Technical Appendix

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A Proofs

A.1 Proof of Theorem 1

Proof. One way to partition the space into 3 partitions, is to have a different class for every unit of space between the two points. Points lying on the x-axis within the interval $(i-1, i)$ should be assigned to the i^{th} class. For example, points on the x-axis within $(0, 1)$ should be assigned to class 1. Succinctly put, we are trying to find y_1 and y_2 such that

$$\arg\max_i \left(\frac{y_{1i}}{d} + \frac{y_{2i}}{3-d} \right) = \lceil d \rceil \ \forall d \in (0, 3) \tag{1}$$

It may also be desirable to have a class's 'influence' decrease monotonically as distance increases along the x-axis away from the center of its corresponding interval. Combining these two objectives results in the following system of inequalities.

$$\begin{cases} \frac{y_{11}}{d} + \frac{y_{21}}{3-d} > \frac{y_{12}}{d} + \frac{y_{22}}{3-d} > \frac{y_{13}}{d} + \frac{y_{23}}{3-d} & 0 < d < 1 \\ \frac{y_{12}}{d} + \frac{y_{22}}{3-d} = \frac{y_{11}}{d} + \frac{y_{21}}{3-d} > \frac{y_{13}}{d} + \frac{y_{23}}{3-d} & d = 1 \\ \frac{y_{12}}{d} + \frac{y_{22}}{3-d} > \frac{y_{11}}{d} + \frac{y_{21}}{3-d} > \frac{y_{13}}{d} + \frac{y_{23}}{3-d} & 1 < d < \frac{3}{2} \\ \frac{y_{12}}{d} + \frac{y_{22}}{3-d} > \frac{y_{11}}{d} + \frac{y_{21}}{3-d} = \frac{y_{13}}{d} + \frac{y_{23}}{3-d} & d = \frac{3}{2} \\ \frac{y_{12}}{d} + \frac{y_{22}}{3-d} > \frac{y_{13}}{d} + \frac{y_{23}}{3-d} > \frac{y_{11}}{d} + \frac{y_{21}}{3-d} & \frac{3}{2} < d < 2 \\ \frac{y_{13}}{d} + \frac{y_{23}}{3-d} = \frac{y_{12}}{d} + \frac{y_{22}}{3-d} > \frac{y_{11}}{d} + \frac{y_{21}}{3-d} & d = 2 \\ \frac{y_{13}}{d} + \frac{y_{23}}{3-d} > \frac{y_{12}}{d} + \frac{y_{22}}{3-d} > \frac{y_{11}}{d} + \frac{y_{21}}{3-d} & 2 < d < 3 \end{cases} \tag{2}$$

Since the labels are probabilistic, the sum of their elements must be one and each element must have a value greater than or equal to 0.

$$\sum_{i=1}^{3} y_{1i} = \sum_{i=1}^{3} y_{2i} = 1$$
$$y_{1i} \geq 0,\ y_{2i} \geq 0 \text{ for } i = 1, 2, 3$$

To simplify this system of equations, we can also require that the label values for the two classes be symmetric with $y_{1i} = y_{2(n-i)}$ for $i = 1, 2, 3$.

$$y_{11} = y_{23}$$
$$y_{12} = y_{22}$$
$$y_{13} = y_{21}$$

$$
\begin{cases}
\frac{y_{11}}{d} + \frac{y_{13}}{3-d} > \frac{y_{12}}{d} + \frac{y_{12}}{3-d} > \frac{y_{13}}{d} + \frac{y_{11}}{3-d} & 0 < d < 1 \\
\frac{y_{12}}{d} + \frac{y_{12}}{3-d} = \frac{y_{11}}{d} + \frac{y_{13}}{3-d} > \frac{y_{13}}{d} + \frac{y_{11}}{3-d} & d = 1 \\
\frac{y_{12}}{d} + \frac{y_{12}}{3-d} > \frac{y_{11}}{d} + \frac{y_{13}}{3-d} > \frac{y_{13}}{d} + \frac{y_{11}}{3-d} & 1 < d < \frac{3}{2} \\
\frac{y_{12}}{d} + \frac{y_{12}}{3-d} > \frac{y_{11}}{d} + \frac{y_{13}}{3-d} = \frac{y_{13}}{d} + \frac{y_{11}}{3-d} & d = \frac{3}{2} \\
\frac{y_{12}}{d} + \frac{y_{12}}{3-d} > \frac{y_{13}}{d} + \frac{y_{11}}{3-d} > \frac{y_{11}}{d} + \frac{y_{13}}{3-d} & \frac{3}{2} < d < 2 \\
\frac{y_{13}}{d} + \frac{y_{11}}{3-d} = \frac{y_{12}}{d} + \frac{y_{12}}{3-d} > \frac{y_{11}}{d} + \frac{y_{13}}{3-d} & d = 2 \\
\frac{y_{13}}{d} + \frac{y_{11}}{3-d} > \frac{y_{12}}{d} + \frac{y_{12}}{3-d} > \frac{y_{11}}{d} + \frac{y_{13}}{3-d} & 2 < d < 3 \\
y_{11} + y_{12} + y_{13} = 1 \\
y_{1i} \geq 0, \text{ for } i = 1, 2, 3
\end{cases}
\tag{3}
$$

