2 Probability

Part II

Permutations and Combinations

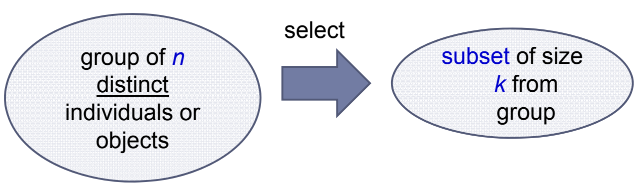
(วิธีเรียงสับเปลี่ยน) (วิธีการจัดหมู่)

Permutations and Combinations

- Consider a group of n distinct individuals or objects

  there is some characteristic that differentiates any particular individual or object from any other

- How many ways are there to select a subset of size k from the group?

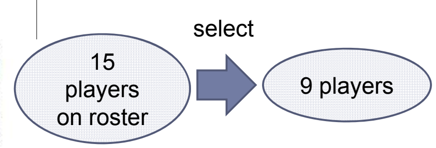


Permutations and Combinations

Example

- if a Little League team has 15 players on its roster, how many ways are there to select 9 players to form a starting lineup?

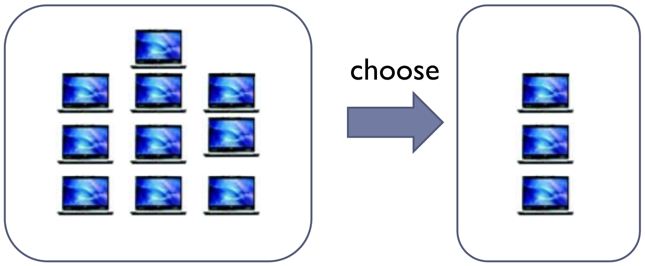
15	Reggie Jackson	G	6-3	208	04/16/90	Boston College/USA	2
47	Grant Jerrett	F	6-10	220	07/08/93	Arizona/USA	R
3	Perry Jones	F	6-11	235	09/24/91	Baylor/USA	1
8	Kyle Kuric	G	6-4	195	08/25/89	Louisville/USA	R
11	Jeremy Lamb	G	6-5	180	05/30/92	Connecticut/USA	1
14	DeAndre Liggins	G	6-6	209	03/31/88	Kentucky/USA	2
33	Daniel Orton	C	6-10	255	08/06/90	Kentucky/USA	3
21	Andre Roberson	F	6-7	190	12/04/91	Colorado/USA	R
44	Andrew Smith	C	6-11	240	09/09/90	Butler/USA	R
2	Michael Snaer	G	6-4	180	06/21/90	Florida State/USA	R
7	Tony Taylor	G	6-0	191	09/09/90	George Washington/USA	R



Permutations and Combinations

Example

- Or if a university bookstore sells ten different laptop computers but has room to display only three of them, in how many ways can three be chosen?



Permutations and Combinations

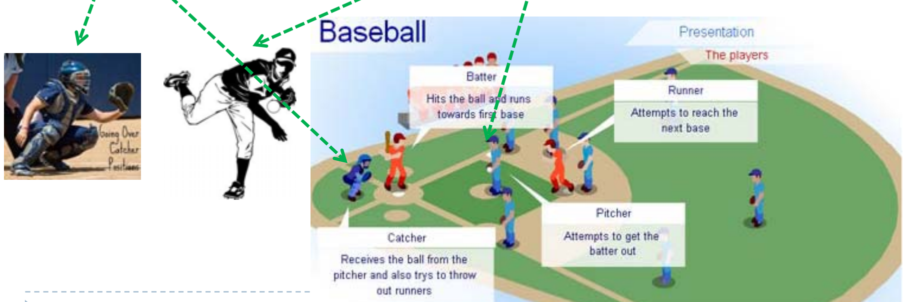
- An answer to the general question just posed requires that we distinguish between two cases.

1) The order of selection is important

2) The order of selection is not important

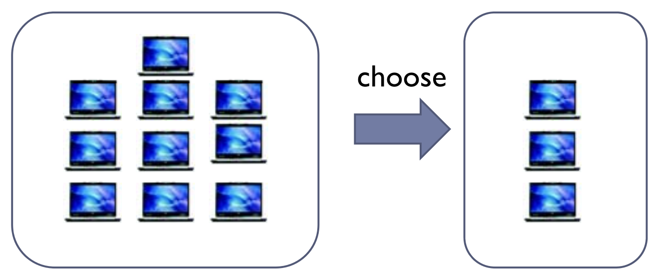
Permutations and Combinations

- In some situations, such as the baseball scenario, the order of selection is important.
  - For example, Angela being the pitcher and Ben the catcher gives a different lineup from the one in which Angela is catcher and Ben is pitcher.



Permutations and Combinations

- Often, though, order is not important and one is interested only in which individuals or objects are selected, as would be the case in the laptop display scenario.



Permutations and Combinations

- **Definition**
- An **ordered subset** is called a **Permutation**.(วิธีเรียงสับเปลี่ยน)
  - Number of permutations of size ***k*** that can be formed from the ***n*** individuals or objects in a group

denoted by ➡ $P_{k,n}$

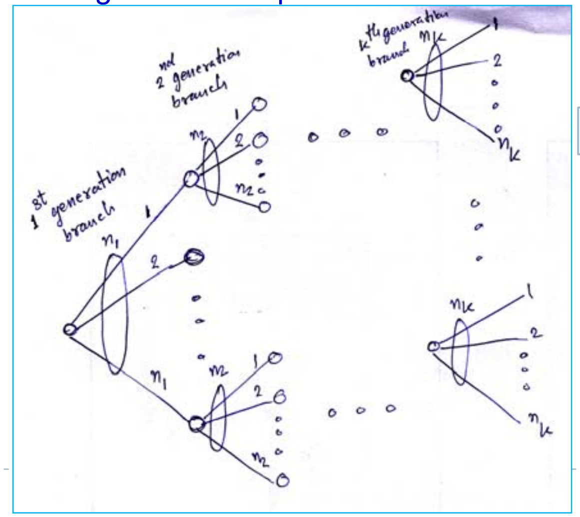
- An **unordered subset** is called a **Combination**. (วิธีการจัดหมู่)
  - Number of combination of size ***k*** that can be formed from the ***n*** individuals or objects in a group

denoted by ➡ $C_{k,n}$ or $\binom{n}{k}$ ➡ read "*n* choose *k*"

This notation is quite common use in probability books.

