# More Gaussian Elimination and Matrix Inversion

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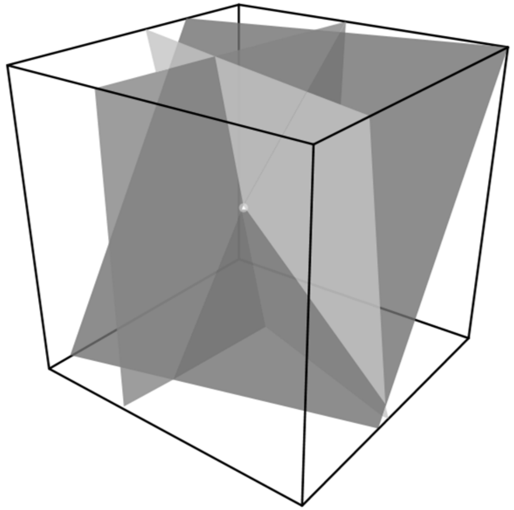
# Opening Remarks



$$Ax = b$$

$$\begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \alpha_{0,2} \\ \alpha_{1,0} & \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,0} & \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$

- **Gaussian Elimination**
- Back substitution
- Check your answer

# When Gaussian Elimination Breaks Down

- When Gaussian Elimination Works
- The Problem
- Permutations
- Gaussian Elimination with Row Swapping (LU factorization with Partial Pivoting)
- When Gaussian Elimination Fails Altogether

# When Gaussian Elimination Breaks Down

- **When Gaussian Elimination Works**

## $Ux = b$

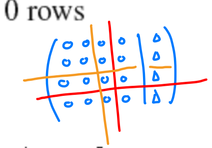
**Algorithm:** $[b] := \text{UTRSV\_UNB\_VAR1}(U, b)$

**Partition** $U \rightarrow \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array}\right)$, $b \rightarrow \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right)$

where $U_{BR}$ is $0 \times 0$, $b_B$ has 0 rows

$B_1 = B_1 - u_{12} b_2$

$B_1 = B_1 / u_{11}$



**while** $m(U_{BR}) < m(U)$ **do**

**Repartition**

ของเราทำแบบขีดได้ใน

นาา = 1 แล้ว ทดไว้..

$$\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \rightarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

$\beta_1 := \beta_1 - u_{12}^T b_2$ การลบกันของตัวเก่า

$\beta_1 := \beta_1 / \upsilon_{11}$

เหมือนทำคนละแบบกับเรา

ที่ทำไปใน assignment

**Continue with**

$$\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \leftarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

**endwhile**

## $Lz = b$

**Algorithm:** $[b] := \text{LTRSV\_UNB\_VAR1}(L, b)$

**Partition** $L \rightarrow \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)$, $b \rightarrow \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right)$

where $L_{TL}$ is $0 \times 0$, $b_T$ has 0 rows

**while** $m(L_{TL}) < m(L)$ **do**

**Repartition**

$$\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \rightarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

where $\lambda_{11}$ is $1 \times 1$, $\beta_1$ has 1 row

$b_2 := b_2 - \beta_1 l_{21}$

**Continue with**

$$\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \leftarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

**endwhile**

# When Gaussian Elimination Breaks Down

- **When Gaussian Elimination Works**

**Homework 7.2.1.1** Let $L \in \mathbb{R}^{1\times 1}$ be a unit lower triangular matrix. $Lx = b$, where $x$ is the unknown and $b$ is given, has a unique solution.

Always/Sometimes/Never

**Homework 7.2.1.2** Give the solution of $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

**Homework 7.2.1.3** Give the solution of $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(Hint: look carefully at the last problem, and you will be able to save yourself some work.)

**Homework 7.2.1.4** Let $L \in \mathbb{R}^{2\times 2}$ be a unit lower triangular matrix. $Lx = b$, where $x$ is the unknown and $b$ is given, has a unique solution.

Always/Sometimes/Never

**Homework 7.2.1.5** Let $L \in \mathbb{R}^{3\times 3}$ be a unit lower triangular matrix. $Lx = b$, where $x$ is the unknown and $b$ is given, has a unique solution.

