More Applications of The Pumping Lemma

The Pumping Lemma: ทวนสรุป

For infinite context-free language L

there exists an integer m such that

for any string $w \in L, \quad |w| \geq m$

we can write $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

and it must be:

$$uv^i xy^i z \in L, \quad \text{for all } i \geq 0$$

เหมือนเดิม อย่าไปใช้กับตัวที่มันไม่ใช่ภาษา cfl → ผลลัพธ์จะแปลก

Non-context free languages

$$\{a^n b^n c^n : n \geq 0\} \qquad \{vv : v \in \{a,b\}\}$$

ไม่สามารถเก็บ stack ให้แสดงประโยคได้

Context-free languages

$$\{a^n b^n : n \geq 0\} \qquad \{ww^R : w \in \{a,b\}^*\}$$

Theorem: The language

$$L = \{vv : v \in \{a,b\}^*\}$$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{vv : v \in \{a, b\}^*\}$$

เริ่มสมมุติว่าปลอมมา

Assume for contradiction that L is context-free

Since L is context-free and infinite we can apply the pumping lemma

$$L = \{vv : v \in \{a,b\}^*\}$$

Pumping Lemma gives a magic number m such that:

Pick any string of L with length at least m

๑ เลือกสตริง

we pick: $a^m b^m a^m b^m \in L$

$$L=\{vv : v\in\{a,b\}^*\}$$

② แบ่งวิธีทำ

We can write: $a^m b^m a^m b^m = uvxyz$

with lengths $|vxy|\le m$ and $|vy|\ge 1$

Pumping Lemma says: โดยพิจารณาทุกกรณี

$uv^i xy^i z \in L$ for all $i\ge 0$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

We examine **all** the possible locations of string vxy in $a^m b^m a^m b^m$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: vxy is within the first a^m

อยู่ในส่วนแรก

$$v = a^{k_1} \qquad y = a^{k_2} \qquad k_1 + k_2 \geq 1$$

$$\overbrace{a \ldots\ldots a}^{m}\ \overbrace{b \ldots\ldots b}^{m}\ \overbrace{a \ldots\ldots a}^{m}\ \overbrace{b \ldots\ldots b}^{m}$$

$$u\ v\ x\ y \qquad z$$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

Case 1: vxy is within the first a^m

$$v = a^{k_1} \qquad y = a^{k_2} \qquad k_1 + k_2 \ge 1$$

เลือก i=2

$$\overbrace{a\,\ldots\ldots\ldots\ldots\,a}^{m+k_1+k_2}\;\overbrace{b\,\ldots\ldots\,b}^{m}\;\overbrace{a\,\ldots\ldots\,a}^{m}\;\overbrace{b\,\ldots\ldots\,b}^{m}$$

$$u \quad v^2 \quad x \quad y^2 \quad z$$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: vxy is within the first a^m

$$a^{m+k_1+k_2} b^m a^m b^m = uv^2xy^2z \quad \notin L$$

$$k_1 + k_2 \geq 1$$

$a^{m+(\geq 1)} b^m a^m b^m \notin L$

ขัดแย้ง

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: vxy is within the first a^m

$$a^{m+k_1+k_2} b^m a^m b^m = uv^2xy^2z \quad \notin L$$

However, from Pumping Lemma: $uv^2xy^2z \in L$

Contradiction!!!

