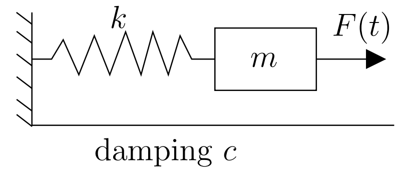
Forced oscillations and resonance

We discuss Forced oscillations and resonance



Let us return back to the example of a mass on a spring. We examine the case of forced oscillations, which we did not yet handle. That is, we consider the equation

$$mx'' + \gamma x' + kx = F(t),$$

for some nonzero $F(t)$. The setup is again: m is mass, γ is friction, k is the spring constant, and $F(t)$ is an external force acting on the mass.

We are interested in periodic forcing, such as noncentered rotating parts, or perhaps loud sounds, or other sources of periodic force.

Undamped forced motion and resonance

First let us consider undamped ($\gamma = 0$) motion. We have the equation

$$mx'' + kx = F_0 \cos(\omega t).$$

This equation has the complementary solution (solution to the associated homogeneous equation)

$$x_c = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t),$$

where $\omega_0 = \sqrt{k/m}$ is the *natural frequency* (angular). It is the frequency at which the system "wants to oscillate" without external interference.

Suppose that $\omega_0 \neq \omega$. We try the solution $x_p = A\cos(\omega t)$ and solve for A. We do not need a sine in our trial solution as after plugging in we only have cosines. If you include a sine, it is fine; you will find that its coefficient is zero (I could not find a second rhyme).

We solve using the method of undetermined coefficients. We find that

$$x_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

We leave it as an exercise to do the algebra required.

The general solution is

$$x = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

Written another way

$$x = C \cos(\omega_0 t - \gamma) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

The solution is a superposition of two cosine waves at different frequencies.

Example 1. *Take*

$$0.5x'' + 8x = 10\cos(\pi t), \qquad x(0) = 0, \qquad x'(0) = 0.$$

Solution: Let us compute. First we read off the parameters: $\omega = \pi$, $\omega_0 = \sqrt{8/0.5} = 4$, $F_0 = 10$, $m = 0.5$. The general solution is

$$x = C_1 \cos(4t) + C_2 \sin(4t) + \frac{20}{16 - \pi^2} \cos(\pi t).$$

Learning outcomes: Write differential equations to model forced oscillators (like masses on springs) Identify when beats, pure resonance, and practical resonance can occur Use proper terminology around transient and steady periodic solutions when discussing these problems.

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Solve for C_1 and C_2 using the initial conditions: $C_1 = \dfrac{-20}{16-\pi^2}$ and $C_2 = 0$. Hence

$$x = \frac{20}{16-\pi^2}\big(\cos(\pi t) - \cos(4t)\big).$$

Notice the "beating" behavior in **Normally a reference to a previous figure goes here.**. First use the trigonometric identity

$$2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right) = \cos B - \cos A$$

to get

$$x = \frac{20}{16-\pi^2}\left(2\sin\left(\frac{4-\pi}{2}t\right)\sin\left(\frac{4+\pi}{2}t\right)\right).$$

The function x is a high frequency wave modulated by a low frequency wave.

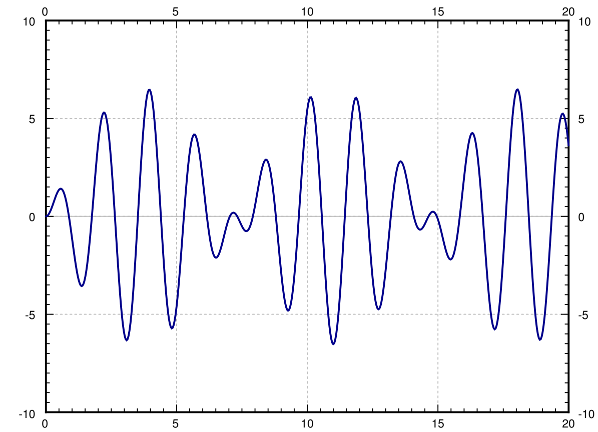

Figure 1: Graph of $\dfrac{20}{16-\pi^2}\big(\cos(\pi t) - \cos(4t)\big)$.

