξ, and η.

Alternatively, if we know ρ as a function of r we can integrate eq. (24.11) to get

$$e^{-\eta} = 1 - \frac{1}{r}\int_0^r \rho\tilde{r}^2\, d\tilde{r} \tag{24.15}$$

assuming e^{η} does not go to zero with r. Then we can find ξ and p from the remaining equations.

To proceed further, we assume ρ is constant. (This may not be very realistic, but it allows us to go ahead and illustrate a development which could be carried out with more technical difficulties under a more realistic assumption.) Now eq. (24.14) becomes

$$-\frac{\xi}{2} = \ln\left(\rho + \frac{p}{c^2}\right) + \text{ const.}$$

and using eq. (24.10) we get

$$Cre^{-\xi/2} = e^{-\eta}(\eta' + \xi') \tag{24.16}$$

where C is an arbitrary constant. With ρ constant eq. (24.15) becomes

$$e^{-\eta} = 1 - \frac{r^2}{r_0^2} \tag{24.17}$$

where $r_0^2 = 3/a\rho$. Finally, substituting (24.17) into (24.16) we get a first-order linear differential equation for $e^{\xi/2}$ with the solution

$$e^{\xi/2} = A - B\left(1 - \frac{r^2}{r_0^2}\right)^{1/2} \tag{24.18}$$

where $A = (C/2)r_0^2$ and B is another arbitrary constant. In eqs. (24.17) and (24.18) we have finally arrived at the form of the coefficients of *the Schwarzschild* *interior line element.* Note that these functions of r are defined only for $0 \leq r < r_0$. With $a = 8\pi\mathbf{g}/c^2$,

$$r_0 = \left(\frac{3}{8\pi\mathbf{g}}\right)^{1/2} c\,\frac{1}{\rho^{1/2}} \approx 0.40 \times 10^{14}\frac{1}{\rho^{1/2}}\ cm$$

Now we have two spherically symmetrical models of spacetime: Schwarzschild's exterior solution due to a point mass, M, Chapter 23, and Schwarzschild's interior solution due to a mass/energy distribution with ρ constant. We can use these two solutions to construct a third model, one in which $\rho = \text{constant} > 0$ for $r \leq \hat{r}$, and $\rho = 0$ (and hence $T = 0$) for $r > \hat{r}$. This could be thought of as a model of a star of "radius" $\hat{r}$. If $\hat{r} < r_0$, then for points $r \leq \hat{r}$ we can use the interior solution we have just found. For $r > \hat{r}$ we assume that the line element of the star is the same as that of some equivalent point mass, M, such that $2\mathbf{g}M/c^2 < \hat{r}$; i.e., such that $\hat{r}$ is greater than the Schwarzschild radius of the point mass.

It would seem desirable that this metric tensor field, or, equivalently, the functions ξ and η, described differently on different parts of spacetime, should be continuous on the common boundary of these parts. Similarly for the fourth function, p. It turns out that we can achieve this by making suitable choices of the available constants. Thus, equating e^η given by (24.17) with the expression for e^η in Section 23.1 at $r = \hat{r}$ we get

$$M = \frac{4}{3}\pi\hat{r}^3\rho \tag{24.19}$$

That is, given ρ and $\hat{r}$, we must choose a point mass M given by (24.19). To make p continuous on the boundary we assume that it goes to zero as r goes to $\hat{r}$ from the interior. Then $\rho a/C = e^{-\xi/2}$ at $r = \hat{r}$, or $e^{\xi/2} = \frac{2}{3}A$ on $r = \hat{r}$. Substituting this into (24.18) we get

$$A = 3B\left(1 - \frac{\hat{r}^2}{r_0^2}\right)^{1/2} \tag{24.20}$$

Finally, equating e^ξ given by (24.18), using (24.20), with the expression for e^ξ given in Section 23.1 at $r = \hat{r}$, we get $B = 1/2$. With these values of A and B we have, on the interior,

$$e^\xi = \frac{1}{4}\left[3\left(1 - \frac{\hat{r}^2}{r_0^2}\right)^{1/2} - \left(1 - \frac{r^2}{r_0^2}\right)^{1/2}\right]^2 \tag{24.21}$$

Now that we have a model, we can try to see what we can get out of it. In particular, since we have made certain simplifying assumptions, it is somewhat reassuring to be able to derive the following interesting and credible result. Note, first of all, that since r is not a distance, the factor $\frac{4}{3}\pi\hat{r}^3$ in (24.19) is not a volume.

We saw, in Section 15.1, that in terms of coordinates r, ϑ, φ the volume form of a $t =$ constant hypersurface is

$$\sqrt{|\det(g_{ij})|}\; dr \wedge d\vartheta \wedge d\varphi = \left(1 - \frac{r^2}{r_0^2}\right)^{1/2} r^2 \sin\vartheta\; dr \wedge d\vartheta \wedge d\varphi$$

in our case, so the volume of the piece between 0 and $\hat{r}$ is

$$\int_0^\pi \int_0^{2\pi} \int_0^{\hat{r}} \left(1 - \frac{r^2}{r_0^2}\right)^{1/2} r^2 \sin\vartheta\; dr\; d\vartheta\; d\varphi = 2\pi r_0^3 \left[\sin^{-1}\frac{\hat{r}}{r_0} - \frac{\hat{r}}{r_0}\left(1 - \frac{\hat{r}^2}{r_0^2}\right)^{1/2}\right]$$

and if we expand the terms in the brackets in power series we get finally,

$$\mathbf{V} = \frac{4}{3}\pi\hat{r}^3\left[1 + \frac{3}{10}\left(\frac{\hat{r}}{r_0}\right)^2 + \text{ higher-order terms}\right] \tag{24.22}$$

If we can indeed interpret ρ as a physical mass density, then the mass of our star is $\rho\mathbf{V}$, which is larger than M. The interesting result, which we find by using the first-order correction for the volume in eq. (24.22), is that

$$\rho\mathbf{V} - M = \frac{\mathbf{\Phi}}{c^2} \tag{24.23}$$

where $\mathbf{\Phi}$ is the potential energy of the mass due to gravity, and $\mathbf{\Phi}/c^2$ is the mass corresponding to the potential energy lost, or kinetic energy acquired, if the sphere were to collapse to a point. (Cf., Problem 24.4.)

PROBLEM 24.1 Show that for our star of "radius" $\hat{r}$, the pressure as a function of r is given by

$$\frac{p}{\rho c^2} = \frac{2}{3 - u^{1/2}} - 1$$

where

$$u = \frac{(r_0/\hat{r})^2 - (r/\hat{r})^2}{(r_0/\hat{r})^2 - 1}$$

PROBLEM 24.2 From Problem 24.1 as well as from eq. (24.21) we see that u must be less than 9. Show $(r_0/\hat{r})^2 = (c^2/2\mathbf{g})(\hat{r}/M)$ and hence deduce the relation

$$\frac{9}{4}\frac{\mathbf{g}}{c^2}M < \hat{r}$$

between M and $\hat{r}$.

PROBLEM 24.3 Fill in the details of the derivation of eq. (24.22).

PROBLEM 24.4 Derive eq. (24.23), using the fact that the gravitational energy loss due to "spherical packing" is $-(16/15)\pi^2\mathbf{g}\rho^2\hat{r}^5$ for a sphere of radius $\hat{r}$. (See Adler et al., p. 475.)

24.2 The form of the Friedmann-Robertson-Walker metric tensor and its properties

To obtain the form of the Friedmann-Robertson-Walker metric tensor we make assumptions about the stress-energy tensor, T.

(i) As in Section 24.1, we assume that the medium is a perfect fluid with no external forces, so that T has the form (24.2). Again we take the velocity field U as a coordinate vector field, $\partial/\partial x^0$, so $U^i = 0$ $(i = 1, 2, 3)$ and we get the conditions on the components of T as before.

(ii) We assume $U = -c^2 \text{ grad } x^0$. With this assumption, on each curve, λ, of U,

$$\frac{dx^0 \circ \lambda}{d\tau} = U^0 = g(U, \text{grad } x^0) = -\frac{g(U,U)}{c^2} = 1$$

so we can choose x^0 to be the proper time for all particles, and we say U *is proper-time synchronizable.*

Theorem 24.1 $U = -c^2\text{grad } x^0 \Rightarrow$ *the integral curves of* U *are geodesics and they have orthogonal hypersurfaces (rest spaces).*

Proof $U = -c^2 \text{ grad } x^0 \Rightarrow \text{ curl } U = 0$ ($g_{\alpha\beta}U^\alpha$ is exact implies $g_{\alpha\beta}U^\alpha$ is closed). So curl $U(X,Y) = 0$ for all X and Y orthogonal to U and U has orthogonal hypersurfaces by Problem 22.3. Further curl $U(U,Z) = g(\nabla_U U, Z) - g(\nabla_Z U, U) = g(\nabla_U, U, Z)$ by eq. (17.12), so curl $U(U,Z) = 0$ for all $Z \Rightarrow \nabla_U U = 0$. □

Corollary *With* U *chosen as a coordinate vector field,* $\partial/\partial x^0$, *and its rest spaces,* $x^0 =$ *constant, chosen as coordinate hypersurfaces, the metric tensor has the form*

$$g = -(dx^0)^2 + g_{ij}dx^i dx^j \quad (i,j = 1,2,3) \tag{24.24}$$

(iii) We assume T *is spatially isotropic*: at each point of each rest space, $x^0 =$ constant, T is invariant under orthogonal transformations of its tangent space. On the basis of this assumption, we infer that *spacetime,* M *is spatially isotropic with respect to* U: at each point there is a set of local isometries which keep the point fixed, and which, given arbitrary unit vectors X and Y in the local rest space at the point, contains an element, ϕ, such that $\phi_* X = Y$ and $\phi_* U = U$. This inference can be given credentials as a form of "Mach's Principle."

From spatial isotropy of M we see that for any two planes $\langle\{X,Y\}\rangle$ and $\langle\{X',Y'\}\rangle$ in the tangent space of the rest space at a point there is an isometry ϕ such that ϕ_* takes $\langle\{X,Y\}\rangle$ into $\langle\{X',Y'\}\rangle$. Under that isometry the sectional curvatures at that point will be the same. That is, spatial isotropy implies that all sectional curvatures of a rest space at a point are the same.

Now, by means of two more standard geometrical results, spatial isotropy leads to a more specific form of eq. (24.24).

Theorem 24.2 (Schur) *If at each point in some region of a pseudo-Riemannian manifold, (M, g), of dimension $n > 2$ the sectional curvatures are all the same, then their common value, $\varkappa$, is constant in the region. (If $\varkappa$ is constant on (M, g), then (M, g) is called* a space of constant curvature.)

Proof By Problem 17.13, at each point we have $R_{ijkl} = \varkappa(g_{ik}g_{jl} - g_{il}g_{jk})$. By the Ricci Lemma, Theorem 16.11, $R_{ijkl,m} = \varkappa_{,m}(g_{ik}g_{jl} - g_{il}g_{jk})$. By the second Bianchi identity, Problem 16.12.

$$\varkappa_{,m}(g_{ik}g_{jl} - g_{il}g_{jk}) + \varkappa_{,k}(g_{il}g_{jm} - g_{im}g_{jl}) + \varkappa_{,l}(g_{im}g_{jk} - g_{ik}g_{jm}) = 0$$

and multiplying this by $g^{ik}g^{jl}$ we get $(n-1)(n-2)\varkappa_{,m} = 0$, so k is constant. □

Theorem 24.3 *If (M, g) is (locally) a space of constant curvature, $\varkappa$, then there exist coordinates $(\mathcal{U}, \mu)$ in terms of which g has components*

$$g_{ij} = \frac{\eta_{ij}d\mu^i d\mu^j}{\left(1 + \dfrac{\varkappa}{4}\eta_{ij}\mu^i\mu^j\right)^2} \tag{24.25}$$

where $\eta_{ij} = \begin{cases} \pm 1 & i = j \\ 0 & i \neq j \end{cases}$.

Proof (1) If (M, g) has coordinates for which g has the form (24.25) where $\varkappa$ is a constant, then direct calculation gives $R_{ijkl} = \varkappa(g_{ik}g_{jl} - g_{il}g_{jk})$ and by Problem 17.13 (M, g) is a space of constant curvature, $\varkappa$.

(2) We will outline a proof of the following statement. If (M, g) has constant curvature, $\varkappa$, in a neighborhood $\mathcal{U}$ of p and coordinates $(\mathcal{U}, \mu)$, and $(M, \bar{g})$ has constant curvature, $\varkappa$, in a neighborhood $\overline{\mathcal{U}}$ of p and coordinates $(\overline{\mathcal{U}}, \bar{\mu})$, then there is a coordinate transformation $\bar{\mu} \circ \mu^{-1}$ such that

$$g_{ij} = \bar{g}_{pq}\frac{\partial\bar{\mu}^p}{\partial\mu^i}\frac{\partial\bar{\mu}^q}{\partial\mu^j} \tag{24.26}$$

That is, locally, there is only one space of constant curvature, $\varkappa$.

(a) Given g_{ij} and $\bar{g}_{pq}$ we have to show there are functions $\bar{\mu}^i$ of μ^j satisfying eq. (24.26).

(b) Functions $\bar{\mu}^i$ of μ^j and p^i_j of μ^k satisfy

$$g_{ij} = \bar{g}_{pq}\frac{\partial\bar{\mu}^p}{\partial\mu^i}\frac{\partial\bar{\mu}^q}{\partial\mu^j}, \quad p^i_j = \frac{\partial\bar{\mu}^i}{\partial\mu^j} \tag{24.27}$$

if and only if $\bar{\mu}^i$ satisfy eq. (24.26).

(c) Let

$$F^i_{jk}(\mu^l, \bar{\mu}^m, p^r_s) = \overline{\left\{ \begin{matrix} t \\ j \quad k \end{matrix} \right\}} p^i_t - \left\{ \begin{matrix} i \\ u \quad v \end{matrix} \right\} p^u_j p^v_k \quad \text{and} \quad f^i_j(\mu^l, \bar{\mu}^m, p^r_s) = p^i_j$$

Then functions $\bar{\mu}^i$ of μ^j and p^i_j of μ^k satisfy

$$\frac{\partial p^i_j}{\partial \mu^k} = F^i_{jk}(\mu^i, \bar{\mu}^m, p^r_s), \quad \frac{\partial \bar{\mu}^i}{\partial \mu^j} = f^i_j(\mu^i, \bar{\mu}^m, p^r_s) \tag{24.28}$$

if and only if they satisfy (24.27).

(d) Equations (24.28) are of the form of eq. (14.1). Applying the integrability conditions, eq. (14.3), to these we get

$$p^r_i R^i_{.jkl} - p^s_j p^t_k p^u_l \bar{R}^r_{.stu} = 0 \tag{24.29}$$

(e) By our hypothesis, $R_{ijkl} = \varkappa(g_{ik}g_{jl} - g_{il}g_{jk})$ and $\bar{R}_{rstu} = \varkappa(g_{rt}g_{su} - g_{ru}g_{st})$, and these satisfy eq. (24.29).

Parts (1) and (2) together give us the required result. □

If we change coordinates by $x^i = \sqrt{|\varkappa|}\ \mu^i$ for $\varkappa \neq 0$, then the right side of eq. (24.25) becomes

$$\frac{1}{|\varkappa|} \frac{\eta_{ij} dx^i dx^j}{\left(1 + \frac{\varkappa_0}{4}\eta_{ij}x^i x^j\right)^2}, \quad \text{where } \varkappa_0 = \pm 1$$

so the metric tensor of each rest space can be written in this form. But $\varkappa$ can change from one rest space to another; that is, $\varkappa$ can be a function of t. If we finally write $1/|\varkappa| = \mathbf{R}^2(t)$, and transform to "spherical" coordinates, r, ϑ, φ, and insert the result in Eq. (24.24), we get the Friedmann-Robertson-Walker metric tensor in the form

$$g = -c^2\ dt^2 + \mathbf{R}^2(t)\ \frac{dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\ d\varphi^2)}{\left(1 + \frac{\varkappa_0}{4}r^2\right)^2} \tag{24.30}$$

where $\varkappa_0 = 0,\ \pm 1$. (The case $\varkappa_0 = 0$ generalizes the case $\varkappa = 0$ by inserting a factor of a function of t. It is sometimes called *a simple cosmological spacetime.*)

The Friedmann-Robertson-Walker metric tensor has a couple of interesting properties.

(1) The metric (24.30) gives rise to a redshift in terms of $\mathbf{R}(t)$. Thus, suppose light travels from a star λ_1 to a star λ_2. Since its path is a null geodesic, along it (from 24.30)

$$dl^2 = \frac{dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\ d\varphi^2)}{\left(1 + \frac{\varkappa_0}{4}r^2\right)^2} = \frac{c^2\ dt^2}{\mathbf{R}^2}$$

Integrating along this geodesic we have

$$\int_{t_1}^{t_2} \frac{c}{\mathbf{R}}\, dt = l(t_2) - l(t_1) \tag{24.31}$$

During the time interval Δt_1 the star λ_1 emits light observed by λ_2 during the time interval Δt_2. So we also have

$$\int_{t_1+\Delta t_1}^{t_2+\Delta t_2} \frac{c}{\mathbf{R}}\, dt = l(t_2 + \Delta t_2) - l(t_1 + \Delta t_1) \tag{24.32}$$

But the right sides of (24.31) and (24.32) are equal since we are using "comoving" coordinates. So

$$\int_{t_1}^{t_1+\Delta t_1} \frac{1}{\mathbf{R}}\, dt = \int_{t_2}^{t_2+\Delta t_2} \frac{1}{\mathbf{R}}\, dt$$

and, if Δt_1 and Δt_2 are small compared to $t_2 - t_1$,

$$\frac{\Delta t_2}{\mathbf{R}(t_2)} = \frac{\Delta t_1}{\mathbf{R}(t_1)} \tag{24.33}$$

If Δt_1 is the period of some radiation emitted at λ_1, then Δt_2 will be the observed period at λ_2. In terms of frequencies, ν_1 and ν_2, (24.33) becomes

$$\frac{\nu_1 - \nu_2}{\nu_2} = \frac{\mathbf{R}(t_2)}{\mathbf{R}(t_1)} - 1 \tag{24.34}$$

Thus, a redshift corresponds to an increasing $\mathbf{R}$ - the universe is expanding.

(2) The metric (24.30) gives rise to Hubble's law. Thus, suppose $t_2 - t_1$ is small, and we expand $1/\mathbf{R}$ in a power series around $t = t_2$. Then substituting the result into (24.31) and (24.34), we get respectively,

$$l(t_2) - l(t_1) = h + \frac{1}{2}\frac{\mathbf{R}_2'}{c}\, h^2 + 0(h^3) \tag{24.35}$$

$$\frac{\nu_1 - \nu_2}{\nu_2} = \frac{\mathbf{R}_2'}{c}\, h + \frac{\mathbf{R}_2}{c^2}\left[\frac{(\mathbf{R}_2')^2}{\mathbf{R}_2} - \frac{\mathbf{R}_2''}{2}\right]h^2 + 0(h^3) \tag{24.36}$$

where $h = c(t_2 - t_1)/\mathbf{R}_2$. Eliminating h between these two equations by solving (24.35) for h and substituting into (24.36), we obtain

$$c\frac{\nu_1 - \nu_2}{\nu_2} = \mathbf{R}_2'\Delta l + \frac{\mathbf{R}_2^{'2}(\Delta l)^2}{2c}\left(1 - \frac{\mathbf{R}_2''\mathbf{R}_2}{\mathbf{R}_2^{'2}}\right) + 0((\Delta l)^3)$$

where $\Delta l = l(t_2) - l(t_1)$. Finally, putting $H = \mathbf{R}'/\mathbf{R}$, the "Hubble constant" and $q = -\mathbf{R}''\mathbf{R}/\mathbf{R}^{'2}$, the "deceleration parameter," we have

$$c\frac{\nu_1 - \nu_2}{\nu_2} = H_2(\mathbf{R}_2\Delta l) + \frac{1}{2c}H_2^2(1 + q_2)(\mathbf{R}_2\Delta l)^2 + 0((\mathbf{R}_2\Delta l)^3) \tag{24.37}$$

$\mathbf{R}_2\Delta l$ can be considered an approximation for the physical distance between λ_1 and λ_2, so (24.37) gives the relative change of the frequency of the radiation in terms of the distance between λ_1 and λ_2. In particular, if we use only the first term on the right we get Hubble's law, if H is approximately constant.

PROBLEM 25.4 Prove that the converse of Theorem 24.1 is valid locally.

PROBLEM 25.5 Fill in some of the details of the proof of Theorem 24.3.

24.3 Friedmann-Robertson-Walker spacetimes

We will look for solutions of eq. (24.1) with $\mathrm{Ric}_{\alpha\beta}$ determined by eq. (24.30) and $T_{\alpha\beta}$ with properties (i), (ii), and (iii) in Section 24.2.

If we put $\mathbf{Q} = [1 + (\varkappa_0/4)r^2]^{-2}$, then the Christoffel symbols for (24.30) are

$$\left\{ {0 \atop 1\ 1} \right\} = \mathbf{RR}'\mathbf{Q}, \quad \left\{ {0 \atop 2\ 2} \right\} = \mathbf{RR}'\mathbf{Q}r^2, \quad \left\{ {0 \atop 3\ 3} \right\} = \mathbf{RR}'\mathbf{Q}r^2 \sin^2 \vartheta$$

$$\left\{ {1 \atop 0\ 1} \right\} = \frac{\mathbf{R}'}{\mathbf{R}}, \quad \left\{ {1 \atop 1\ 1} \right\} = \frac{1}{2}\frac{\mathbf{Q}'}{\mathbf{Q}}, \quad \left\{ {1 \atop 2\ 2} \right\} = -\left(\frac{\mathbf{Q}'}{2\mathbf{Q}} + \frac{1}{r}\right) r^2,$$

$$\left\{ {1 \atop 3\ 3} \right\} = -\left(\frac{\mathbf{Q}'}{2\mathbf{Q}} + \frac{1}{r}\right) r^2 \sin^2 \vartheta$$

$$\left\{ {2 \atop 0\ 2} \right\} = \frac{\mathbf{R}'}{\mathbf{R}}, \quad \left\{ {2 \atop 1\ 2} \right\} = \frac{\mathbf{Q}'}{2\mathbf{Q}} + \frac{1}{r}, \quad \left\{ {2 \atop 3\ 3} \right\} = -\sin\vartheta\cos\vartheta$$

$$\left\{ {3 \atop 0\ 3} \right\} = \frac{\mathbf{R}'}{\mathbf{R}}, \quad \left\{ {3 \atop 1\ 3} \right\} = \frac{\mathbf{Q}'}{2\mathbf{Q}} + \frac{1}{r}, \quad \left\{ {3 \atop 2\ 3} \right\} = \cot\vartheta$$

We saw that property (i) requires that $T_1^1 = T_2^2 = T_3^3$ so that eq. (24.1) in the form

$$\mathrm{Ric}_\beta^\alpha = a(T_\beta^\alpha - \frac{1}{2}T_\lambda^\lambda \delta_\beta^\alpha) \tag{24.38}$$

has only two independent components. From the Christoffel symbols we calculate

$$\mathrm{Ric}_{00} = -3\,\frac{\mathbf{RR}''}{c^2\mathbf{R}^2}$$

$$\mathrm{Ric}_{11} = \mathbf{Q}\left(\frac{\mathbf{RR}'' + 2\mathbf{R}'^2}{c^2} + 2\varkappa_0\right)$$

so eq. (24.38) yields

$$3\,\frac{\mathbf{R}''}{c^2\mathbf{R}} = -\frac{a}{2}\left(\rho + \frac{3p}{c^2}\right) \tag{24.39}$$

$$\frac{\mathbf{R}\mathbf{R}'' + 2\mathbf{R}'^2}{c^2\mathbf{R}^2} + \frac{2\varkappa_0}{\mathbf{R}^2} = \frac{a}{2}\left(\rho - \frac{p}{c^2}\right) \tag{24.40}$$

Forming linear combinations of (24.39) and (24.40) we get

$$\frac{3\mathbf{R}'^2}{c^2\mathbf{R}^2} + \frac{3\varkappa_0}{\mathbf{R}^2} = a\rho \tag{24.41}$$

$$\frac{2\mathbf{R}''}{c^2\mathbf{R}} + \frac{\mathbf{R}'^2}{c^2\mathbf{R}^2} + \frac{\varkappa_0}{\mathbf{R}^2} = -a\frac{p}{c^2} \tag{24.42}$$

Finally, differentiating (24.41) and eliminating $\mathbf{R}''$ between the result and (24.39), we get

$$(\rho\mathbf{R}^3)' + \frac{p}{c^2}(\mathbf{R}^3)' = 0 \tag{24.43}$$

In looking for solutions of eqs. (24.39) - (24.43), an obvious, and historically important, case to consider is the static case, $\mathbf{R}$ = constant. From (24.39) and (24.41) we get $0 = \rho + 3p/c^2$ and $3\varkappa_0/\mathbf{R}^2 = a\rho$. If ρ and p are to be nonnegative, these conditions imply that $p = \rho = \varkappa_0 = 0$, an empty flat universe.

