# Practice for Advanced Linear Algebra

*Why?*

**Exercise 1** *For the following matrices, find a basis for the kernel (nullspace).*

a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 5 \\ 1 & 1 & -4 \end{bmatrix}$ b) $\begin{bmatrix} 2 & -1 & -3 \\ 4 & 0 & -4 \\ -1 & 1 & 2 \end{bmatrix}$ c) $\begin{bmatrix} -4 & 4 & 4 \\ -1 & 1 & 1 \\ -5 & 5 & 5 \end{bmatrix}$ d) $\begin{bmatrix} -2 & 1 & 1 & 1 \\ -4 & 2 & 2 & 2 \\ 1 & 0 & 4 & 3 \end{bmatrix}$

**Exercise 2** *For the following matrices, find a basis for the kernel (nullspace).*

a) $\begin{bmatrix} 2 & 6 & 1 & 9 \\ 1 & 3 & 2 & 9 \\ 3 & 9 & 0 & 9 \end{bmatrix}$ b) $\begin{bmatrix} 2 & -2 & -5 \\ -1 & 1 & 5 \\ -5 & 5 & -3 \end{bmatrix}$ c) $\begin{bmatrix} 1 & -5 & -4 \\ 2 & 3 & 5 \\ -3 & 5 & 2 \end{bmatrix}$ d) $\begin{bmatrix} 0 & 4 & 4 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{bmatrix}$

**Exercise 3** *Suppose a $5 \times 5$ matrix $A$ has rank 3. What is the nullity?*

**Exercise 4** *Consider a square matrix $A$, and suppose that $\vec{x}$ is a nonzero vector such that $A\vec{x} = \vec{0}$. What does the Fredholm alternative say about invertibility of $A$?*

**Exercise 5** *Compute the rank of the matrix $A$ below.*

$$A = \begin{bmatrix} 0 & -3 & 2 & 4 \\ -5 & -4 & -5 & -1 \\ 1 & 4 & -3 & -5 \\ -2 & -3 & -2 & 1 \end{bmatrix}$$

*What does this tell you about the invertibility of $A$? How about the solutions to $A\vec{x} = \vec{0}$?*

**Exercise 6** *Compute the rank of the matrix $A$ below.*

$$A = \begin{bmatrix} 3 & -5 & 5 \\ 2 & -3 & 3 \\ 4 & 0 & -1 \end{bmatrix}$$

*What does this tell you about the invertibility of $A$? How about the solutions to $A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$?*

**Exercise 7** *Consider*

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 2 & ? & ? \\ -1 & ? & ? \end{bmatrix}.$$

*If the nullity of this matrix is 2, fill in the question marks. Hint: What is the rank?*

**Exercise 8** *Suppose the column space of a $9 \times 5$ matrix $A$ of dimension 3. Find*

a) *Rank of $A$.*

b) *Nullity of $A$.*

c) *Dimension of the row space of $A$.*

d) *Dimension of the nullspace of $A$.*

e) *Size of the maximum subset of linearly independent rows of $A$.*

**Exercise 9** *Compute the rank of the given matrices*

a) $\begin{bmatrix} 6 & 3 & 5 \\ 1 & 4 & 1 \\ 7 & 7 & 6 \end{bmatrix}$ b) $\begin{bmatrix} 5 & -2 & -1 \\ 3 & 0 & 6 \\ 2 & 4 & 5 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix}$

**Exercise 10** *Compute the rank of the given matrices*

a) $\begin{bmatrix} 7 & -1 & 6 \\ 7 & 7 & 7 \\ 7 & 6 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 3 & -1 \\ 6 & 3 & 1 \\ 4 & 7 & -1 \end{bmatrix}$

**Exercise 11** *For the matrices in* Exercise , *find a linearly independent set of row vectors that span the row space (they don't need to be rows of the matrix).*

**Exercise 12** *For the matrices in* Exercise , *find a linearly independent set of columns that span the column space. That is, find the pivot columns of the matrices.*

**Exercise 13** *For the matrices in* Exercise , *find a linearly independent set of row vectors that span the row space (they don't need to be rows of the matrix).*