The beating behavior can be experienced even more readily by considering a higher frequency and viewing the resulting function as a sound wave. A sound wave of frequency 440 Hz produces and A4 sound, which is the A above middle C on a piano. This means that the function

$$x_p(t) = \sin(2\pi \cdot 440t)$$

will produce a sound wave equivalent to this A4 sound. In MATLAB, this can be done with the code

Code

```
omega0 = 440*2*pi;
tVals = linspace(0, 5, 5*8192);

testSound = sin(omega0*tVals);
sound(testSound);
```

which will play this pitch for 5 seconds. Now, we want to see what happens if we take a mass-on-a-spring with this natural frequency and apply a forcing function with frequency close to this value. The following code assumes a forcing function of frequency 444 Hz. The multiple of ω_0 in front of the forcing function is only for scaling purposes; otherwise the resulting sound would be too quiet.

Code

```
omega = 444*2*pi;

syms ys(t);
[V] = odeToVectorField(diff(ys, 2) + omega0^2*ys == omega0*cos(omega*t));
MS = matlabFunction(V, 'vars', {'t', 'Y'});
soln = ode45(MS, [0,10], [0,0]);

ySound = deval(soln, tVals);
ySound = ySound(1, :);
sound(ySound);
```

A graph of the solution `ySound` can be found in **Normally a reference to a previous figure goes here.**. This exhibits the beating behavior before on a large scale. The sound played during this code also shows the beating or amplitude modulation that can happen in these sorts of solutions. In terms of tuning instruments, these beats are some of the main things musicians will listen for to know if their instrument is close to the right pitch, but just slightly off.

Now suppose $\omega_0 = \omega$. We cannot try the solution $A\cos(\omega t)$ and then use the method of undetermined coefficients, since we notice that $\cos(\omega t)$ solves the associated homogeneous equation. Therefore, we try $x_p = At\cos(\omega t) + Bt\sin(\omega t)$. This time we need the sine term, since the second derivative of $t\cos(\omega t)$ contains sines. We write the equation

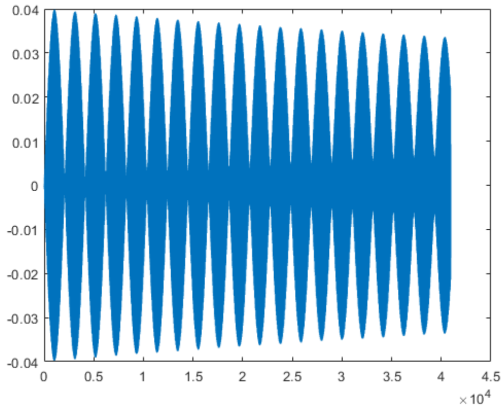

Figure 2: Plot of `ySound` illustrating the beating behavior of interacting sound waves.

$$x'' + \omega^2 x = \frac{F_0}{m} \cos(\omega t).$$

Plugging x_p into the left-hand side we get

$$2B\omega \cos(\omega t) - 2A\omega \sin(\omega t) = \frac{F_0}{m} \cos(\omega t).$$

Hence $A = 0$ and $B = \frac{F_0}{2m\omega}$. Our particular solution is $\frac{F_0}{2m\omega}\, t \sin(\omega t)$ and our general solution is

$$x = C_1 \cos(\omega t) + C_2 \sin(\omega t) + \frac{F_0}{2m\omega}\, t \sin(\omega t).$$

The important term is the last one (the particular solution we found). This term grows without bound as $t \to \infty$. In fact it oscillates between $\frac{F_0 t}{2m\omega}$ and $\frac{-F_0 t}{2m\omega}$. The first two terms only oscillate between $\pm\sqrt{C_1^2 + C_2^2}$, which becomes smaller and smaller in proportion to the oscillations of the last term as t gets larger. In **Normally a reference to a previous figure goes here.** we see the graph with $C_1 = C_2 = 0$, $F_0 = 2$, $m = 1$, $\omega = \pi$.

By forcing the system in just the right frequency we produce very wild oscillations. This kind of behavior is called *resonance* or perhaps *pure resonance*. Sometimes resonance is desired. For example, remember when as a kid you could start swinging by just moving back and forth on the swing seat in the "correct frequency"? You were trying to achieve resonance. The force of each one of your moves was small, but after a while it produced large swings.

