

Elimination

We discuss Elimination

Linear systems of equations

One application of matrices is to solve systems of linear equations¹. Consider the following system of linear equations

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 2, \\ x_1 + x_2 + 3x_3 &= 5, \\ x_1 + 4x_2 + x_3 &= 10. \end{aligned} \tag{1}$$

There is a systematic procedure called *elimination* to solve such a system. In this procedure, we attempt to eliminate each variable from all but one equation. We want to end up with equations such as $x_3 = 2$, where we can just read off the answer.

We write a system of linear equations as a matrix equation:

$$A\vec{x} = \vec{b}.$$

The system (1) is written as

$$\underbrace{\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 4 & 1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}}_{\vec{b}}.$$

If we knew the inverse of A , then we would be done; we would simply solve the equation:

$$\vec{x} = A^{-1}A\vec{x} = A^{-1}\vec{b}.$$

Well, but that is part of the problem, we do not know how to compute the inverse for matrices bigger than 2×2 . We will see later that to compute the inverse we are really solving $A\vec{x} = \vec{b}$ for several different \vec{b} . In other words, we will need to do elimination to find A^{-1} . In addition, we may wish to solve $A\vec{x} = \vec{b}$ even if A is not invertible, or perhaps not even square.

Let us return to the equations themselves and see how we can manipulate them. There are a few operations we can perform on the equations that do not change the solution. First, perhaps an operation that may seem stupid, we can swap two equations in (1):

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 5, \\ 2x_1 + 2x_2 + 2x_3 &= 2, \\ x_1 + 4x_2 + x_3 &= 10. \end{aligned}$$

Clearly these new equations have the same solutions x_1, x_2, x_3 . A second operation is that we can multiply an equation by a nonzero number. For example, we multiply the third equation in (1) by 3:

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 2, \\ x_1 + x_2 + 3x_3 &= 5, \\ 3x_1 + 12x_2 + 3x_3 &= 30. \end{aligned}$$

Finally we can add a multiple of one equation to another equation. For example, we add 3 times the third equation in (1) to the second equation:

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 2, \\ (1+3)x_1 + (1+12)x_2 + (3+3)x_3 &= 5+30, \\ x_1 + 4x_2 + x_3 &= 10. \end{aligned}$$

Learning outcomes: Write a system of linear equations in matrix form Use row reduction to put a matrix into row echelon form or reduced row echelon form Determine whether a system of linear equations has no solution, one solution, or infinitely many solutions.

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¹Although perhaps we have this backwards, quite often we solve a linear system of equations to find out something about matrices, rather than vice versa.

The same x_1, x_2, x_3 should still be solutions to the new equations. These were just examples; we did not get any closer to the solution. We must do these three operations in some more logical manner, but it turns out these three operations suffice to solve every linear equation.

The first thing is to write the equations in a more compact manner. Given

$$A\vec{x} = \vec{b},$$

we write down the so-called *augmented matrix*

$$[A \mid \vec{b}],$$

where the vertical line is just a marker for us to know where the “right-hand side” of the equation starts. For example, for the system (1) the augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array} \right].$$

The entire process of elimination, which we will describe, is often applied to any sort of matrix, not just an augmented matrix. Simply think of the matrix as the 3×4 matrix

$$\left[\begin{array}{cccc} 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array} \right].$$

Row echelon form and elementary operations

We apply the three operations above to the matrix. We call these the *elementary operations* or *elementary row operations*.

Definition 1. *The elementary row operations on a matrix are:*

- (a) *Swap two rows.*
- (b) *Multiply a row by a nonzero number.*
- (c) *Add a multiple of one row to another row.*

Note that these are the same three operations that we could do with equations to try to solve them earlier in this section. We run these operations until we get into a state where it is easy to read off the answer, or until we get into a contradiction indicating no solution.

More specifically, we run the operations until we obtain the so-called *row echelon form*. Let us call the first (from the left) nonzero entry in each row the *leading entry*. A matrix is in *row echelon form* if the following conditions are satisfied:

- (a) The leading entry in any row is strictly to the right of the leading entry of the row above.
- (b) Any zero rows are below all the nonzero rows.
- (c) All leading entries are 1.²

A matrix is in *reduced row echelon form* if furthermore the following condition is satisfied.

- (a) All the entries above a leading entry are zero.

