The Pumping Lemma for Context-Free Languages

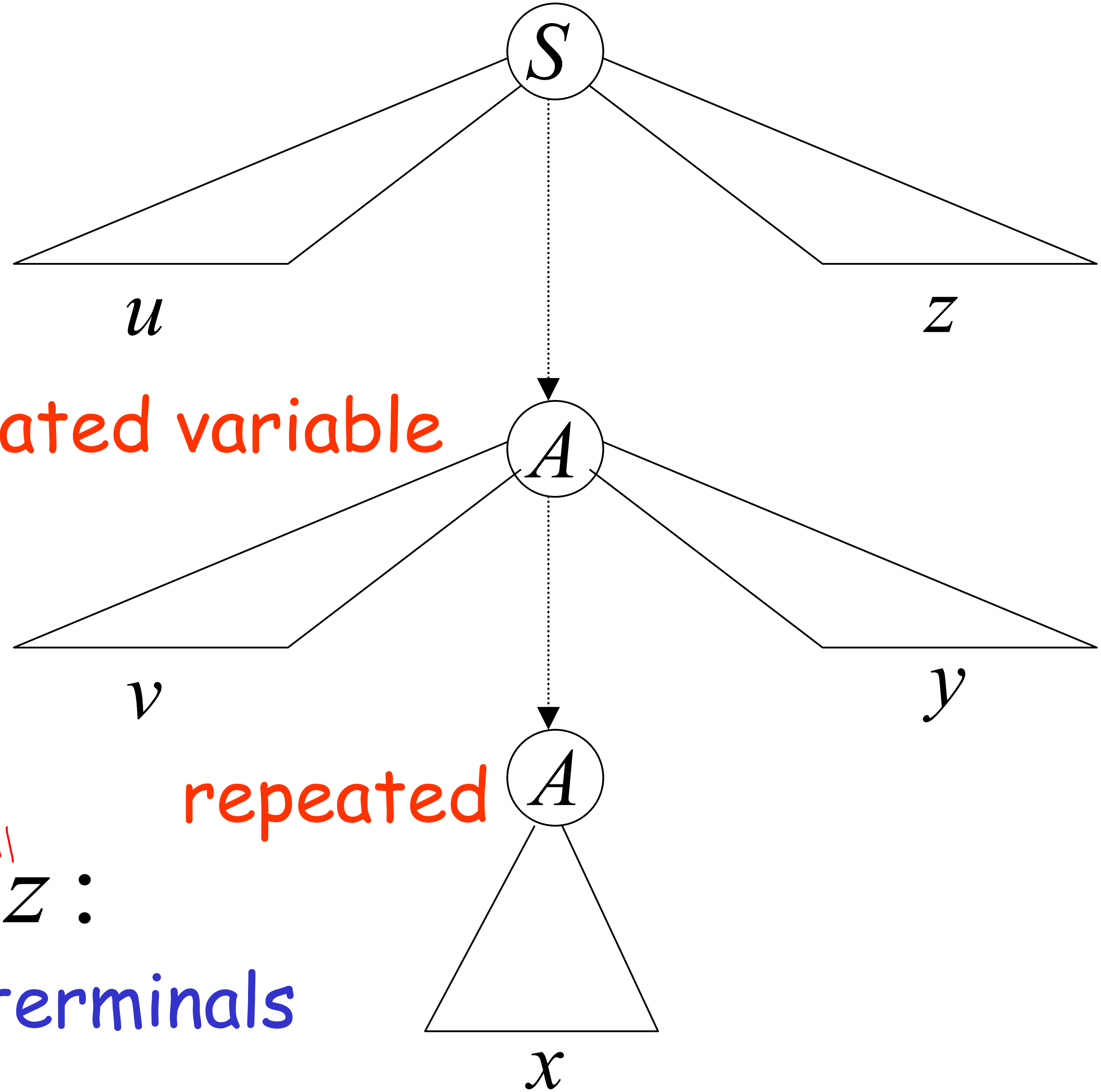
ใช้พิสูจน์ว่าไม่ใช่ cfl

Derivation tree of string w

$S \to uAz$

$A \to vAy$

$A \to x$



พี่ใน DFA → xyz

$\boxed{w = uvxyz}$

derive ออกมาให้ได้เยอะๆ

u, v, x, y, z :