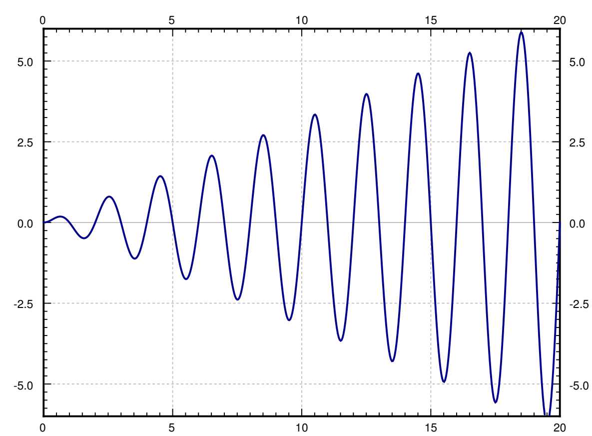

Figure 3: Graph of $\frac{1}{\pi} t \sin(\pi t)$.

On the other hand resonance can be destructive. In an earthquake some buildings collapse while others may be relatively undamaged. This is due to different buildings having different resonance frequencies. So figuring out the resonance frequency can be very important.

A common (but wrong) example of destructive force of resonance is the Tacoma Narrows bridge failure. It turns out there was a different phenomenon at play[1].

[1] K. Billah and R. Scanlan, *Resonance, Tacoma Narrows Bridge Failure, and Undergraduate Physics Textbooks*, American Journal of Physics, 59(2), 1991, 118–124, `http://www.ketchum.org/billah/Billah-Scanlan.pdf`

Damped forced motion and practical resonance

In real life things are not as simple as they were above. There is, of course, some damping. Our equation becomes

$$mx'' + \gamma x' + kx = F_0 \cos(\omega t), \tag{1}$$

for some $\gamma > 0$. We solved the homogeneous problem before. We let

$$p = \frac{\gamma}{2m}, \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

We replace equation (1) with

$$x'' + 2px' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t).$$

The roots of the characteristic equation of the associated homogeneous problem are $r_1, r_2 = -p \pm \sqrt{p^2 - \omega_0^2}$. The form of the general solution of the associated homogeneous equation depends on the sign of $p^2 - \omega_0^2$, or equivalently on the sign of $\gamma^2 - 4km$, as before:

$$x_c = \begin{cases} C_1 e^{r_1 t} + C_2 e^{r_2 t} & \text{if } \gamma^2 > 4km, \\ C_1 e^{-pt} + C_2 t e^{-pt} & \text{if } \gamma^2 = 4km, \\ e^{-pt}\big(C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)\big) & \text{if } \gamma^2 < 4km, \end{cases}$$

where $\omega_1 = \sqrt{\omega_0^2 - p^2}$. In any case, we see that $x_c(t) \to 0$ as $t \to \infty$.

Let us find a particular solution. There can be no conflicts when trying to solve for the undetermined coefficients by trying $x_p = A\cos(\omega t) + B\sin(\omega t)$, because the solution to the homogeneous problem will always have exponential factors (since we have damping) and so there is no ω where this will exactly match the form of the homogeneous solution. Let us plug in and solve for A and B. We get (the tedious details are left to reader)

$$\big((\omega_0^2 - \omega^2)B - 2\omega p A\big)\sin(\omega t) + \big((\omega_0^2 - \omega^2)A + 2\omega p B\big)\cos(\omega t) = \frac{F_0}{m}\cos(\omega t).$$

We solve for A and B:

$$A = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2},$$
$$B = \frac{2\omega p F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2}.$$

We also compute $C = \sqrt{A^2 + B^2}$ to be

$$C = \frac{F_0}{m\sqrt{(2\omega p)^2 + (\omega_0^2 - \omega^2)^2}}.$$

Thus our particular solution is

$$x_p = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2}\cos(\omega t) + \frac{2\omega p F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2}\sin(\omega t).$$

Or in the alternative notation we have amplitude C and phase shift δ where (if $\omega \neq \omega_0$)

$$\tan\delta = \frac{B}{A} = \frac{2\omega p}{\omega_0^2 - \omega^2}.$$

Hence,

$$x_p = \frac{F_0}{m\sqrt{(2\omega p)^2 + (\omega_0^2 - \omega^2)^2}}\cos(\omega t - \delta).$$

If $\omega = \omega_0$, then $A = 0$, $B = C = \dfrac{F_0}{2m\omega p}$, and $\delta = \pi/2$.

