Rocket Project - Xirui Huang

Hand Calculations

Fluids Derivations

Conservation of Energy:

W: work	$\Delta x, \Delta r$: displacement
k: kinetic	P: pressure
U: potential	A: Area
F: force	ρ: density
v: velocity	V: volume

we know $W = \Delta K + \Delta U$

$$W = \vec{F} \cdot \Delta \vec{r} \Rightarrow W = F \cdot \Delta x$$

$$W_i = W_f$$

$$F = P \cdot A$$

so $W_1 + k_1 + U_1 = W_2 + k_2 + U_2$

$$P_1 A_1 \Delta x_1 + \frac{1}{2} m v_1^2 + m g y_1 = P_2 A_2 \Delta x_2 + \frac{1}{2} m v_2^2 + m g y_2$$

since density $\Rightarrow \rho = \frac{m}{V} \Rightarrow m = \rho V$ and volume $V = A \cdot \Delta x$, we get

$$P_1 V + \frac{1}{2}\rho V v_1^2 + \rho V g y_1 = P_2 V + \frac{1}{2}\rho V v_2^2 + \rho V g y_2$$

Bernoulli's Equation: $P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$

From Continuity of flow: $Q_1 = Q_2 \Rightarrow A_1 v_1 = A_2 v_2$ where $Q = \frac{dV}{dt}$ (m^3/s)

From Bernoulli's and Continuity, we get:

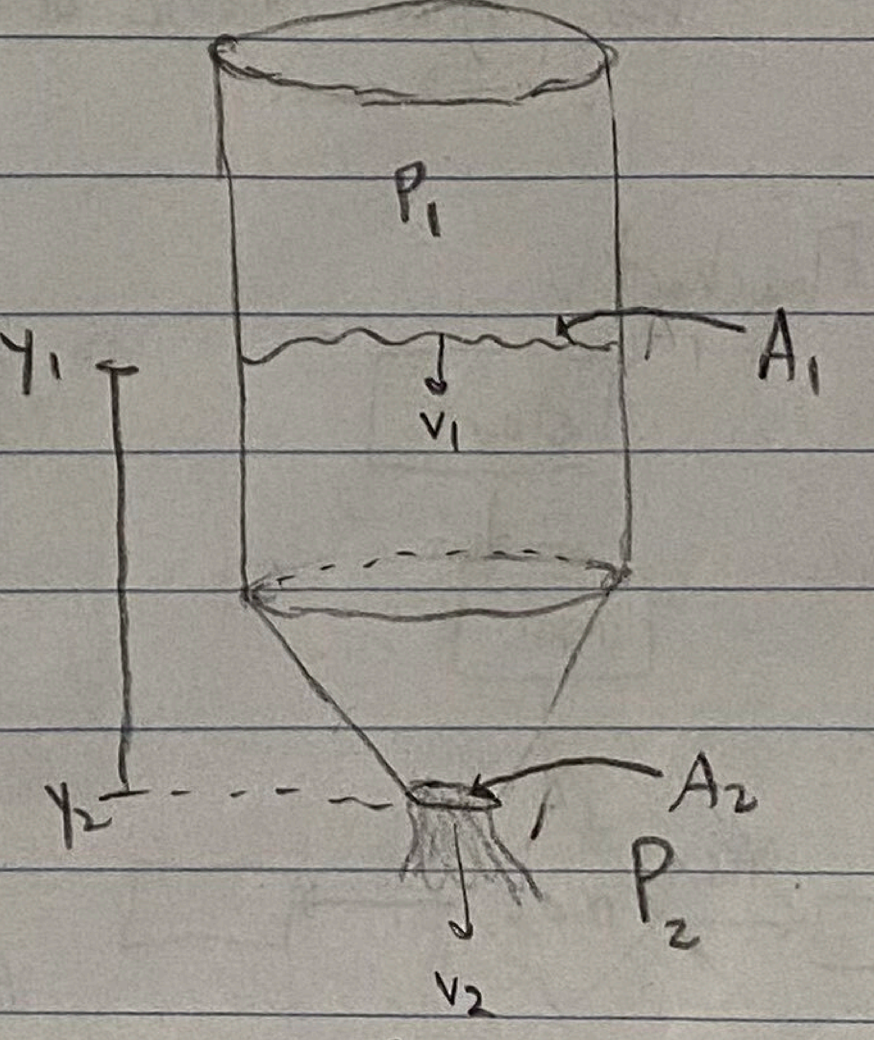
let $y_2 = 0$

$$P_1 + \frac{1}{2}\rho\left(\frac{A_2 v_2}{A_1}\right)^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \cancel{\rho g y_2}$$

$$\frac{\rho v_2^2 A_2^2}{2A_1^2} - \frac{\rho v_2^2}{2} = P_2 - P_1 - \rho g y_1$$

$$v_2^2\left(\frac{\rho A_2^2}{2A_1^2} - \frac{\rho A_1^2}{2A_1^2}\right) = P_2 - P_1 - \rho g y_1$$

$$\boxed{v_2 = \sqrt{\frac{(P_2 - P_1 - \rho g y_1) 2A_1^2}{\rho (A_2^2 - A_1^2)}}}$$



