Introduction

- Across many fields it is often necessary to count objects
  - Number of times instructions in a loop are executed
  - Number of possible passwords given for a given scheme
  - Number of unique IP addresses
  - Historically used in games and gambling
  - ...
- Counting is essential for calculating probabilities
- We will discuss a number of common techniques used to count objects
  - Note, some of the topics, definitions, and theorems discussed will be slightly out of order from that of the book, however, references are always given.

Some material from the Teaching Company Lecture series on Discrete Mathematics

Introduction, cont.

- How do you count the number of people in a crowded room?

Every counting problem comes down to determining the size of a finite set.

Claim: You can often find the size of one set by finding the size of a related set.

Several general rules will be presented, often the challenge for counting problems is selecting the appropriate rules to apply.

Two basic useful rules for counting:

- *Sum Rule*: if you have m choices for some event and n choices for another event and the events are disjoint, then there are $n + m$ choices for **either** event to occur.
- *Product Rule*: if you have m choices for some event and n choices for another event, then there are mn choices for **both** events to occur.

Example 1 Consider a coffee shop with 8 types of coffee and 6 types of muffins, then there are

- $8 + 6 = 14$ choices from the shop, and
- $8 \cdot 6 = 48$ choices of a coffee and muffin.

Sum Rule

Sum Rule - Set Formulation: If A and B are disjoint, finite sets, then

$$|A \cup B| = |A| + |B|.$$

Sum Rule - Task Formulation: If an activity can be done in either one of n_1 ways or in one of n_2 ways, where none of the set of n_1 and n_2 ways are the same, then there are $n_1 + n_2$ ways to do the task.

Example 2 Suppose a either a physics professor or physics student is chosen to be on a committee. If there are 15 physics professors and 73 physics students and no one is both a professor and a student, how many different choices are their for the committee representative?

73 + 15 = 88

Slot

Generalized Sum Rule - Set Formulation: If $A_1, A_2, \ldots, A_n$ are disjoint finite sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

Generalized Sum Rule - Task Formulation: If a task can be broken up into t activities, where task i can be done in n_i ways and none of the n_i ways is the same as the set of n_j ways of completing an activity, for all pairs of i and j with $1 \leq i < j \leq t$, then the task can be done in $n_1 + n_2 + \cdots + n_t$ ways.

Example 3 Alex needs to decide what to wear. He can wear a black suit (of which he owns two), a navy suit (of which he owns three), or khakis and a button-down (of which he has five options). In how many different ways can he get dressed?

$\underbrace{2 + 3 + 5}_{\text{ways to dress}} = 10$ options

Product Rule - Set Formulation: If A and B are finite sets, then

$$|A \times B| = |A| \cdot |B|.$$

Product Rule - Task Formulation: If a procedure can be broken down into a sequence of two tasks and there are n_1 ways to complete task 1 and n_2 ways to complete task 2, then there are $n_1 n_2$ ways to complete the procedure.

Example 4 Seats at a hockey game are labeled by letter (A-Z) and number (1-25). How many different seat labels are possible?

$$\underbrace{26}_{\text{Letter}} \cdot \underbrace{25}_{\text{Number}} = 650 \text{ ways}$$

Generalized Product Rule - Set Formulation: If $A_1, A_2, \ldots, A_n$ are finite sets, then

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|.$$

Generalized Product Rule - Task Formulation: If a procedure can be broken down into a sequence of m tasks and there are n_1 ways to complete task 1 and n_2 ways to complete task 2, ..., and n_m ways to complete task m, then there are $n_1 \cdot n_2 \cdots n_m$ ways to complete the procedure.

Example 5 How many unique license plates are available if the plate sequence follows the pattern of 3 letters and 3 numbers? the pattern of 3 letters and 4 numbers?

$\underset{L}{\underline{26}} \cdot \underset{L}{\underline{26}} \cdot \underset{L}{\underline{26}} \cdot \underset{\#}{\underline{10}} \cdot \underset{\#}{\underline{10}} \cdot \underset{\#}{\underline{10}} = 26^3 \cdot 10^3 = 17.58$ million

$\underset{L}{\underline{26}} \cdot \underset{L}{\underline{26}} \cdot \underset{L}{\underline{26}} \cdot \underset{\#}{\underline{10}} \cdot \underset{\#}{\underline{10}} \cdot \underset{\#}{\underline{10}} \cdot \underset{\#}{\underline{10}} = 26^3 \cdot 10^4 \sim 175$ M