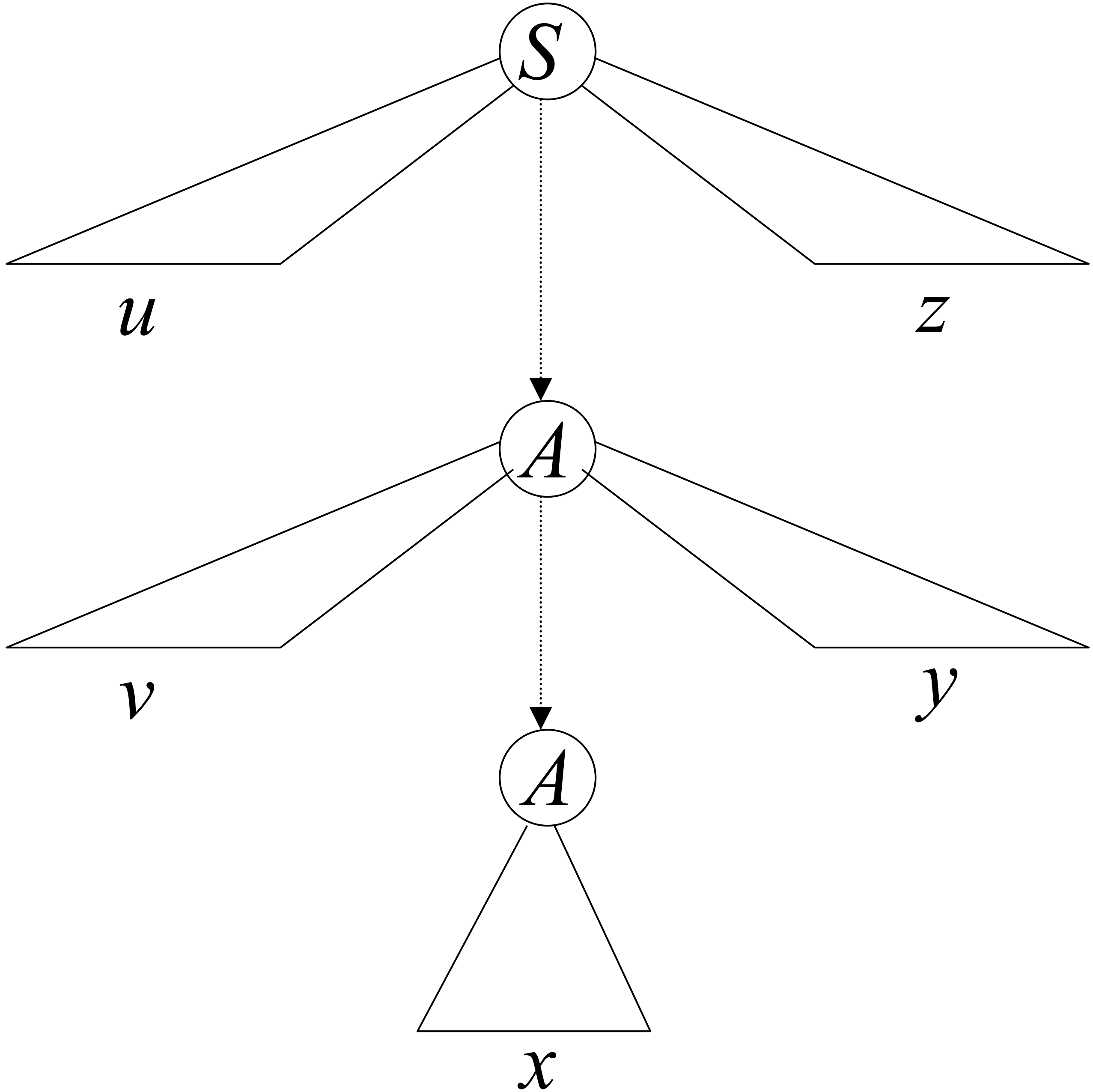
Strings of terminals

Possible derivations:

$$S \overset{*}{\Rightarrow} uAz$$

$$A \overset{*}{\Rightarrow} vAy$$

$$A \overset{*}{\Rightarrow} x$$



We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uxz$$

$$uv^0xy^0z$$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvxyz$$

The original $w = uv^1xy^1z$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvvAyyz \overset{*}{\Rightarrow} uvvxyyz$$

$$uv^2xy^2z$$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \Rightarrow uvAyz \overset{*}{\Rightarrow} uvvAyyz \overset{*}{\Rightarrow}$$

