Self-Organized Criticality & The Abelian Sandpile Model

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Abstract

This paper intends to introduce the reader to the concept of Self-Organized Criticality. Key concepts and results are illustrated via the The Abelian Sandpile Model which was introduced in the Bak-Tang-Wiesenfeld 1987 paper *Self-Organized Criticality: An Explanation of 1/f Noise.* [1].

We characterize the statistically stationary state of the model and show that the mean number of grain sand per lattice site follows a normal distribution with mean of approximately 2.06. We also show that avalanche properties follow a power distribution and calculate these coefficients via simulations. Furthermore, simulations show that height values 0, 1, 2, 3 converge to a normal distribution in the stationary state assuming the addition of a sand grain is randomly uniformly distributed over all sites. From this we show that in the stationary state, the proportion of the grid sub-critical is approximately 0.4143 for a 20x20 lattice. From this we demonstrate that the times between avalanches in the stationary state are geometrically distributed with rate 0.4143.

We conclude our discussion by introducing the General Wildfire Model which is a common extension of the Abelian Sandpile Model. We build upon the extensions given by F. Schwab1 B. Drossel [2] by introducing a wind component and we show that global behaviour of the system influenced by wind can be achieved by only a local definition. We discuss fire cluster density formation and how this wind component extension directly affects this property.

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Chapter 1

Self-Organized Criticality & The Abelian Sandpile Model

1.1 Introduction

1.1.1 What is Self-Organized Criticality?

Self-Organized Criticality (SOC) is a property of certain dynamical systems in which they evolve towards a critical point irrespective of initial conditions. Their global behaviour displays certain spatial and/or temporal scale-invariance characteristics of their critical point without needing to tune parameters.

This concept was first published in the Bak-Tang-Wiesenfeld 1987 paper *Self-Organized Criticality: An Explanation of 1/f Noise.* [1]. Ever since the concept of Self-Organized Criticality has been applied in various fields.

There are various models which display SOC, however, according to Vespignanni & Zapperi 1998 [3], these models fall into two main categories: The first category is called the stochastic models which contain systems with a stochastic dynamic within a deterministic environment. The second category contains systems with deterministic dynamics within a stochastic environment. For the rest of this exposition we will be focusing on the first category, more notably The Abelian Sandpile Model.

1.1.2 The Abelian Sandpile Model

The Abelian Sandpile Model is the most influential model to exhibit Self-Organized Criticality which was first introduced in 1987 by Bak, Tang, Wiesenfeld [1]. The Abelian Sandpile Model is an example of a celluar automation model in which like mentioned before is a model which operates within a fixed determined environment however its evolution rules are governed by a random dynamic. What makes The Abelian Sandpile Model so special is that it is one of few models to exhibit Self-Organized Criticality, meaning it converges to a crtical point without the need to finely tune parameters. This is the model which has formed the basis for much research over the last 35 years. We provide an explicit formulation of The Abelian Sandpile Model below.

Let L_N^d be a d dimensional square lattice with side length N. Hence we can identify this lattice with the d dimensional quotient set $(\mathbb{Z}/\mathbb{Z}(N-1))^d$. This lattice is the environment in which our sand pile model will operate. Define a map $p : L_N^d \longrightarrow \mathbb{N}_0$ send a point in the lattice to some non-negative integer representing the number of sand grains on that lattice site. Define a stability threshold constant $k \in \mathbb{Z}_{\geq 4}$. This constant defines the point at which a site of sand piles becomes critical, meaning it is ready to topple.

Hence we can define a stable lattice as one in which $\forall x \in L_N^d$ we have that $p(x) < k$. By extension an unstable lattice is one in which $\forall x \in L_N^d$ there exists a site $x \in L_N^d$ such $p(x) \geq k$

The model evolves by adding a sand grain to an arbitrary site each time step. If this addition of a sand grain does not cause a site to become critical then we initiate the next time step, again by adding a sand grain randomly. If however, the addition of a sand grain causes a site to become critical then we initiate a toppling of that site and by extension we define an avalanche as a chain of topples which takes a lattice from critical to stable.

For an unstable lattice we introduce a toppling matrix which defines a toppling of a critical site.

$$\triangle(x,y) = \begin{cases} -2d & \text{x = y} \\ 1 & \text{x,y nearest neighbours} \\ 0 & \text{else} \end{cases}$$

This toppling matrix $\triangle(x,y)$ takes as input a critical site x and updates site x and its nearest neighbour sites y as described above. The idea here is that once a site is critical it should collapse and redistribute the collapsing grains to nearest neighbours. For a d dimensional lattice we would expect a critical site to collapse sending 2 grains in each dimension, one for the positive direction and one for hte negative direction. This defined a symmetric redistribution of one grain sand in every orthogonal direction for an arbitrary critical site collapse. We also note that this definition of a toppling matrix implies the stability threshold k unique up to the dimension of the lattice. Since the stability threshold k was defined as the minimum integer such that a site is critical we see that for a lattice of dimension d the stability threshold k must be $2d$.

We now want to define a chain of topples which we call an avalanche. This chain of topples should map a critical lattice to a stable lattice via a finite composition of the toppling matrix $\triangle(x,y)$ acting on a critical lattice.

We define the toppling operator T_x which maps a a lattice configuration p to a new lattice configuration p'. This new lattice configuration is the same as the last except with site x toppled.

$$(T_x p)(y) = \begin{cases} p(y) + \triangle(x,y) & p(x) \geq \triangle(x,y) \\ p(y) & \text{else} \end{cases}$$

Claim: T_x and $T_{x'}$ commute for unstable configurations.

Proof: Assuming the definition of the toppling operator T_x as above we get;

$$(T_x \circ T_{x'} p)(y) = \begin{cases} T_x(p)(y) + \triangle(x',y) & p(x') \geq 2d \\ T_x(p)(y) & p(x') < 2d \end{cases}$$

$$= \begin{cases} p(y) + \triangle(x,y) + \triangle(x',y) & p(x) \geq 2d \ \&\ p(x') \geq 2d \\ p(y) + \triangle(x',y) & p(x) < 2d \ \&\ p(x') \geq 2d \\ p(y) & p(x) < 2d \ \&\ p(x') < 2d \end{cases}$$

We apply the swap $x' \leftrightarrow x$ for the middle term. We can do this because y being an orthogonal nearest neighbour of x' with x' critical gives $\triangle(x',y) = 1$ which is equivalent to y being an orthogonal nearest neighbour to x with x critical giving $\triangle(x,y) = 1$. All other cases give $\triangle(x',y) = \triangle(x,y) = 0$ for y not a nearest neighbour of x and x' and $\triangle(x',y) = \triangle(x,y) = 1$ for y being an orthogonal nearest neighbour of both x and x' which also allow this swap. And top swap trivial due to commutativity of addition.

$$= \begin{cases} p(y) + \triangle(x',y) + \triangle(x,y) & p(x') \geq 2d \ \&\ p(x) \geq 2d \\ p(y) + \triangle(x,y) & p(x') < 2d \ \&\ p(x) \geq 2d \\ p(y) & p(x) < 2d \ \&\ p(x') < 2d \end{cases}$$

$$= \begin{cases} T_{x'}(p)(y) + \triangle(x,y) & p(x) \geq 2d \\ T_{x'}(p)(y) & p(x) < 2d \end{cases}$$

$$= (T_{x'} \circ T_x p)(y))$$

$\square$

Claim: Avalanches are path connected

Proof: Let L_N^d be a d dimensional lattice with side length N. Let $p(L_N^d)$ be an arbitrary stable configuration of the lattice with respect to a stability threshold $k \in \mathbb{N}$. Hence, $\forall x \in L_N^d$ $p(x) \in \{0, 1, 2, ..., k-1\}$. Suppose we add a sand grain to the lattice which induces an avalanche. Then we only have a single critical site on the lattice, the site that the sand grain landed on. Let $x \in L_N^d$ be this critical site. Hence $p(x) = k$. Applying the topple operator we get the following mappings $\forall x' \in L_N^d$:

$$p(x') \longrightarrow \begin{cases} p(x') - k & \text{x' = x} \\ p(x') + 1 & \text{such x' is an orthogonal nearest neighbour to x} \\ p(x') & \text{else} \end{cases}$$

The avalanche ends here if all the orthogonal nearest neighbours are not critical after this mapping. If this happens then we have a path connected component of length 1 and we are done.

