1. Consider the following picture of "Timmy":

$f\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}0\\-2\end{pmatrix}$

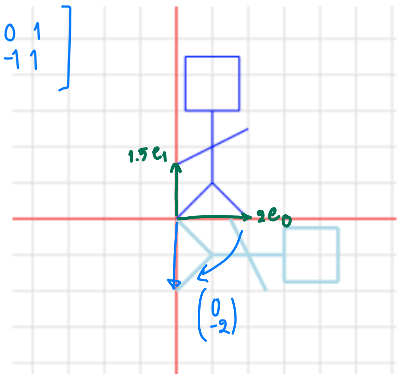
$2f(e_0) = 2\begin{pmatrix}0\\-1\end{pmatrix}$

$f(e_0) = \begin{pmatrix}0\\-1\end{pmatrix}$

$f\begin{pmatrix}0\\1.5\end{pmatrix} = \begin{pmatrix}1.5\\0\end{pmatrix}$

$1.5\, f(e_1) = \begin{pmatrix}1\\0\end{pmatrix}$

$\begin{bmatrix}0 & 1\\-1 & 1\end{bmatrix}$



$f(2e_0) = \begin{pmatrix}0\\-2\end{pmatrix}$

$f(e_0) = \begin{pmatrix}0\\-1\end{pmatrix} = a_0$

$f(1.5e_1) = \begin{pmatrix}1.5\\0\end{pmatrix}$

$f(e_1) = \begin{pmatrix}1\\0\end{pmatrix}$

(a) What matrix, A, transforms Timmy into the target?

$$A = \begin{pmatrix}\boxed{0} & \boxed{1}\\ \boxed{-1} & \boxed{0}\end{pmatrix}$$ ✓

(b) What matrix, B, transforms the target into the original Timmy?

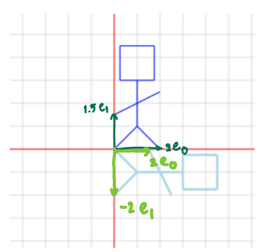
(Hint: You may want to look at how the vector $\begin{pmatrix}0\\-1\end{pmatrix}$ is transformed. How does this relate to how $\begin{pmatrix}0\\1\end{pmatrix}$ is transformed?)

$f(1.5e_0) = \begin{pmatrix}0\\1.5\end{pmatrix}$

$e_0 = \begin{pmatrix}0\\1\end{pmatrix}$

$f(-2e_1) = \begin{pmatrix}2\\0\end{pmatrix}$

$e_1 = \begin{pmatrix}-1\\0\end{pmatrix}$



$$B = \begin{pmatrix}\boxed{0} & \boxed{-1}\\ \boxed{1} & \boxed{0}\end{pmatrix}$$

$B = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$

(c) With the resulting matrices A and B that you computed, evaluate BA.

(Before you compute it by brute force, conjecture what the answer should be by thinking through how multiplication is defined. Then check what you think is the answer by computing it with the matrices.)

$$BA = \begin{pmatrix}\boxed{1} & \boxed{0}\\ \boxed{0} & \boxed{1}\end{pmatrix}$$

2. Determine the matrix A so that

$$\boxed{A\begin{pmatrix}0\\1\end{pmatrix}} = \begin{pmatrix}1\\2\\-1\end{pmatrix} \quad \text{and} \quad \boxed{A\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\3\\0\end{pmatrix}}.$$

$$\boxed{\begin{aligned} A(e_0) &= f(e_0) = a_0 \\ A(e_1) &= a_1 \end{aligned}}$$

vq a_0

$$A\left(\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}-1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\3\\0\end{pmatrix}$$

$$A\begin{pmatrix}0\\1\end{pmatrix} - A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\3\\0\end{pmatrix}$$

$$\begin{pmatrix}1\\2\\-1\end{pmatrix} - \begin{pmatrix}1\\3\\0\end{pmatrix} = A\begin{pmatrix}1\\0\end{pmatrix}$$

$$A(e_0) = \begin{pmatrix}0\\-1\\-1\end{pmatrix}$$

$$-1\,A\begin{pmatrix}-1\\1\end{pmatrix} = A\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-1\\-3\\0\end{pmatrix}$$

$$A\begin{pmatrix}1\\0\end{pmatrix} = A\begin{pmatrix}1\\-1\end{pmatrix} + A\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}-1\\-3\\0\end{pmatrix} + \begin{pmatrix}1\\2\\-1\end{pmatrix}$$

$$= \begin{pmatrix}0\\-1\\-1\end{pmatrix} = a_0$$

$$\therefore \begin{bmatrix}0 & 1\\-1 & 2\\-1 & -1\end{bmatrix}$$

3. For each of the following functions $f : \mathbb{R}^3 \to \mathbb{R}^2$, indicate whether it is a linear transformation (circle TRUE) or not (circle FALSE). Justify your answer.

