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Modelling Count Data by Random Effect Poisson Model

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Abstract

It is well known that count data show overdispersion compared to the Poisson distribution, which is extensively used for the analysis of discrete data. In order to account for the unobserved heterogeneity, in this paper we introduce an additive and a multiplicative random effect Poisson model. The random effect is modelled by the gamma distribution and the inverse Gaussian distribution and both univariate as well as multivariate models are developed. Expressions for the various conditionals and marginal distributions are obtained and the correlation introduced by sharing a common random effect is studied. Some computational aspects, of the models developed , are presented to illustrate the results. Thus the purpose of this paper is to provide some alternative models that can be used for analysing data which shows overdispersion.

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1 Introduction

It is well known that count data show overdispersion compared to the Poisson distribution, which is extensively used for the analysis of discrete data. For example, for counts of the number of epileptic seizures, there is a very large individual variation in the seizure rate, see Hougaard et al. (1997). In certain applications involving count data, it is sometimes found that the zeros are observed with frequency significantly higher than predicted by the assumed model. Examples of such applications are cited in the literature from engineering, manufacturing, economics, public health, epidemiology, psychology, sociology, political science, agriculture, road safety, species abundance, horticulture and criminology. In these cases, the mean variance relationship of the Poisson model is quite restrictive. It underestimates the observed dispersion, which may be caused by extra zeros. Mullahy (1997) has demonstrated that the unobserved heterogeneity commonly assumed to be the source of overdispersion in the count data models has predictable implications for the probability structures of such models. In the context of survival analysis, Aalen (1992) has observed that individual risks differ in, possibly, unknown ways and this heterogeneity may be difficult to assess, but is nevertheless of great importance.

Böhning et al. (1999) note that the phenomenon of overdispersion occurs because a single parameter λ (mean of the Poisson distribution) is often insufficient to describe the population, see for example McCullagh and Nelder (1989), Aitken et al. (1989) and Campbell et al. (1991). In fact in many cases, it can be suspected that population heterogeneity, which has not been accounted for, is causing the overdispersion. In this regard, Mullahy (1997) remarks that overdispersion is some form of parameter heterogeneity. One approach to model the heterogeneity is by means of mixture models. Another way to look at it is to assume that the original Poisson assumption is a misspecification of the true model and to consider alternate specifications like zero-inflated, hurdle or negative binomial models. The tendency to observe excess zeros as well as heavy upper tails in the data turns out to be a strict implication of unobserved heterogeneity, see Mullahy (1997) for further discussion on this subject. The excess zero models have attracted a great deal of attention in the last 10 years. More recently, Gupta et al. (2004) have derived a score test for zero-inflated generalized Poisson regression models, see also Gupta et al. (1995, 1996) and the references there-in.

As has been mentioned above, one way to take care of this heterogeneity is by way of mixture models. In the case of Poisson distribution, the mean θ of the Poisson distribution is considered as a random variable with an appropriate probability structure. The simplest choice of the distribution of θ is the gamma density resulting in a negative binomial distribution. Some generalizations of this concept have been studied by employing a generalized gamma distribution resulting in a generalized form of the negative binomial distribution, see Gupta and Ong (2004). Another choice of the distribution of θ is taken as the inverse Gaussian distribution or generalized inverse Gaussian distribution giving rise to Sichel distribution, see Ord and Whitmore (1986) and Atkinson and Yeh (1982). It is a long tailed distribution that is suitable for highly skewed data. For other choices of the distribution of θ, the reader is referred to Johnson et al.(1992), see also Hougaard et al.(1997).

In survival analysis, one method of taking the extra variation into account is through observed covariates. However, it is not possible to include all risk functions and all possible forms of inter-individual heterogeneity. The heterogeneity unexplained by observed covariates can be a serious problem because it can lead to misleading conclusions. A well known model, particularly used for handling heterogeneity left unexplained by observed covariates, is the " frailty model" introduced by Vaupel et al. (1979). They introduced an unobserved quantity V called frailty and proposed that the death rate or hazard rate at age t for a person with frailty v be taken as $v\lambda_0(t)$, where the basic rate $\lambda_0(t)$ is independent of v and describes the age effect. The frailty from person to person varies and the population of frailties is modelled by a frailty random variable. We refer to Aalen (1987, 1988, 1992), Gupta and Kirmani (2004), Hougaard (1984, 1991, 1995, 2000) and Vaupel et al. (1979) for various aspects of frailty models and related statistical analyses.

The additive random effect model also has its kernels in the survival analysis literature. For example, Andersen and Veath (1989) presented semiparametric and non-parametric models for excess and relative mortality. Zahl (1997) points out that the excess hazard rate, as described by Andersen and Veath (1989) may be derived from a model with two competing risks, the risk of dying from the disease under study and the risk of dying from other causes. Andersen and Veath (1989) apply this model to the data on 205 patients (126 women and 79 men) operated for malignant melanoma at a hospital in Denmark. Elandt-Johnson (1980) considers the model $\lambda(t; z_0) = \lambda(t) + h(t)z_0$, where z_0 is normally distributed. Notice that the hazard rate function is of the additive form. Kim and Lee (1998) studied the additive model $\lambda(t; z) = \lambda_0(t) + \beta_0 z$ and presented two simple goodness of fit tests for this model with censored observations.

In the case of Poisson distribution, we proceed as in survival analysis and modify the Poisson parameter θ in two ways: (i) adding an unobserved random variable η or (ii) multiplying by an unobserved random variable η as in the case of frailty models mentioned earlier. In the first case, we call it a random additive model and in the second case, a random multiplicative model. As an explanation of such modelling, consider a telephone company where the number of telephone calls received by the telephone exchange per unit time follows a Poisson distribution. Imagine that the company has advertised certain incentives for its customers so that the average rate is modified by the random effect due to advertising. The random effect can be modelled by an appropriate probability distribution. Note the difference between the ordinary mixture model described earlier and random effect model considered here. In the ordinary mixture model, the baseline parameter is randomized while in the random effect model, the baseline parameter is modified by a random effect. It will be seen that in both cases, the variance exceeds the mean of the model which takes care of the overdispersion problem.