Suppose the avalanche does not end here. Then after the before mentioned mapping, at least one of the orthogonal nearest neighbour sites y_i for $i \in \{1, 2, ..., 2d\}$ are critical. Suppose y' is a critical site. Then after applying the toppling operator we get the before mentioned mapping which gives sends $p(y')$ to $p(y') - k$ and $p(y)$ to $p(y) + 1$ for y an orthogonal nearest neighbour of y'. Hence this mapping gives another step in our avalanche path. We continue this process until the toppling operator gives a non-critical configuration. At this point we have a series of topplings whereby each toppling occurred as an orthogonal nearest neighbour of the previous toppling. But this is just a connected path in L_N^d. $\square$

In order to properly define an avalanche we need to topple sites until we reach a stable configuration. If there is a single critical site on the lattice then a single iteration of the toppling matrix will be sufficient to define an avalanche, the trivial avalanche consisting of only a single topple. However, in general, there are always multiple critical sites which can only become stable through a chain of topples.

Hence we define the stabilization operator which maps a critical configuration to a stable configuration through a finite composition of the toppling matrix $\triangle(x, y)$.

$$\Gamma : U_{L_N^d} \longrightarrow S_{L_N^d}$$

where $U_{L_N^d}$ is the space of all unstable configurations and $S_{L_N^d}$ is the space of all stable configurations.

Letting x_i be a critical site for an i^{th} composition of the toppling operator we can explicitly define the stabilisation operator as follows, where M is the number of instabilities of the avalanche.

$$\Gamma = \prod_{i=1}^{M} T_{x_i}$$

Claim: Γ is well defined.

Proof: Using the stabilisation operator definition above we get:

$\Gamma = \prod_{i=1}^{M} T_{x_i} = T_{(i_M^1, i_M^2, ..., i_M^d)} \circ T_{(i_{M-1}^1, i_{M-1}^2, ..., i_{M-1}^d} \circ ... \circ T_{(i_1^1, i_1^2, ..., i_1^d}(p)$ (1)

where $\forall \alpha \in \{2, 3, ..., M\}$ exactly one and only one of the following d-many equalities must be true;

$i_\alpha^1 = i_{\alpha-1}^1$, $i_\alpha^2 = i_{\alpha-1}^2$, $i_\alpha^3 = i_{\alpha-1}^3$, ..., $i_\alpha^d = i_{\alpha-1}^d$ (2)

(1) Is true due to expansion of product of operators.

(2) Is true due to the path connectedness of avalanches property proved earlier. This is because the expansion of (1) gives a path connected avalanche in L_N^d. This means every topple is an orthogonal nearest neighbour of the previous with the exception of the catalyst (avalanche initiating) topple. Hence an equality of one of the coordinates of (2) is the equality of the common axial direction of two orthogonal nearest neighbour topples in L_N^d.

Furthermore, by the path connectedness property proved earlier, avalanches are unique up to path connectedness. Meaning, the ending configuration of an avalanche is unique with respect to the initial topple location and this uniqueness holds for any path this avalanche takes assuming this path is connected. Hence we conclude that (1) is also unique up to rearrangement of operator such that condition (2) is holds. Hence we conclude that Γ is well defined. $\square$

Key Concept 1: The Abelian Sandpile Model is Ergodic

We claim that the abelian sand pile model is ergodic once it enters its statistically stationary state. This is because, once in the stationary state, all system observables are stationary in distribution. Hence sampling the system at any point in the stationary state is no different to sampling an ensemble of the same system also in the stationary state. This is because the distribution of system observables in the stationary state is fixed with constant mean. Hence, a hundred samples taken from a single system in this stationary state has the same probabilistic properties of taking a hundred samples from a hundred systems in stationary state. The idea here is that it is impossible to tell whether those hundred samples came from the same system or a hundred systems.

Key Concept 2: The Abelian Sandpile Model exhibits Self-Organized Criticality For a system to exhibit Self-Organized Criticality, it must converge to a critical state with respect to time independent of initial conditions. In the context of The Abelian Sandpile Model, the system will enter a statistically stationary state with probability one if given enough time. This convergence is independent of initial configurations and topology of lattice assuming finite boundaries. We show this via simulations in the sections following.

Key Concept 3: Finite Boundaries imply Stationarity

If we did not have finite boundaries then the system would never enter a statistically stationary state. This is because, without finite boundaries, our lattice area is infinite and hence sand grains will disperse over this area over time. Because of this, the mean number of sand grains will continually increase with respect to time. The rate of this increase will be dependent on the topology of the boundaries. If however we had finite boundaries, then there would be an amount of sand grains falling off the edge every time a sand grain is added to the system. This amount could be zero, however, after many additions of sand grains, eventually this amount will become non-zero. We must first note that with finite boundaries, the mean number of sand grains cannot increase indefinitely. This is because the most amount of sand grain in the lattice at any one time is bounded above by $(k-1)$ multiplied by the number of lattice sites, where k is the critical threshold. The mean may increase but eventually there will be amount of sand grains falling off the edge associated with the addition of a sand grain.

1.2 Simulation Results

1.2.1 Statistical Stationarity & The Power Law Relationship

One of the special properties of The Abelian Sandpile Model is that it will enter a state of statistical stationarity if given enough time. In this state all system observables will follow a stationary distribution. Furthermore, specific correlated avalanche properties will demonstrate power law distributions in this statistical stationary state. We first introduce some definitions in order for us to investigate these concepts.

Definition (Steady State):

A system is in a Steady State when properties of the system are unchanging over time.

Definition (Equilibrium State):

A system is in an Equilibrium State when there is zero variation in the system over time.

Definition (Statistically Stationary State):

A system is in a Statistical Stationary State when all system observable properties are distributed with constant parameters.

Definition (Power Law Relationship):

A power law is a functional relationship between two quantities whereby a relative change in one quantity results in a proportional relative change in the other quantity independent of the initial size of those quantities: Meaning one quantity varies as a power of the other quantity.

If a system exhibits scale invariance then the system looks the same over varying scales, meaning properties deduced from the system are scale invariant. So if f is some property of the system given the current state of the system and some scaling c then variation in this scaling constant c will produce a relative variation in f up to a power of c. This is so that the property is invariant under scaling. But this is exactly the definition of a power law.

We expect the abelian sand pile model to reach a statistically stationary state if given enough time. We do not expect the system to enter an equilibrium state. For an equilibrium state we would expect that system observables are in equilibrium, meaning they are constant with respect to time. We do not expect the system to enter a steady state. For a steady state, the system would have to have system observables such as the number of sand grains fixed with respect to time. For this to occur the addition of a sand grain(s) would have equal the sand grain(s) lost to the boundaries. In the next section we explain how to identify the statistical stationary state.

