Summary of Current Work - Vaidya Black Hole in f(R) Gravity

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This is a summary of work done by J. P. Hou, NUAA. One main topic is included:

1) We investigate the thermodynamics properties of Vaidya black holes in f(R) gravity

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The action of f(R) gravity is

$$S = \frac{1}{16\pi G}\int d^4 s\sqrt{-g}f(R) + S_{matter} \tag{.1}$$

Varying the action, one gets the equation of gravitational field

$$f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} + g_{\mu\nu}\Box f_R = 8\pi G T_{\mu\nu} \tag{.2}$$

I. STATIC

For simplicity, we fisrtly consider the static situation. A general form of static spherical symmetric spacetime can be written as

$$ds^2 = -F(r)dt^2 + \frac{1}{F(r)}dr^2 + b^2(r)(d\theta^2 + \sin^2 d\phi^2) \tag{I.1}$$

Assuming the metric (I.1) is non-extreme black holes, that is, $F(r)|_{r=r_+} = 0$ while $F'(r)|_{r=r_+} \neq 0$, where $r = r_+$ is the horizon. From the definition

$$\kappa = \frac{1}{2\sqrt{-h}}\partial_a(\sqrt{-h}h^{ab}\partial_b r) \tag{I.2}$$

one could obtain the temperature of the black hole

$$T = \frac{\kappa}{2\pi}|_{r=r_+} = \frac{1}{4\pi}F'(r_+)b'(r_+) \tag{I.3}$$

The components of (.2) with metric (I.1) are

$$T^t_t = \frac{1}{2b}(-bf - 2f_R b'F' + 4Fb'f'_R - bf_R F'' - 2bFf''_R) \tag{I.4}$$

$$T^r_r = \frac{1}{2b}(-bf - 2f_R b'F' + 4Fb'f'_R - bf_R F'' - 4Ff_R b'') \tag{I.5}$$

Here, $'$ is derivative with respect to r. Note that $F(r_+) = 0$, thus

$$T^t_t|_{r=r_+} = T^r_r|_{r=r_+} = \frac{1}{2b}(-bf - 2f_R b'F' + 4Fb'f'_R - bf_R F'')|_{r=r_+} \tag{I.6}$$

It is obviously that $T^r_r|_{r=r_+}$ can be treated as pressure, as a result

$$4\pi G b^2 P dr_+ = \frac{b}{4G}(-bf - 2f_R b'F' + 4Fb'f'_R - bf_R F'')dr_+ \tag{I.7}$$

In f(R) gravity, the entropy is

$$S = \frac{\pi b^2}{G}f_R \tag{I.8}$$

Inserting TdS into (I.7)

$$PdV = TdS - \frac{1}{4G}(b'b^2(f-R) + 2b')dr_+ \tag{I.9}$$

To investigate this equation further, we calculate the Misner-Shape energy. One can define a Kodama vector in a spherical symmetric spacetime and it is apparent that it satisfies the following constraints

$$\nabla_\mu \nabla_\nu f_R \nabla^\mu K^\nu = 0 \tag{I.10}$$

Then, one may calculate the Misner-Shape energy with

$$dE_{eff} = dE_{eff} = A\Phi_a dx^a + WdV = A(t,r)dt + B(t,r)dr \tag{I.11}$$

where $A = V^k_{n-2} b^{n-2}$ and $V = V^k_{n-2} b^{n-1}/(n-1)$ are area and volume of $(n-2)$ dimension sphere with radius r, and

$$A(t,r) = 0,\ \ B(t,r) = A(FT_{rr} + 2W) \tag{I.12}$$

Inserting T_{rr} into the equation above, one has

$$dE_{eff} = \frac{b^2}{2G}(\frac{1}{2}f + f_R \frac{b'}{b}F' - 2F\frac{b'}{b}f'_R - \frac{1}{2}F'f'_R + \frac{1}{2b}f_R F'' - Ff''_R)b'dr_+ \tag{I.13}$$

After integrated, the Misner-Shape energy is

$$E_{eff} = \frac{b}{2G}((1 - h^{ab}\partial_a b \partial_b b)f_R + \frac{b^2}{6}(f - f_R R) - bh^{ab}\partial_a f_R \partial_b b) \tag{I.14}$$

$$-\frac{1}{2G}\int dr(-\frac{1}{2}b^2 b'^2 F' + b - b'^2 F b - \frac{1}{6}b^3 R - b^2 b'' F)f'_R \tag{I.15}$$

