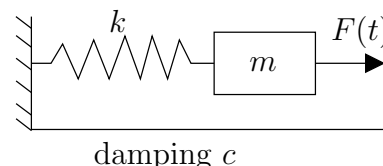


Mechanical vibrations

We discuss Mechanical vibrations

In the last few sections, we have discussed all of the different possible solutions to constant coefficient second order differential equations, whether the roots of the characteristic polynomial real and distinct, complex, or repeated. Now, we want to look at applications of these equations, now that we know how to solve them. Since Newton's Second Law $F = ma$ involves the second derivative of position (acceleration), it is reasonable that a lot of physical systems will be defined by second order differential equations.

Our first example is a mass on a spring. Suppose we have a mass $m > 0$ (in kilograms) connected by a spring with spring constant $k > 0$ (in newtons per meter) to a fixed wall. There may be some external force $F(t)$ (in newtons) acting on the mass. Finally, there is some friction measured by $c \geq 0$ (in newton-seconds per meter) as the mass slides along the floor (or perhaps a damper is connected).



Let x be the displacement of the mass ($x = 0$ is the rest position), with x growing to the right (away from the wall). The force exerted by the spring is proportional to the compression of the spring by Hooke's law. Therefore, it is kx in the negative direction. Similarly the amount of force exerted by friction is proportional to the velocity of the mass. By Newton's second law we know that force equals mass times acceleration and hence $mx'' = F(t) - cx' - kx$ or

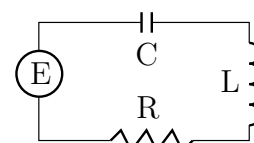
$$mx'' + cx' + kx = F(t).$$

This is a linear second order constant coefficient ODE. We say the motion is

- (a) *forced*, if $F \not\equiv 0$ (if F is not identically zero),
- (b) *unforced* or *free*, if $F \equiv 0$ (if F is identically zero),
- (c) *damped*, if $c > 0$, and
- (d) *undamped*, if $c = 0$.

This system appears in lots of applications even if it does not at first seem like it. Many real-world scenarios can be simplified to a mass on a spring. For example, a bungee jump setup is essentially a mass and spring system (you are the mass). It would be good if someone did the math before you jump off the bridge, right? Let us give two other examples.

Here is an example for electrical engineers. Consider the pictured RLC circuit. There is a resistor with a resistance of R ohms, an inductor with an inductance of L henries, and a capacitor with a capacitance of C farads. There is also an electric source (such as a battery) giving a voltage of $E(t)$ volts at time t (measured in seconds). Let $Q(t)$ be the charge in coulombs on the capacitor and $I(t)$ be the current in the circuit. The relation between the two is $Q' = I$. By elementary principles we find $LI' + RI + Q/C = E$. Since $Q' = I$, this means that $I' = Q''$, and we can write this equation as



$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t).$$

We can also write this a different way by differentiating the entire equation in t to get a second order equation for $I(t)$:

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = E'(t).$$

This is a nonhomogeneous second order constant coefficient linear equation. As L , R , and C are all positive, this system behaves just like the mass and spring system. Position of the mass is replaced by current. Mass is replaced by inductance, damping is replaced by resistance, and the spring constant is replaced by one over the capacitance. The change in voltage becomes the forcing function—for constant voltage this is an unforced motion.

Learning outcomes: Write second-order differential equations to model physical situations Classify a mechanical oscillation as undamped, underdamped, critically damped, or overdamped Use the solution to a differential equation to describe the resulting physical motion.

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Our next example behaves like a mass and spring system only approximately. Suppose a mass m hangs on a pendulum of length L . We seek an equation for the angle $\theta(t)$ (in radians). Let g be the force of gravity. Elementary physics mandates that the equation is

$$\theta'' + \frac{g}{L} \sin \theta = 0.$$

Let us derive this equation using Newton's second law: force equals mass times acceleration. The acceleration is $L\theta''$ and mass is m . So $mL\theta''$ has to be equal to the tangential component of the force given by the gravity, which is $mg \sin \theta$ in the opposite direction. So $mL\theta'' = -mg \sin \theta$. The m curiously cancels from the equation.

Now we make our approximation. For small θ we have that approximately $\sin \theta \approx \theta$. This can be seen by looking at the graph. In **Normally a reference to a previous figure goes here.** we can see that for approximately $-0.5 < \theta < 0.5$ (in radians) the graphs of $\sin \theta$ and θ are almost the same.