(a) $f\left(\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} x_0 - x_1 + 2x_2 \\ 2x_0 + 3x_1 - x_2 \end{pmatrix}$ TRUE/FALSE

$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ x \end{pmatrix}$

$f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(b) $f\left(\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ TRUE/FALSE

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_{2\times 2} \begin{pmatrix} x \\ y \end{pmatrix}_{2\times 1}$

$\begin{pmatrix} 0 \\ x \end{pmatrix}$

(c) $f\left(\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ TRUE/FALSE

มีการบวกต่อเวกเตอร์

4. (15 points - 3 points each for (a) and (b), 9 points for (c))

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) Let $D = \begin{pmatrix} \delta_0 & 0 & 0 \\ 0 & \delta_1 & 0 \\ 0 & 0 & \delta_2 \end{pmatrix}$ where δ_0,δ_1, and δ_2 are scalars. Compute DD (matrix D multiplied by itself).

$$DD = \begin{pmatrix} \delta_0^2 & 0 & 0 \\ 0 & \delta_1^2 & 0 \\ 0 & 0 & \delta_2^2 \end{pmatrix}$$

(c) For square matrix A define A^n as $A^0 = I$ (the identity) and $A^{n+1} = A^n A$ for $n \geq 0$.

Let D again be defined as $D = \begin{pmatrix} \delta_0 & 0 & 0 \\ 0 & \delta_1 & 0 \\ 0 & 0 & \delta_2 \end{pmatrix}$. Give a proof by induction to show that

$D^n = \begin{pmatrix} \delta_0^n & 0 & 0 \\ 0 & \delta_1^n & 0 \\ 0 & 0 & \delta_2^n \end{pmatrix}$ for $n \geq 0$.

(You may want to use the next blank page)

$$A^0 = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{n+1} = A^n A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} ; \; n = 0$$

$$A^1 = A^0 A = \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{pmatrix}$$

$$\text{So, } D^1 = A^0 D$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta_0 & j & k \\ l & \delta_1 & m \\ n & 0 & \delta_2 \end{pmatrix}$$

เนื่องจาก I คูณ Matrix อะไร ค่าจะเปลี่ยนแค่ตำแหน่ง (1,1), (2,2), (3,3) ดังนั้น D^n ไม่ว่า n จะเป็นเท่าไร ค่าก็จะเปลี่ยน แค่ตำแหน่งข้างต้นเป็นค่านั้นยกกำลัง n ดังสมการ

$$D^n = \begin{pmatrix} \delta_0^n & 0 & 0 \\ 0 & \delta_1^n & 0 \\ 0 & 0 & \delta_2^n \end{pmatrix}$$

5.

```
function [ y_out ] = matvec( A, x, y )

n = size( A, 1 );
for j = 1:n

  for i = 1:j-1
    y( i ) = A( i,j ) * x( j ) + y( i );
  end

  y( j ) = A( j,j ) * x( j ) + y( j );

  for i = j+1:n
    y( i ) = A( i,j ) * x( j ) + y( i );
  end

end

y_out = y;

return
```

row

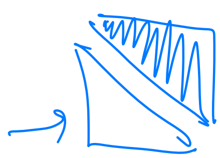
2 column

วนตามทุกแถว

ตามแถว -1

n x n.

$A^T = A$.