Practice Problems

Example 6 A new company is formed with two employees rents a office floor with 13 offices. How many ways are there to assign different offices to the two employees?

$$\underset{\text{Emp 1}}{\underline{13}} \cdot \underset{\text{Emp 2}}{\underline{12}} = 156$$

Example 7 A card in the standard deck has one of 13 values (ace to King) and one of four suits (♡, ♢, ♣, ♠). How many cards are there?

$$\underset{\text{Suits}}{\underline{4}} \cdot \underset{\text{Rank}}{\underline{13}} = 52$$

Assume upper case letters

Example 8 Consider strings of 3 letters from the English alphabet.

a) How many different strings are there? $\underline{26}\cdot\underline{26}\cdot\underline{26} = 26^3$

b) How many different strings with no repetition of letters?

$\underline{26}\cdot\underline{25}\cdot\underline{24}$

c) How many different strings ~~with~~ only one A?

$A\cdot 25\cdot 25 + 25\,A\,25 + 25\cdot 25A = 3\cdot 25^2$

d) How many different strings contain only 1 character from the set $\{A, B, C\}$?

$\underline{3}\cdot\underline{23}\cdot\underline{23} + \underline{23}\cdot\underline{3}\cdot\underline{23} + \underline{23}\cdot\underline{23}\cdot\underline{3} = 3^2\cdot 23^2$

Example 9 How many 8-bit strings

a) are there? 2^8 $\underline{2}\cdot\underline{2}\cdot\underline{2}\cdot\underline{2}\cdot\underline{2}\cdot\underline{2}\cdot\underline{2}\cdot\underline{2} = 2^8$

b) begin and end with a 0? $\underline{0}\ \underline{2}\cdot\underline{2}\cdot\underline{2}\cdot\underline{2}\cdot\underline{2}\cdot\underline{2}\ 0\quad 2^6$

c) begin with 1100? $1\,1\,0\,0\ \underline{2}\cdot\underline{2}\cdot\underline{2}\cdot\underline{2} = 2^4$

Inclusion-Exclusion Principle

Inclusion-Exclusion Principle - Set Formulation: Let A and B be finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Inclusion-Exclusion Principle - Task Formulation: Suppose an activity can be completed in n_1 or n_2 ways, but that some of the set n_1 ways are the same as the set n_2 ways to do the activity. The sum rule can not be used, because some ways will be counted twice.

Instead, add each of the number of ways to do the activity and subtract duplicates.

Example 10 How many 8-bit strings start with a 1 or end with two 00s?

$1 \; 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7$

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \; 0 \; 0 = 2^6$

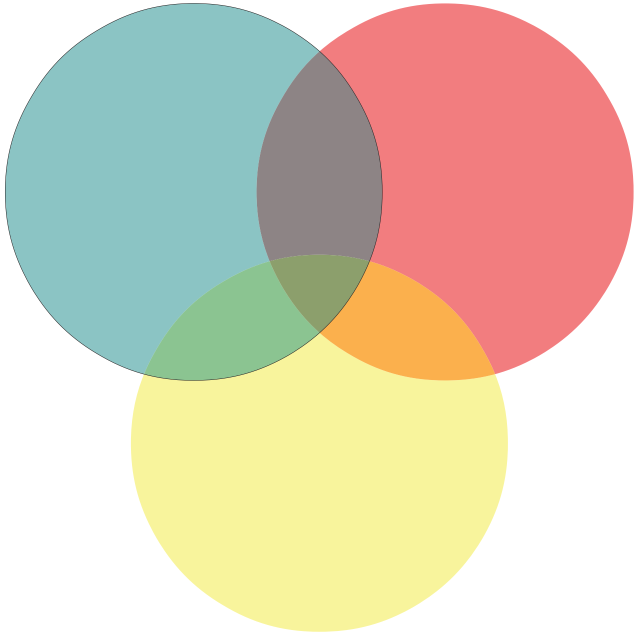
$1 \; 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \; 0 \; 0 = 2^5$

$2^7 + 2^6 - 2^5$

Generalized Inclusion-Exclusion Principle

The size of the union of three sets, A, B, and C is:

$$\begin{aligned}|A \cup B \cup C| = |A| + |B| + |C| \\ - |A \cap B| - |A \cap C| - |B \cap C| \\ + |A \cap B \cap C|\end{aligned}$$



Example 11 Among the numbers 1-100, how many are divisible by 2?