The organization of this paper is as follows: In Section 2, we present the univariate additive model and in Section 3, we extend it to the multivariate case which we call the shared additive model. The correlation introduced by sharing a common random effect is derived. As examples, the gamma and the inverse Gaussian random effects are considered and expressions for the various conditionals and marginal distributions are obtained. Similarly, the univariate multiplicative model is considered in Section 4 which is extended to the multivariate multiplicative model in Section 5. Finally, in Section 6, for illustration purpose, we consider the data on the distribution of Corbet's Malayan Butterflies, see Blumer (1974). This data was originally fitted by Poisson-lognormal model by Blumer (1974). We have fitted this data with the four models developed in this paper. It turns out that the additive model using gamma random effect provides the best fit. It also beats the fit obtained by the Poisson-lognormal model. Thus, the purpose of this paper is to provide some alternative models that can be used for analysing data which shows overdispersion.

2 Univariate Additive Model

Let X be a discrete random variable having Poisson distribution with mean θ. If the mean is modified to $\theta+\eta$, then the probability mass function (pmf) of the modified random variable Y is given by

$$P(Y=y|\eta)=\frac{e^{-(\theta+\eta)}(\theta+\eta)^y}{y!}, \quad y=0,1,2,\ldots, \tag{2.1}$$

where η is considered as a random variable with certain probability structure. It is clear that $E(Y|\eta)=\theta+\eta$ and $Var(Y|\eta)=\theta+\eta$. Also, conditionally Y has a Poisson distribution. Unconditionally, $E(Y)=\theta+E(\eta)$ and

$$\begin{aligned} Var(Y) &= E(Var(Y|\eta))+Var(E(Y|\eta)) \\ &= \theta+E(\eta)+Var(\eta) \quad = \quad E(Y)+Var(\eta). \end{aligned} \tag{2.2}$$

Thus $Var(Y) > E(Y)$ and the mean- variance relationship of the Poisson distribution does not hold.

2.1. Probability Mass Function of Y. Denoting the probability density function of η by $g(\eta)$, the pmf of Y can be written as

$$P(Y=y)=\int_0^\infty \frac{e^{-(\theta+\eta)}(\theta+\eta)^y}{y!}g(\eta)d\eta=\frac{e^{-\theta}\theta^y}{y!}\sum_{r=0}^{r=y}\int_0^\infty \frac{\binom{y}{r}}{\theta^r}e^{-\eta}\eta^r g(\eta)d\eta. \tag{2.3}$$

2.2. Probability Generating Function of Y. The probability generating function (pgf) of the conditional distribution is given by

$$G(z|\eta) = e^{\theta(z-1)+\eta(z-1)}$$

Therefore, the unconditional pgf is given by

$$G(z) = \int_0^\infty G(z|\eta)g(\eta)d\eta = e^{\theta(z-1)}M_\eta(z-1), \tag{2.4}$$

where $M_\eta(z-1)$ is the mgf of the mixing pdf $g(\eta)$.

Now we present two examples.

EXAMPLE 1. Suppose η has a gamma distribution with pdf

$$g(\eta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\eta^{\alpha-1}e^{-\beta\eta}, \ \eta > 0, \alpha > 0, \beta > 0.$$

In this case the pmf of Y is given by

$$P(Y=y) = \frac{e^{-\theta}\theta^y}{y!}\sum_{r=0}^{y}\frac{\binom{y}{r}}{\theta^r}\frac{\beta^\alpha}{\Gamma(\alpha)}\frac{\Gamma(r+\alpha)}{(\beta+1)^{r+\alpha}}, \ y = 0, 1, 2, \ldots \tag{2.5}$$

Note that the above expression can be written as the convolution of a Poisson and a negative binomial distribution as follows

$$P(Y=y) = \sum_{r=0}^{y}\frac{e^{-\theta}\theta^r}{r!}\frac{(\alpha)_{y-r}}{(y-r)!}\frac{1}{(1+\beta)^{y-r}}(\frac{\beta}{1+\beta})^\alpha, \tag{2.6}$$

where $\alpha_{(r)} = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+r-1)$.
This formula is used to compute the probabilities in section 6.

EXAMPLE 2. Suppose η has an inverse Gaussian distribution with pdf

$$g(\eta) = \left(\frac{1}{2\pi a\eta^3}\right)^{1/2}\exp(-(b\eta-1)^2/2a\eta), \ \eta > 0, a > 0, b > 0.$$

In this case the pmf of Y is given by

$$\begin{aligned} P(Y{=}y) &= \frac{e^{-\theta}\theta^y}{y!}\sum_{r=0}^{y}\frac{\binom{y}{r}}{\theta^r}\int_0^\infty\frac{1}{(2\pi a\eta^3)^{1/2}}\exp(-(b\eta-1)^2/2a\eta)e^{-\eta}\eta^r d\eta \\ &= \frac{e^{-\theta}\theta^y}{y!}e^{-\left(\frac{\sqrt{b^2+2a}-b}{a}\right)}\sum_{r=0}^{y}\frac{\binom{y}{r}}{\theta^r}\int_0^\infty\frac{\eta^r}{(2\pi a\eta^3)^{1/2}}\exp[-h(\eta)]d\eta \quad (2.7)\end{aligned}$$

where $h(\eta) = \frac{((\sqrt{b^2+2a})\eta-1)^2}{2a\eta}$. Note that the integral in the above expression is the r th moment of the inverse Gaussian distribution with parameters a and $\sqrt{b^2+2a}$, see Chhikara and Folks (1989).