For reasons we will explain in a moment, we call x_c the *transient solution* and denote it by x_{tr}. We call the x_p from above the *steady periodic solution* and denote it by x_{sp}. The general solution is

$$x = x_c + x_p = x_{tr} + x_{sp}.$$

The transient solution $x_c = x_{tr}$ goes to zero as $t \to \infty$, as all the terms involve an exponential with a negative exponent. So for large t, the effect of x_{tr} is negligible and we see essentially only x_{sp}. Hence the name *transient*. Notice that x_{sp} involves no arbitrary constants, and the initial conditions only affect x_{tr}. Thus, the effect of the initial conditions is negligible after some period of time. We might as well focus on the steady periodic solution and ignore the transient solution. See **Normally a reference to a previous figure goes here.** for a graph given several different initial conditions.

The speed at which x_{tr} goes to zero depends on p (and hence γ). The bigger p is (the bigger γ is), the "faster" x_{tr} becomes negligible. So the smaller the damping, the longer the "transient region". This is consistent with the observation that when $\gamma = 0$, the initial conditions affect the behavior for all time (i.e. an infinite "transient region").

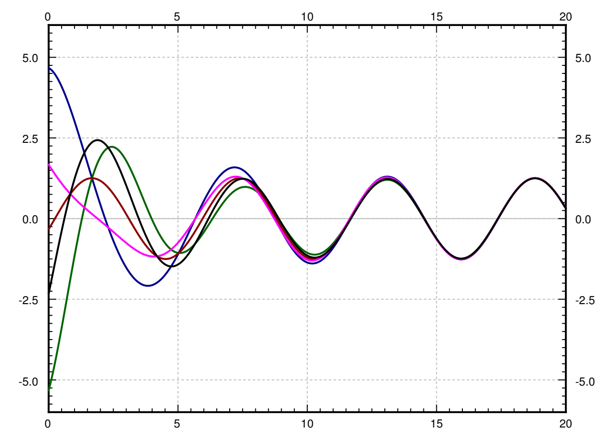

Figure 4: Solutions with different initial conditions for parameters $k = 1$, $m = 1$, $F_0 = 1$, $\gamma = 0.7$, and $\omega = 1.1$.

Let us describe what we mean by resonance when damping is present. Since there were no conflicts when solving with undetermined coefficient, there is no term that goes to infinity. We look instead at the maximum value of the amplitude of the steady periodic solution. Let C be the amplitude of x_{sp}. If we plot C as a function of ω (with all other parameters fixed), we can find its maximum. We call the ω that achieves this maximum the *practical resonance frequency*. We call the maximal amplitude $C(\omega)$ the *practical resonance amplitude*. Thus when damping is present we talk of *practical resonance* rather than pure resonance. A sample plot for three different values of γ is given in **Normally a reference to a previous figure goes here.**. As you can see the practical resonance amplitude grows as damping gets smaller, and practical resonance can disappear altogether when damping is large.

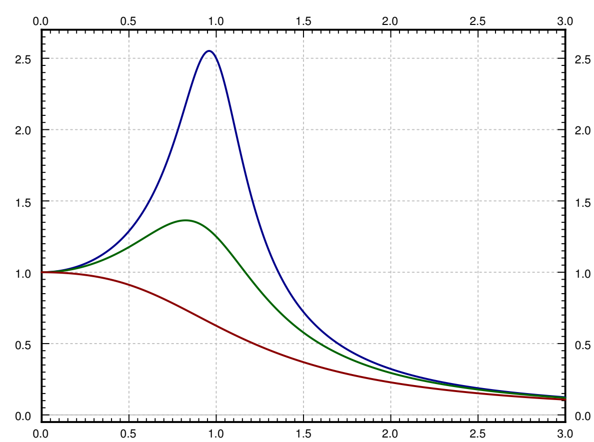

Figure 5: Graph of $C(\omega)$ showing practical resonance with parameters $k = 1$, $m = 1$, $F_0 = 1$. The top line is with $\gamma = 0.4$, the middle line with $\gamma = 0.8$, and the bottom line with $\gamma = 1.6$.