Identifying the Statistical Stationary State.

In order for us to identify the stationary state we must first define a system observable which will imply the stationary state. For the abelian sand pile model, we use the mean number of sand grains on the lattice as this system observable. We assume this as an appropriate metric to imply stationarity because we know that in the stationary state all system observables will follow a stationary distribution. We define the mean number of sand grains on the lattice L_N^d at any time t as follows:

$$\langle p_t \rangle = \frac{1}{N^2} \sum_{i,j} p_t(i,j)$$

where we sum over all sites $(i,j) \in L_N^d$ for some time t.

We hypothesis that in the stationary state our mean number of sand grains will fluctuate round 2 or just above 2. This is because, in the stationary state, we may assume that the lattice is dense. Meaning almost every site has at least one sand grain except for one site. This site which does not have the sand grain is the site where the last topple occurred. By this assumption, we average over the site values 1,2,3 in order to get a value of 2.

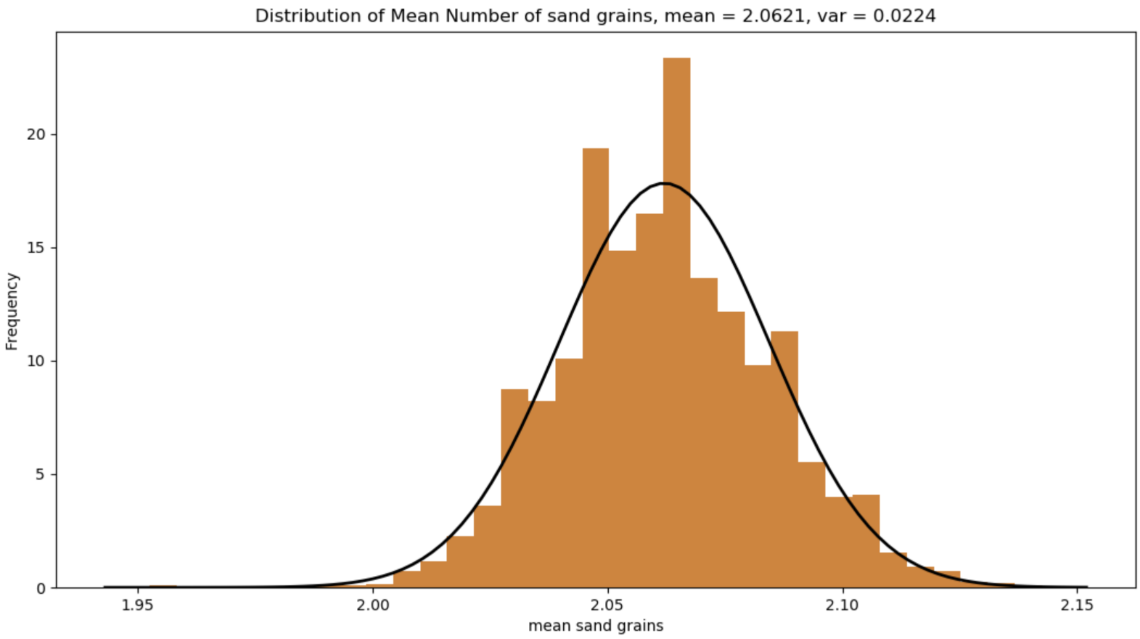


Figure 1.1: Frequency Distribution of mean number of sand grains on each lattice site.
20x20 Grid 10000 iterations.

Figure 1.1 shows the distribution of the mean number of sand grains on each lattice site with 10000 avalanche occurrences in the stationary state on a 20x20 lattice. From this we can see that the mean number of sand grains per site follows a normal distribution with mean 2.0621 and variance 0.0224. We conclude that there is strong evidence to suggest that the mean number of sand grains is an appropriate metric to conclude stationarity of the abelian sand pile model. Furthermore, this result satisfies our hypothesis that the mean number of sand grains in the stationary state will fluctuate around or just above 2. Now that we have found a way to characterize the statistical stationary state we will discuss power distributions of avalanche properties in the next section.

Identifying the power law coefficients of avalanche properties.

Here we study four properties of avalanches in The Abelian Sandpile Model. We define them as follows:

Topples: The positive integer which represents the number of compositions of the topple operator T_x for the avalanche.

Area: The positive integer which represents the number of lattice sites which an avalanche covered.

Length: The positive real number which represents the length of the avalanche. Here we define length as the supremum of the length of all site pairs in an avalanche, whereby we assume the L_2 norm. We assume the L_2 norm because this norm is the most appropriate when we want to scale our model in the spatial limit.

Loss: The non-negative integer which represents the number of sand grains leaving the lattice via toppling off an edge.

Given the above definitions we suspect correlations between the properties Topples, Area and Length. This is because if the number of Topples were to increase then we would suspect the number of unique sites covered by these topples to also increase and hence the area of the avalanche. Similarly, the increasing of the number of unique sites visited increases the number of site pair lengths we consider and taking the supremum of these we would expect length of avalanche to also increase.

We do not suspect correlation between Loss and the other properties. This is because an avalanche occurring near a boundary is more likely to have a higher loss count than an avalanche occurring near the center of the lattice despite the Area, Topples and Length of avalanche.

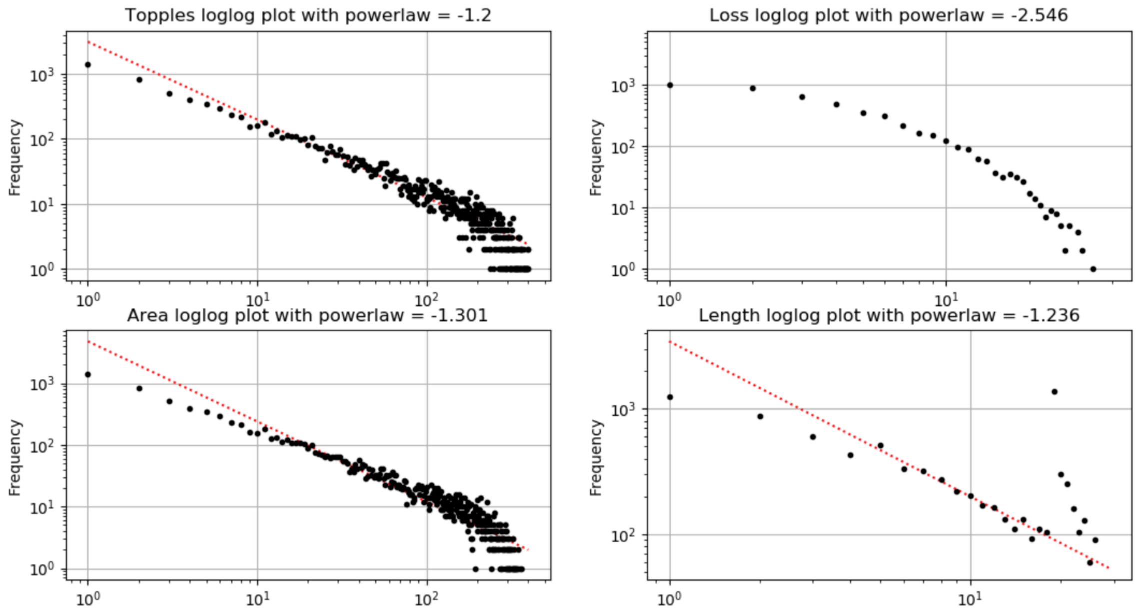


Figure 1.2: Avalanche Property Distributions - Power Law Exponents. 20x20 Grid 10000 iterations.

