Terminology

A proposition is a statement that is either true or false but not both.

A theorem is a statement that can be shown to be true (an important proposition).

A proof is the sequence of statements that form a valid argument that a theorem is true.

An axiom is a statement accepted as true to be used in a proof.

A lemma is a less important theorem used in the proof of another theorem.

A corollary is a theorem that can be derived directly from another proven theorem.

A conjecture is a statement that is proposed to be true, but a proof has not been established.

A direct proof shows that a conditional statement $p \rightarrow q$ is true.

To show the conditional is true, you want to show that if p is true then q can not be false (show that if p is true, that q must be true).

General method - Assume p is true, use knowledge of domain (definitions), rules of inference, and equivalences to show q is also true.

In the examples, use the following definitions

Definition 1. The integer n is even if there is exists an integer k such that $n = 2k$, and n is odd if there exists an integer k such that $n = 2k + 1$.

$\forall x\, (P(x) \to Q(x))$

$p \to q$

Example 1. Prove: If n is even, then n^2 is even.

Proof:

Assumption Definitions

Assume n is even. By definition of even, there exists an integer k such that $n = 2k$.

Apply

Compute n^2:

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2) = 2k'.$$

The value n^2 is of the form of an even integer.

Conclude

Consequently, if n is even, then n^2 is even.

Example of Direct Proof

Example 1. Prove: If n is even, then n^2 is even.

Proof: Assume n is even. Then, by definition of even, there exists an integer k such that $n = 2k$. Compute n^2 (to show it is also even):

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2) = 2k'$$

from this you can see n^2 is in the form that fits the definition of an even integer.

Consequently, if n is even, n^2 is even.

$p \rightarrow q$

Example 2. Prove: If n is odd, then n^2 is odd.

Proof:

Suppose n is odd. By definition of odd, then there exists an integer k such that $n = 2k+1$. Compute n^2:

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$
$$= 2k' + 1,$$

from above we can see n^2 is of the form of an odd integer.

Therefore, if n is odd, then n^2 is odd.

Example 2. Prove: If n is odd, then n^2 is odd.

Proof: Assume n is odd. By definition of odd, there exists an integer k such that $n = 2k + 1$. Compute n^2:

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2k' + 1$$

from this you can see n^2 is in the form that fits the definition of an odd integer.

Consequently, if n is odd, then n^2 is odd.

Example 3. Prove: The sum of any two odd integers is even.

If we have two odd integers, a and b, then $a+b$ is even.

Proof:

Let a and b be odd integers. By definition, there exists an integer K_a and an integer K_b such that $a = 2K_a + 1$ and $b = 2K_b + 1$. Compute $a+b$:

$$\begin{aligned} a+b &= 2K_a + 1 + 2K_b + 1 \\ &= 2K_a + 2K_b + 2 \\ &= 2(K_a + K_b + 1) = 2K' \end{aligned}$$

We can see $a+b$ is of the form of an even integer. Therefore, we have shown the sum of two odd integers is even.

Example 3. Prove: The sum of any two odd integers is even.

Let a and b be two odd integers. Then by definition of odd, there exists a k_a and a k_b such that $a = 2k_a + 1$ and $b = 2k_b + 1$. Examine $a + b$

$$\begin{aligned} a + b &= 2k_a + 1 + 2k_b + 1 \\ &= 2k_a + 2k_b + 2 \\ &= 2(k_a + k_b + 1) \\ &= 2(k_c) \end{aligned}$$

Here $a + b$ is of the form, $2k_c$, for even numbers.
Consequently, we have shown the sum of any two odd integers is even.

Proof by Contraposition

A direct proof method is often preferred however, other proof techniques are available.

Proof by contraposition uses the logical equivalence

$$p \to q \equiv \neg q \to \neg p.$$

In this method assume $\neg q$ is true and show $\neg q \to \neg p$.

Example Proofs by Contraposition

p

Example 4. Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

q

Example 4. Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Proof (direct): Assume that $3n + 2$ is odd. This means that $3n + 2 = 2k + 1$ for some integer k. How can this be used to show that n is odd?

$3n+2 = 2K+1$

$3n = 2K-1$

$n = \frac{2K-1}{3}$

$n = 2(K') + 1$

If $3n+2$ is odd, then n is odd.

Proof by contraposition

Assume n is even. By definition, there exists an integer k such that $n = 2k$. Compute $3n+2$

$$3(2k)+2 = 6k+2 = 2(3k+1)$$
$$= 2k'$$

Hence, $3n+2$ is of the form of an even integer.

Therefore, by contraposition, if $3n+2$ is odd, n is odd.

Example 4. Prove that if n is an integer and $3n+2$ is odd, then n is odd.

Proof (direct): Assume that $3n+2$ is odd. This means that $3n+2 = 2k+1$ for some integer k (definition of odd). How can this be used to show that n is odd?

