A geometric framework for time-dependent contact systems

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b-Day

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Outline

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1. Cocontact structures
2. Cocontact Hamiltonian systems
3. Cocontact Lagrangian systems
4. More examples

Cocontact manifolds

Definition

Let M be a $(2n+2)$-dimensional manifold. A **cocontact structure** on M is a pair (τ, η) of one-forms on M such that

1. $\mathrm{d}\tau = 0$,
2. $\tau \wedge \eta \wedge (\mathrm{d}\eta)^{\wedge n}$ is a volume form on M.

In this case, (M, τ, η) is a cocontact manifold.

Note that

- $\ker \tau$ is an integrable distribution giving a foliation of M with contact leaves.
- $\ker \eta$ is a non-integrable distribution.

Examples

Example

Let (P, η_0) be a contact manifold. The product $M = \mathbb{R} \times P$ is a cocontact manifold with cocontact structure $(\mathrm{d}t, \eta)$ where t denotes the canonical coordinate of $\mathbb{R}$ and η is the pull-back of η_0 to M.

Example

Let $(P, \tau, -\mathrm{d}\theta_0)$ be an exact cosymplectic manifold. The product $M = P \times \mathbb{R}$ is a cocontact manifold with cocontact structure $(\tau, \mathrm{d}s - \theta)$, where s denotes the canonical coordinate of $\mathbb{R}$ and θ is the pull-back of θ_0 to M.

Canonical cocontact manifold

Consider an n-dimensional manifold Q with coordinates (q^i) and its cotangent bundle T^*Q with induced natural coordinates (q^i, p_i). Consider the product manifolds $\mathbb{R} \times \mathrm{T}^*Q$ with coordinates (t, q^i, p_i), $\mathrm{T}^*Q \times \mathbb{R}$ with coordinates (q^i, p_i, s) and $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$ with coordinates (t, q^i, p_i, s) and the projections

$$
\begin{tikzcd}
 & \mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R} \arrow[dl, "\rho_1"'] \arrow[dd, "\pi"] \arrow[dr, "\rho_2"] & \\
\mathbb{R} \times \mathrm{T}^*Q \arrow[dr, "\pi_2"'] & & \mathrm{T}^*Q \times \mathbb{R} \arrow[dl, "\pi_1"] \\
 & \mathrm{T}^*Q &
\end{tikzcd}
$$

Let $\theta_0 \in \Omega^1(\mathrm{T}^*Q)$ be the canonical 1-form of T^*Q given by $\theta_0 = p_i \mathrm{d}q^i$. If $\theta_2 = \pi_2^*\theta$, we have that $(\mathrm{d}t, \theta_2)$ is a cosymplectic structure in $\mathbb{R} \times \mathrm{T}^*Q$.

Denoting by $\theta_1 = \pi_1^*\theta_0$, we have that $\eta_1 = \mathrm{d}s - \theta_1$ is a contact form in $\mathrm{T}^*Q \times \mathbb{R}$.

Finally, consider the 1-form $\theta = \rho_1^*\theta_2 = \rho_2^*\theta_1 = \pi^*\theta_0 \in \Omega^1(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R})$ and let $\eta = \mathrm{d}s - \theta$. Then, $(\mathrm{d}t, \eta)$ is a cocontact structure in $\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R}$. The local expression of the 1-form η is

$$\eta = \mathrm{d}s - p_i \mathrm{d}q^i\,.$$

The $\flat$ isomorphism

Proposition

Let (M, τ, η) be a cocontact manifold. We have the following isomorphism of vector bundles:

$$\begin{array}{rccl} \flat = \flat_{\tau,\eta}\colon & \mathrm{T}M & \longrightarrow & \mathrm{T}^*M \\ & v & \longmapsto & (\iota_v\tau)\tau + \iota_v \mathrm{d}\eta + (\iota_v\eta)\,\eta \end{array}$$

Proof.