We can get something more interesting if we add the cosmological constant, Λ, which is what Einstein did - reluctantly. It turns out that now we must have $\varkappa_0 = 1$ *and* $\mathbf{R} = 2/[a(\rho + p/c^2)]$. With $a = 8\pi\mathbf{g}/c^2$ one gets $\mathbf{R}$, "the radius of the universe" of the order of 10^{10} light years.

The static models were proposed before the redshift was discovered. Hence the fact that they predict a zero redshift is a serious flaw.

An early nonstatic solution of Einstein's equation was obtained by de Sitter (1917) by including the cosmological constant. With notable prescience he assumed $\mathbf{R}'/\mathbf{R}$ = constant. The equations then require $\varkappa_0 = 0$ (a simple cosmological spacetime), and again, unfortunately, $\rho = p = 0$.

Now we look for solutions of eqs. (24.39) - (24.43) with $p = 0$, incoherent dust. Eliminating ρ between (24.39) and (24.41) we get

$$2\mathbf{R}\mathbf{R}'' + \mathbf{R}'^2 + c^2\varkappa_0 = 0 \tag{24.44}$$

which is integrated to

$$\mathbf{R}'^2 + c^2\varkappa_0 = \frac{D_0 c^2}{\mathbf{R}} \tag{24.45}$$

where D_0 is an arbitrary constant. Putting (24.45) into (24.44) we get

$$\mathbf{R}'' = -\frac{D_0 c^2}{2\mathbf{R}^2} \tag{24.46}$$

and putting this into (24.39) (with $p = 0$) we get

$$D_0 = \frac{a}{3}\,\rho\mathbf{R}^3 \tag{24.47}$$

This is consistent with eq. (24.39), which in this case, for $\rho \neq 0$, requires $\mathbf{R}'' < 0$, so $D_0 > 0$, and is also consistent with eq. (24.43), which gives, in this case, $\mathbf{R}^3\rho =$ constant.

Finally, for the deceleration parameter, q, $\mathbf{R}'' < 0 \Rightarrow q > 0$. Moreover, from eqs. (24.45) and (24.46) we get

$$\frac{\varkappa_0 \mathbf{R}}{D_0} = \frac{2q-1}{2q}$$

so $q < \frac{1}{2}$ for $\varkappa_0 = -1$, $q = \frac{1}{2}$ for $\varkappa_0 = 0$, and $q > \frac{1}{2}$ for $\varkappa_0 = 1$.

Now consider the cases $\varkappa_0 = 1$, $\varkappa_0 = 0$, and $\varkappa_0 = -1$ (see Fig. 24.1).

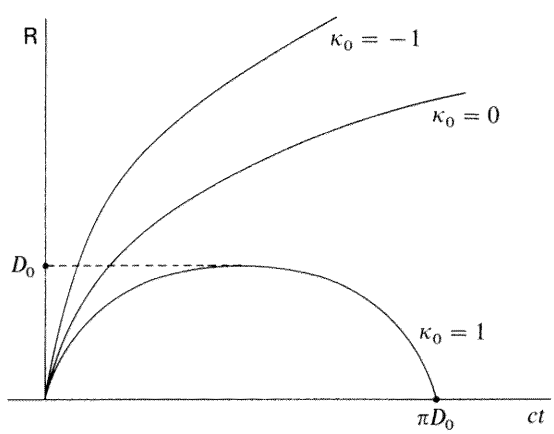

Figure 24.1

(i) If $\varkappa_0 = 1$, and we introduce a parameter ω by

$$ct = \frac{D_0}{2}(2\omega - \sin 2\omega),$$

then eq. (24.25) has a solution of the form

$$\begin{cases} ct = \dfrac{D_0}{2}(2\omega - \sin 2\omega) \\ \mathbf{R} = \dfrac{D_0}{2}(1 - \cos 2\omega) \end{cases}$$

This is a cycloid. The slope is infinite at $t = 0$, which could correspond to the "big bang." However, $p = 0$ is not a good assumption for small t.

(ii) If $\varkappa_0 = 0$, then integrating (24.45), with $\mathbf{R} = 0$ when $t = 0$,

$$\mathbf{R} = (\frac{3}{2}c)^{3/2}\ D_0^{1/3}\ t^{2/3}.$$

Again we get a big bang at $t = 0$. This solution was obtained by A. Friedmann in 1922. Nowadays it is called *Einstein-de Sitter spacetime.* It is another example of a simple cosmological spacetime.

(iii) If $\varkappa_0 = -1$, and we introduce a parameter ω by $ct = (D_0/2)(\sinh 2\omega - 2\omega)$, then eq. (24.45) has a solution of the form

$$\begin{cases} ct = \dfrac{D_0}{2}(\sinh 2\omega - \sin 2\omega) \\ \mathbf{R} = \dfrac{D_0}{2}(\cosh 2\omega - 1) \end{cases}$$

and as $t \to \infty$, $\mathbf{R}' \to c$.

The quantities ρ, H, and q in the equations above are subject to astronomical measurement. One might hope that such information could sort out of the three cases, the model that best corresponds to reality. As we have seen, a precise enough determination of q alone would select one of these cases. Alternatively, if we knew H and ρ with enough accuracy we could determine the choice of $\varkappa_0$ from eq. (24.41). Unfortunately, the range within which all these quantities are presently known is too wide to make a definite determination. This is currently an area of active investigation.

PROBLEM 24.7 Show that for $\varkappa_0 \neq 0$, D_0 is obtained from H and q by

$$D_0 = \frac{2q}{|2q-1|^{3/2}}\ \frac{c}{H}$$

and for $\varkappa_0 = 0$, ρ and H are related by $a\rho = (3/c^2)H^2$.

25

LIE GROUPS

We have mentioned Lie groups in passing, in Sections 13.2 and 19.2. To apply tensors and manifolds to current physics a deeper look is indispensable. In particular, we will describe a Lie algebra of vector fields (cf., Section 11.1) on a Lie group, G, whose properties strongly reflect those of G itself.

25.1 Definition and examples

A Lie group is a differentiable manifold with a differentiable group structure; i.e., the group multiplication, $M \times M \to M$, and the inverse map $M \to M$ are differentiable.

There are the following standard examples of Lie groups.

1. The manifold $\mathbb{R}^n$ with the group structure that of the direct sum of n copies of $\mathbb{R}$ is an n-dimensional Lie group.
2. On the quotient manifold (cf., Section 9.1), $\mathbb{R}^n/H$, where $H = \{(a_0^1, \ldots, a_0^n) : a_0^i \text{ are integers}\}$ we can define a group structure by $[a] + [b] = [a+b]$ and $-[a] = [-a]$, where $[a], [b] \in \mathbb{R}^n/H$ and $a, b \in \mathbb{R}^n$.
3. Since the manifold S^1 is a subset of the nonzero complex numbers, it can be given the group structure of the multiplicative group of the nonzero complex numbers.
4. The product manifold $S^1 \times S^1 \times \cdots \times S^1$ with $(p_1, \ldots, p_n) \cdot (q_1, \ldots, q_n) = (p_1q_1, \ldots, p_nq_n)$ and $(p_1, \ldots, p_n)^{-1} = (p_1^{-1}, \ldots, p_n^{-1})$. This Lie group is the same as Example 2. It is called the *n-Torus, T^n.*
5. Given any two Lie groups G, H we can construct a Lie group with the product manifold structure and with $(g_1, h_1) \cdot (g_2, h_2) = (g_1g_2, h_1h_2)$ and $(g, h)^{-1} = (g^{-1}, h^{-1})$.
6. The set of all $n \times n$ matrices with real (complex) entries is an n^2-dimensional Lie group under matrix addition.
7. $GL(n, \mathbb{R})$, the set of all nonsingular real matrices is an n^2-dimensional Lie group under matrix multiplication. The fact that $GL(n, \mathbb{R})$ is an n^2-dimensional manifold comes from the facts that the map det: $GL(n, \mathbb{R}) \to \mathbb{R}$ is continuous and that the set $\{\det(GL(n, \mathbb{R})) \neq 0\}$ is open so that $GL(n, \mathbb{R})$ is an open subset of $\mathbb{R}^n$. $GL(n, \mathbb{R})$ is *a general linear group.*

We have exactly corresponding results for $GL(n,\mathbb{C})$, matrices with complex entries. More generally, these matrices are representations of nonsingular linear transformations of a vector space, V, and they form a group, Aut V, under compositions.

8. Lie subgroups of a given Lie group. If we think of a Lie subgroup as a subset of a given Lie group, then we need the criterion of Theorem 10.1 for its manifold structure. For the subsets $SL(n,\mathbb{R}), O(r,s)$, and $SO(r,s)$ of $GL(n,\mathbb{R})$ we actually have more - they are differentiable varieties of a submersion (Section 9.4).
   (i) *The special linear group*, $SL(n,\mathbb{R})$ is the subset of $GL(n,\mathbb{R})$ with determinant 1. Let $\phi = \det : GL(n,\mathbb{R}) \to \mathbb{R}$. Then $SL(n,\mathbb{R}) = \phi^{-1}(1)$, and all we have to do is show that ϕ has rank 1. Thinking of ϕ as a function on $\mathbb{R}^{n^2}$ we have $\phi(a^i_j) = \sum_\pi (\text{sgn } \pi) a_1^{\pi(1)} \cdots a_n^{\pi(n)}$ and $\phi_* = d\phi$ is a linear combination of the da^i_j with coefficients being the cofactors of a^i_j in (a^i_j). These are not all zero since $\det(a^i_j) \neq 0$, and since da^i_j are linearly independent, $d\phi \neq 0$. By Problem 10.6, dim $SL(n,\mathbb{R}) = n^2 - 1$.
   (ii) In Section 5.4 we saw that the matrix, (a^i_j), of an orthogonal transformation satisfies

$$(a^i_j)^{tr} \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix} (a^i_j) = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix} \tag{5.22}$$

The set of all such matrices is *the* $r+s$*-dimensional orthogonal group*, $O(r,s)$. $O(r,s)$ is a subgroup of $GL(n,\mathbb{R})$ and has determinant ± 1. Let ϕ be a map of $GL(n,\mathbb{R})$ given by

$$\phi(a^i_j) = (a^i_j)^{tr} \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix} (a^i_j) - \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix}. \tag{25.1}$$

Since $\phi(a^i_j)$ is symmetric, this is a map $\phi : GL(n,\mathbb{R}) \to \mathbb{R}^{n(n+1)/2}$. Then $O(r,s) = \phi^{-1}(0)$, and this will be a differentiable variety if the rank of ϕ is $n(n+1)/2$, or equivalently if ϕ_* is surjective.

Thinking of ϕ as a map from $\mathbb{R}^{n^2}$ to $\mathbb{R}^{n(n+1)/2}$ we write eq. (25.1) as

$$\phi^{pq}(a_{ij}) = \sum_\lambda e_\lambda a_{\lambda p} a_{\lambda q} - e_p \delta_{pq} \tag{25.2}$$

$$e_\lambda, e_p = \begin{cases} 1 & \text{for} \quad \lambda, p = 1, \ldots, r \\ -1 & \text{for} \quad \lambda, p = r+1, \ldots, r+s \end{cases}$$

(see eq. (20.35)), and if v is a tangent at (a_{ij}) and w is a tangent at $\phi^{pq}(a_{ij})$, then $w^{pq} = \frac{\partial \phi^{pq}}{\partial a_{ij}} v^{ij}$, so from eq. (25.2)

$$w^{pq} = \sum_i e_i a_{ip} v^{iq} + \sum_i e_i v^{ip} a_{iq}$$

Now, given any w^{pq}, $v^{ik} = e_r a_{ir} w^{rk}/2$ satisfies this equation, so ϕ_* is surjective. By Problem 10.6

$$\dim\ O(r,s) = n^2 - \frac{(n+1)n}{2} = \frac{n(n-1)}{2}.$$

(Recall that our terminology and notation slightly generalizes the frequent usage in which the scalar product is positive definite).

(iii) $O(r,s)$ has 2 diffeomorphic components, one of which contains the identity and all of whose members have det 1. This component is *the special orthogonal group*, $SO(r,s)$.

$SO(n)$ is frequently called *the n-dimensional rotation group*. In Section 18.3 we have already seen $SO(3)$, the 3-dimensional configuration space of a rigid body with a fixed point. As a manifold $SO(3)$ is $P^3(\mathbb{R})$ (Choquet-Bruhat and DeWitt-Morette, p. 188), and $SO(2) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$ is S^1, or equivalently, the subset $\{e^{i\vartheta}\}$ of the complex plane.

The 6-dimensional groups $O(3,1)$ and $SO(3,1)$, *the Lorentz group* and *the proper Lorentz group*, respectively, have already appeared in Section 20.4.

Finally, we note that $GL(n,\mathbb{C})$ has subgroups which in some sense correspond to those of $GL(n,\mathbb{R})$ which we have discussed. The unitary and special unitary groups, in particular, play a central role in the "standard model" in particle physics.

PROBLEM 25.1 Prove that the two differentiability conditions in the definition of a Lie group can be replaced by the one condition: the map $M \times M \to M$ given by $(p,q) \mapsto pq^{-1}$ is differentiable.

PROBLEM 25.2 Prove that the components of a Lie group are diffeomorphic and the component containing the identity is a Lie group.

25.2 Vector fields on a Lie group

The group operation $G\times G \to G$ of a Lie group gives rise to two partial mappings, $G \to G$:

(i) for each $g \in G$ we have $L_g : h \mapsto gh$, *a left translation of G by g*

(ii) for each $g \in G$ we have $R_g : h \mapsto hg$, *a right translation of G by g*

Definition If $X \in \mathfrak{X}G$ is a vector field on G, it is *left invariant* if it is invariant under all left translations; i.e.,

$$L_{g*}X(h) = X(L_g h) \quad \forall\ g, h \in G$$

(cf., Section 11.4).

Theorem 25.1 *Left invariant vector fields are complete.*

Proof If X is left invariant, then it follows from Problem 13.4 that its flow, Θ, has the property

$$\Theta(t, L_g h) = L_g \Theta(t, h) \quad \forall\ g, h,\ |t| < \varepsilon$$

Then we can define Θ for all $g \in G$ and $|t| < \varepsilon$ by $\Theta(t, g) = L_g \Theta(t, e)$ and then by Problem 13.6, X is complete. □

Theorem 25.2 *Linear combinations and the Lie bracket of left invariant vector fields are left invariant vector fields, so that the left invariant vector fields form a subalgebra of the Lie algebra $\mathfrak{X}G$* (Theorem 11.12) *called* the Lie algebra of G, *and denoted by* $\mathfrak{g}$.

Proof The property of linear combinations is clear, and for the Lie bracket we have Theorem 11.24. □

Theorem 25.3 *As a vector space $\mathfrak{g}$ is naturally isomorphic to T_eG, the tangent space of G at the identity, and, in particular,* $\dim \mathfrak{g} = \dim G$.

Proof The mapping $\xi : X \to X(e)$ is clearly linear. If $X(e) = Y(e)$, then $X(g) = L_{g*}X(e) = L_{g*}Y(e) = Y(g)$ so ξ is 1-1. If $v \in T_eG$, define a vector field X by $X(g) = L_{g*}v$, then $X(e) = v$ and ξ is onto. Moreover, X is left invariant. □

If $(X_1, \ldots, X_n)$ is a basis of $\mathfrak{g}$, then the numbers c_{ij}^k defined by $[X_i, X_j] = c_{ij}^k X_k$ are called *the structure constants of* G. From the properties (i)-(iii) of a Lie algebra, Section 11.1, we get, respectively,

(i) the c_{ij}^k are components of a $(1, 2)$ tensor

(ii) $c_{ij}^k = -c_{ji}^k$

(iii) $c_{ij}^\ell c_{k\ell}^m + c_{ki}^\ell c_{i\ell}^m + c_{jk}^\ell c_{i\ell}^m = 0$

Definition A curve $\gamma : \mathbb{R} \to G$ is *a 1-parameter subgroup of* G if

$$\begin{aligned} \gamma(t) \cdot \gamma(s) &= \gamma(t + s) \\ \gamma(0) &= e \end{aligned} \tag{25.3}$$

Theorem 25.4 *There is a 1-1 correspondence between the left invariant vector fields on G and the 1-parameter subgroups of G.*

Proof If γ is a 1-parameter subgroup of G, then $\gamma(t) \cdot \gamma(s) = \gamma(s+t)$ or

$$L_{\gamma(t)}\gamma(s) = \gamma(s+t)$$

Then, for each fixed t, the tangent of γ at $s+t$ is

$$(L_{\gamma(t)*} \circ \dot{\gamma})(s) = \dot{\gamma}(s+t)$$

and for $s = 0$

$$L_{\gamma(t)*}\dot{\gamma}(0) = \dot{\gamma}(t) \tag{25.4}$$

Now, let X be the left invariant vector field corresponding to $\dot{\gamma}(0)$ according to Theorem 25.3. Then

$$L_{\gamma(t)*}\dot{\gamma}(0) = X(\gamma(t))$$

so with (25.4) we have

$$\dot{\gamma}(t) = X(\gamma(t)) \tag{25.5}$$

Equation (25.5) says that γ is the integral curve through e of X, and hence unique. Conversely, each left invariant vector field has an integral curve through e with the properties (25.3) by Theorem 13.6. □

Note that a 1-parameter subgroup of G is not a subset of G (cf., Section 10.1) so that, in particular Theorem 25.4 does not relate left invariant vector fields with subsets of G. We can, however, say more.

Lemma *If γ_X is the 1-parameter subgroup of G corresponding to the left invariant vector field X, then*

$$\gamma_{aX}(t) = \gamma_X(at) \quad (a \in \mathbb{R})$$

Proof Let $\quad \bar{\gamma}(at) = \gamma_A(t)$

Then $\quad \dot{\bar{\gamma}}(u)a = \dot{\gamma}_A(t) \qquad u = at$

$$= A(\gamma_A(t)) = A(\bar{\gamma}(u))$$

So $\quad \dot{\bar{\gamma}}(u) = A/a(\bar{\gamma}(u))$

But $\quad \dot{\gamma}_{A/a}(u) = A/a(\gamma_{A/a}(u))$

Hence, by the uniqueness of solutions of this differential equation

$$\gamma_{A/a}(u) = \bar{\gamma}(u)$$

or,

$$\gamma_{A/a}(at) = \gamma_A(t)$$

or, putting $X = A/a$

$$\gamma_X(at) = \gamma_{aX}(t)$$

□

This lemma says that, if we think of a curve of G as a subset of G, then the elements of the 1-dimensional subspace, aX, of $\mathfrak{g}$ all correspond to the same curve of G with different parameters.

Definition *The exponential map*, $\exp: \mathfrak{g} \to G$ is defined by $X \mapsto \gamma_X(1)$.

Theorem 25.5 (i) $\exp\ aX = \gamma_X(a)$
(ii) $\exp(a+b)X = (\exp\ aX)(\exp\ bX)$
(iii) $\exp(-a)X = (\exp\ aX)^{-1}$

Proof Problem 25.3 □

Since $\mathfrak{g}$ is an n-dimensional vector space it can be given the standard manifold structure and then one can show that exp is differentiable. Moreover, it gives a diffeomorphism of a neighborhood of 0 in $\mathfrak{g}$ with a neighborhood of e in G (Warner, p. 103). In particular, the 1-parameter subgroups fill a neighborhood of e, and can be used to define *normal coordinates* in that neighborhood (cf., Section 16.4).

Theorem 25.6 *If G and H are Lie groups and $\phi: G \to H$ is a homomorphism, then the tangent map ϕ_* maps T_eG to T_eH. Identifying T_eG and $\mathfrak{g}$ and T_eH and $\mathfrak{h}$*
(i) *ϕ_* is a Lie algebra homomorphism.*
(ii) *The diagram*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\phi_*"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\phi"'] & H \end{tikzcd}$$

is commutative.

Proof (i) Since X_i is ϕ-related to $\bar{X}_i = \phi_* \cdot X_i$, $i = 1, 2$, implies $[X_1 X_2]$ is ϕ-related to $[\bar{X}_1, \bar{X}_2]$, we have $[\bar{X}_1, \bar{X}_2] = \phi_*[X_1, X_2] = \overline{[X_1 X_2]}$, so the Lie bracket operation is preserved.

(ii) If $X \in \mathfrak{g}$, then $\gamma: t \mapsto \phi(\exp\ tX)$ is a 1-parameter subgroup of H with $\dot{\gamma}(0) = \phi_* X$. But $t \to \exp(t\ \phi_* X)$ is the unique 1-parameter subgroup of H whose tangent at e is $\phi_* X$ so $\phi(\exp X) = \exp\ \phi_* X$. □

It is enlightening to examine the exponential map in the case $G = GL(n, \mathbb{R})$. But first recall some properties of functions of $n \times n$ matrices, In particular, if A is an $n \times n$ matrix we define

$$e^A = \sum_0^\infty \frac{A^n}{n!}$$

This series converges for every A (actually, uniformly in bounded regions) so the definition makes sense. Further, e^A has the properties:

(i) $e^A \in GL(n, \mathbb{R})$

(ii) $\det e^A = e^{trA}$

(iii) $e^{A+B} = e^A \cdot e^B$ if A and B commute

(iv) $\frac{d}{dt} e^{tA}\big|_0 = A$

Now, given an $n \times n$ matrix A, $\gamma : t \mapsto e^{tA}$ is a 1-parameter subgroup of $GL(n, \mathbb{R})$, by (i) and (iii). Moreover, the tangent to γ at e is $\dot\gamma(0) = D_0 \gamma^{ij} \frac{\partial}{\partial \pi^{ij}} = a^{ij} \frac{\partial}{\partial \pi^{ij}} = v(A)$ by (iv). $v(A)$ is in the Lie algebra of G. So, by uniqueness, its 1-parameter subgroup $\gamma_{v(A)}$ must be γ. Hence, $e^A = \gamma(1) = \gamma_{v(A)}(1) = \exp v(A)$. Finally, invoking the natural isomorphism $\alpha : v(A) \mapsto A$ between the vector spaces $T_e\mathcal{M}_n$ and $\mathcal{M}_n$, (Problem 9.14 (ii)) we get $e^A = \exp A$. Thus what we have shown is that $n \times n$ matrices are in the Lie algebra of $GL(n, \mathbb{R})$ and for such elements the map exp is the same as the ordinary exponential.