Always/Sometimes/Never

**Homework 7.2.1.6** Let $L \in \mathbb{R}^{n\times n}$ be a unit lower triangular matrix. $Lx = b$, where $x$ is the unknown and $b$ is given, has a unique solution.

Always/Sometimes/Never

# When Gaussian Elimination Breaks Down

- **When Gaussian Elimination Works**

**Homework 7.2.1.8** Let $L \in \mathbb{R}^{n\times n}$ be a unit lower triangular matrix. $Lx = 0$, where 0 is the zero vector of size $n$, has the unique solution $x = 0$.

Always/Sometimes/Never

**Homework 7.2.1.9** Let $U \in \mathbb{R}^{1\times 1}$ be an upper triangular matrix with no zeroes on its diagonal. $Ux = b$, where $x$ is the unknown and $b$ is given, has a unique solution.

Always/Sometimes/Never

**Homework 7.2.1.10** Give the solution of $\begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

**Homework 7.2.1.11** Give the solution of $\begin{pmatrix} -2 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

**Homework 7.2.1.12** Let $U \in \mathbb{R}^{2\times 2}$ be an upper triangular matrix with no zeroes on its diagonal. $Ux = b$, where $x$ is the unknown and $b$ is given, has a unique solution.

Always/Sometimes/Never

**Homework 7.2.1.13** Let $U \in \mathbb{R}^{3\times 3}$ be an upper triangular matrix with no zeroes on its diagonal. $Ux = b$, where $x$ is the unknown and $b$ is given, has a unique solution.

Always/Sometimes/Never

**Homework 7.2.1.14** Let $U \in \mathbb{R}^{n\times n}$ be an upper triangular matrix with no zeroes on its diagonal. $Ux = b$, where $x$ is the unknown and $b$ is given, has a unique solution.

Always/Sometimes/Never

# When Gaussian Elimination Breaks Down

- **The Problem**

A simple example where Gaussian elimination and LU factorization break down involves the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In the first step, the multiplier equals $1/0$, which will cause a "division by zero" error.

Now, $Ax = b$ is given by the set of linear equations

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_0 \end{pmatrix}$$

so that $Ax = b$ is equivalent to

$$\begin{pmatrix} \chi_1 \\ \chi_0 \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

**Algorithm:** $[b] := \text{UTRSV\_UNB\_VAR1}(U, b)$

**Partition** $U \rightarrow \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array}\right)$, $b \rightarrow \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right)$

where $U_{BR}$ is $0 \times 0$, $b_B$ has 0 rows

**while** $m(U_{BR}) < m(U)$ **do**

**Repartition**

$$\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \rightarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

$\beta_1 := \beta_1 - u_{12}^T b_2$

$\beta_1 := \beta_1 / \upsilon_{11}$

**Continue with**

$$\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \leftarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

**endwhile**

# When Gaussian Elimination Breaks Down

- **The Problem**

**Homework 7.2.2.1** Solve the following linear system, via the steps in Gaussian elimination that you have learned so far.

$$
\begin{aligned}
2\chi_0 + \quad 4\chi_1 + (-2)\chi_2 &= -10 \\
4\chi_0 + \quad 8\chi_1 + \quad 6\chi_2 &= 20 \\
6\chi_0 + (-4)\chi_1 + \quad 2\chi_2 &= 18
\end{aligned}
$$

ลองทำดูจะเห็น $-\frac{16}{0} \rightarrow \infty$

Mark all that are correct:

(a) The process breaks down.

(b) There is no solution.

(c) $\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$

**Homework 7.2.2.2** Perform Gaussian elimination with

$$
\begin{aligned}
0\chi_0 + \quad 4\chi_1 + (-2)\chi_2 &= -10 \\
4\chi_0 + \quad 8\chi_1 + \quad 6\chi_2 &= 20 \\
6\chi_0 + (-4)\chi_1 + \quad 2\chi_2 &= 18
\end{aligned}
$$

# When Gaussian Elimination Breaks Down

- **Permutations**

**Homework 7.2.3.1** Compute

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} -2 & 1 & 2 \\ 3 & 2 & 1 \\ -1 & 0 & -3 \end{pmatrix}}_{A} =$$

