Practice for Euler Method

Why?

Exercise 1 *Consider* $\frac{dx}{dt} = (2t - x)^2$, $x(0) = 2$. *Use Euler's method with step size* $h = 0.5$ *to approximate* $x(1)$.

Exercise 2 *Consider the differential equation* $\frac{dy}{dt} = t^2 - 3y + 1$ *with* $y(1) = 4$. *Approximate the solution at* $t = 3$ *using Euler's method with a step size of* $h = 1$ *and* $h = 0.5$. *Compare these values with the actual solution at* $t = 3$.

Exercise 3 *Consider the differential equation* $\frac{dy}{dt} = 2ty + y^2$ *with* $y(0) = 1$. *Approximate the solution at* $t = 2$ *using Euler's method with a step size of* $h = 1$ *and* $h = 0.5$.

Exercise 4 *Consider* $\frac{dx}{dt} = t - x$, $x(0) = 1$.

a) *Use Euler's method with step sizes* $h = 1, 1/2, 1/4, 1/8$ *to approximate* $x(1)$.

b) *Solve the equation exactly.*

c) *Describe what happens to the errors for each* h *you used. That is, find the factor by which the error changed each time you halved the interval.*

Exercise 5 *Let* $x' = \sin(xt)$, *and* $x(0) = 1$. *Approximate* $x(1)$ *using Euler's method with step sizes 1, 0.5, 0.25. Use a calculator and compute up to 4 decimal digits.*

Exercise 6 *Approximate the value of* e *by looking at the initial value problem* $y' = y$ *with* $y(0) = 1$ *and approximating* $y(1)$ *using Euler's method with a step size of* 0.2.

Exercise 7 *Let* $x' = 2t$, *and* $x(0) = 0$.

a) *Approximate* $x(4)$ *using Euler's method with step sizes 4, 2, and 1.*

b) *Solve exactly, and compute the errors.*

c) *Compute the factor by which the errors changed.*

Exercise 8 *Let* $x' = xe^{xt+1}$, *and* $x(0) = 0$.

a) *Approximate* $x(4)$ *using Euler's method with step sizes 4, 2, and 1.*

b) *Guess an exact solution based on part a) and compute the errors.*

Learning outcomes:

Exercise 9 *Example of numerical instability: Take $y' = -5y$, $y(0) = 1$. We know that the solution should decay to zero as x grows. Using Euler's method, start with $h = 1$ and compute y_1, y_2, y_3, y_4 to try to approximate $y(4)$. What happened? Now halve the interval. Keep halving the interval and approximating $y(4)$ until the numbers you are getting start to stabilize (that is, until they start going towards zero). Note: You might want to use a calculator.*

There is a simple way to improve Euler's method to make it a second order method by doing just one extra step. Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, and a step size h. What we do is to pretend we compute the next step as in Euler, that is, we start with (x_i, y_i), we compute a slope $k_1 = f(x_i, y_i)$, and then look at the point $(x_i + h, y_i + k_1 h)$. Instead of letting our new point be $(x_i + h, y_i + k_1 h)$, we compute the slope at that point, call it k_2, and then take the average of k_1 and k_2, hoping that the average is going to be closer to the actual slope on the interval from x_i to $x_i + h$. And we are correct, if we halve the step, the error should go down by a factor of $2^2 = 4$. To summarize, the setup is the same as for regular Euler, except the computation of y_{i+1} and x_{i+1}.

$$k_1 = f(x_i, y_i), \qquad x_{i+1} = x_i + h,$$

$$k_2 = f(x_i + h, y_i + k_1 h), \qquad y_{i+1} = y_i + \frac{k_1 + k_2}{2} h.$$

Exercise 10 *Consider $\frac{dy}{dx} = x + y$, $y(0) = 1$.*

a) *Use the improved Euler's method (see above) with step sizes $h = 1/4$ and $h = 1/8$ to approximate $y(1)$.*

b) *Use Euler's method with $h = 1/4$ and $h = 1/8$.*

c) *Solve exactly, find the exact value of $y(1)$.*

d) *Compute the errors, and the factors by which the errors changed.*

The simplest method used in practice is the *Runge–Kutta method*. Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, and a step size h. Everything is the same as in Euler's method, except the computation of y_{i+1} and x_{i+1}.

$$\begin{aligned} k_1 &= f(x_i, y_i), \\ k_2 &= f\big(x_i + h/2, y_i + k_1 (h/2)\big), \\ k_3 &= f\big(x_i + h/2, y_i + k_2 (h/2)\big), \\ k_4 &= f(x_i + h, y_i + k_3 h). \end{aligned} \qquad \begin{aligned} x_{i+1} &= x_i + h, \\ y_{i+1} &= y_i + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} h, \end{aligned}$$

Exercise 11 *Consider $\frac{dy}{dx} = yx^2$, $y(0) = 1$.*

a) *Use Runge–Kutta (see above) with step sizes $h = 1$ and $h = 1/2$ to approximate $y(1)$.*

b) *Use Euler's method with $h = 1$ and $h = 1/2$.*

c) *Solve exactly, find the exact value of $y(1)$, and compare.*