Remark *The inequalities in Equation 3 are symmetric about* $d = \frac{3}{2}$. *We can reduce them by considering only the cases where* $d \leq \frac{3}{2}$.

$$
\begin{cases}
\frac{y_{11}}{d} + \frac{y_{13}}{3-d} > \frac{y_{12}}{d} + \frac{y_{12}}{3-d} > \frac{y_{13}}{d} + \frac{y_{11}}{3-d} & 0 < d < 1 \\
\frac{y_{12}}{d} + \frac{y_{12}}{3-d} = \frac{y_{11}}{d} + \frac{y_{13}}{3-d} > \frac{y_{13}}{d} + \frac{y_{11}}{3-d} & d = 1 \\
\frac{y_{12}}{d} + \frac{y_{12}}{3-d} > \frac{y_{11}}{d} + \frac{y_{13}}{3-d} > \frac{y_{13}}{d} + \frac{y_{11}}{3-d} & 1 < d < \frac{3}{2} \\
\frac{y_{12}}{d} + \frac{y_{12}}{3-d} > \frac{y_{11}}{d} + \frac{y_{13}}{3-d} & d = \frac{3}{2} \\
y_{11} + y_{12} + y_{13} = 1 \\
y_{1i} \geq 0, \textit{ for } i = 1, 2, 3
\end{cases}
\tag{4}
$$

We can further simplify this system.

$$
\begin{cases}
(3-d)y_{11} + dy_{13} > 3y_{12} > (3-d)y_{13} + dy_{11} & 0 < d < 1 \\
3y_{12} = 2y_{11} + y_{13} > 2y_{13} + y_{11} & d = 1 \\
3y_{12} > (3-d)y_{11} + dy_{13} > (3-d)y_{13} + dy_{11} & 1 < d < \frac{3}{2} \\
4y_{12} > 2y_{11} + 2y_{13} & d = \frac{3}{2} \\
y_{11} + y_{12} + y_{13} = 1 \\
y_{1i} \geq 0, \text{ for } i = 1, 2, 3
\end{cases}
$$

$$
\begin{cases}
3y_{12} = 2y_{11} + y_{13} > 1 \\
y_{11} > y_{12} > y_{13} \geq 0 \\
y_{11} + y_{12} + y_{13} = 1
\end{cases}
$$

□

A.2 Proof of Corollary 2

Proof. We proceed similarly to the previous proof. One way to partition the space into 3 partitions, is to have a different class for every unit of space between the two points; each interval should have length $\frac{c}{3}$. Points lying on the x-axis within the interval $((i-1)\frac{c}{3}, (i)\frac{c}{3})$ should be assigned to the i^{th} class. For example, points on the x-axis within $(0, \frac{c}{3})$ should be assigned to class 1. Succinctly put, we are trying to find y_1 and y_2 such that

$$\arg\max_i \left(\frac{y_{1i}}{(d)\frac{c}{3}} + \frac{y_{2i}}{(3-d)\frac{c}{3}} \right) = \lceil (d) \rceil \frac{c}{3} \;\forall d \in (0,3) \tag{5}$$

However, this is exactly equal to Equation 1 and the rest of the proof follows as above. □

A.3 Proof of Theorem 3

Proof. One way to select such labels is to use the same labels for each pair as in Theorem 1. This results in $y_1, ..., y_{M-1}$ each having a label distribution containing two non-zero values: $\frac{3}{5}$ (associated with its main class) and $\frac{2}{5}$ (associated with the class created between itself and x_0). Meanwhile, y_0 contains one element with value $\frac{3}{5}$ (associated with its own class) and M-1 elements with value $\frac{2}{5}$ (each associated with a unique class created between x_0 and each one of the other points). To get all probabilistic labels, we can normalize y_0 to instead have values $\frac{3}{2M+1}$ and $\frac{2}{2M+1}$.