Note that the above expression may be written as the convolution of a Poisson and a Poisson-inverse Gaussian (PiG) as follows:

$$P(Y=y) = \sum_{r=0}^{y} \frac{e^{-\theta}\theta^r}{r!} P_{PiG}(y-r), \tag{2.8}$$

where $P_{PiG}(y)$ is the pmf of PiG. For PiG model, see Ord and Whitmore (1986).

This formula is used to compute the probabilities in section 6.

3 Shared Additive Multivariate Model

Suppose $Y_1, Y_2, \ldots, Y_p$ are conditionally independent Poisson random variables given η with means $\theta_i + \eta, i = 1, 2, \ldots, p$. Then the probability generating function (pgf) of $Y_1, Y_2, \ldots, Y_p$ is given by

$$\begin{aligned} \psi(t_1, t_2, \ldots, t_p) &= \int_0^\infty \prod_1^p e^{-(\theta_i+\eta)(1-t_i)} g(\eta) d\eta \qquad (3.1) \\ &= e^{-\sum_1^p \theta_i(1-t_i)} \int_0^\infty e^{-\eta \sum_1^p (1-t_i)} g(\eta) d\eta \\ &= \phi(t_1, t_2, \ldots, t_p) M_\eta \left(-\sum_1^p (1-t_i) \right), \end{aligned}$$

where $\phi(t_1, t_2, \ldots, t_p)$ is the joint pgf of the Poisson random variables with means $\theta_1, \theta_2, \ldots, \theta_p$ and $M_\eta(.)$ is the moment generating function (mgf)of η.

Two special cases.

(1) Suppose η has a gamma distribution with mgf given by

$$M_\eta(t) = (1 - t/\beta)^{-\alpha}.$$

Then the joint pgf is given by

$$\psi(t_1, t_2, \ldots, t_p) = \phi(t_1, t_2, \ldots, t_p) \left(\frac{\beta}{\beta + p - \sum_1^p t_i} \right)^\alpha. \tag{3.2}$$

(2) Suppose η has an inverse Gaussian distribution with mgf given by

$$M_\eta(t) = \exp\left[\frac{b}{a} \left(1 - \left(1 - \frac{2at}{b^2} \right)^{1/2} \right) \right].$$

Then the joint pgf is given by

$$\psi(t_1,t_2,\ldots,t_p)=\phi(t_1,t_2,\ldots,t_p)\exp\left\{\frac{b}{a}\left[1-\left(1+\frac{2a}{b^2}\sum_1^p(1-t_i)\right)^{1/2}\right]\right\}. \tag{3.3}$$

The expressions for the probability mass functions are complex and will be obtained later. We now obtain the pgf of the conditional distribution of $Y_{r+1},Y_{r+2},\ldots,Y_p$ given $Y_1,Y_2,\ldots.Y_r$. The above pgf is given by

$$\psi^\star(t_{r+1},t_{r+2},\ldots,t_p)=\frac{\psi(1,1,\ldots,1,t_{r+1},t_{r+2},\ldots,t_p)}{P(Y_1=y_1,Y_2=y_2,\ldots,Y_r=y_r)}. \tag{3.4}$$

(1) For the gamma distribution, the above equation reduces to

$$\psi^\star(t_{r+1},t_{r+2},\ldots,t_p)=\frac{e^{-\sum_{r+1}^p\theta_i(1-t_i)}\left[\frac{\beta}{\beta+p-r-\sum_{r+1}^p t_i}\right]}{P(Y_1=y_1,Y_2=y_2,\ldots,Y_r=y_r)}. \tag{3.5}$$

(2) For the inverse Gaussian distribution, equation (3.4) reduces to

$$\psi^\star(t_{r+1},t_{r+2},\ldots,t_p)=\frac{e^{-\sum_{r+1}^p\theta_i(1-t_i)+\frac{b}{a}(1-(1+\frac{2a}{b^2}\sum_{r+1}^p(1-t_i))^{1/2})}}{P(Y_1=y_1,Y_2=y_2,\ldots,Y_r=y_r)}. \tag{3.6}$$

We now want to study the correlation induced in the case of $p=2$.

Induced Dependence. As explained before $E(Y_i)=\theta_i+E(\eta)$ and $Var(Y_i)$ $=\theta_i+E(\eta)+Var(\eta), i=1,2$. Also

$$\begin{aligned}Cov(Y_1,Y_2) &= E[E(Y_1Y_2|\eta)]-E[E(Y_1|\eta)]E[E(Y_2|\eta)]\\ &= E[(\theta_1+\eta)(\theta_2+\eta)]-E(\theta_1+\eta)E(\theta_2+\eta) \quad = \quad Var(\eta).\end{aligned}$$

Thus the correlation coefficient between Y_1 and Y_2 is given by

$$\rho=\frac{Var(\eta)}{\sqrt{(\theta_1+E(\eta)+Var(\eta)}\sqrt{\theta_2+E(\eta)+Var(\eta)}}. \tag{3.7}$$

Note that the case $Var(\eta)=0$ corresponds to independence and high values of $Var(\eta)$ corresponds to high positive correlation between the variables.

(1) For the gamma distribution, the correlation coefficient is given by

$$\rho=\frac{\alpha}{(\sqrt{\theta_1\beta^2+\alpha\beta+\alpha})(\sqrt{\theta_2\beta^2+\alpha\beta+\alpha})}. \tag{3.8}$$

(2) For the inverse Gaussian distribution, the correlation coefficient is given by

$$\rho = \frac{a}{(\sqrt{\theta_1 b^3 + b^2 + a})(\sqrt{\theta_2 b^3 + b^2 + a})}. \tag{3.9}$$

3.1 Joint Probability Mass Functions. In this section, we shall obtain the joint probability mass functions of $Y_1, Y_2, \ldots, Y_p$ and the conditional mass functions of $Y_{r+1}, Y_{r+2}, \ldots, Y_p$ given $Y_1, Y_2, \ldots, Y_r$ in the case of gamma random effect as well as in the case of inverse Gaussian random effect.