ทำให้สลับแถว

Permutation matrix

# When Gaussian Elimination Breaks Down

- **Permutations**

**Definition 7.1** *A vector with integer components*

$$p = \begin{pmatrix} k_0 \\ k_1 \\ \vdots \\ k_{n-1} \end{pmatrix}$$

*is said to be a permutation vector if*

- $k_j \in \{0, \ldots, n-1\}$ , *for* $0 \le j < n$; *and*
- $k_i = k_j$ *implies* $i = j$.

$$p\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = p\begin{pmatrix} e_1^T \\ e_0^T \\ e_2^T \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

*In other words, p is a rearrangement of the numbers* $0, \ldots, n-1$ *(without repetition).*

We will often write $(k_0, k_1, \ldots, k_{n-1})^T$ to indicate the column vector, for space considerations.

# When Gaussian Elimination Breaks Down

- **Permutations**

**Definition 7.2** *Let $p = (k_0, \ldots, k_{n-1})^T$ be a permutation vector. Then*

$$P = P(p) = \begin{pmatrix} e_{k_0}^T \\ e_{k_1}^T \\ \vdots \\ e_{k_{n-1}}^T \end{pmatrix}$$

*is said to be a permutation matrix.*

In other words, $P$ is the identity matrix with its rows rearranged as indicated by the permutation vector $(k_0, k_1, \ldots, k_{n-1})$. We will frequently indicate this permutation matrix as $P(p)$ to indicate that the permutation matrix corresponds to the permutation vector $p$.

# When Gaussian Elimination Breaks Down

- **Permutations**

$0 \quad 0 \quad 1$
$1 \quad 0 \quad 0$
$0 \quad 1 \quad 0$

**Homework 7.2.3.2** For each of the following, give the permutation matrix $P(p)$:

- If $p = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$ then $P(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,
- If $p = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$ then $P(p) = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & 0 & \end{pmatrix}$

**Homework 7.2.3.3** Let $p = (2, 0, 1)^T$. Compute

- $P(p) \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} =$
- $P(p) \begin{pmatrix} -2 & 1 & 2 \\ 3 & 2 & 1 \\ -1 & 0 & -3 \end{pmatrix} =$

**Homework 7.2.3.4** Let $p = (2, 0, 1)^T$ and $P = P(p)$. Compute

$0 \quad 0 \quad 1$
$1 \quad 0 \quad 0$
$0 \quad 1 \quad 0$

$$\begin{pmatrix} -2 & 1 & 2 \\ 3 & 2 & 1 \\ -1 & 0 & -3 \end{pmatrix} P^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 3 & 2 \\ -3 & -1 & 0 \end{pmatrix}$$

ทรสลับหลัก

if $P(A)$ = rowSwap
if $AP^t$ = colSwap

# When Gaussian Elimination Breaks Down

- **Permutations**

สรุป อยากสลับแถวทำ $P \cdot A$ ; $P = \begin{pmatrix} e_n^t \\ e_m^t \\ e_y^t \\ \vdots \end{pmatrix}$

อยากสลับ col ทำ $A \cdot P^t$

**Homework 7.2.3.5** Let $p = (k_0, \ldots, k_{n-1})^T$ be a permutation vector. Consider

$$x = \begin{pmatrix} \chi_0 \\ \hline \chi_1 \\ \hline \vdots \\ \hline \chi_{n-1} \end{pmatrix}.$$

Applying permuation matrix $P = P(p)$ to $x$ yields

$$Px = \begin{pmatrix} \chi_{k_0} \\ \hline \chi_{k_1} \\ \hline \vdots \\ \hline \chi_{k_{n-1}} \end{pmatrix}.$$

**Homework 7.2.3.6** Let $p = (k_0, \ldots, k_{n-1})^T$ be a permutation. Consider

$$A = \begin{pmatrix} \tilde{a}_0^T \\ \hline \tilde{a}_1^T \\ \hline \vdots \\ \hline \tilde{a}_{n-1}^T \end{pmatrix}.$$

Applying $P = P(p)$ to $A$ yields

$$PA = \begin{pmatrix} \tilde{a}_{k_0}^T \\ \hline \tilde{a}_{k_1}^T \\ \hline \vdots \\ \hline \tilde{a}_{k_{n-1}}^T \end{pmatrix}.$$

**Homework 7.2.3.7** Let $p = (k_0, \ldots, k_{n-1})^T$ be a permutation, $P = P(p)$, and $A = \left( a_0 \mid a_1 \mid \cdots \mid a_{n-1} \right)$.
$AP^T = \left( a_{k_0} \mid a_{k_1} \mid \cdots \mid a_{k_{n-1}} \right)$.