Permutations and Combinations

- Number of permutations can be determined by using counting rule for *k*-tuples.



Tree diagram

เกิดในที่นี้มีความแตกต่างของการตำแหน่ง :: Permutation is used

Permutations and Combinations

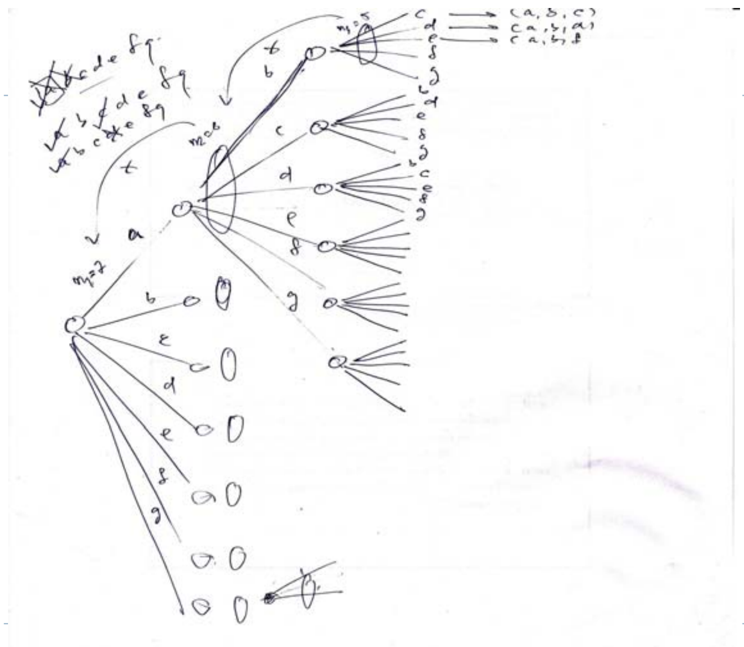
- Suppose, for example, that a college of engineering has seven departments, which we denote by *a*, *b*, *c*, *d*, *e*, *f*, and *g*.
- Each department has one representative on the college's student council.
- From these **seven representatives**,
  - one is to be chosen **chair**,
  - another is to be selected **vice-chair**, and
  - a third will be **secretary**.

- How many ways are there to select the three officers?

That is, how many permutations of size 3 can be formed from 7 representatives?

Permutations and Combinations

- To answer this question, think of forming a **triple (3-tuple)**
  - The first element is the chair,
  - The second is the vice-chair, and
  - The third is the secretary.
- One such triple is (*a*, *g*, *b*), another is (*b*, *g*, *a*), and yet another is (*d*, *f*, *b*).
  - Now **the chair** can be selected in any of $n_1 = 7$ **ways**.
- For each way of selecting the chair, there are $n_2 = 6$ **ways** to select **the vice-chair**, and hence $7 \times 6 = 42$ (chair, vice-chair) pairs.



Permutations and Combinations

- Finally, for each way of selecting a chair and vice-chair, there are n_3 = 5 ways of choosing the secretary.
- This gives

$$P_{3,7} = 7 \times 6 \times 5 = 210$$

Number of permutations of size 3 that can be formed from 7 distinct individuals

- Tree diagram representation would show three generations of branches.
- Expression for $P_{3,7}$ can be rewritten with the aid of *factorial notation*.

Recall that $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

"7 factorial" is compact notation for descending product of integers

Permutations and Combinations

- More generally, for any positive integer *m*,

$$m! = m \times (m-1) \times (m-2) \times (m-3) \times \cdots \times 3 \times 2 \times 1$$

This gives **1! = 1**, and we also define **0! = 1**. Then

$$P_{3,7} = 7 \times 6 \times 5 = 7 \times (7-1)(7-(3-1))$$

$$= 7 \times 6 \times 5 \times \left(\frac{4!}{4!}\right) = 7 \times 6 \times 5 \times \left(\frac{4!}{(7-3)!}\right) = \frac{7 \times 6 \times 5 \times 4!}{(7-3)!} = \frac{7!}{4!}$$

- More generally,

$$P_{k,n} = n \times (n-1) \times (n-2) \times (n-3) \cdots (n-(k-2)) \times (n-(k-1))$$

Permutations and Combinations

$$P_{k,n} = n \times (n-1) \times (n-2) \times (n-3) \cdots (n-(k-2)) \times (n-(k-1))$$

- Multiplying and dividing this by **(*n* – *k*)!** gives a compact expression for the number of permutations.

$$P_{k,n} = n \times (n-1) \times (n-2) \times (n-3) \cdots (n-(k-2)) \times (n-(k-1)) \times \frac{(n-k)!}{(n-k)!}$$

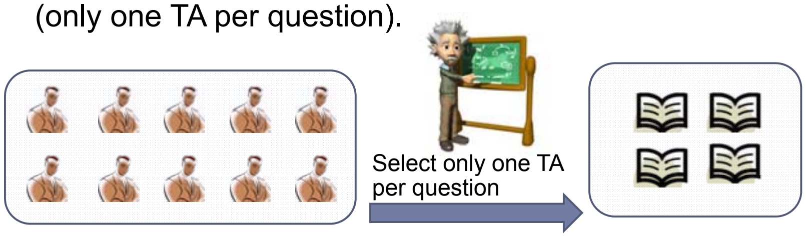
$$= \frac{n \times (n-1) \times (n-2) \times (n-3) \cdots (n-(k-2)) \times (n-(k-1)) \times (n-k)!}{(n-k)!}$$