Let us now find the Lie algebras of some of the examples of Lie groups described above.

1. For the Lie group $\mathbb{R}^n$, $L_p q = p + q$. If we write a vector field, X, in terms of its components X^i, then $L_{p*}X(q) = X^i(q) \frac{\partial (L_p)^j}{\partial \pi^i} \frac{\partial}{\partial \pi^j} = X^i(q)\delta_i^j \frac{\partial}{\partial \pi^j} = X^i(q) \frac{\partial}{\partial \pi^i}$. So, if $L_{p*}X(q) = X(L_p q) = X(p+q)$, then $X^i(q) = X^i(p+q)$; i.e., the X^i are constant.

2. Let $\mathfrak{g}$ be the Lie algebra of $GL(n, \mathbb{R})$. Then by Theorem 25.3, $\mathfrak{g}$ is naturally isomorphic to $T_eGL(n, \mathbb{R})$. Moreover, if $\mathcal{M}_n(\mathbb{R})$ is the vector space of $n \times n$ matrices with real entries (Section 2.1), then $T_e\mathcal{M}_n(\mathbb{R}) = T_eGL(n, \mathbb{R})$. Finally, we can again invoke the natural isomorphism between a vector space and its tangent spaces. Thus, we have $\alpha : T_e\mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ given by $v_e \mapsto (a^{ij})$ where $a^{ij} = v_e \pi^{ij}$ and π^{ij} are the natural coordinate functions on $\mathbb{R}^{n^2}$. Under the composition,

$$\beta : \mathfrak{g} \to T_eGL(n, \mathbb{R}) = T_e\mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$$

where $X \mapsto X_e \pi^{ij}$, then as vector spaces, $\mathfrak{g}$ is isomorphic to $\mathcal{M}_n(\mathbb{R})$. To show $\mathfrak{g}$ is the Lie algebra, $g\ell(n, \mathbb{R})$, $\mathcal{M}_n(\mathbb{R})$ with the product defined in Problem 11.10, we have to show $\beta : [X, Y] \mapsto \beta(X)\beta(Y) - \beta(Y)\beta(X)$.

But

$$\beta[X,Y] = [X,Y]_e\pi^{ij} = X_e(Y\pi^{ij}) - Y_e(X\pi^{ij}) \tag{25.6}$$

by Eq. (11.14), and

$$\beta(X)\beta(Y) - \beta(Y)\beta(X) = \sum_k X_e\pi^{ik}Y_e\pi^{kj} - \sum_k Y_e\pi^{ik}X_e\pi^{kj} \tag{25.7}$$

and since X and Y are left invariant, the right sides of (25.6) and (25.7) are equal, Problem 25.5.

We have exactly corresponding results for $GL(n,\mathbb{C})$, in which case the Lie algebra consists of matrices, $g\ell(n,\mathbb{C})$, with complex entries. More generally, the Lie algebra of Aut$V$ is End$V$, the set of linear transformations of V with the Lie algebra operations.

3. One can use properties of the exponential map to obtain the Lie algebras of the various subgroups of $GL(n,\mathbb{R})$. Thus, if $g \in SL(n,\mathbb{R})$ in a neighborhood of e, then $g = \exp X$ and $\det g = \det\exp X = 1$. But $\det\exp X = \exp\ trX$ (Property (ii) above) so $trX = 0$. Similarly, recalling that the condition (5.22) on $O(r,s)$ comes from (5.21), differentiating the latter along a 1-parameter subgroup of $GL(n,\mathbb{R})$ gives

$$X^{tr}\begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix} = -\begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix} X$$

the corresponding condition on $g\ell(n,\mathbb{R})$ for the Lie algebra of $O(r,s)$. Finally, we note that $SO(r,s)$ has the same Lie algebra as $O(r,s)$.

We have seen how the partial maps L_g coming from the group operation sort out an important subalgebra of vector fields from the Lie algebra of C^∞ vector fields. Clearly, we could have used R_g to get similar results. Now combining L_g and R_g we get a third important set of maps of $G \to G$.

Definition For each $g \in G$, $C_g = R_{g^{-1}} \circ L_g = L_g \circ R_{g^{-1}}$ is *an inner automorphism* or *conjugation* of G.

Theorem 25.7 (i) *For each $g \in G$, C_g is an automorphism of G (in contrast to L_g and R_g) and in particular maps e to e.*

(ii) *The map $G \to$ Aut G given by $g \mapsto C_g$ is a homomorphism.*

(iii) $C_{g*}\big|_{T_e(G)}: T_eG \to T_eG$ *is an automorphism of T_eG.*

(iv) $Ad: G \to$ Aut (T_eG) *given by* $Ad: g \mapsto C_{g*}\big|_{T_eG}$ *is a C^∞ homomorphism.*

Proof The arguments are all straightforward except to show Ad is C^∞ which is done by choosing a basis of T_eG and writing an element of Aut (T_eG) in terms of this basis, cf., Boothby, p. 242. □

Ad defined in Theorem 25.7 (iv) is called *the adjoint representation of G on* $\mathfrak{g}$. Identifying $\mathfrak{g}$ and T_eG we usually write $Ad: G \to \text{Aut } \mathfrak{g}$.

Note that $Ad_{g*}\, X = (R_{g^{-1}} \circ L_g)_*\, X = R_{g^{-1}*} \circ L_{g*}\, X$ so if X is left invariant

$$R_{g^{-1}*}\, X = Ad_{g*}\, X \tag{25.8}$$

Note also that if G is abelian, $Ad\; G \subset \text{Aut } \mathfrak{g}$ is just the identity.

The relation between (i) and (iii), i.e., between C_g and Ad_g, is a special case of the commutative relation shown in Theorem 25.6. Again, the tangent map of Ad takes T_eG to $T_e(\text{Aut } G)$, or, equivalently $Ad_*: \mathfrak{g} \to \text{End } \mathfrak{g}$ and by Theorem 25.6 $ad = Ad_*$ is a Lie algebra homomorphism and

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "ad"] \arrow[d, "\exp"'] & \text{End } \mathfrak{g} \arrow[d, "\exp"] \\
G \arrow[r, "Ad"] & \text{Aut } \mathfrak{g}
\end{tikzcd}$$

PROBLEM 25.3 Prove Theorem 25.5.

PROBLEM 25.4 Verify the 4 listed properties of the ordinary exponential function. (Cf., Warner, p. 105.)

PROBLEM 25.5 Show that the right sides of (25.6) and (25.7) are equal. (Cf., Chevalley, p. 105.)

PROBLEM 25.6 $\gamma_{Ad_g X} = g\, \gamma_X\, g^{-1}$.

PROBLEM 25.7 If $G = GL(n, \mathbb{R})$, $g \in GL(n, \mathbb{R})$, and $h \in g\ell(n, \mathbb{R})$ then $Ad_g h = g\, h\, g^{-1}$.

25.3 Differential forms on a Lie group

Just as the requirement of left invariance on the Lie algebra, $\mathfrak{X}M$, of vector fields sorts out the subalgebra of left invariant vector fields, we can get an important subalgebra from the algebra of differential forms (Section 11.3).

Definition *A differential form ω on G is left invariant* if it is invariant under all left translations; i.e.,

$$L_g^*\, \omega(L_g h) = \omega(h) \quad \forall\, g, h \in G$$

(cf., Section 11.4).

Linear combinations and the wedge product of left invariant differential forms are left invariant so that they form a subalgebra of the algebra of differential forms. Moreover, this subalgebra is isomorphic to the exterior algebra $\bigwedge T_e^*G$ at $e \in G$.

For 1-forms we have the duals of results for vector fields.

Theorem 25.8 *The left invariant 1-form on G forms a vector space isomorphic with T_e^*G. Hence, we will denote the space of left invariant 1-forms on G by $\mathfrak{g}^*$.*

Theorem 25.9 *If X is a left invariant vector field and ω is a left invariant 1-form, then $\langle \omega, X \rangle$ is constant.*

Proof For any X, ω and $\phi : h \mapsto \phi(h)$, $\langle \phi^*\omega(\phi(h)), X(h) \rangle = \langle \omega(\phi(h)), \phi_* X(h) \rangle$. So, for $\phi = L_g : g \mapsto L_g h$ and X and ω left invariant

$$\langle \omega(h), X(h) \rangle = \langle \omega(L_g h), X(L_g h) \rangle$$

for all $g, h \in G$. □

Corollary *If X, Y, and ω are all left invariant, then*

$$d\omega = -\frac{1}{2}\,\omega[X, Y]$$

Definition If $\{X_i\}$ is a basis of $\mathfrak{g}$, then the basis $\{\omega^i\}$ of $\mathfrak{g}^*$ such that $\langle \omega^i, X_j \rangle = \delta^i_j$ is *the dual basis of* $\{X_i\}$.

Theorem 25.10 The Maurer-Cartan structure equation: *If $\{\omega^i\}$ is a basis of $\mathfrak{g}^*$*

$$d\omega^i = -\frac{1}{2} c^i_{jk} \omega^j \wedge \omega^k \tag{25.9}$$

where the c^i_{jk} are the structure constants of G.

Proof $d\omega^i = a^i_{jk}\omega^j \wedge \omega^k$ for some a^i_{jk} with $a^i_{jk} = -a^i_{kj}$. If we evaluate both sides on X_ℓ and X_m in the dual basis, $d\omega^i(X_\ell, X_m) = -\frac{1}{2}\omega^i[X_\ell, X_m]$ (by the corollary) $= -\frac{1}{2}\omega^i c^j_{\ell m} X_j = -\frac{1}{2} c^i_{\ell m}$ and

$$\begin{aligned} a^i_{jk}\omega^j \wedge \omega^k(X_\ell, X_m) &= a^j_{jk}\frac{1}{2}\det(\langle \omega^\ell, X_m \rangle) \quad \text{(using eqs. (6.7) and (6.9))} \\ &= a^i_{jk}\frac{1}{2}(\delta^j_\ell \delta^k_m - \delta^j_m \delta^k_\ell) \\ &= a^i_{\ell m} \end{aligned}$$

□

This is an appropriate point at which to discuss a generalization of some of our previous results - some comments on vector-valued s-forms. Thus, if V and W are vector spaces, we can construct the vector space $\bigwedge^s(V^*, W)$ of skew-symmetric multilinear maps $A: V\times\cdots\times V \to W$ (generalizing the notation of Theorem 5.12). Then for each $p \in M$, we have $\bigwedge^s(T_p^*, W)$ and we can make $\bigwedge^s(T^*M, W) = \bigcup_{p\in M}\bigwedge^s(T_p^*, W)$ a differentiable manifold, and then a differentiable map $\theta : M \to \bigwedge^s(T^*M, W)$ such that $\theta(p) \in \bigwedge^s(T_p^*, W)$ is *a W-valued s-form on M*. (It is possible to go further and construct a Lie algebra of vector-valued forms by defining the product of p and q forms, but for the moment we will forgo this additional technicality, see Bleecker.)

If we pick a basis $\{E_i\}$ in W, we can express a W-valued s-form in terms of its values on vector fields $X_1, \ldots, X_s$ on M by

$$\theta(X_1, \ldots, X_s) = \theta^i(X_1, \ldots, X_s)E_i \tag{25.10}$$

(i.e., $\theta(X_1, \ldots, X_s)(p) = \theta^i(X_1, \ldots, X_s)(p)E_i$) where θ^i are real-valued s-forms, the "components" of θ in the basis $\{E_i\}$. Moreover, we can define the pull-back, $\phi^*\theta$, of θ by ϕ as before, and

$$\phi^*\theta(X_1, \ldots, X_s) = \phi^*\theta^i(X_1, \ldots, X_s)E_i \tag{25.11}$$

We have already sporadically encountered vector-valued forms (e.g., Chapter 16) where they have arisen as alternative interpretations of various tensor product spaces which come from the natural vector space isomorphisms. We can now go in the opposite direction, for if $\sigma \in W^*$, then by (25.10),

$$\theta(X_1, \ldots, X_s)\cdot\sigma = \theta^i(X_1, \ldots, X_s)(E_i\cdot\sigma) = \theta^i(X_1\wedge\cdots\wedge X_s)(E_i\cdot\sigma) \quad \text{by eq. (6.8)}$$

$$= E_i \otimes \theta^i(\sigma, X_1\wedge\cdots\wedge X_s) \quad \text{by our definition in Section 3.1,}$$

so we can write $\theta = E_i \otimes \theta^i$, a sum of tensor products.

Definition *The Maurer-Cartan 1-form on* G, θ, takes $g \in G$ to $\theta(g) \in \bigwedge^1(T_g^*G, T_eG)$ where $\theta(g)$ is defined by $\theta(g)\cdot v_g = L_{g*}^{-1}v_g \in T_eG$. That is, θ takes the value of each vector field at g back to its value at T_eG.

This form can be written in terms of a basis of T_eG.

Theorem 25.11 *If $\{E_i\}$ is a basis of T_eG, then the Maurer-Cartan form on G is given by $\theta(X) = \theta^i(X)E_i$ where θ^i are the duals of the vector fields X_i corresponding to E_i.*

Proof $X_g = v_g^jX_j$, so $\theta(X_g) = \theta^i(v_g^jX_j)E_i = v_g^iE_i$. □

From this result we have that θ is left invariant, since θ^i are. Further, for the Maurer-Cartan 1-form we have

Theorem 25.12 $R_h^*\theta(X) = Ad_{h^{-1}}\theta(X)$.

Proof At a given point g, $R_h^*\theta(v_{gh^{-1}}) = \theta(R_{h*}v_{gh^{-1}})$ (definition of pull-back) $= L_{g*}^{-1}(R_{h*}v_{gh^{-1}})$ (definition of θ) $= L_{h*}^{-1} \circ L_{gh^{-1}*}^{-1}(R_{h*}v_{gh^{-1}})$ $= L_{h*}^{-1} \circ R_{h*}(L_{gh^{-1}*}^{-1}v_{gh^{-1}})) = L_{h*}^{-1} \circ R_{h*}\theta(v_{gh^{-1}}) = Ad_{h^{-1}}\theta(v_{gh^{-1}})$. □

25.4 The action of a Lie group on a manifold

Definition If G is a Lie group, M is a differentiable manifold, and $\mathcal{R}: M \times G \to M$ is a differentiable map such that the partial maps $\mathcal{R}_g$ satisfy

$$\mathcal{R}_{gh} = \mathcal{R}_h \circ \mathcal{R}_g$$

$$\mathcal{R}_e(x) = x \quad \text{for all} \quad x \in M$$

then $\mathcal{R}$ is *a right action of G on M* (G acts on M to the right).

Note that the conditions on $\mathcal{R}$ imply that the set of transformations $\{\mathcal{R}_g : g \in G\}$ is a group, a subgroup of Diff(M), the group of diffeomorphisms of M, and $\rho: G \to \{\mathcal{R}_g\}$ is a homomorphism. If ρ is an isomorphism, then *G acts effectively on M*, and we can identify G and $\{\mathcal{R}_g\}$. Further, a right action, $\mathcal{R}$, decomposes M into equivalence classes, $O_x = \{z \in M : z = \mathcal{R}_g x \text{ for } g \in G\}$, *the orbit of x,* and we have the natural projection $\pi: M \to M/G$ given by $\pi: x \mapsto O_x$ (cf., Example 7, Section 9.1).

Similar definitions and properties exist for left actions. Clearly, the structures and concepts introduced here can be thought of as generalizations of those introduced in the previous sections where M was G.

Now, given a right action of G on M, instead of the partial maps $\mathcal{R}_g$, we can consider the partial maps $\mathcal{R}_x : G \to M$ given by $g \mapsto \mathcal{R}(x, g)$ for $x \in M$. In particular, $\mathcal{R}_x(e) = x$ for all $x \in M$, and $\mathcal{R}_{x*} : T_eG \to T_xM$. So for each $v \in T_eG$ we have a vector field on M

$$X_v : M \to TM$$

defined by $X_v : x \mapsto \mathcal{R}_{x*}(v)$.

Then, if $\Xi \in \mathfrak{g}$, the Lie algebra of G, and $v = \Xi(e)$, $\Xi \mapsto v \mapsto X_v$ where X_v is given as above, defines a map

$$\lambda : \mathfrak{g} \to \mathfrak{X}M$$

Definition If $\Xi \in \mathfrak{g}$, $\lambda(\Xi)$ is called *the fundamental vector field (on M) of Ξ* and

$$\lambda(\Xi)(x) = X_v(x) = \mathcal{R}_{x*}(v) \quad (v = \Xi(e)) \tag{25.12}$$

λ is sometimes called "the infinitesimal version" of the action of G on M (see Fig. 25.1).

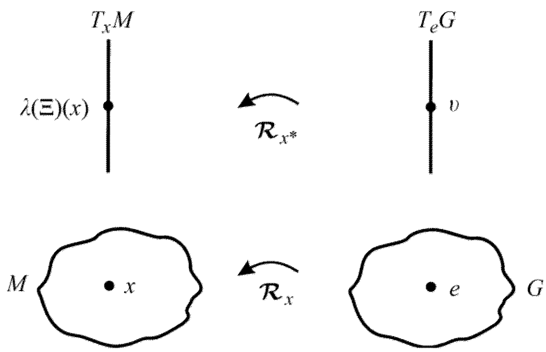

Figure 25.1

We can also obtain the map λ via the 1-parameter subgroups of G. Thus given Ξ we get the 1-parameter subgroup γ_Ξ of G given by $\gamma_\Xi(t) = \exp t\,\Xi$. Now let $\mathcal{A}: \mathbb{R} \times M \to M$ be the 1-parameter group action on M (Section 13.2) given by $\mathcal{A}_t(x) = \mathcal{R}_{\gamma_\Xi(t)}x$. Then the curve $\mathcal{A}_x$ is the image of γ_Ξ under $\mathcal{R}_x$ and the tangent of $\mathcal{A}_x$ at $t = 0$ is the image of Ξ under $\mathcal{R}_{x*}$; i.e., $\lambda(\Xi)x$.

Theorem 25.13 *λ is a Lie algebra homomorphism.*

Proof Problem 25.8. □

We will need to pin down λ a little more, which we can do if we impose conditions on $\mathcal{R}$.

Definitions *G acts freely on M* if there exist $x \in M$ and $g \in G$ such that $\mathcal{R}_g x = x$ then $g = e$. The *x-isotropy subgroup of G* is $G_x = \{g \in G : \mathcal{R}_g x = x\}$.

Theorem 25.14 (i) *G acts freely on $M \Leftrightarrow G_x = e$ for all x.*

(ii) *If x and $\bar{x}$ are in the same orbit, then G_x and $G_{\bar{x}}$ are conjugate subgroups.*

Proof Problem 25.9. □

Theorem 25.15 (i) *λ is an isomorphism of $\mathfrak{g}$ and $\lambda(\mathfrak{g})$ if G acts effectively on M.*

(ii) *$\lambda(\Xi)(x) = 0$ implies $\Xi = 0$ if G acts freely on M.*

Proof (i) If $\lambda(\Xi_1) = \lambda(\Xi_2)$, then their flows are the same; i.e., $\mathcal{R}_{\gamma_{\Xi_1}(t)}x = \mathcal{R}_{\gamma_{\Xi_2}(t)}x$ for all x, t. So $\mathcal{R}_{\gamma_{\Xi_1}(t)} = \mathcal{R}_{\gamma_{\Xi_2}(t)}$, and if G acts effectively on M, then $\gamma_{\Xi_1}(t) = \gamma_{\Xi_2}(t)$ and hence $\Xi_1 = \Xi_2$.

(ii) If $\lambda(\Xi)(x) = 0$, then the tangent of $\mathcal{A}_x$ at x is zero, which implies $\mathcal{A}_x(t) = x$ for all t. But $\mathcal{A}_t(x) = x$ for all t implies $R_{\gamma_\Xi(t)}x = x$, and if G acts freely on M, $\gamma_\Xi(t) = e$, so $\Xi = 0$. □

Corollary *If G acts freely on M, then $\mathcal{R}_{x*}:T_eG \to T_xM$ is 1-1.*

We will need the following important result showing how maps of vector fields Ξ of $\mathfrak{g}$ by the adjoint representation of G produce maps of their fundamental vector fields on M.

Theorem 25.16 *Given $g \in G$, $x \in M$ and $\Xi \in \mathfrak{g}$*

$$\mathcal{R}_{g*}(\lambda(\Xi)(x)) = \lambda(Ad_{g^{-1}}\Xi)(\mathcal{R}_g(x)) \tag{25.13}$$

In words, the image at $\mathcal{R}_g(x)$, of the vector at x of the fundamental field of Ξ, is the vector at $\mathcal{R}_g(x)$ of the fundamental field of $Ad_{g^{-1}}\Xi$ (Fig. 25.2).

Proof Given Ξ and g, if γ_Ξ is the 1-parameter subgroup corresponding to Ξ, then $g^{-1}\gamma_\Xi g$ is the 1-parameter subgroup corresponding to $Ad_{g^{-1}}\Xi$, by Problem 25.6. With $g^{-1}\gamma_\Xi g$ we have the 1-parameter group action on M given by $R_{g^{-1}\gamma_\Xi(t)g} = R_g \circ R_{\gamma_\Xi(t)} \circ R_{g^{-1}}$ with infinitesimal generator $\lambda(Ad_{g^{-1}}\Xi)$. But, by Problem 13.4 the vector field of this flow is $R_{g*}(\lambda(\Xi))$ since $\lambda(\Xi)$ is the infinitesimal generator of $R_{\gamma_\Xi(t)}$. □

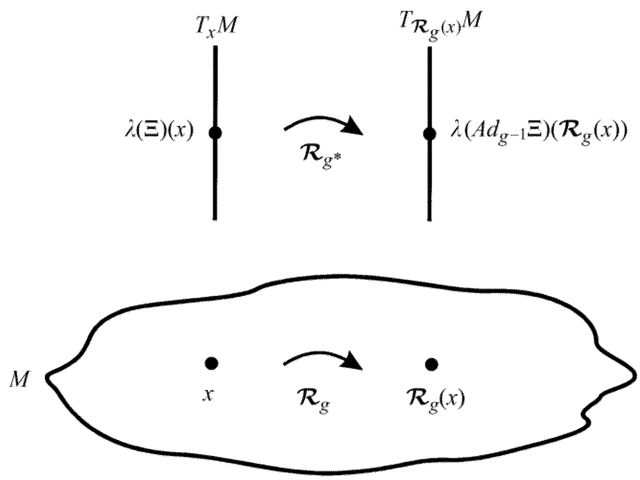

Figure 25.2

There is an analogue of this result if G acts on M to the left; namely,

$$\mathcal{L}_{g*}(\lambda(\Xi)(x)) = \lambda(Ad_g\Xi)(\mathcal{L}_g(x)) \tag{25.14}$$

where $\mathcal{L}_g$ is a partial map of the left action, $\mathcal{L}$.

