Vectors in Linear Algebra

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Vector

- **Notation**
  - Definition
    - a one-dimensional array of n numbers : a vector of size n

$$x = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix}$$

- n : size of the vector
- $\chi_i \in \mathbb{R}$
- $x \in \mathbb{R}^n$

Vector

- A vector has a direction and a length:
  - Direction : draw an arrow from the origin to the point $(\chi_0, \chi_1, \dots, \chi_{n-1})$
  - Its length is given by the Euclidean length of this arrow

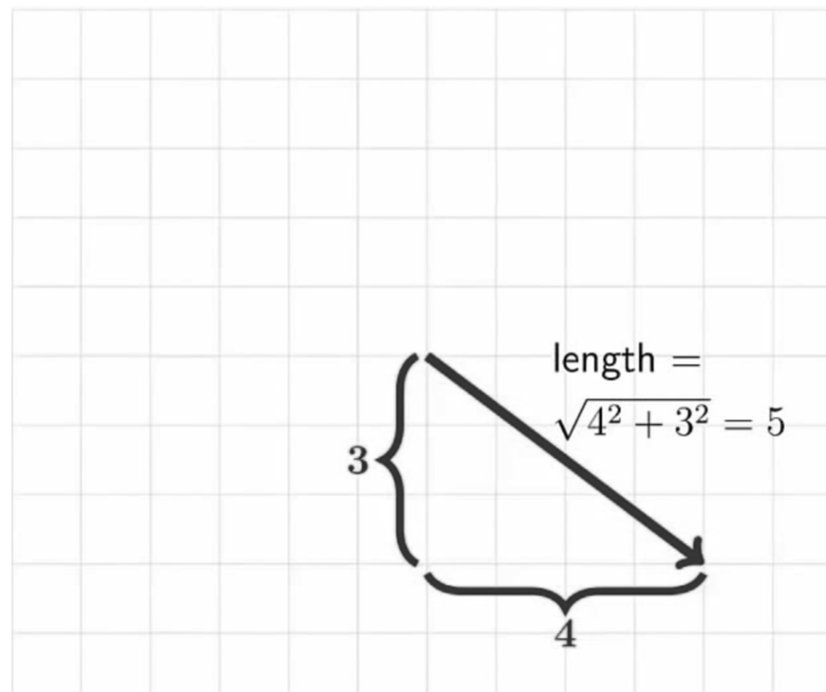
$$\sqrt{\chi_0^2 + \chi_1^2 + \cdots + \chi_{n-1}^2}$$

    » It is denoted by $\|x\|_2$ called the two-norm (the magnitude of the vector)

- A vector does not have a location

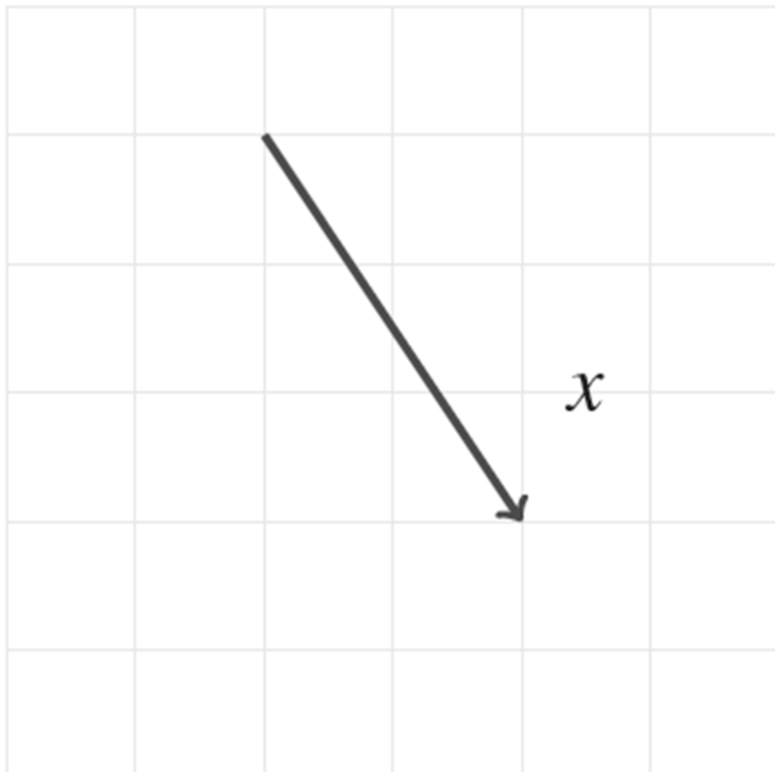
Example

- Consider $x = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$
  - Component indexed with 0 : $\chi_0 = 4$
  - Component indexed with 1 : $\chi_1 = -3$
  - The vector is of size 2
    - $x \in \mathbb{R}^2$
  - Length (magnitude) = 5