$$\overset{*}{\Rightarrow} uvvvAyyyz \overset{*}{\Rightarrow} uvvvxyyyz$$

$$uv^3xy^3z$$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvvAyyz \overset{*}{\Rightarrow}$$

$$\overset{*}{\Rightarrow} uvvvAyyyz \overset{*}{\Rightarrow} \ldots$$

$$\overset{*}{\Rightarrow} uvvv \cdots vAy \cdots yyyz \overset{*}{\Rightarrow}$$

$$\overset{*}{\Rightarrow} uvvv \cdots vxy \cdots yyyz$$

$$uv^i xy^i z$$

Therefore, any string of the form

$$uv^i xy^i z \qquad i \geq 0$$

is generated by the grammar G

Therefore,

knowing that $uvxyz \in L(G)$

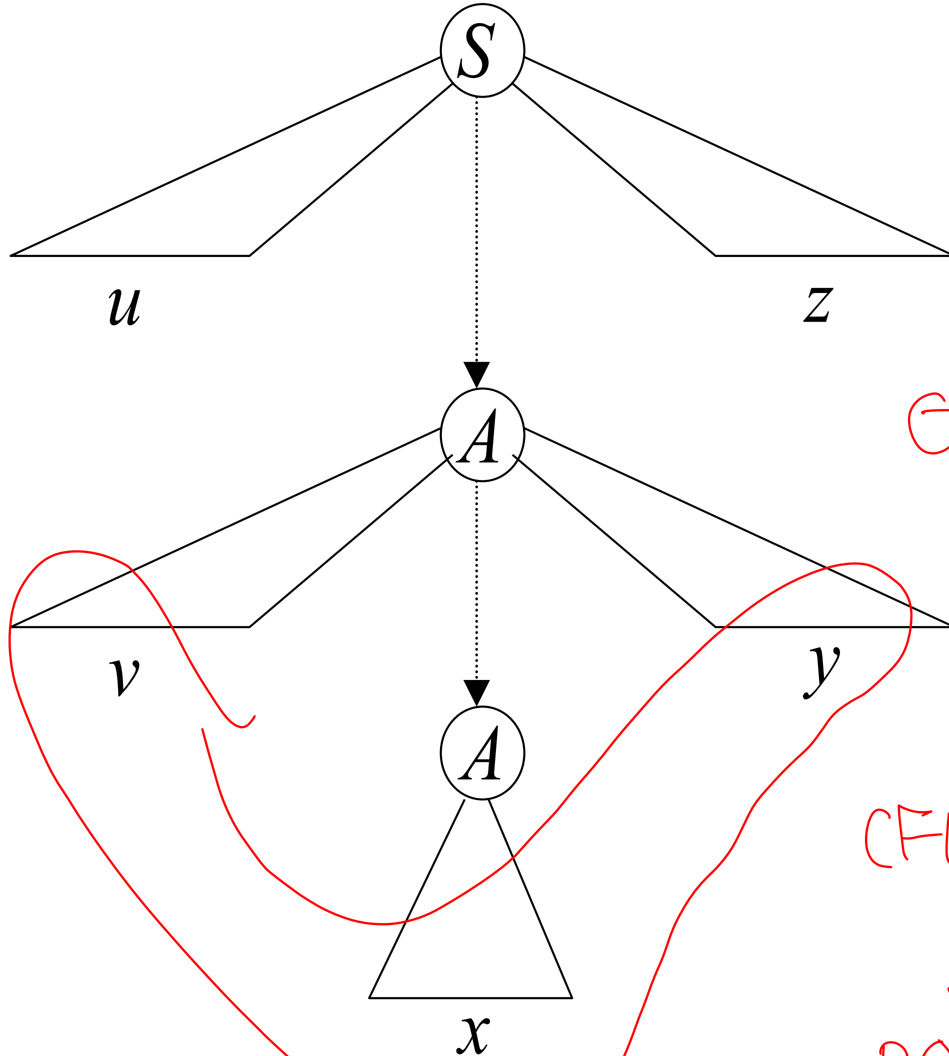
ปั๊มกี่รอบก็ตาม → อยู่ใน L(G)