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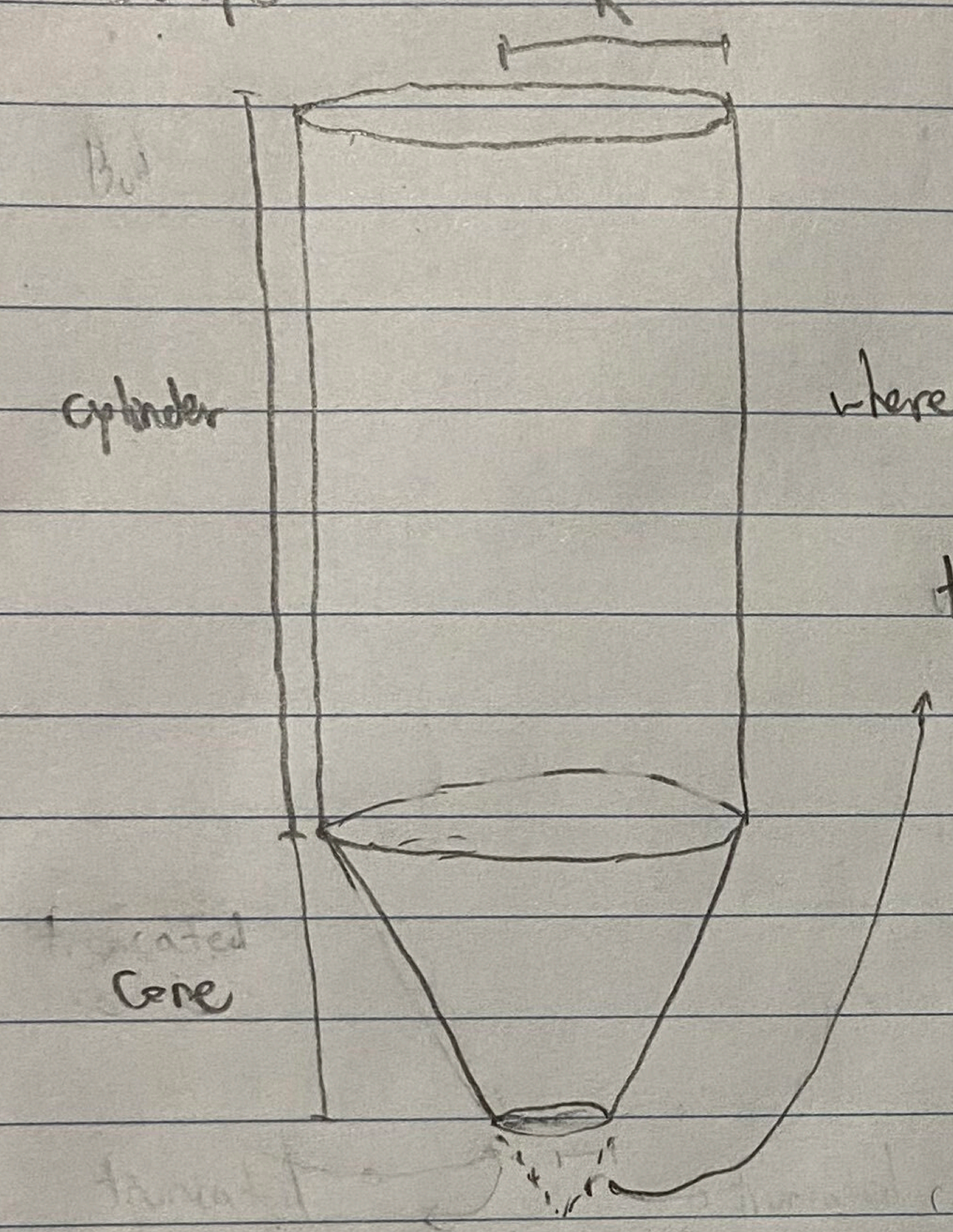
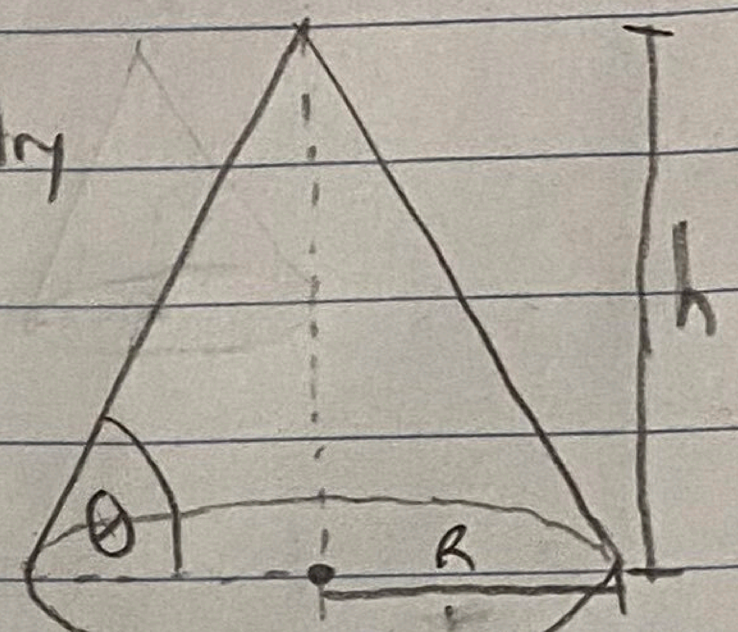
Geometry and water bottle data