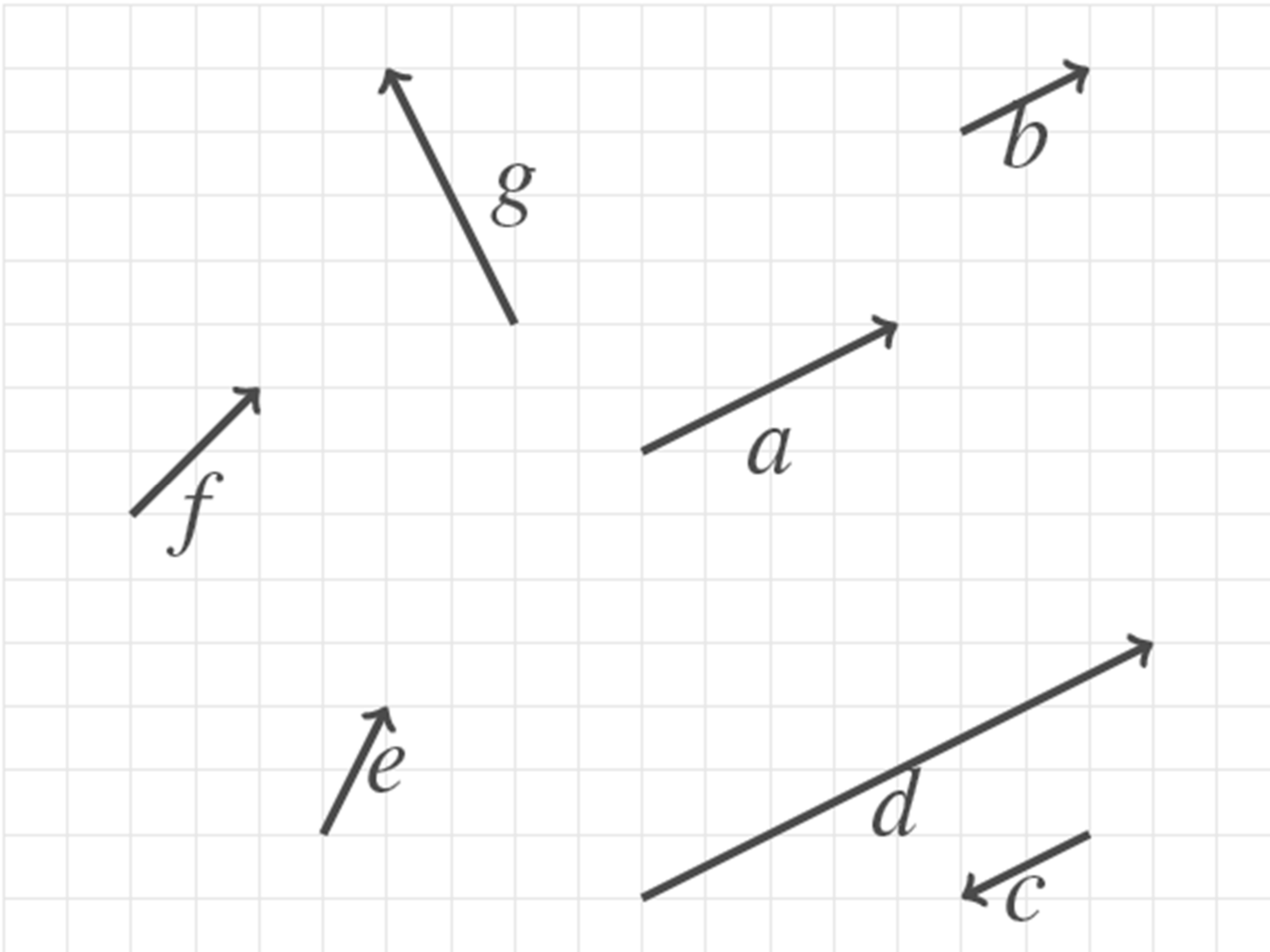
Exercises

- **Consider the following picture**



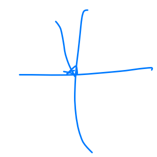
Exercises

- **Consider the following picture**



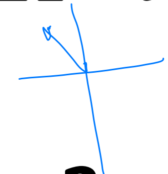
Exercises

- Write each of the following as a vector:
  - The vector represented geometrically in $\mathbb{R}^2$ by an arrow from point (-1, 2) to point (0, 0)



$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

  - The vector represented geometrically in $\mathbb{R}^2$ by an arrow from point (0, 0) to point (-1, 2).

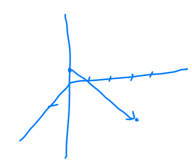


$\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

  - The vector represented geometrically in $\mathbb{R}^3$ by an arrow from point (-1, 2, 4) to point (0, 0, 1).

$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

  - The vector represented geometrically in $\mathbb{R}^3$ by an arrow from point (1, 0, 0) to point (4, 2, -1).



$\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

Vector

- Unit Basis Vectors
  - Definition
    - An important set of vectors is the set of unit basis vectors given by

$$e_j = \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array}\right) \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} j \; zeros \\ \longleftarrow \text{component indexed by } j \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} n - j - 1 \; zeros \end{array}$$

แทน unit vector
ตัวๆ

    - where the “1” appears as the component indexed by j.

Vector

- The set $\{e_0, e_1, \dots, e_{n-1}\} \subset \mathbb{R}^n$ given by

$$e_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_1 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

2 dimension

$e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

  - Unit basis vectors of $x \in \mathbb{R}^2$?
  - Unit basis vectors of $x \in \mathbb{R}^3$?

Exercises

- **Which of the following is not a unit basis vector?**
  - $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ ✓
  - $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ✓
  - $\begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$
  - $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ✓
  - **None of these are unit basis vectors.**

Simple Vector Operations

- Equality (=), Assignment (:=), and Copy
- Vector Addition (ADD)
- Scaling (SCAL)
- Vector Subtraction

Simple Vector Operations

- Equality (=)

$$x = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} \text{ and } y = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{n-1} \end{pmatrix}$$

– Definition

dimention เท่ากัน , components n เท่ากัน

- Two vectors $\boxed{x, y \in \mathbb{R}^n}$ are equal if all their components are element-wise equal:

$$x = y \textit{ if and only if } \chi_i = \psi_i, \textit{for all } 0 \le i < n$$

Simple Vector Operations

- Assignment ($:=$) or Copy
  - Algorithm
    - The following algorithm copies vector $x \in \mathbb{R}^n$ into vector $y \in \mathbb{R}^n$, performing the operation $y := x$

$$\begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{n-1} \end{pmatrix} := \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix}$$

```
for i = 0, ..., n − 1
    ψ_i := χ_i
endfor
```

Exercises

- Decide if the two vectors are equal.
  - The vector represented geometrically in $\mathbb{R}^2$ by an arrow from point $(-1,2)$ to point $(0,0)$ and the vector represented geometrically in $\mathbb{R}^2$ by an arrow from point $(1,-2)$ to point $(2,-1)$ are equal.
  - The vector represented geometrically in $\mathbb{R}^3$ by an arrow from point $(1,-1,2)$ to point $(0,0,0)$ and the vector represented geometrically in $\mathbb{R}^3$ by an arrow from point $(1,1,-2)$ to point $(0,2,-4)$ are equal.