PROBLEM 25.8 Prove Theorem 25.13. (To show λ preserves the Lie bracket, write it as a Lie derivative; cf., Kobayashi and Nomizu, p. 42.)

PROBLEM 25.9 Prove Theorem 25.14.

PROBLEM 25.10 Prove eq. (25.14).

26

FIBER BUNDLES

In physics one has various fields defined over regions of spacetime, M. Such fields have their values in manifolds, E, such as the tangent manifold, the cotangent manifold, and manifolds of tensor spaces. These manifolds have the property that there exists a differentiable map, π, of E onto M taking all the points of a vector space at $x \in M$ to x. A manifold, E, of the form $E = M \times F$, where F is a manifold clearly has this property. Moreover, the product $M \times F$ also has a projection to F.

Now, in Section 11.1 we have seen that TM looks locally like $\mathcal{U} \times \mathbb{R}^n$, but there is no projection of TM to $\mathbb{R}^n$. We use this property - E looks locally like $\mathcal{U} \times \mathbb{R}^n$ - to generalize the concept of a product manifold.

26.1 Principal fiber bundles

Definitions Given manifolds M, E, and a differentiable map $\pi : E \to M$ onto M, we say *E (or π) is locally trivial* if every $x \in M$ has a neighborhood, $\mathcal{U}$, and a diffeomorphism.

$$\psi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times F$$

of the form

$$\psi(p) = (\pi(p), \phi(p)) \tag{26.1}$$

for $p \in \pi^{-1}(\mathcal{U})$, and for some manifold, F, and for some differentiable map, $\phi : \pi^{-1}(\mathcal{U}) \to F$. $(\mathcal{U}, \psi)$ is *a local trivialization of* E, F is *the fiber*, and $\pi^{-1}(x)$ is *the fiber at* x.

Condition (26.1) can be written $\pi = \boldsymbol{\pi}_1 \circ \psi$ for points in $\pi^{-1}(\mathcal{U})$, where $\boldsymbol{\pi}_1$ is the projection $\mathcal{U} \times F \to \mathcal{U}$. Moreover, condition (26.1) implies that $\phi|_{\pi^{-1}(x)}$ is $1-1$ onto F, and $\psi(\pi^{-1}(x)) = (x, \phi(\pi^{-1}(x)))$, so

$$\phi\Big|_{\pi^{-1}(x)} = \boldsymbol{\pi}_2 \circ \psi\Big|_{\pi^{-1}(x)} \tag{26.2}$$

Local triviality has certain important consequences.

Definition If π is a differentiable map of E onto M and $\mathcal{U}$ is open in M, a differentiable map $\sigma : \mathcal{U} \to E$ such that $\pi \circ \sigma = id$ is *a local section of* E.

Theorem 26.1 *If E (or π) is locally trivial, then*
(i) *there exist local sections of E*
(ii) *for each $x \in M$, $\pi^{-1}(x)$ is a closed imbedded submanifold of E,*
(iii) *for each $x \in M$, $\phi\big|_{\pi^{-1}(x)}$ is a diffeomorphism of $\pi^{-1}(x)$ with F.*

Proof (i) $\sigma : \mathcal{U} \to E$ according to $x \mapsto \psi^{-1}(x, i(x))$ for differentiable i, is a local section.

(ii) Since $\boldsymbol{\pi}_1$ is a submersion and ψ is a diffeomorphism $\pi = \boldsymbol{\pi}_1 \circ \psi$ is a submersion, so by (a slight extension of) Theorem 10.7, $\pi^{-1}(x)$ is a closed embedded submanifold of E.

(iii) Under $\psi|_{\pi^{-1}(x)}$ the manifold $\pi^{-1}(x)$ is diffeomorphic with the manifold $\{x\} \times F$ which is in turn diffeomorphic with F, so by (26.2) $\phi|_{\pi^{-1}(x)}$ is a diffeomorphism. □

We will first consider an important class of fiber bundles from which we will subsequently construct a more general object. Let E, M, and π be as in the above definition, and let E be locally trivial.

Definition Suppose that $F = G$, a Lie group, and there exists a free right action $\mathcal{R} : E \times G \to E$ of G on E, such that $\pi(p) = \pi(q)$ iff $p \sim q$ in the equivalence relation determined by $\mathcal{R}$ (Section 25.4). That is, the fiber $\pi^{-1}(\pi(p))$ at $x = \pi(p)$ is the orbit, O_p, of p (of $\mathcal{R}$). Finally, suppose that on $\pi^{-1}(\mathcal{U})$, ϕ and $\mathcal{R}$ are related by

$$\phi(\mathcal{R}_g(p)) = \phi(p)g \tag{26.3}$$

(equivariance). Then $(E, M, \pi, G, \mathcal{R})$ is *a (differentiable) principal fiber bundle over M with group G* (Fig. 26.1).

We will replace E in this case by P, usually put $\mathcal{R}_g(p) = pg$, and denote the fiber bundle $(P, M, \pi, G, \mathcal{R})$ by P.

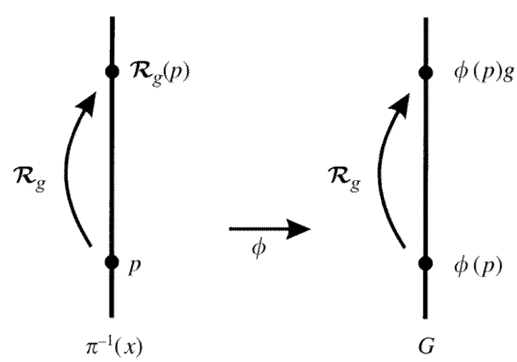

Figure 26.1

Note that (1), with (26.3), $\mathcal{R}$ is transitive on fibers (if p, q are in $\pi^{-1}(x)$, there is a g such that $p = qg$), and with each local section, σ, of P, we get a local trivialization $(\mathcal{U}, \psi)$ given by $\psi(\sigma(x)g) = (x, g)$ such that $\sigma(x) = \psi^{-1}(x, e)$, and (2) for each $p \in P$, $\mathcal{R}_p : G \to P$ given by $g \mapsto \mathcal{R}_p(g) = \mathcal{R}_g(p)$ is a diffeomorphism of G and $\pi^{-1}(\pi(p))$.

With local triviality we can choose *a trivialization of* P: i.e., a covering $\{\mathcal{U}_\alpha\}$ of M, and a set of local trivializations, $\{(\mathcal{U}_\alpha, \psi_\alpha)\}$, with $\psi_\alpha = (\pi, \phi_\alpha)$. Now for $p \in \pi^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$

$$(\phi_\alpha(pg))(\phi_\beta(pg))^{-1} = \phi_\alpha(p)g(\phi_\beta(p)g)^{-1} = \phi_\alpha(p)(\phi_\beta(p))^{-1}$$

for all g, so $(\phi_\alpha(p))(\phi_\beta(p))^{-1}$ is the same for all p on a fiber. Therefore, for each α, β we can define a map

$$\phi_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G$$

by

$$\phi_{\alpha\beta}(x) = \phi_\alpha(\pi^{-1}(x))(\phi_\beta(\pi^{-1}(x)))^{-1}.$$

$\phi_{\alpha\beta}$ are called *transition functions* (of the trivialization).

Theorem 26.2 *The transition functions, $\phi_{\alpha\beta}$, have the property*

$$\phi_{\alpha\beta}(x) \cdot \phi_{\beta\gamma}(x) = \phi_{\alpha\gamma}(x) \quad \text{(compatibility)} \tag{26.4}$$

for $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$. In particular, $\phi_{\alpha\alpha}(x) = e$, and $\phi_{\alpha\beta}(x) = (\phi_{\beta\alpha}(x))^{-1}$.

Proof Problem 25.2. □

Denoting the restrictions $\phi_\alpha|_{\pi^{-1}(x)}$ by $\phi_{\alpha x}$ we get the following relation between the transition functions $\phi_{\alpha\beta}$ and the maps $\phi_{\alpha x}$ and $\phi_{\beta x}$. If $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ and $g \in G$, then for any $p \in \pi^{-1}(x)$ such that $\phi_\beta(p) = g$, $\phi_{\alpha\beta}(x) = \phi_{\alpha x}(p)g^{-1}$ and $(\phi_{\alpha x} \circ \phi_{\beta x}^{-1})(g) = \phi_{\alpha x}(p)$ so

$$\phi_{\alpha\beta}(x)g = (\phi_{\alpha x} \circ \phi_{\beta x}^{-1})(g) \tag{26.5}$$

With the relation (26.5) and the relation

$$\phi_x(\sigma(x)g) = g \tag{26.6}$$

between local trivializations and local sections, we get a relation between transition functions and local sections.

Theorem 26.3 *If σ_α and σ_β are local sections on $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$, respectively, and $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$, then*

$$\sigma_\beta(x) = \sigma_\alpha(x)\phi_{\alpha\beta}(x)$$

Proof For $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$, let $\gamma(x) \in G$ be given by $\mathcal{R}_{\gamma(x)}\sigma_\alpha(x) = \sigma_\beta(x)$; i.e.,

$$\sigma_\beta(x) = \sigma_\alpha(x)\gamma(x) \tag{26.7}$$

Then from (26.5) and (26.6)

$$\begin{aligned}\phi_{\alpha\beta}(x)g &= \phi_{\alpha x}(\phi_{\beta x}^{-1}(g)) = \phi_{\alpha x}(\sigma_\beta(x)g)\\ &= \phi_{\alpha x}(\sigma_\alpha(x)\gamma(x)g) = \phi_{\alpha x}(\sigma_\alpha(x)(g\gamma(x))) = g\gamma(x)\end{aligned}$$

So from (26.7), $\sigma_\beta(x) = \sigma_\alpha(x)g^{-1}\phi_{\alpha\beta}(x)g$ and we can put $g = e$ to get our result. □

With two coverings $\{\mathcal{U}_{\bar{\alpha}}\}$ and $\{\mathcal{U}_{\bar{\bar{\alpha}}}\}$ of M of a given principal fiber bundle, P has corresponding trivializations and their transition functions. We can replace these by trivializations $\{(\mathcal{V}_\alpha, \phi_\alpha)\}$ and $\{(\mathcal{V}_\alpha, \phi'_\alpha)\}$ with transition functions $\phi_{\alpha\beta}$ and $\phi'_{\alpha\beta}$ with a common covering $\{\mathcal{V}_\alpha = \mathcal{U}_{\bar{\alpha}} \cap \mathcal{U}_{\bar{\bar{\alpha}}}\}$.

Theorem 26.4 *If $\{(\mathcal{V}_\alpha, \phi_\alpha)\}$ and $\{(\mathcal{V}_\alpha, \phi'_\alpha)\}$ are trivializations of a principal fiber bundle, then there exist mappings $\lambda_\alpha : \mathcal{V}_\alpha \to G$ such that*

$$\phi'_{\alpha\beta}(x) = \lambda_\beta(x)\phi_{\alpha\beta}(x)(\lambda_\alpha(x))^{-1} \quad x \in \mathcal{V}_\alpha \cap \mathcal{V}_\beta \tag{26.8}$$

Proof For each α there is a map λ_α such that $\phi'_\alpha(p) = \lambda_\alpha(p)\phi_\alpha(p)$ for $p \in \pi^{-1}(\mathcal{V}_\alpha)$. So

$$\begin{aligned}\phi'_\alpha(p)\phi'_\beta(p)^{-1} &= \lambda_\alpha(p)\phi_\alpha(p)\big(\lambda_\beta(p)\phi_\beta(p)\big)^{-1}\\ &= \lambda_\alpha(p)\big(\phi_\alpha(p)(\phi_\beta(p))^{-1}\big)(\lambda_\beta(p))^{-1}\\ &\qquad \text{for} \quad p \in \pi^{-1}(\mathcal{V}_\alpha \cap \mathcal{V}_\beta)\end{aligned} \tag{26.9}$$

Furthermore, for any $\mathcal{V} \in \{\mathcal{V}_\alpha\}$, $\lambda(p)\phi(pg) = \lambda(p)\phi(p)g = \phi'(p)g = \phi'(pg) = \lambda(pg)\phi(pg)$ so $\lambda(p) = \lambda(pg)$. Hence λ depends only on $x = \pi(p)$. With this, (26.9) becomes (26.8). □

The importance of transition functions arises from the fact that they can be used to *construct* (principal) fiber bundles. That is, when we put together pieces of M, they tell us how the fibers fit together. More specifically, given M, G, a covering $\{\mathcal{U}_\alpha\}$, and a set of transition functions $\{\phi_{\alpha\beta}\}$ satisfying eq. (26.4), we can construct $P, \mathcal{R}$, and π with the required properties and with these transition functions (cf. Kobayashi and Nomizu, p. 52, or Auslander and MacKenzie, p. 162).

Principal bundles constructed in this way are called *principal coordinate bundles.* Different coverings and different transition functions will give different coordinate bundles. With a suitable definition of equivalence of coordinate bundles, it turns out that the condition of Theorem 26.4 is necessary and sufficient for two principal coordinate bundles to be equivalent, so we can, alternatively, define a principal fiber bundle to be an equivalence class of principal coordinate bundles (cf., Steenrod, p. 12).

PROBLEM 26.1 Given M and G, a principal bundle, P, is *trivial* if $P = M \times G$. Prove P is trivial iff there is a section, σ, defined and differentiable on all of M.

PROBLEM 26.2 Prove Theorem 26.2.

26.2 Examples

1. Let $P = \mathbb{R}^{n+1} - 0$, $G = \mathbb{R} - 0 = GL(1, \mathbb{R})$ and $\mathcal{R}: P \times G \to P$ by $(p, g) \mapsto pg$ ($p = (p_1, \ldots, p_{n+1})$ and $pg = (p_1 g, \ldots, p_{n+1} g)$). This gives us equivalence classes $[p] = O_p$ (lines through the origin of $\mathbb{R}^{n+1}$) which can be made into a manifold $P^n(\mathbb{R})$, real projective space (cf., Problems 9.5 and 9.8). Then let $M = P^n(\mathbb{R})$ and $\pi: \mathbb{R}^{n+1} - 0 \to M$ by $p \mapsto [p]$.

Now, in our definition of principal fiber bundle, $\mathcal{R}$ and π are required to be differentiable. Clearly $\mathcal{R}$ is, and if we introduce coordinates $(\mathcal{U}_i, \mu_i)$, $i = 1, \ldots x+1$, in $P^n(\mathbb{R})$ as we did in Problem 9.8, then we see that π is differentiable also. Further, in a coordinate neighborhood of $P^n(\mathbb{R})$, $\psi_i: \pi^{-1}(\mathcal{U}_i) \mapsto \mathcal{U}_i \times G$ given by $p \mapsto (x, p_i)$, where $x = \pi(p)$, (that is, $\phi_i(p) = p_i$) is a diffeomorphism and for $x \in \mathcal{U}_i \cap \mathcal{U}_j$, the transition functions are $\phi_{ij}(x) = p_i/p_j$. Thus, we have constructed a principal fiber bundle.

Finally, note that this is a nontrivial principal fiber bundle. This can be easily seen in the 1-dimensional case where $P^1(\mathbb{R})$ can be visualized as the set of lines in the plane through the origin and the values of a section as a curve intersecting each line precisely once. To be defined everywhere, the curve would have to go through the origin.

2. There is an important class of principal fiber bundles which we will need to make use of to fit the concepts of "connections" which we described in Chapters 16 and 17 into a more general concept, which we will describe in Chapter 27.

Let M be an n-dimensional manifold, and in each of its tangent spaces, T_x, let F_x be the set of all ordered bases, $(e_1, \ldots, e_n)_x$, *frames* of T_x. $FM = \bigcup_{x \in M} F_x$ can be made into a manifold, *the frame manifold of* M, as we did for TM in Section 11.1. That is, if $(\mathcal{U}, \mu)$ is a chart on M, then choose $(\pi^{-1}(\mathcal{U}), \nu)$ to be a chart on FM, where $\pi : (e_1, \ldots, e_n)_x \mapsto x$ and $\nu : \pi^{-1}(\mathcal{U}) \to \mathbb{R}^{n+n^2}$ by $p \mapsto (\mu^i(x), e^j_k(x))$ where $e_k(x) = e^j_k(x)\frac{\partial}{\partial \mu^j}(x)$.

Definition *The bundle of (linear) frames* of M* consists of the following ingredients. Given M, let $P = FM$, and $\pi: FM \to M$ as above. Let $G = GL(n, \mathbb{R})$ and $\mathcal{R}: P \times G \to P$ by $((e_1, \ldots, e_n)_x, (a^i_j)) \mapsto (e_i a^i_1, \ldots, e_i a^i_n)_x$.

Note that $\pi(p) = \pi(q)$ iff $p \sim q$, and from their coordinate representations it is clear that π and $\mathcal{R}$ are differentiable. Further, on $\mathcal{U}$, $\psi: \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times G$ is given by $(e_1, \ldots, e_n)_x \mapsto (x, e^i_j(x))$. That is $\phi(p) = e^i_j(\pi(p))$. On $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, $e^i_j \frac{\partial}{\partial \mu^i} = \bar{e}^p_j \frac{\partial}{\partial \bar{\mu}^p}$ so

$$\phi_{\alpha\beta}(x) = (e^i_j(x))(\bar{e}^p_q(x))^{-1} = \left(\frac{\partial \mu^i}{\partial \bar{\mu}^j}(x)\right) \tag{26.10}$$

(cf., Section 16.1).

On the frame bundle there exists an $\mathbb{R}^n$ valued 1-form which will be needed when we get to linear connections in Section 27.3. Note that there is a 1-1 correspondence between frames $p \in F_x$ and linear maps

$$\mathbf{u}_p : \mathbb{R}^n \to T_x M$$

given by $(a^1, \ldots, a^n) \mapsto a^i e_i$. Clearly $\mathbf{u}_p$ is a vector space isomorphism.

Definition *The solder form, Σ, of FM* is an $\mathbb{R}^n$ valued 1-form on FM given by $\Sigma_p(\xi_p) = \mathbf{u}_p^{-1}(\pi_*(\xi_p))$, $\xi_p \in T_p FM$.

Theorem 26.5 (i) $\Sigma(\xi) = 0$ *when ξ is in the tangent space of $\pi^{-1}(x)$. (Σ is horizontal)*

(ii) $\Sigma_{pg} \cdot \mathcal{R}_{g*}(\xi_p) = g^{-1} \cdot \Sigma_p(\xi_p)$. *($\Sigma$ is $\mathbb{R}^n$-equivariant)*

Note that since in this case $G = GL(n, \mathbb{R})$, $g^{-1} \cdot \Sigma(\xi)$ denotes the operation of the matrix g^{-1} on $\Sigma(\xi)$.

Proof (i) $\pi_*(\xi) = 0$ when ξ is in the tangent space of $\pi^{-1}(x)$ (cf., Problem 11.7).

(ii) $\Sigma_{pg}\mathcal{R}_{g*}(\xi_p) = \mathbf{u}_{pg}^{-1}\pi_*(\mathcal{R}_{g*}(\xi_p)) = (\mathbf{u}_p g)^{-1}\pi_*(\xi_p) = g^{-1}\Sigma_p(\xi_p)$. □

*This is sometimes called *the bundle of bases* – the terminology *bundle of frames* and notation FM being reserved for orthonormal frames when the spaces T_x, have a metric structure.

PROBLEM 26.3 Construct a principal fiber bundle with $P = R$ and $G = Z$, the additive group of integers. (See Example (2) in Section 25.1.)

26.3 Associated bundles

Theorem 26.6 *Given a principal bundle $(P, M, \pi, G, \mathcal{R})$, a manifold F, and a left action $\mathcal{L}: G \times F \to F$. $\mathcal{R}': (P \times F) \times G \to P \times F$ given by $((p, f), g) \mapsto (\mathcal{R}_g p, \mathcal{L}_{g^{-1}} f)$ is a right action on $P \times F$.*

Proof $\mathcal{R}'_h \circ \mathcal{R}'_g(p, f) = \mathcal{R}'_h(\mathcal{R}_g p,\ \mathcal{L}_{g^{-1}} f) = (\mathcal{R}_h(\mathcal{R}_g p),\ \mathcal{L}_{h^{-1}}(\mathcal{L}_{g^{-1}} f)) = (\mathcal{R}_h \circ \mathcal{R}_g(p),\ \mathcal{L}_{h^{-1}} \circ \mathcal{L}_{g^{-1}}(f)) = (\mathcal{R}_{gh} p, \mathcal{L}_{h^{-1}g} f) = \mathcal{R}'_{gh}(p, f)$ and $\mathcal{R}'$ is differentiable since both $\mathcal{R}$ and $\mathcal{L}$ are. □

With $P \times F, G$, and M we can construct

(i) $E = (P \times F)/G$, the quotient set of $P \times F$ with respect to G with projection $\bar{\pi}: P \times F \to E$ and elements $[(p, f)]$ with $(p, f) \sim (pg, g^{-1} f)$. (Again, we will usually abbreviate $\mathcal{R}_g p$ by pg and also $\mathcal{L}_g f$ by gf.)

(ii) A map $\pi_E : E \to M$ given by $\pi_E \circ \bar{\pi} = \pi \circ \boldsymbol{\pi}_1$, i.e., according to the diagram

$$\begin{tikzcd} P \arrow[d, "\pi"'] & P \times F \arrow[l, "\boldsymbol{\pi}_1"'] \arrow[d, "\bar{\pi}"] \\ M & E \arrow[l, "\pi_E"] \end{tikzcd}$$

Theorem 26.7 *We can define a differentiable structure in E such that*

(1) π_E is differentiable,

(2) for each $x \in M$, there is a neighborhood $\mathcal{U}$ and a diffeomorphism $\Psi: \pi_E^{-1}(\mathcal{U}) \to \mathcal{U} \times F$ of the form

$$\Psi(\zeta) = (\pi_E(\zeta),\ \Phi(\zeta)) \tag{26.11}$$

for $\zeta \in \pi_E^{-1}(\mathcal{U})$, and a differentiable map $\Phi: \pi_E^{-1}(\mathcal{U}) \to F$.

(3) $\Phi_{ij} = \Phi_i\big|_{\pi_E^{-1}(x)} \circ \Phi_j\big|_{\pi_E^{-1}(x)}^{-1}$ is a differentiable function of $x \in \mathcal{U}_i \cap \mathcal{U}_j$, and $\Phi_{ij}(x) \subset \{\mathcal{L}_g : g \in G\} \subset \mathrm{Diff}(F)$; that is, $\Phi_{ij}(x)$ is in $\{\mathcal{L}_g\}$ for some $g \in G$.

Proof We first define for a given $\mathcal{U}$, a bijection $\Psi : \pi_E^{-1}(\mathcal{U}) \to \mathcal{U} \times F$; namely, for $\zeta \in \pi_E^{-1}(\mathcal{U})$, $\Psi(\zeta) = (\pi_E(\zeta),\ \Phi(\zeta))$ where $\Phi(\zeta) = \mathcal{L}_{\phi(p)} f$ and ϕ comes from $\psi = (\pi, \phi)$. Note that Φ, indeed, only depends on ζ. If we define a map $\eta : \mathcal{U} \times F \to \pi_E^{-1}(\mathcal{U})$ by $(x, f) \mapsto [(\psi^{-1}(x, e), f)]$ then $\Psi \circ \eta$ and $\eta \circ \Psi$ are both identities so Ψ is bijective (cf., Poor, p. 29), so we can make it a diffeomorphism, and we have (2).

