Linear Transformations and Matrices

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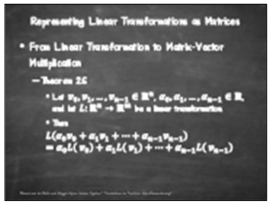

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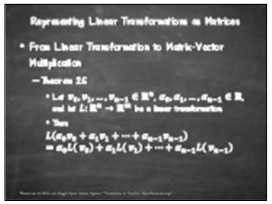
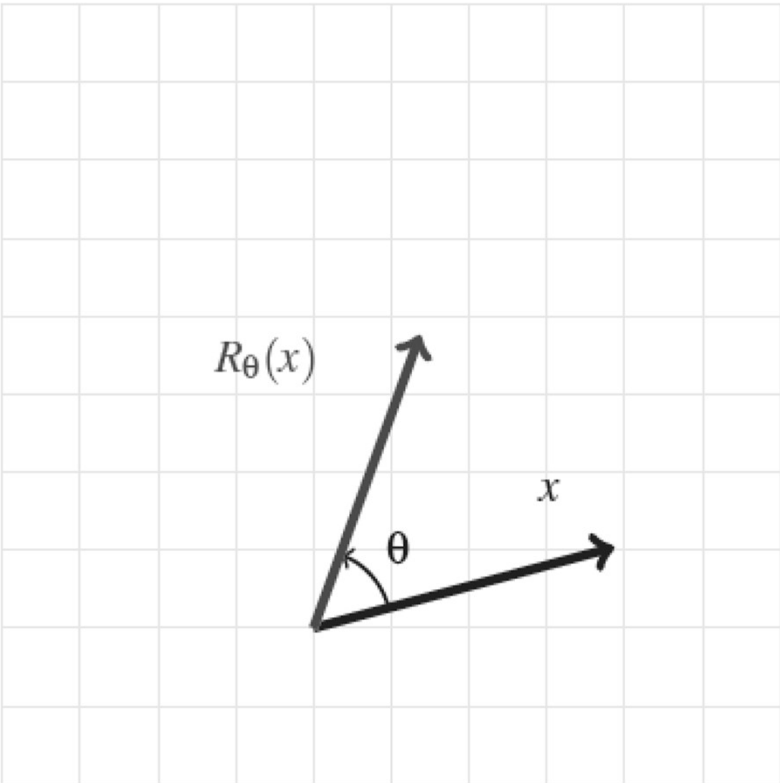


Rotating in 2D

- Let $R_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the function that rotates an input vector through an angle θ



- $R_\theta(\alpha x)$
- $\alpha R_\theta(x)$
- $R_\theta(x + y)$
- $R_\theta(x) + R_\theta(y)$

Example

- A reflection with respect to a 45 degree line is illustrated by
  - Think of the dashed green line as a mirror and $M: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as the vector function that maps a vector to its mirror image.
  - If $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, then $\mathbf{M}(\alpha x) = \alpha \mathbf{M}(x)$ and $\mathbf{M}(x + y) = \mathbf{M}(x) + \mathbf{M}(y)$ (in other words, M is a linear transformation).
    - True
    - False

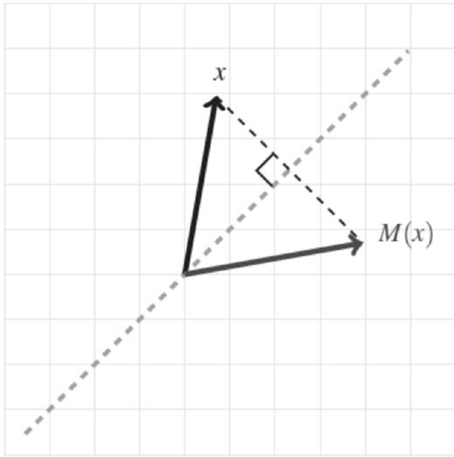
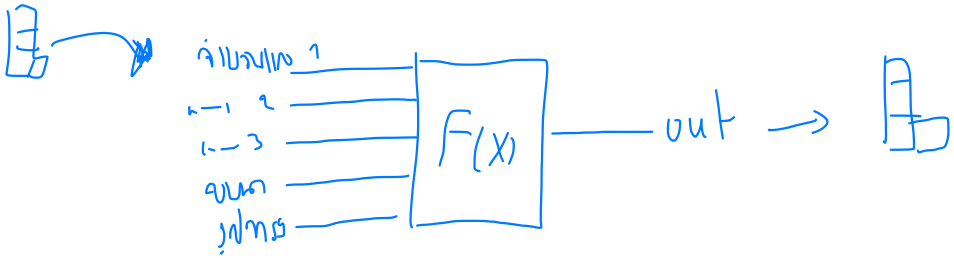