Therefore, when the swings are small, θ is small and we can model the behavior by the simpler linear equation

$$\theta'' + \frac{g}{L} \theta = 0.$$

The errors from this approximation build up. So after a long time, the state of the real-world system might be substantially different from our solution. Also we will see that in a mass-spring system, the amplitude is independent of the period. This is not true for a pendulum. Nevertheless, for reasonably short periods of time and small swings (that is, only small angles θ), the approximation is reasonably good.

In real-world problems it is often necessary to make these types of simplifications. We must understand both the mathematics and the physics of the situation to see if the simplification is valid in the context of the questions we are trying to answer.

Free undamped motion

In this section we only consider free or unforced motion, as we do not know yet how to solve nonhomogeneous equations. Let us start with undamped motion where $c = 0$. The equation is

$$mx'' + kx = 0.$$

We divide by m and let $\omega_0 = \sqrt{k/m}$ to rewrite the equation as

$$x'' + \omega_0^2 x = 0.$$

The general solution to this equation is

$$x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t).$$

By a trigonometric identity that we discussed previously in § ??,

$$A \cos(\omega_0 t) + B \sin(\omega_0 t) = C \cos(\omega_0 t - \delta),$$

for two constants C and γ . Earlier, we found that we can compute these constants as $C = \sqrt{A^2 + B^2}$ and $\tan \delta = B/A$. Therefore, we let C and δ be our arbitrary constants and write $x(t) = C \cos(\omega_0 t - \delta)$.

Exercise 1 Justify the identity $A \cos(\omega_0 t) + B \sin(\omega_0 t) = C \cos(\omega_0 t - \delta)$ and verify the equations for C and δ . Hint: Start with $\cos(\alpha - \beta) = \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta)$ and multiply by C . Then what should α and β be?

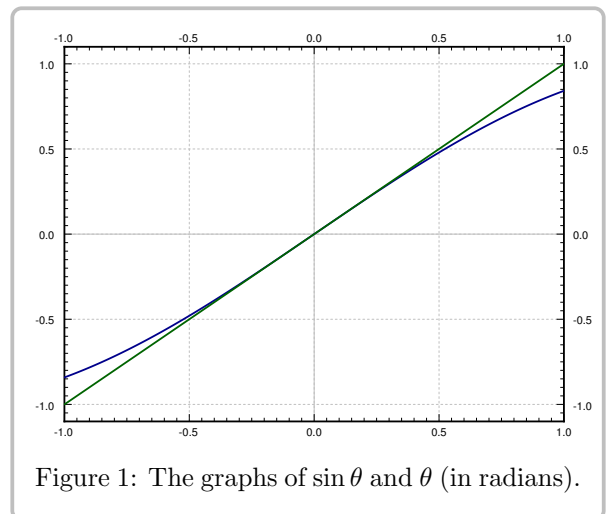
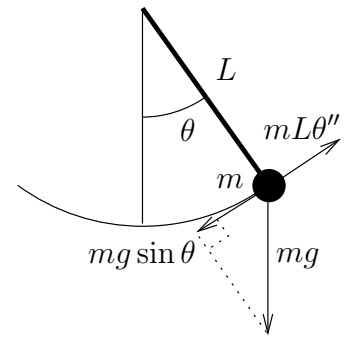


Figure 1: The graphs of $\sin \theta$ and θ (in radians).

While it is generally easier to use the first form with A and B to solve for the initial conditions, the second form is much more natural to use for interpretation of physical systems, since the constants C and δ have nice physical interpretation. Write the solution as

$$x(t) = C \cos(\omega_0 t - \delta).$$

This is a pure-frequency oscillation (a sine wave). The *amplitude* is C , ω_0 is the (angular) *frequency*, and δ is the so-called *phase shift*. The phase shift just shifts the graph left or right. We call ω_0 the *natural (angular) frequency*. This entire setup is called *simple harmonic motion*.

Let us pause to explain the word *angular* before the word *frequency*. The units of ω_0 are radians per unit time, not cycles per unit time as is the usual measure of frequency. Because one cycle is 2π radians, the usual frequency is given by $\frac{\omega_0}{2\pi}$. It is simply a matter of where we put the constant 2π , and that is a matter of taste.