Now, using the map η we get for $\zeta \in \pi_E^{-1}(x)$

$$\Phi_i(\zeta) \circ \Phi_j^{-1}(\zeta) f = \mathcal{L}_{\phi_i(p)\phi_j^{-1}(p)} f \tag{26.12}$$

which shows that Φ_{ij} depends only on x, and that $\Phi_{ij}(x)$ is in $\{\mathcal{L}_g : g \in G\}$ and, in particular, is a diffeomorphism, so we have (3).

Finally, since $\Phi_{ij} \circ \Phi_{jk} = \Phi_{ik}$ (compatibility) we can put the pieces $\pi_E^{-1}(\mathcal{U})$ together to make a coordinate bundle (Auslander and MacKenzie, p. 162, or Poor, p. 9) out of E, so that, in particular, it has a differentiable structure and π_E is differentiable. □

Definitions With the differentiable structure on $E = (P \times F)/G$ of Theorem 26.7, E is *a fiber bundle associated with P with fiber F*. The Φ_{ij} are again called *transition functions*. Eq. (26.12) is often described by saying the Φ_{ij} are *the associated transition functions of ϕ_{ij} via the "representation" $\rho : G \to \{\mathcal{L}_g\}$.*

Notice that with parts (1) and (2) of Theorem 26.7 we have the ingredients and conditions described at the very beginning of Section 26.1. Thus, local triviality gives local sections, the fibers at x are closed imbedded submanifolds of E, and for each $x \in M$, $\Phi\big|_{\pi_E^{-1}(x)}$ is a diffeomorphism of $\pi_E^{-1}(x)$ with F. Moreover, we now have, for each $p \in P$, a diffeomorphism $\bar{\pi}_p : F \to \pi_E^{-1}(\pi(p))$ given by the partial map $\bar{\pi}_p : f \mapsto [(p, f)]$, and for $\bar{\pi}_p$ we have the commutativity $\bar{\pi}_p(gf) = \bar{\pi}_{pg}(f)$ (Fig. 26.2).

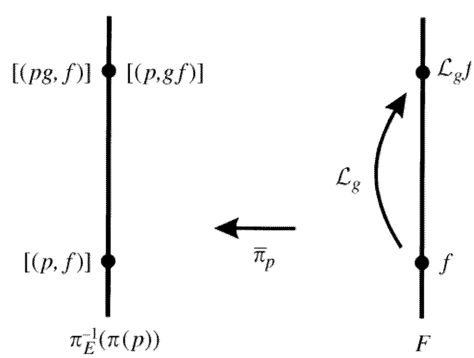

Figure 26.2

Theorem 26.8 *The group of automorphisms of each fiber at x is isomorphic to G.*

Proof By definition an automorphism of a fiber is a map of the form $\bar{\pi}_{pg} \circ \bar{\pi}_p^{-1}$. □

Now we make the following definition.

Definition *A fiber bundle with structure group G and fiber F* consists of a Lie group, G, manifolds E (not necessarily of the form $P \times F/G$), M, and F, a map of E onto M, and a left action $\mathcal{L}: G \times F \to F$ which together satisfy the conditions of Theorem 26.7.

According to this definition every fiber bundle associated with a principal bundle is a fiber bundle.

We also have the converse, in the following sense. Given a fiber bundle, we can define its "underlying" principal bundle by taking $F = G$, and $\mathcal{L}$ to be left multiplication. Then we can define a right action by G on E, and E will be a principal bundle. Then, the bundle associated with it with fiber F will correspond to the given fiber bundle. That is, every fiber bundle is a bundle associated with its "underlying" principal bundle (cf., Steenrod, p. 36 and Poor, p. 31).

PROBLEM 26.4 Prove that the diagram above Theorem 26.7 defines a map from E onto M.

PROBLEM 26.5 If σ_i, σ_j are local sections of E, then on $\mathcal{U}_i \cap \mathcal{U}_j$, $\Phi_i \circ \sigma_i(x) = \Phi_{ij}(\Phi_j \circ \sigma_j(x))$ (cf., Theorem 26.3).

26.4 Examples of associated bundles

1. The Möbius strip. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ with center at the origin, and let G be the group $\{e, g\} = Z_2$ where e is the identity and $g \cdot g = e$. Then S^n/G is the projective space $P^n(\mathbb{R})$. Now let $P = S^1$ and $M = P^1(\mathbb{R}) = S^1/G$ where $G = Z_2$. Let $\mathcal{R}: S^1 \times Z_2 \to S^1$ be given by $\mathcal{R}_e: p \mapsto p$ and $\mathcal{R}_g: p \mapsto -p$. $(P, M, \pi, G, \mathcal{R})$ will be a principal fiber bundle if P has a local trivialization. Let us use the covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of $P^1(\mathbb{R})$ of Problem 9.8 (for the case $n = 1$). We can define local trivializations $(\mathcal{U}_1, \psi_1)$ and $(\mathcal{U}_2, \psi_2)$ by defining ϕ_1 and ϕ_2 on points $p \in S^1$ such that $\pi(p) \in \mathcal{U}_1$ and $\pi(p) \in \mathcal{U}_2$, respectively, and in particular, on the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$.

Specifically, let $\phi_1(p) = e$ and $\phi_2(p) = g$. Then on $\mathcal{U}_1 \cap \mathcal{U}_2$ the transition function $\phi_{12}(x) = \phi_1(p) \cdot (\phi_2(p))^{-1} = g$.

With this principal bundle and with $F = I$, the interval $(-1, 1)$ and $\mathcal{L}: G \times F \to F$ given by $(e, r) \mapsto r$, and $(g, r) \mapsto -r$ we have an associated bundle of P, the Möbius strip. The transition function Φ_{12}, from Eq. (26.12), on $\mathcal{U}_1 \cap \mathcal{U}_2$ maps r to $\mathcal{L}_{\phi_1(p)\phi_2^{-1}(p)} r = -r$, which we can describe by saying that a "twist" occurs on $\mathcal{U}_1 \cap \mathcal{U}_2$.

2. The most important class of associated bundles for geometry and physics are the vector bundles.

Definition A fiber bundle associated with P with fiber F is *a vector bundle* if F is a vector space, V, and the group $\{\mathcal{L}_g\}$ is a subgroup of Aut V. If $V = \mathbb{R}^m$, then $\{\mathcal{L}_g\}$ is a subgroup of $GL(m, \mathbb{R})$*.

Vector bundles can be thought of as generalizations of structures we have already dealt with quite extensively; namely, tangent manifolds, TM, and the various other tensor manifolds T^*M, T^r_sM, $\bigwedge^s(T^*M)$, etc. (Section 11.3).

(i) The vector bundle associated with the frame bundle FM with fiber $F = \mathbb{R}^n$ ($n = \dim M$), and $\{\mathcal{L}_g\} = GL(n, \mathbb{R})$ is called *the tangent bundle of* M. There is a 1-1 correspondence between points of $\pi_E^{-1}(x)$ and T_x given by $[((e_1, \ldots, e_n)_x, a)] \longleftrightarrow (x, a^i e_i)$ where $a = (a^1, \ldots, a^n) \in \mathbb{R}^n$ so we identify this bundle with the tangent manifold TM.

(ii) With P and G the same as in (i) with fiber $F =$ the tensor product $(\mathbb{R}^n)^r_s$ we get *a tensor bundle* which we call T^r_sM because of the 1-1 correspondence given by $[((e_1, \ldots, e_n)_x, A)] \longleftrightarrow (x, A^{i_1\cdots i_r}_{j_1\cdots j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_s})$ where $A = \{A^{i_1\cdots i_r}_{j_1\cdots j_s}\} \in (\mathbb{R}^n)^r_s$ between points of $\pi_E^{-1}(x)$ and points of $(T_x)^r_s$ thus enabling us to identify this bundle with the tensor manifold T^r_sM in Section 11.3. Similarly, we get a bundle, $\bigwedge^s(T^*M)$, which we identify with the manifold of exterior s-forms.

Finally, with each given vector bundle E we can construct new vector bundles.

Definitions Let E_x^* be the dual vector space of the fiber E_x. Then $E^* = \bigcup_{x \in M} E_x^*$ is *the dual bundle of* E. *The tensor product bundle of* E_1 *and* E_2, $E_1 \otimes E_2$, is the vector bundle whose fibers are $E_{1x} \otimes E_{2x}$. Continuing, we have *the tensor product bundles, exterior s-form bundles*, etc.

Note also that for a vector bundle any local section, σ, can be written as a linear combination of local sections called *a basis of local sections*.

PROBLEM 26.6 In the light of the map between $\pi_E^{-1}(x)$ and T_x in the definition of tangent bundle, the map $\mathbf{u}_p$ in Section 26.2 used to define the canonical form of FM is a special case of $\Phi\big|^{-1}_{\pi_E^{-1}(\pi(p))}$ occurring in Section 26.3.

PROBLEM 26.7 In a vector bundle each fiber at x is a vector space isomorphic with the fiber, V.

*Notice that we have defined a vector bundle as a fiber bundle associated with a principal bundle. One frequently defines a vector bundle as a particular kind of fiber bundle, without reference to a principal bundle. (Cf., Poor, p. 12, Choquet-Bruhat and DeWitt-Morette, p. 125.)

27

CONNECTIONS ON FIBER BUNDLES

In Section 17.1, we noted that in general there is no a priori relation between the fibers of the tangent bundle of a manifold, and that an additional structure "a connection" provides such a relation. The same is true for fiber bundles generally. We will now introduce an additional structure on fiber bundles which relates the fibers in the sense that to each given curve in M and point in E, there is a unique curve in E along which we say a point is parallelly propagated (translated, transported, or displaced). Once we have a connection, we can define the curvature of the connection.

27.1 Connections on principal fiber bundles

Definition *A connection on a principal fiber bundle, P, with group G,* is a differentiable map (Fig. 27.1)

$$\Gamma : P \to \mathcal{L}(T_xM, T_pP) \qquad x = \pi(p)$$

such that

(1) $\pi_* \circ \Gamma_p = id$ for all p

(2) $\Gamma_{pg}(v) = \mathcal{R}_{g*} \circ \Gamma_p(v)$ where $v \in T_xM$, $g \in G$, and $pg = \mathcal{R}_g p$.

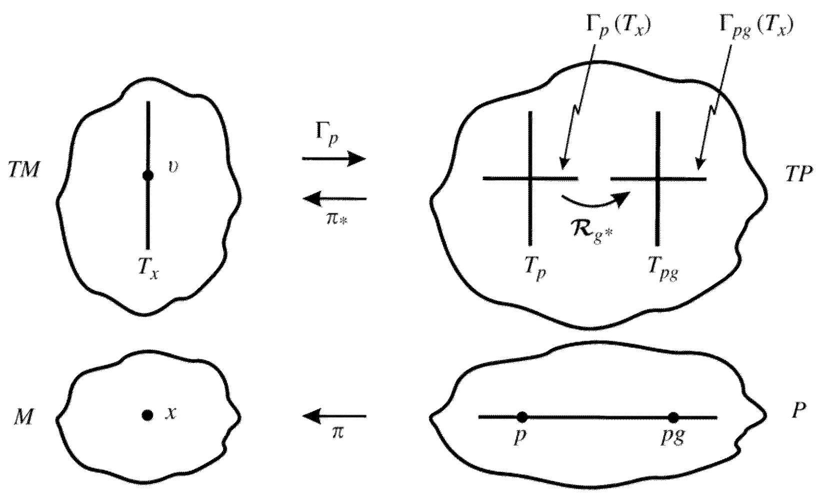

Figure 27.1

Property (1) says that the images of T_xM are n-dimensional vector spaces ($n = \dim M$), and property (2) says that for points on the same fiber the images of T_xM map onto one another by the tangent map of $\mathcal{R}_g$.

Definitions $H_p = \Gamma_p(T_xM)$ is called *the horizontal subspace of* T_pP. If X is a vector field on M, then $\bar{X} : p \mapsto \Gamma_p(X(\pi(p)))$ is a vector field on P which at each point lies in H_p and is called *the horizontal lift of* X (of the connection Γ).

In terms of X, property (2) says that $\bar{X}$ is invariant under the right action of G on P.

Now, given a curve, γ, (an immersion) in M it has a tangent field, $\dot{\gamma}:\mathbb{R} \to TM$, which can be extended to a local vector field X on M (Section 17.2) of which γ is an integral curve. Let $\bar{X}$ be the horizontal lift of X of a connection on P. If p_0 is such that $\pi(p_0)$ is a point of γ, then

$$\dot{\bar{\gamma}}(t) = \bar{X}(\bar{\gamma}(t)) \tag{27.1}$$

has a unique local solution through p_0.

Definitions The solution $\bar{\gamma}_{p_0}$ of eq. (27.1) through p_0 is *the horizontal lift of the curve* γ through $\pi(p_0)$ and points on $\bar{\gamma}$ are said to be *parallelly transported* (*translated, displaced, or propagated*) *along* $\bar{\gamma}$.

Note that we have defined a local object. With some additional effort we can piece these together to get the horizontal lift of an entire curve.

Theorem 27.1 *If $\bar{\gamma}_p$ is a horizontal lift of γ, then $\mathcal{R}_g \circ \bar{\gamma}_p$ is a horizontal lift of γ for all $g \in G$.*

Proof Immediate from eq. (27.1) and property (2) of a connection. □

Theorem 27.1 says that $\bar{\gamma}_{pg}(q) = \mathcal{R}_g \circ \bar{\gamma}_p(q)$. That is, the action of G on P commutes with parallel displacements of points. Thus, with a curve γ in M, a connection gives mappings, τ_γ, *of fibers, the parallel transport of fibers,* defined by the parallel transport of points along any horizontal lift, $\bar{\gamma}$.

We will return to the concept of parallel transport of fibers in Section 27.4 where we will need it in the context of vector bundles. For now we will simply mention, in passing, a very important special application.

If γ is a closed curve through $x \in M$ then the τ_γ are diffeomorphisms of $\pi^{-1}(x)$ onto itself. If C_x is the set of all (piecewise C^∞) curves through $x \in M$, the corresponding set of diffeomorphisms of $\pi^{-1}(x)$ with $\tau_{\gamma_1} \cdot \tau_{\gamma_2} = \tau_{\gamma_1 \circ \gamma_2}$ and $\tau_\gamma^{-1} = \tau_{\gamma^{-1}}$ form a group, $\mathcal{H}_x$, *the holonomy group of* Γ *at* x.

Given a point $p \in \pi^{-1}(x)$, there is a homomorphism $\mathcal{H}_x \to G$ given by $\tau_\gamma(p) = pg$. The image is a subgroup of G, G_p, called *the holonomy group of* Γ *at* p. Clearly, $g \in G_p$ iff pg and p are on the same horizontal lift of some γ; i.e., iff pg and p can be joined by a horizontal curve.

Theorem 27.2 *If $p, q \in \pi^{-1}(x)$, then G_p and G_q are conjugate subgroups, and if, in particular, p and q are on the same horizontal lift of some γ, then $G_p = G_q$.*

Proof Write $p \sim q$ if p and q can be joined by a horizontal curve. Let $p \sim pb$. Also $q = pa$ for some a. Then $q = pa \sim pba$. So $q \sim pba$. But $p = qa^{-1}$. So $q \sim qa^{-1}ba$. The special case follows from the transitivity of $\sim$. □

The set of all points of P that can be joined to a given point p by a horizontal curve is called *the holonomy bundle*, $P(p)$, *at* p. Clearly, Γ decomposes P into the disjoint union of holonomy bundles.

Theorem 27.3 *The holonomy group is the same subgroup of G for all points of a holonomy bundle.*

Proof Problem 27.2. □

We make a couple of general observations illustrating the significance of holonomy groups without going into details (see Kobayashi and Nomizu p. 83 ff., or Choquet-Bruhat and DeWitt-Morette p. 381 ff.).

1. It is often of interest to know whether a given principal bundle has a "reduced bundle"; i.e., a "bundle homomorphism" with a bundle over M whose group, G', is a subgroup of G. It turns out that each holonomy bundle of P is a reduced bundle whose group is its holonomy group.

2. In Section 17.3 we saw that what happened to a vector in TM when it was parallelly transported along a closed curve in M depended on the curvature. One can compare this to an important theorem (Ambrose-Singer) that says that the Lie algebra of the holonomy group of a holonomy bundle depends on the values of the curvature of connection (Section 27.2).

Going back to the beginning of this section we note that the definition given there of a connection can be expressed in a slightly different way. Thus, instead of looking at the map Γ, we can look at the set of images $H_p = \Gamma_p(T_xM)$ in T_pP. That is, we have an n-dimensional (differentiable) *horizontal distribution*, $\mathcal{D}_H : P \to \bigcup_{p\in P} H_p$. Recall, (Problem 11.7) that we also have a *vertical distribution*, $\mathcal{D}_V$, whose vector spaces V_p, are the m-dimensional tangent spaces of the fibers ($m = \dim G$). Since, if H_p and V_p are horizontal and vertical spaces, respectively, then $H_p \cap V_p = 0$, we have $T_pP = H_p \oplus V_p$. Hence, we can alternatively define a connection as an (n-dimensional) distribution $\mathcal{D}_H$, on P such that for each $p \in P$, $g \in G$

(1′) $T_pP = H_p \oplus V_p$

(2′) $H_{pg} = \mathcal{R}_{g*}H_p$

Having a connection on a principal bundle, P, we can construct other objects on P in terms of it. In particular, we can construct a 1-form on P with values in the Lie algebra of G which is both determined by and also uniquely determines a connection, Γ.

Recall, Section 25.4, that with a right action, $\mathcal{R}$, of G on M we have a Lie algebra homomorphism $\lambda : \mathfrak{g} \to \mathfrak{X}M$ defined in terms of the tangent maps $\mathcal{R}_{x*}$ from T_eG to T_xM, so that, in particular, we have these structures on principal bundles. Further, since $\mathcal{R}_p$ is a diffeomorphism from G to $\pi^{-1}(\pi(p))$, $\mathcal{R}_{p*}$ are isomorphisms. We define a map (for each p) $\lambda_p : \mathfrak{g} \to T_pP$ by $\lambda_p : \Xi \mapsto \lambda(\Xi)(p)$, the value of the fundamental field of Ξ at p, so by eq. (25.12) λ_p is an isomorphism of $\mathfrak{g}$ with $T_p(\pi^{-1}(\pi(p))) = V_p$.

Now, if P has a connection, then by (1′) we have the projections $\mathbf{H}_p : T_pP \to H_p$ and $\mathbf{V}_p : T_pP \to V_p$ and we can define maps, ω_p, from T_pP to $\mathfrak{g}$ by

$$\omega_p = \lambda_p^{-1} \circ \mathbf{V}_p \tag{27.2}$$

and a differentiable vector-valued 1-form on P

$$\omega : P \to \bigwedge\nolimits^1(T^*P, \mathfrak{g})$$

by $\omega : p \mapsto \omega_p$ (cf., Section 25.3). ω is called *the connection form* of the connection on P.

Theorem 27.4 *ω has the properties* (1) $\omega_p(\xi) = 0$ *if* $\xi \in H_p$ (ω *is* a vertical form)

(2) *If* $\xi = \lambda_p(\Xi)$, *then* $\omega_p(\xi) = \Xi$

(3) *For* $\xi \in T_pP$,

$$\omega_{pg}(\mathcal{R}_{g*}\xi) = Ad_{g^{-1}}\ \omega_p(\xi) \quad (\omega \textit{ is } \mathfrak{g} - \text{equivariant}) \tag{27.3}$$

Proof (1) Immediate from the definition (27.2).

(2) $\xi \in V_p$ so $\omega_p(\xi) = \lambda_p^{-1}(\xi) = \Xi$.

(3) If $\xi \in H_p$, then $\mathcal{R}_{g*}\xi \in H_{pg}$ by (2′) so both sides of (27.3) vanish. If $\xi \in V_p$ there is a unique Ξ such that $\xi = \lambda_p(\Xi)$, and we apply Theorem 25.16. Thus from eq. (25.13), $\omega_{pg}\mathcal{R}_{g*}(\lambda(\Xi)(p)) = \omega_{pg}(\lambda(Ad_{g^{-1}}\Xi)(pg))$. So, $\omega_{pg}\mathcal{R}_{g*}(\xi) = Ad_{g^{-1}}\omega_p(\xi)$ since $\omega_{pq} \circ \lambda_{pq} = id$, and by property (2). □

Theorem 27.5 *Given a $\mathfrak{g}$-valued form, ω, on P satisfying conditions* (2) *and* (3) *of Theorem 27.4, there is a unique distribution $\mathcal{D}_H$ on P satisfying* (1′) *and* (2′).

Proof Define $H_p = \{\xi \in T_pP : \omega(\xi) = 0\}$. Then $H_p \cap V_p = 0$. Now if $\xi \in T_pP$, let $\omega(\xi) = \Xi$. Let $\eta \in V_p$ be such that $\eta = \lambda(\Xi)$, then by (2), $\omega(\eta) = \Xi$. Hence $\omega(\xi) - \omega(\eta) = 0$, or $\xi - \eta \in H_p$ so every element of T_pP is the sum of an element of V_p and and an element of H_p.

From (3) if $\omega(\xi) = 0$, then $\omega_{pg}(\mathcal{R}_{g*}\xi) = 0$ which implies $\mathcal{R}_{g*}\xi \in H_{pg}$ so $\mathcal{R}_{g_I} H_p \subset H_{pg}$ and hence $\mathcal{R}_{g*}H_p = H_{pg}$ □

To summarize the results of this section, we have shown that a connection on a principal fiber bundle is determined by either a map, Γ, a distribution, $\mathcal{D}_H$, or a $\mathfrak{g}$-valued form, ω.

PROBLEM 27.1 Show that locally $\bar{\gamma}$ is a horizontal curve in P ($\bar{\gamma}(t) \in H_p$) such that $\pi(\bar{\gamma}(t)) = \lambda(t)$ iff $\bar{\gamma}$ satisfies eq. (27.1).

PROBLEM 27.2 Prove Theorem 27.3.

PROBLEM 27.3 The difference of two connection forms is horizontal, and the set, $\mathcal{C}$, of all connection forms on P is an affine space over the vector space of horizontal equivariant forms.

27.2 Curvature

For vector-valued s-forms we generalize some of our former terminology.

1. A $\mathfrak{g}$-valued s-form, θ, on P which satisfies condition (3) of Theorem 27.4 is called $\mathfrak{g}$-*equivariant*. (More generally, for W-valued forms we can write a condition like eq. (27.3) with $Ad_{g^{-1}}$ replaced by a linear transformation, $\mathcal{L}_g$, of W.)
2. An s-form which vanishes when any one of its arguments is vertical is called *horizontal.*
3. *The exterior differential (derivative) of* θ is the $s+1$ form $d\theta$ given by $d\theta(X_1, \ldots, X_{s+1}) = d\theta^i(X_1, \ldots, X_{s+1})E_i$ where $\{E_i\}$ is a basis of W, and θ^i are the components of θ in that basis (see Section 25.3). Such a representation defines a form since $d\theta^i(X_1, \ldots, X_{s+1})E_i$ is the same for all bases. Note also (Section 25.3) that we can write $d\theta = E_i \otimes d\theta^i$.
4. In the presence of a connection, we say an s-form is *vertical* if it vanishes when any of its arguments are horizontal.