It is clear that $\ker \flat = 0$, since M is a cocontact manifold. Hence, it follows that $\flat$ has to be an isomorphism. □

This isomorphism can be extended to an isomorphism of $\mathscr{C}^\infty(M)$-modules:

$$\begin{array}{rccl} \flat\colon & \mathfrak{X}(M) & \longrightarrow & \Omega^1(M) \\ & X & \longmapsto & (\iota_X\tau)\tau + \iota_X \mathrm{d}\eta + (\iota_X\eta)\,\eta \end{array}$$

Reeb vector fields

Proposition

*Given a cocontact manifold (M, τ, η), there exist two unique vector fields $R_t, R_s \in \mathfrak{X}(M)$, called **Reeb vector fields**, satisfying the conditions*

$$\begin{aligned} \iota_{R_t}\tau &= 1\,, & \iota_{R_t}\eta &= 0\,, & \iota_{R_t}\mathrm{d}\eta &= 0\,, \\ \iota_{R_s}\tau &= 0\,, & \iota_{R_s}\eta &= 1\,, & \iota_{R_s}\mathrm{d}\eta &= 0\,. \end{aligned}$$

*R_t is the **time Reeb vector field** and R_s is the **contact Reeb vector field**.*

Using the isomorphism $\flat$, we can redefine the Reeb vector fields:

$$R_t = \flat^{-1}(\tau)\,, \quad R_s = \flat^{-1}(\eta)\,.$$

Figure: Georges Reeb

Darboux coordinates

Theorem (Darboux theorem for cocontact manifolds)

Let (M, τ, η) be a cocontact manifold. Then, for every point $p \in M$, there exists a local chart $(U; t, q^i, p_i, s)$ around p such that

$$\tau|_U = \mathrm{d}t \,, \quad \eta|_U = \mathrm{d}s - p_i \mathrm{d}q^i \,.$$

These coordinates are called ***canonical*** *or* ***Darboux*** *coordinates. Moreover, in Darboux coordinates, the Reeb vector fields are*

$$R_t|_U = \frac{\partial}{\partial t} \,, \quad R_s|_U = \frac{\partial}{\partial s} \,.$$

Figure: Gaston Darboux

Note that if $(M, \tau = \mathrm{d}t, \eta)$ is a cocontact manifold, then

$$\Omega = \mathrm{d}(e^{-t}\eta) = e^{-t}(\eta \wedge \tau + \mathrm{d}\eta)$$

is an exact symplectic form on M.

Jacobi manifolds

A **Jacobi manifold** is a triple (M, Λ, E) where Λ is a bivector field on M and E is a vector field on E such that

$$[\Lambda, \Lambda] = 2E \wedge \Lambda\,, \qquad \mathscr{L}_E \Lambda = [E, \Lambda] = 0\,.$$

We can define a bilinear map on $\mathscr{C}^\infty(M)$ given by

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) + fE(g) - gE(f)\,.$$

This bracket
is called the **Jacobi bracket** and has the following properties:

- is bilinear,
- is skew-symmetric,
- satisfies
  the Jacobi identity: $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$,
- safisfies the weak Leibniz rule: $\operatorname{Supp}\{f, g\} \subseteq \operatorname{Supp} f \cap \operatorname{Supp} g$.

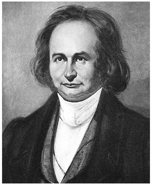


Figure: Carl Gustav Jakob Jacobi

We have the morphism $\widehat{\Lambda} : \alpha \in \Omega^1(M) \mapsto \Lambda(\alpha, \cdot) \in \mathfrak{X}(M)$.

Given a smooth function $f \in \mathscr{C}^\infty(M)$, we define the **Hamiltonian vector field** relative to f as

$$X_f = \widehat{\Lambda}(\mathrm{d}f) + fE\,.$$

Examples of Jacobi manifolds

Example

A **Poisson manifold** is a manifold M with a Lie bracket on $\mathscr{C}^\infty(M)$ satisfying the Leibniz rule

$$\{fg, h\} = f\{g, h\} + \{f, h\}g\,.$$

Equivalently, one can define a Poisson manifold as a pair (M, Λ) where Λ is a bivector on M such that

$$[\Lambda, \Lambda] = 0\,.$$

One can prove that a Jacobi manifold (M, Λ, E) is Poisson if, and only if, $E = 0$. Thus, the Jacobi bracket becomes a Poisson bracket and satisfies the Leibniz rule. Since symplectic and cosymplectic manifolds are Poisson, they are also Jacobi.

Example

Let (M, τ, η) be a cocontact manifold. Then,

$$\Lambda(\alpha, \beta) = -\mathrm{d}\eta(\flat^{-1}\alpha, \flat^{-1}\beta)\,, \qquad E = -R_s$$

define a Jacobi structure on M.