A → 

$[y] := \text{MATVEC\_UNB\_VAR2}(A, x, y)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$, $x \rightarrow \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right)$, $y \rightarrow \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right)$

where A_{TL} is 0×0, x_T has 0 rows, y_T has 0 rows

$\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\end{pmatrix}$ $\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$ $\begin{pmatrix}x_0\\x_1\\x_2\end{pmatrix}$

while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right),$$

$$\left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \rightarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \rightarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

where α_{11} is 1×1, χ_1 has 1 row, ψ_1 has 1 row

$y_0 := \chi_1 a_{01} + y_0$

$\psi_1 := \chi_1 \alpha_{11} + \psi_1$

$y_2 := \chi_1 a_{21} + y_2$

$y_0 = \chi_2 \begin{pmatrix}7\\8\end{pmatrix} + \begin{pmatrix}x_0+4x_1\\2x_0+5x_1\end{pmatrix}$

$\psi_1 = 9(\chi_2) + (3x_0+6x_1)$

$y = \begin{bmatrix} x_0+4x_1+7x_2 \\ 2x_0+5x_1+8x_2 \\ 3x_0+6x_1+9x_2 \end{bmatrix}$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right),$$

$$\left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \leftarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \leftarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

endwhile

Both the MATLAB code on the left and the algorithm (expressed with FLAME notation) to its right implement the computation $y := Ax + y$ (matrix-vector multiplication).

You get a choice. Either

- Modify the code on the left to implement $y := Ax + y$ where A is symmetric, stored only in the lower triangular part of array `A`, **or**
- Modify the code on the right to implement $y := Ax + y$ where A is symmetric and stored only in the lower triangular part of array A.

For an extra 2 points: do both.

ให้มาแค่ได้คเป็นสามเหลี่ยมล่าง

5.

row

2 column

```
function [ y_out ] = matvec( A, x, y )

n = size( A, 1 );
for j = 1:n

  for i = 1:j-1
    y( i ) = A( i,j ) * x( j ) + y( i );
  end

  y( j ) = A( j,j ) * x( j ) + y( j );

  for i = j+1:n
    y( i ) = A( i,j ) * x( j ) + y( i );
  end

end

y_out = y;

return
```

วิ่งตามทุกแถว

ตามแถว -1

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \quad \begin{bmatrix} y_0 \\ \psi_1 \\ y_2 \end{bmatrix} \begin{matrix} \}\, y_0 \\ \}\, \psi_1 \\ \}\, y_2 \end{matrix}$$

$$y_1 = \begin{bmatrix} x_0 \\ 4x_0 \\ 7x_0 \end{bmatrix}$$

$[y] := \text{MATVEC\_UNB\_VAR2}(A, x, y)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$, $x \rightarrow \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right)$, $y \rightarrow \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right)$

where A_{TL} is 0×0, x_T has 0 rows, y_T has 0 rows

while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \quad \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \rightarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \quad \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \rightarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

where α_{11} is 1×1, χ_1 has 1 row, ψ_1 has 1 row

$$\begin{aligned} y_0 &:= \chi_1 a_{01} + y_0 \\ \psi_1 &:= \chi_1 \alpha_{11} + \psi_1 \\ y_2 &:= \chi_1 a_{21} + y_2 \end{aligned}$$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \quad \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \leftarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \quad \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \leftarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

endwhile

Both the MATLAB code on the left and the algorithm (expressed with FLAME notation) to its right implement the computation $y := Ax + y$ (matrix-vector multiplication).

$$y_2 = \begin{bmatrix} 3x_2 + 2x_1 \\ 6x_2 + 5x_1 \\ 5x_1 + 9x_2 \end{bmatrix}$$

You get a choice. Either

- Modify the code on the left to implement $y := Ax + y$ where A is symmetric, stored only in the lower triangular part of array `A`, **or**
- Modify the code on the right to implement $y := Ax + y$ where A is symmetric and stored only in the lower triangular part of array A.

For an extra 2 points: do both.

ให้มาแต่โค้ดเป็นสามเหลี่ยมล่าง

6. Compute

(a) $(-1)\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

(b) $2\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

(e) $\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -5 & -1 \\ 3 & 2 \end{pmatrix}$

(f) $\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}^T \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}^T = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

$$= \begin{pmatrix} -5 & 3 \\ -1 & 2 \end{pmatrix}$$

7. (a) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 2 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \\ 0 & 4 \end{pmatrix}$

(e) $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}^T \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} =$ 0

(f) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} =$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

1 ยาว

(g) Is $x = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ a unit vector? $\|X\|$

TRUE/FALSE

(Why?)

$$\|x\|_2 = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}$$

$$= 1$$

เป็น เพราะ (NORM 2) = 1