Simple Vector Operations

- Vector Addition (ADD)
  - Definition
    - Vector addition $x + y$ (sum of vectors) is defined by

$$x + y = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} + \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{n-1} \end{pmatrix} = \begin{pmatrix} \chi_0 + \psi_0 \\ \chi_1 + \psi_1 \\ \vdots \\ \chi_{n-1} + \psi_{n-1} \end{pmatrix}$$

Exercises

- $\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \end{pmatrix} =$

พิสูจน์ $x+y = y+x$

- $\begin{pmatrix} -3 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} =$
- For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$
  - Always ✓
  - Sometime
  - Never

$\begin{pmatrix} \psi_0 \\ \psi_{\ldots} \end{pmatrix} \quad \begin{pmatrix} \chi_0 \\ \chi_{\ldots} \end{pmatrix}$

Exercises

- $\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \left(\begin{pmatrix} -3 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) =$
- $\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \end{pmatrix} \right) + \begin{pmatrix} 1 \\ 2 \end{pmatrix} =$
- For $x, y, z \in \mathbb{R}^n$, $(x + y) + z = x + (y + z)$
  - Always
  - Sometime
  - Never

Exercises

- $\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} =$
- For $x \in \mathbb{R}^n$, $x + 0 = x$, where 0 is zero vector of appropriate size
  - Always
  - Sometime
  - Never

Simple Vector Operations

- Vector Addition (ADD)
  - Algorithm
    - The following algorithm assigns the sum of vectors x and y (of size n and stored in arrays x and y) to vector z (of size n and stored in array z), computing $z := x + y$

$$\begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \vdots \\ \zeta_{n-1} \end{pmatrix} := \begin{pmatrix} \chi_0 + \psi_0 \\ \chi_1 + \psi_1 \\ \vdots \\ \chi_{n-1} + \psi_{n-1} \end{pmatrix}$$

$$\boxed{\begin{array}{l} \texttt{for } i = 0, \ldots, n-1 \\ \quad \zeta_i := \chi_i + \psi_i \\ \texttt{endfor} \end{array}}$$

Simple Vector Operations

- **Scaling (SCAL)**
  - Definition
    - Multiplying vector x by scalar α yields a new vector, αx, in the same direction as x, but scaled by a factor α. Scaling a vector by α means each of its components, χ_i is multiplied by α

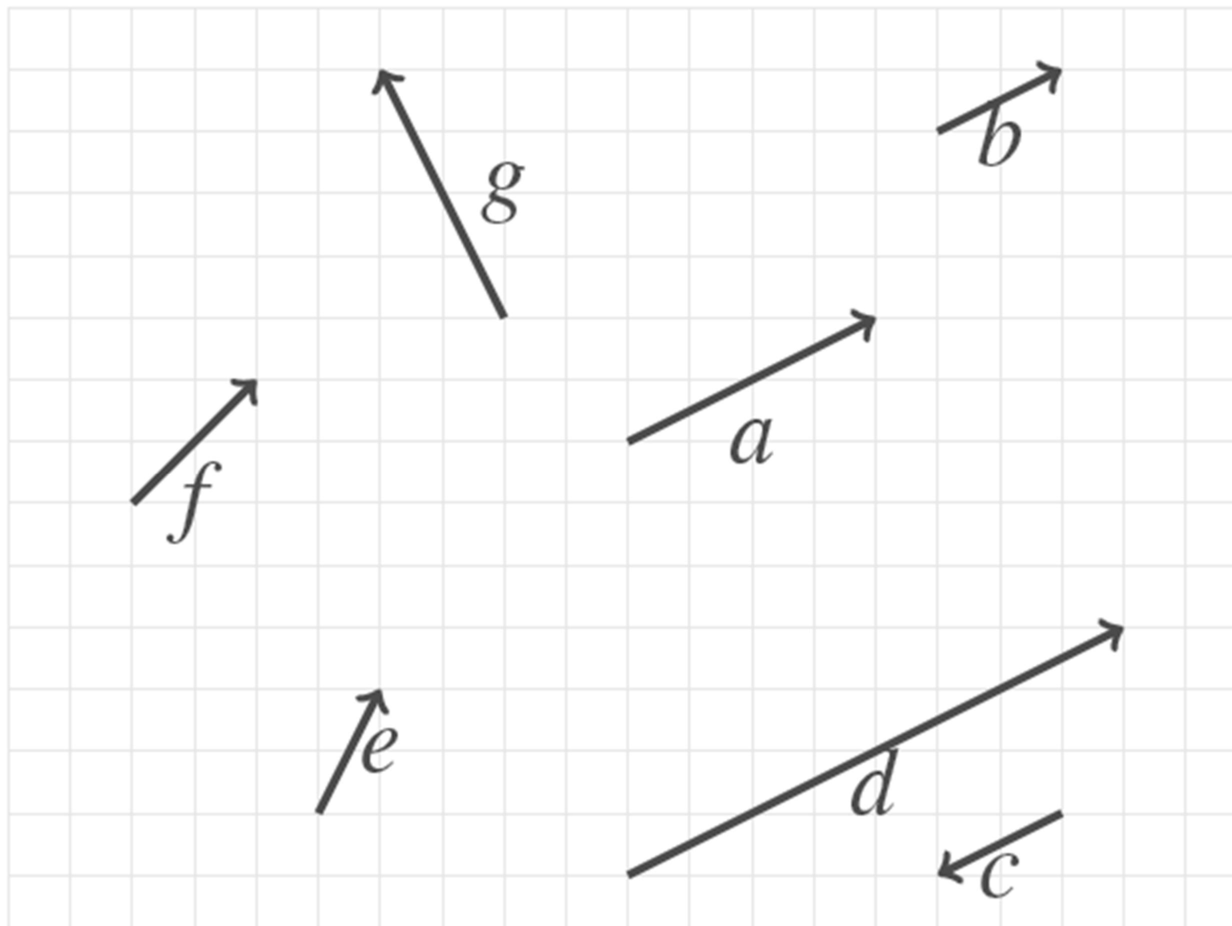
$$\alpha x = \alpha \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} = \begin{pmatrix} \alpha\chi_0 \\ \alpha\chi_1 \\ \vdots \\ \alpha\chi_{n-1} \end{pmatrix}$$

Exercises

- $\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right) + \begin{pmatrix} -1 \\ 2 \end{pmatrix} =$
- $3 \begin{pmatrix} -1 \\ 2 \end{pmatrix} =$

Exercises

- Consider the following picture
  - Which vector equals
    - $2a$?;
    - $(1/2)a$? ;
    - $-(1/2)a$?