figure out how to prove.

Linear Transformations

- What Makes Linear Transformations so Special?
  - Many problems in science and engineering involve vector functions such as: $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$. Given such a function, one often wishes to do the following:
    - Given vector $x \in \mathbb{R}^n$, evaluate $f(x)$; or
    - Given vector $y \in \mathbb{R}^m$, find x such that $f(x) = y$; or
    - Find scalar λ and vector x such that $f(x) = \lambda x$ (only if $m = n$).



Linear Transformations

- What is a Linear Transformation?
  - Definition 2.1

    คุณสมบัติของ function's linear transformation

    - A vector function $L: \mathbb{R}^n \to \mathbb{R}^m$ is said to be a linear transformation, if for all $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$
      - Transforming a scaled vector is the same as scaling the transformed vector:

        $$L(\alpha x) = \alpha L(x)$$

      - Transforming the sum of two vectors is the same as summing the two transformed vectors:

        $$L(x + y) = L(x) + L(y)$$

How to prove vector → linear transform func

Example

- The transformation $f\left(\begin{pmatrix}\chi_0\\ \chi_1\end{pmatrix}\right) = \begin{pmatrix}\chi_0+\chi_1\\ \chi_0\end{pmatrix}$

  is a linear transformation?

$$f\left(\alpha\begin{pmatrix}x_0\\ x_1\end{pmatrix}\right) = f\begin{pmatrix}\alpha x_0\\ \alpha x_1\end{pmatrix} = \begin{pmatrix}\alpha x_0+\alpha x_1\\ \alpha x_0\end{pmatrix}$$

$$\alpha f\begin{pmatrix}x_0\\ x_1\end{pmatrix} = \alpha\begin{pmatrix}x_0+x_1\\ x_0\end{pmatrix} = \begin{pmatrix}\alpha x_0+\alpha x_1\\ \alpha x_0\end{pmatrix}$$

$$f\begin{pmatrix}x_0\\ x_1\end{pmatrix} + f\begin{pmatrix}y_0\\ y_1\end{pmatrix} = \begin{pmatrix}x_0+x_1\\ x_0\end{pmatrix} + \begin{pmatrix}y_0+y_1\\ y_0\end{pmatrix} = \begin{pmatrix}x_0+x_1+y_0+y_1\\ x_0+y_1\end{pmatrix}$$

$$f\begin{pmatrix}x_0+y_0\\ x_1+y_1\end{pmatrix} = \begin{pmatrix}x_0+y_0+x_1+y_1\\ x_0+y_0\end{pmatrix}$$

เป็น linear transformation.

Example

- The transformation $f\left(\begin{pmatrix}\chi \\ \psi\end{pmatrix}\right) = \begin{pmatrix}\chi + \psi \\ \chi + 1\end{pmatrix}$ is a linear transformation?

ไม่ใช่

$$f\begin{pmatrix}\alpha x \\ \alpha y\end{pmatrix} = \begin{pmatrix}\alpha x + \alpha y \\ \alpha x \underbrace{+1}\end{pmatrix}$$

ก็อบบ $\alpha f\begin{pmatrix}x \\ y\end{pmatrix}$ มีความไม่เท่า)

Exercises

- The vector function $f\left(\begin{pmatrix}\chi \\ \psi\end{pmatrix}\right) = \begin{pmatrix}\chi\psi \\ \chi\end{pmatrix}$ is a linear transformation. ✗
  - TRUE / FALSE