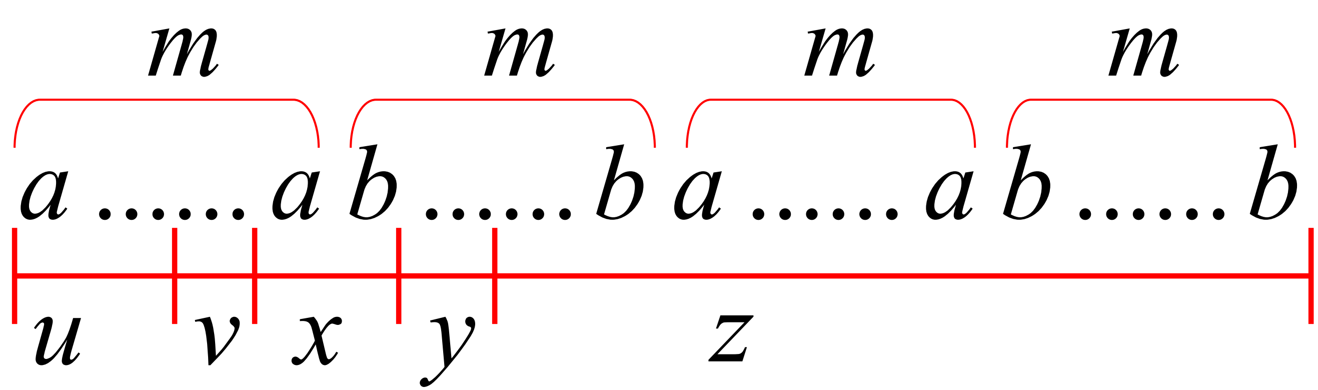
$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

Case 2: v is in the first a^m

y is in the first b^m

$$v = a^{k_1} \qquad y = b^{k_2} \qquad k_1 + k_2 \ge 1$$



$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: v is in the first a^m

y is in the first b^m

i=2

$$v = a^{k_1} \qquad y = b^{k_2} \qquad k_1 + k_2 \geq 1$$

$$\overbrace{a \ldots\ldots\ldots\ldots a}^{m+k_1}\ \overbrace{b \ldots\ldots\ldots\ldots b}^{m+k_2}\ \overbrace{a \ldots\ldots a}^{m}\ \overbrace{b \ldots\ldots b}^{m}$$

$$u \quad v^2 \quad x \quad y^2 \quad z$$

$$L = \{vv : v \in \{a, b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: v is in the first a^m

y is in the first b^m

ต้องมีสักอันที่ไม่เท่า

$$a^{m+k_1} b^{m+k_2} a^m b^m = uv^2xy^2z \quad \notin L$$

$$k_1 + k_2 \geq 1$$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

Case 2: v is in the first a^m
y is in the first b^m

$$a^{m+k_1} b^{m+k_2} a^m b^m = uv^2xy^2z \quad \notin L$$

However, from Pumping Lemma: $uv^2xy^2z \in L$

Contradiction!!!

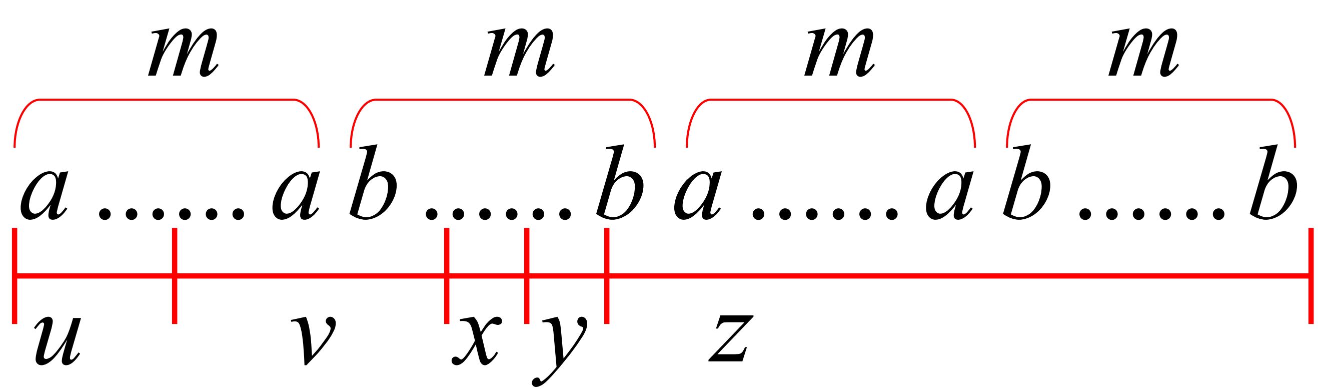
$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