Aways/Sometimes/Never

# When Gaussian Elimination Breaks Down

- **Permutations**

**Definition 7.3** *Let us call the special permutation matrix of the form*

P จิ๋ว

สลับแค่ π กับ 0

$$\widetilde{P}(\pi) = \begin{pmatrix} \boxed{e_\pi^T} \\ e_1^T \\ \vdots \\ e_{\pi-1}^T \\ \boxed{e_0^T} \\ e_{\pi+1}^T \\ \vdots \\ e_{n-1}^T \end{pmatrix} = \left(\begin{array}{c|c|c|c|c|c|c|c} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \hline 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{array}\right)$$

$\widetilde{P}(7)$

สลับแค่ 0 กับ 7

*a pivot matrix.*

# When Gaussian Elimination Breaks Down

- **Permutations**

**Homework 7.2.3.9** Compute

การสลับแต่ 1 กับ 0

$$\widetilde{P}(1)\begin{pmatrix}-2\\3\\-1\end{pmatrix} = \quad \text{and} \quad \widetilde{P}(1)\begin{pmatrix}-2 & 1 & 2\\ 3 & 2 & 1\\ -1 & 0 & -3\end{pmatrix} = .$$

$$P\begin{pmatrix}1\\0\\2\end{pmatrix}\left(\boxed{\phantom{0}}\right) = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\left(\boxed{\phantom{0}}\right)$$

**Homework 7.2.3.10** Compute

$$\begin{pmatrix}-2 & 1 & 2\\ 3 & 2 & 1\\ -1 & 0 & -3\end{pmatrix}\widetilde{P}(1) = .$$

# When Gaussian Elimination Breaks Down

- Gaussian Elimination with Row Swapping (LU factorization with Partial Pivoting)

ตรงนี้ไม่เกิด break down process

**Homework 7.2.4.1** Compute

- $$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right)\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)\left(\begin{array}{c|cc} 2 & 4 & -2 \\ \hline 4 & 8 & 6 \\ 6 & -4 & 2 \end{array}\right) =$$
- $$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 3 & 1 & 0 \\ 2 & 0 & 1 \end{array}\right)\left(\begin{array}{c|cc} 2 & 4 & -2 \\ \hline 0 & -16 & 8 \\ 0 & 0 & 10 \end{array}\right) =$$
- What do you notice?

| $i$ | $L_i$ | $\tilde{P}$ | $A$ | $p$ |
| --- | --- | --- | --- | --- |
| 0 | | $\begin{array}{c\|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}$ | not zero <br> $\begin{array}{c\|cc} 2 & 4 & -2 \\ \hline 4 & 8 & 6 \\ 6 & -4 & 2 \end{array}$ | 0 <br> · <br> · |
| | $\begin{array}{c\|cc} 1 & 0 & 0 \\ \hline \boxed{-2} & 1 & 0 \\ \boxed{-3} & 0 & 1 \end{array}$ <br> แทนค่า tranform | | $\begin{array}{c\|cc} 2 & 4 & -2 \\ \hline 4 & 8 & 6 \\ 6 & -4 & 2 \end{array}$ | 0 <br> · <br> · |
| 1 | | $\tilde{p}(1)$ <br> $\begin{array}{c\|c\|c} & & \\ \hline & 0 & 1 \\ \hline & 1 & 0 \end{array}$ | $\begin{array}{cc\|c\|c} & 2 & 4 & -2 \\ \hline 2 & & 0 & 10 \\ \hline 3 & & -16 & 8 \end{array}$ <br> need to do permutation (row 0 กับ row) | 0 <br> 1 <br> · |
| | [illegible] <br> $\begin{array}{c\|c\|c} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & -0 & 1 \end{array}$ | | $\begin{array}{cc\|c\|c} & 2 & 4 & -2 \\ \hline 3 & & -16 & 8 \\ \hline 2 & & 0 & 10 \end{array}$ | 0 <br> 1 <br> · |
| 2 | | | $\begin{array}{ccc\|c} & 2 & 4 & -2 \\ 3 & & -16 & 8 \\ \hline 2 & & 0 & 10 \end{array}$ | 0 <br> 1 <br> 0 |