αx
αy $\quad \begin{pmatrix}\alpha^2 xy \\ \alpha x\end{pmatrix} \neq \alpha\begin{pmatrix}xy \\ x\end{pmatrix}$

- The vector function $f\left(\begin{pmatrix}\chi_0 \\ \chi_1 \\ \chi_2\end{pmatrix}\right) = \begin{pmatrix}\chi_0 + 1 \\ \chi_1 + 2 \\ \chi_2 + 3\end{pmatrix}$ is a linear transformation. ✗
  - TRUE / FALSE

$\begin{matrix}\alpha x_0 \\ \alpha x_1 \\ \alpha x_1\end{matrix} \Rightarrow \begin{matrix}\alpha x_1 + 1 \\ \text{"} \\ \text{"}\end{matrix} \neq \begin{pmatrix}\alpha x_1 + \alpha \\ \alpha x_2 + \alpha 2 \\ \alpha x_3 + \alpha 3\end{pmatrix}$

- The vector function $f\left(\begin{pmatrix}\chi_0 \\ \chi_1 \\ \chi_2\end{pmatrix}\right) = \begin{pmatrix}\chi_0 \\ \chi_0 + \chi_1 \\ \chi_0 + \chi_1 + \chi_2\end{pmatrix}$ is a linear transformation. ✓
  - TRUE / FALSE

$\begin{pmatrix}\alpha x_0 \\ \alpha x_0 + \alpha x_1 \\ \alpha x_0 + \alpha x_1 + \alpha x_2\end{pmatrix} = \alpha\begin{pmatrix}\bigcirc \\ \triangle \\ \square\end{pmatrix}$

ลัดก้อนก็ได้เร

Exercises

- Find an example of a function f such that $f(\alpha x) = \alpha f(x)$, but for some x, y it is the case that $f(x+y) \neq f(x) + f(y)$. (This is pretty tricky!)

  ดูใน error

  ดูใน error
  + [illegible]

αx_1
αx_0

- The vector function $f\left(\begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix}\right) = \begin{pmatrix} \chi_1 \\ \chi_0 \end{pmatrix}$ is a linear transformation.
  - TRUE / FALSE

$f\begin{pmatrix} \alpha x_0 \\ \alpha y_0 \end{pmatrix} = \begin{pmatrix} \alpha y_0 \\ \alpha x_0 \end{pmatrix}$

$\alpha f\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \alpha \begin{pmatrix} y_0 \\ x_0 \end{pmatrix}$

$f\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) = \begin{pmatrix} y_0 + y_1 \\ x_0 + x_1 \end{pmatrix}$

$\begin{pmatrix} y_0 \\ x_0 \end{pmatrix} + \begin{pmatrix} y_1 \\ x_1 \end{pmatrix} =$

Linear Transformations

- Of Linear Transformations and Linear Combinations
  - Lemma 2.4
    - $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if (iff) for all $u, v \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$$

  - Lemma 2.5
    - Let $v_0, v_1, \ldots, v_{k-1} \in \mathbb{R}^n$ and let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then

$$L(v_0 + v_1 + \cdots + v_{k-1}) \\ = L(v_0) + L(v_1) + \cdots + L(v_{k-1})$$

Mathematical Induction

- What is the Principle of Mathematical Induction?
  - The Principle of Mathematical Induction (weak induction) says that if one can show that
    - (Base case) a property holds for $n = k_b$; and
    - (Inductive step) if it holds for $n = K$, where $\mathrm{K} \geq k_b$, then it is also holds for $n = K + 1$,
  - then one can conclude that the property holds for all integers $n \geq k_b$.
  - Often $k_b = 0$ or $k_b = 1$.