In Figure 1.2 we can see four loglog scale plots demonstrating power law distributions, except for the Loss property. We notice that the Topples property has a power law exponent of -1.2, Area property with -1.3 and Length property with -1.236. These exponents are very close and verify our hypothesis that these properties are indeed correlated. We also note the outlier points in the length property power distribution. Several points above the value 11-12 are accumulated and do not fit the power curve line. This can be attributed to the boundaries influencing the length of the avalanches.

For the Loss property we achieved a power law exponent of -2.546 which is significantly different to the other properties. Moreover, the distribution of the Loss property does not match that of a power law distribution since there is a significant curve to the fit, hence we did not fit the power law line through it.

1.2.2 Independence of Initial Conditions

One of the special properties of The Abelian Sandpile Model is that it exhibits Self-Organized Criticality [1]. This means that the system will converge to criticality without the need to fine tune its parameters. In the context of The Abelian Sandpile Model, this means that the model will converge to a state of statistical stationarity independence of initial conditions. Before we discussed how we can identify the stationary state and showed that the mean number of sand grains per site is an appropriate measure. Here we show that our simulation results demonstrate convergence in mean number of sand grains per site despite varying in initial configurations.

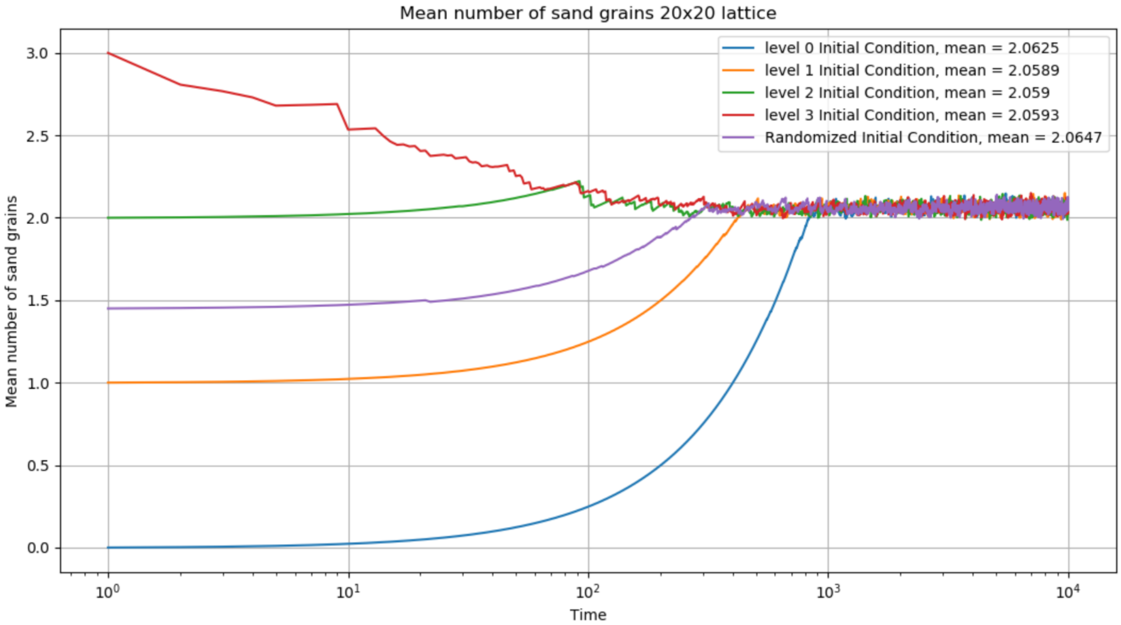


Figure 1.3: Mean sand grains per lattice site for each initial condition. 25x25 Grid over 3000 iterations.

In Figure 1.3 we can see the convergence in mean number of grain sands for five different initial configurations. The purple line is the randomized initial configuration on the set $\{0, 1, 2, 3\}$. The other four lines are level set configurations where by level i implies that at time $t = 0$ we have $\forall x \in L_N^d \; p(x) = i$. These results demonstrate that The Abelian Sandpile Model does indeed exhibit Self-Organized Criticality. The system converged to statistical stationarity despite difference in initial configurations. In fact, the convergence to stationarity is statistically identical. All five initial configurations converged to a stationary mean number of sand grains of approximately 2.06 with negligible variation which can be attributed to stochastic noise. This mean sand grain per site of 2.06 also agrees with the mean of the normal distribution of the mean number of sand grains per site we attained previously which was 2.0621.

1.2.3 Timings between avalanches are Geometrically distributed

Much of the research in the literature is focused on the power law relationships of models exhibiting Self-Organized Criticality, however, not much attention is paid to the timing between critical configurations of the system. Below we demonstrate through simulations that for a 20x20 2-dimensional grid The Abelian Sandpile Model as described by Bak, Tang, Wiesenfeld 1987 [1] has a geometric distribution in timings between avalanches.

In order to study the timings between avalanches in the statistically stationary state we must first understand what causes an avalanche. As discussed earlier, the addition of a sand grain onto a lattice which causes a site to become critical will induce an avalanche. Hence, the sand grain must land on a sub-critical site (ie, a site with value 3 assuming critical threshold of $k = 4$). Assuming the location of the addition of the sand grain is uniformly randomized among the possible lattice sites then knowing the probability distribution of the proportion of grid in a sub-critical state will allow us to predict if an addition of a sand grain will cause an avalanche.

As shown in the previous section, The Abelian Sandpile Model will approach a statistically stationary state if given enough time. In this stationary state all system observables become stationary in distribution. In the context of The Abelian Sandpile Model, we showed that the mean number of sand grains on the lattice becomes a stationary distribution with mean [enter mean here] and is a satisfactory metric to imply stationarity.

In this section we show that, through simulations, the site value distributions for site values 0, 1, 2, 3 are normally distributed. Furthermore, we show that once in the statistically stationary state, the site value distribution for site value 3 is normally distributed with mean 0.4143. This is the key result which will allow us to predict avalanche timings in the stationary state.

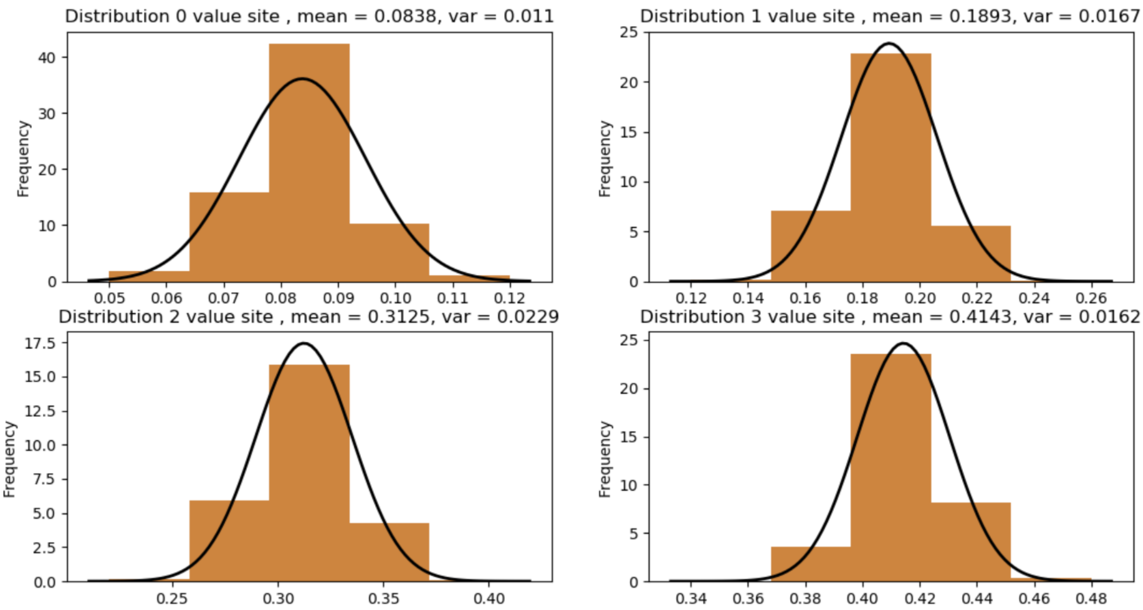


Figure 1.4: Distributions of site values 0, 1, 2, 3 of system during stationary state with fitted normal curve. 20x20 Grid 10000 iterations.