Definition If θ is a W-valued s-form on a principal bundle with a connection, then $D\theta$ defined by

$$(D\theta)(X_1, \ldots, X_{s+1}) = d\theta \circ \mathbf{H}(X_1, \ldots, X_{s+1}) = d\theta(\mathbf{H}X_1, \ldots, \mathbf{H}X_{s+1})$$

is *the exterior covariant derivative* of θ.

Theorem 27.6 (i) *$D\theta$ is horizontal.*

(ii) *If θ is equivariant, then $d\theta$ and $D\theta$ are equivariant.*

Proof (i) Obvious.

(ii) We can write eq. (27.3) as $(\mathcal{R}_g^*\theta_{pg})(\xi_1, \ldots, \xi_g) = Ad_{g^{-1}}\theta_p(\xi_1, \ldots, \xi_g)$. Then by Theorem 12.4, $d\theta$ and $D\theta$ satisfy the same equation. □

Definition If ω is a connection form on P, then $\varkappa = D\omega$ is *the curvature form of the connection.*

By Theorem 27.6, the curvature form is horizontal and $\mathfrak{g}$-equivariant.

In order to prove Cartan's Structural Equation for the curvature form we need the following lemma.

Lemma If X is horizontal and Y is vertical, then $[X, Y]$ is horizontal.

Proof Write $[Y, X] = L_{\lambda(A)}X$, the Lie derivative of X with respect to $\lambda(A)$. The flow of $\lambda(A)$ is given by $\mathcal{A}_t = \mathcal{R}_{\gamma_A(t)}$, so in this case the values of $X_p^\#$ are $(\mathcal{R}^*_{\gamma_A(t)}X)(p)$. But $\mathcal{R}^*_{\gamma_A(t)}$ maps horizontal vectors to horizontal vectors, so the derivative, $D_0 X_p^\#$ is horizontal. □

Theorem 27.7 (The Structural Equation) *If ω is a connection form on P, and $\varkappa$ is its curvature form, then*

$$\varkappa(X, Y) = \frac{1}{2}[\omega(X), \omega(Y)] + d\omega(X, Y) \tag{27.4}$$

for $X, Y \in \mathfrak{X}P$.

Proof We prove eq. (27.4) pointwise for the three cases: X, Y both horizontal, X, Y both vertical, and X horizontal and Y vertical.

(i) For horizontal X, Y, the 1st term on the right vanishes and the two sides are the same.

(ii) For vertical X, Y, $\varkappa(X, Y) = 0$. Further, looking at the components of the right side in a basis of $\mathfrak{g}$ we can apply Theorem 12.2(ii). Thus, writing $X = \lambda(A)$, $Y = \lambda(B)$, the right side of 27.4 becomes

$$\frac{1}{2}[\omega(\lambda(A)), \omega(\lambda(B))] + \frac{1}{2}\Big(\lambda(A)\omega(\lambda(B)) - \lambda(B)\omega(\lambda(A)) - \omega([\lambda(A), \lambda(B)])\Big)$$

$$= \frac{1}{2}[A, B] + \frac{1}{2}(\lambda(A)B - \lambda(B)A - [A, B])$$

since λ is a Lie algebra homomorphism (Theorem 25.13). Finally A and B are vector fields wtih constant components, so $\lambda(A)B$ and $\lambda(B)A$ are both zero.

(iii) For X horizontal and Y vertical, now $\varkappa(X,Y)$ and $[\omega(X),\omega(Y)]$ both vanish and using Theorem 12.2(ii) again $d\omega(X,Y) = \frac{1}{2}(X\omega(Y) - Y\omega(X) - \omega([X,Y]))$. But $X\omega(Y) = 0$ because $\omega(Y)$ has constant components, $Y\omega(X) = 0$ because $\omega(X) = 0$ and $\omega([X,Y]) = 0$ by previous lemma. □

Corollary If X and Y are both horizontal eq. (27.4) becomes

$$\varkappa(X,Y) = -\frac{1}{2}\omega([X,Y]) \tag{27.5}$$

Equation (27.5) has an interesting consequence. Namely, if X and Y are horizontal vector fields, $[X,Y]$ is horizontal iff $\varkappa = 0$. That is, the distribution $\mathcal{D}_H$ is involutive and P has an n-dimensional integral submanifold iff $\varkappa = 0$ (cf., Section 14.1). In this case, we say the connection is *flat* (cf., Section 15.3, Section 17.2 and Section 17.3).

It is often convenient to write eq. (27.4) in terms of wedge products of ordinary real-valued forms. We use the general relation

$$\begin{aligned} &\theta^i \wedge \eta^j(\xi_1,\ldots,\xi_{p+q})[e_i,e_j] \\ &\quad = \frac{1}{(p+q)!}\sum_\pi \text{sgn}\ \pi[\theta(\xi_{\pi(1)},\ldots,\xi_{\pi(p)}),\eta(\xi_{\pi(p+1)},\ldots,\xi_{\pi(p+q)})] \end{aligned} \tag{27.6}$$

where θ^i and η^i are the components of the vector-valued forms θ and η in a basis $\{e_j\}$ (cf., Theorem 6.2). In the special case $\theta = \eta = \omega$ with $[e_i,e_j] = c^k_{ij}e_k$, eq. (27.4) becomes

$$\varkappa^k = \frac{1}{2}c^k_{ij}\omega^i \wedge \omega^j + d\omega^k. \tag{27.7}$$

Corollary For vertical vector fields on P, (27.7) reduces to $d\omega^k = -\frac{1}{2}c^k_{ij}\omega^i\wedge\omega^j$, which is the same as the Maurer-Cartan Structure Equation, (25.9), for 1-forms on a Lie group.

Corollary The Bianchi Identity

$$D\varkappa = 0 \tag{27.8}$$

Proof If we differentiate eq. (27.7), we get $d\varkappa^k = \frac{1}{2}c^k_{ij}(d\omega^i \wedge \omega^j - \omega^i \wedge d\omega^j)$. But to get $D\varkappa(X,Y,Z)$ we need only evaluate $d\varkappa$ on horizontal vectors, and on these, the right side vanishes. □

Finally, we note that corresponding to eq. (27.4), there is an equation for the exterior covariant derivative of any *horizontal* $\mathfrak{g}$-equivariant s-form.

Theorem 27.8 *If ω is a connection form on P, and θ is a horizontal $\mathfrak{g}$-equivariant s-form on P, then*

$$D\theta(X_1, \ldots, X_{s+1}) = \frac{1}{(s+1)!} \sum_{\pi} \text{sgn}\ \pi[\omega(X_i), \theta(X_1, \ldots, \hat{X}_i, \ldots X_{s+1}]$$

$$+ d\theta(X_1, \ldots, X_{s+1})$$

Proof See Bishop and Crittenden, p. 86, or Bleecker, p. 45. □

PROBLEM 27.4 Prove eq. (27.6).

27.3 Linear Connections

Recall that in Section 26.2 we described a particular principal fiber bundle, FM, over M called the bundle of linear frames of M.

Definition A connection on the bundle of linear frames of M is called *a linear connection on M.*

In a bundle of linear frames with a connection, it is possible to define certain additional structure in terms of which additional details of the bundle can be described.

(1) In the case of a linear connection on M, we can sort out a special set of n linearly independent horizontal vector fields from the distribution, $\mathcal{D}_H$, as follows. Given a point $a \in \mathbb{R}^n$, then for each $p \in FM$ we get an element of T_xM, $\mathbf{u}_p(a) = a^i e_i$, and from the isomorphisms, $\Gamma_p : T_xM \to H_p$, of the connection, $\mathbf{u}_p(a) \mapsto B_a(p) \in H_p$. So B_a is a horizontal vector field, and

$$\Gamma_p \mathbf{u}_p(a) = B_a(p) \tag{27.9}$$

or, $\mathbf{u}_p(a) = \pi_*(B_a(p))$, or in terms of the solder form, Σ,

$$a = \Sigma_p(B_a(p)) \tag{27.10}$$

for all $p \in FM$.

Definition B_a is called *the basic (or, standard) vector field corresponding to a.*

Theorem 27.9

$$\mathcal{R}_{g*}(B_a(p)) = B_{g^{-1}a}(pg) \tag{27.11}$$

for $p \in FM,\ a \in \mathbb{R}^n$, *and* $g \in GL(n, \mathbb{R})$ (cf., Theorem 25.16).

Proof By eq. (27.9),

$$\begin{aligned} \mathbb{R}_{g*}(B_a(p)) &= \mathbb{R}_{g*}(\Gamma_p \mathbf{u}_p(a)) \\ &= \Gamma_{pg}(\mathbf{u}_p(a)) \quad \text{by the definition of } \Gamma \\ &= \Gamma_{pg}(\mathbf{u}_{pg}(g^{-1}a)) = B_{g^{-1}a}(pg) \quad \text{by (27.9)} \end{aligned}$$

□

Now, let Σ^i be the components of Σ in the basis $\{e_i = (0, \ldots, 1, \ldots, 0)\}$ of $\mathbb{R}^n$. That is, $\Sigma(X) = \Sigma^i(X)e_i$. Then we have n horizontal vector fields, B_{e_j}, with $\Sigma^i(B_{e_j}) = \delta^i_j$.

Further, recall that for the bundle of linear frames $G = GL(n, \mathbb{R})$, so $\mathfrak{g} = gl(n, \mathbb{R})$, the algebra of $n \times n$ matrices (Section 25.2). Let $E^i_j = \begin{array}{cc} \begin{pmatrix} 0 & \cdots & 0 \\ & 1 & \\ 0 & \cdots & 0 \end{pmatrix} & \begin{matrix} \\ j \\ \\ \end{matrix} \\ \begin{matrix} & i & \end{matrix} & \end{array}$,

and let ω^i_j be the components of ω in the basis $\{E^i_j\}$; i.e., $\omega(X) = \omega^j_i(X)E^i_j$. So the fundamental vector fields $\lambda(E^\alpha_\beta)$ satisfy $\omega(\lambda(E^\alpha_\beta)) = \omega^j_i(\lambda(E^\alpha_\beta))E^i_j$, and since $\omega(\lambda(A)) = A$ for $A \in \mathfrak{g}$, we have $\omega^j_i(\lambda(E^\alpha_\beta)) = \delta^i_\alpha \delta^j_\beta$.

We summarize these results, leaving the details to the reader, by the following.

Theorem 27.10 *The fundamental vector fields,* $\lambda(E^i_j)$, *and the basic vector fields,* B_{e_j}, *constitute a set of* $n^2 + n$ *linearly independent non-vanishing vector fields on* FM. *Moreover,* $\{\lambda(E^i_j), B_{e_j}\}$ *and* $\{\omega^i_j, \Sigma^j\}$ *are dual bases of* T_p *and* T^*_p, *respectively, at each point.*

(2) With a connection on a bundle of linear frames we can define an $\mathbb{R}^n$-valued 2-form in terms of the solder form, Σ.

Definition $\tau = D\Sigma$ is called *the torsion form of the connection.*

Theorem 27.11 τ *is horizontal and* $\mathbb{R}^n$*-equivariant.*

Proof Immediate from Theorem 27.6. □

Theorem 27.12 *The Structure Equation for the torsion form.*

$$\tau(X,Y) = \frac{1}{2}(\omega(X)\cdot\Sigma(Y) - \omega(Y)\cdot\Sigma(X)) + d\Sigma(X,Y) \tag{27.12}$$

Proof We proceed pointwise in three cases as in Theorem 27.7, the only non-trivial case being when one argument is horizontal and one is vertical. (Alternatively, there is a generalization of this result for $\mathbb{R}^n$-equivariant forms on the bundle of linear frames analogous to Theorem 27.8, cf., Bishop and Crittenden, p. 100.) □

For linear connections eqs. (27.12) and (27.4) are often referred to as (Cartan's) 1st and 2nd structural equations, respectively.

For any principal bundle with $G = GL(n,\mathbb{R})$ and in particular for a linear connection, using the basis $\{E^i_j\}$ of $\mathfrak{g}$ the structural equation in terms of wedge products of real-valued forms, eq. (27.7), reduces to

$$\varkappa^\alpha_\beta = \omega^\alpha_\gamma \wedge \omega^\gamma_\beta + d\omega^\alpha_\beta \tag{27.13}$$

if we write (27.7) in terms of this basis as $\varkappa^\alpha_\beta = \frac{1}{2}c^{\alpha i_2 j_2}_{\beta i_1 j_1}\,\omega^{i_1}_{i_2} \wedge \omega^{j_1}_{j_2} + d\omega^\alpha_\beta$. Then multiplying both sides of $c^{\alpha i_2 j_2}_{\beta i_1 j_1}E^\beta_\alpha = \delta^{i_2}_{j_1}E^{j_2}_{i_1} - \delta^{j_2}_{i_1}E^{i_2}_{j_1}$ by $\omega^{i_1}_{i_2} \wedge \omega^{j_1}_{j_2}$ we get (27.13).

We can also write eq. (27.12) in terms of wedge products of real-valued forms. Again we use a general results corresponding to eq. (27.6) in which now θ is a form whose values are in a space of linear operators with basis $\{E^i_j\}$, η is a vector-valued form, and the bracket is replaced by $\cdot$, the linear operation. In this case $\theta = \omega$, $\eta = \Sigma$, and $[e_i, e_j]$ becomes $E^i_j \cdot e_k = \delta^i_k e_j$. Using this relation, the first term on the right side of (27.12) becomes $\omega^j_k \wedge \Sigma^k(X,Y)e_j$ and eq. (27.12) becomes

$$\tau^j = \omega^j_k \wedge \Sigma^k + d\Sigma^j \tag{27.14}$$

For linear connections there is a formula for $D\tau$ called *Bianchi's 1st Identity*;

$$D\tau(X,Y,Z) = \frac{1}{3}(\varkappa(X,Y)\cdot\Sigma(Z) + \varkappa(Y,Z)\cdot\Sigma(X) + \varkappa(Z,X)\Sigma(Y)) \tag{27.15}$$

Equation (27.8) is now called *Bianchi's 2nd Identity*

To get eq. (27.15) we differentiate eq. (27.14), use the fact that ω is vertical, $\Sigma(\mathbf{H}X) = \Sigma(X)$, $d\omega(\mathbf{H}X, \mathbf{H}Y) = \varkappa(X,Y)$, and use the result described above eq. (27.14) to go back from an equation for real-valued forms to one for vector-valued forms.

PROBLEM 27.5 Fill in the details of the proof of Theorem 27.10.

PROBLEM 27.6 Derive the relation described above eq. (27.14) for linear operators and vector-valued forms corresponding to eq. (27.6).

PROBLEM 27.7 Fill in the details for the derivation of Bianchi's 1st Identity.

27.4 Connections on vector bundles

In Section 27.1, we defined the concept of a connection for principal fiber bundles. When we introduce this concept for a vector bundle, we can then introduce the "covariant derivative". We will first say what we mean by a connection on a general fiber bundle.

Let E be a fiber bundle associated with P with fiber F. Given $f_1 \in F$ we get a partial map $\bar{\pi}_{f_1} : P \to E$ with $p \mapsto \zeta = [(p, f_1)]$. Then the tangent map, $\bar{\pi}_{f_1 *}$ at p takes the horizontal space of P to $H'_\zeta \subset T_\zeta E$. With $f_2 = gf_1 \in F$ we get a partial map $\bar{\pi}_{f_2} : P \to E$ with $pg \mapsto \zeta$. But H_p under $\bar{\pi}_{f_2}$ has the same image, H'_ζ, since $H_{pg} = \mathcal{R}_{g*} H_p$. So the map $\bar{\pi}_{f*} : H_p \to H'_\zeta$ is independent of the choice of $f \in F$.

Definitions H'_ζ is *the horizontal* subspace of $T_\zeta E$ associated with H_p. The distribution $\mathcal{D}'_H : E \to \bigcup_{\zeta \in E} H'_\zeta$ is *the associated connection* of the associated bundle, E.

Theorem 27.13 *$T_\zeta E = H'_\zeta \oplus V'_\zeta$ where V'_ζ is the tangent space of the fiber, $\pi_E^{-1}(x)$, of E at x.*

Proof Problem 27.8. □

We saw that a vector field X on M has a horizontal lift $\bar{X}$ on P, and now this, in turn under $\bar{\pi}_{f*}$, induces a *horizontal lift*, X' on E. Moreover, the integral curves of this vector field are the images of the integral curves of $\bar{X}$ under $\bar{\pi}_f$ for any f. Thus, we obtain the horizontal lifting of curves in M to curves in E along which we say points are parallelly transported. Finally, since two points on a given fiber of P are mapped into the same fiber of E under $\bar{\pi}_f$, the parallel transport, τ_γ, of fibers of P gives parallel transport τ_γ, of fibers of E.

To complete the picture one can define a connection on a fiber bundle by abstracting properties we found for associated bundles; namely, *a connection on a fiber bundle, E,* is a distribution on E, which complements the vertical distribution on E and which has a unique integral curve corresponding to every given curve in M. (Cf., Sternberg, p. 308.)

Now, suppose E is a vector bundle; i.e., F is a vector space, V, and $\{\mathcal{L}_g\}$ is a subgroup of Aut V. Let γ be a curve in M through x_0 and let X_0 be the tangent to γ at x_0. If ψ is a section of E whose domain includes the range of γ, then $\psi(\gamma(t))$ is in the fiber $\pi_E^{-1}(\gamma(t))$. Now let τ_t denote the parallel transport of $\psi(\gamma(t))$ to a point $\tau_t(\psi(\gamma(t)))$ in the fiber $\pi_E^{-1}(x_0)$. We can then form

$$\nabla_{X_0} \psi(x_0) = \lim_{t \to 0} \frac{\tau_t(\psi(\gamma(t))) - \psi(x_0)}{t} \tag{27.16}$$

Definition $\nabla_{X_0}\psi(x_0) \in \pi_E^{-1}(x_0)$ is *the covariant derivative* of ψ *in the* X_0 *direction at* x_0.

The existence and properties of $\nabla_{X_0}\psi(x_0)$ can be proved by expressing $\nabla_{X_0}\psi(x_0)$ in a trivialization of E (Choquet-Bruhat and DeWitt-Morette, p. 370). Alternatively, these can be inferred from a different construction for $\nabla_{X_0}\psi(x_0)$. To make this construction we need the following important lemma.

Lemma *If E is an associated bundle of P with fiber F, there is a 1-1 correspondence between differentiable maps $\bar{\psi}: P \to F$ with the property $\bar{\psi}(pg) = g^{-1}\bar{\psi}(p)$, and sections $\psi: M \to E$ given by the diagram*

$$\begin{tikzcd} P \arrow[r, "\bar{\psi}"] \arrow[d, "\pi"'] & F \arrow[d, "\bar{\pi}_p"] \\ M \arrow[r, "\psi"] & E \end{tikzcd}$$

Proof Given a section ψ, then

$$\bar{\psi} = \bar{\pi}_p^{-1} \circ \psi \circ \pi \tag{27.17}$$

is a map from P to F. Moreover, since $\bar{\pi}_p(gf) = \bar{\pi}_{pg}(f)$, $\bar{\psi}$ has the required property. Conversely, if $\bar{\psi}$ has the required property, then (27.17) has the same value at p and pg, so (27.17) defines ψ as a map on M. □

Theorem 27.14 *If $\bar{X}_0$ is the horizontal lift of the vector X_0 above in the definition of the covariant derivative, and ψ is the image of $\bar{\psi}$ in the lemma, then $\nabla_{X_0}\psi(x_0)$ is the image of $\bar{X}_0\bar{\psi}$; i.e.,*

$$\nabla_{X_0}\psi(x_0) = \bar{\pi}_{p_0}(\bar{X}_0\bar{\psi}) \tag{27.18}$$

Proof If $\bar{\gamma}$ is the horizontal lift through p_0 with tangent X_0 of γ then $\bar{X}_0\bar{\psi} = D_0\bar{\psi} \circ \bar{\gamma} = \lim\limits_{t \to 0} \dfrac{\bar{\psi}(\bar{\gamma}(t)) - \bar{\psi}(p_0)}{t}$ since F is a vector space. Then

$$\begin{aligned} \bar{\pi}_{p_0}\bar{X}_0\bar{\psi} &= \lim_{t \to 0} \frac{\bar{\pi}_{p_0}\bar{\psi}(\bar{\gamma}(t)) - \psi(x_0)}{t} \\ &= \lim_{t \to 0} \frac{\bar{\pi}_{p_0}(\bar{\pi}_p^{-1}\psi(\gamma(t)) - \psi(x_0)}{t}. \end{aligned} \tag{27.19}$$

Let $f = \bar{\pi}_p^{-1}\psi(\gamma(t))$, $\pi(p) = \gamma(t)$. Since $\bar{\gamma}$ is horizontal in P, $\bar{\gamma}f$ is horizontal in E. The points $\bar{\pi}_p\bar{\pi}_p^{-1}\psi(\gamma(t)) = \psi(\gamma(t))$ and $\bar{\pi}_{p_0}\bar{\pi}_p^{-1}\psi(\gamma(t))$, are on $\bar{\gamma}f$, so $\tau_t\psi(\gamma(t)) = \bar{\pi}_{p_0}\bar{\pi}_p^{-1}\psi(\gamma(t))$, and the right side of (27.19) is $\nabla_{X_0}\psi(x_0)$ (Fig. 27.2). □

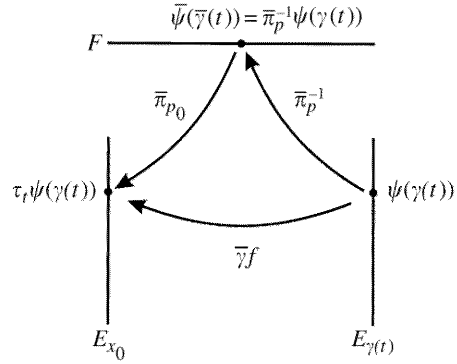

Fig. 27.2

From this characterization, it is clear that the covariant derivative depends only on the tangent of γ at x_0 (and not otherwise on γ).

Definitions If ψ is a section of E, and X is a vector field on M, then $\nabla_X\psi$, *the covariant derivative of ψ in the X direction* is a section of E given by $(\nabla_X\psi)(x) = \nabla_{X(x)}\psi(x)$. The operation ∇_X on sections of E taking ψ to $\nabla_X\psi$ is called *covariant differentiation in the X direction.*

From its definition or the characterization (27.18), the following properties of $\nabla_X\psi$ are evident.

Theorem 27.15 *If ψ, ψ_1 and ψ_2 are sections of E, and X and Y are vector fields on M, and f is a real-valued function on M, then*
(1) $\nabla_{X+Y}\psi = \nabla_X\psi + \nabla_Y\psi$.
(2) $\nabla_{fX}\psi = f\nabla_X\psi$.
(3) $\nabla_X(\psi_1 + \psi_2) = \nabla_X\psi_1 + \nabla_X\psi_2$.
(4) $\nabla_X f\psi = f\nabla_X\psi + (Xf)\psi$.