Simple Vector Operations

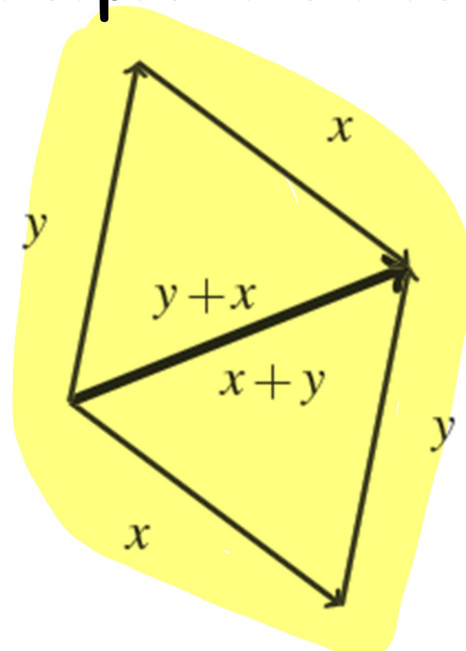
- **Scaling (SCAL)**
  - **Algorithm**
    - The following algorithm scales a vector $x \in \mathbb{R}^n$ by α, overwriting x with the result αx

$$\begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \vdots \\ \zeta_{n-1} \end{pmatrix} := \begin{pmatrix} \alpha\chi_0 \\ \alpha\chi_1 \\ \vdots \\ \alpha\chi_{n-1} \end{pmatrix}$$

for $i = 0, \ldots, n-1$
  $\zeta_i := \alpha\chi_i$
endfor

Simple Vector Operations

- **Vector Subtraction**
  - Recall the geometric interpretation for adding two vectors, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$



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  - Subtracting $\boldsymbol{y}$ from $\boldsymbol{x}$ is defined as

$$\boldsymbol{x} - \boldsymbol{y} = \boldsymbol{x} + (-\boldsymbol{y})$$

Simple Vector Operations

- Now computing $x - y$ when $x, y \in \mathbb{R}^n$ is a simple matter of subtracting components of y off the corresponding components of x

$$x - y = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} - \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{n-1} \end{pmatrix} = \begin{pmatrix} \chi_0 - \psi_0 \\ \chi_1 - \psi_1 \\ \vdots \\ \chi_{n-1} - \psi_{n-1} \end{pmatrix}$$

Exercises

- For $x \in \mathbb{R}^n$, $x - x = 0$
  - Always
  - Sometime
  - Never
- For $x, y \in \mathbb{R}^n$, $x - y = y - x$
  - Always
  - Sometime
  - Never

Advanced Vector Operations

- Scaled Vector Addition (AXPY)
- Linear Combinations of Vectors
- Dot or Inner Product (DOT)
- Vector Length (NORM2)
- Vector Functions
- Vector Functions that Map a Vector to a Vector

Advanced Vector Operations

- Scaled Vector Addition (AXPY)
  - Definition
    - One of the most commonly encountered operations when implementing more complex linear algebra operations is the scaled vector addition, which (given $x, y \in \mathbb{R}^n$) computes $y := \alpha\, x + y$

$$\alpha x + y = \alpha \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} + \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha\chi_0 \\ \alpha\chi_1 \\ \vdots \\ \alpha\chi_{n-1} \end{pmatrix} + \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{n-1} \end{pmatrix} = \begin{pmatrix} \alpha\chi_0 + \psi_0 \\ \alpha\chi_1 + \psi_1 \\ \vdots \\ \alpha\chi_{n-1} + \psi_{n-1} \end{pmatrix}$$

Advanced Vector Operations

– Algorithm

- Obviously, one could copy x into another vector, scale it by a, and then add it to y. Usually, however, vector y is simply updated one element at a time:

$$\begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{n-1} \end{pmatrix} := \begin{pmatrix} \alpha\chi_0 + \psi_0 \\ \alpha\chi_1 + \psi_1 \\ \vdots \\ \alpha\chi_{n-1} + \psi_{n-1} \end{pmatrix}$$

```
for i = 0, ..., n − 1
    ψ_i := αχ_i + ψ_i
endfor
```

Advanced Vector Operations

- Linear Combinations of Vectors
  - Definition
    - Let $u, v \in \mathbb{R}^m$ and $\alpha, \beta \in \mathbb{R}$. Then $\alpha u + \beta v$ is said to be a linear combination of vectors u and v

$$\alpha u + \beta v = \alpha \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{pmatrix} + \beta \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{m-1} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha u_0 \\ \alpha u_1 \\ \vdots \\ \alpha u_{m-1} \end{pmatrix} + \beta \begin{pmatrix} \beta v_0 \\ \beta v_1 \\ \vdots \\ \beta v_{m-1} \end{pmatrix} = \begin{pmatrix} \alpha u_0 + \beta v_0 \\ \alpha u_1 + \beta v_1 \\ \vdots \\ \alpha u_{m-1} + \beta v_{m-1} \end{pmatrix}$$

Exercises

- $3\begin{pmatrix} 2 \\ 4 \\ -1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} =$

- $-3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$

- $\alpha\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \gamma\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

  $\alpha =$ $\quad$ $\beta =$ $\quad$ $\gamma =$

Advanced Vector Operations

- Algorithm
  - Given $v_0, v_1, \ldots, v_{n-1} \in \mathbb{R}^m$ and $\chi_0, \chi_1, \ldots, \chi_{n-1} \in \mathbb{R}$ the linear combination

$$w = \chi_0 v_0 + \chi_1 v_1 + \cdots + \chi_{n-1} v_{n-1}$$

  - can be computed by first setting the result vector w to the zero vector of size m, and then performing n AXPY operations

$w = 0$ (the zero vector of size m)
for $j = 0, \ldots, n-1$
  $w := \chi_j v_j + w$
endfor

An important example

- Given any $x \in \mathbb{R}^n$ with $x = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix}$, this vector can always be written as the linear combination of the unit basis vectors given by

$$x = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} = \chi_0 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + \chi_1 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + \cdots + \chi_{n-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$= \chi_0 e_0 + \chi_1 e_1 + \cdots + \chi_{n-1} e_{n-1} = \sum_{i=0}^{n-1} \chi_i e_i$$

  - Shortly, this will become really important as we make the connection between linear combinations of vectors, linear transformations, and matrices.

