# Linear equations and the integrating factor

*Stuff about Linear equations and the integrating factor*

One of the most important types of equations we will learn how to solve are the so-called *linear equations*. In fact, the majority of the course is about linear equations. In this section we focus on the *first order linear equation*.

**Definition 1.** *A first order equation is* linear *if we can put it into the form:*

$$y' + p(x)y = f(x). \tag{1}$$

The word linear means linear in $y$ and $y'$; no higher powers nor functions of $y$ or $y'$ appear. The dependence on $x$ can be more complicated.

Solutions of linear equations have nice properties. For example, the solution exists wherever $p(x)$ and $f(x)$ are defined, and has the same regularity (read: it is just as nice). We'll see this in detail in § **??**. But most importantly for us right now, there is a method for solving linear first order equations. In § **??**, we saw that we could easily solve equations of the form

$$\frac{dy}{dx} = f(x)$$

because we could directly integrate both sides of the equation, since the left hand side was the derivative of something (in this case, $y$) and the right side was only a function of $x$. We want to do the same here, but the something on the left will not be the derivative of just $y$.

The trick is to rewrite the left-hand side of (1) as a derivative of a product of $y$ with another function. Let $r(x)$ be this other function, and we can compute, by the product rule, that

$$\frac{d}{dx}\Big[r(x)y\Big] = r(x)y' + r'(x)y.$$

Now, if we multiply (1) by the function $r(x)$ on both sides, we get

$$r(x)y' + p(x)r(x)y = f(x)r(x)$$

and the first term on the left here matches the first term from our product rule derivative. To make the second terms match up as well, we need that

$$r'(x) = p(x)r(x).$$

This equation is separable! We can solve for the $r(x)$ here by separating variables to get that

$$\frac{dr}{r} = p(x)\,dx$$

so that

$$\ln|r| = \int p(x)\,dx$$

or

$$r(x) = e^{\int p(x)\,dx}.$$

With this choice of $r(x)$, we get that

$$r(x)y' + r(x)p(x)y = \frac{d}{dx}\Big[r(x)y\Big],$$

so that if we multiply (1) by $r(x)$, we obtain $r(x)y' + r(x)p(x)y$ on the left-hand side, which we can simplify using our product rule derivative above to obtain

$$\frac{d}{dx}\Big[r(x)y\Big] = r(x)f(x).$$

Learning outcomes: Identify a linear first-order differential equation and write a first-order linear equation in standard form Solve initial value problems for first-order linear differential equations by integrating factors Write solutions to first-order linear initial value problems in integral form if needed.

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Now we integrate both sides. The right-hand side does not depend on $y$ and the left-hand side is written as a derivative of a function. Afterwards, we solve for $y$. The function $r(x)$ is called the *integrating factor* and the method is called the *integrating factor method.*

This method works for any first order linear equation, no matter what $p(x)$ and $f(x)$ are. In general, we can compute:

$$\begin{aligned} y' + p(x)y &= f(x), \\ e^{\int p(x)\,dx} y' + e^{\int p(x)\,dx} p(x) y &= e^{\int p(x)\,dx} f(x), \\ \frac{d}{dx}\left[ e^{\int p(x)\,dx} y \right] &= e^{\int p(x)\,dx} f(x), \\ e^{\int p(x)\,dx} y &= \int e^{\int p(x)\,dx} f(x)\,dx + C, \\ y &= e^{-\int p(x)\,dx} \left( \int e^{\int p(x)\,dx} f(x)\,dx + C \right). \end{aligned}$$

Advice: Do not try to remember the formula itself, that is way too hard. It is easier to remember the process and repeat it.

Of course, to get a closed form formula for $y$, we need to be able to find a closed form formula for the integrals appearing above.

**Example 1.** *Solve*

$$y' + 2xy = e^{x-x^2}, \qquad y(0) = -1.$$

**Solution:** First note that $p(x) = 2x$ and $f(x) = e^{x-x^2}$. The integrating factor is $r(x) = e^{\int p(x)\,dx} = e^{x^2}$. We multiply both sides of the equation by $r(x)$ to get

$$\begin{aligned} e^{x^2} y' + 2xe^{x^2} y &= e^{x-x^2} e^{x^2}, \\ \frac{d}{dx}\left[ e^{x^2} y \right] &= e^{x}. \end{aligned}$$

We integrate

$$\begin{aligned} e^{x^2} y &= e^{x} + C, \\ y &= e^{x-x^2} + Ce^{-x^2}. \end{aligned}$$

Next, we solve for the initial condition $-1 = y(0) = 1 + C$, so $C = -2$. The solution is

$$y = e^{x-x^2} - 2e^{-x^2}.$$

Note that we do not care which antiderivative we take when computing $e^{\int p(x)dx}$. You can always add a constant of integration, but those constants will not matter in the end.