Mathematical Induction

- Examples

$$\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}; n \geq 1$$

  - Base case : $n = 1$
  - Inductive step : Inductive Hypothesis (IH)
    - Assume that the result is true for $n = k$ where $k \geq 1$
    - Show that the result is then also true for $n = k + 1$
  - Find : $2\sum_{i=0}^{n-1} i$

Exercises

- Let $n \geq 1$. Then $\sum_{i=1}^{n} i = n(n+1)/2$.
  - Always / Sometimes / Never
- Let $n \geq 1$. Then $\sum_{i=0}^{n-1} 1 = n$.
  - Always / Sometimes / Never
- Let $n \geq 1$ and $x \in \mathbb{R}^m$. Then $\sum_{i=0}^{n-1} x = x + x + \cdots + x = nx$.
  - Always / Sometimes / Never
- Let $n \geq 1$. $\sum_{i=0}^{n-1} i^2 = (n-1)n(2n-1)/6$.
  - Always / Sometimes / Never

Representing Linear Transformations as Matrices

- From Linear Transformation to Matrix-Vector Multiplication
  - Theorem 2.6
    - Let $v_0, v_1, \ldots, v_{n-1} \in \mathbb{R}^n$, $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$, and let $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation.
    - Then

$$L(\alpha_0 v_0 + \alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1}) \\ = \alpha_0 L(v_0) + \alpha_1 L(v_1) + \cdots + \alpha_{n-1} L(v_{n-1})$$

Exercises

Homework 2.4.1.2 Let L be a linear transformation such that

$$L\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \text{and} \quad L\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

แยกเพื่อจะหาความสัมพันธ์ของ 2 $L(x)=l$

เราแยกมันเป็น vector ย่อยๆ

Then $L\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) =$ $2L\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \end{pmatrix} + \begin{pmatrix} 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \end{pmatrix}$

For the next three exercises, let L be a linear transformation such that

$$L\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \text{and} \quad L\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 4 \end{pmatrix}.$$

Homework 2.4.1.3 $L\left(\begin{pmatrix} 3 \\ 3 \end{pmatrix}\right) =$ $\begin{matrix} 15 \\ 12 \end{matrix}$

Homework 2.4.1.4 $L\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) =$ $\begin{matrix} -3 \\ -5 \end{matrix}$

หาผลลัพธ์

Homework 2.4.1.5 $L\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) =$ $3L\begin{pmatrix} 1 \\ 1 \end{pmatrix} - L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{matrix} 12 \\ 7 \end{matrix}$

Representing Linear Transformations as Matrices

Now we are ready to link linear transformations to matrices and matrix-vector multiplication. Recall that any vector $x \in \mathbb{R}^n$ can be written as

สามารถเขียน vector ใดๆ
เป็น linear ได้

$$x = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} = \chi_0 \underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{e_0} + \chi_1 \underbrace{\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}}_{e_1} + \cdots + \chi_{n-1} \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}}_{e_{n-1}} = \sum_{j=0}^{n-1} \chi_j e_j.$$

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Given $x \in \mathbb{R}^n$, the result of $y = L(x)$ is a vector in $\mathbb{R}^m$. But then

$$y = L(x) = L\left(\sum_{j=0}^{n-1} \chi_j e_j.\right) = \sum_{j=0}^{n-1} \chi_j L(e_j) = \sum_{j=0}^{n-1} \chi_j a_j,$$

L(x)

Matrix e_j Basis

m×n n×1

m×1 $\boxed{a_j}$

where we let $a_j = L(e_j)$.

> **The Big Idea.** The linear transformation L is completely described by the vectors
>
> $$a_0, a_1, \ldots, a_{n-1}, \quad \text{where } a_j = L(e_j)$$
>
> because for any vector x, $L(x) = \sum_{j=0}^{n-1} \chi_j a_j$.

By arranging these vectors as the columns of a two-dimensional array, which we call the matrix A, we arrive at the observation that the matrix is simply a representation of the corresponding linear transformation L.

Representing Linear Transformations as Matrices

– Definition 2.7 ($\mathbb{R}^{m\times n}$)

- The set of all $m \times n$ real valued matrices is denoted by $\mathbb{R}^{m\times n}$.
- Thus, $A \in \mathbb{R}^{m\times n}$ means that A is a real valued matrix of size $m \times n$.