$$\left\lfloor \frac{100}{2} \right\rfloor = 50$$

Example 12 Among the numbers 1-100, how many are divisible by 3?

$$\left\lfloor \frac{100}{3} \right\rfloor = 33$$

Example 13 Among the numbers 1-100, how many are divisible by 5?

$$\left\lfloor \frac{100}{5} \right\rfloor = 20$$

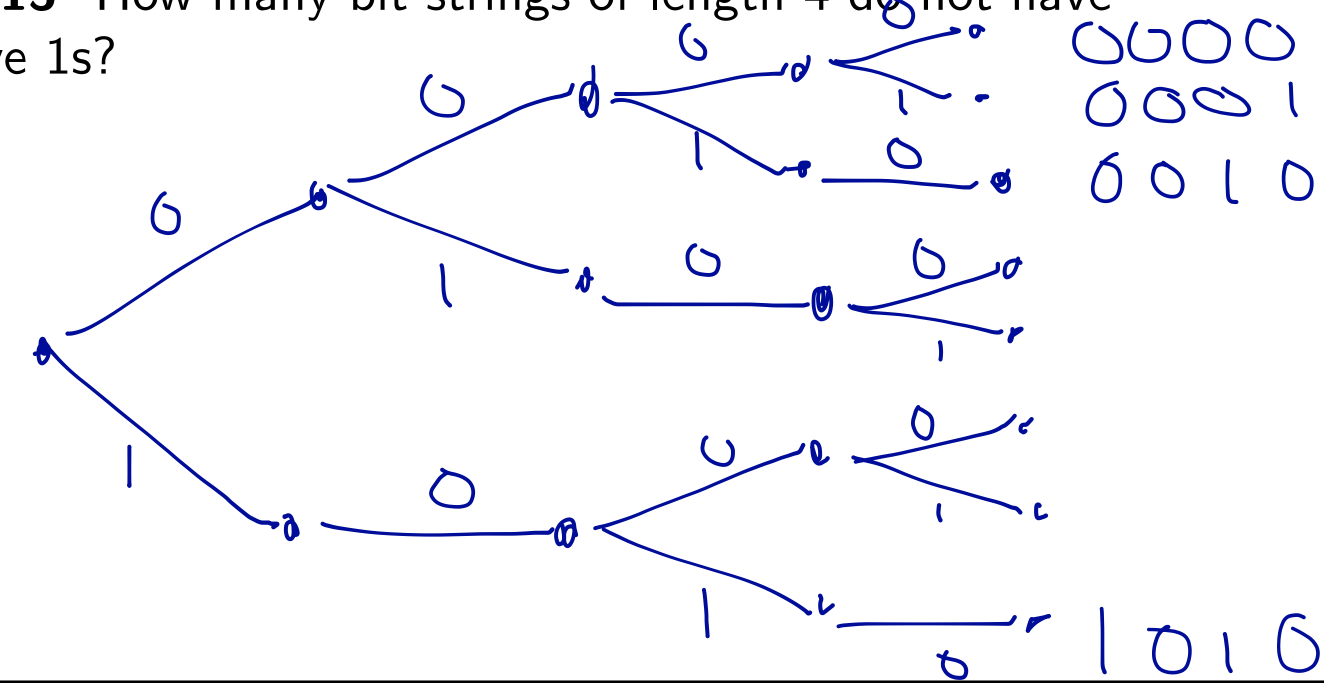
Example 14 Among the numbers 1-100, how many are divisible by 2, 3, or 5?

$$d(2 \vee 3 \vee 5) = d2 + d3 + d5$$
$$- d(2 \wedge 3) - d(2 \wedge 5) - d(3 \wedge 5)$$
$$+ d(2 \wedge 3 \wedge 5)$$
$$= 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74$$

Counting with Tree Diagrams

- A tree starts with a root node, a number of branches leaving the root, and additional branches leaving from the endpoints of other branches.
- Each branch represents a choice.
- Possible outcomes are read from the terminal leaves.

Example 15 How many bit strings of length 4 do not have consecutive 1s?



Example 16 A manufacturer offers desktops in 5 configurations and laptops in 3 configurations.

a) How many ways are there to choose a computer? sum 8

b) How many ways are there to choose a desktop and a laptop?

$3 \cdot 5 = 15$

Example 17 How many strings of length 5 consisting of the letters of the English alphabet are there? upper case

a) How many of those strings begin with the letter B?

$B\,\underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{26} = 26^4$

b) What is no repetitions are allowed?