Proof (indirect): First assume that the conclusion of the conditional is false. Namely, assume n is even. Then $n = 2k$ for some integer k (def. of even). Substitute $2k$ for n.

$$3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1).$$

Then, $3n+2$ is even, and therefore it is not odd. This is the negation of the hypothesis of the theorem.
Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true.
Therefore, you proved "If $3n+2$ is odd, n is odd."

Example Proofs by Contraposition

Example 5. Prove for all natural numbers a and b, and non-negative real numbers n, if $n = ab$ then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

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Proof (direct): Assume a and b are natural numbers, n is a non-negative real number, and $n = ab$. *Now what?*
At this point in proofs, usually try to "compute" something, but there is not much available here.

Example 5. Prove for all natural numbers a and b, and non-negative real numbers n, if $n = ab$ then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Proof (direct): Assume a and b are natural numbers, n is a non-negative real number, and $n = ab$. *Now what?*
At this point in proofs, usually try to "compute" something, but there is not much available here.

Proof (contrapositive): Let a and b be natural numbers and let n be a non-negative real number. We must show that,

$$\text{if } a > \sqrt{n} \text{ and } b > \sqrt{n}, \text{ then } n \neq ab.$$

Suppose $a > \sqrt{n}$ and $b > \sqrt{n}$. Note that for any non-negative reals or integers, if $x > z$ and $y > z$, then $xy > z^2$. It follows that $ab > n$, so we may conclude that $ab \neq n$. Therefore, with proof by contraposition we have proved if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Vacuous and Trivial Proofs

Suppose you are trying to prove $p \to q$.

- If p is false then the theorem is vacuously true by a vacuous proof.
- If q is true then the theorem is trivially true by a trivial proof.

Examples:

- If a is in $\overset{F}{\text{the}}$ set $\{\}$, then a is in every set whose size is greater than the size of $\{\}$. - Vacuous
- If there are more than 1 million prime numbers, then the size of the $\underbrace{\text{set of prime numbers}}_{T}$ is non-zero. - Trivial

Examples of Vacuous and Trivial Proofs

Show that the proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$, then $n^2 > n$" and the domain of n is the integers.

Show that the proposition $P(0)$ is true, where $P(n)$ is "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$." where the domain of n is all integers.

Examples of Vacuous and Trivial Proofs

Show that the proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$, then $n^2 > n$" and the domain of n is the integers.

Solutions:
$P(0)$ corresponds to "If $0 > 1$, then $0^2 > 0$. The hypothesis is false $(0 > 1)$ therefore, $P(0)$ is true. – **Vacuous Proof**

Show that the proposition $P(0)$ is true, where $P(n)$ is "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$." where the domain of n is all integers.

Solutions:
$P(0)$ is "If $a \geq b$, then $a^0 \geq b^0$." Both a^0 and $b^0 = 1$, and the conclusion of the conditional statement is true. Hence the conditional statement $P(0)$ is true. – **Trivial Proof**

In a proof by contradiction, we want to prove a proposition $p \rightarrow q$. Start by assuming that p and $\neg q$ are true and show that $(p \wedge \neg q) \rightarrow q$ or $(p \wedge \neg q) \rightarrow \neg p$, thereby contradicting the assumptions.

The contradiction comes from if you assume $p \wedge \neg q$, you can show that either p is false or q is true. The propositions p and $\neg q$ cannot be true at the same time. As a result, if p is true, then q must be true, too.

This type of proof is also referred to as *Reductio Ad Absurdum*.

Example 6. Prove: If $3n + 2$ is odd, then n is odd.

Proof by contradiction:

Suppose $3n+2$ is odd and n is even.

By definition, there exists an integer k such that $n = 2k$ (def. of even). Compute $3n+2$

$$3(2k)+2 = 6k+2 = 2(3k+1) = 2k'$$

This shows $3n+2$ is of the form of an even integer. This leads to a contradiction.

Therefore, by contradiction, if $3n+2$ is odd, then n is also odd.

Example 6. Prove: If $3n + 2$ is odd, then n is odd.

Proof by contradiction: Suppose $3n + 2$ is odd, and n is even. Then there is an integer k such that $n = 2k$ (def. of even). Compute

$$3n + 2 = 3(2k) + 2 = 2(3k + 1).$$

This shows $3n + 2$ is even. That **contradicts** the fact that $3n + 2$ is assumed to be odd. Therefore, proof by contradiction shows if $3n + 2$ is odd, then n is also odd.

Example Proofs by Contradiction

Example 7. Prove: If $x^2 + y = 13$ and $y \neq 4$, then $x \neq 3$.

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Proof by contradiction: Suppose $x^2 + y = 13$ and $y \neq 4$ and $x = 3$. Compute

$$\begin{aligned} x^2 + y &= 13 \\ 3^2 + y &= 13 \\ 9 + y &= 13 \\ y &= 4 \end{aligned}$$

This contradicts the fact that $y \neq 4$. Therefore, we have proved if $x^2 + y = 13$ and $y \neq 4$, x can not equal 3.