- **Proposition** $$P_{k,n} = \frac{n!}{(n-k)!}$$

Number of permutations of size k that can be formed from n distinct individuals

Example 21 (permutations)

- There are ten teaching assistants (10 TAs) available for grading papers in a calculus course at a large university.
- First exam consists of four questions, and the professor wishes to select a different TA to grade each question (only one TA per question).



n = group size = 10 *k* = subset size = 4

- How many ways can the TAs be chosen for grading?

cont'd

Example 21

- The number of permutations is

$\frac{10\times9\times8\times7 \times 6!}{4\times 6!}$

$$P_{k,n} = \frac{n!}{(n-k)!}$$

$\binom{10}{4}4!$

$$P_{4,10} = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 10\times9\times8\times7 = 5{,}040$$

- That is, the professor could give 5040 different four-question exams without using the same assignment of graders to questions, by which time all the teaching assistants would hopefully have finished their degree programs!

Permutations and Combinations

- Now let's move on to combinations (i.e., unordered subsets).
- Again refer to the student council scenario, and suppose that three of the seven representatives are to be selected to attend a statewide convention.

$\Downarrow$

order of selection is not important

- The number of combinations of size 3 that can be formed from the 7 individuals is

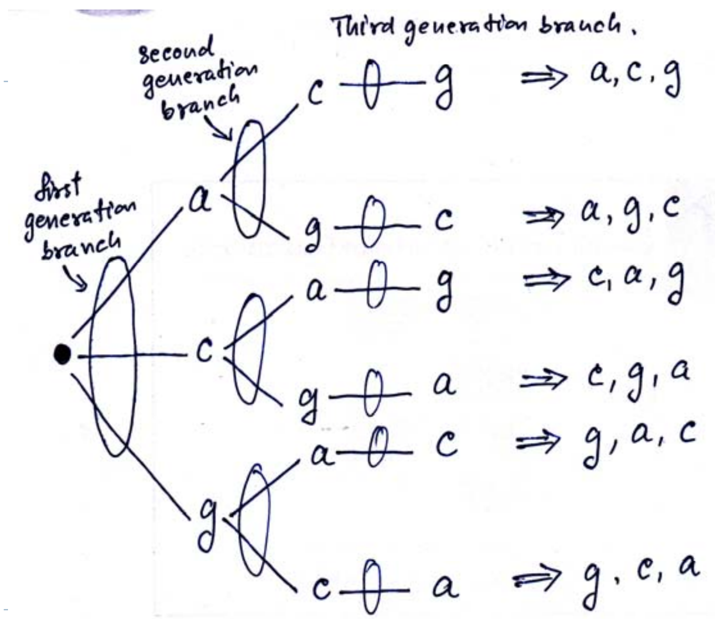
"*n* choose *k*" $\Rightarrow \binom{7}{3}$

Permutations and Combinations

- Consider for a moment the combination *a*, *c*, *g*.
- These three individuals can be ordered in 3! = 6 ways to produce permutations:

a, *c*, *g*   *a*, *g*, *c*   *c*, *a*, *g*   *c*, *g*, *a*   *g*, *a*, *c*   *g*, *c*, *a*

- Similarly, there are 3! = 6 ways to order the combination *b*, *c*, *e* to produce permutations, and
- in fact 3! ways to order any particular combination of size 3 to produce permutations.



Permutations and Combinations

- This implies the following relationship between the number of combinations and the number of permutations:

$$P_{3,7} = (3!)\binom{7}{3} \quad \Rightarrow \quad \binom{7}{3} = \frac{P_{3,7}}{3!} = \frac{7!}{(3!)(4!)} = \frac{7\times6\times5}{3\times2\times1} = 35$$

- It would not be too difficult to list the 35 combinations, but there is no need to do so if we are interested only in how many there are.
- Notice that the number of permutations 210 far exceeds the number of combinations; the former is larger than the latter by a factor of 3! since that is how many ways each combination can be ordered.

generalizing $P_{k,n} = k!\binom{n}{k}$

เน้นการ combination × k! ดีกว่า

Permutations and Combinations

- Generalizing foregoing line of reasoning gives a simple relationship between the number of permutations and the number of combinations that yields a concise expression for latter quantity.

$$P_{k,n} = k!\binom{n}{k}$$

- **Proposition**

$$\binom{n}{k} = \frac{P_{k,n}}{k!} = \frac{n!}{k!\times(n-k)!}$$

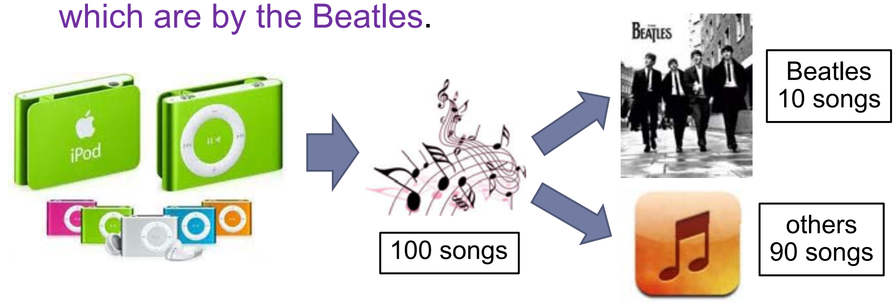
- Notice that

$\binom{n}{n} = 1$ and $\binom{n}{0} = 1$ Since there is only one way to choose a set of (all) n elements or of no elements

$\binom{n}{1} = n$ Since there are n subsets of size 1

Example 22 : Permutation

- A particular iPod playlist contains 100 songs, 10 of which are by the Beatles.



- Suppose the shuffle feature is used to play the songs in random order (the randomness of the shuffling process is investigated in “Does Your iPod *Really* Play Favorites?”