$\underline{26} \cdot \underline{25} \cdot \underline{24} \cdot \underline{23} \cdot \underline{22}$

Example Problems

Example 18 How many different 5-digit zip codes are there?

$\underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10}$ $\quad 10^5$

Example 19 How many different zip codes contain only even digits?

$\underline{5} \cdot \underline{5} \cdot \underline{5} \cdot \underline{5} \cdot \underline{5} = 5^5$

Example 20 How many zip codes contain at least one odd digit?
Solve using Strategy of Complements

think about answering the opposite question

"How many zip-codes have all even digits?"

\# all zip-codes − # zip-codes w/ all even =
\# w/ at least one odd

$10^5 - 5^5 = 96{,}875$

Consider the following famous puzzle:

> A drawer in a dark room contains 20 red socks, 20 green socks, and 20 blue socks. How many socks must you withdraw to be sure that you have a matching pair?

Pick 1? Not a pair.

Pick 2? Not guaranteed to get a pair.

Pick 3? Still no guarantee; may get:

$$SOCK, SOCK, SOCK$$

Consider the following famous puzzle:

> A drawer in a dark room contains 20 red socks, 20 green socks, and 20 blue socks. How many socks must you withdraw to be sure that you have a matching pair?

Pick 4? YES, guaranteed to get a pair.

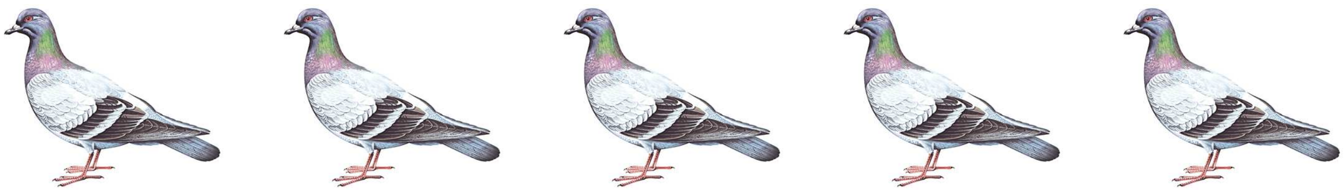
You don't know what the color with the pair is, but a pair will exist. For example, each of the following are possible:

$$\begin{gathered} {\color{green}SOCK}, {\color{red}SOCK}, {\color{blue}SOCK}, {\color{green}SOCK} \\ {\color{green}SOCK}, {\color{red}SOCK}, {\color{blue}SOCK}, {\color{red}SOCK} \\ {\color{green}SOCK}, {\color{red}SOCK}, {\color{blue}SOCK}, {\color{blue}SOCK} \\ {\color{green}SOCK}, {\color{green}SOCK}, {\color{blue}SOCK}, {\color{green}SOCK} \\ \vdots \end{gathered}$$

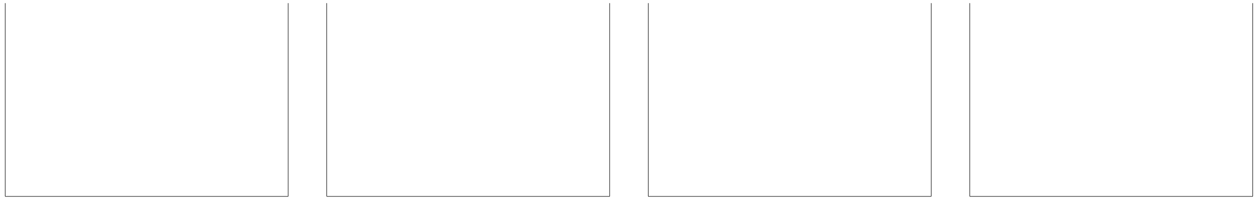
Pigeonhole Principle

The Pigeonhole Principle: If $k + 1$ or more objects are placed in k boxes then there is at least one box containing two or more objects.