Advanced Vector Operations

- Dot or Inner Product (DOT)
  - Definition
    - The other commonly encountered operation is the dot (inner) product. It is defined by

$$dot(x, y) = \sum_{i=0}^{n-1} \chi_i \psi_i \quad = \quad (x^t)(y)$$

$$= \chi_0 \psi_0 + \chi_1 \psi_1 + \cdots + \chi_{n-1} \psi_{n-1}$$

Advanced Vector Operations

- Dot or Inner Product (DOT)
  - Alternative notation

$$x^T y = dot(x, y) = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix}^T \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} \chi_0 & \chi_1 & \cdots & \chi_{n-1} \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_{n-1} \end{pmatrix}$$

$$= \chi_0 \psi_0 + \chi_1 \psi_1 + \cdots + \chi_{n-1} \psi_{n-1}$$

Exercises

- $\begin{pmatrix} 2 \\ 5 \\ -6 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} =$

- $\begin{pmatrix} 2 \\ 5 \\ -6 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} =$

- $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 2 \\ 5 \\ -6 \\ 1 \end{pmatrix} =$

- For $x, y \in \mathbb{R}^n, x^T y = y^T x$
  - Always
  - Sometime
  - Never

Exercises

- $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}^T \left(\begin{pmatrix} 2 \\ 5 \\ -6 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right) =$ $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ -3 \\ 5 \end{pmatrix}$ $= \cancel{3} + 7 \cancel{-3} + 5$
  $= 12$

- $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 2 \\ 5 \\ -6 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} =$ 12

- $\left(\begin{pmatrix} 2 \\ 5 \\ -6 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right)^T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} =$ คิดเวรเยอะครับไม่ไหวไม่ไหวไหว

Exercises

$x_0 y_0 + x_0 z_0$
$x_1 y_1 + x_1 z_1$
$=$

$(x_0\ x_1)\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} + \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}$

- For $x, y, z \in \mathbb{R}^n$, $x^T(y+z) = x^Ty + x^Tz$
  - Always / Sometime / Never
- For $x, y, z \in \mathbb{R}^n$, $(x+y)^Tz = x^Tz + x^Tz$
  - Always / Sometime / Never

$x_0^2 + x_1^2$ $\quad 2(x_0y_0 + x_1y_1) + y_0^2 + y_1^2$

- For $x, y, z \in \mathbb{R}^n$, $(x+y)^T(x+y) = x^Tx + 2x^Ty + y^Ty$
  - Always / Sometime / Never

$\begin{pmatrix} x_0+y_0 \\ x_1+y_1 \end{pmatrix}^T \begin{pmatrix} x_0+y_0 \\ x_1+y_1 \end{pmatrix}$

$(x_0+y_0)(x_0+y_0) + (x_1+y_1)(x_1+y_1)$

$x_0^2 + 2x_0y_0 + y_0^2 + x_1^2 + 2x_1y_1 + y_1^2$

- For $x, y, z \in \mathbb{R}^n$. When $x^Ty = 0$, x or y is zero vector.
  - Always / Sometime / Never

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad 1 + (-1) = 0$

- For $x \in \mathbb{R}^n$, $e_i^Tx = x^Te_i = \chi_i$, where χ_i equals the ith component of x.
  - Always / Sometime / Never

Advanced Vector Operations

- Dot or Inner Product (DOT)
  - Algorithm

```
α := 0
for i = 0, ..., n − 1
    α := χ_j ψ_j + α
endfor
```

$$
\begin{array}{l}
\alpha := 0 \\
\texttt{for } i = 0, \ldots, n-1 \\
\quad \alpha := \chi_j \psi_j + \alpha \\
\texttt{endfor}
\end{array}
$$

Advanced Vector Operations

- Vector Length (NORM2)
  - Definition
    - Let $x \in \mathbb{R}^n$. Then the (Euclidean) length of a vector x (the two-norm) is given by

$$\|x\|_2 = \sqrt{\chi_0^2 + \chi_1^2 + \cdots + \chi_{n-1}^2} = \sqrt{\sum_{i=0}^{n-1} \chi_i^2}$$

$$= \sqrt{\mathrm{dot}(x, x)} = \sqrt{x^T x}$$

    - Here $\|x\|_2$ notation stands for "the two norm of x", which is another way of saying "the length of x"
  - A vector of length one is said to be a unit vector.

Exercises

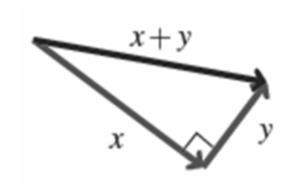
- **Compute the lengths of the following vectors**
- $\left\| \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\| =$
- $\begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} =$
- $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} =$
- $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} =$

Exercises

- Let $x \in \mathbb{R}^n$. The length of x is less than zero: $\|x\|_2 < 0$.
  – Always / Sometime / Never
- If x is a unit vector then x is a unit basis vector.
  wtf mam.
  – TRUE / FALSE
- If x is a unit basis vector then x is a unit vector.
  – TRUE / FALSE

Exercises

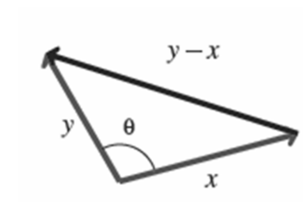
- If x and y are perpendicular (orthogonal) then $x^T y = 0$.



$|x||y|\cos\theta \Rightarrow 90$

0

  - TRUE / FALSE
- Let $x, y \in \mathbb{R}^n$ be nonzero vectors and let the angle between them equal θ. Then $\cos\theta = \dfrac{x^T y}{\|x\|_2 \|y\|_2}$



  - Always / Sometime / Never
- Let $x, y \in \mathbb{R}^n$ be nonzero vectors. Then $x^T y = 0$ if and only if x and y are orthogonal (perpendicular).
  - TRUE / FALSE

Advanced Vector Operations

- Vector Length (NORM2)
  - Algorithm
    - Clearly, $\|x\|_2 = \sqrt{x^T x}$, so that the DOT operation can be used to compute this length

Advanced Vector Operations

- Vector Functions
  - Definition
    - A vector(-valued) function is a mathematical functions of one or more scalars and/or vectors whose output is a vector.