Example 22

- What is **probability** that the first Beatles song heard is the fifth song played?

NB NB NB NB B

- In order for this event to occur, it must be the case that
  - First four songs played are not Beatles' songs (NBs) and
  - Fifth song is by the Beatles (B).
- Random shuffle assumption implies that any particular set of 5 songs from amongst the 100 has the same chance of being selected as the first five played as does any other set of five songs; each outcome is equally likely.

Example 22

cont'd

- Therefore desired probability is

$$\frac{\textit{Number of outcomes for which event of interest occurs}}{\textit{Number of possible outcomes}}$$

$$\frac{95 \times 94 \times 93 \times 92 \times \binom{5}{1}}{\binom{100}{5}}$$

Example 23 : Combination

- University warehouse has received a shipment of 25 printers, of which
  - 10 are laser printers and
  - 15 are inkjet model

Laser printer

Inkjet printer

- If 6 of these 25 are selected at random to be checked by a particular technician,
- What is the probability that exactly 3 of those selected are laser printers (so that the other 3 are inkjets?

Example 23 : Combination (cont.)

- Let D_3 = {exactly 3 of 6 selected are inkjet printers}
- Assuming that any particular set of 6 printers is as likely to be chosen as is any other set of 6, we have equally likely outcome, so

$$P(D_3) = \frac{N(D_3)}{N}$$

$N(D_3)$: Number of way of choosing 3 laser printers and 3 Inkjet models

- No. of way of choosing 3 of 15 Inkjet models $\Rightarrow \binom{15}{3}$
- No. of way of choosing 3 of 15 Inkjet models $\Rightarrow \binom{10}{3}$

N: Number of way of choosing 6 printers from the 25 $\Rightarrow \binom{25}{6}$

$$P(D_3) = \frac{N(D_3)}{N} = \frac{\binom{15}{3}\binom{10}{3}}{\binom{25}{6}} = \frac{\frac{15!}{3!12!} \cdot \frac{10!}{3!7!}}{\frac{25!}{6!19!}} = 0.3083$$

Example 23 : Combination (cont.)

- What is the probability that at least 3 inkjet printers are selected?
- Let
  - D_3 = {exactly 3 of 6 selected are inkjet printers}
  - D_4 = {exactly 4 of 6 selected are inkjet printers}
  - D_5 = {exactly 5 of 6 selected are inkjet printers}
  - D_6 = {exactly 6 of 6 selected are inkjet printers}

$$P(D_3 \cup D_4 \cup D_5 \cup D_6) = P(D_3) + P(D_4) + P(D_5) + P(D_6)$$
$$= \frac{\binom{15}{3}\binom{10}{3}}{\binom{25}{6}} + \frac{\binom{15}{4}\binom{10}{2}}{\binom{25}{6}} + \frac{\binom{15}{5}\binom{10}{1}}{\binom{25}{6}} + \frac{\binom{15}{6}\binom{10}{0}}{\binom{25}{6}}$$
$$= 0.8530$$

2.4 Conditional Probability

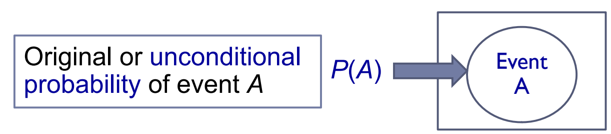
(ความน่าจะเป็นแบบมีเงื่อนไข)

Conditional Probability

- Probabilities assigned to various events depend on what is known about experimental situation when assignment is made
- Subsequent to initial assignment, partial information relevant to outcome of experiment may become available.
- Such information may cause us to revise some of our probability assignments.

Conditional Probability

- For particular event A, we have used $P(A)$ to represent probability, assigned to A; we now think of ***P*(*A*)** as **original**, or **unconditional probability of the event *A***



- In this section, we examine how the information "an event B has occurred" affects probability assigned to A

Conditional Probability

For example

Event A → Individual having a particular disease in the presence of certain symptoms

blood test is performed on the individual

Event B → result is negative (*B* = 5 negative blood test)

probability of having the disease will change

(it should decrease, but not usually to zero, since blood tests are not infallible)

Conditional Probability

***P*(*A* | *B*)** → **Conditional probability of *A* given that Event *B* has occurred**

B is the "conditioning event"

Example

Event *A* → Randomly selected student at your university obtained all desired classes during previous term's registration cycle
Presumably P(A) is not very large

Event *B* → Selected student is athlete who gets special registration priority

P(A|B) should be substantially larger than P(A), although perhaps still not close to 1

Example 24

- Complex components are assembled in a plant that uses two different assembly lines, *A* and *A′*

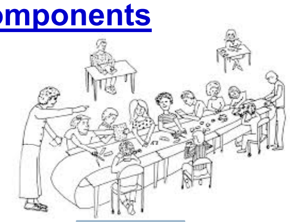
Line *A*

Line *A′*

- Line *A* uses older equipment than *A′*
- Line *A* is somewhat slower and less reliable than *A′*

Example 24

- Suppose on a given, line A and A' has **assembled 18 components**



Line *A*

Line *A′*

- 8 components
  - 2 defective (*B*)
  - 6 nondefective (*B′*)
- 10 components
  - 1 defective
  - 9 nondefective

Example 24

cont'd

- This information is summarized in the accompanying table

		Condition	
		B (defective)	B' (nondefective)
Line	A	2	6
	A'	1	9

A: 8 components

A': 10 components

- Unaware of this information, sales manager randomly selects 1 of these 18 components for a demonstration.