Proof Problem 27.9. □

The operation, ∇_X which we have defined here on sections of E is clearly a generalization of $\nabla_X : \mathfrak{X}M \to \mathfrak{X}M$ defined in Section 17.2 on vector fields. In Section 17.2 we had $\nabla_X Y \in \mathfrak{X}M$. Now $\nabla_X\psi$ is a section of E. If s is a section of E^*, we can form $\nabla_X\psi(s)$, so we have a bilinear map, $\nabla\psi$, from sections of E^* and vector fields to functions on M given by $\nabla\psi(s, X) = \nabla_X\psi(s)$. That is, $\nabla\psi$ is a section of $E \otimes T^*M$. Then

$$\nabla : \text{sections of } E \to \text{ sections of } E \otimes T^*M = \textstyle\bigwedge^1(T^*M, E)$$

(cf., Section (25.3)) given by $\nabla : \psi \to \nabla\psi$ satisfies

$$\nabla f\psi = \psi \otimes df + f\nabla\psi \tag{27.20}$$

by Theorem 27.15 (4), and $df(X) = Xf$ and thus is a generalization of the connection defined in Section 17.2.

Definitions* $\nabla\psi$ is called *the covariant derivative of* ψ of *the connection*, ∇, *on* E.

Proceeding as in Section 17.2 we can choose a local basis of sections of E, namely, $\{\bar{e}_1, \ldots, \bar{e}_m\}$, $m = \dim E$. Then $\psi = \psi^\alpha \bar{e}_\alpha$ and $\nabla\psi = \bar{e}_\alpha \otimes d\psi^\alpha + \psi^\alpha \nabla \bar{e}_\alpha$. Let

$$\nabla \bar{e}_\alpha = \Gamma^\beta_{\alpha k}\ \bar{e}_\beta \otimes \varepsilon^k \qquad \alpha, \beta = 1, \ldots, m,\ k = 1, \ldots, n \tag{27.21}$$

where $\{\varepsilon^k\}$ is a local basis of $\mathfrak{X}^*M$ and $\Gamma^\beta_{\alpha\varkappa}$ are functions on M. If $\{\bar{\varepsilon}^\alpha\}$ and $\{e_k\}$ are dual bases of $\{\bar{e}_\alpha\}$ and $\{\varepsilon^k\}$, respectively, then $\Gamma^\beta_{\alpha k} = \nabla \bar{e}_\alpha(\bar{\varepsilon}^\beta, e_k)$, and

$$\nabla\psi = (\langle d\psi^\beta, e_k\rangle + \psi^\alpha \Gamma^\beta_{\alpha k})\bar{e}_\beta \otimes \varepsilon^k \tag{27.22}$$

and

$$\nabla_X \psi = (X\psi^\beta + \psi^\alpha \Gamma^\beta_{\alpha k} X^k)\bar{e}_\beta \tag{27.23}$$

Finally, if $\{\varepsilon^k\}$ is any local basis of $\mathfrak{X}^*M$, we put

$$\varpi^\beta_\alpha = \Gamma^\beta_{\alpha k}\varepsilon^k \tag{27.24}$$

and get $m \times m$ 1-forms on M. In terms of ϖ^β_α the expression (27.22) for $\nabla\psi$ becomes

$$\nabla\psi = \bar{e}_\beta \otimes (d\psi^\beta + \psi^\alpha \varpi^\beta_\alpha) \tag{27.25}$$

The matrix (ϖ^β_α) is called *the connection matrix of* E *with respect to* $\{\bar{e}_\beta\}$. Since the Lie algebra of $GL(m, \mathbb{R})$ consists of $m \times m$ matrices, $\varpi = (\varpi^\beta_\alpha)$ can be considered a $\mathfrak{g}$-valued 1-form. Moreover, since (ϖ^β_α) is skew-symmetric, ϖ is in the Lie algebra of $SO(m)$.

Now we want to define the curvature of the connection, ∇, on E. Since we already have the curvature, $\varkappa$, of the connection of the principal bundle underlying E, it would seem that our definition for ∇ should be related to $\varkappa$. We first define, in terms of $\varkappa$, a 2-form, whose value at each point , $x \in M$ is a linear transformation of $\pi_E^{-1}(p)$.

Let $X, Y \in T_xM$, and $\bar{X}, \bar{Y} \in T_pP$ with $\pi(p) = x$, $\pi_*(\bar{X}) = X$, and $\pi_*(\bar{Y}) = Y$. Then $2\varkappa(\bar{X}, \bar{Y}) \in \mathfrak{g}$, and can be considered to be a linear transformation on V, the fiber of E. So, if $\psi_x \in E_x$, we define

$$\tilde{R}(X, Y)\psi_x = \bar{\pi}_p((2\varkappa(\bar{X}, \bar{Y}))(\bar{\pi}_p^{-1}\psi)) \in E_x$$

*Recall that *a connection on a fiber bundle* was defined at the beginning of this section, while what we are now calling *a connection on E* is a generalization of *a Koszul connection* which we defined in Section 17.2. For the relation between these and other related concepts see Poor, pp. 74-78.

That is, we have a map of E_x depending on the given vectors X, Y. If we show that this map is linear, doesn't depend on the choice of $p, \bar{X}, \bar{Y}$ and $\tilde{R}(X,Y) = -\tilde{R}(Y,X)$, and if we let x vary so that X, Y become vector fields and ψ is a section of E, then $\tilde{R}(X,Y)$, *the curvature transformation of* E, is a linear operator on sections of E, and $\tilde{R}$ is a linear operator-valued 2-form (cf., Section 17.3).

Clearly, $\tilde{R}(X,Y)$ depends on the curvature, $\varkappa$, of the principal bundle. That it depends on ∇, and hence called *the curvature transformation of* ∇, is shown by the following result.

Theorem 27.16 *If X, Y are vector fields and ψ is a section of E then*

$$\tilde{R}(X,Y)\cdot\psi = \nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi - \nabla_{[X,Y]}\psi \tag{27.26}$$

Proof At each point, p, both sides are equal to $\bar{\pi}_p((\mathbf{v}[\bar{X},\bar{Y}])\bar{\psi})$ where $\bar{X}, \bar{Y}$ are horizontal vectors at p. See Kobayashi and Nomizu, p. 134, for details. □

Definitions $\tilde{R}$ is *the curvature* of the connection, ∇. The 2-forms, R^α_β, defined by $\tilde{R}(X,Y)\cdot\bar{e}_\beta = R^\alpha_\beta(X,Y)\bar{e}_\alpha$ are *the curvature forms* of the connection.

Note that the definition of R^α_β is a generalization of eq. (16.30). This corresponds to the fact that eq. (16.30) and all of Chapters 16 and 17 are a special case of our present development when $E = TM$. By slightly generalizing the arguments in Chapters 16 and 17 we get, in particular

$$R^\alpha_\beta = 2(\varpi^\alpha_\delta \wedge \varpi^\delta_\beta + d\varpi^\alpha_\beta) \tag{27.27}$$

PROBLEM 27.8 Prove Theorem 27.13.

PROBLEM 27.9 Prove Theorem 27.15.

PROBLEM 27.10 If $\{\bar{e}_\alpha\}$ and $\{\bar{e}'_\alpha\}$ are local bases of sections of E, then on the overlap

$$\varpi^\delta_\alpha p^\beta_\delta = p^\varepsilon_\alpha \varpi'^\beta_\varepsilon + dp^\beta_\alpha \tag{27.28}$$

where $\bar{e}_\alpha = p^\beta_\alpha \bar{e}'_\beta$. (See Problem 16.4.)

PROBLEM 27.11 Verify the properties listed above which are needed to define $\tilde{R}(X,Y)$ as the curvature transformation of E.

28

GAUGE THEORY

We will discuss gauge transformations on fiber bundles and how they act on connections and their curvatures. Then we will briefly indicate how the concepts we have introduced relate to the gauge theory of elementary particle physics.

28.1 Gauge transformation of a principal bundle

Among the various possible mappings of principal fiber bundles the most important are *the principal bundle homomorphisms.* These mappings take fibers into fibers, or equivalently, are given by a mapping of their base manifolds. We further restrict ourselves to a single principal fiber bundle and a mapping of it onto itself.

Definitions A diffeomorphism $F : P \to P$ of a principal fiber bundle such that $F(pg) = F(p)g,\ p \in P,\ g \in G$ is called *a principal bundle automorphism.* (It is clearly a bundle homomorphism.) If the corresponding mapping of M is the identity it is called *a gauge transformation of* P (or, a vertical automorphism of P). The set of all gauge transformations from a group, G_P, *the gauge group of* P.

There are several different characterizations of gauge transformations.

Consider the associated bundle, B_G, *the gauge bundle* with fiber G and left action $\mathcal{L}$ given by $\mathcal{L}_g g' = gg'g^{-1}$ (conjugation).

Lemma *There is a 1-1 correspondence between the set of gauge transformations and the set of sections of* B_G.

Proof We will use the lemma in Section 27.4 and exhibit a 1-1 correspondence between elements of G_P and maps $\bar{\psi} : P \to G$ such that $\bar{\psi}(pg) = \mathcal{L}_{g^{-1}}\bar{\psi}(p)$ $(= g^{-1}\bar{\psi}(p)g)$. Thus, given $\bar{\psi}$, let

$$F(p) = p\bar{\psi}(p) \tag{28.1}$$

Then $F(pg) = pg\bar{\psi}(pg) = pgg^{-1}\bar{\psi}(p)g = p\bar{\psi}(p)g = F(p)g$ so $F \in G_P$. Conversely, if F is given such that $F(p) = p\bar{\psi}(p)$, then $pg\bar{\psi}(pg) = F(pg) = F(p)g = p\bar{\psi}(p)g$, so $\bar{\psi}(pg) = g^{-1}\bar{\psi}(p)g$, and $\bar{\psi}$ has the required property. □

Now we can define the product $\bar{\psi}_1 \circ \bar{\psi}_2$ of the maps $\bar{\psi} : P \to G$ of the lemma of Section 27.4 by $\bar{\psi}_1 \circ \bar{\psi}_2(p) = (p\bar{\psi}_1(p))\bar{\psi}_2(p)$, and we can define the product of ψ's by that of the $\bar{\psi}$'s. So we have, in short, the following result.

Theorem 28.1 *The group, G_P, of gauge transformations of P is isomorphic to the group of sections of B_G and to the group of maps $\bar{\psi}$ of the lemma of Section 27.4.*

Theorem 28.2 *With a given trivialization $\{(\mathcal{U}_\alpha, \psi_\alpha)\}$ of P with transition functions $\phi_{\alpha\beta}$ there is a 1-1 correspondence between gauge transformations of P and trivializations of P.*

Proof On the one hand, suppose $F(p) = p'$. Define a trivialization $\{(\mathcal{U}_\alpha, \psi'_\alpha)\}$ by $\phi'_\alpha(p) = \phi_\alpha(p')$. Now there exist $\lambda_\alpha : \mathcal{U}_\alpha \to G$ such that $\phi_\alpha(p') = \lambda_\alpha(x)\phi_\alpha(p)$ so $\phi'_\alpha(p) = \lambda_\alpha(x)\phi_\alpha(p)$ for all α. Then for $p \in \pi^{-1}(x)$, $\lambda_\alpha(x)\phi_{\alpha\beta}(x)\phi_\beta(p) = \lambda_\alpha(x)\phi_\alpha(p) = \phi'_\alpha(p) = \phi'_{\alpha\beta}(x)\phi'_\beta(p) = \phi'_{\alpha\beta}(x)\lambda_\beta(x)\phi_\beta(p)$ so

$$\lambda_\alpha(x)\phi_{\alpha\beta}(x) = \phi'_{\alpha\beta}(x)\lambda_\beta(x) \tag{26.8}$$

This is the condition of Theorem 26.4 for $\{(\mathcal{U}_\alpha, \psi'_\alpha)\}$ to be a trivialization of P.

On the other hand, with a given trivialization $\{(\mathcal{U}_\alpha, \psi'_\alpha)\}$ define $F(p) = p'$ with p' given by $\phi_\alpha(p') = \phi'_\alpha(p)$ for some α. If λ_α satisfy (26.8), then starting with (26.8) and following the circle of equalities above in the other direction we get that $\phi_\beta(p') = \phi'_\beta(p)$ for all p so that p' is well-defined. □

Because of this result, a gauge transformation is often described as a change of trivialization (Bourguignon and Lawson, p. 398). Further, since there is a natural correspondence between local trivializations and local sections of P, a gauge transformation is also frequently described as a change of local sections (Choquet-Bruhat and DeWitt-Morette, p. 405). Finally, one often says a gauge transformation is given by a family, $\{\lambda_\alpha\}$, of mappings satisfying (26.8) (Daniel and Viallet, p. 192).

Now with a gauge transformation, F, each connection form, ω, on P has a pull-back, $F^*\omega$, by F.

Theorem 28.3 *If ω is a connection form on P and F is a gauge transformation of P, then $F^*\omega$ is a connection form on P.*

Proof We have to show that $F^*\omega$ satisfies the three properties of Theorem 27.4

(1) $F^*\omega$ is vertical since F_* maps horizontal vectors to horizontal vectors.

(2) In the definition of the fundamental field in Section 25.4 the curve $\mathcal{A}_p$ in P goes to $\mathcal{A}_{F(p)}$ under F so the fundamental field A^*_p at p of A goes to $A^*_{F(p)}$ under F_*, so $F^*\omega_p(A^*_p) = \omega_p F_*(A^*_p) = \omega_{F(p)}(A^*_{F(p)}) = A$.

(3) $F^*\omega_{pg}\mathcal{R}_{g^*}(\xi) = (\mathcal{R}^*_g(F^*\omega))_p(\xi) = (F^*(\mathcal{R}^*_g\omega))_p(\xi)$ (since $F(pg) = F(p)g) =$ $(F^*Ad_{g^{-1}}\omega)(\xi) = Ad_{g^{-1}}(F^*\omega)_p(\xi)$. □

Corollary *The pullback by F of the curvature of ω is the curvature of $F^*\omega$.*

Theorem 28.4 *If θ is a horizontal $\mathfrak{g}$-invariant s-form on P and F is a gauge transformation of P, then $F^*\theta$ is a horizontal $\mathfrak{g}$-invariant s-form on P.*

Proof Problem 28.1. □

Note that we actually have more than we proved in Theorems 28.3 and 28.4; namely, the pull-back by G_P is an action on the affine space, $\mathcal{C}_P$ of all connection forms on P and on the vector space of horizontal $\mathfrak{g}$-invariant s-forms on P.

Lemma *If F is a gauge transformation, then the tangent map of F is given by*

$$F_*(\xi) = \mathcal{R}_{\bar{\psi}(p)_*}(\xi) + \mathcal{R}_{p*}(w) \tag{28.2}$$

where $\mathcal{R} : P \times G \to P$ by $(p, \bar{\psi}(p)) \mapsto F_{(p)}$, $\xi \in T_pP$ and $w = \bar{\psi}_{*p}(\xi) \in T_{\bar{\psi}(p)}G$.

Proof For any right action $\mathcal{R} : P \times G \to P$ we can write $\mathcal{R}(P, \bar{\psi}(p)) = \mathcal{R}_{\bar{\psi}(p)}p$. But by eq. (28.1) $\mathcal{R}_{\bar{\psi}(p)}p = F(p)$ so $\mathcal{R}(p, \bar{\psi}(p)) = F(p)$ and

$$\mathcal{R}_*(\xi, w) = F_*(\xi) \tag{28.3}$$

where $\xi \in T_pP$ and $w = \bar{\psi}_{*p}(\xi) \in T_{\bar{\psi}(p)}G$.

Now by Leibniz's Formula (cf. Kobayashi and Nomizu, p. 11)

$$\mathcal{R}_*(\xi, w) = \mathcal{R}_{\bar{\psi}(p)*}(\xi) + \mathcal{R}_{p*}(w)$$

and with (28.3) we get our result. □

Theorem 28.5 *If F is a gauge transformation, and ω is a connection form on P, then ω and $F^*\omega$ are related by*

$$F^*\omega_p(\xi) = Ad_{\bar{\psi}(p)^{-1}}\ \omega_p(\xi) + L^{-1}_{\bar{\psi}(p)*}(\bar{\psi}_{*p}(\xi)) \tag{28.4}$$

Proof We apply $\omega_{p\bar{\psi}(p)}$ to both sides of eq. (28.2).

$$\omega_{p\bar{\psi}(p)} F_*(\xi) = F^*\omega_p(\xi)$$

$$\omega_{p\bar{\psi}(p)} \mathcal{R}_{\bar{\psi}(p)*}(\xi) = Ad_{\bar{\psi}(p)^{-1}}\ \omega_p(\xi) \quad \text{by eq. (27.3)}$$

$$\omega_{p\bar{\psi}(p)} \mathcal{R}_{p*}(w) = L^{-1}_{\bar{\psi}(p)*}(\bar{\psi}_{*p}(\xi))$$

since $\mathcal{R}_{p*}(w)$ is the image in $T_{p\bar{\psi}(p)}P$ of a vector w at $\bar{\psi}(p) \in G$ of the vector field whose value at T_eG is $L^{-1}_{\bar{\psi}(p)*}(\bar{\psi}_{*p}(\xi))$. □

Theorem 28.6 *If F is a gauge transformation and θ is a horizontal $\mathfrak{g}$-invariant s-form on P, then θ and $F^*\theta$ are related by*

$$F^*\theta_p(\xi_1, \ldots, \xi_s) = Ad_{\bar{\psi}(p)^{-1}}\theta_p(\xi_1, \ldots, \xi_s) \tag{28.5}$$

Proof From the lemma and the fact that $\mathcal{R}_{p*}(w)$ is a vertical vector. □

Corollary

$$F^*\varkappa = Ad_{\bar{\psi}(p)^{-1}}\varkappa \tag{28.6}$$

If we think of a gauge transformation as a change of local sections, we can describe transformation of forms under gauge transformations in terms of transformations of the projections

$$\omega_\alpha = \sigma_\alpha^*\omega$$

of ω on $\mathcal{U}_\alpha \subset M$ under a change of local sections, or, equivalently, in terms of transformations of the ω_α in overlapping open sets of a single trivialization.

We might expect the formulas to be related to (28.4) - (28.6) in Theorems 28.5 and 28.6, and they are. If we replace $F(p) = \mathcal{R}_{\bar{\psi}(p)}p$ by $\sigma_\beta(x) = \mathcal{R}_{\phi_{\alpha\beta}(x)}\sigma_\alpha(x)$ where $\sigma_\alpha, \sigma_\beta$ are local sections and $\phi_{\alpha\beta}$ are transition functions of a trivialization of P (Theorem 26.3), then a proof exactly analogous to that in the lemma gives

$$\sigma_{\beta*}(X) = \mathcal{R}_{\phi_{\alpha\beta}(x)*}\ \sigma_{\alpha*}(X) + \mathcal{R}_{\sigma_\alpha *}(\phi_{\alpha\beta*}(X))$$

$X \in T_*M$, from which there follows.

Theorem 28.7 *At $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$*

$$(1) \quad \omega_\beta(X) = Ad_{\phi_{\alpha\beta}^{-1}}\ \omega_\alpha(X) + L^{-1}_{\phi_{\alpha\beta^*}}(\phi_{\alpha\beta^*}(X)) \tag{28.7}$$

$$(2) \quad \theta_\beta(X_1, \ldots, X_s) = Ad_{\phi_{\alpha\beta}^{-1}}\ \theta_\alpha(X_1, \ldots, X_s) \tag{28.8}$$

where $\theta_\alpha = \sigma_\alpha^*\theta$

$$(3) \quad \varkappa_\beta(X_1, X_2) = Ad_{\phi_{\alpha\beta}^{-1}}\ \varkappa_\alpha(X_1, X_2) \tag{28.9}$$

where $\varkappa_\alpha = \sigma_\alpha^*\varkappa$

PROBLEM 28.1 Prove Theorem 28.4.

PROBLEM 28.2 If G is a group of matrices, then the $\omega_\alpha, \omega_\beta$ are Lie algebra of G-valued forms on $\mathcal{U}_\alpha$, $\mathcal{U}_\beta$ in M and eq. (28.7) becomes

$$\phi_{\alpha\beta}\ \omega_\beta = \omega_\alpha\ \phi_{\alpha\beta} + d\phi_{\alpha\beta} \tag{28.10}$$

See Problem 25.7. (Also, cf., eq. (27.28).)

PROBLEM 28.3 The projections ω_α and $\varkappa_\alpha$ are related by

$$\varkappa_\alpha(X,Y) = \frac{1}{2}[\omega_\alpha(X), \omega_\alpha(Y)] + d\omega_\alpha(X,Y)$$

(cf. eq. (27.4)).

PROBLEM 28.4 The projections of the pull-backs by F of ω, θ, and $\varkappa$ in terms of their projections ω_α, θ_α, $\varkappa_\alpha$ are given, respectively, by

(1) $\sigma_\alpha^* F^* \omega(X) = Ad_{(\bar\psi\circ\sigma_\alpha)^{-1}}\ \omega_\alpha(X) + L^{-1}_{(\bar\psi\circ\sigma_\alpha)*}(\bar\psi \circ \sigma_\alpha(X))$

(2) $\sigma_\alpha^* F^* \theta(X_1, \ldots, X_s) = Ad_{(\bar\psi\circ\sigma_\alpha)^{-1}}\ \theta_\alpha(X_1, \ldots, X_s)$

(3) $\sigma_\alpha^* F^* \varkappa(X_1, X_2) = Ad_{(\bar\psi\circ\sigma_\alpha)^{-1}}\ \varkappa_\alpha(X_1, X_2)$

28.2 Gauge transformations of a vector bundle

Definitions A differentiable map, φ, of vector bundles E_1 and E_2 over the same base which preserves fibers and is linear on each fiber is *a vector bundle homomorphism.* If $E_1 = E_2$ and the map is an automorphism on each fiber, $\varphi\big|_{\pi^{-1}(x)} \in \text{Aut } E_x$, then it is *a vector bundle automorphism.*

If V is the fiber of E, then Aut $E_x \cong$ Aut V. *A gauge transformation,* Γ, *of a vector bundle,* E is a vector bundle automorphism, φ, such that $\varphi\big|_{\pi^{-1}(x)} \in \{\mathcal{L}_g\} \subset$ Aut V. Under composition, the set of gauge transformations forms a group, G_E, *the gauge group of* E.

The concepts described above generalize corresponding concepts for vector spaces. Similarly, in terms of their fibers, one can construct vector bundles in terms of which, in particular, we can describe the group of gauge transformations.

Definition *The homomorphism bundle from* E_1 *to* E_2, $\text{Hom}(E_1, E_2)$, is the vector bundle whose fibers are $\mathcal{L}(E_{1x}, E_{2x})$ the vector spaces of linear mappings from E_{1x} to E_{2x}.

The sections of $\text{Hom}(E_1, E_2)$ are precisely the vector bundle homomorphisms from E_1 to E_2. Moreover, from the isomorphism $\mathcal{L}(V, W) \cong V^* \otimes W$ for vector spaces, we get

$$\text{Hom}(E_1, E_2) = E_1^* \otimes E_2 \tag{28.11}$$

If $E_1 = E_2$, and the fibers are the groups, Aut E_x, then the homomorphism bundle is an *automorphism bundle*, Aut E, and if the fibers are the group $\{\mathcal{L}_g\}$, then it is a *gauge bundle*, Gauge E. Thus, G_E, the gauge group of E, can be thought of as the group, under composition, of sections of the gauge bundle of E.

Let $\mathcal{C}_E$ be the affine space of connections on E. We now define a certain action by G_E on $\mathcal{C}_E$.