Frontie

| $i$ | $L_i$ | $\tilde{P}$ | $A$ | $p$ |
| --- | --- | --- | --- | --- |
| 0 | | $\begin{array}{c\|cc} 0 & 1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}$ | $\begin{array}{c\|cc} 0 & 4 & -2 \\ \hline 4 & 8 & 6 \\ 6 & -4 & 2 \end{array}$ | 1 <br> · <br> · |
| | $\begin{array}{c\|cc} 1 & 0 & 0 \\ \hline \boxed{0} & 1 & 0 \\ \boxed{-\frac{6}{4}} & 0 & 1 \end{array}$ | | $\begin{array}{c\|cc} 4 & 8 & 6 \\ \hline 0 & 4 & -2 \\ 6 & -4 & 2 \end{array}$ | 1 <br> · <br> · |
| 1 | | $\begin{array}{c\|c\|c} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}$ | $\begin{array}{c\|c\|c} 4 & 8 & 6 \\ \hline 0 & 4 & -2 \\ \hline 0 & -16 & -7 \end{array}$ | 1 <br> 0 <br> · |
| | $\begin{array}{c\|c\|c} & & \\ \hline & 1 & 0 \\ \hline & 4 & 1 \end{array}$ | | $\begin{array}{c\|c\|c} 4 & 8 & 6 \\ \hline 0 & 4 & -2 \\ \hline 0 & 0 & -15 \end{array}$ | 1 <br> 0 <br> · |
| 2 | | | $\begin{array}{cc\|c} 4 & 8 & 6 \\ 0 & 4 & -2 \\ \hline 0 & 0 & -15 \end{array}$ | 1 |

# When Gaussian Elimination Breaks Down

**Algorithm:** $[A,p] := \text{LU\_PIV}(A,p)$

**Partition** $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right), p \rightarrow \left(\begin{array}{c} p_T \\ \hline p_B \end{array}\right)$

where $A_{TL}$ is $0 \times 0$ and $p_T$ has 0 components

**while** $m(A_{TL}) < m(A)$ **do**

**Repartition**

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} p_T \\ \hline p_B \end{array}\right) \rightarrow \left(\begin{array}{c} p_0 \\ \hline \pi_1 \\ \hline p_2 \end{array}\right)$$

$$\pi_1 = \text{PIVOT}\left(\left(\begin{array}{c} \alpha_{11} \\ \hline a_{21} \end{array}\right)\right)$$

$$\left(\begin{array}{c|c|c} a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right) := P(\pi_1)\left(\begin{array}{c|c|c} a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

$a_{21} := a_{21}/\alpha_{11}$ ($a_{21}$ now contains $l_{21}$)

$$\left(\begin{array}{c} a_{12}^T \\ \hline A_{22} \end{array}\right) = \left(\begin{array}{c} a_{12}^T \\ \hline A_{22} - a_{21}a_{12}^T \end{array}\right)$$

**Continue with**

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} p_T \\ \hline p_B \end{array}\right) \leftarrow \left(\begin{array}{c} p_0 \\ \hline \pi_1 \\ \hline p_2 \end{array}\right)$$

**endwhile**

**Algorithm:** $b := \text{APPLY\_PIV}(p,b)$

**Partition** $p \rightarrow \left(\begin{array}{c} p_T \\ \hline p_B \end{array}\right), b \rightarrow \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right)$

where $p_T$ and $b_T$ have 0 components

**while** $m(b_T) < m(b)$ **do**

**Repartition**

$$\left(\begin{array}{c} p_T \\ \hline p_B \end{array}\right) \rightarrow \left(\begin{array}{c} p_0 \\ \hline \pi_1 \\ \hline p_2 \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \rightarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

$$\left(\begin{array}{c} \beta_1 \\ \hline b_2 \end{array}\right) := P(\pi_1)\left(\begin{array}{c} \beta_1 \\ \hline b_2 \end{array}\right)$$

