# Practice for Eigenvalues

*Why?*

**Exercise 1** *Find the eigenvalues and corresponding eigenvectors of the matrix*

$$\begin{bmatrix} -8 & -18 \\ 4 & 10 \end{bmatrix}$$

**Exercise 2** *Find the eigenvalues and corresponding eigenvectors of the matrix*

$$\begin{bmatrix} -2 & 0 \\ 8 & -4 \end{bmatrix}$$

**Exercise 3** *Find the eigenvalues and corresponding eigenvectors of the matrix*

$$\begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix}$$

**Exercise 4** *Find the eigenvalues and corresponding eigenvectors of the matrix*

$$\begin{bmatrix} -3 & 5 \\ -8 & 9 \end{bmatrix}$$

**Exercise 5** *Find the eigenvalues and corresponding eigenvectors of the matrix*

$$\begin{bmatrix} 0 & 2 \\ -1 & -2 \end{bmatrix}$$

**Exercise 6** *Find the eigenvalues and corresponding eigenvectors of the matrix*

$$\begin{bmatrix} -4 & 1 \\ -8 & 0 \end{bmatrix}$$

**Exercise 7** *Find the eigenvalues and corresponding eigenvectors of the matrix*

$$\begin{bmatrix} 0 & -8 \\ 2 & 8 \end{bmatrix}$$

**Exercise 8** *Find the eigenvalues and corresponding eigenvectors of the matrix*

$$\begin{bmatrix} 1 & -2 \\ 8 & -7 \end{bmatrix}$$

**Exercise 9** *Find the eigenvalues and corresponding eigenvectors of the matrix*

$$\begin{bmatrix} 4 & 0 & 0 \\ -4 & 2 & 1 \\ -6 & 0 & 1 \end{bmatrix}$$

**Exercise 10** *Find the eigenvalues and corresponding eigenvectors of the matrix*

$$\begin{bmatrix} -4 & 9 & 9 \\ -3 & 6 & 9 \\ 3 & -7 & -10 \end{bmatrix}$$

**Exercise 11** *Find the eigenvalues and corresponding eigenvectors of the matrix*

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 6 \\ 6 & -3 & -2 \end{bmatrix}$$

**Exercise 12** *Find the eigenvalues and corresponding eigenvectors of the matrix*

$$\begin{bmatrix} 5 & 3 & 6 \\ 2 & 2 & 2 \\ -3 & -2 & -3 \end{bmatrix}$$

**Exercise 13** *Find the eigenvalues and eigenvectors for the matrix below. Compute generalized eigenvectors if needed to get to a total of two vectors.*

$$\begin{bmatrix} -11 & -9 \\ 12 & 10 \end{bmatrix}$$

**Exercise 14** *Find the eigenvalues and eigenvectors for the matrix below. Compute generalized eigenvectors if needed to get to a total of two vectors.*

$$\begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$$

**Exercise 15** *This exercise will work through the process of finding the eigenvalues and corresponding eigenvectors of the matrix*

$$A = \begin{bmatrix} -2 & 0 & -3 \\ 12 & 5 & 12 \\ 0 & -1 & 1 \end{bmatrix}.$$

a) *Find the characteristic polynomial of this matrix by computing* $\det(A - \lambda I)$ *using any method from this section.*

b) *This polynomial can be rewritten as* $-(\lambda - r_1)^2(\lambda - r_2)$ *where* $r_1$ *and* $r_2$ *are the eigenvalues of* $A$*. What are the eigenvalues? What is each of their algebraic multiplicity? (Hint: One of these roots is 2)*

c) *Find an eigenvector for eigenvalue* $r_2$ *above. What is the geometric multiplicity of this eigenvalue?*

d) *Find an eigenvector for eigenvalue* $r_1$*. What is the geometric multiplicity of this eigenvalue?*

e) *There is only one possible eigenvector for* $r_1$*, which means it is defective. Find a solution to the equation* $(A - r_1 I)\vec{w} = \vec{v}$*, where* $\vec{v}$ *is the eigenvector you found in the previous part. This is the generalized eigenvector for* $r_1$*.*

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**Exercise 16** *We say that a matrix* $A$ *is* diagonalizable *if there exist matrices* $D$ *and* $P$ *so that* $PDP^{-1} = A$*. This really means that* $A$ *can be represented by a diagonal matrix in a different basis (as opposed to the standard basis). One way this can be done is with eigenvalues.*

a) *Consider the matrix* $A$ *given by*

$$A = \begin{bmatrix} -4 & 6 \\ -1 & 1 \end{bmatrix}.$$

*Find the eigenvalues and corresponding eigenvectors of this matrix.*

b) *Form two matrices,* $D$*, a diagonal matrix with the eigenvalues of* $A$ *on the diagonal, and* $E$*, a matrix whose columns are the eigenvectors of* $A$ *in the* same order *as the eigenvalues were put into* $D$*. Write out these matrices.*

c) *Compute* $E^{-1}$*.*

d) *Work out the products* $EDE^{-1}$ and $E^{-1}AE$*. What do you notice here?*

*This shows that, in the case of a* $2 \times 2$ *matrix, if we have two distinct real eigenvalues, that matrix is diagonalizable, using the eigenvectors.*

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**Exercise 17** *Follow the process outlined in Exercise  to attempt to diagonalize the matrix*

$$\begin{bmatrix} 13 & 14 & 12 \\ -6 & -4 & -6 \\ -3 & -6 & -2 \end{bmatrix}$$

*Hint: 1 is an eigenvalue.*

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**Exercise 18** *The diagonalization process decribed in Exercise  works for any case where there are real and distinct eigenvalues, as well as complex eigenvalues (but the algebra with the complex numbers gets complicated). It may or may not work in the case of repeated eigenvalues, and it fails whenever there are defective eigenvalues. Consider the matrix*

$$\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$$

a) *Find the eigenvalue(s) of this matrix, and see that we have a repeated eigenvalue.*

b) *Find the eigenvector for that eigenvalue, as well as a generalized eigenvector.*

c) *Build a matrix* $E$ *like before, but this time put the eigenvector in the first column and the generalized eigenvector in the second. Compute* $E^{-1}$*.*

d) *Find the product* $E^{-1}AE$*. Before, this gave us a diagonal matrix, but what do we get now?*

*The matrix we get here is almost diagonal, but not quite. It turns out that this is the best we can do for matrices with defective eigenvalues. This matrix is often called $J$ and is the* Jordan Form *of the matrix $A$.*

**Exercise 19** *Follow the process in Exercise  to find the Jordan Form of the matrix*

$$\begin{bmatrix} -7 & 5 & 5 \\ -4 & 5 & 7 \\ -6 & 3 & 1 \end{bmatrix}.$$