It can be shown that this configuration of labels leads to $2M-1$ classes as each point has its own class and every pair creates one additional one. Without loss of generality, assume X_0 has position $(0,0)$ and one of the remaining points $x_i, i > 0$ has position $(p,0)$. Since all the points are equidistant to x_0, the nearest 2 points to any location between $(0,0)$ and $(\frac{p}{2}, 0)$ must be x_0 and x_i. Let d be the distance from x_0 to some arbitrary point $q = (d,0)$ within this interval (then $p-d$ is the distance from q to x_i). Let a denote the index of the main class of x_0 (i.e. the class with the highest label value in y_0), b denote the index of the main class of x_i, and c denote the index of the class they share (i.e. the only other non-zero class in y_1). Using the distance-weighted SLaPkNN rule, q would be classified as follows.

$$\begin{cases} a \text{ if } \frac{y_{0,a}}{d} > \frac{y_{i,b}}{p-d}, \frac{y_{0,c}}{d} + \frac{y_{i,c}}{p-d} \\ b \text{ if } \frac{y_{i,b}}{p-d} > \frac{y_{0,a}}{d}, \frac{y_{0,c}}{d} + \frac{y_{i,c}}{p-d} \\ c \text{ if } \frac{y_{0,c}}{d} + \frac{y_{i,c}}{p-d} > \frac{y_{0,a}}{d}, \frac{y_{i,b}}{p-d} \end{cases} \tag{6}$$

Plugging in the label values described above, we get the following system.

$$\begin{cases} a \text{ if } \frac{3}{(2M+1)d} > \frac{3}{5(p-d)}, \frac{2}{(2M+1)d} + \frac{2}{5(p-d)} \\ b \text{ if } \frac{3}{5(p-d)} > \frac{3}{(2M+1)d}, \frac{2}{(2M+1)d} + \frac{2}{5(p-d)} \\ c \text{ if } \frac{2}{(2M+1)d} + \frac{2}{5(p-d)} > \frac{3}{(2M+1)d}, \frac{3}{5(p-d)} \end{cases} \tag{7}$$

This can be further simplified.

$$\begin{cases} a \text{ if } \frac{1}{(2M+1)d} > \frac{2}{5(p-d)} \\ b \text{ if } \frac{1}{5(p-d)} > \frac{2}{(2M+1)d} \\ c \text{ if } \frac{1}{(2M+1)d} < \frac{2}{5(p-d)} < \frac{4}{(2M+1)d} \end{cases} \tag{8}$$

$$\begin{cases} a \text{ if } d < \frac{5p}{4M+7} \\ b \text{ if } d > \frac{10p}{2M+11} \\ c \text{ if } \frac{5p}{4M+7} < d < \frac{10p}{2M+11} \end{cases} \tag{9}$$

Clearly, there are valid values of d that can result in each of these classifications. $\square$

A.4 Proof of Proposition 4

Proof. Theorem 1 shows that a pair of neighboring prototypes can define their own respective classes as well as induce a third class between them. Theorem 3 shows that a single point can belong to multiple such pairs that each generate a unique 'third' class. If this is the case, then by taking the pair of vertices corresponding to each edge in an M-sided regular polygon we can create one class for every vertex and one for every edge of the polygon for a total of $2M$ classes.