Proof of Equivalence

To prove theorems of the form $p \leftrightarrow q$ show

$$(p \rightarrow q) \land (q \rightarrow p)$$

Example Proof of Equivalence

Example 8. Rosen Example 12, p. 87
Prove that "If n is a positive integer, then n is odd if and only if n^2 is odd."

Solution

Example 8. Rosen Example 12, p. 87
Prove that "If n is a positive integer, then n is odd if and only if n^2 is odd."

Solution
First, parse the sentence to understand what to show. The first if can be removed,

"If n is a positive integer, ..."

is equivalent to

"For all positive integers n, ..."

So now the goal is to prove

"For all positive integers n, n is odd if and only if n^2 is odd."

The "if and only if" phrase means the biconditional $\leftrightarrow$.

Example 8. Rosen Example 12, p. 87
Prove that "If n is a positive integer, then n is odd if and only if n^2 is odd."

Solution, cont.
Let $P(x)$ be "x is odd" and $Q(x)$ is "x^2 is odd." We are trying to show

$$\forall n\,(P(n) \leftrightarrow Q(n))$$

This can be accomplished by showing both directions of the arrow, that is

$$\forall n\,(P(n) \rightarrow Q(n)) \land \forall n\,(Q(n) \rightarrow P(n))$$

Prove separately:
(a) If n is odd, then n^2 is odd.
(b) If n^2 is odd, then n is odd.
This type of proof we have done.

Counterexample

To prove a theorem of the form $\forall x\, P(x)$ is false, by showing $\exists x\, \neg P(x)$. This is called a proof by counterexample.

Example 9. Disprove a statement with a counterexample.

Show that "Every positive integer is the sum of squares of two integers" is false.

Solution:

Example using Counterexample

Example 9. Disprove a statement with a counterexample.

Show that "Every positive integer is the sum of squares of two integers" is false.

Solution: Find a particular integer that is not the sum of squares of two integers.

For example, 3 can not be written as the sum of squares of two integers. The only squares not exceeding 3 are $0^2 = 0$ and $1^2 = 1$. No combination of two of these sums equals 3. Consequently the statement is false.

To prove a conditional statement of the form

$$(p_1 \vee p_2 \vee \ldots \vee p_n) \to q$$

it is easier to prove

$$(p_1 \to q) \wedge (p_2 \to q) \wedge \ldots \wedge (p_n \to q).$$

Example 10. Prove that if n is an integer, then $n^2 \geq n$.

Solution:

Example of Proof by Cases

Example 10. Prove that if n is an integer, then $n^2 \geq n$.

Solution: Prove $n^2 \geq n$ for every integer considering three cases: $n = 0$, $n \geq 1$, and $n \leq -1$.

Example of Proof by Cases

Example 10. Prove that if n is an integer, then $n^2 \geq n$.

Solution: Prove $n^2 \geq n$ for every integer considering three cases: $n = 0$, $n \geq 1$, and $n \leq -1$.

Case (i): When $n = 0$, because $0^2 = 0$, it is true that $0 \geq 0$ therefore, $n^2 \geq n$ is true in this case.

Case (ii): When $n \geq 1$, multiply both sides of the inequality $n \geq 1$ by positive integer n to get $n \cdot n \geq n \cdot 1$. This implies that $n^2 \geq n$ for $n \geq 1$.

Case (iii): In this case $n \leq -1$ but $n^2 \geq 0$ therefore, $n^2 \geq n$.

The inequality $n^2 \geq n$ holds in all three cases, we can conclude that if n is an integer, then $n^2 \geq n$.

Exhaustive Proof

Some theorems can be proved by examining a small number of examples, these are examples of exhaustive proofs. An exhaustive proof is a special type of proof by cases where each case checks a single example.

Example of Exhaustive Proof

Example 11. Prove that $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.

Solution:

Example of Exhaustive Proof

Example 11. Prove that $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.

Solution: Use a proof by exhaustion. Verify the expression for $n =$ 1, 2, 3, and 4.

For $n = 1$, we have $(n+1)^3 = 2^3 = 8$ and $3^n = 3^1 = 3$.

For $n = 2$, we have $(n+1)^3 = 3^3 = 27$ and $3^n = 3^2 = 9$.

For $n = 3$, we have $(n+1)^3 = 4^3 = 64$ and $3^n = 3^3 = 27$.

For $n = 4$, we have $(n+1)^3 = 5^3 = 125$ and $3^n = 3^4 = 81$.

In each of the four cases, we see $(n+1)^3 \geq 3^n$.

Existence Proof

A theorem of the form $\exists x\ P(x)$ can be proven by showing one instance of x where $P(x)$ is true. This is an existence proof.

Example of Existence Proof

Example 12. Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

Solution

Example of Existence Proof

Example 12. Show that there is a positive integer that can be written as the sum of two cubes of positive integers in two different ways.

Solution

Do a search over integers to determine,

$$1729 = 10^3 + 9^3 = 12^3 + 1^3$$

Showing a single case of a positive integer that can be written as the sum of cubes in two different ways completes the proof.