The *period* of the motion is one over the frequency (in cycles per unit time) and hence $\frac{2\pi}{\omega_0}$. That is the amount of time it takes to complete one full cycle.

Example 1. Suppose that $m = 2\text{ kg}$ and $k = 8\text{ N/m}$. The whole mass and spring setup is sitting on a truck that was traveling at 1 m/s . The truck crashes and hence stops. The mass was held in place 0.5 meters forward from the rest position. During the crash the mass gets loose. That is, the mass is now moving forward at 1 m/s , while the other end of the spring is held in place. The mass therefore starts oscillating. What is the frequency of the resulting oscillation? What is the amplitude? The units are the mks units (meters-kilograms-seconds).

Solution: The setup means that the mass was at half a meter in the positive direction during the crash and relative to the wall the spring is mounted to, the mass was moving forward (in the positive direction) at 1 m/s . This gives us the initial conditions.

So the equation with initial conditions is

$$2x'' + 8x = 0, \quad x(0) = 0.5, \quad x'(0) = 1.$$

We directly compute $\omega_0 = \sqrt{k/m} = \sqrt{4} = 2$. Hence the angular frequency is 2. The usual frequency in Hertz (cycles per second) is $2/2\pi = 1/\pi \approx 0.318$.

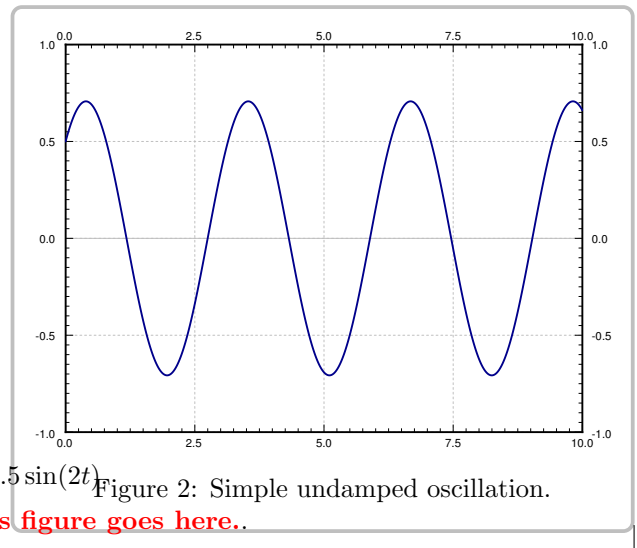
The general solution is

$$x(t) = A \cos(2t) + B \sin(2t).$$

Letting $x(0) = 0.5$ means $A = 0.5$. Then $x'(t) = -2(0.5)\sin(2t) + 2B\cos(2t)$. Letting $x'(0) = 1$ we get $B = 0.5$. Therefore, the amplitude is $C = \sqrt{A^2 + B^2} = \sqrt{0.25 + 0.25} = \sqrt{0.5} \approx 0.707$. The solution is

$$x(t) = 0.5 \cos(2t) + 0.5 \sin(2t).$$

A plot of $x(t)$ is shown in [Normally a reference to a previous figure goes here.](#)



In general, for free undamped motion, a solution of the form

$$x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t),$$

corresponds to the initial conditions $x(0) = A$ and $x'(0) = \omega_0 B$. Therefore, it is easy to figure out A and B from the initial conditions. The amplitude and the phase shift can then be computed from A and B . In the example, we have already found the amplitude C . Let us compute the phase shift. We know that $\tan \delta = B/A = 1$. We take the arctangent of 1 and get $\pi/4$ or approximately 0.785. We still need to check if this δ is in the correct quadrant (and add π to δ if it is not). Since both A and B are positive, then δ should be in the first quadrant, $\pi/4$ radians is in the first quadrant, so $\delta = \pi/4$.

Note: Many calculators and computer software have not only the `atan` function for arctangent, but also what is sometimes called `atan2`. This function takes two arguments, B and A , and returns a δ in the correct quadrant for you.