Representing Linear Transformations as Matrices

– Definition 2.8 (Matrix-vector multiplication or product)

- Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{x} \in \mathbb{R}^{n}$ with

$$A = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,0} & \cdots & \alpha_{0,n-1} \\ \alpha_{1,0} & \alpha_{1,0} & \cdots & \alpha_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,0} & \cdots & \alpha_{m-1,n-1} \end{pmatrix} \quad \mathit{and} \quad x = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix}$$

$a \quad b \qquad n \quad 1$

$a_\phi \; 1$

- then

$$\begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,n-1} \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0}\chi_0 + & \alpha_{0,1}\chi_1 + & \cdots + & \alpha_{0,n-1}\chi_{n-1} \\ \alpha_{1,0}\chi_0 + & \alpha_{1,1}\chi_1 + & \cdots + & \alpha_{1,n-1}\chi_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{m-1,0}\chi_0 + & \alpha_{m-1,1}\chi_1 + & \cdots + & \alpha_{m-1,n-1}\chi_{n-1} \end{pmatrix}$$

Robert van de Geijn and Maggie Myers. Linear Algebra - Foundations to Frontiers. https://www.edx.org/

Representing Linear Transformations as Matrices

- **Practice with Matrix-Vector Multiplication**

Homework 2.4.2.1 Compute Ax when $A = \begin{pmatrix} -1 & 0 & 2 \\ -3 & 1 & -1 \\ -2 & -1 & 2 \end{pmatrix}$ and $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. $\begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix}$

Homework 2.4.2.2 Compute Ax when $A = \begin{pmatrix} -1 & 0 & 2 \\ -3 & 1 & -1 \\ -2 & -1 & 2 \end{pmatrix}$ and $x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

Homework 2.4.2.3 If A is a matrix and e_j is a unit basis vector of appropriate length, then $Ae_j = a_j$, where a_j is the jth column of matrix A.

Always/Sometimes/Never

Homework 2.4.2.4 If x is a vector and e_i is a unit basis vector of appropriate size, then their dot product, $e_i^T x$, equals the ith entry in x, χ_i.

Always/Sometimes/Never

Homework 2.4.2.5 Compute

$$\underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T}_{1\times3} \left(\underbrace{\begin{pmatrix} -1 & 0 & 2 \\ -3 & 1 & -1 \\ -2 & -1 & 2 \end{pmatrix}}_{3\times3} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{3\times1} \right) = -2$$

Representing Linear Transformations as Matrices

- It Goes Both Ways
  - Theorem 2.9
    - Let $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be defined by $L(x) = Ax$ where $A \in \mathbb{R}^{m \times n}$. Then L is a linear transformation.
    - A function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation if and only if it can be written as a matrix-vector multiplication.

Exercises

Homework 2.4.3.1 Give the linear transformation that corresponds to the matrix

$$\begin{pmatrix} 2 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

$A \in R^{2\times 4}$

$L : R^4 \to R^2$

$X \in R^4$

2×4

$L(X) = A(X)$

Homework 2.4.3.2 Give the linear transformation that corresponds to the matrix

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}_{4\times 2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{2\times 1} = \begin{pmatrix} 2x_1 + x_2 \\ x_2 \\ x_1 \\ x_1 + x_2 \end{pmatrix}_{4\times 1}$$

$A \in R^{4\times 2}$

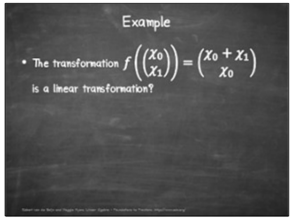
$L : R^2 \to R^4$

$X = R^2$

Representing Linear Transformations as Matrices

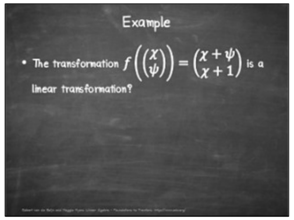
$f(e_1) = (\ )_1$
$f(e_2) = (\ )_2$ $\rightarrow Ax = ((\ )_1 (\ )_2)$

- Example 2.10 (from 2.2)
  - The transformation $f\left(\begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix}\right) = \begin{pmatrix} \chi_0 + \chi_1 \\ \chi_0 \end{pmatrix}$ is a linear transformation.



- Example 2.11 (from 2.3)
  - The transformation $f\left(\begin{pmatrix} \chi \\ \psi \end{pmatrix}\right) = \begin{pmatrix} \chi + \psi \\ \chi + 1 \end{pmatrix}$ is not a linear transformation.