3.1.1. Gamma Random Effect.

$$\begin{aligned} P(Y_1 = y_1, Y_2 = y_2) & \qquad\qquad (3.10)\\ = \int_0^\infty \frac{e^{-(\theta_1+\eta)}(\theta_1+\eta)^{y_1}}{y_1!} \frac{e^{-(\theta_2+\eta)}(\theta_2+\eta)^{y_2}}{y_2!} g(\eta)d\eta \\ = \frac{e^{-\theta_1}\theta_1^{y_1}}{y_1!}\frac{e^{-\theta_2}\theta_2^{y_2}}{y_2!}\sum_{r_2=0}^{y_2}\sum_{r_1=0}^{y_1}\int_0^\infty \frac{\binom{y_1}{r_1}\binom{y_2}{r_2}}{\theta_1^{r_1}\theta_2^{r_2}} e^{-2\eta}\eta^{r_1+r_2} g(\eta)d\eta \\ = \frac{e^{-\theta_1}\theta_1^{y_1}}{y_1!}\frac{e^{-\theta_2}\theta_2^{y_2}}{y_2!}\frac{\beta^\alpha}{\Gamma(\alpha)}\sum_{r_2=0}^{y_2}\sum_{r_1=0}^{y_1}\frac{\binom{y_1}{r_1}\binom{y_2}{r_2}}{\theta_1^{r_1}\theta_2^{r_2}}\frac{\Gamma(r_1+r_2+\alpha)}{(2+\beta)^{r_1+r_2+\alpha}}. \end{aligned}$$

Generalizing the above expression, the joint probability mass function of $Y_1, Y_2, \ldots, Y_p$ is given by

$$\begin{aligned} P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_p = y_p) & \qquad\qquad (3.11)\\ = \prod_{i=1}^{p}\frac{e^{-\theta_i}\theta_i^{y_i}}{y_i!}\frac{\beta^\alpha}{\Gamma(\alpha)}\sum_{r_p=0}^{y_p}\sum_{r_{p-1}=0}^{y_{p-1}}\cdots\sum_{r_1=0}^{y_1}\left(\prod_{i=1}^{p}\frac{\binom{y_i}{r_i}}{\theta_i^{r_i}}\right)\frac{\Gamma(\sum_1^p r_i+\alpha)}{(p+\beta)^{\sum_1^p r_i+\alpha}}. \end{aligned}$$

The conditional pmf of $Y_{r+1}, Y_{r+2}, \ldots, Y_p$ given $Y_1, Y_2, \ldots, Y_r$ is given by

$$\begin{aligned} P(Y_{r+1} = y_{r+1}, Y_{r+2} = y_{r+2}, \ldots, Y_p = y_p | Y_1 = y_1, Y_2 = y_2, \ldots, Y_r = y_r) \\ = \frac{\prod_{r+1}^{p}\frac{e^{-\theta_i}\theta_i^{y_i}}{y_i!}\sum_{r_p=0}^{y_p}\cdots\sum_{r_1=0}^{y_1}\left(\prod_{i=1}^{p}\frac{\binom{y_i}{r_i}}{\theta_i^{r_i}}\right)\frac{\Gamma(\sum_{i=1}^p r_i+\alpha)}{(p+\beta)^{\sum_{i=1}^p r_i+\alpha}}}{\sum_{r_r=0}^{y_r}\cdots\sum_{r_1=0}^{y_1}\left(\prod_{i=1}^{r}\frac{\binom{y_i}{r_i}}{\theta_i^{r_i}}\right)\frac{\Gamma(\sum_{i=1}^r r_i+\alpha)}{(r+\beta)^{\sum_1^r r_i+\alpha}}}. \end{aligned} \tag{3.12}$$

3.1.2. Inverse Gaussian Random Effect

$$\begin{aligned}
P(Y_1 = y_1, Y_2 = y_2) \\
&= \int_0^\infty \frac{e^{-(\theta_1+\eta)}(\theta_1+\eta)^{y_1}}{y_1!}\frac{e^{-(\theta_2+\eta)}(\theta_2+\eta)^{y_2}}{y_2!} g(\eta)d\eta \qquad (3.13) \\
&= \frac{e^{-\theta_1}\theta_1^{y_1}}{y_1!}\frac{e^{-\theta_2}\theta_2^{y_2}}{y_2!}\sum_{r_2=0}^{y_2}\sum_{r_1=0}^{y_1}\int_0^\infty \frac{\binom{y_1}{r_1}\binom{y_2}{r_2}}{\theta_1^{r_1}\theta_2^{r_2}}\frac{\eta^{r_1+r_2}e^{-2\eta}}{\sqrt{2\pi a\eta^3}}e^{-(b\eta-1)^2/2a\eta}d\eta \\
&= \frac{e^{-\theta_1}\theta_1^{y_1}}{y_1!}\frac{e^{-\theta_2}\theta_2^{y_2}}{y_2!}\sum_{r_2=0}^{y_2}\sum_{r_1=0}^{y_1}\frac{\binom{y_1}{r_1}\binom{y_2}{r_2}}{\theta_1^{r_1}\theta_2^{r_2}}e^{-(\sqrt{b^2+4a}-b)/a}\times \\
&\int_0^\infty \frac{\eta^{r_1+r_2}}{\sqrt{2\pi a\eta^3}}e^{-((\sqrt{b^2+4a})\eta-1)^2/2a\eta}d\eta.
\end{aligned}$$

Note that the integral in the above expression is the $(r_1 + r_2)th$ moment of the inverse Gaussian distribution with parameters $\sqrt{b^2 + 4a}$ and a.