Free damped motion

Let us now focus on damped motion. Let us rewrite the equation

$$mx'' + \gamma x' + kx = 0,$$

as

$$x'' + 2px' + \omega_0^2 x = 0,$$

where

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad p = \frac{\gamma}{2m}.$$

The characteristic equation is

$$r^2 + 2pr + \omega_0^2 = 0.$$

Using the quadratic formula we get that the roots are

$$r = -p \pm \sqrt{p^2 - \omega_0^2}.$$

The form of the solution depends on whether we get complex or real roots. We get real roots if and only if the following number is nonnegative:

$$p^2 - \omega_0^2 = \left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m} = \frac{\gamma^2 - 4km}{4m^2}.$$

The sign of $p^2 - \omega_0^2$ is the same as the sign of $\gamma^2 - 4km$. Thus we get real roots if and only if $\gamma^2 - 4km$ is nonnegative, or in other words if $\gamma^2 \geq 4km$. If these look familiar, that is not surprising, as they are the same as the conditions we had for the different types of roots in second order constant coefficient equations.

Overdamping

When $\gamma^2 - 4km > 0$, the system is *overdamped*. In this case, there are two distinct real roots r_1 and r_2 . Both roots are negative: As $\sqrt{p^2 - \omega_0^2}$ is always less than p , then $-p \pm \sqrt{p^2 - \omega_0^2}$ is negative in either case.

The solution is

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

Since r_1, r_2 are negative, $x(t) \rightarrow 0$ as $t \rightarrow \infty$. Thus the mass will tend towards the rest position as time goes to infinity. For a few sample plots for different initial conditions, see **Normally a reference to a previous figure goes here..**

No oscillation happens. In fact, the graph crosses the x -axis at most once. To see why, we try to solve $0 = C_1 e^{r_1 t} + C_2 e^{r_2 t}$. Therefore, $C_1 e^{r_1 t} = -C_2 e^{r_2 t}$ and using laws of exponents we obtain

$$\frac{-C_1}{C_2} = e^{(r_2 - r_1)t}.$$

This equation has at most one solution $t \geq 0$. For some initial conditions the graph never crosses the x -axis, as is evident from the sample graphs.

Example 2. Suppose the mass is released from rest. That is $x(0) = x_0$ and $x'(0) = 0$. Then

$$x(t) = \frac{x_0}{r_1 - r_2} (r_1 e^{r_2 t} - r_2 e^{r_1 t}).$$

It is not hard to see that this satisfies the initial conditions.

Critical damping

When $\gamma^2 - 4km = 0$, the system is *critically damped*. In this case, there is one root of multiplicity 2 and this root is $-p$. Our solution is

$$x(t) = C_1 e^{-pt} + C_2 t e^{-pt}.$$

The behavior of a critically damped system is very similar to an overdamped system. After all a critically damped system is in some sense a limit of overdamped systems. Even though our models are only approximations of the real world problem, the idea of critical damping can be helpful in optimizing systems.

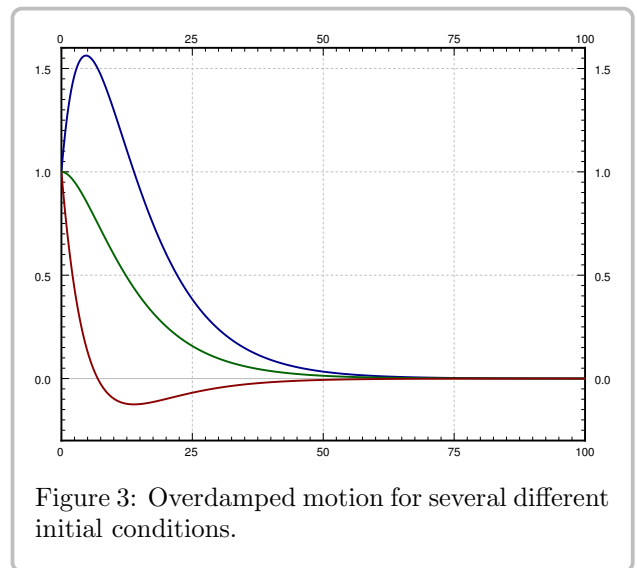
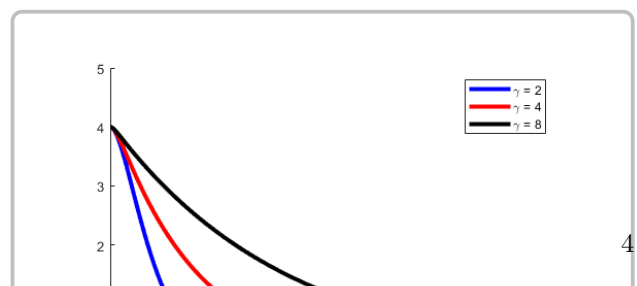


Figure 3: Overdamped motion for several different initial conditions.