In this case, every point belongs to two pairs, so the system will be somewhat different from the one found in the proof of Theorem 3 since every point will now need to have its label normalized. We use T to denote the number of pairs each point participates in, which in this case would just be two.

$$\begin{cases} a \text{ if } \frac{3}{(2T+3)d} > \frac{3}{(2T+3)(p-d)}, \frac{2}{(2T+1)d} + \frac{2}{(2T+3)(p-d)} \\ b \text{ if } \frac{3}{(2T+3)(p-d)} > \frac{3}{(2T+3)d}, \frac{2}{(2T+3)d} + \frac{2}{(2T+3)(p-d)} \\ c \text{ if } \frac{2}{(2T+3)d} + \frac{2}{(2T+3)(p-d)} > \frac{3}{(2T+3)d}, \frac{3}{(2T+3)(p-d)} \end{cases} \tag{10}$$

This can be further simplified.

$$\begin{cases} a \text{ if } \frac{1}{d} > \frac{2}{(p-d)} \\ b \text{ if } \frac{1}{(p-d)} > \frac{2}{d} \\ c \text{ if } \frac{1}{d} < \frac{2}{(p-d)} < \frac{4}{d} \end{cases} \tag{11}$$

$$\begin{cases} a \text{ if } d < \frac{p}{3} \\ b \text{ if } d > \frac{2p}{3} \\ c \text{ if } \frac{p}{3} < d < \frac{2p}{3} \end{cases} \tag{12}$$

$\square$

A.5 Proof of Theorem 5

Proof. We have already shown that every pair of neighboring prototypes can define their own respective classes as well as induce a third class between them. However, we now arrange M soft-label prototypes to be the vertices and centroid of an $(M-1)$-sided regular polygon. Using our previous method, we now have $2(M-1)$ classes from the perimeter. In addition, the prototype in the middle induces its own class as well as another class with each of the $M-1$ other points. Thus this configuration allows us to divide the space into $3M-2$ partitions. $\square$

A.6 Proof of Lemma 6

Proof. We proceed in the same manner as in Theorem 1 .
One way to partition the space into 4 partitions, is to have a different class for every unit of space between the two prototypes. Points lying on the x-axis within the interval $(i-1, i)$ should be assigned to the i^{th} class. For example, points on the x-axis within $(0, 1)$ should be assigned to class 1. Succinctly put, we are trying to find y_1 and y_2 such that

$$\arg\max_i \left(\frac{y_{1i}}{d} + \frac{y_{2i}}{4-d} \right) = \lceil d \rceil \; \forall d \in (0, 4) \tag{13}$$

It may also be desirable to have a class's 'influence' decrease monotonically as distance increases along the x-axis away from the center of its corresponding interval. Combining these two objectives results in a system of inequalities that are symmetric about $d = 2$. We can reduce them by considering only the cases where $d \leq 2$.

$$\begin{cases} \frac{y_{11}}{d} + \frac{y_{14}}{4-d} > \frac{y_{12}}{d} + \frac{y_{13}}{4-d} > \frac{y_{13}}{d} + \frac{y_{12}}{4-d} > \frac{y_{14}}{d} + \frac{y_{11}}{4-d} & 0 < d < 1 \\ \frac{y_{11}}{d} + \frac{y_{14}}{4-d} = \frac{y_{12}}{d} + \frac{y_{13}}{4-d} > \frac{y_{13}}{d} + \frac{y_{12}}{4-d} > \frac{y_{14}}{d} + \frac{y_{11}}{4-d} & d = 1 \\ \frac{y_{12}}{d} + \frac{y_{13}}{4-d} > \frac{y_{11}}{d} + \frac{y_{14}}{4-d} > \frac{y_{13}}{d} + \frac{y_{12}}{4-d} > \frac{y_{14}}{d} + \frac{y_{11}}{4-d} & 1 < d < 1.5 \\ \frac{y_{12}}{d} + \frac{y_{13}}{4-d} > \frac{y_{11}}{d} + \frac{y_{14}}{4-d} = \frac{y_{13}}{d} + \frac{y_{12}}{4-d} > \frac{y_{14}}{d} + \frac{y_{11}}{4-d} & d = 1.5 \\ \frac{y_{12}}{d} + \frac{y_{13}}{4-d} > \frac{y_{13}}{d} + \frac{y_{12}}{4-d} > \frac{y_{11}}{d} + \frac{y_{14}}{4-d} > \frac{y_{14}}{d} + \frac{y_{11}}{4-d} & 1.5 < d < 2 \\ \frac{y_{12}}{d} + \frac{y_{13}}{4-d} = \frac{y_{13}}{d} + \frac{y_{12}}{4-d} > \frac{y_{11}}{d} + \frac{y_{14}}{4-d} = \frac{y_{14}}{d} + \frac{y_{11}}{4-d} & d = 2 \\ y_{11} + y_{12} + y_{13} + y_{14} = 1 \\ y_{1i} \geq 0, \text{ for } i = 1, 2, 3, 4 \end{cases}$$