Generalizing the above expression, the joint probability mass function of $Y_1, Y_2, \ldots, Y_p$ is given by

$$\begin{aligned}
P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_p = y_p) \qquad (3.14) \\
= \prod_{i=1}^{p}\frac{e^{-\theta_i}\theta_i^{y_i}}{y_i!}e^{-(\sqrt{b^2+2pa}-b)/a)}\sum_{r_p=0}^{y_p}\sum_{r_{p-1}}^{y_{p-1}}\cdots\sum_{r_1=0}^{y_1}\prod_{i=1}^{p}\frac{\binom{y_i}{r_i}}{\theta_i^{r_i}}\times \\
\int_0^\infty \frac{\eta^{\sum_1^p r_i}}{\sqrt{2\pi a\eta^3}}e^{-((\sqrt{b^2+2pa})\eta-1)^2/2a\eta}d\eta.
\end{aligned}$$

As before the integral in the above expression is the $\sum_1^p r_i$ th moment of the inverse Gaussian distribution with parameters a and $\sqrt{b^2 + 2pa}$.

The conditional pmf of $Y_{r+1}, Y_{r+2}, \ldots, Y_p$ given $Y_1, Y_2, \ldots, Y_r$ is given by

$$\begin{aligned}
P(Y_{r+1} = y_{r+1}, Y_{r+2} = y_{r+2}, \ldots, Y_p{=}y_p|Y_1{=}y_1, Y_2{=}y_2, \ldots, Y_r{=}y_r) \qquad (3.15) \\
= \frac{e^{-\left[\frac{\sqrt{b^2+2pa}-\sqrt{b^2+2ra}}{a}\right]}\prod_{i=r+1}^{p}\frac{e^{-\theta_i}\theta_i^{y_i}}{y_i!}\sum_{r_p=0}^{y_p}\sum_{r_{p-1}=0}^{y_{p-1}}\cdots\sum_{r_1=0}^{y_1}\prod_{i=1}^{p}\frac{\binom{y_i}{r_i}}{\theta_i^{r_i}}\times \int_0^\infty \frac{\eta^{\sum_{i=1}^p r_i}}{\sqrt{2\pi a\eta^3}}e^{-((\sqrt{b^2+2pa})\eta-1)^2/2a\eta}d\eta}{\sum_{r_r=0}^{y_r}\sum_{r_{r-1}}^{y_{r-1}}\cdots\sum_{r_1=0}^{y_1}\prod_{i=1}^{r}\frac{\binom{y_i}{r_i}}{\theta_i^{r_i}}\int_0^\infty \frac{\eta^{\sum_1^r r_i}}{\sqrt{2\pi a\eta^3}}e^{-((\sqrt{b^2+2ra})\eta-1)^2/2a\eta)}d\eta}.
\end{aligned}$$

4 Univariate Multiplicative Model

Let X be a discrete random variable having Poisson distribution with mean θ. If the mean is modified to $\theta\eta$, then the probability mass function (pmf) of the modified random variable Y is given by

$$P(Y = y|\eta) = \frac{e^{-\theta\eta}(\theta\eta)^y}{y!}, \quad y = 0, 1, 2, \ldots, \tag{4.1}$$

where η is considered as a random variable with certain probability structure. It is clear that

$E(Y|\eta) = \theta\eta$ and $Var(Y|\eta) = \theta\eta$. Also conditionally Y has a Poisson distribution. Unconditionally,

$$\begin{aligned} E(Y) &= \theta E(\eta) \\ \text{and } Var(Y) &= E(Var(Y|\eta)) + Var(E(Y|\eta)) = \theta E(\eta) + \theta^2 Var(\eta). \end{aligned} \tag{4.2}$$

Thus $Var(Y) > E(Y)$ and the mean -variance relationship of the Poisson distribution does not hold.

4.1 Probability Mass Function of Y. The pmf of Y can be written as

$$P(Y{=}y){=}\int_0^\infty \frac{e^{-\theta\eta}(\theta\eta)^y}{y!} g(\eta)d\eta{=}\frac{\theta^y}{y!}\int_0^\infty e^{-\theta\eta}\eta^y g(\eta)d\eta, y{=}0,1,2,\ldots \tag{4.3}$$

4.2 Probability Generating Function of Y. The pgf of the conditional distribution is given by

$$G(z|\eta) = e^{\theta\eta(z-1)}$$

Then the unconditional pgf is given by

$$G(z) = \int_0^\infty G(z|\eta)g(\eta)d\eta = M_\eta(\theta(z-1)),$$

where $M_\eta(t)$ is the moment generating function of the mixing pdf $g(\eta)$.

Now we present two examples.

EXAMPLE 3. Suppose η has a gamma distribution. In this case the pmf of Y is given by

$$\begin{aligned} P(Y = y) &= \frac{\theta^y}{y!}\int_0^\infty e^{-\theta\eta}\eta^y \frac{\beta^\alpha \eta^{\alpha-1} e^{-\eta\beta}}{\Gamma(\alpha)} d\eta \\ &= \frac{\Gamma(\alpha+y)}{y!\Gamma(\alpha)}\left(\frac{\theta}{\theta+\beta}\right)^y \left(\frac{\beta}{\theta+\beta}\right)^\alpha, \quad y = 0, 1, 2, \ldots \end{aligned} \tag{4.4}$$

Note that Y has a negative binomial distribution.