Example 1. *The following matrices are in row echelon form. The leading entries are marked:*

$$\left[\begin{array}{cccc} 1 & 2 & 9 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad \left[\begin{array}{ccc} 1 & -1 & -3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{array} \right] \quad \left[\begin{array}{ccc} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right] \quad \left[\begin{array}{cccc} 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Note that the definition applies to matrices of any size. None of the matrices above are in reduced row echelon form. For example, in the first matrix none of the entries above the second and third leading entries are zero; they are 9, 3, and 5.

²Some books do not require this and allow the entries to be any non-zero number, putting this as a requirement for reduced row echelon form. I like leaving this here because it makes the process of row reduction seem more defined and algorithmic. They are equivalent though and with or without the 1 requirement can be used to answer the same questions.

The following matrices are in reduced row echelon form. The leading entries are marked:

$$\begin{bmatrix} 1 & 3 & 0 & 8 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The procedure we will describe to find a reduced row echelon form of a matrix is called *Gauss–Jordan elimination*. The first part of it, which obtains a row echelon form, is called *Gaussian elimination* or *row reduction*. For some problems, a row echelon form is sufficient, and it is a bit less work to only do this first part.

To attain the row echelon form we work *systematically*. We go column by column, starting at the first column. We find topmost entry in the first column that is not zero, and we call it the *pivot*. If there is no nonzero entry we move to the next column. We swap rows to put the row with the pivot as the first row. We divide the first row by the pivot to make the pivot entry be a 1. Now look at all the rows below and subtract the correct multiple of the pivot row so that all the entries below the pivot become zero.

After this procedure we forget that we had a first row (it is now fixed), and we forget about the column with the pivot and all the preceding zero columns. Below the pivot row, all the entries in these columns are just zero. Then we focus on the smaller matrix and we repeat the steps above.

It is best shown by example, so let us go back to the example from the beginning of the section. We keep the vertical line in the matrix, even though the procedure works on any matrix, not just an augmented matrix. We start with the first column and we locate the pivot, in this case the first entry of the first column.

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array} \right]$$

We multiply the first row by $1/2$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array} \right]$$

We subtract the first row from the second and third row (two elementary operations).

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 3 & 0 & 9 \end{array} \right]$$

We are done with the first column and the first row for now. We almost pretend the matrix doesn't have the first column and the first row.

$$\left[\begin{array}{ccc|c} * & * & * & * \\ * & 0 & 2 & 4 \\ * & 3 & 0 & 9 \end{array} \right]$$

OK, look at the second column, and notice that now the pivot is in the third row.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 3 & 0 & 9 \end{array} \right]$$

We swap rows.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 9 \\ 0 & 0 & 2 & 4 \end{array} \right]$$

And we divide the pivot row by 3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 4 \end{array} \right]$$

We do not need to subtract anything as everything below the pivot is already zero. We move on, we again start ignoring the second row and second column and focus on

$$\left[\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & 2 & 4 \end{array} \right].$$

We find the pivot, then divide that row by 2:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 4 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right].$$

The matrix is now in row echelon form.

The equation corresponding to the last row is $x_3 = 2$. We know x_3 and we could substitute it into the first two equations to get equations for x_1 and x_2 . Then we could do the same thing with x_2 , until we solve for all 3 variables. This procedure is called *backsubstitution* and we can achieve it via elementary operations. We start from the lowest pivot (leading entry in the row echelon form) and subtract the right multiple from the row above to make all the entries above this pivot zero. Then we move to the next pivot and so on. After we are done, we will have a matrix in reduced row echelon form.

We continue our example. Subtract the last row from the first to get

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right].$$

The entries above the pivot in the third row is already zero. So we move onto the next pivot, the one in the second row. We subtract this row from the top row to get

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right].$$

The matrix is in reduced row echelon form.

If we now write down the equations for x_1, x_2, x_3 , we find

$$x_1 = -4, \quad x_2 = 3, \quad x_3 = 2.$$

In other words, we have solved the system.

Example 2. Solve the following system of equations using row reduction:

$$\begin{aligned} -x_1 + x_2 + 3x_3 &= 7 \\ -3x_1 &+ x_3 = -5 \\ -2x_1 - x_2 &= -4 \end{aligned}$$

Solution: In order to solve this problem, we need to set up the augmented matrix for this system, which is

$$\left[\begin{array}{ccc|c} -1 & 1 & 3 & 7 \\ -3 & 0 & 1 & -5 \\ -2 & -1 & 0 & -4 \end{array} \right]$$

To carry out the process, we need to get a 1 in the top left corner, then work from there. We multiply the first row by -1 to get

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & -7 \\ -3 & 0 & 1 & -5 \\ -2 & -1 & 0 & -4 \end{array} \right].$$