Bottle shape



Cone geometry



$h = R\tan(\theta)$

$V_{cone} = \frac{\pi R^3 h}{3}$

where $R_{cylinder} = R_{cone}$

truncated cone where radius $= r_{trunc}$, and height $= l$

Volume of cone:

$$\frac{\pi R^3 h}{3} - \frac{\pi r^3 l}{3}$$

$$\frac{\pi R^3 (R\tan(\theta))}{3} - \frac{\pi r^3 (r\tan\theta)}{3}$$

with measured $R = 0.054$ m   we get $\text{volume}_{cone} = 0.36$ L, *Poured in 800 mL (0.8 L) water, so

$r = 0.011$ m

$\theta = 65.8°$ water in cone $= 0.36$ L, water in cylinder $= 0.8 - 0.36 = 0.44$ L

height of cone: $R\tan\theta = 0.054\tan(65.8) = 0.12$ m

height of water in cylinder initially:

we know $V_{cylinder} = \pi R^2 \cdot h$, so $0.44 = \pi (0.054)^2 \cdot h \Rightarrow h = 0.047$ m

therefore, height of water = height of cone + height of water in cylinder initially

$= 0.12 + 0.047 = 0.0167$ m

* Initial water is 800 mL, but volume changes as follows:

$V_{new} = V_{initial} - Q\,dt$ where Q = flow rate; volume of water leaving bottle

Q = Area of Cap * velocity of exhaust $= A_2 V_2$

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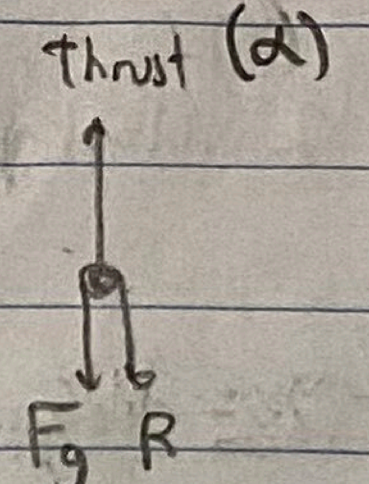
Rocket Equation & Derivations

we know $R = \frac{1}{2} A_{crossec} \rho_{air} D v_{Net}^2$ and FBD:



so $F_{net} = \alpha - F_g - R$

$ma = \alpha - mg - \frac{1}{2}A\rho D v^2 \Rightarrow a = \frac{\alpha - D\rho A v^2}{2m} - g$ for acceleration data

→ Main Rocket Equation:

$$\vec{F}_{Net} = m\cdot\vec{a} = m\left(\frac{d\vec{v}}{dt}\right) = \frac{d\vec{p}}{dt} \quad \text{where } \vec{p} = m\cdot\vec{v} \text{ (momentum)}$$

since $F_{Net} = -F_g - F_{Air}$, we get $-mg - Dv^2 = \frac{d\vec{p}}{dt} = \frac{P_{sys}(t+dt) - P_{sys}}{dt}$ ⇒ from derivative definition

$$-mg\,dt - Dv^2\,dt = (m+dm)(v+dv) + (-dm)(v+dv-u) - mv$$, where u = exhaust speed

$$-mg\,dt - Dv^2\,dt = \cancel{mv} + m\,dv + \cancel{v\,dm} + \cancel{dm\,dv} - \cancel{v\,dm} - \cancel{dm\,dv} + u\,dm - \cancel{mv} = m\,dv + u\,dm$$

$$m\,dv = -u\,dm - mg\,dt - Dv^2\,dt$$

$$dv = \frac{-u}{m}\,dm - g\,dt - \frac{Dv^2}{m}\,dt \Rightarrow \boxed{\int dv = -\int \frac{u}{m}\,dm - \int g\,dt - \int \frac{Dv^2}{m}\,dt}$$