- Prior to the demonstration

$$P(\text{line } A \text{ component selected} = P(A) = \frac{N(A)}{N} = \frac{8}{18} = 0.44$$

Example 24

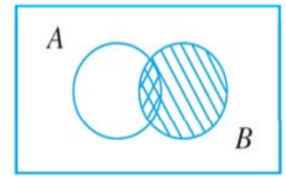
cont'd

However, if chosen component turns out to be defective

Event B has occurred

So component must have been 1 of 3 in B column of table

- Since these 3 components are equally likely among themselves after B has occurred,



		Condition	
		B	B'
Line	A	2	6
	A'	1	9

$$P(A \mid B) = \frac{2}{3} = \frac{2/18}{3/18} = \frac{P(A \cap B)}{P(B)} = 0.667 \quad (2.2)$$

Conditional Probability

- In Equation (2.2), conditional probability is expressed as a ratio of unconditional probabilities:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Numerator is probability of intersection of two events

Denominator is the probability of conditioning event B

- A Venn diagram illuminates this relationship

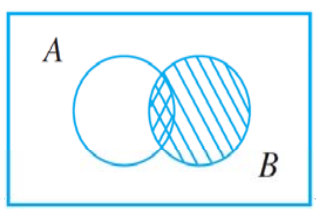


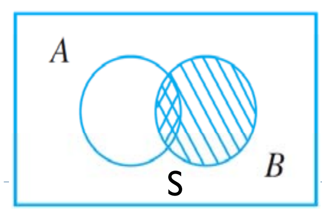
Figure 2.8 Motivating the definition of conditional probability

Conditional Probability

- Given that ***B*** has occurred, the relevant sample space is no longer ***S*** but consists of outcomes in B;

- A has occurred if and only if one of outcomes in intersection occurred, so conditional probability of A given B is proportional to $P(A \cap B)$.

- Proportionality constant $1/P(B)$ is used to ensure that the probability $P(B \mid B)$ of the new sample space B equals 1.



$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Definition of Conditional Probability

(นิยามของความน่าจะเป็นอย่างมีเงื่อนไข)

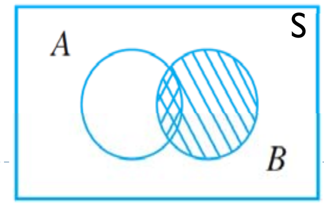
Definition of Conditional Probability

Definition

ต้องมี B เกิดขึ้นก่อนแน่นอน

For any two events A and B with $P(B) > 0$, the **conditional probability of *A* given that *B* has occurred** is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad (2.3)$$



Example 25

- Suppose that of all individuals buying a certain digital camera,
  - 60% include an optional memory card in their purchase,
  - 40% include an extra battery, and
  - 30% include both a card and battery

- Consider randomly selecting a buyer and let

  A = {memory card purchased} and
  B = {battery purchased}

  Then

  $P(A) = 0.60$,
  $P(B) = 0.40$,
  $P(\text{both purchased}) = P(A \cap B) = 0.30$

Example 25

cont'd

- Given that selected individual purchased an **extra battery (event B)**, the probability that **optional memory card (event A)** was also purchased is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.30}{0.40} = 0.75$$

- That is, of all those purchasing an extra battery, 75% purchased an optional memory card.
- Similarly,

$$P(battery \mid memory\ card) = P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.30}{0.60} = 0.50$$

- Notice that

$$P(A \mid B) \neq P(A) \ and \ P(B \mid A) \neq P(B)$$

Event whose probability is desired might be union or intersection of other events, and the same could be true of the conditioning event

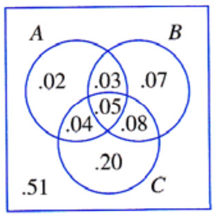
Example

- News magazine publishes three columns entitled
  - "Art" (A),
  - "Books" (B), and
  - "Cinema" (C)

การสุ่มเลือกผู้อ่าน

- Reading habits of randomly selected reader with respect to these columns are

Read regularly	A	B	C	A∩B	A∩C	B∩C	A∩B∩C
Probability	0.14	0.23	0.37	0.08	0.09	0.13	0.05



Venn diagram

We thus have $P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{0.08}{0.23}=0.348$

$$P(A|B\cup C)=\frac{P(A\cap(B\cup C))}{P(B\cup C)}=\frac{0.04+0.05+0.03}{0.03+0.05+0.07+0.08+0.04+0.2}=\frac{0.12}{0.47}=0.255$$

$$P(A|\textit{reads at least one})=P(A|(A\cup B\cup C))=\frac{P(A\cap(A\cup B\cup C))}{P(A\cup B\cup C)}=\frac{P(A)}{P(A\cup B\cup C)}=\frac{0.14}{0.49}=0.286$$

and

$$P(A\cup B|C)=\frac{P((A\cup B)\cap C)}{P(C)}=\frac{0.04+0.05+0.08}{0.37}=0.459$$

Multiplication Rule for $P(A \cap B)$

Multiplication Rule for $P(A \cap B)$

- Definition of conditional probability yields the following result, obtained by multiplying both sides of Equation (2.3) by $P(B)$.

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$

conditional probability of A given that B has occurred

Multiplication Rule

$$P(A\cap B)=P(A|B)\cdot P(B)$$

given something
multiply something

This rule is important because it is often the case that $P(A \cap B)$ is desired, whereas both $P(B)$ and $P(A|B)$ can be specified from the problem description.

Consideration of $P(B|A)$ gives $P(A\cap B)=P(B|A)\cdot P(A)$

Example 27

- Four individuals have responded to a request by a blood bank for blood donations.
- None of them has donated before, so their blood types are unknown
- Suppose only type O+ is desired and only one of four actually has this type.
- If the potential donors are selected in random order for typing,
- What is probability that **at least three individuals** must be typed to obtain the desired type?