**Continue with**

$$\left(\begin{array}{c} p_T \\ \hline p_B \end{array}\right) \leftarrow \left(\begin{array}{c} p_0 \\ \hline \pi_1 \\ \hline p_2 \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \leftarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

**endwhile**

# When Gaussian Elimination Breaks Down

- **When Gaussian Elimination Fails Altogether**
  - Gaussian elimination (LU factorization) with $Ax = b$ where $A$ is a square matrix, one of three things can happen:
    - The process completes with no zeroes on the diagonal of the resulting matrix $U$. Then $A = LU$ and $Ax = b$ has a unique solution, which can be found by solving $Lz = b$ followed by $Ux = z$.
    - The process requires row exchanges, completing with no zeroes on the diagonal of the resulting matrixU. Then $PA=LU$ and $Ax = b$ has a unique solution, which can be found by solving $Lz = Pb$ followed by $Ux = z$.
    - The process requires row exchanges, but at some point no row can be found that puts a nonzero on the diagonal, at which point the process fails (unless the zero appears as the last element on the diagonal, in which case it completes, but leaves a zero on the diagonal).

# The Inverse Matrix

- Inverse Functions in 1D
- Back to Linear Transformations
- Simple Examples
- More Advanced (but Still Simple) Examples
- Properties

# The Inverse Matrix

- Inverse Functions in 1D
  - $f: \mathbb{R} \rightarrow \mathbb{R}$ maps a real to a real; and
  - it is a bijection (both one-to-one and onto)
- then
  - $f(x) = y$ has a unique solution for all $\mathbf{y} \in \mathbb{R}$.
  - The function that maps $\mathbf{y}$ to $\mathbf{x}$ so that $g(y) = x$ is called the inverse of $f$.
  - It is denoted by $f^{-1}: \mathbb{R} \rightarrow \mathbb{R}$.
  - Importantly, $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

# The Inverse Matrix

- Back to Linear Transformations
  - Theorem 7.5
    - Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a vector function. Then $f$ is one-to-one and onto (a bijection) implies that $m = n$. The proof of this hinges on the dimensionality of $\mathbb{R}^m$ and $\mathbb{R}^n$. We won't give it here.
  - Corollary 7.6
    - Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a vector function that is a bijection. Then there exists a function $f\colon \mathbb{R}^n \to \mathbb{R}^n$, which we will call its inverse, such that $f\left(f^{-1}(x)\right) = f^{-1}\left(f\,(x)\right) = x$.

# The Inverse Matrix

- Back to Linear Transformations
  - Theorem 7.7
    - Let $L: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation, and let $A$ be the matrix that represents $L$. If there exists a matrix $LB$ such that $AB = BA = I$, then $L$ has an inverse, $L^{-1}$, and $B$ equals the matrix that represents that linear transformation.

# The Inverse Matrix

- Back to Linear Transformations
  - Definition 7.8
    - A matrix $A$ is said to be invertible if the inverse, $A^{-1}$, exists. An equivalent term for invertible is nonsingular.
- The following statements are equivalent statements about $\mathbf{A} \in \mathbb{R}^{n \times n}$:
  - $A$ is nonsingular.
  - $A$ is invertible.
  - $A^{-1}$ exists.
  - $AA^{-1} = A^{-1}A = I$.
  - $A$ represents a linear transformation that is a bijection.
  - $Ax = b$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^n$.
  - $Ax = 0$ implies that $x = 0$.

# The Inverse Matrix

- Simple Examples
  - General principles
    - $AB = BA = I$
  - Inverse of the Identity matrix

$f(x) = Fx$

$g(y) = g(Fx)$

$g(y) = f^{-1}(x)$

**Homework 7.3.3.1** If $I$ is the identity matrix, then $I^{-1} = I$. True/False

  - Inverse of a diagonal matrix

**Homework 7.3.3.2** Find

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}^{-1} =$$

# The Inverse Matrix

- Simple Examples
  - Inverse of a Gauss transform

**Homework 7.3.3.4** Find

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}^{-1} =$$

Important: read the answer!