**Exercise 14** *For the matrices in* Exercise , *find a linearly independent set of columns that span the column space. That is, find the pivot columns of the matrices.*

**Exercise 15** *Compute the rank of the matrix*

$$\begin{bmatrix} 10 & -2 & 11 & -7 \\ -5 & -2 & -5 & 5 \\ 1 & 0 & -4 & -4 \\ 1 & 2 & 2 & -1 \end{bmatrix}$$

**Exercise 16** *Compute the rank of the matrix*

$$\begin{bmatrix} 4 & -2 & 0 & -4 \\ 3 & -5 & 2 & 0 \\ 1 & -2 & 0 & 1 \\ -1 & 1 & 3 & -3 \end{bmatrix}$$

**Exercise 17** *Find a linearly independent subset of the following vectors that has the same span.*

$$\begin{bmatrix}-1\\1\\2\end{bmatrix},\quad \begin{bmatrix}2\\-2\\-4\end{bmatrix},\quad \begin{bmatrix}-2\\4\\1\end{bmatrix},\quad \begin{bmatrix}-1\\3\\-2\end{bmatrix}$$

**Exercise 18** *Find a linearly independent subset of the following vectors that has the same span.*

$$\begin{bmatrix}0\\0\\0\end{bmatrix},\quad \begin{bmatrix}3\\1\\-5\end{bmatrix},\quad \begin{bmatrix}0\\3\\-1\end{bmatrix},\quad \begin{bmatrix}-3\\2\\4\end{bmatrix}$$

**Exercise 19** *For the following sets of vectors, determine if the set is linearly independent. Then find a basis for the subspace spanned by the vectors, and find the dimension of the subspace.*

a) $\begin{bmatrix}1\\1\\1\end{bmatrix},\ \begin{bmatrix}-1\\-1\\-1\end{bmatrix}$

b) $\begin{bmatrix}1\\0\\5\end{bmatrix},\ \begin{bmatrix}0\\1\\0\end{bmatrix},\ \begin{bmatrix}0\\-1\\0\end{bmatrix}$

c) $\begin{bmatrix}-4\\-3\\5\end{bmatrix},\ \begin{bmatrix}2\\3\\3\end{bmatrix},\ \begin{bmatrix}2\\0\\2\end{bmatrix}$

d) $\begin{bmatrix}1\\3\\0\end{bmatrix},\ \begin{bmatrix}0\\2\\2\end{bmatrix},\ \begin{bmatrix}-1\\-1\\2\end{bmatrix}$

e) $\begin{bmatrix}1\\3\end{bmatrix},\ \begin{bmatrix}0\\2\end{bmatrix},\ \begin{bmatrix}-1\\-1\end{bmatrix}$

f) $\begin{bmatrix}3\\1\\3\end{bmatrix},\ \begin{bmatrix}2\\4\\-4\end{bmatrix},\ \begin{bmatrix}-5\\-5\\-2\end{bmatrix}$

**Exercise 20** *For the following sets of vectors, determine if the set is linearly independent. Then find a basis for the subspace spanned by the vectors, and find the dimension of the subspace.*

a) $\begin{bmatrix}1\\2\end{bmatrix},\ \begin{bmatrix}1\\1\end{bmatrix}$

b) $\begin{bmatrix}1\\1\\1\end{bmatrix},\ \begin{bmatrix}2\\2\\2\end{bmatrix},\ \begin{bmatrix}1\\1\\2\end{bmatrix}$

c) $\begin{bmatrix}5\\3\\1\end{bmatrix},\ \begin{bmatrix}5\\-1\\5\end{bmatrix},\ \begin{bmatrix}-1\\3\\-4\end{bmatrix}$

d) $\begin{bmatrix}2\\2\\4\end{bmatrix},\ \begin{bmatrix}2\\2\\3\end{bmatrix},\ \begin{bmatrix}4\\4\\-3\end{bmatrix}$

e) $\begin{bmatrix}1\\0\end{bmatrix},\ \begin{bmatrix}2\\0\end{bmatrix},\ \begin{bmatrix}3\\0\end{bmatrix}$

f) $\begin{bmatrix}1\\0\\0\end{bmatrix},\ \begin{bmatrix}2\\0\\0\end{bmatrix},\ \begin{bmatrix}0\\1\\2\end{bmatrix}$