$P(O+)$

$\frac{1}{4}$

Example 27 cont'd

- Making the identification

  B = {first type not O+} and
  A = {second type not O+}, $P(B)=\frac{3}{4}$

- Given that first type is not O+, two of the three individuals left are not O+, so $\Rightarrow P(A \mid B)=\frac{2}{3}$

- Multiplication rule now gives

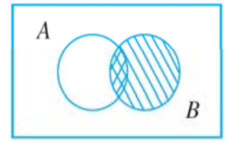
$$\begin{aligned} P(\textit{at least three individuals are typed}) &= P(A \cap B) \\ &= P(A \mid B)\cdot P(B) \\ &= \frac{2}{3}\cdot\frac{3}{4}=\frac{6}{12} \\ &= 0.5 \end{aligned}$$

Example 27 cont'd

- Multiplication rule is most useful when experiment consists of several stages in succession.
- Conditioning event B then describes outcome of first stage and A outcome of the second, so that $P(A \mid B)$ —conditioning on what occurs first—will often be known

conditional probability of A given that B has occurred $\Rightarrow P(A \mid B)=\frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B)=P(A \mid B)\cdot P(B)$

Multiplication Rule for $P(A \cap B)$



- Rule is easily extended to experiments involving more than two stages.

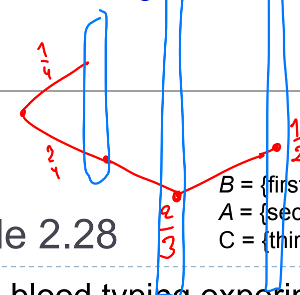
Example 27 $P(A \mid B)=\frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B)=P(A \mid B)\cdot P(B)$ cont'd

- For example,

$$P(A_1 \cap A_2 \cap A_3) = P(A_3 \mid A_1 \cap A_2)\cdot P(A_1 \cap A_2)$$

$$= P(A_3 \mid A_1 \cap A_2)\cdot P(A_2 \mid A_1)\cdot P(A_1) \qquad (2.4)$$

where A_1 occurs **first**, followed by A_2, and finally A_3.



B = {first type not O+} and
A = {second type not O+},
C = {third type is O+}

Example 2.28

- For the blood typing experiment of Example 2.27

$$P(A \cap B \cap C) = P(C \mid B \cap A)\cdot P(A \mid B)\cdot P(B)$$

$$\begin{aligned} P(\textit{third type is O+}) &= P(\textit{third is} \mid \textit{first isn't} \cap \textit{second isn't}) \\ &\quad \cdot P(\textit{second isn't} \mid \textit{first isn't})\cdot P(\textit{first isn't}) \\ &= \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}=\frac{1}{4}=0.25 \end{aligned}$$

- When experiment of interest consists of a sequence of several stages, it is convenient to represent these with a tree diagram.
- Once we have an appropriate tree diagram, probabilities and conditional probabilities can be entered on the various branches; this will make repeated use of the multiplication rule quite straightforward

Example 2.29

- A chain of video stores sells three different brands of DVD players.
- Of its DVD player sales,
  - 50% are brand 1 (the least expensive),
  - 30% are brand 2, and
  - 20% are brand 3.
- Each manufacturer offers a 1-year warranty on parts and labor.
- It is known that
  - 25% of brand 1's DVD players require warranty repair work,
  - whereas the corresponding percentages for brands 2 and 3 are 20% and 10%, respectively.

Example 2.29

1. What is probability that a randomly selected purchaser has bought a brand 1 DVD player that will need repair while under warranty?

   P(repair | band1)

2. What is probability that a randomly selected purchaser has a DVD player that will need repair while under warranty?

3. If a customer returns to store with DVD player that needs warranty repair work,

   what is probability that it is

   a brand 1 DVD player

   a brand 2 DVD player"

   a brand 3 DVD player?

Example 2.29 (cont.)

- The first stage of the problem involves a customer selecting one of the three brands of DVD player
- Let A_i={brand i is purchased}, for i=1, 2, and 3.
- Then $P(A_1)=0.50$, $P(A_2)=0.30$, and $P(A_3)=0.20$.
- Once a brand of DVD player is selected, the second stage involves observing whether the selected DVD player needs warranty repair.
- With B = {needs repair} and B' = {doesn't need repair}, the given information implies that

  $P(B|A_1)=0.25$, $P(B|A_2)=0.20$, and $P(B|A_3)=0.10$

Example 2.29 (cont.)

- Tree diagram representing this experimental situation is shown in Figure 2.10.

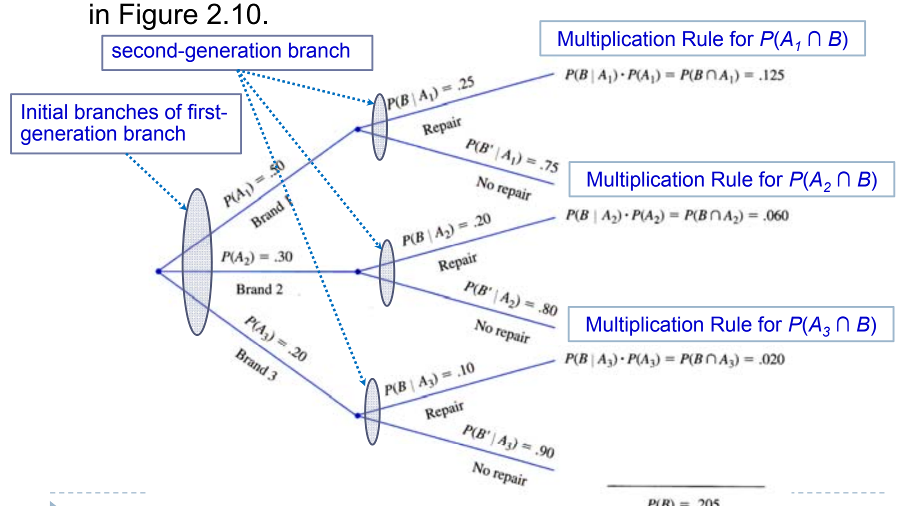


Figure 2.10 Tree diagram for example 2.29

Example 2.29 (cont.)