  - Inverse of a permutation

**Homework 7.3.3.7** Find

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} =$$

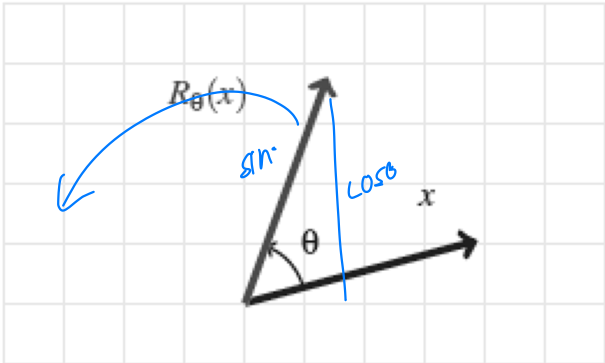
**Homework 7.3.3.8** Find

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} =$$

# The Inverse Matrix

- Simple Examples
  - Inverting a 2D rotation

**Homework 7.3.3.11** Recall from Week 2 how $R_\theta(x)$ rotates a vector $x$ through angle $\theta$:




$R_\theta$ is represented by the matrix

$$R = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

$$R\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

What transformation will "undo" this rotation through angle $\theta$? (Mark all correct answers)

(a) $R_{-\theta}(x)$

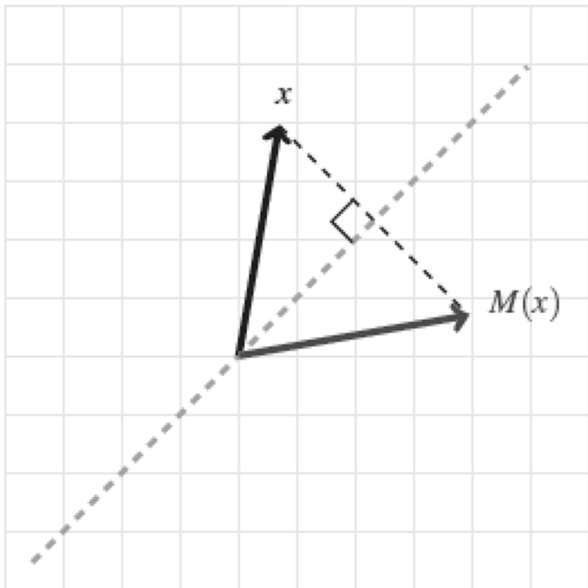
(b) $Ax$, where $A = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$

(c) $Ax$, where $A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$

# The Inverse Matrix

- Simple Examples
  - Inverting a 2D reflection

**Homework 7.3.3.12** Consider a reflection with respect to the 45 degree line:




If $A$ represents the linear transformation $M$, then

(a) $A^{-1} = -A$

(b) $A^{-1} = A$

(c) $A^{-1} = I$

(d) All of the above.

# The Inverse Matrix

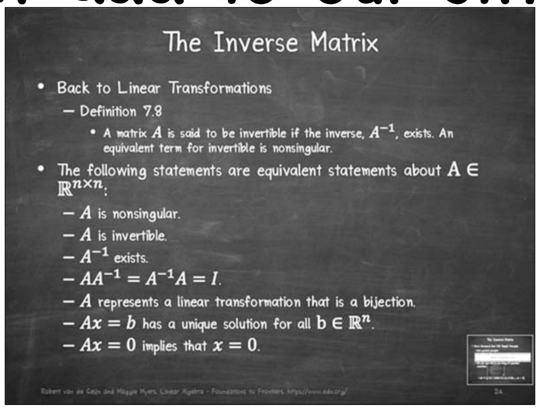
- More Advanced (but Still Simple) Examples
  - More general principles

Notice that $AA^{-1} = I$. Let's label $A^{-1}$ with the letter $B$ instead. Then $AB = I$. Now, partition both $B$ and $I$ by columns. Then

$$A \left( b_0 \mid b_1 \mid \cdots \mid b_{n-1} \right) = \left( e_0 \mid e_1 \mid \cdots \mid e_{n-1} \right)$$

and hence $Ab_j = e_j$. So.... the $j$th column of the inverse equals the solution to $Ax = e_j$ where $A$ and $e_j$ are input, and $x$ is output.