Next, we want to use row 1 to cancel out the -3 and -2 in column 1. To do this, we add three copies of row 1 to row 2, and two copies of row 1 to row 3 to get the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & -7 \\ 0 & -3 & -8 & -26 \\ 0 & -3 & -6 & -18 \end{array} \right].$$

Normally, the next step would be to divide the second row by -3 in order to put a 1 in that pivot spot. However, since both the second and third rows have a -3 in the second column, we can combine these two rows directly without dividing by -3 first. We subtract row 2 from row 3 to get

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & -7 \\ 0 & -3 & -8 & -26 \\ 0 & 0 & 2 & 8 \end{array} \right]$$

and we can now use this to solve the system. The bottom row says that $2x_3 = 8$, so that $x_3 = 4$. The second row says that $-3x_2 - 8x_3 = -26$, since $x_3 = 4$, we have that $-3x_2 = -26 + 32 = 6$, so $x_2 = -2$. Finally, the first row of the augmented matrix says that $x_1 - x_2 - 3x_3 = -7$. Plugging in our values for x_2 and x_3 gives $x_1 = -7 - 2 + 12 = 3$. Therefore, the solution is

$$x_1 = 3 \quad x_2 = -2 \quad x_3 = 4.$$

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Non-unique solutions and inconsistent systems

It is possible that the solution of a linear system of equations is not unique, or that no solution exists. Suppose for a moment that the row echelon form we found was

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array} \right].$$

Then the last row gives the equation $0x_1 + 0x_2 + 0x_3 = 1$, or $0 = 1$. That is impossible and the equations are *inconsistent*. There is no solution to $A\vec{x} = \vec{b}$.

On the other hand, if we find a row echelon form of

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right],$$

then there is no issue with finding solutions. In fact, we will find way too many. Let us continue with backsubstitution (subtracting 3 times the second row from the first) to find the reduced row echelon form and let's mark the pivots.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The last row is all zeros; it just says $0 = 0$ and we ignore it. The two remaining equations are

$$x_1 + 2x_2 = -5, \quad x_3 = 3.$$

Let us solve for the variables that corresponded to the pivots, that is x_1 and x_3 as there was a pivot in the first column and in the third column:

$$\begin{aligned} x_1 &= -2x_2 - 5, \\ x_3 &= 3. \end{aligned}$$

The variable x_2 can be anything you wish and we still get a solution. The x_2 is called a *free variable*. There are infinitely many solutions, one for every choice of x_2 . For example, if we pick $x_2 = 0$, then $x_1 = -5$, and $x_3 = 3$ give a solution. But we also get a solution by picking say $x_2 = 1$, in which case $x_1 = -7$ and $x_3 = 3$, or by picking $x_2 = -5$ in which case $x_1 = 5$ and $x_3 = 3$.

The general idea is that if any row has all zeros in the columns corresponding to the variables, but a nonzero entry in the column corresponding to the right-hand side \vec{b} , then the system is inconsistent and has no solutions. In other words, the system is inconsistent if you find a pivot on the right side of the vertical line drawn in the augmented matrix. Otherwise, the system is consistent, and at least one solution exists.

If the system is consistent:

- (a) If every column corresponding to a variable has a pivot element, then the solution is unique.
- (b) If there are columns corresponding to variables with no pivot, then those are *free variables* that can be chosen arbitrarily, and there are infinitely many solutions.

Another way to interpret this idea of free variables is that at the beginning, before you look at the system of equations, all of the variables can be anything, and there are no constraints on them. The equations then give us constraints on these variables, because they give us rules that the variables must satisfy. When we have a row of the augmented matrix that becomes all zeros, it means that the equation that was there is redundant and doesn't add any constraints to the equations. This may result in an *underdetermined* system, which will likely have free variables.

Example 3. Solve the following two systems of equations, or determine that no solution exists, using row reduction:

$$\begin{aligned}x_1 - x_2 - 3x_3 &= -3 \\ -x_1 - 2x_2 + 4x_3 &= 6 \\ x_1 + 5x_2 - 5x_3 &= -9\end{aligned}$$

$$\begin{aligned}x_1 - x_2 - 3x_3 &= -3 \\ -x_1 - 2x_2 + 4x_3 &= 6 \\ x_1 + 5x_2 - 5x_3 &= 1\end{aligned}$$

Solution: For the first of these systems, we will set up the augmented matrix and proceed through the process like normal. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & -3 \\ -1 & -2 & 4 & 6 \\ 1 & 5 & -5 & -9 \end{array} \right].$$