$$\begin{cases} \frac{y_{11}}{d} + \frac{y_{14}}{4-d} > \frac{y_{12}}{d} + \frac{y_{13}}{4-d} > \frac{y_{13}}{d} + \frac{y_{12}}{4-d} > \frac{y_{14}}{d} + \frac{y_{11}}{4-d} & 0 < d < 1 \\ 3y_{11} + y_{14} = 3y_{12} + y_{13} > 3y_{13} + y_{12} > 3y_{14} + y_{11} & d = 1 \\ \frac{y_{12}}{d} + \frac{y_{13}}{4-d} > \frac{y_{11}}{d} + \frac{y_{14}}{4-d} > \frac{y_{13}}{d} + \frac{y_{12}}{4-d} > \frac{y_{14}}{d} + \frac{y_{11}}{4-d} & 1 < d < 1.5 \\ 5y_{12} + 3y_{13} > 5y_{11} + 3y_{14} = 5y_{13} + 3y_{12} > 5y_{14} + 3y_{11} & d = 1.5 \\ \frac{y_{12}}{d} + \frac{y_{13}}{4-d} > \frac{y_{13}}{d} + \frac{y_{12}}{4-d} > \frac{y_{11}}{d} + \frac{y_{14}}{4-d} > \frac{y_{14}}{d} + \frac{y_{11}}{4-d} & 1.5 < d < 2 \\ y_{12} + y_{13} > y_{11} + y_{14} & d = 2 \\ y_{11} + y_{12} + y_{13} + y_{14} = 1 \\ y_{1i} \geq 0, \text{ for } i = 1, 2, 3, 4 \end{cases}$$

$$\begin{cases} y_{11} + y_{14} = 2y_{13} \\ 2y_{11} + 10y_{13} = 3 \\ y_{12} + 3y_{13} = 1 \\ \frac{1}{2} > y_{11} > \frac{5}{12} > y_{12} > \frac{1}{4} > y_{13} > \frac{1}{6} > y_{14} \geq 0 \end{cases} \tag{14}$$

□

A.7 Proof of Theorem 7 (Main Theorem)

Proof. The $n = 1$ and $n = 2$ cases are trivial. We have already shown that this holds for $n = 3$ and $n = 4$ in Theorem 1 and Lemma 6 . In order to show the statement holds for

$n \in [5, \infty)$ we proceed similarly. Assume that two prototypes are positioned distance n apart in a two-dimensional Euclidean space. Without loss of generality, suppose that point $x_1 = (0, 0)$ and point $x_2 = (n, 0)$ have probabilistic labels y_1 and y_2 respectively. We denote the i^{th} element of each label by y_{1i} and y_{2i}. One way to partition the space into n partitions, is to have a different class for every unit of space between the two points. Points lying on the x-axis within the interval $[i-1, i)$ belong to the i^{th} class. The general objective is to find y_1 and y_2 satisfying the following equation.

$$\arg\max_i \left(\frac{y_{1i}}{d} + \frac{y_{2i}}{n-d} \right) = \lceil d \rceil$$

We require that the label values for the two classes be symmetric with $y_{1i} = y_{2(n-i)}$ for $i = 1, 2, ..., n$. We also require that a class's 'influence' decrease monotonically as distance away from the center of its interval increases. Just as before, we can equivalently consider only the equations where $d \leq \frac{n}{2}$. The resulting system is described in Equation 15.

$$\begin{aligned} \forall i &= 1, 2, ..., \lceil \frac{n}{2} \rceil - 1 \\ \forall j &= 2i, 2i+1, ..., n-1 \\ \frac{y_{1i}}{\frac{j}{2}} + \frac{y_{1(n-i+1)}}{n - \frac{j}{2}} &= \frac{y_{1(j-i+1)}}{\frac{j}{2}} + \frac{y_{1(n-j+i)}}{n - \frac{j}{2}} \end{aligned} \tag{15}$$

Unfortunately, this system is overdetermined for $n > 6$. In fact, the number of equations in this system for a given n is described in Equation 16.