Then, one may get $dE_{eff}|r_{r=r_+}$ readily

$$dE_{eff} = \frac{b^2}{2G}(\frac{1}{2}f + f_R \frac{b'}{b}F' - 2F\frac{b'}{b}f'_R - \frac{1}{2}F'f'_R + \frac{1}{2b}f_R F'' - Ff''_R)b'dr_+ \tag{I.16}$$

Finally, we have the thermodynamics relation at the horizon

$$dE = TdS - PdV + \frac{1}{4G}((b^2 R - 2)(1 - f_R) - b^2 F' f'_R)b'dr_+ \tag{I.17}$$

II. DYNAMIC

The metric we chosen is

$$ds^2 = -F(t,r)dt^2 + 2dtdr + r^2(d\theta^2 + \sin^2 d\phi^2) \tag{II.1}$$

Some the non-zero component of the motion equation are

$$\begin{aligned}
8\pi G T_{tt} = &\ \frac{1}{2r}(frF + 2f_R F F_{,r} - 4F^2 f_{R,r} - rFF_{,r}f_{R,r} \\
&-2rF^2 f_{R,rr} - 2f_R F_{,t} - rf_{R,r}F_{,r} - 4Ff_{R,t} + rF_{,r}f_{R,t} \\
&-4rFf_{R,tr} - 2rf_{R,tt}) \qquad \text{(II.2)} \\
8\pi G T_{tr} = &\ \frac{1}{2G}(-fr - 2f_R F_{,r} + 4Ff_{R,r} + rF_{,r}f_{R,r} - f_R rF_{,rr} \\
&+2rFf_{R,rr} + 4f_{R,t} + 2rf_{R,tr}) \qquad \text{(II.3)} \\
8\pi G T_{rr} = &\ -f_{R,rr} \qquad \text{(II.4)} \\
8\pi G T_{\theta\theta} = &\ \frac{1}{2}(2f_R - fr^2 - f_R F - 2f_R rF_{,r} + 2rFf_{R,r} + 2rf_{R,t} \\
&2r^2 F_{,r}f_{R,r} + 2r^2 Ff_{R,rr} + 4r^2 f_{R,tr}) \qquad \text{(II.5)} \\
8\pi G T_{\phi\phi} = &\ \frac{1}{2}(2f_R - fr^2 - 2f_R F - 2f_R rF_{,r} + 2rFf_{R,r} + 2rf_{R,t}) \\
&2r^2 F_{,r}f_{R,r} + 2r^2 Ff_{R,rr} + 4r^2 f_{R,tr})\sin^2\theta \qquad \text{(II.6)}
\end{aligned}$$

We follow the aforementioned steps to investigate its thermodynamics. However, constraint of Kodama vector dose not satisfy automatically. Now, the Kodama vector is simply

$$k^\mu = (\frac{\partial}{\partial t})^\mu \qquad \text{(II.7)}$$

and the constraint turns out to be

$$-2f_R F_{,t} - 4rf_{R,r}F_{,t} - r^2 f_{R,rr}F_{,t} - 4rf_R F_{,tr} - 2r^2 f_{R,r}F_{,tr} - r^2 f_R F_{,trr} = 0 \qquad \text{(II.8)}$$

Solving the differential equation, one obtains

$$f_R F_{,t} = \frac{C_1}{r} + \frac{C_2}{r^2} \qquad \text{(II.9)}$$

Assuming this holds, we can still apply the unified first law

$$\begin{aligned}
\delta Q &= A\Phi_a dx^2 \\
&= A(T_{tt} + FT_{tr})dt \\
&= \frac{r_A}{4G}(r_A f_{R,t}F_{,r} - 2f_R F_{,t} - r_A f_{R,r}F_{,t} - 2r_A f_{R,tt})dt \qquad \text{(II.10)}
\end{aligned}$$

Notice that, $F(t, r_A(t)) = 0$, and thus, $r_{A,t} = -F_{,t}/F_{,r}$, is used. While now, T takes the form $F_{,r}/(4|pi)$ and S are the same as the static situation since it has nothing to do with metric. Thus,

$$TdS = \frac{r_A}{4G}(r_A f_{R,t}F_{,r} - 2f_R F_{,t} - r_A f_{R,r}F_{,t})dt \qquad \text{(II.11)}$$