Example

- $f(\alpha,\beta) = \begin{pmatrix} \alpha+\beta \\ \alpha-\beta \end{pmatrix}$ so that $f(-2,1) = \begin{pmatrix} -2+1 \\ -2-1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$
- $f\left(\alpha, \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) = \begin{pmatrix} \chi_0+\alpha \\ \chi_1+\alpha \\ \chi_2+\alpha \end{pmatrix}$ so that $f\left(-2, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 1+(-2) \\ 2+(-2) \\ 3+(-2) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$
- The AXPY and DOT vector functions are other functions that we have already encountered
- $f\left(\alpha, \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) = \begin{pmatrix} \chi_0+\chi_1 \\ \chi_1+\chi_2 \end{pmatrix}$ so that $f\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 1+2 \\ 2+3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

Exercises

- If $f\left(\alpha, \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) = \begin{pmatrix} \chi_0 + \alpha \\ \chi_1 + \alpha \\ \chi_2 + \alpha \end{pmatrix}$, find
- $f\left(1, \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}\right) =$ $\begin{pmatrix} 7 \\ 3 \\ 4 \end{pmatrix}$
- $f\left(\alpha, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) =$ $\begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix}$
- $f\left(0, \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) =$ $\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}$
- $f\left(\beta, \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) =$ $\begin{pmatrix} \chi_0 + \beta \\ \chi_1 + \beta \\ \chi_2 + \beta \end{pmatrix}$
- $\alpha f\left(\beta, \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) =$
- $f\left(\beta, \alpha \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) =$

$\neq$

- $f\left(\alpha, \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} + \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \end{pmatrix}\right) =$
- $f\left(\alpha, \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) + f\left(\alpha, \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \end{pmatrix}\right) =$

$\neq$

Advanced Vector Operations

- **Vector Functions that Map a Vector to a Vector**
  - **Example**

$$f(\alpha,\beta) = \begin{pmatrix} \alpha+\beta \\ \alpha-\beta \end{pmatrix} \text{ so that } f(-2,1) = \begin{pmatrix} -2+1 \\ -2-1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

We can define

$$g\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right) = \begin{pmatrix} \alpha+\beta \\ \alpha-\beta \end{pmatrix} \text{ so that } g\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -2+1 \\ -2-1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

  - **Example**

$$f\left(\alpha, \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) = \begin{pmatrix} \chi_0+\alpha \\ \chi_1+\alpha \\ \chi_2+\alpha \end{pmatrix} \text{ so that } f\left(-2, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 1+(-2) \\ 2+(-2) \\ 3+(-2) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

We can define

$$g\left(\begin{pmatrix} \alpha \\ \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} \end{pmatrix}\right) = g\begin{pmatrix} \alpha \\ \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \chi_0+\alpha \\ \chi_1+\alpha \\ \chi_2+\alpha \end{pmatrix} \text{ so that } g\begin{pmatrix} -2 \\ 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2+1 \\ -2-1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

Exercises

- If $f\left(\begin{pmatrix}\chi_0\\ \chi_1\\ \chi_2\end{pmatrix}\right) = \begin{pmatrix}\chi_0+1\\ \chi_1+2\\ \chi_2+3\end{pmatrix}$, evaluate
- $f\left(\begin{pmatrix}6\\ 2\\ 3\end{pmatrix}\right) =$
- $f\left(\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right) =$
- $f\left(2\begin{pmatrix}\chi_0\\ \chi_1\\ \chi_2\end{pmatrix}\right) =$
- $2f\left(\begin{pmatrix}\chi_0\\ \chi_1\\ \chi_2\end{pmatrix}\right) =$
- $f\left(\alpha\begin{pmatrix}\chi_0\\ \chi_1\\ \chi_2\end{pmatrix}\right) =$
- $\alpha f\left(\begin{pmatrix}\chi_0\\ \chi_1\\ \chi_2\end{pmatrix}\right) =$

$\neq$

- $f\left(\begin{pmatrix}\chi_0\\ \chi_1\\ \chi_2\end{pmatrix} + \begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\end{pmatrix}\right) =$
- $f\left(\begin{pmatrix}\chi_0\\ \chi_1\\ \chi_2\end{pmatrix}\right) + f\left(\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\end{pmatrix}\right) =$

$\neq$

Exercises

- If $f\left(\begin{pmatrix}\chi_0\\ \chi_1\\ \chi_2\end{pmatrix}\right)=\begin{pmatrix}\chi_0\\ \chi_0+\chi_1\\ \chi_0+\chi_1+\chi_2\end{pmatrix}$, evaluate
- $f\left(\begin{pmatrix}6\\ 2\\ 3\end{pmatrix}\right)=$
- $f\left(\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\right)=$
- $f\left(2\begin{pmatrix}\chi_0\\ \chi_1\\ \chi_2\end{pmatrix}\right)=$
- $2f\left(\begin{pmatrix}\chi_0\\ \chi_1\\ \chi_2\end{pmatrix}\right)=$
- $f\left(\alpha\begin{pmatrix}\chi_0\\ \chi_1\\ \chi_2\end{pmatrix}\right)=$
- $\alpha f\left(\begin{pmatrix}\chi_0\\ \chi_1\\ \chi_2\end{pmatrix}\right)=$
- $f\left(\begin{pmatrix}\chi_0\\ \chi_1\\ \chi_2\end{pmatrix}+\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\end{pmatrix}\right)=$
- $f\left(\begin{pmatrix}\chi_0\\ \chi_1\\ \chi_2\end{pmatrix}\right)+f\left(\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\end{pmatrix}\right)=$

Exercises

- If $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$, then $f(0) = 0$
  - Always / Sometime / Never
- If $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$, then $f(\lambda x) = \lambda f(x)$
  - Always / Sometime / Never

LAFF Package Development: Vectors

- Starting the Package
- A Copy Routine (copy)
- A Routine that Scales a Vector (scal)
- A Scaled Vector Addition Routine (axpy)
- An Inner Product Routine (dot)
- A Vector Length Routine (norm2)

LAFF Package Development: Vectors

- Starting the Package

Operation Abbrev.	Definition	Function	M-script intrinsic	Approx. cost	
				flops	memops
Vector-vector operations					
Copy (COPY)	$y := x$	`y = laff_copy( x, y )`	`y = x`	0	$2n$
Vector scaling (SCAL)	$x := \alpha x$	`x = laff_scal( alpha, x )`	`x = alpha * x`	n	$2n$
Scaled addition (AXPY)	$y := \alpha x + y$	`y = laff_axpy( alpha, x, y )`	`y = alpha x + y`	$2n$	$3n$
Dot product (DOT)	$\alpha := x^T y$	`alpha = laff_dot( x, y )`	`alpha = x' * y`	$2n$	$2n$
Length (NORM2)	$\alpha := \|x\|_2$	`alpha = laff_norm2( x )`	`alpha = norm2( x )`	$2n$	n

LAFF Package Development: Vectors

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad x = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$$

- In M-script (MATLAB), this can be entered as

```
A = [1 2 3; 4 5 6; 7 8 9]
```

- Now, a (column) vector can be entered as

```
x = [1; 4; 7]
```

- and is hence stored just like a matrix with only one column. Similarly, a row vector can be entered as

```
x = [1 4 7]
```