$$\sum_{i=1}^{\lceil \frac{n}{2} \rceil - 1} \sum_{j=2i}^{n-1} 1 = \sum_{i=1}^{\lceil \frac{n}{2} \rceil - 1} n - 2i = (\lceil \frac{n}{2} \rceil - 1)(n - \lceil \frac{n}{2} \rceil) \tag{16}$$

Lemma A 1 *The system described in Equation 15 has only $n-2$ linearly independent equations.*

Proof. First, we rewrite the system slightly to remove the denominators.

$$\begin{aligned} \forall i &= 1, 2, ..., \lceil \frac{n}{2} \rceil - 1 \\ \forall j &= 2i, 2i+1, ..., n-1 \\ (n - \frac{j}{2}) y_{1i} + (\frac{j}{2}) y_{1(n-i+1)} &= (n - \frac{j}{2}) y_{1(j-i+1)} + (\frac{j}{2}) y_{1(n-j+i)} \end{aligned}$$

It may be helpful to visualize this system as the equations arranged onto a grid corresponding to different values of i and j. Note that for $i = 1$, there are $n-2$ equations and they are clearly linearly independent. In addition, for $i > 1$ each equation can be rewritten as a linear combination of these first $n-2$ equations. We denote the equation associated with $i = a$, $j = b$ by $eqn_{a,b}$.

$$\begin{aligned} eqn_{i,j} = &- \frac{n+i-j-1}{n-1}(eqn_{1,i}) + \frac{n-i}{n-1}(eqn_{1,(j-i+1)}) \\ &+ \frac{i}{n-1}(eqn_{1,(j-i)}) - \frac{j-i}{n-1}(eqn_{1,(n-i)}) \end{aligned}$$

Thus, the system has exactly $n-2$ linearly independent equations. $\square$

Finally, we add our familiar restrictions: making the labels probabilistic and setting the last label value to zero. These additional equations increase the rank of the system to n.

$$\sum_{k=1}^{n} y_{1k} = 1,\ y_{1n} = 0 \tag{17}$$

The resulting system always has one solution, and this solution allows us to separate n classes using two points. $\square$

Remark *We can find the exact solution to the system in Equation 15 combined with the equations from Equation 17 for a given n.*

$$y_{1i} = y_{2(n-i)} = \frac{\sum_{j=i}^{n-1} j}{\sum_{j=1}^{n-1} j^2} = \frac{n(n-1) - i(i-1)}{2\sum_{j=1}^{n-1} j^2},\ i = 1, 2, ..., n$$

Remark *The two points do not have to be n units apart. The same result can be shown to hold for two points any distance part by dividing that distance into n intervals of equal length and scaling the associated constants accordingly.*

A.8 Proof of Theorem 8

Proof. We can rephrase our objective as the following. Find the minimum number of prototypes required on each circle, such that the nearest prototype to any point along the circumference of the i^{th} circle, is also on the i^{th} circle.

Let the number of prototypes on the i^{th} circle (with radius $i * c$ for some positive constant c) be denoted by n_i. Since the prototypes must be evenly spread out along the circumference of each circle, the furthest point on circle i from any prototype of circle i, must be the point located at the arc midpoint of two neighboring prototypes. In that case, a sufficient condition is that the shortest distance between two neighboring circles (c), be longer than the distance between neighboring prototypes and arc midpoints on the i^{th} circle. Since c is constant, we can only modify the distance between prototypes and arc midpoints. We can do so by changing the number of prototypes found on the i^{th} circle. The Euclidean distance between a prototype and its neighboring arc midpoint is $d_i = r_i\sqrt{2 - 2cos(\frac{l_i}{r_i})}$ where $r_i = i * c$ is the radius of the i^{th} circle, and $l_i = \frac{2\pi r}{2n}$ is the arclength between neighboring prototypes and arc midpoints. Thus $d_i = ic\sqrt{2 - 2cos(\frac{\pi}{n_i})}$ and we want $d_i < c\ \forall i = 1, 2, ...$ Solving for n_i we get the following inequality.

$$n_i > \frac{\pi}{cos^{-1}(1 - \frac{1}{2i^2})} \tag{18}$$

We can use the approximation $cos^{-1}(1-y) \approx \sqrt{2y}$ to get that $\frac{\pi}{cos^{-1}(1-\frac{1}{2i^2})} \approx \frac{\pi}{\frac{1}{i})} = i\pi$ $\square$