**Exercise 21** *Suppose that $X$ is the set of all the vectors of $\mathbb{R}^3$ whose third component is zero. Is $X$ a subspace? And if so, find a basis and the dimension.*

**Exercise 22** *Consider a set of 3 component vectors.*

a) *How can it be shown if these vectors are linearly independent?*

b) *Can a set of 4 of these 3 component vectors be linearly independent? Explain your answer.*

c) *Can a set of 2 of these 3 component vectors be linearly independent? Explain.*

d) *How would it be shown if these vectors make up a spanning set for all 3 component vectors?*

e) *Can 4 vectors be a spanning set? Explain.*

f) *Can 2 vectors be a spanning set? Explain.*

**Exercise 23** *Consider the vectors*

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

*Let $A$ be the matrix with these vectors as columns and $\vec{b}$ the vector* $[1\ 0\ 0]$.

a) *Compute the rank of $A$ to determine how many of these vectors are linearly independent.*

b) *Determine if $\vec{b}$ is in the span of the given vectors by using row reduction to try to solve $A\vec{x} = \vec{b}$.*

c) *Look at the columns of the row-reduced form of $A$. Is $\vec{b}$ in the span of those vectors?*

d) *What do these last two parts tell you about the span of the columns of a matrix, and the span of the columns of the row-reduced matrix?*

e) *Now, build a matrix $D$ with these vectors as rows. Row-reduce this matrix to get a matrix $D_2$.*

f) *Is $\vec{b}$ in the span of the rows of $D_2$? You can't check this in using the matrix form; instead, just brute force it based on the form of $D_2$. What does this potentially say about the span of the rows of $D$ and the rows of $D_2$?*

**Exercise 24** *Complete* Exercise *with*

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -6 \\ 2 \\ 3 \\ -1 \end{bmatrix} \quad \begin{bmatrix} -13 \\ 3 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v}_4 \begin{bmatrix} 11 - 1 \\ -5 \\ -1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

**Exercise 25** *Compute the inverse of the given matrices*

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

**Exercise 26** *Compute the inverse of the given matrices*

a) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 4 & 0 \\ 2 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix}$

**Exercise 27** *By computing the inverse, solve the following systems for $\vec{x}$.*

a) $\begin{bmatrix} 4 & 1 \\ -1 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 13 \\ 26 \end{bmatrix}$ b) $\begin{bmatrix} 3 & 3 \\ 3 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

**Exercise 28** *By computing the inverse, solve the following systems for $\vec{x}$.*

a) $\begin{bmatrix} -1 & 1 \\ 3 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 7 \\ 1 & 6 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

**Exercise 29** *For each of the following matrices below:*

a) *Compute the trace and determinant of the matrix, and*

b) *Find the eigenvalues of the matrix and verify that the trace is the sum of the eigenvalues and the determinant is the product.*

$$(i)\ \begin{bmatrix} -4 & 2 \\ -9 & 5 \end{bmatrix} \qquad (ii)\ \begin{bmatrix} 2 & -3 \\ 6 & -4 \end{bmatrix} \qquad (iii)\ \begin{bmatrix} -10 & -12 \\ 6 & 8 \end{bmatrix}. \qquad (iv)\ \begin{bmatrix} -7 & -9 \\ 1 & -1 \end{bmatrix}$$

**Exercise 30** *For each of the following matrices below:*

a) *Compute the trace and determinant of the matrix, and*

b) *Find the eigenvalues of the matrix and verify that the trace is the sum of the eigenvalues and the determinant is the product.*

$$(i)\ \begin{bmatrix} -1 & -16 & -4 \\ 1 & 6 & 1 \\ -2 & -4 & 1 \end{bmatrix} \qquad (ii)\ \begin{bmatrix} 1 & 2 & 0 \\ -12 & -13 & -4 \\ 16 & 14 & 3 \end{bmatrix} \qquad (iii)\ \begin{bmatrix} 10 & -7 & -14 \\ 0 & 5 & 6 \\ 7 & -8 & -14 \end{bmatrix}$$