Case 3: v overlaps the first $a^m b^m$

y is in the first b^m

$$v = a^{k_1} b^{k_2} \qquad y = b^{k_3} \qquad k_1, k_2 \ge 1$$



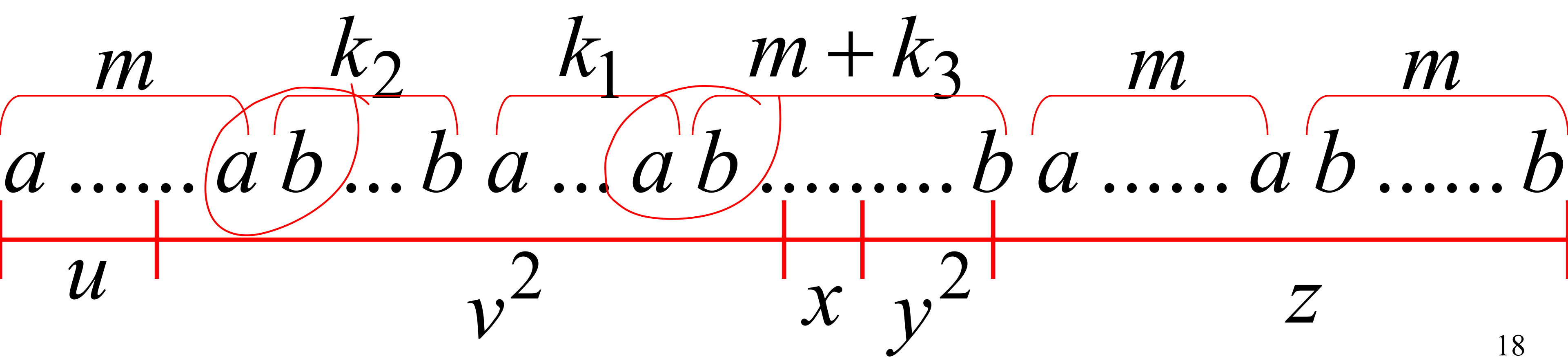
$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

Case 3: v overlaps the first $a^m b^m$

y is in the first b^m

$$v = a^{k_1} b^{k_2} \qquad y = b^{k_3} \qquad k_1, k_2 \ge 1$$



$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: v overlaps the first $a^m b^m$

y is in the first b^m

$$a^m b^{k_2} a^{k_1} b^{m+k_3} a^m b^m = uv^2xy^2z \quad \notin L$$

$$k_1, k_2 \geq 1$$

ตรงแรกนี่ไม่เหมือนกันแน่นอน

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: v overlaps the first $a^m b^m$
y is in the first b^m

$$a^m b^{k_2} a^{k_1} b^{k_3} a^m b^m = uv^2xy^2z \quad \notin L$$

However, from Pumping Lemma: $uv^2xy^2z \in L$

Contradiction!!!

$$L = \{vv : v \in \{a,b\}^*\}$$

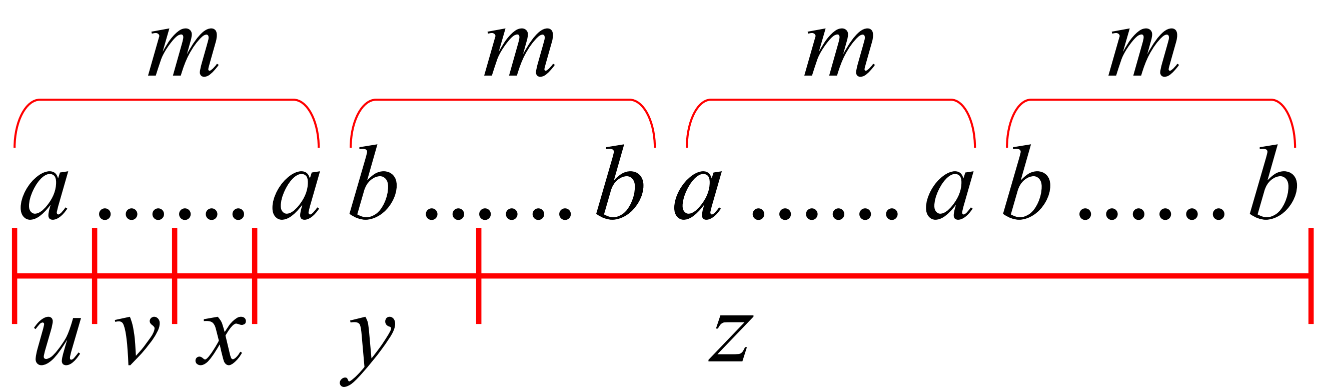
$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