Analogously, a contact manifold (M, η) is also Jacobi.

Outline

Cocontact Hamilton equations for curves

Definition

A **cocontact Hamiltonian system** is a tuple (M, τ, η, H), where (M, τ, η) is a cocontact manifold and $H \in \mathscr{C}^\infty(M)$ is a Hamiltonian function.
The **cocontact Hamilton equations for a curve** $\psi : I \subset \mathbb{R} \longrightarrow M$ are

$$\begin{cases} \iota_{\psi'} \mathrm{d}\eta = (\mathrm{d}H - (R_s H)\eta - (R_t H)\tau) \circ \psi \,, \\ \iota_{\psi'} \eta = -H \circ \psi \,, \\ \iota_{\psi'} \tau = 1 \,, \end{cases}$$

where $\psi' : I \subset \mathbb{R} \longrightarrow \mathrm{T}M$ denotes the canonical lift of ψ to $\mathrm{T}M$.

In Darboux coordinates, these equations read

$$\dot{t} = 1 \,, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i} \,, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i} - p_i \frac{\partial H}{\partial s} \,, \qquad \dot{s} = p_i \frac{\partial H}{\partial p_i} - H \,.$$

Cocontact Hamilton equations for vector fields

Definition

Let (M, τ, η, H) be a cocontact Hamiltonian system. The **cocontact Hamilton equations for a vector field** $X \in \mathfrak{X}(M)$ are

$$\begin{cases} \iota_X \mathrm{d}\eta = \mathrm{d}H - (R_s H)\eta - (R_t H)\tau \,, \\ \iota_X \eta = -H \,, \\ \iota_X \tau = 1 \,, \end{cases}$$

or equivalently $\flat(X) = \mathrm{d}H - (R_s H + H)\eta + (1 - R_t H)\tau$.

Thus it is clear that for every Hamiltonian function $H \in \mathscr{C}^\infty(M)$ there exists a unique $X_H \in \mathfrak{X}(M)$, called the **cocontact Hamiltonian vector field**, satisfying the Hamilton equations for the function H. In Darboux coordinates, it reads

$$X_H = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial s}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial s} \,.$$

Given a vector field $X \in \mathfrak{X}(M)$, its integral curves satisfy the cocontact Hamilton equations for curves if, and only if, X satisfies the cocontact Hamilton equations for vector fields.

Example: A time-dependent system with central force and friction (I)

Consider the Kepler problem in the case where the mass of the particle subjected to the central force is a non-vanishing function of time $m(t)$. The motion of the particle is on a plane, so the configuration space is $Q = \mathbb{R}^2 \setminus \{0\}$ with coordinates (r, φ).

The phase bundle $\mathbb{R} \times \mathrm{T}^* Q \times \mathbb{R}$ with coordinates $(t, r, \varphi, p_r, p_\varphi, s)$ has a natural cocontact structure given by

$$\tau = \mathrm{d}t \,, \qquad \eta = \mathrm{d}s - p_r \mathrm{d}r - p_\varphi \mathrm{d}\varphi \,.$$

The Reeb vector fields are $R_t = \dfrac{\partial}{\partial t}$ and $R_s = \dfrac{\partial}{\partial s}$.

Example: A time-dependent system with central force and friction (II)

Consider the Hamiltonian function $H \in \mathscr{C}^\infty(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R})$ given by

$$H(t, r, \varphi, p_r, p_\varphi, s) = \frac{p_r^2}{2m(t)} + \frac{p_\varphi^2}{2m(t)r^2} + \frac{k}{r} + \gamma s\,.$$

The vector field $X \in \mathfrak{X}(\mathbb{R} \times \mathrm{T}^*Q \times \mathbb{R})$ satisfying Hamilton's equations reads

$$\begin{aligned} X = \frac{\partial}{\partial t} + \frac{p_r}{m(t)}\frac{\partial}{\partial r} + \frac{p_\varphi}{m(t)r^2}\frac{\partial}{\partial \varphi} + \left(\frac{p_\varphi^2}{m(t)r^3} + \frac{k}{r^2} - \gamma p_r\right)\frac{\partial}{\partial p_r} \\ - \gamma p_\varphi \frac{\partial}{\partial p_\varphi} + \left(\frac{p_r^2}{2m(t)} + \frac{p_\varphi^2}{2m(t)r^2} - \frac{k}{r} - \gamma s\right)\frac{\partial}{\partial s}\,. \end{aligned}$$