Since we already have a 1 in the top-left corner of this matrix, we can use it to cancel the entries in the rest of column 1. We add one copy of row 1 to row 2, and subtract row 1 from row 3 to get the next augmented form matrix as

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & -3 \\ 0 & -3 & 1 & 3 \\ 0 & 6 & -2 & -6 \end{array} \right].$$

Looking at the matrix here, we see that row 3 is -2 times row 2. Therefore, if we add two copies of row 2 to row 3, we get the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & -3 \\ 0 & -3 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Therefore, we have a situation where there are only two pivot columns, and the last row is all zeros. Since there are three variables and column 3 is not a pivot column, we can take x_3 as a free variable. If we do that, the second equation tells us that $-3x_2 + x_3 = 3$, or, since we are taking x_3 as a free variable, we can write $x_2 = -1 + \frac{1}{3}x_3$. We can then take the first equation, which says that $x_1 - x_2 - 3x_3 = -3$ or, by rearranging

$$x_1 = -3 + x_2 + 3x_3 = -3 + \left(-1 + \frac{1}{3}x_3\right) + 3x_3 = -4 + \frac{10}{3}x_3.$$

This means that for any value of t , our solution is determined by

$$\begin{aligned}x_1 &= -4 + \frac{10}{3}t \\ x_2 &= -1 + \frac{1}{3}t \\ x_3 &= t\end{aligned}$$

The use of t here is just to separate it from the variable x_3 . For example, we could pick $t = 3$, in which case we would get $x_1 = 6$, $x_2 = 0$, $x_3 = 3$.

For the second version of the problem, we again set up the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & -3 \\ -1 & -2 & 4 & 6 \\ 1 & 5 & -5 & 1 \end{array} \right].$$

Since the left side matrix part is the same as the previous version, the process of row reducing the matrix is identical to what was done previously. When we carry out this process we get the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & -3 \\ 0 & -3 & 1 & 3 \\ 0 & 0 & 0 & 10 \end{array} \right].$$

In this case, we see that the last row corresponds to the equation $0 = 10$ so these equations are inconsistent and do not have a solution. └

The point of the above example is to illustrate the fact that whether or not a system is inconsistent or has free variables in the solution depends on the right-hand side of the equation, even if the left-hand side has the same coefficients. We'll see more about why this is in § ??.

When $\vec{b} = \vec{0}$, we have a so-called *homogeneous matrix equation*

$$A\vec{x} = \vec{0}.$$

There is no need to write an augmented matrix in this case. As the elementary operations do not do anything to a zero column, it always stays a zero column. Moreover, $A\vec{x} = \vec{0}$ always has at least one solution, namely $\vec{x} = \vec{0}$. Such a system is always consistent. It may have other solutions: If you find any free variables, then you get infinitely many solutions. As mentioned in the last section, this is directly connected to linear independence of the columns of A . If there are other solutions, then there are other linear combinations that give $\vec{0}$, and so the columns of A are linearly dependent. Otherwise, if there are no such solutions outside of $\vec{x} = \vec{0}$, then the columns of A are linearly independent.

How would we determine this fact? We don't need to include the zero column on the far right, but we can apply the same operations to the matrix A alone. In this case, we either get a pivot column in every column for the row echelon form, in which case the only solution is $\vec{x} = \vec{0}$, or we get at least one non-pivot column, which means that there are free variables, implying that non-zero solutions exist. This leads to a first equivalence statement we can make about the solution to homogeneous matrix equations.

Theorem 1. *Let A be a matrix. The following statements are equivalent (meaning if any one of them is true, so are all of the other ones):*

- (a) *The only solution to the matrix equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.*
- (b) *The row echelon form of A has a pivot element in every column.*
- (c) *The reduced row echelon form of A is an identity matrix, potentially with rows of zero on the bottom.*
- (d) *The columns of A are linearly independent.*

The set of solutions of $A\vec{x} = \vec{0}$ comes up quite often so people give it a name. It is called the *nullspace* or the *kernel* of A . One place where the kernel comes up is invertibility of a square matrix A . If the kernel of A contains a nonzero vector, then it contains infinitely many vectors (there was a free variable). But then it is impossible to invert $\vec{0}$, since infinitely many vectors go to $\vec{0}$, so there is no unique vector that A takes to $\vec{0}$. So if the kernel is nontrivial, that is, if there are any nonzero vectors, in other words, if there are any free variables, or in yet other words, if the row echelon form of A has columns without pivots, then A is not invertible. We will return to this idea later.