- Initial branches correspond to different brands of DVD players;
- there are two second-generation branches emanating from the tip of each initial branch,
  - one for "needs repair" and
  - the other for "doesn't need repair."
- Probability $P(A_i)$ appears on the i^{th} initial branch, whereas conditional probabilities $P(B|A_i)$ and $P(B'|A_i)$ appear on second generation branches.
- To the right of each second-generation branch corresponding to the occurrence of B, we display the product of probabilities on branches leading out to that point.
- This is simply multiplication rule in action

Example 2.29 (cont.)

What is probability that a randomly selected purchaser has bought a brand 1 DVD player that will need repair while under warranty?

Answer to question posed in 1 is thus

$$P(A_1 \cap B) = P(B | A_1) \cdot P(A_1) = 0.125$$

What is probability that a randomly selected purchaser has a DVD player that will need repair while under warranty?

Answer to question 2 is

$$\begin{aligned} P(B) &= P[(brand\ 1\ and\ repair)\ or\ (brand\ 2\ and\ repair)\ or\ (brand\ 3\ and\ repair)] \\ &= P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) \\ &= 0.125 + 0.060 + 0.020 = 0.205 \end{aligned}$$

Example 2.29 (cont.)

If a customer returns to store with DVD player that needs warranty repair work,

what is probability that it is

a brand 1 DVD player

a brand 2 DVD player"

a brand 3 DVD player?

- Finally,

$$P(A_1 | B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{0.125}{0.205} = 0.61$$

$$P(A_2 | B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{0.060}{0.205} = 0.29$$

- and

$$P(A_3 | B) = 1 - P(A_1 | B) - P(A_2 | B) = 0.10$$

- Initial or prior probability of brand 1 is 0.50.
  - Once it is known that the selected DVD player needed repair, the posterior probability of brand 1 increases to 0.61.
- This is because brand 1 DVD players are more likely to need warranty repair than are the other brands.
- The posterior probability of brand 3 is $P(A_3|B)=0.10$, which is much less than the prior probability $P(A_3)=0.20$

Bayes' Theorem

(ทฤษฎีของเบย์)

Bayes' Theorem

- Computation of posterior probability $P(A_j \mid B)$ from
  - given prior probabilities $P(A_i)$ and
  - conditional probabilities $P(B \mid A_i)$ occupies a central position in elementary probability.

- General rule for such computations, which is really just a simple application of multiplication rule, goes back to Reverend Thomas Bayes, who lived in the eighteenth century.
- To state it we first need another result.
  - Recall that events $A_1, \ldots, A_k$ are mutually exclusive if no two have any common outcomes.
  - Events are *exhaustive* if one A_i must occur, so that

$$A_1 \cup A_2 \cup \cdots \cup A_k = S$$

Bayes' Theorem

The Law of Total Probability

- Let $A_1, \ldots, A_k$ be mutually exclusive and exhaustive events. Then for any other event B,

$$\begin{aligned} P(B) &= P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + \cdots + P(B \mid A_k)P(A_k) \\ &= \sum_{i=1}^{k} P(B \mid A_i)P(A_i) \end{aligned} \quad (2.5)$$

เหตุการณ์ B เกิดขึ้นทั้งหมด

Bayes' Theorem

Proof

- Because the A_i s are mutually exclusive and exhaustive, if B occurs it must be in conjunction with exactly one of the A_j's
- That is, $B = (A_1 \cap B) \cup \cdots \cup (A_k \cap B)$
- where events $A_i \cap B$ are mutually exclusive
- This "partitioning of B" is illustrated in Figure 2.11. Thus

$$P(B) = \sum_{i=1}^{k} P(A_i \cap B) = \sum_{i=1}^{k} P(B \mid A_i)P(A_i)$$

as desired.

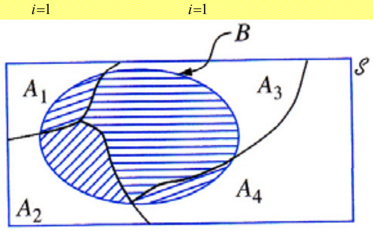


Figure 2.11 Partition of B by mutually exclusive and exhaustive A_i's

Bayes' Theorem

- An example of the use of Equation (2.5) appeared in answering question 2 of Example 2.29,
- where

  A_1 = {brand 1},
  A_2 = {brand 2},
  A_3 = {brand 3}, and B = {repair}

Law of Total Probability

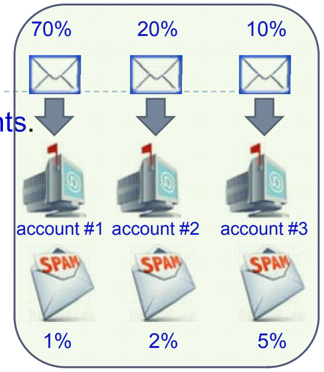
$$\begin{aligned} P(B) &= P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + \cdots + P(B \mid A_k)P(A_k) \\ &= \sum_{i=1}^{k} P(B \mid A_i)P(A_i) \end{aligned}$$

Equation (2.5)

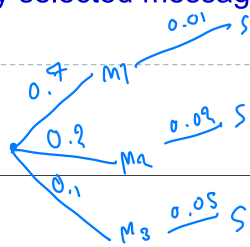
answering question 2 of Example 2.29

$$\begin{aligned} P(B) &= P[(brand\ 1\ and\ repair)\ or\ (brand\ 2\ and\ repair)\ or\ (brand\ 3\ and\ repair)] \\ &= P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) \\ &= 0.125 + 0.060 + 0.020 = 0.205 \end{aligned}$$

Example 30



- An individual has 3 different email accounts.
- Most of her messages, in fact
  - 70%, come into account #1, whereas
  - 20% come into account #2 and
  - the remaining 10% into account #3
- Of messages into account #1, only 1% are spam, whereas the corresponding percentages for accounts #2 and #3 are 2% and 5%, respectively.
- What is probability that a randomly selected message is spam?