we also know that $uv^i xy^i z \in L(G)$

$$L(G) = L - \{\lambda\}$$

$$\Downarrow$$

$$uv^i xy^i z \in L$$



$G: S \to uAz$

$A \to vAy$

$A \Rightarrow x$

↓

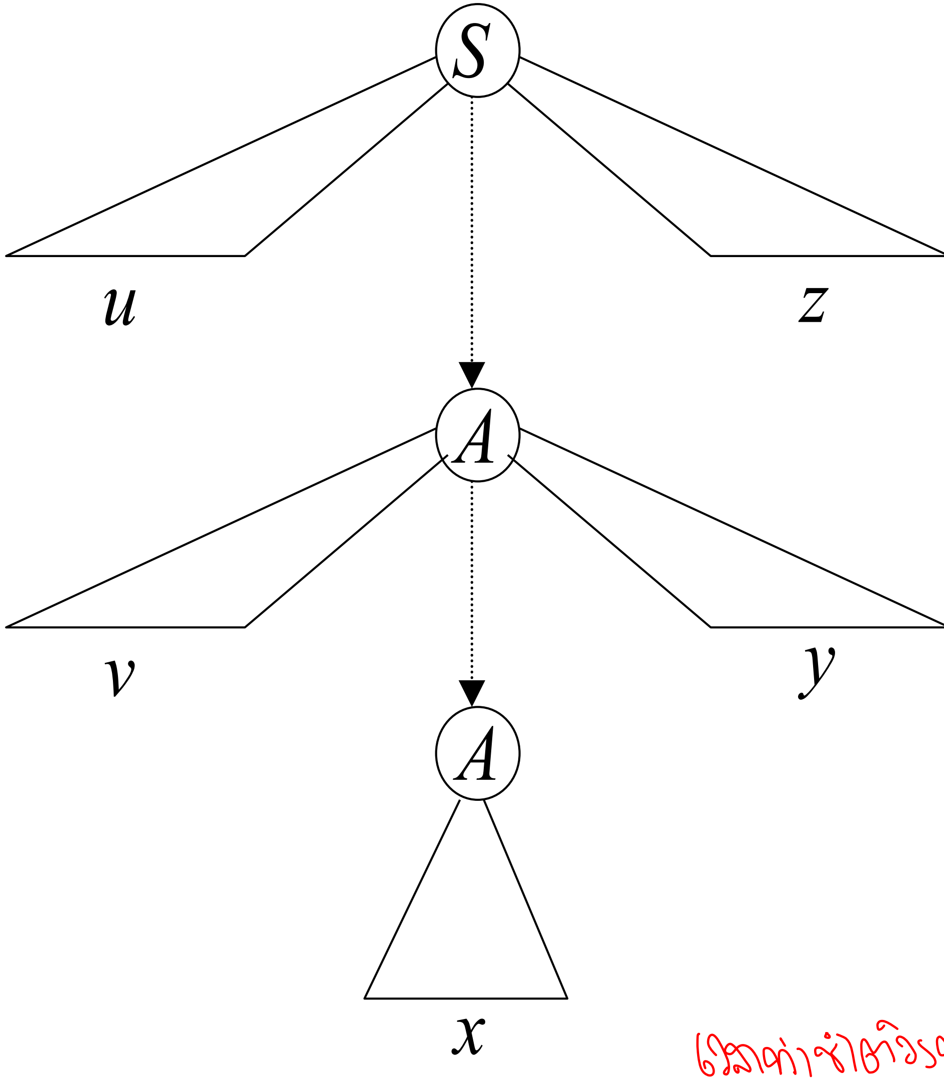
CFL → $L(G)$

↓

PDA (m: number of state)

Observation: $|vxy| \leq m$

Since A is the last repeated variable



แปลว่าซ้ำต้องทำได้ $|vy| \geq 1$

Observation: $|vy| \geq 1$

Since there are no unit or λ-productions

The Pumping Lemma:

สรุป

For infinite context-free language L

there exists an integer m such that

for any string $w \in L, \quad |w| \geq m$

we can write $w = uvxyz$ 1

2 เขาเลือก

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

and it must be: 3 เราเลือกค่า i

ก็เจอ $w \notin L$ → เป็น contra

$$uv^i xy^i z \in L, \quad \text{for all } i \geq 0$$

Applications
of
The Pumping Lemma

Non-context free languages

$$\{a^n b^n c^n : n \geq 0\}$$

ตอนนี้เราจะproveให้ดู

Context-free languages

$$\{a^n b^n : n \geq 0\}$$

Theorem: The language

$$L = \{a^n b^n c^n : n \geq 0\}$$

assume → is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{a^n b^n c^n : n \geq 0\}$$