Theorem 28.8 *If ∇ is a connection on E, and Γ is a gauge transformation of E, then ∇^{Γ} given by*

$$\nabla^{\Gamma}_X \psi = \Gamma \; \nabla_X \; \Gamma^{-1} \; \psi \tag{28.12}$$

is a connection on E.

Proof ∇^{Γ}_X satisfies Theorem 27.15 since ∇_X does and Γ is linear. □

Theorem 28.9 *With a given basis, $\{\bar{e}_\alpha\}$ of local sections of E, the connection matrix, $(\varpi^{\Gamma\alpha}_{\ \ \beta})$, of ∇^{Γ} is related to the connection matrix, (ϖ^α_β), of ∇ by*

$$\varpi^{\Gamma\alpha}_{\ \ \delta} \; \Gamma^\delta_\beta = \; \Gamma^\alpha_\delta \; \varpi^\delta_\beta - d \, \Gamma^\alpha_\beta \tag{28.13}$$

where $\Gamma^\alpha_\beta \; \bar{e}_\alpha = \bar{e}^{\Gamma}_\beta$.

Proof Express both sides of eq. (28.13) in terms of the right side of eq. (27.23). □

Theorem 28.10 *The curvature transformation $\widetilde{R}(X,Y)$ of a connection ∇ transforms under a gauge transformation, Γ, according to*

$$\widetilde{R}^{\Gamma}(X,Y)\psi = \; \Gamma \widetilde{R}(X,Y) \, \Gamma^{-1} \; \psi \tag{28.14}$$

Proof This is a straightforward application of (28.12) to the definition of $\widetilde{R}(X,Y)$ in Section 27.4. □

An explanation for the choice of the particular action given by eq. (28.12) may be given by the fact that if we define a metric on Aut E with the value trace $(\mathcal{A}^{tr} \circ \mathcal{B})$ for $\mathcal{A}$, $\mathcal{B} \in$ Aut E, then the norm of the curvature transformation of E will be the same for all equivalent connections, or, briefly, the norm of the curvature is invariant under the gauge group (Lawson, p. 39).*

*Another explanation of the choice of the connection ∇^{Γ} in eq. (28.12) may be given in terms of parallel translation in E by the requirement that gauge transformations and parallel translations should commute. (Cf., Göckeler and Schücker, pp. 47-51.)

There is an instructive description of the curvature of ∇ which appears when we make certain natural generalizations of the concepts we have been using. First, we generalize the idea of a vector-valued s-form: $\omega^s : M \to \bigwedge^s(T^*M, W)$ such that $\omega^s(x) \in \bigwedge^s(T_x^*, W)$ (Section 25.3).

Definition *A bundle-valued s-form, Ω^s, of a vector bundle, E, is a section of* $\mathrm{Hom}(\bigwedge^s TM, E)$. That is

$$\Omega^s : M \to \mathrm{Hom}(\bigwedge^s TM, E) \tag{28.15}$$

such that $\Omega^s(x) \in \mathcal{L}(\bigwedge^s T_x, E_x)$, the fiber of $\mathrm{Hom}(\bigwedge^s TM, E)$ at x. In particular, $\Omega^0 = \psi$, a section of E.

Note that from the duality in tensor algebras, by an extension of eq. (6.8),

$$\mathcal{L}(\bigwedge^s T_x, E_x) = \bigwedge^s (T_x^*, E_x)$$

so we can also describe the fiber of $\mathrm{Hom}(\bigwedge^s TM, E)$ at x by the expression on the right.

Finally, from (28.11), $\mathrm{Hom}(\bigwedge^s TM, E) = \bigwedge^s T^*M \otimes E$ so that we have, alternatively,

$$\Omega^s : M \to \bigwedge^s T^*M \otimes E \tag{28.16}$$

Theorem 28.11 *If E is a vector bundle with fiber V and underlying principal bundle P, there is a 1-1 correspondence between V-valued horizontal equivariant (tensorial) forms on P and bundle-valued form on M. (This is a generalization of the Lemma in Section 27.4)*

Proof Problem 28.5. □

Now we define an operation on the set of all bundle-valued s-forms for all s. This will simultaneously generalize two operations we have already defined and have been using: exterior differentiation in Section 12.1, and covariant differentiation which we first encountered in Section 16.1 for vector fields, and generalized to sections of vector bundles as $d\psi \cdot X = \nabla_X \psi$.

Definition *The covariant exterior differential, d^∇,* (or, exterior covariant differential) of a bundle-valued s-form, Ω^s, takes Ω^s to a bundle-valued $s+1$ form $d^\nabla \Omega^s$ according to

$$(d^\nabla \Omega^s)(X_1,\ldots,X_{s+1}) = \frac{1}{s+1}\left(\sum_1^{s+1}(-1)^{i+1}\nabla_{X_i}\Omega^s(X_1,\ldots,\widehat{X}_i,\ldots,X_{s+1}) + \sum_{i<j}(-1)^{i+j}\Omega^s([X_i,X_j],X_1,\ldots,\widehat{X}_j,\ldots,X_{s+1})\right)$$

$$X_i \in \mathfrak{X}M. \tag{28.17}$$

This is clearly a generalization of Theorem 12.2 and of $d\psi \cdot X = \nabla_X \psi$. When $s = 1$, eq. (28.17) is

$$(d^\nabla \Omega^1)(X_1,X_2) = \frac{1}{2}\left(\nabla_{X_1}\Omega^1(X_2) - \nabla_{X_2}\Omega^1(X_1) - \Omega^1[X_1,X_2]\right) \tag{28.18}$$

Theorem 28.12

$$\widetilde{R}^\nabla(X,Y)\psi = 2(d^\nabla d^\nabla \psi)(X,Y). \tag{28.19}$$

Proof Put $\Omega^1 = d^\nabla\psi$ in eq. (28.18) and use the definition of $\widetilde{R}^\nabla(X,Y)$ in Section 27.4. □

On the basis of Theorem 28.12 one can express the curvature $\widetilde{R}^\nabla$ in terms of the exterior covariant derivative as

$$\widetilde{R}^\nabla = 2d^\nabla \circ d^\nabla$$

PROBLEM 28.5 Prove Theorem 28.11. (Cf., Kobayashi and Nomizu, p. 76.)

PROBLEM 28.6 Prove that, with the metric given above, the norm of the curvature is invariant under a gauge transformation.

PROBLEM 28.7 Given a vector bundle, E, let $\mathfrak{g}(E)$ be the bundle over M whose fibers are isomorphic to the Lie algebra of $\{\mathcal{L}_g\}$ (thinking of $\{\mathcal{L}_g\} \subset (\text{Aut } E)_x$ for each $x \in M$). Then, if ∇_1 and ∇_2 are two connections on E

(1) $A = \nabla_2 - \nabla_1$ is a section of Hom $(TM, \mathfrak{g}(E))$

(2) if $v \in T_xM$, $Av = \nabla_{2v} - \nabla_{1v} \in \mathfrak{g}(E)_x$

(3) $\widetilde{R}^{\nabla_2}(X,Y)\psi = \widetilde{R}^{\nabla_1}(X,Y)\psi + (d^\nabla A(X,Y))\psi + [A,A](X,Y)(\psi)$

28.3 How fiber bundles with connections form the basic framework of the Standard Model of elementary particle physics

Physicists have experimentally identified a variety of particles and forces acting on them. According to the "Standard Model" which attempts to organize these observations, these forces are "mediated" by particles, so we have matter particles (or, fields), called fermions, and gauge particles (or, fields), called bosons.

The matter particles, electrons, protons, etc., have been around for a long time. The motivation for introducing gauge particles comes from invariance requirements for Schrödinger's equation and the Lagrangian of the system according to the following explanation.

The fact that the solution of the Schrödinger equation, the state of a matter particle, is not an observable physical quantity but some function of it (it's "magnitude") is, leads to the idea of a matter particle having an internal structure. The requirement that the Schrödinger equation or the Lagrangian of the system should depend only on the particle and not its internal structure, that is, that they should have a certain invariance can be achieved by inserting another quantity into these expressions which transforms properly under changes of the internal structure of the matter particle. The requirement that the Lagrangian is invariant under transformation of the new quantity leads, in turn, by Noether's Theorem to the existence of a conserved current.

Here we started with a matter particle and ended up with a current which we should be able to find. In practice, one may observe a conserved current, and then, with a given matter particle, reversing the above argument, Noether's Theorem suggests the existence of a quantity under whose transformation the Lagrangian is invariant.

The major justification for the general theory described above is that it fits the specific case of the electron in an electromagnetic field. A big boost to its acceptance was given by the Yang-Mills theory of the nucleon. Today, with vast technical additions and modifications (including that of the Yang-Mills theory) it is the basis of the Standard Model.

Let us look at the electromagnetic case. The matter particle is the electron. It is represented by ψ, a complex-valued function on spacetime.* It can go through a set of phases given by $e^{i\lambda}$ where λ is a real-valued function on spacetime. The quantity we insert in Schrödinger's equation (and in the Lagrangian) is the potential, A, we found in Section 21.3. We insert it by changing the partial derivatives that occur to covariant derivatives, that is, replace $\frac{\partial \psi}{\partial x^i}$ by $\frac{\partial \psi}{\partial x^i} + \psi A_i$ (cf., eq. (27.25)). Now when

$$\psi \to e^{i\lambda}\psi \tag{28.20}$$

if

*Our electron is the scalar, or spin-zero electron. The Dirac electron has values in $\mathbb{C}^2$.

$$A \to A - d\lambda \tag{28.21}$$

then the form of the Schrödinger equation remains the same as it was.

The physical significance of the potential was described by Weyl (1950) in 1930: "The influence of an electromagnetic field on a particle of charge - e can be expressed by replacing p_α by $p_\alpha + e\ \phi_\alpha$ in the equations of motion for a free particle." (Here $p_\alpha = \frac{h}{i}\ \frac{\partial}{\partial x_\alpha}$ and ϕ_α is the potential of the electromagnetic field.) That is, ϕ_α corresponds to the force acting on the particle due to an electromagnetic field. This is confirmed by the fact that the different representations of the potential under the transformation (28.21) required to make the Schrödinger equation invariant give, by our derivation in Section 21.3, the same electromagnetic field.

A small conceptual step permits us to associate the new introduced quantity, in this case, the electromagnetic potential, with a "gauge particle", in this case a photon, which mediates the force.

To generalize the electromagnetic case, in addition to matter particles with internal structure and gauge particles which transform suitably under changes of the internal structure of the matter particles we must have a generalization of the electromagnetic field.

The following table shows how the ingredients of the physical systems we have been describing can be modeled by the ingredients of fiber bundles with connections:

Elementary particles	**Fiber bundles**
(the state of) a matter particle (field)	a section of a complex vector bundle, E
the internal structure of a matter particle	the group, $\{\mathcal{L}g\} \subset GL(m, \mathbb{C})$, of a vector bundle
a gauge particle(field)	a connection 1-form with values in the Lie algebra of $\{\mathcal{L}g\} \subset GL(m, \mathbb{C})$
the required group of transformations for the gauge particle	the gauge group, G_E, of the vector bundle
a generalization of the electromagnetic field	the curvature of the connection (a 2-form with values in the Lie algebra of $\{\mathcal{L}g\} \subset GL(m, \mathbb{C})$)

For the electromagnetic case, on which the general theory is modeled, the state of the electron, ψ, is a section of a complex vector bundle with fiber $\mathbb{C}$. The set of phases given by e^{ir} is the group $U(1)$, the 1-dimensional unitary group, the unit circle in the complex plane. The potential, A, is the connection 1-form ϖ, the special case of (27.24) where $m = 1$. ϖ has values $i\lambda(x)$ in the 1-dimensional Lie algebra of $U(1)$ for each $x \in M$. The covariant derivative in this case is the special case, $d\psi + \psi\varpi$, of (27.25).

Equation (28.20) is a special case of $\psi^{\bar{g}}(x) = \bar{\pi}_p e^{i\lambda(x)} \bar{\pi}_p^{-1}(\psi(x))$ for the transformation of ψ or, briefly,

$$\psi^{\bar{g}}(x) = e^{i\lambda(x)}\psi(x) \tag{28.22}$$

where $\bar{g} \in \{\mathcal{L}_g\}$ and $i\lambda(x)$ is in the Lie algebra of $\{\mathcal{L}_g\}$, and

$$\varpi^{\Gamma}{}^{\alpha}_{\delta} = \Gamma^{\alpha}_{\gamma}\, \varpi^{\gamma}_{\beta}\, \Gamma^{-1}{}^{\beta}_{\delta} - d\,\Gamma^{\alpha}_{\beta}\, \Gamma^{-1}{}^{\beta}_{\delta} \tag{28.13}$$

for the transformation of the connection under the gauge group reduces to (28.21), since $U(1)$ is abelian.

Finally, the electromagnetic field is $2d\varpi$, a special case of eq. (27.27) for the curvature.

In the case of the Yang-Mills nucleon, the state of the nucleon, ψ, is a section of a complex vector bundle with fiber $\mathbb{C}^2$. The group of the bundle is $SU(2)$, the special unitary group which preserves the Hermitian inner product and orientation. The connection form is $(\varpi^{\beta}_{\alpha})$ in (27.24) with $\alpha, \beta = 1, 2$, and $(\varpi^{\beta}_{\alpha})$ has its value in the 3-dimensional Lie algebra of $SU(2)$. The generalization of the electromagnetic field is given by eq. (27.27) with $\alpha, \beta = 1, 2$. Note that terms that dropped out in eq. (28.13) for the transformation of the connection and in eq. (27.27) for the curvature in the electromagnetic case do not do so now.

As indicated above, the original Yang-Mills theory of the nucleon has been modified in that the nucleon is no longer thought of as a single elementary particle, but is made of "more elementary" particles. These elementary particles (including electrons) can be thought of as having more than one kind of internal symmetry corresponding to the fact that they respond to more than one kind of force.

Other than the gravitational force there are the strong force mediated by 8 *gluons*, the weak force mediated by 3 *intermediate vector bosons*, and the electromagnetic force mediated by a *photon*, corresponding respectively to the symmetries, the color group, $SU(3)$, the isotopic spin group, $SU(2)$, and the phase group, $U(1)$. In the Standard Model, a matter particle, ψ, is a section of a vector bundle whose group is a product of these groups, depending on the forces to which it responds.

At the beginning of this section we referred to the Lagrangian of a system of elementary particles, and our subsequent discussion was motivated by the idea that it should be invariant under simultaneous changes of the state of the matter particles and that of the gauge particles.

The Lagrangian is important because it describes a given system according to the doctrine that "the equations of motion" of the system are given by a variational principal: their solutions are critical points with respect to variations of the arguments of an integral of the Lagrangian. This elegant general principal generalizes the basic mechanics case we described in Theorem 19.5.

In Section 18.4, the Lagrangian in mechanics was a real-valued function on the tangent bundle of a manifold, M. For a given system, it was written, locally, in terms of the coordinates, q^i, of the system, and $\dot{q}^i$ which can be thought of as representing the tangent map of some curve in M. For our purposes we replace M by E, the curve by a section of E represented by ψ^α, and $\dot{q}^i$ by the tangent map of ψ^α represented $d\psi^\alpha + \psi^\beta \varpi^\alpha_\beta$ (cf., eq. (27.25)). Thus, locally, the Lagrangian is a function of ψ^α and $d\psi^\alpha + \psi^\beta \varpi^\alpha_\beta$. This is the Lagrangian of a matter particle (or, particles), ψ. If there is a (force) field present so that there are gauge particles "interacting with" the matter particles we must add it to the Lagrangian of ψ, so we must add a term involving the curvature given locally by R^α_β (cf., eq. (27.27)).

In terms of a local coordinate basis of $\mathfrak{X}^* M$

$$d\psi^\alpha + \psi^\beta \varpi^\alpha_\beta = \left(\frac{\partial \psi^\alpha}{\partial \mu^i} + \psi^\beta \varpi^\alpha_{\beta i} \right) d\mu^i$$

and

$$\varpi^\alpha_\delta \wedge \varpi^\delta_\beta + d\varpi^\alpha_\beta = \left(\varpi^\alpha_{\delta i} \varpi^\delta_{\beta j} + \frac{\partial \varpi^\alpha_{\beta j}}{\partial \mu^i} \right) d\mu^i \wedge d\mu^j$$

Thus, we can think of our total Lagrangian as being a function of ψ^α, $\dot{\psi}^\alpha_i \equiv \frac{\partial \psi^\alpha}{\partial \mu^i} + \psi^\beta \varpi^\alpha_{\beta i}$, $\varpi^\alpha_{\beta j}$, $\frac{\partial \varpi^\alpha_{\beta j}}{\partial \mu^i}$.

Now, if we apply the variational principal to this Lagrangian we are looking for critical points with respect to variations of both ψ^α and ϖ^α_β. However, such a critical point will be a critical point under variations of ψ^α alone and under variations of ϖ^α_β alone. Thus we have the two conditions, schematically,

$$\delta_\psi \int L_\psi(\psi^\alpha, \dot{\psi}^\alpha_i) = 0$$

and

$$\delta_\varpi \int L_\varpi \left(\varpi^\alpha_{\beta j}, \frac{\partial \varpi^\alpha_{\beta j}}{\partial \mu^i} \right) = 0$$

with corresponding Euler-Lagrange equations

$$\frac{\partial L_\psi}{\partial \psi^\alpha} - \sum_i \frac{\partial}{\partial \mu^i} \left(\frac{\partial L_\psi}{\partial \dot{\psi}^\alpha_i} \right) = 0 \tag{28.23}$$

$$\frac{\partial L_\varpi}{\partial \varpi^\alpha_{\beta j}} - \sum_i \frac{\partial}{\partial \mu^i} \frac{\partial L_\varpi}{\partial (\frac{\partial \varpi^\alpha_{\beta j}}{\partial \mu^i})} = 0 \tag{28.24}$$

Notice that in L_ψ only terms involving ψ^α and $\dot{\psi}^\alpha_i$ occur whereas in L_ϖ in addition to terms coming from the curvature (force field) there is a term involving $\dot{\psi}^\alpha_i$.

In order to proceed to derive the differential equations governing the motion of ψ^α and ϖ^α_β from the Euler-Lagrange equations we have to describe the Lagrangian L_ψ and L_ϖ more precisely. This requires, in particular, defining inner products or norms on M, E, and the relevant Lie algebra. The general case is technically quite complicated and we don't wish to go into the details here.

In the simplest case, electromagnetic,

$$L_\psi(\psi^\alpha,\ \dot{\psi}^\alpha_i) = \frac{1}{2}(g^{ij}\dot{\psi}_i\dot{\psi}_j + m^2\bar{g}_{ij}\psi^i\psi^j)$$

where ψ^i, $i = 1, 2$ are the real and imaginary parts of ψ, and $g, \bar{g}$ are, respectively, metrics on M and E with fiber $\mathbb{C} = \mathbb{R}^2$. Then

$$\frac{\partial L_\psi}{\partial \psi} = (m^2\bar{g}_{1j}\psi^j, m^2\bar{g}_{2j}\psi^j) = (m^2\psi^1, m^2\psi^2) = m^2\psi$$

$\dfrac{\partial L_\psi}{\partial \dot{\psi}_i} = g^{ij}\dot{\psi}_j$ and if $\dfrac{\partial g^{ij}}{\partial \mu^k} = 0$, then

$$\frac{\partial}{\partial \mu^i}\left(\frac{\partial L_\psi}{\partial \dot{\psi}_i}\right) = g^{ij}\frac{\partial}{\partial \mu^i}\dot{\psi}_j = g^{ij}\left(\frac{\partial}{\partial \mu^i}\frac{\partial \psi}{\partial \mu^j} + \frac{\partial \psi}{\partial \mu^i}\ \varpi_j\right)$$

So eq. (28.23) gives

$$g^{ij}\frac{\partial^2 \psi}{\partial \mu^i \partial \mu^j} + g^{ij}\frac{\partial \psi}{\partial \mu^i}\ \varpi_j - m^2\psi = 0$$

Note that this is the Klein-Gordon equation with an additionl term due to the presence of the connection, ϖ.

Further, in this case

$$L_\varpi\left(\varpi^\alpha_{\beta j},\ \frac{\partial \varpi^\alpha_{\beta j}}{\partial \mu^i}\right) = \frac{1}{2}\left(g^{ij}\dot{\psi}_i\dot{\psi}_j + g^{ip}g^{jq}\left(\frac{\partial \varpi_i}{\partial \mu^j} - \frac{\partial \varpi_j}{\partial \mu^i}\right)\left(\frac{\partial \varpi_p}{\partial \mu^q} - \frac{\partial \varpi_q}{\partial \mu^p}\right)\right)$$

If we let $\mathcal{F}_{ij} = \frac{\partial \varpi_i}{\partial \mu^j} - \frac{\partial \varpi_j}{\partial \mu^i}$, then

$$\frac{\partial L_\varpi}{\partial(\frac{\partial \varpi_k}{\partial \mu^l})} = \frac{1}{2}\left(g^{pk}g^{ql}\mathcal{F}_{pq} - g^{pl}g^{qk}\mathcal{F}_{pq}\right) = \mathcal{F}^{kl}$$

and $\dfrac{\partial}{\partial \mu^l}\left(\dfrac{\partial L_\varpi}{\partial(\frac{\partial \varpi_k}{\partial \mu^l})}\right) = \dfrac{\partial}{\partial \mu^l}\mathcal{F}^{kl}$

Moreover,

$$\frac{\partial L_\varpi}{\partial \varpi_k} = \dot{\psi}^k\psi$$

So if we identify $\dot{\psi}^k\psi$ with a charge-current density, Z^k, from eq. (28.24) we get back our eq. (21.31) derived from Maxwell's equations. Note that this current comes from the interaction term, so there is no current unless the system contains both matter and gauge particles.

Finally, to see how the invariance of the Lagrangian leads to a conserved current, we go back to the derivation of (28.23). The variation $\delta_\psi \int_M L_\psi(\psi^\alpha, \dot{\psi}_i^\alpha)$ can be written, by integration by parts as the sum of a term containing the left side of (28.23) and a term $\displaystyle\int_M \frac{\partial}{\partial\mu^i}\left(\frac{\partial L_\psi}{\partial\dot{\psi}_i^\alpha}\delta\psi^\alpha\right)$ where $\delta\psi^\alpha$ is a variation of ψ^α. Now suppse ψ^α satisfies (28.23) and it transforms by the 1-parameter group $\psi^\alpha(t) = e^{tE_\beta^\alpha}\psi^\beta$ under which the Lagrangian is invariant. Then $\delta\psi^\alpha = E_\beta^\alpha\psi^\beta$ and $\displaystyle\int_M \frac{\partial}{\partial\mu^i}\left(\frac{\partial L_\psi}{\partial\dot{\psi}_i^\alpha}E_\beta^\alpha\psi^\beta\right) = 0$. Since this is valid for any compact submanifold of M, we get

$$\frac{\partial}{\partial\mu^i}\left(\sum_\alpha \dot{\psi}_i^\alpha E_\beta^\alpha\psi^\beta\right) = 0$$

That is, the internal symmetry reflected in the invariance of the Lagrangian leads to a "conserved" vector field $Z^i = \sum_\alpha \dot{\psi}_i^\alpha E_\beta^\alpha\psi^\beta$.

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NOTATION

1. SPACES

3. MAPPINGS AND OTHER NOTATION

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