LAFF Package Development: Vectors

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad x = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$$

```
A = [1 2 3
4 5 6
7 8 9
]
x = [1
4
7
]
A(2,1) => 4
```

LAFF Package Development: Vectors

```
>> S = sin(180)

S =

   -0.8012

>> C = cos(180)

C =

   -0.5985

>> T = tan(180)

T =

    1.3387
```

```
>> [S, C, T] = trigon(180)

S =

   1.2246e-16


C =

    -1


T =

  -1.2246e-16
```

deg2rad.m

```
function [rad] = deg2rad(deg)
rad = deg * pi / 180;
end
```

trigon.m

```
function [x, y, z] = trigon(d)
r = deg2rad(d);
x = sin(r);
y = cos(r);
z = tan(r);
end
```

LAFF Package Development: Vectors

- A Copy Routine (copy)
  - Implement the function laff_copy that copies a vector into another vector. The function is defined as

    function [y_out] = laff_copy(x, y)

  - Where
    - x and y must each be either an $n \times 1$ array (column vector) or a $1 \times n$ array (row vector);
    - y_out must be the same kind of vector as y (in other words, if y is a column vector, so is y_out and if y is a row vector, so is y_out).
    - The function should “transpose” the vector if x and y do not have the same “shape” (if one is a column vector and the other one is a row vector).
    - If x and/or y are not vectors or if the size of (row or column) vector x does not match the size of (row or column) vector y, the output should be ’FAILED’.

LAFF Package Development: Vectors

- A Routine that Scales a Vector (scal)
  - Implement the function laff scal that scales a vector x by a scalar a. The function is defined as

    function [x_out] = laff_scal(alpha, x)

  - where
    - x must be either an $n \times 1$ array (column vector) or a $1 \times n$ array (row vector);
    - x_out must be the same kind of vector as x; and
    - If x or alpha are not a (row or column) vector and scalar, respectively, the output should be 'FAILED'.

LAFF Package Development: Vectors

- A Scaled Vector Addition Routine (axpy)
  - Implement the function laff_axpy that computes αx+y given scalar a and vectors x and y. The function is defined as

    function [y_out] = laff_axpy(alpha, x, y)

  - where
    - x and y must each be either an $n \times 1$ array (column vector) or a $1 \times n$ array (row vector);
    - y out must be the same kind of vector as y; and
    - If x and/or y are not vectors or if the size of (row or column) vector x does not match the size of (row or column) vector y, the output should be 'FAILED'.
    - If alpha is not a scalar, the output should be 'FAILED'.

LAFF Package Development: Vectors

- An Inner Product Routine (dot)
  - Implement the function laff_dot that computes the dot product of vectors x and y. The function is defined as

    function [alpha] = laff_dot(x, y)

  - where
    - x and y must each be either an $n \times 1$ array (column vector) or a $1 \times n$ array (row vector);
    - If x and/or y are not vectors or if the size of (row or column) vector x does not match the size of (row or column) vector y, the output should be 'FAILED'.

LAFF Package Development: Vectors

- A Vector Length Routine (norm2)
  - Implement the function laff_norm2 that computes the length of vector x. The function is defined as

    function [alpha] = laff_norm2(x)

  - where
    - x is an $n \times 1$ array (column vector) or a $1 \times n$ array (row vector);
    - If x is not a vector the output should be 'FAILED'.

Slicing and Dicing

- Slicing and Dicing: Dot Product
  - Theorem
    - Let $x \in \mathbb{R}^n$ and partition (Slice and Dice) these vectors as

$$x = \left(\begin{array}{c} x_0 \\ \hline x_1 \\ \hline \vdots \\ \hline x_{N-1} \end{array} \right) \text{ and } y = \left(\begin{array}{c} y_0 \\ \hline y_1 \\ \hline \vdots \\ \hline y_{N-1} \end{array} \right)$$

    - Where $x_i, y_i \in \mathbb{R}^{n_i}$ with $\sum_{i=0}^{N-1} n_i = n$. Then

$$x^T y = x_0{}^T y_0 + x_1{}^T y_1 + \cdots x_{N-1}{}^T y_{N-1} = \sum_{i=0}^{N-1} x_i{}^T y_i$$

Slicing and Dicing

- **Algorithms with Slicing and Redicing: Dot Product**

Algorithm: $[\alpha] := \text{DOT}(x, y)$

Partition $x \rightarrow \left(\frac{x_T}{x_B}\right)$, $y \rightarrow \left(\frac{y_T}{y_B}\right)$

where x_T and y_T have 0 elements

$\alpha := 0$

while $m(x_T) < m(x)$ **do**

Repartition

$$\left(\frac{x_T}{x_B}\right) \rightarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\frac{y_T}{y_B}\right) \rightarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

where χ_1 has 1 row, ψ_1 has 1 row

$\alpha := \chi_1 \times \psi_1 + \alpha$

Continue with

$$\left(\frac{x_T}{x_B}\right) \leftarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\frac{y_T}{y_B}\right) \leftarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

endwhile

Slicing and Dicing

- Coding with Slicing and Redicing: Dot Product
  - Follow along with the video to implement the routine

    Dot_unb(x, y)

  - The "Spark webpage" can be found at http://edx-org-utaustinx.s3.amazonaws.com/UT501x/Spark/index.html

Slicing and Dicing

- Slicing and Dicing: axpy
  - Theorem
    - Let $\alpha \in \mathbb{R}$, $x, y \in \mathbb{R}^n$ and partition (Slice and Dice) these vectors as

$$x = \left(\begin{array}{c} x_0 \\ \hline x_1 \\ \hline \vdots \\ \hline x_{N-1} \end{array} \right) \text{ and } y = \left(\begin{array}{c} y_0 \\ \hline y_1 \\ \hline \vdots \\ \hline y_{N-1} \end{array} \right)$$

    - Where $x_i, y_i \in \mathbb{R}^{n_i}$ with $\sum_{i=0}^{N-1} n_i = n$. Then

$$\alpha x + y = \alpha \left(\begin{array}{c} x_0 \\ \hline x_1 \\ \hline \vdots \\ \hline x_{N-1} \end{array} \right) + \left(\begin{array}{c} y_0 \\ \hline y_1 \\ \hline \vdots \\ \hline y_{N-1} \end{array} \right) = \left(\begin{array}{c} \alpha x_0 + y_0 \\ \hline \alpha x_1 + y_1 \\ \hline \vdots \\ \hline \alpha x_{N-1} + y_{N-1} \end{array} \right)$$