Figure 1.4 shows the distributions of site values 0, 1, 2, 3 in the statistically stationary state of the Abelian Sandpile Model. From this we can see that the frequency bars follow a normal distribution and when fitted produces a normally generated mean and variance as shown. The key result we are looking for is the mean for the 3-value site which based on 10000 iterations of a 20x20 grid gave a mean of 0.4143. This means that in the stationary state, on average, 41.43% of the grid is sub-critical.

If given an arbitrary sub-critical lattice then one can easily predict if the random addition of a sand grain will induce an avalanche. However, what if we don't know what the current lattice configuration is? Clearly in this case we cannot predict bases on the proportion of configuration being sub-critical. Hence we make the assumption that in the stationary state on average 0.4143 of the lattice is sub-critical which was verified above by simulation.

Definition (Geometric Distribution):
The Geometric distribution gives the probability that the first occurrence of success requires k independent Bernoulli trials, each with probability p. If the probability of success on each trial is p, then the probability that the k^{th} trial is the first success is

$$P(X = k) = p(1 - p)^{k-1}$$

Within the context of avalanche timings we can assume that the addition of a sand grain is a Bernoulli trial where it either initiates an avalanche (Success) or does not initiate an avalanche (Fail). We not that the addition of a sand grain is independent of the addition of other sand grains and that the addition is uniform random on the lattice. Since the probability of causing an avalanche in the stationary state is on average 0.4143 (as shown before), we hypothesis that our avalanche timings are geometrically distributed with rate 0.4143.

Figure 1.5: Distribution of times between avalanches in stationary state. 20x20 Grid 10000 avalanche occurrences.

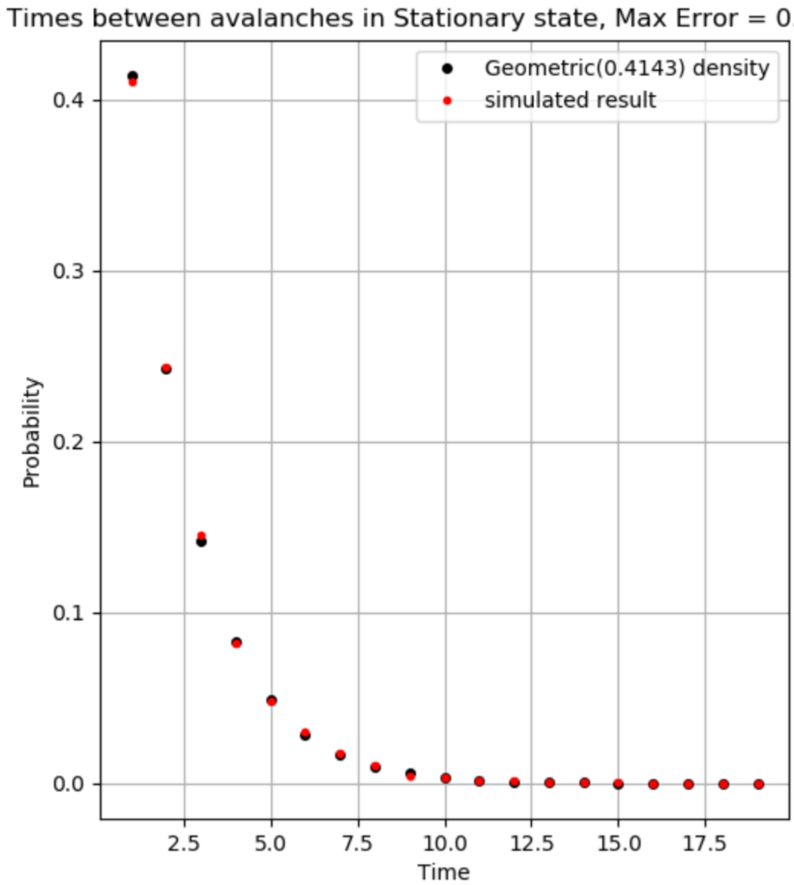


Figure 1.5 shows the distribution of times between avalanches for a 20x20 grid with 10000 stationary avalanche occurrences. This is plotted against an actual geometric probability distribution with parameter rate 0.4143. From this simulation we achieved a maximum error of 0.00418 between the plotted geometric distribution density points and the simulated density points. Hence we have strong evidence that the timings between avalanches are geometrically distributed with parameter given by the mean density of 3-value sites in the stationary state of the system.

This concludes this section, however, we stress that much more research can be completed by looking into the avalanche timings for more sophisticated lattice topologies. Some of our simulated results demonstrated that the 3-value site mean of 0.4143 is unique to the 20x20 lattice and hence largely dependent on the lattice size.

1.3 Computational Implementation

1.3.1 Evolution of Sandpile Model Flow Diagram

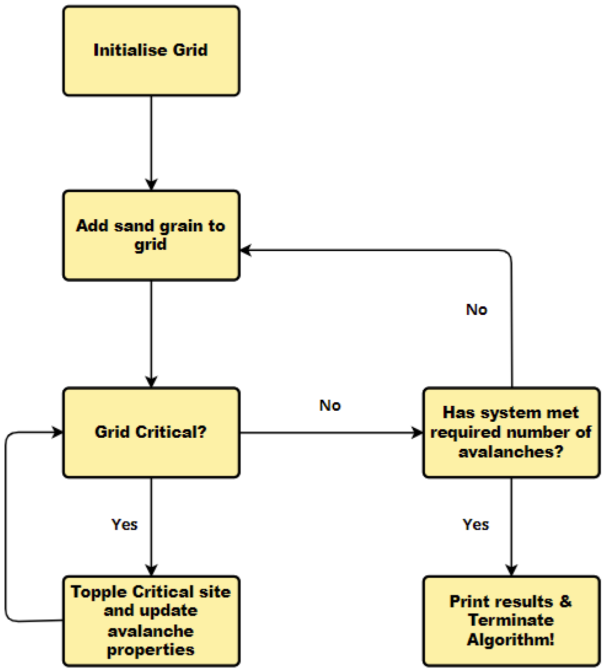


Figure 1.6: Global Algorithm Flow Chart.

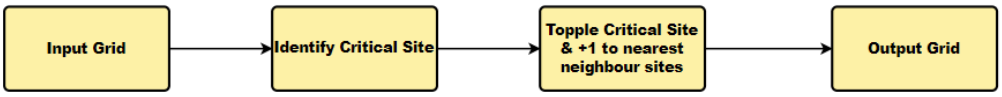


Figure 1.7: Flow Chart Representing the Toppling Matrix.

1.3.2 Computational Complexity in the Models scale limits

There are various ways in which The Abelian Sandpile Model can be implemented as a computer simulation. Some implementations will be superior to others in terms of their computational efficiency. There are two major factors which will limit an algorithms ability to produce a result in a palatable amount of time and memory efficiency. This being its order in time and order in space.