Case 4: v in the first a^m

y Overlaps the first $a^m b^m$

ແນວໃດ

Analysis is similar to case 3



Other cases: vxy is within $a^m \boxed{b^m} a^m b^m$

or

$a^m b^m \boxed{a^m} b^m$

or

$a^m b^m a^m \boxed{b^m}$

Analysis is similar to case 1:

$\boxed{a^m} b^m a^m b^m$

More cases: vxy overlaps $a^m \boxed{b^m a^m} b^m$

or

$a^m b^m \boxed{a^m b^m}$

Analysis is similar to cases 2,3,4:

$\boxed{a^m b^m} a^m b^m$

There are no other cases to consider

Since $|vxy| \leq m$, it is impossible vxy to overlap:

$$\boxed{a^m b^m a^m} b^m$$

nor

$$a^m \boxed{b^m a^m b^m}$$

nor

$$\boxed{a^m b^m a^m b^m}$$

In all cases we obtained a contradiction

Therefore: The original assumption that

$$L = \{vv : v \in \{a,b\}^*\}$$

is context-free must be wrong

เพราะเราคิดแย้งได้

ถ้าเราเลือก $a^m a^m$

ไม่สมการนล

ไปดูเอง

Conclusion: L is not context-free

Non-context free languages

$$\{a^n b^n c^n : n \geq 0\}$$

$$\{ww : w \in \{a,b\}\}$$

$$\{a^{n!} : \underline{n \geq 0}\}$$

ไม่เป็น regular & ไม่เป็น cfr

Context-free languages

$$\{a^n b^n : n \geq 0\}$$

$$\{ww^R : w \in \{a,b\}^*\}$$

Theorem: The language

$$L = \{a^{n!} : n \geq 0\}$$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{a^{n!} : n \geq 0\}$$

ขัดแย้ง

Assume for contradiction that L is context-free

Since L is context-free and infinite we can apply the pumping lemma

$$L = \{a^{n!} : n \geq 0\}$$

Pumping Lemma gives a magic number m such that:

Pick any string of L with length at least m

1 เราเลือก

we pick: $a^{m!} \in L$

$$L = \{a^{n!} : n \geq 0\}$$

We can write: $a^{m!} = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

Pumping Lemma says:

$uv^i xy^i z \in L$ for all $i \geq 0$

$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

๒ เลือกรใทหญ

We examine **all** the possible locations of string vxy in $a^{m!}$

There is only one case to consider

$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

มีกรณีเดียว

$$\overbrace{a \ldots\ldots\ldots\ldots\ldots a}^{m!}$$

$$\underbrace{}_{u}\underbrace{}_{v}\underbrace{}_{x}\underbrace{}_{y}\underbrace{}_{z}$$

$|v|+|y| \leq m$ อยู่น

$$v = a^{k_1} \qquad y = a^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$

ซึ่งจริง

$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

ধরি = 2

$$\overbrace{a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots a}^{m! + k_1 + k_2}$$

$$u \quad v^2 \quad x \quad y^2 \quad z$$

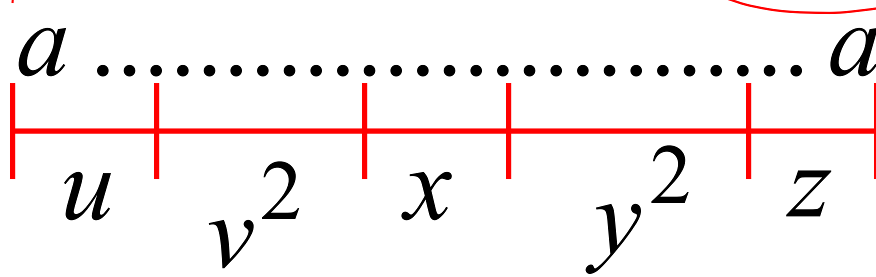
$$v = a^{k_1} \qquad y = a^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$