Slicing and Dicing

- **Algorithms with Slicing and Redicing: axpy**

Algorithm: $[y] := \text{AXPY}(\alpha, x, y)$

Partition $x \rightarrow \left(\frac{x_T}{x_B}\right)$, $y \rightarrow \left(\frac{y_T}{y_B}\right)$
where x_T and y_T have 0 elements

while $m(x_T) < m(x)$ **do**

Repartition

$$\left(\frac{x_T}{x_B}\right) \rightarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\frac{y_T}{y_B}\right) \rightarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

where χ_1 has 1 row, ψ_1 has 1 row

$$\psi_1 := \alpha \times \chi_1 + \psi_1$$

Continue with

$$\left(\frac{x_T}{x_B}\right) \leftarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\frac{y_T}{y_B}\right) \leftarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

endwhile

Slicing and Dicing

- Coding with Slicing and Redicing: axpy
  - Implement the routine

    Axpy_unb(alpha, x, y).

  - The "Spark webpage" can be found at http://edx-org-utaustinx.s3.amazonaws.com/UT501x/Spark/index.html

Enrichment

- **Learn the Greek Alphabet**
  - Lowercase Greek letters (α, β, etc.) are used for scalars.
  - Lowercase (Roman) letters (a, b, etc) are used for vectors.
  - Uppercase (Roman) letters (A, B, etc) are used for matrices.

  Exceptions include the letters i, j, k, l, m, and n, which are typically used for integers.

Enrichment

Matrix	Vector	Scalar			Note
		Symbol	\LaTeX	Code	
A	a	α	`\alpha`	`alpha`	
B	b	β	`\beta`	`beta`	
C	c	γ	`\gamma`	`gamma`	
D	d	δ	`\delta`	`delta`	
E	e	ε	`\epsilon`	`epsilon`	$e_j = j$th unit basis vector.
F	f	ϕ	`\phi`	`phi`	
G	g	ξ	`\xi`	`xi`	
H	h	η	`\eta`	`eta`	
I					Used for identity matrix.
K	k	κ	`\kappa`	`kappa`	
L	l	λ	`\lambda`	`lambda`	
M	m	μ	`\mu`	`mu`	$m(\cdot) =$ row dimension.
N	n	ν	`\nu`	`nu`	ν is shared with V. $n(\cdot) =$ column dimension.
P	p	π	`\pi`	`pi`	
Q	q	θ	`\theta`	`theta`	
R	r	ρ	`\rho`	`rho`	
S	s	σ	`\sigma`	`sigma`	
T	t	τ	`\tau`	`tau`	
U	u	υ	`\upsilon`	`upsilon`	
V	v	ν	`\nu`	`nu`	ν shared with N.
W	w	ω	`\omega`	`omega`	
X	x	χ	`\chi`	`chi`	
Y	y	ψ	`\psi`	`psi`	
Z	z	ζ	`\zeta`	`zeta`	

Enrichment

- $|\alpha|$: the magnitude of α

$$|\alpha| = \begin{cases} \alpha & if \alpha \geq 0 \\ -\alpha & otherwish \end{cases}$$

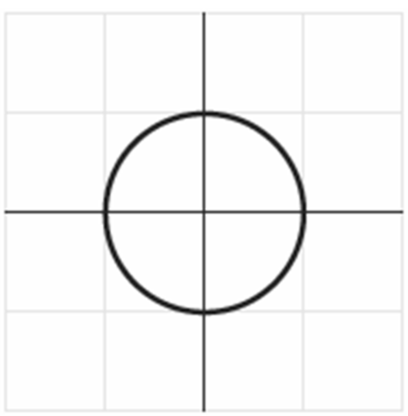
- $\|x\|_2$ (2-norms) : the magnitude of vectors x

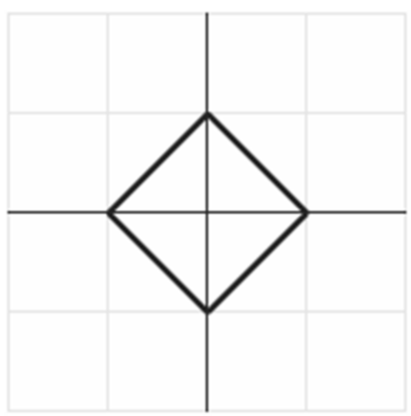
$$\|x\|_2 = \sqrt{\sum_{i=0}^{n-1} \chi_i^2}$$

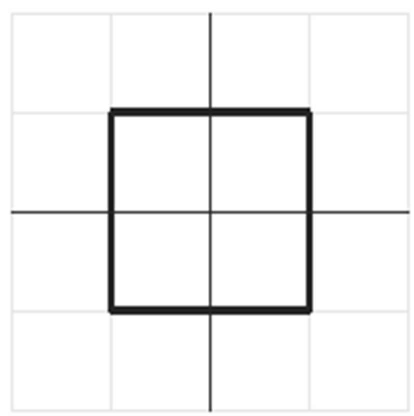
- Function $\|\cdot\|: \mathbb{R}^n \rightarrow \mathbb{R}$ is a norm if and only if the following properties hold for all $x, y \in \mathbb{R}^n$:
  - $\|x\| \geq 0$; and
  - $\|x\| = 0$ if and only if $x = 0$; and
  - $\|x + y\| \leq \|x\| + \|y\|$ (the triangle inequality).

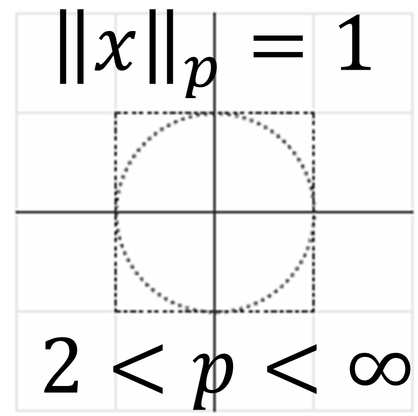
Example

- The vectors with norm equal to one are often of special interest. Below we plot the points to which vectors x with $\|x\|_2 = 1$ point (when those vectors start at the origin, (0,0)). (E.g., the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ points to the point (1,0) and that vector has 2-norm equal to one, hence the point is one of the points to be plotted.)
- Similarly, below we plot all points to which vectors x with $\|x\|_1 = 1$ point (starting at the origin)
- Similarly, below we plot all points to which vectors x with $\|x\|_\infty = 1$ point









Questions and Answers