Example: A time-dependent system with central force and friction (III)

Then, the integral curves $(t, r, \varphi, p_r, p_\varphi, s)$ satisfy

$$\begin{cases} \dot{t} = 1\,, \\ m(t)\dot{r} = p_r\,, \\ m(t)r^2\dot{\varphi} = p_\varphi\,, \\ \dot{p}_r = \dfrac{p_\varphi^2}{m(t)r^3} + \dfrac{k}{r^2} - \gamma p_r\,, \\ \dot{p}_\varphi = -\gamma p_\varphi\,, \\ \dot{s} = \dfrac{p_r^2}{2m(t)} + \dfrac{p_\varphi^2}{2m(t)r^2} - \dfrac{k}{r} - \gamma s\,. \end{cases}$$

Hence, the integral curves must fulfill the system of second-order equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(m(t)\dot{r}\right) = m(t)r\dot{\varphi}^2 + \dfrac{k}{r^2} - \gamma m(t)\dot{r}\,, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(m(t)r^2\dot{\varphi}\right) = -\gamma m(t)r^2\dot{\varphi}\,. \end{cases}$$

Note that if $\gamma = 0$ we get the usual angular momentum conservation law.

Outline

Euler–Lagrange equations

The Hamiltonian formulation has a Lagrangian counterpart. It is based in **Herglotz variational principle**. Given a manifold Q, consider a Lagrangian function $L : \mathbb{R} \times \mathrm{T}Q \times \mathbb{R} \longrightarrow \mathbb{R}$. We can modify Hamilton's variational principle to only consider curves on $\mathbb{R} \times \mathrm{T}Q \times \mathbb{R}$ of the form $\sigma(t) = (t, q^i(t), \dot{q}^i(t), s(t))$ such that

$$\dot{s}(t) = L(t, q^i(t), \dot{q}^i(t), s(t)) \,.$$

We obtain the so-called **Herglotz–Euler–Lagrange equations**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = \frac{\partial L}{\partial s}\frac{\partial L}{\partial v^i} \,, \qquad \dot{s} = L \,.$$

These Lagrangians are called *action-dependent Lagrangians*.

There exists a Poincaré–Cartan-like formulation of these equations so that they can be written as

$$\begin{cases} \iota_{X_L}\mathrm{d}\eta_L = \mathrm{d}E_L - (R_t^L E_L)\mathrm{d}t - (R_s^L E_L)\eta_L \,, \\ \iota_{X_L}\eta_L = -E_L \,, \\ \iota_{X_L}\mathrm{d}t = 1 \,. \end{cases}$$

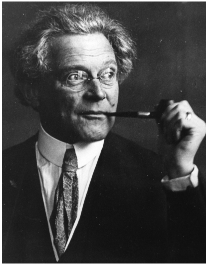


Figure: Gustav Herglotz

Example: Duffing's equation (I)

The Duffing equation, named after G. Duffing, is a non-linear second-order differential equation which can be used to model certain damped and forced oscillators. The Duffing equation is

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos\omega t\,,$$

where $\alpha, \beta, \gamma, \delta, \omega$ are constant parameters. Note that

- if $\gamma = 0$, i.e. the system does not depend on time, we are in the case of contact mechanics.
- if $\delta = 0$, namely there is no damping, we have a cosymplectic system.
- if $\beta = \delta = \gamma = 0$, we obtain the equation of a simple harmonic oscillator.

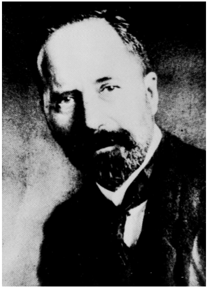


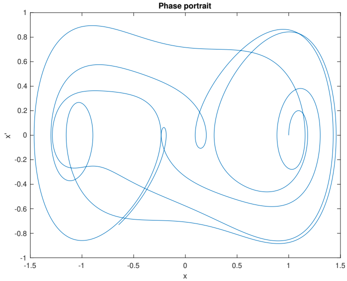
Figure: Georg Duffing

Duffing's equation models a damped forced oscillator with a stiffness different from the one obtained by Hooke's law.