The main takeaways from **Normally a reference to a previous figure goes here.** is that the amplitude can be larger than 1, which is the idea of resonance in this case. Based on Hooke's law, we know that a constant force of magnitude F_0 will stretch (or compress) a spring with constant k a length of F_0/k. If we take $F_0 = 1$ and $k = 1$, as is done in **Normally a reference to a previous figure goes here.**, then the resulting magnitude should be 1. However, if we don't use a constant force of magnitude F_0, but instead use an oscillatory force with frequency ω of the form $F(t) = F_0 \cos(\omega t)$, we get an amplitude of $C(\omega)$. This graph indicates how the forcing frequency changes the amplitude of the resulting oscillation. Since the amplitude "should" be 1 based on F_0/k, if $C(\omega) > 1$, then the frequency chosen is causing an increase in the amplitude, which is the idea of practical resonance.

To find the maximum, or determine if there is a maximum, we need to find the derivative $C'(\omega)$. Computation shows

$$C'(\omega) = \frac{-2\omega(2p^2 + \omega^2 - \omega_0^2)F_0}{m\big((2\omega p)^2 + (\omega_0^2 - \omega^2)^2\big)^{3/2}}.$$

This is zero either when $\omega = 0$ or when $2p^2 + \omega^2 - \omega_0^2 = 0$. In other words, $C'(\omega) = 0$ when

$$\omega = \sqrt{\omega_0^2 - 2p^2} \quad \text{or} \quad \omega = 0.$$

If $\omega_0^2 - 2p^2$ is positive, then there is a positive value of ω, namely $\omega = \sqrt{\omega_0^2 - 2p^2}$ where the amplitude attains a maximum value. Since we know that the amplitude is $F_0/(m\omega_0^2)$ or F_0/k when $\omega = 0$, the maximum will be larger than this. As described above, this value, F_0/k is the expected amplitude, that is, the amplitude you would get with no oscillation, so that if the amplitude is larger than this for some value of ω, this means that the oscillation at frequency ω is resonating with the system to create a larger oscillation. This is the idea of *practical resonance*. It is practical because there is damping, so the situation is more physically relevant (to contrast with pure resonance), and still results in larger amplitudes of oscillation.

Our previous work indicates that a system will exhibit practical resonance for some values of ω whenever $\omega_0^2 - 2p^2$ is positive, and the frequency where the amplitude hits the maximum value is at $\sqrt{\omega_0^2 - 2p^2}$. This follows by the first derivative test for example as then $C'(\omega) > 0$ for small ω in this case. If on the other hand $\omega_0^2 - 2p^2$ is not positive, then $C(\omega)$ achieves its maximum at $\omega = 0$, and there is no practical resonance since we assume $\omega > 0$ in our system. In this case the amplitude gets larger as the forcing frequency gets smaller.

If practical resonance occurs, the peak frequency is smaller than ω_0. As the damping γ (and hence p) becomes smaller, the peak practical resonance frequency goes to ω_0. So when damping is very small, ω_0 is a good estimate of the peak practical resonance frequency. This behavior agrees with the observation that when $\gamma = 0$, then ω_0 is the resonance frequency.

Another interesting observation to make is that when $\omega \to \infty$, then $C \to 0$. This means that if the forcing frequency gets too high it does not manage to get the mass moving in the mass-spring system. This is quite reasonable intuitively. If we wiggle back and forth really fast while sitting on a swing, we will not get it moving at all, no matter how forceful. Fast vibrations just cancel each other out before the mass has any chance of responding by moving one way or the other.

The behavior is more complicated if the forcing function is not an exact cosine wave, but for example a square wave. A general periodic function will be the sum (superposition) of many cosine waves of different frequencies. The reader is encouraged to come back to this section once we have learned about the ideas of Fourier series.