Assume for contradiction that L

is context-free

ถ้าสมมติว่า l เป็น cfl แล้วหามาขัดแย้งได้

ไม่เป็นจริง

so l is not cfl

Since L is context-free and infinite

we can apply the pumping lemma

$$L = \{a^n b^n c^n : n \geq 0\}$$

Pumping Lemma gives a magic number m such that:

Pick any string $w \in L$ with length $|w| \geq m$

We pick: $\underline{w = a^m b^m c^m}$ เราเลือก

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

② แจกแจง

We can write: $w = uvxyz$

ต้องขยายเริ่มไป

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = u\boxed{vxy}z \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Pumping Lemma says:

$$uv^i xy^i z \in L \quad \text{for all} \quad i \geq 0$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

We examine **all** the possible locations

of string vxy in w

u, z ไม่มีกฎเกณฑ์อะไร → เป็นอะไรก็ได้

ดูทุกทางของรูปสิ่งตำแหน่ง vxy ที่เป็นไปได้

x, ω

→ พิสูจน์ทุกกรณีที่เป็นไปได้

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

กรณีแรก

Case 1: vxy is within a^m

$$\underbrace{aa}_{u}\underbrace{a...a}_{vxy}\underbrace{a \; bbb...bbb \; ccc...ccc}_{z}$$

$$\overbrace{aaa...aaa}^{m}\overbrace{bbb...bbb}^{m}\overbrace{ccc...ccc}^{m}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: v and y consist from only a

$vy = a^k; k \geq 1$

$$\underbrace{aa}_{u}\underbrace{a\ldots aa}_{vxy}\underbrace{a\;bbb\ldots bbb\;ccc\ldots ccc}_{z}$$

$$\overbrace{aaa\ldots aaa}^{m}\;\overbrace{bbb\ldots bbb}^{m}\;\overbrace{ccc\ldots ccc}^{m}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: Repeating v and y

$$k \geq 1$$

ให้ $i=2$

$$\overbrace{\underbrace{aa}\ \ \underbrace{a^k}\ \underbrace{a^k}}^{m}\ \underbrace{aabb..cc..c}_{z}$$

$m + k$ มีเพิ่มมา k ตัว

$$\underbrace{\overbrace{\underbrace{aa}_{u}\underbrace{aaaa...aaaaa}_{v^2xy^2}\underbrace{a}_{}}^{m+k}\ \overbrace{bbb...bbb}^{m}\ \overbrace{ccc...ccc}^{m}}$$

$$\underbrace{a\,bbb...bbb\,ccc...ccc}_{z}$$

a^{k+k+k}

a^k เพิ่มมา

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: From Pumping Lemma: $uv^2xy^2z \in L$

$$k \geq 1$$

$$\overbrace{\underbrace{aa}_{u}\underbrace{aaaa...aaaaa}_{v^2xy^2}a}^{m+k}\;\overbrace{bbb...bbb}^{m}\;\overbrace{ccc...ccc}^{m}$$

$$z = a\,bbb...bbb\,ccc...ccc$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: From Pumping Lemma: $uv^2xy^2z \in L$

$$k \geq 1$$

หลุดไอสัส

However: $uv^2xy^2z = a^{m+k} b^m c^m \notin L$

ขัดแย้ง → ∴ L is not cfl

Contradiction!!!

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: vxy is within b^m

เหมือนกรณีที่ 1 อ้ารเลย !

$$\overbrace{aaa...aaa}^{m}\;\overbrace{bbb...bbb}^{m}\;\overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa...aaa\;bbb}_{u}\underbrace{b...bb}_{vxy}\underbrace{b\;ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: Similar analysis with case 1

$$\underbrace{\overbrace{aaa...aaa}^{m}\,bbb}_{u}\underbrace{...bb}_{vxy}\underbrace{b\,\overbrace{ccc...ccc}^{m}}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: vxy is within c^m

$$\underbrace{\overbrace{aaa...aaa}^{m}\,\overbrace{bbb...bbb}^{m}\,cc}_{u}\overbrace{\underbrace{c...c}_{vxy}\,\underbrace{ccc}_{z}}^{m}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: Similar analysis with case 1

$$\overbrace{aaa...aaa}^{m}\ \overbrace{bbb...bbb}^{m}\ \overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa...aaa\ bbb...bbb\ c}_{u}\ \underbrace{cc...c}_{vxy}\ \underbrace{cc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