Time Complexity Refers to the amount of time an algorithm takes to complete its instructions.

Space Complexity Refers to the amount of memory required for an algorithm to complete its instructions.

For our results we implemented our system using multiple nested for and while loops. To be precise the highest nested loop reached order 4. Given that the addition of two orders of the same order gives the same order, our algorithm ran in order of time of $O(M^4)$.

In terms of space order, our algorithm used a two dimensional array to represent the two dimensional lattice. Given that we use a side length of N we have an order in space of $O(N^2)$.

We would encourage others to try to implement the model using a connected spanning tree for the avalanche. This would decrease both the memory needed and the time needed to complete the algorithm.

Chapter 2

Extending the Model - Wildfires

2.1 Introduction

In 1990, Bak and Chen [3] introduced a wildfire model which they assumed to exhibit self-organized critical behaviour. The wildfire model they constructed was based on a stochastic cellular automation defined on some finite dimensional hypercube lattice. Each site on this lattice is occupied by either a tree, a burning tree or an empty site. The rules for updating the lattice are as follows:

1) An empty site grows a tree with some probability, p.

2) A tree becomes a burning tree if there is a neighbouring tree which is burning.

3) A burning tree becomes an empty site.

The General Wildfire Model has been updated and improved on many times over. Namely, in 1993 B.Dossel and F. Schwabl [2] constructed a more general wildfire model improving on the works of Bak, Chen and Tang [1]. Their model assumed the same lattice updating rules, however they introduced a new probability to the step when a tree becomes a burning tree. That is, a tree next to a fire is not guaranteed to become burning but will burn in the next step according to this new probability. This was to capture the idea that not every tree next to a fire will catch on fire at the same time. Further works by, B.Dossel and F. Schwabl [4] allowed for the possibility that a tree can catch on fire without being next to a fire because of a lightning strike. They modelled this by incorporating another time step rule being that an arbitrary tree will catch fire with some lightning probability f.

Moving forward, we take a different perspective of the generalisation of B.Dossel and F. Schwabl [4]. We show that by introducing a probability associated with the sensitivity of a tree to catching fire is equivalent to the extension of [4]. Building on this, we introduce a wind tuple describing a wind direction and wind strength in order to make the model more realistic. With this adjustment, our results show that the lattice experiences fire cluster density formation in the direction of the wind by only adjusting the local site transition probabilities.

2.2 Revisiting the Results of B.Drossel and F.Schwabl

2.2.1 An Equivalent set up and Results

We assume a two dimensional square lattice L of side length N. Each site in L will be either an empty site, a site containing a tree or a site with a burning tree. Hence each site takes values in the set $S_L = \{0, 1, 2\}$, where we denote 0 as an empty site, 1 as a tree site, and 2 as a burning tree site.

We introduce transition probabilities $p_g, p_f, p_l, p_e \in [0, 1]$ on these site states via the following

- p_g $\quad 0 \longmapsto 1$ (probability an empty site grows a tree)
- p_f $\quad 1 \longmapsto 2$ (probability an a tree catches fire if on boundary of fire cluster)
- p_l $\quad 1 \longmapsto 2$ (probability an arbitrary tree catches fire by a lightning strike)
- p_e $\quad 2 \longmapsto 0$ (probability that a tree on fire burns out and becomes an empty site)

A critical variation in our set up is that we introduced a tree sensitivity probability p_f instead of a tree immunity probability g as done in [2]. However, there is a clear equivalence since the probability that a tree catches fire on the boundary of a fire cluster p_f is equal to the complementary probability that a tree does not catch fire if on the boundary of a fire cluster g.

We conclude that the equivalence between our set up and that of 1993 B. Drossel and F. Schwabl [2] is

$$p_f = 1 - g$$

2.2.2 Dynamics of Wildfires

Previous research shows that in general, the behaviour of wildfires is heavily influences by its environment [5]. Many of these factors include,

- Wind.
- Type of tree's and brush.
- Humidity and moisture.

Unfortunately, the General Wildfire Model does not accurately model and predict the behaviour of real wildfires. We aim to improve on the current research by introducing a wind component. The reason behind this is that, in general, a wildfire will preferentially consume land in the direction of the wind. We implement this observation into the model by adjusting the local site transition probabilities.

2.3 Extending the General Model - Wind Component

2.3.1 The Set Up and Expectations

In order to improve the General Wildfire Model we must accommodate for the factors described earlier. In this section we will focus on the wind component. We attempt to show that by incorporating wind into the model we attain a more accurate picture of real wildfires, hence improving the accuracy of our model.

As discussed earlier, we expect a wildfire to consume more land in the direction of a wind. Hence in order to incorporate this factor into our model we will adjust the probability p_f to become a new probability $p_{f,W}$, where,

- $W = (d, s)$ is a tuple such d denotes the wind direction on N,E,S,W and $s \in (0, 1)$ is a wind strength constant.

$$p_{f,W} = \begin{cases} p_f & \text{if in direction of wind} \\ p_f \cdot (1-s)^2 & \text{if against direction of wind} \\ p_f \cdot (1-s) & \text{else} \end{cases}$$

Based on the above probability transformation we now have new transition probabilities on the local sites whereby a tree next to a fire is more likely to catch fire if it is in the direction of the wind with respect to the position of the fire. Equivalently, trees not in the immediate direction of the wind with respect to the position of the fire, are less likely to catch fire as demonstrated by the $(1-s)$ and $(1-s)^2$ factors, with the trees in the opposite direction of the fire most affected.

Given the new transition probabilities we hypothesize that our fire will converge in the direction of the wind, independent of initial state. This is because there is a higher probability of a tree catching fire in the direction of the wind. Furthermore, we expect that, in the long run time limit, there will be a higher density of trees on fire in the direction of the wind as opposed to trees in the opposite direction of the wind.

2.3.2 Fire Cluster Density Formation

Our results show that for probability values $p_f = 0.8$, $p_g = 0.4$, $p_e = 0.4$ and a lighting probability of $p_l = 0.0003$ we attain the following fire cluster distributions on the lattice after 200 iterations. This was with an initial tree density of 50% randomized over the possible sites and a wind strength constant of 0.5.

From Figures 2.1 and 2.2 we can see that our hypothesis was correct. The fire formations tends towards the direction of the wind. The figures show a higher fire formations density in the direction of the wind and a lower density formation in the opposite direction.

Given these results we believe this adjustment to the general wildfire model is accurate to what we observe in reality, that being, wildfires consuming proportionately more land in the direction of a wind. However, we need to also show that these density formations also hold in higher grid dimensions so that this adjustment becomes a viable candidate for future models. We discuss fire density formation in the spatial scaling limit in the next section.