ไม่ต้องคิดแทนด้วย matrix



Exercises

Homework 2.4.3.3 Let f be a vector function such that $f\left(\begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix}\right) = \begin{pmatrix} \chi_0^2 \\ \chi_1 \end{pmatrix}$ Then

- (a) f is a linear transformation. $\alpha^2 \chi_0^2$
- (b) f is not a linear transformation. $\alpha \chi_1$
- (c) Not enough information is given to determine whether f is a linear transformation.

How do you know? not [illegible], [illegible] (+)

Homework 2.4.3.4 For each of the following, determine whether it is a linear transformation or not:

- $f\left(\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) = \begin{pmatrix} \chi_0 \\ 0 \\ \chi_2 \end{pmatrix}$. ✓
- $f\left(\begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix}\right) = \begin{pmatrix} \chi_0^2 \\ 0 \end{pmatrix}$. ✗

Representing Linear Transformations as Matrices

$R_{(\theta)}$ เป็น $: \mathbb{R}^n \to \mathbb{R}^n$

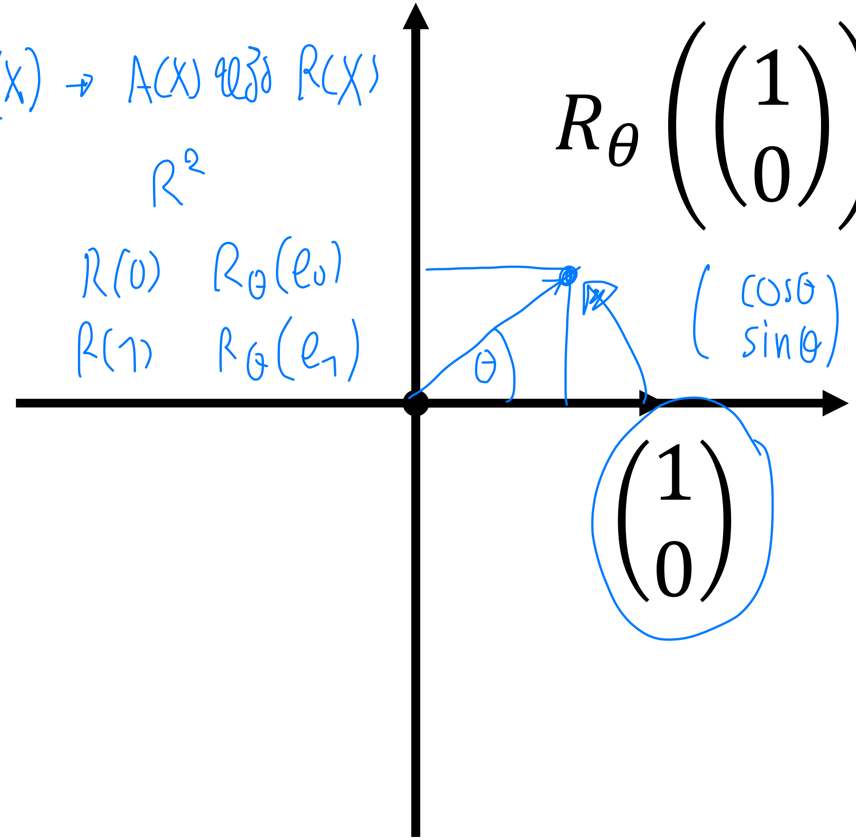
- **Rotations and Reflections, Revisited**