กรณีที่ overlaps

Case 4: vxy overlaps a^m and b^m

มองทุกกรณี 3 possible in overlap

ซึ่งและจะเกิดกรณีย่อยอีก 3 กรณี

$$\overbrace{aaa...aaa}^{m}\;\overbrace{bbb...bbb}^{m}\;\overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{...aaa\;bbb}_{vxy}\underbrace{...bbb\;ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 1: v contains only a

y contains only b

$$\overbrace{aaa...aaa}^{m}\ \overbrace{bbb...bbb}^{m}\ \overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{...a\underbrace{a}_{v}a\ bb\underbrace{b}_{y}}_{vxy}\underbrace{..bbb\ ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 1: v contains only a

y contains only b

$k_1 + k_2 \geq 1$ เลือก i=2

ของ a

ของ b

$$\overbrace{aaa\ldots aaaaaaa}^{m+k_1}\;\overbrace{bbbbbbb\ldots bbb}^{m+k_2}\;\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa}_{u}\;\underbrace{\ldots aaaaaaa\;bbbbbbb\ldots}_{v^2xy^2}\;\underbrace{bbb\;ccc\ldots ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$$k_1 + k_2 \geq 1$$

$$\overbrace{aaa\ldots aaaaaaa}^{m+k_1}\,\overbrace{bbbbbbb\ldots bbb}^{m+k_2}\,\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{\ldots aaaaaaa\,bbbbbbb}_{v^2xy^2}\underbrace{\ldots bbb\,ccc\ldots ccc}_{z}$$

$$L=\{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$$k_1 + k_2 \geq 1$$

ไขยตามเท่าเดิม

However: $uv^2xy^2z = a^{m+k_1} b^{m+k_2} c^m \notin L$

Contradiction!!!

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 2: v contains a and b

y contains only b

$$\overbrace{aaa...aaa}^{m}\;\overbrace{bbbb...bbb}^{m}\;\overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{...aaa\,bbbb}_{vxy}\underbrace{...bbb\,ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

However: $k_1 + k_2 + k \geq 1$

$$uv^2xy^2z = a^m b^{k_1} a^{k_2} b^{m+k} c^m \notin L$$

Contradiction!!!

$$a^3 b^3 c^3$$

$$\underbrace{aa}_{u}\ \underbrace{ab}_{v}\ \underbrace{b}_{x}\ \underbrace{b}_{y}\ \underbrace{ccc}_{z}$$

$i=2$ $\quad uv^2xy^2z \quad aa\ abab\ b\ bb\ ccc$

$$\downarrow$$

$$a^3 b^1 a^1 b^{3+1} c^n$$

$$\downarrow$$

$$a^n b^{k_1} a^{k_2} b^{n+k} c^n$$

$k_1 + k_2 + k \geq 1$ หลุดแน่นอน

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 3: v contains only a
y contains a and b

$$\overbrace{aaa...aaa}^{m}\,\overbrace{bbb...bbb}^{m}\,\overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa}_{u}\,\underbrace{...aaa\,bbb}_{vxy}\,\underbrace{...bbb\,ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 3: v contains only a

y contains a and b

Similar analysis with Possibility 2

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

ย้ายมาอีกที่

Case 5: vxy overlaps b^m and c^m

$$\overbrace{aaa\ldots aaa}^{m}\ \overbrace{bbb\ldots bbb}^{m}\ \overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa\ldots aaa\ bbb\ldots}_{u}\underbrace{bbb\ ccc\ldots}_{vxy}\underbrace{ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 5: Similar analysis with case 4

$$\overbrace{aaa\ldots aaa}^{m}\,\overbrace{bbb\ldots bbb}^{m}\,\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa\ldots aaa\,bbb\ldots}_{u}\underbrace{bbb\,ccc\ldots}_{vxy}\underbrace{ccc}_{z}$$

There are no other cases to consider

ทุก case หมดแล้ว

(since $|vxy| \leq m$, string vxy cannot

overlap a^m, b^m and c^m at the same time)

In all cases we obtained a contradiction

เราสามารถ pump ออกได้ทุกกรณี

Therefore: The original assumption that

ขัดแย้งกับ

$$L = \{a^n b^n c^n : n \geq 0\}$$

is context-free must be wrong

สรุป

Conclusion: L is not context-free