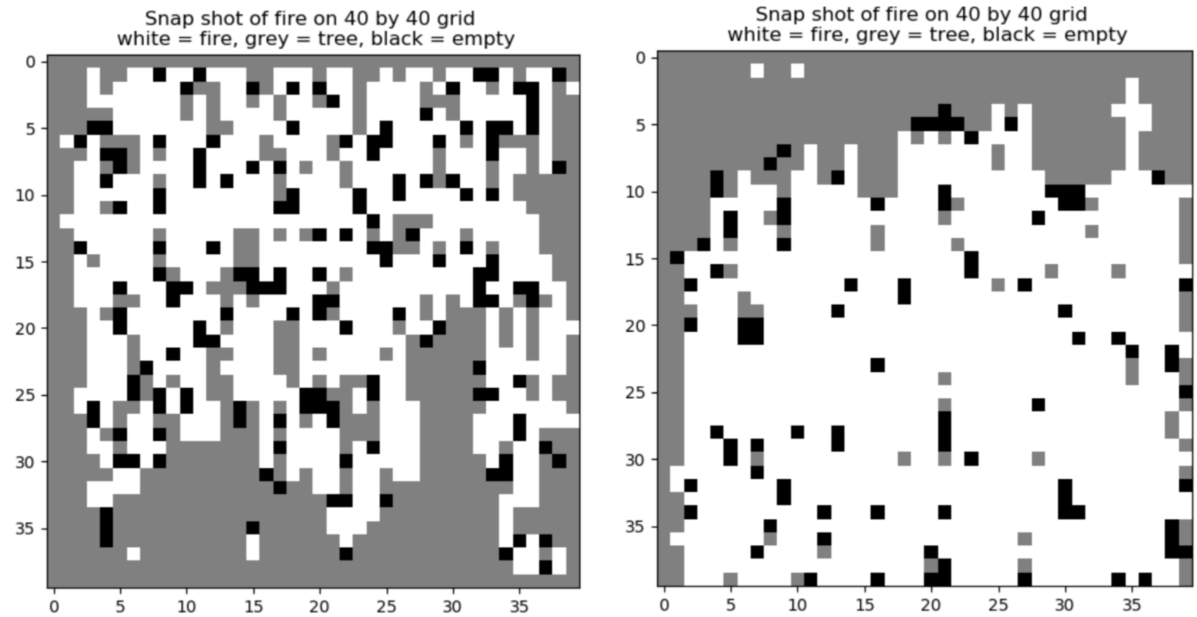


Figure 2.1: Left: Northern Wind. Right: Southern Wind

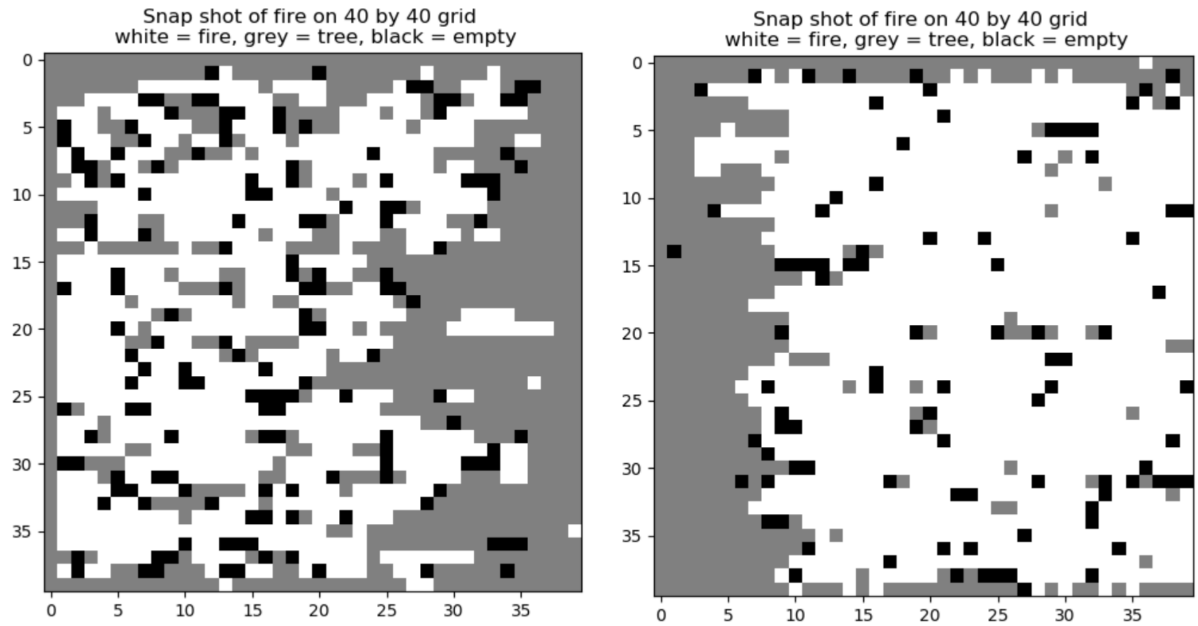


Figure 2.2: Left: Western Wind. Right: Eastern Wind

2.3.3 Fire Cluster Density Formation in the Spatial Scaling Limit

A key property that we want to capture in this wind component extension is its ability to exist in on all spatial lengths for some two dimensional lattice. This means we would expect that the fire cluster density formation to be higher in the direction of the wind for all space lengths of some lattice.

The results shown in the figures that follow show that fire cluster density formation is indeed higher in the direction of the wind. Although, this seems to be more apparent in the 25x25 lattice as opposed to the 75x75 lattice. A possible cause for this is that the 25x25 lattice will experience fewer lightning strikes because of its smaller area. Hence the fire cluster formation will be proportionately smaller than the larger lattice. Despite this, we still see that the fire cluster density formation is higher in the direction of the wind for the larger 75x75 lattice. Unfortunately, the density does not seem to be consistent with the 50x50 and 25x25 lattice. We would encourage readers interested in building on our results to attempt to find a better implementation of the wind component whereby the density formation is proportionately consistent on all lattice length scales.

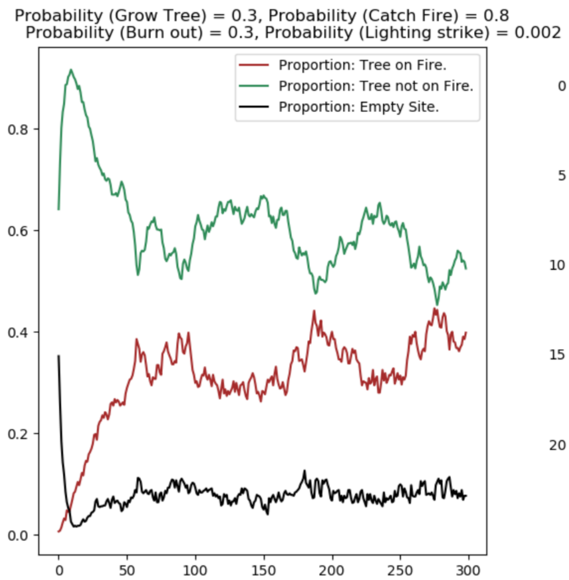
Furthermore, our results show that the General Wildfire Model with the wind component also approaches a stationary state for lattice configurations 25x25, 50x50 and 75x75. However, this stationary state mean is different for each lattice. For example, the 25x25 lattice has a stationary mean of trees on fire of approximately 0.34. While the 50x50 lattice had a stationary state mean of trees on fire of approximately 0.5 and the 75x75 lattice with 0.55. We argue that this difference is mostly due to the fact that the larger lattice's will experience more lightning strikes and hence disconnected fire clusters will form up more often and combine into larger clusters much faster.

Figures 2.3, 2.4, 2.5 show the fire density formation with a northern wind with wind strength constant $s = 0.5$ for grid sizes 25x25, 50x50 and 75x75. This is over 300 iterations of the wildfire model with an initial tree density of 50% randomized over the possible sites.