EXAMPLE 4. Suppose η has an inverse Gaussian distribution as before. Then the pmf of Y is given by

$$\begin{aligned} P(Y=y) &= \int_0^\infty \frac{e^{-\theta\eta}(\theta\eta)^y}{y!}\frac{1}{(2\pi a\eta^3)^{1/2}}e^{-(b\eta-1)^2/2a\eta}d\eta \qquad (4.5) \\ &= \frac{\theta^y}{y!}e^{-(\sqrt{b^2+2a\theta}-b)/a}\int_0^\infty \eta^y\frac{1}{(2\pi a\eta^3)^{1/2}}e^{-((\sqrt{b^2+2a\theta})\eta-1)^2/2a\eta}d\eta. \end{aligned}$$

Note that the integral in the above expression is the y th moment of inverse Gaussian distribution with parameters a and $\sqrt{b^2+2a\theta}$.

It can be noticed that the above expression has the following recurrence formula

$$P(y+1) = \delta^2\theta[\theta P(y-1)/y + a(2y-1)P(y)]/(y+1), \ y = 1,2,3,\ldots$$

where $\delta = 1/\sqrt{b^2+2a\theta}$and $P(y) = P(Y = y)$, see for instance Ord and Whitmore (1986).

5 Shared Multiplicative Multivariate Model

Suppose $Y_1, Y_2, \ldots, Y_p$ are conditionally independent Poisson random variables with means $\theta_i\eta$, $i = 1,2,\ldots,p$. Then the pgf of $Y_1, Y_2, \ldots, Y_p$ is given by

$$\begin{aligned} \psi(t_1,t_2,\ldots,t_p) &= \int_0^\infty \prod_{i=1}^p e^{-\theta_i\eta(1-t_i)}g(\eta)d\eta \qquad (5.1) \\ &= \int_0^\infty e^{-\eta\sum_{i=1}^p \theta_i(1-t_i)}g(\eta)d\eta \ = \ M_\eta\left(-\sum_{i=1}^p \theta_i(1-t_i)\right), \end{aligned}$$

where $M_\eta(.)$ is the moment generating function of η.

Two special cases.

(1) Suppose η has a gamma distribution with mgf given by

$$M_\eta(t) = (1-t/\beta)^{-\alpha}.$$

Then the joint pgf is given by

$$\psi(t_1,t_2,\ldots,t_p) = M_\eta\left(-\sum_{i=1}^p \theta_i(1-t_i)\right) = \left(1+\frac{\sum_{i=1}^p \theta_i(1-t_i)}{\beta}\right)^{-\alpha}. \quad (5.2)$$

(2) Suppose η has an inverse Gaussian distribution with mgf given by

$$M_\eta(t) = \exp\left[\frac{b}{a}\left(1-\left(1-\frac{2at}{b^2}\right)^{1/2}\right)\right].$$

Then the joint pgf is given by

$$\psi(t_1, t_2, \ldots, t_p) = \exp\left(\frac{b}{a}\left(1-\left(1+\frac{2a\sum_{i=1}^{p}\theta_i(1-t_i)}{b^2}\right)^{1/2}\right)\right). \quad (5.3)$$

As before the expressions for the probability mass functions are complex and will be obtained later.

We now obtain the pgf of the conditional distribution of $Y_{r+1}, Y_{r+2}, \ldots, Y_p$ given $Y_1, Y_2, \ldots . Y_r$. This pgf is given by

$$\psi^\star(t_{r+1}, t_{r+2}, \ldots, t_p) = \frac{\psi(1, 1, \ldots, 1, t_{r+1}, t_{r+2}, \ldots, t_p)}{P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_r = y_r)}. \quad (5.4)$$

(1) For the gamma distribution, the above equation reduces to

$$\psi^\star(t_{r+1}, t_{r+2}, \ldots, t_p) = \frac{(1+\sum_{i=r+1}^{p}\frac{\theta_i(1-t_i)}{\beta})^{-\alpha}}{P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_r = y_r)}. \quad (5.5)$$

(2) For the inverse Gaussian distribution, equation (5.4) reduces to

$$\psi^\star(t_{r+1}, t_{r+2}, \ldots, t_p) = \frac{\exp\{\frac{b}{a}[1-(1+\frac{2a}{b^2}\sum_{i=r+1}^{p}\theta_i(1-t_i))^{1/2}]\}}{P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_r = y_r)}. \quad (5.6)$$

We now want to study the correlation induced in the case of p=2.

Induced Dependence. As explained before $E(Y_i) = \theta_i E(\eta)$ and $Var(Y_i) = \theta_i E(\eta) + \theta_i^2 Var(\eta), i = 1, 2$.

Also

$$\begin{aligned} Cov(Y_1, Y_2) &= E[E(Y_1 Y_2|\eta)] - E[E(Y_1|\eta)]E[E(Y_2|\eta)] \\ &= \theta_1\theta_2 E(\eta^2) - \theta_1\theta_2(E(\eta))^2 = \theta_1\theta_2 Var(\eta). \end{aligned}$$

Thus the correlation coefficient between Y_1 and Y_2 is given by

$$\begin{aligned} \rho &= \frac{\theta_1\theta_2 Var(\eta)}{\sqrt{\theta_1 E(\eta) + \theta_1^2 Var(\eta)}\sqrt{\theta_2 E(\eta) + \theta_2^2 Var(\eta)}} \quad (5.7) \\ &= \frac{1}{\sqrt{\frac{1}{\theta_1\theta_2}\left(\frac{E(\eta)}{Var(\eta)}\right)^2 + \left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)\frac{E(\eta)}{Var(\eta)} + 1}}, \end{aligned}$$

assuming $Var(\eta) \neq 0$. The case $Var(\eta) = 0$ corresponds to independence.

(1) For the gamma distribution, the correlation coefficient is given by

$$\rho = \frac{1}{\sqrt{(1+\frac{\beta}{\theta_1})(1+\frac{\beta}{\theta_2})}}. \tag{5.8}$$

Notice that in this case, the correlation coefficient is independent of α.