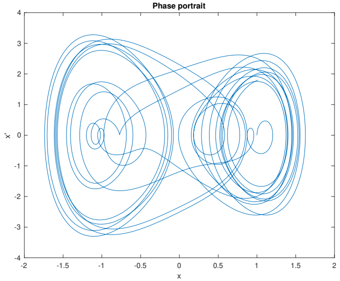
It can be derived from the Lagrangian function

$$L(t, x, v, s) = \frac{1}{2}v^2 - \frac{1}{2}\alpha x^2 - \frac{1}{4}\beta x^4 - \delta s + \gamma x\cos\omega t\,.$$

Example: Duffing's equation (II)



(a) $\alpha = -1$, $\beta = 1$, $\gamma = 0.44$, $\delta = 0.3$, $\omega = 1.2$, $x(0) = 1$, $\dot{x}(0) = 0$



(b) $\alpha = 1$, $\beta = 5$, $\gamma = 8$, $\delta = 0.02$, $\omega = 0.5$, $x(0) = 1$, $\dot{x}(0) = 0$

Outline

Damped pendulum with variable length (I)

Consider a damped pendulum of mass m with time-dependent length $\ell(t)$. Its position in the plane can be described using polar coordinates (r, θ).

The constraint $r = \ell(t)$ will be introduced in the Lagrangian function via a Lagrange multiplier. The phase space of this system is the bundle $\mathbb{R} \times \mathrm{T}\mathbb{R}^3 \times \mathbb{R}$, equipped with coordinates $(t, r, \theta, \lambda, \dot{r}, \dot{\theta}, \dot{\lambda}, s)$.

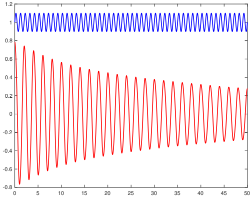
The Lagrangian function describing this system is

$$L(t, r, \theta, \lambda, \dot{r}, \dot{\theta}, \dot{\lambda}, s) = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr(1 - \cos\theta) + \lambda(r - \ell(t)) - \gamma s \in \mathscr{C}^\infty(\mathbb{R} \times \mathrm{T}\mathbb{R}^3 \times \mathbb{R}),$$

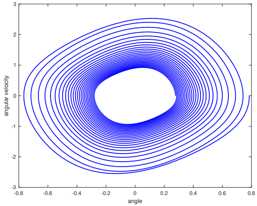
where λ is the Lagrange multiplier.

Note that this is a singular Lagrangian.

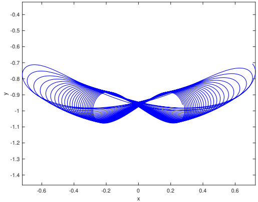
Damped pendulum with variable length (II)



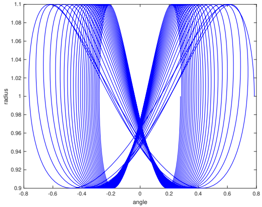
(a) Radius (blue) and angle θ (red) with respect to time



(b) Phase portrait of the pendulum (θ and $\dot{\theta}$)



(c) Trajectory of the pendulum in the plane XY



(d) Radius with respect to the angle θ

Figure: Plots of the pendulum with friction coefficient $\gamma = 0.5$ and $\ell(t) = 1 + 0.1\sin(2\pi t)$

System with time-dependent mass and quadratic drag

Consider a system of time-dependent mass with an engine providing an ascending force $F > 0$ and subjected to a drag proportional to the square of the velocity.

Let $Q = \mathbb{R}$ with coordinate (y) be the configuration manifold of our system and consider the Lagrangian function

$$L\colon \mathbb{R} \times \mathrm{T}Q \times \mathbb{R} \longrightarrow \mathbb{R}$$

given by

$$L(t, y, \dot{y}, s) = \frac{1}{2} m(t)\dot{y}^2 + \frac{m(t)g}{2\gamma}\left(e^{-2\gamma y} - 1\right) - 2\gamma \dot{y} s + \frac{1}{2\gamma} F\,,$$

where γ is the drag coefficient and the mass is given by the monotone decreasing function $m(t)$.

The Herglotz–Euler–Lagrange for this Lagrangian function are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m(t)\dot{y}\right) = F - m(t)g - \gamma m(t)\dot{y}^2\,, \qquad \dot{s} = L\,.$$

References

Thanks for your attention and happy *b*-day!

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