→ Liftoff happens when thrust > Fg (ignore resistance because at liftoff, v=0). With that, we get

$$dv = \frac{-u}{m}\,dm - g\,dt \Rightarrow \frac{dv}{dt} = \frac{-u\,dm}{m\,dt} - g \Rightarrow a = \frac{U\alpha}{m} - g \Rightarrow ma = U\alpha - mg$$

setting $F_{Net} = 0$ at liftoff; $0 = U\alpha - mg$

$$U\alpha = (m_i - \alpha t)g \quad \text{so} \quad \boxed{t_{liftoff} = \frac{m_i}{\alpha} - \frac{U}{g}}$$

*mass at liftoff

$m_{lift} = m_i - \alpha t_{lift}$

$m_{lift} = m_i - \alpha\left(\frac{m_i}{\alpha} - \frac{U}{g}\right)$

$m_{lift} = \frac{\alpha U}{g}$

where $\alpha = U\frac{dm}{dt}$, but from Bernoulli's, and with $m = \rho V$, we get

$$\alpha = \left|U\rho\frac{dV}{dt}\right| = |U\rho Q| = V_{exhaust}\,\rho A V_{exhaust} = A_2 \rho v_2^2 = \boxed{\frac{2A_1^2 A_2 (P_2 - P_1 - \rho g y_1)}{A_2^2 - A_1}}$$

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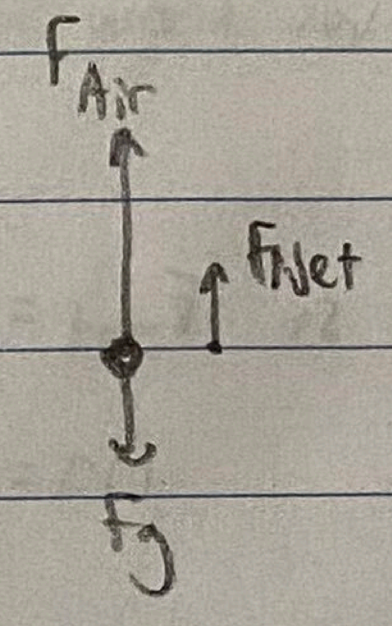
Hand Calculations

Landing time & velocity when Air Resistance force is proportional to the velocity squared

Let λ = drag coefficient and $F_{Air} = \lambda v^2$



$$F_{Net} = m \cdot a = F_g - F_{Air}$$

$$mg - \lambda v^2 = m\frac{dv}{dt}$$

$$(mg - \lambda v^2)\frac{dy}{dt} = mv\frac{dv}{dt}$$

$$\int_0^{y_f} dy = \int_0^{v_f} \frac{mv\,dv}{mg - \lambda v^2}$$

with u-sub, let $u = g - \frac{\lambda v^2}{m}$. So, if $du = \frac{-2\lambda v}{m}dv$, then $v\,dv = \frac{-m}{2\lambda}du$

$$y_f - 0 = \int_{v=0}^{v=v_f} -\frac{m}{2\lambda u}du$$

must always be (+)

$$y_f = \frac{-m}{2\lambda}\ln\left|g - \frac{\lambda v^2}{m}\right|_{v=0}^{v=v_f} = \frac{-m}{2\lambda}\ln\left(\frac{\overbrace{\left|g - \frac{\lambda v_f^2}{m}\right|}}{g}\right) = \frac{-m}{2\lambda}\ln\left(1 - \frac{\lambda v^2}{mg}\right)$$

Then, $e^{\frac{-2\lambda}{m}} = 1 - \frac{\lambda v^2}{mg}$, so with that: $\frac{\lambda v_f^2}{mg} = 1 - e^{\frac{-2\lambda y_f}{m}}$

isolating for v_f, we get $\boxed{v_f = \sqrt{\frac{mg}{\lambda}\left(1 - e^{\frac{-2\lambda y_f}{m}}\right)}}$ for landing velocity

As for flight duration equation;

from $v_f = \sqrt{\frac{mg}{\lambda}\left(1 - e^{\frac{-2\lambda y_f}{m}}\right)} = \frac{dy}{dt}$

$$\int_{y_i}^{y_f} \frac{dy}{\sqrt{\frac{mg}{\lambda}\left(1 - e^{\frac{-2\lambda y}{m}}\right)}} = \int_0^{t_f} dt \text{, so } \boxed{t = \left[\frac{\quad ?}{\sqrt{\frac{mg}{\lambda}\left(1 - e^{\frac{-2\lambda y}{m}}\right)}}\right]_{y_i}^{y_f}}$$

t =