(2) For the inverse Gaussian distribution, the correlation coefficient is given by

$$\rho = \frac{1}{\sqrt{(1+\frac{b^2}{a\theta_1})(1+\frac{b^2}{a\theta_2})}}. \tag{5.9}$$

5.1 Joint probability mass functions. In this section, we shall obtain the joint probability mass functions of $Y_1, Y_2, \ldots, Y_p$ and the conditional mass function of $Y_{r+1}, Y_{r+2}, \ldots, Y_p$ given $Y_1, Y_2, \ldots, Y_r$ in the case of gamma random effect as well as in the case of inverse Gaussian random effect.

5.1.1. Gamma random effect.

$$\begin{aligned} &P(Y_1 = y_1, Y_2 = y_2) \qquad (5.10)\\ &= \int_0^\infty \frac{e^{-\theta_1\eta}(\theta_1\eta)^{y_1}}{y_1!}\frac{e^{-\theta_2\eta}(\theta_2\eta)^{y_2}}{y_2!}g(\eta)d\eta \\ &= \frac{\theta_1^{y_1}}{y_1!}\frac{\theta_2^{y_2}}{y_2!}\int_0^\infty e^{-(\theta_1+\theta_2)\eta}\eta^{y_1+y_2}\frac{\beta^\alpha}{\Gamma(\alpha)}e^{-\beta\eta}\eta^{\alpha-1}d\eta \\ &= \frac{\Gamma(y_1+y_2+\alpha)}{y_1!y_2!\Gamma(\alpha)}\left(\frac{\theta_1}{\theta_1+\theta_2+\beta}\right)^{y_1}\left(\frac{\theta_2}{\theta_1+\theta_2+\beta}\right)^{y_2}\left(\frac{\beta}{\theta_1+\theta_2+\beta}\right)^{\alpha}, \\ &\quad y_1, y_2 = 0, 1, 2, \ldots. \end{aligned}$$

Generalizing the above expression, the joint probability mass function of $Y_1, Y_2, \ldots, Y_p$ is given by

$$\begin{aligned} &P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_p = y_p) \qquad (5.11)\\ &= \frac{\Gamma(\sum_{i=1}^p y_i + \alpha)}{\prod_{i=1}^p y_i!\Gamma(\alpha)}\prod_{i=1}^p\left(\frac{\theta_i}{\sum_{i=1}^p\theta_i+\beta}\right)^{y_i}\left(\frac{\beta}{\sum_{i=1}^p\theta_i+\beta}\right)^{\alpha}, \\ &\quad y_i = 0, 1, 2, \ldots;\ i = 1, 2, \ldots, p. \end{aligned}$$

The conditional pmf of $Y_{r+1}, Y_{r+2}, \ldots, Y_p$ given $Y_1, Y_2, \ldots, Y_r$ is given by

$$\begin{aligned} P(Y_{r+1} &= y_{r+1},,...Y_p = y_p | Y_1 = y_1, ..., Y_r = y_r) \qquad (5.12) \\ &= \binom{\sum_{i=1}^p y_i + \alpha - 1}{y_{r+1}, y_{r+2}, ..., y_p} \left(\frac{\sum_{j=1}^r \theta_j + \beta}{\sum_{j=1}^p \theta_j + \beta} \right)^{\alpha + \sum_{i=1}^r y_i} \times \\ &\quad \left(\frac{\theta_{r+1}}{\sum_{j=1}^p \theta_j + \beta} \right)^{y_{r+1}} \cdots \left(\frac{\theta_p}{\sum_{j=1}^p \theta_j + \beta} \right)^{y_p}, \\ y_{r+1}, y_{r+2}, ..., y_p &= 0, 1, 2, ... \end{aligned}$$

5.1.2. Inverse Gaussian random effect.

$$\begin{aligned} P(Y_1 &= y_1, Y_2 = y_2) = \int_0^\infty \frac{e^{-\theta_1 \eta}(\theta_1 \eta)^{y_1}}{y_1!} \frac{e^{-\theta_2 \eta}(\theta_2 \eta)^{y_2}}{y_2!} g(\eta) d\eta \\ &= \frac{\theta_1^{y_1} \theta_2^{y_2}}{y_1! y_2!} \int_0^\infty e^{-(\theta_1 + \theta_2)\eta} \eta^{y_1 + y_2} \frac{e^{-(b\eta - 1)^2 / 2a\eta}}{(2a\pi\eta^3)^{1/2}} d\eta \\ &= e^{[b - \sqrt{b^2 + 2a(\theta_1 + \theta_2)}]/a} \left(\frac{\theta_1^{y_1} \theta_2^{y_2}}{y_1! y_2!} \right) \\ \times \quad & \int_0^\infty \frac{\eta^{y_1 + y_2}}{(2a\pi\eta^3)^{1/2}} e^{-((\sqrt{b^2 + 2a(\theta_1 + \theta_2)})\eta - 1)^2 / 2a\eta} d\eta. \qquad (5.13) \end{aligned}$$

Generalizing the above expression, the joint probability mass function of $Y_1, Y_2, \ldots, Y_p$ is given by

$$\begin{aligned} & P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_p = y_p) \qquad (5.14) \\ &= e^{[b - \sqrt{b^2 + 2a \sum_{i=1}^p \theta_i}]/a} \left(\prod_{i=1}^p \frac{\theta_i^{y_i}}{y_i!} \right) \times \\ &\quad \int_0^\infty \frac{\eta^{\sum_1^p y_i}}{(2a\pi\eta^3)^{1/2}} e^{-((\sqrt{b^2 + 2a \sum_{i=1}^p \theta_i})\eta - 1)^2 / 2a\eta} d\eta. \end{aligned}$$

Note that the integral in the above expression is the $\sum_{i=1}^p y_i th$ moment of the inverse Gaussian distribution with parameters a and $\sqrt{b^2 + 2a \sum_{i=1}^p \theta_i}$.