**Exercise 1** *Try it! Add a constant of integration to the integral in the integrating factor and show that the solution you get in the end is the same as what we got above.*

Since we cannot always evaluate the integrals in closed form, it is useful to know how to write the solution in definite integral form. A definite integral is something that you can plug into a computer or a calculator. Suppose we are given

$$y' + p(x)y = f(x), \qquad y(x_0) = y_0.$$

Look at the solution and write the integrals as definite integrals.

$$y(x) = e^{-\int_{x_0}^{x} p(s)ds} \left( \int_{x_0}^{x} e^{\int_{x_0}^{t} p(s)\,ds} f(t)dt + y_0 \right). \tag{2}$$

You should be careful to properly use dummy variables here. If you now plug such a formula into a computer or a calculator, it will be happy to give you numerical answers.

**Exercise 2** *Check that $y(x_0) = y_0$ in formula (2).*

**Example 2.** *Solve the initial value problem*

$$ty' + 4y = t^2 - 1 \qquad y(1) = 3.$$

**Solution:** In order to solve this equation, we want to put the equation in standard form, which is

$$y' + \frac{4}{t}y = t - \frac{1}{t}.$$

In this form, the coefficient $p(t)$ of $y$ is $p(t) = \frac{4}{t}$, so that the integrating factor is

$$r(t) = e^{\int p(t)\ dt} = e^{\int \frac{4}{t}\ dt} = e^{4\ln(t)}.$$

Since $4\ln(t) = \ln(t^4)$, we have that $r(t) = t^4$. Multiplying both sides of the equation by $t^4$ gives

$$t^4y' + 4t^3y = t^5 - t^3$$

where the left hand side is $\frac{d}{dt}(t^4y)$. Therefore, we can integrate both sides of the equation in $t$ to give

$$t^4y = \frac{t^6}{6} - \frac{t^4}{4} + C$$

and we can solve out for $y$ as

$$y(t) = \frac{t^2}{6} - \frac{1}{4} + \frac{C}{t^4}.$$

To solve for $C$ using the initial condition, we plug in $t = 1$ to get that we need

$$3 = \frac{1}{6} - \frac{1}{4} + C \qquad C = \frac{37}{12}.$$

Therefore, the solution to the initial value problem is

$$y(t) = \frac{t^2}{6} - \frac{1}{4} + \frac{37/12}{t^4}.$$

**Example 3.** *Solve the initial value problem*

$$y' + 2xy = 3 \qquad y(0) = 4.$$

**Solution:** This equation is already in standard form. Since the coefficient of $y$ is $p(x) = 2x$, we know that the integrating factor is

$$r(x) = e^{\int p(x)\ dx} = e^{x^2}.$$

We can multiply both sides of the equation by this integrating factor to give

$$y'e^{x^2} + 2xe^{x^2}y = 3e^{x^2}$$

and then want to integrate both sides. The left-hand side of the equation is $\frac{d}{dx}[e^{x^2}y]$, so the antiderivative of that side is just $ye^{x^2}$. For the right-hand side, we would need to compute

$$\int 3e^{x^2}\ dx,$$

which does not have a closed-form expression. Therefore, we need to represent this as a definite integral. Since our initial condition gives the value of $y$ at zero, we want to use zero as the bottom limit of the integral. Therefore, we can write the solution as

$$ye^{x^2} = \int_0^x 3e^{s^2}\ ds + C$$

and so can solve for $y$ as

$$y(x) = e^{-x^2} \int_0^x 3e^{s^2}\, ds + Ce^{-x^2}.$$

Plugging in the initial condition gives that

$$y(0) = 4 = e^{-0} \int_0^0 3e^{s^2}\, ds + Ce^{-0} = C.$$

Therefore, the solution to the initial value problem is

$$y(x) = e^{-x^2} \int_0^x 3e^{s^2}\, ds + 4e^{-x^2}.$$

**Exercise 3** *Write the solution of the following problem as a definite integral, but try to simplify as far as you can. You will not be able to find the solution in closed form.*

$$y' + y = e^{x^2 - x}, \qquad y(0) = 10.$$