  - We can now add to our string of equivalent conditions:




    - $Ax = e_j$ has a solution for all $j \in \{0, \ldots, n-1\}$.

# The Inverse Matrix

- More Advanced (but Still Simple) Examples
  - Inverse of a triangular matrix

**Homework 7.3.4.1** Compute $\begin{pmatrix} -2 & 0 \\ 4 & 2 \end{pmatrix}^{-1} =$

**Homework 7.3.4.2** Find

$$\begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}^{-1} =$$

# The Inverse Matrix

- More Advanced (but Still Simple) Examples
  - Inverting a $2 \times 2$ matrix

**Homework 7.3.4.7** Find

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} =$$

**Homework 7.3.4.8** If $\alpha_{0,0}\alpha_{1,1} - \alpha_{1,0}\alpha_{0,1} \neq 0$ then

$$\begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} \\ \alpha_{1,0} & \alpha_{1,1} \end{pmatrix}^{-1} = \frac{1}{\alpha_{0,0}\alpha_{1,1} - \alpha_{1,0}\alpha_{0,1}} \begin{pmatrix} \alpha_{1,1} & -\alpha_{0,1} \\ -\alpha_{1,0} & \alpha_{0,0} \end{pmatrix}$$

(Just check by multiplying... Deriving the formula is time consuming.)

True/False

**Homework 7.3.4.9** The $2 \times 2$ matrix $A = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} \\ \alpha_{1,0} & \alpha_{1,1} \end{pmatrix}$ has an inverse if and only if $\alpha_{0,0}\alpha_{1,1} - \alpha_{1,0}\alpha_{0,1} \neq 0$.

True/False

# The Inverse Matrix

The expression $\alpha_{0,0}\alpha_{1,1} - \alpha_{1,0}\alpha_{0,1} \neq 0$ is known as the *determinant* of

$$\begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} \\ \alpha_{1,0} & \alpha_{1,1} \end{pmatrix}.$$

This $2 \times 2$ matrix has an inverse if and only if its determinant is nonzero. We will see how the determinant is useful again late in the course, when we discuss how to compute eigenvalues of small matrices. The determinant of a $n \times n$ matrix can be defined and is similarly a condition for checking whether the matrix is invertible. For this reason, we add it to our list of equivalent conditions:

The following statements are equivalent statements about $A \in \mathbb{R}^{n \times n}$:

- $A$ is nonsingular.
- $A$ is invertible.
- $A^{-1}$ exists.
- $AA^{-1} = A^{-1}A = I$.
- $A$ represents a linear transformation that is a bijection.
- $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$.
- $Ax = 0$ implies that $x = 0$.
- $Ax = e_j$ has a solution for all $j \in \{0, \ldots, n-1\}$.
- The determinant of $A$ is nonzero: $\det(A) \neq 0$.

# The Inverse Matrix

- Properties
  - Inverse of product

**Homework 7.3.5.1** Let $\alpha \neq 0$ and $B$ have an inverse. Then

$$(\alpha B)^{-1} = \frac{1}{\alpha} B^{-1}.$$

True/False

**Homework 7.3.5.2** Which of the following is true regardless of matrices $A$ and $B$ (as long as they have an inverse and are of the same size)?

(a) $(AB)^{-1} = A^{-1}B^{-1}$

(b) $(AB)^{-1} = B^{-1}A^{-1}$

(c) $(AB)^{-1} = B^{-1}A$

(d) $(AB)^{-1} = B^{-1}$

**Homework 7.3.5.3** Let square matrices $A, B, C \in \mathbb{R}^{n \times n}$ have inverses $A^{-1}$, $B^{-1}$, and $C^{-1}$, respectively. Then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

Always/Sometimes/Never

# The Inverse Matrix

- Properties
  - Inverse of transpose

**Homework 7.3.5.4** Let square matrix $A$ have inverse $A^{-1}$. Then $(A^T)^{-1} = (A^{-1})^T$.

Always/Sometimes/Never

  - Inverse of inverse

**Homework 7.3.5.5**

$$(A^{-1})^{-1} = A$$

Always/Sometimes/Never

# Questions and Answers