The conditional pmf of $Y_{r+1}, Y_{r+2}, \ldots, Y_p$ given $Y_1, Y_2, \ldots, Y_r$ is given by

$$\begin{aligned} P(Y_{r+1} &= y_{r+1}, \ldots, Y_p = y_p | Y_1 = y_1, Y_2 = y_2, \ldots, Y_r = y_r) \\ &= e^{-[\sqrt{b^2 + 2a \sum_{i=1}^p \theta_i} - \sqrt{b^2 + 2a \sum_{i=1}^r \theta_i}]} \times I, \qquad (5.15) \end{aligned}$$

where

$$I = \prod_{i=r+1}^{p} \left(\frac{\theta_i^{y_i}}{y_i!} \right) \frac{\int_0^\infty \frac{\eta^{\sum_1^p y_i}}{(2a\pi\eta^3)^{1/2}} e^{-\left(\eta\sqrt{b^2+2a\sum_{i=1}^{p}\theta_i}-1\right)^2 / 2a\eta} d\eta}{\int_0^\infty \frac{\eta^{\sum_1^r y_i}}{(2a\pi\eta^3)^{1/2}} e^{-\left(\eta\sqrt{b^2+2a\sum_{i=1}^{r}\theta_i}-1\right)^2 / 2a\eta} d\eta}. \tag{5.16}$$

6 An Example

As an example, we consider the data on Malayan butterflies, which was fitted by Poisson-lognormal distribution by Blumer (1974). We examine the fitting of the four random effect models in Examples 1 to 4 to this data set. As a comparison, the fit of Blumer (1974) with the Poisson-lognormal distribution is also presented.

Maximum likelihood (ML) estimation is used to obtain the parameter estimates of the models. Due to the complicated likelihood function, the ML estimates are determined numerically by simulated annealing, a global optimization procedure. The probabilities of the additive models are computed from formulas (2.6) and (2.8). The MLE's are given by

POISSON-NEGATIVE BINOMIAL: $\hat{\theta} = 0.15109$, $\hat{\alpha} = 0.26231$, $\hat{\beta} = 0.02605$.
(ADDITIVE WITH GAMMA RANDOM EFFECT)

POISSON-POISSON INVERSE GAUSSIAN: $\hat{\theta} = .77884$, $\hat{a} = 100.0$, $\hat{b} = 0.01752$.
(ADDITIVE WITH INVERSE GAUSSIAN RANDOM EFFECT)

NEGATIVE BINOMIAL: $\hat{\theta} = 11.26006$, $\hat{\alpha} = 0.31644$, $\hat{\beta} = 0.52417$.
(MULTIPLICATIVE WITH GAMMA RANDOM EFFECT)

POISSON-INVERSE GAUSSIAN: $\hat{\theta} = 8.03419$, $\hat{a} = 10.2100$, $\hat{b} = 0.14711$.
(MULTIPLICATIVE WITH INVERSE GAUSSIAN RANDOM EFFECT)

The fits of the five models is given in Table 1. It can be seen that the fit by the additive Poisson-NB model is apparently better than the fits by the Poisson. It should be noticed that the chi-square value and the parameter estimates for the multiplicative Poisson inverse Gaussian model agree with those given by Ord and Whitmore (1986) for the Poisson inverse Gaussian after taking into consideration the multiplicative parameter θ.

TABLE 1. DISTRIBUTION OF CORBET'S BUTTERFLIES WITH ZEROS (BLUMER, 1974)

	Obs.	Poi.LogN (Blumer)	Additive Poi-NB	Additive Poi-PiG	Mult.NB	Mult.PiG
0	304	295.0	303.10	266.19	345.06	267.34
1	118	127.4	123.28	168.76	104.33	167.68
2	74	74.6	62.83	95.01	65.62	94.50
3	44	50.7	43.29	58.39	48.41	58.24
4	24	37.5	33.73	39.51	38.36	39.49
5	29	29.3	27.77	28.76	31.64	28.79
6	22	23.7	23.61	22.06	26.79	22.10
7	20	19.7	20.51	17.59	23.10	17.63
8	19	16.7	18.10	14.44	20.18	14.48
9	20	14.4	16.16	12.12	17.82	12.16
10	15	12.6	14.57	10.36	15.86	10.40
11	12	11.1	13.23	8.99	14.22	9.02
12	14	9.9	12.10	7.89	12.81	7.93
13	6	8.9	11.10	7.00	11.60	7.03
14	12	8.1	10.24	6.27	10.54	6.30
15	6	7.3	9.49	5.65	9.61	5.68
16	9	6.7	8.81	5.13	8.79	5.16
17	9	6.2	8.21	4.69	8.06	4.71
18	6	5.7	7.67	4.30	7.41	4.32
19	10	5.3	7.19	3.97	6.83	3.99
20	10	4.9	6.74	3.67	6.30	3.69
21	11	4.6	6.34	3.41	5.82	3.43
22	5	4.3	5.97	3.18	5.39	3.20
23	3	4.0	5.63	2.98	5.00	2.99
24	3	3.8	5.32	2.79	4.64	2.81
25+	119	131.3	118.99	120.84	69.78	120.92
Total	924	923.7	923.98	923.95	923.97	923.99
Chi.sq.		36.8	19.46	100.55	64.86	98.57
df		23	22	22	22	22

7 Some Conclusions and Remarks

It is well known that the unobserved heterogeneity is a major source of overdispersion in count data models. One way to take care of this heterogeneity is by way of mixture models. In this paper, we have introduced an additive and a multiplicative random effect Poisson model. The random effect has been modelled by the gamma distribution and the inverse Gaussian distribution. This way we have proposed four alternative models for overdispersed Poisson data.These models provide alternatives for the data analyst when there is some evidence that the assumptions of the model are satisfied, see our example of telephone company in the introduction. The example presented in the paper illustrates that one of those models gives a better fit than the other three and in fact better than the Poisson-lognormal model.

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