$R_{(\theta)}(X) \to A(X)$ เหมือน $R(X)$

$\mathbb{R}^2$

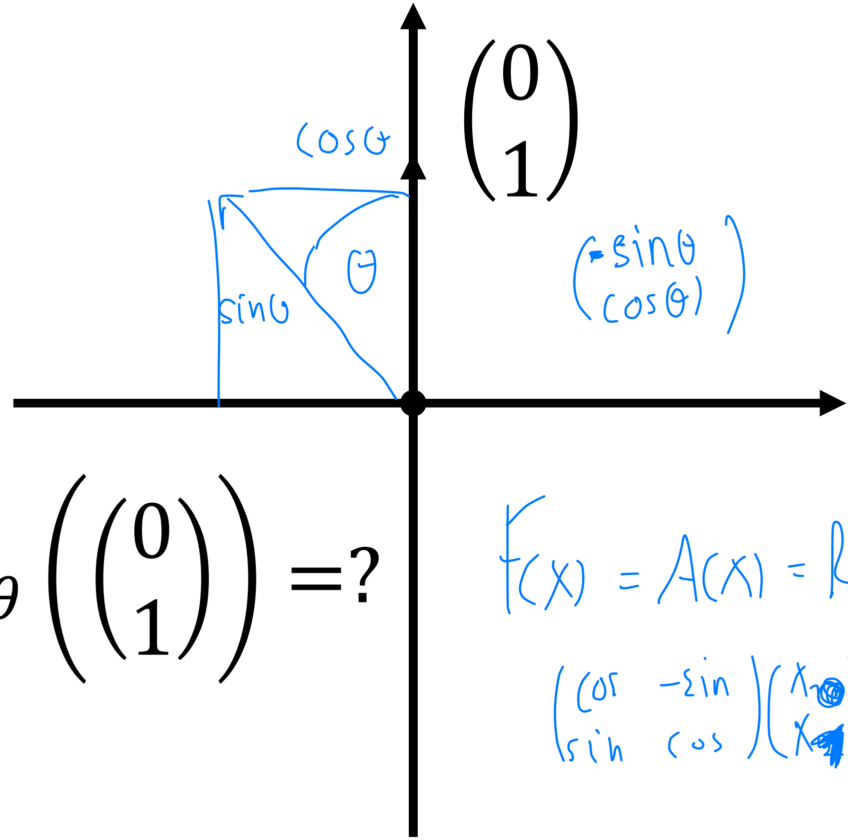
$R(0) \quad R_\theta(e_0)$

$R(1) \quad R_\theta(e_1)$



$$R_\theta\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = ?$$

$$\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$



$$R_\theta\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = ?$$

$$\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

$F(X) = A(X) = R(X)$

$$\begin{pmatrix}\cos & -\sin\\ \sin & \cos\end{pmatrix}\begin{pmatrix}X_0\\ X_1\end{pmatrix}$$

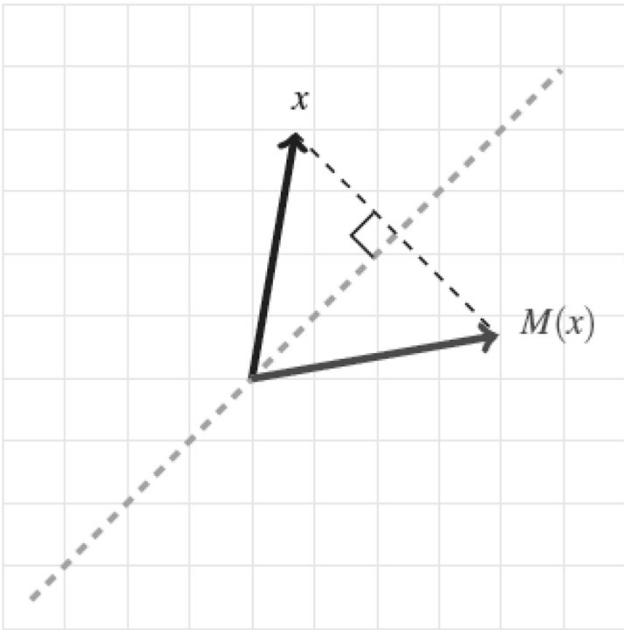
$$\text{Matrix } R = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

Representing Linear Transformations as Matrices

สรุป : ๆ ถ้ามีคุณสมบัติ $\square$ ได้ linear combination สามารถขยายได้

เอาไปแทนค่า $e_1, e_2 e_3$ แล้วเขียน $A(x)$ ได้

Homework 2.4.4.2 A reflection with respect to a 45 degree line is illustrated by



Again, think of the dashed green line as a mirror and let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector function that maps a vector to its mirror image. Compute the matrix that represents M (by examining the picture).

Questions and Answers