$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

สามารถเขียนอยู่ใน รูป ได้ไหม

$$m! + k \qquad k = k_1 + k_2$$



ก็เป็น รูป แฟคตอเรียล
ๆน L

$$v = a^{k_1} \qquad y = a^{k_2} \qquad 1 \leq k \leq m$$

$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$a^{m!+k} = uv^2xy^2z$$

$$1 \leq k \leq m$$

Since $1 \le k \le m$, for $m \ge 2$ we have:

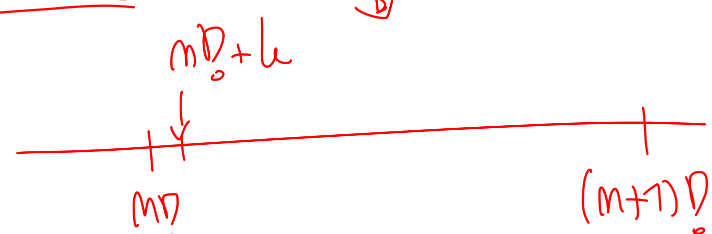
$$m! + k \le m! + m$$ เหมือนเดิม

$$< m! + m!m$$ m! เพิ่มๆ

$$= m!(1+m)$$

$$= (m+1)!$$ ที่เท่าไหร่

$$\Downarrow$$



$$m! < m! + k < (m+1)!$$

ไม่มีทางเขียนอยู่ในรูปจำนวนเต็มได้เลยสักตัว

$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$m! < m! + k < (m+1)!$$

$$\Downarrow$$

$$a^{m!+k} = uv^2xy^2z \quad \notin L$$

$$L = \{a^{n!} : n \geq 0\}$$

$$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

However, from Pumping Lemma: $uv^2xy^2z \in L$

$$a^{m!+k} = uv^2xy^2z \quad \notin L$$

Contradiction!!!

We obtained a contradiction

Therefore: The original assumption that

$$L = \{a^{n!} : n \geq 0\}$$

is context-free must be wrong

Conclusion: L is not context-free

Non-context free languages

$$\{a^n b^n c^n : n \geq 0\}$$

$$\{ww : w \in \{a,b\}\}$$

$$\{a^{n^2} b^n : n \geq 0\}$$

$$\{a^{n!} : n \geq 0\}$$

Context-free languages

$$\{a^n b^n : n \geq 0\}$$

$$\{ww^R : w \in \{a,b\}^*\}$$

Theorem: The language

$$L = \{a^{n^2} b^n : n \geq 0\}$$

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Proof: Use the Pumping Lemma for context-free languages

$$L = \{a^{n^2} b^n : n \geq 0\}$$

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Since L is context-free and infinite we can apply the pumping lemma

$$L = \{a^{n^2} b^n : n \geq 0\}$$

Pumping Lemma gives a magic number m such that:

Pick any string of L with length at least m

เลือก string ขึ้นมา

we pick: $a^{m^2} b^m \in L$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

We can write: $a^{m^2} b^m = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

Pumping Lemma says:

$uv^i xy^i z \in L$ for all $i \geq 0$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

We examine **all** the possible locations

of string vxy in $a^{m^2} b^m$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