Normally a reference to a previous figure goes here. shows how the solution to

$$x'' + \gamma x' + x = 0$$

for different values of γ and initial conditions $x(0) = 4$ and $x'(0) = 0$. This solution is critically damped if $\gamma = 2$, as that will give us a repeated root in the characteristic equation. Comparing these solutions, we see that the critically damped solution gets back to equilibrium faster than any of the more overdamped solution. When trying to design a system, if we want it to settle back to the zero point as quickly as possible, then we should try to get as close to critically damped as possible. Even though we are always a little bit underdamped or a little bit overdamped, getting as close as possible will give the best possible result for returning to equilibrium.

Underdamping

When $\gamma^2 - 4km < 0$, the system is *underdamped*. In this case, the roots are complex.

$$\begin{aligned} r &= -p \pm \sqrt{p^2 - \omega_0^2} \\ &= -p \pm \sqrt{-1} \sqrt{\omega_0^2 - p^2} \\ &= -p \pm i\omega_1, \end{aligned}$$

where $\omega_1 = \sqrt{\omega_0^2 - p^2}$. Our solution is

$$x(t) = e^{-pt} (A \cos(\omega_1 t) + B \sin(\omega_1 t)),$$

or

$$x(t) = C e^{-pt} \cos(\omega_1 t - \delta).$$

An example plot is given in **Normally a reference to a previous figure goes here.** Note that we still have that $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

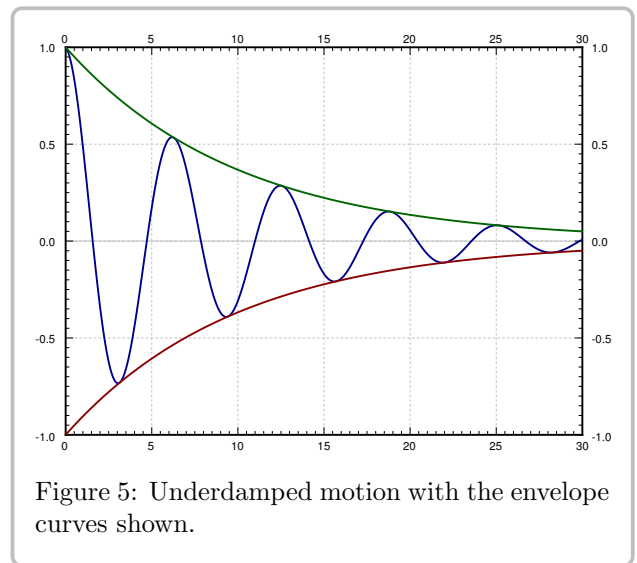


Figure 5: Underdamped motion with the envelope curves shown.

The figure also shows the *envelope curves* Ce^{-pt} and $-Ce^{-pt}$. The solution is the oscillating line between the two envelope curves. The envelope curves give the maximum amplitude of the oscillation at any given point in time. For example, if you are bungee jumping, you are really interested in computing the envelope curve as not to hit the concrete with your head.

The phase shift δ shifts the oscillation left or right, but within the envelope curves (the envelope curves do not change if δ changes).

Notice that the angular *pseudo-frequency*¹ or *quasi-frequency* becomes smaller when the damping γ (and hence p) becomes larger. This makes sense. When we change the damping just a little bit, we do not expect the behavior of the solution to change dramatically. If we keep making γ larger, then at some point the solution should start looking like the solution for critical damping or overdamping, where no oscillation happens. So if γ^2 approaches $4km$, we want ω_1 to approach 0.

Since $\omega_1 = \sqrt{\omega_0^2 - p^2}$ with $p = \frac{\gamma}{2m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$, we have that

$$\omega_1 = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}} = \sqrt{\frac{4mk - \gamma^2}{4m^2}},$$

which does go to zero as γ^2 gets closer to $4mk$.

On the other hand, when γ gets smaller, ω_1 approaches ω_0 (ω_1 is always smaller than ω_0), and the solution looks more and more like the steady periodic motion of the undamped case. The envelope curves become flatter and flatter as γ (and hence p) goes to 0.

¹We do not call ω_1 a frequency since the solution $x(t)$ is not really a periodic function.