Example: If there are more pigeons,



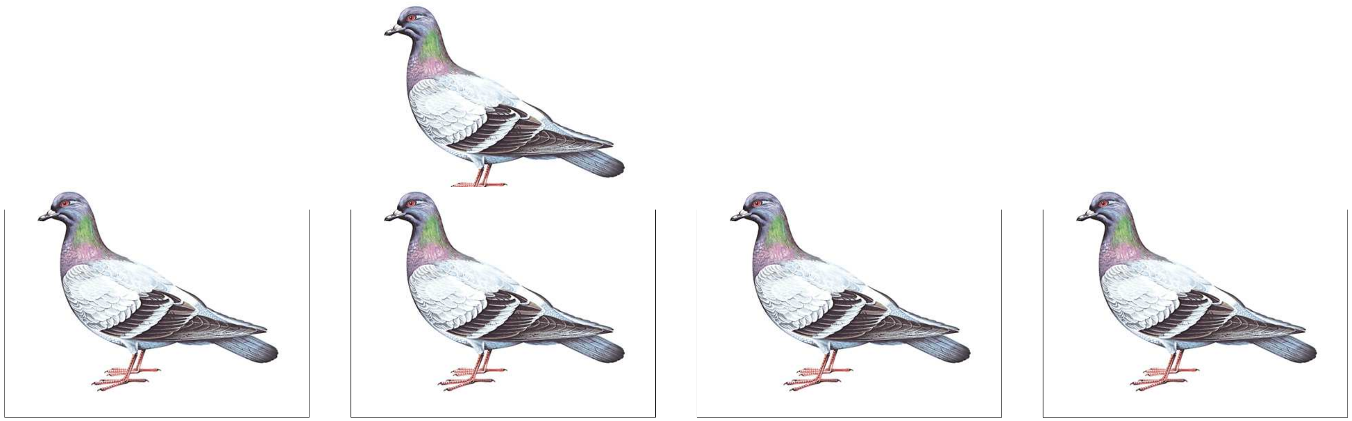
than pigeonholes



Pigeonhole Principle

Example: If there are more pigeons, than pigeonholes . . .

Then, some hole must contain more than one pigeon!



The Pigeonhole Principle: If $k+1$ or more objects are placed in k boxes then there is at least one box containing two or more objects.

Example 21 Among any group of 367 people, there must be at least two with the same birthday.

Example 22 In any group of 27 English words, there are at least 2 that begin with the same letter.

Generalized Pigeonhole Principle: If N object are placed into k boxes, then there is at least one box with at least $\left\lceil \dfrac{N}{k} \right\rceil$ objects.

Example 23 If you are given 21 numbers from the set $\{0 - 9\}$ there must be 3 numbers of the same digit.

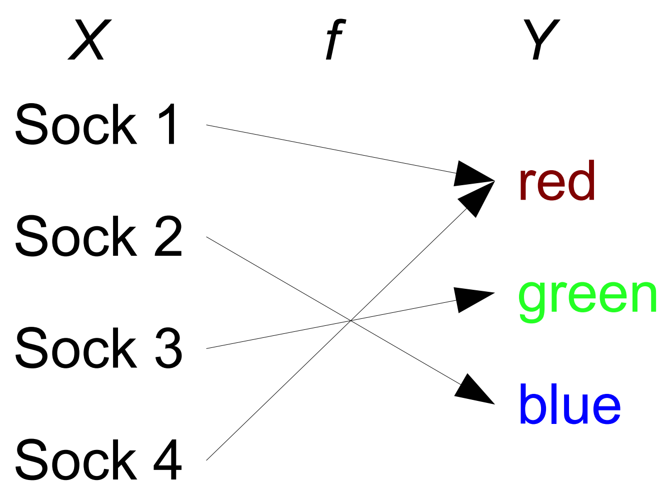
$$\left\lceil \frac{21}{10} \right\rceil = 3$$

Example 24 Among 100 people there are at least $\lceil \frac{100}{12} \rceil = 9$ who were born in the same month.

Let's rethink of the pigeonhole principles in terms of functions:

Pigeonhole Principle: If $|X| > |Y|$, then for every total function $f\colon X \to Y$, there exists at least two different elements of X that are mapped to the same element of Y.

Generalized Pigeonhole Principle: If $|X| > k \cdot |Y|$, then every total function $f\colon X \to Y$ maps at least $k+1$ different elements of X to the same element of Y.



Pigeonhole Example Problems

Example 25 What is the minimum number of people required in a math class to make sure 6 people receive the same grade if there are 5 possible grades: A, B, C, D, F.

$$\left\lceil \frac{N}{5} \right\rceil = 6 \qquad N = 26$$

Example 26 How many cards must be selected from a deck of 52 cards to guarantee that at least 3 cards of the same suit are chosen?

$$\left\lceil \frac{N}{4} \right\rceil = 3 \qquad N = 9$$

Example 27 How many cards need to be selected to guarantee 3 clubs are selected?

Worst-case analysis: select all ♡ ◇ ♠

$3 \cdot 13 = 39$

$+ 3$ clubs $= 42$