ควรไปไล่ให้ดูแต่ที่ซับซ้อน

Most complicated case: v is in a^m

y is in b^m

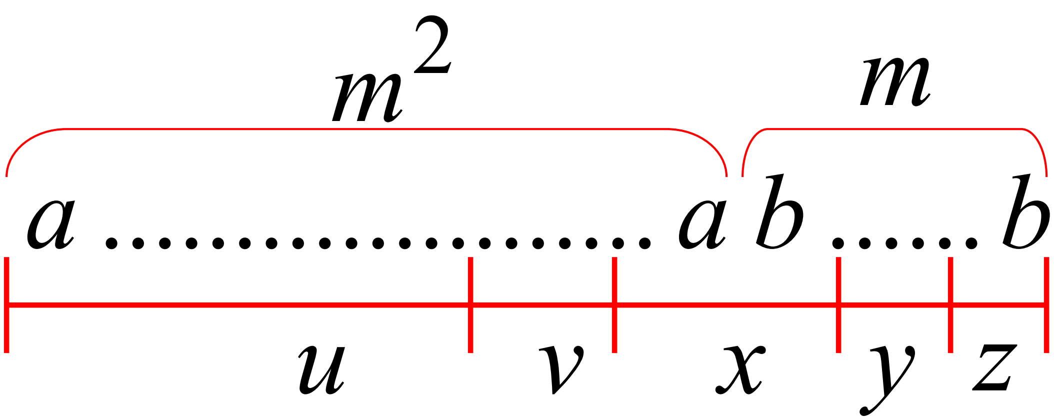
$$\overbrace{a \ldots\ldots\ldots\ldots\ldots\ldots a}^{m^2}\ \overbrace{b \ldots\ldots b}^{m}$$

$u \quad v \quad x \quad y \quad z$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$v = a^{k_1} \qquad y = b^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$



$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

ใช้ในกรณีนี้

Most complicated sub-case: $k_1 \neq 0$ and $k_2 \neq 0$

$$v = a^{k_1} \qquad y = b^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$

$$\overbrace{\underbrace{a \ldots\ldots}_{u}\ \underbrace{\ldots\ldots}_{v}\ \underbrace{\ldots a}_{x}}^{m^2}\overbrace{\underbrace{b \ldots}_{x}\ \underbrace{\ldots}_{y}\ \underbrace{\ldots b}_{z}}^{m}$$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Most complicated sub-case: $k_1 \neq 0$ and $k_2 \neq 0$

$$v = a^{k_1} \qquad y = b^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$

$i=0$

$$\overbrace{a \ldots\ldots\ldots\ldots\ldots a}^{m^2-k_1}\;\overbrace{b \ldots b}^{m-k_2}$$

$$\underbrace{a \ldots\ldots\ldots}_{u}\;\underbrace{}_{v^0}\;\underbrace{\ldots a\, b}_{x}\;\underbrace{}_{y^0}\;\underbrace{\ldots b}_{z}$$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Most complicated sub-case: $k_1 \neq 0$ and $k_2 \neq 0$

$$v = a^{k_1} \qquad y = b^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$

$$a^{m^2 - k_1} b^{m - k_2} = uv^0 xy^0 z$$

$k_1 \neq 0$ and $k_2 \neq 0$ $\qquad 1 \leq k_1 + k_2 \leq m$

เลือกค่าเป็นแล้วยกกำลัง 2 มันจะได้เท่า m^2-1 หรือมาก

$$(m-k_2)^2 \leq (m-1)^2$$

$k_2 \neq 0$ จะ >1 ได้

$$= \underbrace{m^2 - 2m + 1}$$

$m^2 - \underbrace{(2m+1)}_{\text{linear}}$

$$< m^2 - k_1$$

$m^2 - \underbrace{k_1}_{\text{constant}}$

$$m^2 - k_1 \neq (m - k_2)^2$$

ค่าน้อยกว่า

$m^2 - (2m+1)$

ค่าที่มากกว่า

ไม่เท่ากันแน่นอน

linear > constant

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$m^2 - k_1 \neq (m - k_2)^2$$

$$a^{m^2 - k_1} b^{m - k_2} = uv^0xy^0z \quad \notin L$$

ไม่เท่ากันแสดงว่าหลุดเซตแอล

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

However, from Pumping Lemma: $uv^0xy^0z \quad \in L$

$$a^{m^2-k_1} b^{m-k_2} = uv^0xy^0z \quad \notin L$$

Contradiction!!!

When we examine the rest of the cases we also obtain a contradiction

In all cases we obtained a contradiction

Therefore: The original assumption that

$$L = \{a^{n^2} b^n : n \geq 0\}$$

is context-free must be wrong

Conclusion: L is not context-free