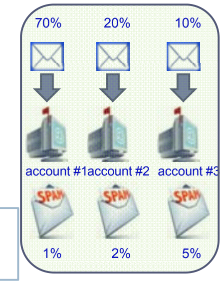


Example 30 cont'd

- To answer this question, let's first establish some notation:

  A_i = {message is from account # i} for i = 1, 2, 3,

  B = {message is spam}



- Then the given percentages imply that

  $P(A_1) = 0.70$,

  $P(A_2) = 0.20$,

  $P(A_3) = 0.10$

$P(B \mid A_1) = .01$, $P(B \mid A_2) = .02$, $P(B \mid A_3) = .05$

messages into account #1, 1% are spam

messages into account #2, 2% are spam

messages into account #3, 5% are spam

Example 30 cont'd

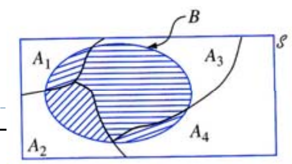
- Now it is simply a matter of substituting into the equation for the **law of total probability**:

$$P(B) = P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + \cdots + P(B \mid A_k)P(A_k) = \sum_{i=1}^{k} P(B \mid A_i)P(A_i) \qquad \text{Equation (2.5)}$$

$P(B) = (0.01)(0.70) + (0.02)(0.20) + (0.05)(0.10) = 0.016$

In the long run, 1.6% of this individual's messages will be spam.

Bayes' Theorem



Bayes' Theorem

Let $A_1, A_2, \ldots, A_k$ be a collection of k mutually exclusive and exhaustive events with *prior* probabilities $P(A_i)$ $(i = 1,2,\ldots,k)$.

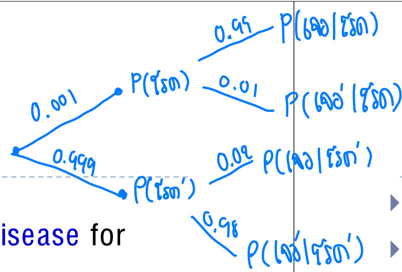
Then for any other event B for which $P(B) > 0$, the *posterior* probability of A_j given that B has occurred is

Multiplication Rule for $P(A \cap B)$

$$P(A_j \mid B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B \mid A_j)P(A_j)}{\sum_{i=1}^{k} P(B \mid A_i).P(A_i)} \qquad j = 1,2,\ldots,k \qquad (2.6)$$

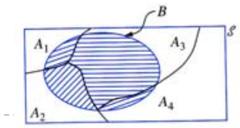
law of total probability

Example



- Incidence of a rare disease.
- Only 1 in 1,000 adults in afflicted with a rare disease for which a diagnostic test has been developed.
- The test is such that when
  an individual actually has the disease, a positive result will occur **99%** of the time,
  whereas
  an individual without the disease will show a positive test result only **2%** of the time.
- If a randomly selected individual is tested and the **result is positive**, what is **probability** that **individual has the disease?**

Example (cont.)



- To use Bayes' theorem,

$$P(A_j \mid B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B \mid A_j)P(A_j)}{\sum_{i=1}^{k} P(B \mid A_i).P(A_i)} \qquad j = 1,2,\dots,k$$

- Let
  - A_1 = {individual has the disease},
  - A_2 = {individual does not have the disease}, and
  - B = {positive test result}
- Then
  - $P(A_1) = 0.001$, Only 1 in 1,000 adults in afflicted with rare disease
  - $P(A_2) = 0.999$,
  - $P(B|A_1) = 0.99$, and individual has the disease, a positive result will occur **99%**
  - $P(B|A_2) = 0.02$ individual without disease will show positive test result only **2%**

Example (cont.)

- Then tree diagram for this problem is in Figure 2.12

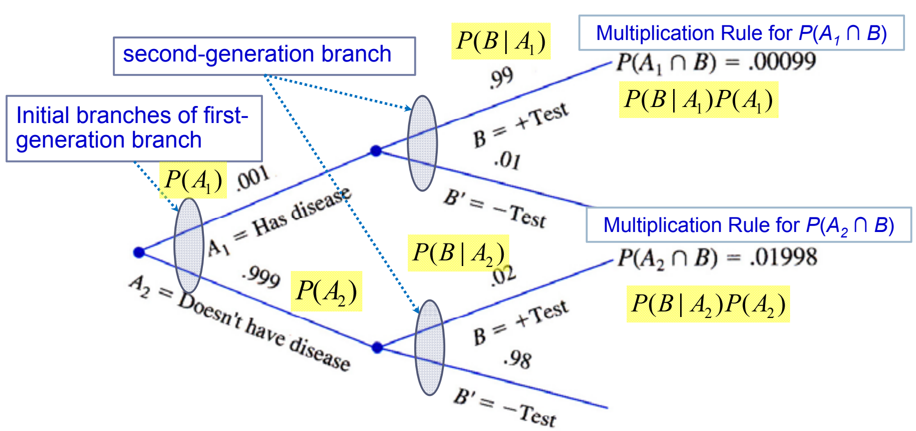
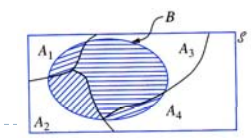


Figure 2.12 Tree diagram for the rare-disease problem

Example (cont.)



- Next to each branch corresponding to a positive test result, multiplication rule yields the recorded probabilities
- Therefore,

law of total probability

$$P(B) = P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) = \sum_{i=1}^{2} P(B \mid A_i)P(A_i) \qquad \text{Equation (2.5)}$$

$$P(B) = (0.99 \times 0.001) + (0.02 \times 0.999)$$

$$P(B) = 0.00099 + 0.01998 = 0.02097$$

- From which we have

$$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{0.00099}{0.02097} = 0.047$$

If a randomly selected individual is tested and the **result is positive**, what is **probability** that **individual has the disease?**