Our simulation used the following transition probabilities:

- $p_g = 0.3$ (Probability site grows tree)
- $p_f = 0.8$ (Probability tree catches fire)
- $p_e = 0.3$ (Probability tree burns out)
- $p_l = 0.002$ (Probability tree catches fire by lightning strike)

Figure 2.3: Left: Grid Timeline Right: Snap Shot after 300 iterations



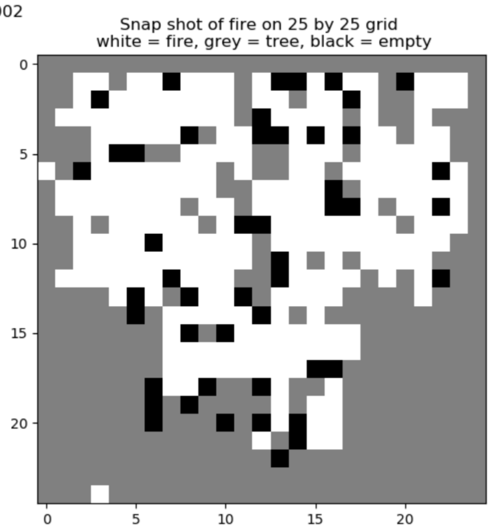


Figure 2.4: Left: Grid Timeline Right: Snap Shot after 300 iterations

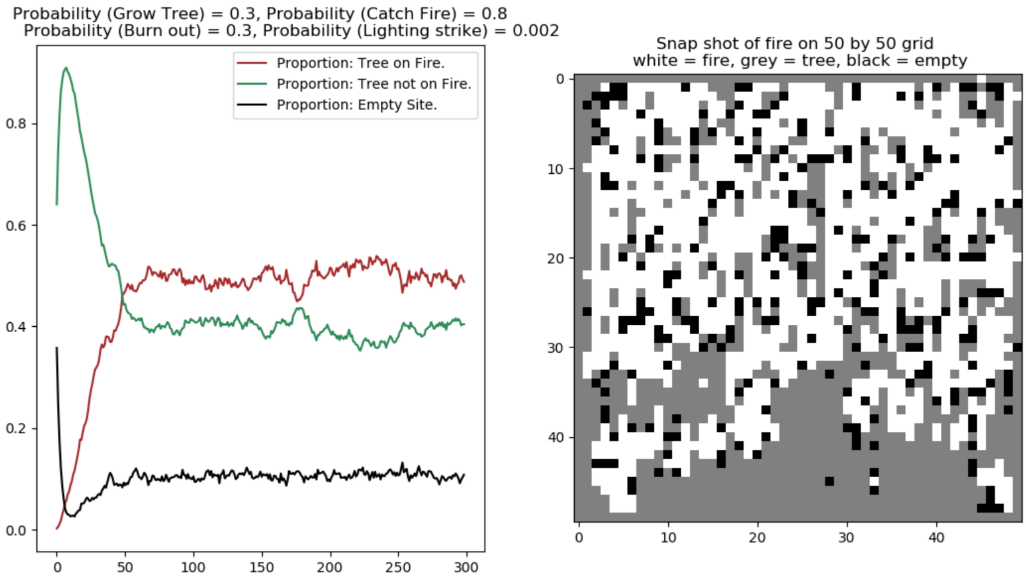
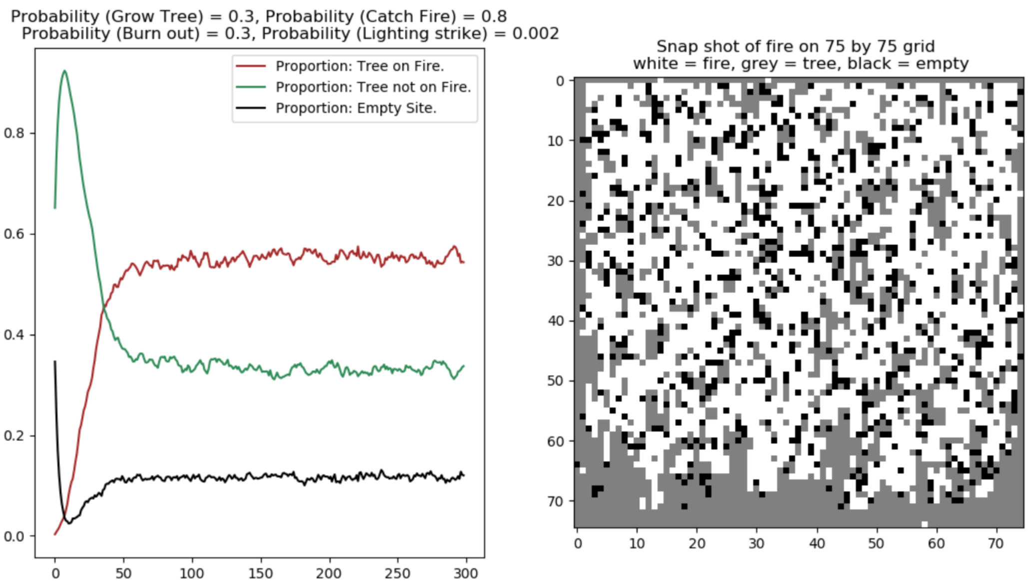


Figure 2.5: Left: Grid Timeline Right: Snap Shot after 300 iterations



Chapter 3

Conclusion

3.1 Concluding Remarks

In conclusion, we introduced the concept of Self-Organized Criticality and demonstrated its manifestation with The Abelian Sandpile Model. We showed that The Abelian Sandpile Model satisfies Self-Organized Criticality. We characterized the stationary state and calculated the mean number of grains sand per site to be approximately 2.06 as well as produced power law distributions in avalanche properties. Furthermore, our most unique result was that of the timings between avalanches in the stationary state being geometrically distributed with parameter rate of 0.4143 for a 20x20 lattice.

Building on The Abelian Sandpile Model, we introduced the General Wildfire Model and extended it by adding a wind factor. This addition demonstrated fire cluster density formation in the direction of the wind. Furthermore, this density formation held while increasing the length scale of the model. This shows that our approach to incorporating wind into the model is a possible candidate for a formalism of this type of extension. Unfortunately, this density formation was not proportionately consistent over varying lattice sizes.

3.2 Recommendations for Future Work

Readers looking to further the results in this paper are encouraged to do so. The main unique result achieved in this paper was the formulation of the timings between avalanches in the stationary state. Extending on this result would entail finding the distribution of timings between avalanches for higher dimensional lattices and lattices with different topologies. Furthermore, formalising the general wildfire model with the wind component is necessary. We would encourage others to try a stochastic wind component or redefine the local probabilities which constitute the wind factor. Ultimately, the goal is to find a general wildfire model which mathematically satisfies Self-Organized Criticality while also being able to accurately replicate real wildfires on all length scales. We were unable to produce a proportionately consistent fire cluster density formation for varying lattice sized given some wind direction. In order to fix this, one would need to redefine the local state transition probabilities. We believe that making these probabilities also a function of the lattice size will induce some sense of scale invariance of the fire cluster density formations.

Bibliography

[1] Chao Tang Per Bak and Kurt Wiesenfeld. Self-organized criticality: An explanation of the 1/f noise. *Phys. Rev.*, 59:381–384, 1987.

[2] F. Schwab1 B. Drossel. Forest-fire model with immune trees. *Physica A*, 199:183–197, 1993.

[3] Alessandro Vespignani and Stefano Zapperi. How self-organized criticality works: A unified mean-field picture. *Phys. Rev.*, E 57:6345, 1988.

[4] F. Schwab1 B. Drossel. Self-organized critical forest-fire model. *Physical Review Letters*, 69, 1992.

[5] Todd J. Hawbaker Jeffrey P. Prestemon. Wildfire ignitions: A review of the science and recommendations for empirical modeling. *Gen. Tech. Rep. SRS-171. Asheville, NC: U.S. Department of Agriculture Forest Service, Southern Research Station, 20 p.*, 2013.