1. Let $L_A : \mathbb{R}^3 \to \mathbb{R}^2$ and $L_B : \mathbb{R}e^3 \to \mathbb{R}^3$ be linear transformations with

$$L_B\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = \begin{pmatrix}3\\1\\0\end{pmatrix}, \boxed{L_B\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = \begin{pmatrix}-2\\-1\\1\end{pmatrix}}, L_B\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}0\\1\\2\end{pmatrix}$$

and

$$L_A\left(\begin{pmatrix}3\\1\\0\end{pmatrix}\right) = \begin{pmatrix}2\\1\end{pmatrix}, L_A\left(\begin{pmatrix}-2\\-1\\1\end{pmatrix}\right) = \begin{pmatrix}0\\1\end{pmatrix}, L_A\left(\begin{pmatrix}0\\1\\2\end{pmatrix}\right) = \begin{pmatrix}1\\0\end{pmatrix}$$

(a) Let B equal the matrix that represents the linear transformation L_B. (In other words, $Bx = L_B(x)$ for all $x \in \mathbb{R}^3$). Then

$$B = \begin{pmatrix}3 & -2 & 0\\1 & -1 & 1\\0 & 1 & 2\end{pmatrix}$$

ตัวอื่นทำเหมือนกัน

$L_C(\cancel{x}) = L_A(L_B(x))$

(b) Let C equal the matrix such that $Cx = L_A(L_B(x))$ for all $x \in \mathbb{R}^3$.

- What are the row and column sizes of C?

$L_B(X) = \begin{pmatrix}3 & -2 & 0\\1 & -1 & 1\\0 & 1 & 2\end{pmatrix}$

$\begin{pmatrix}3\\1\\0\end{pmatrix}$

$L_C\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = L_A\left(L_B\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\right)$

$= \begin{pmatrix}2\\1\end{pmatrix}$

$\begin{pmatrix}2\\1\end{pmatrix}$

- Then

$$C = \begin{pmatrix}2 & 0 & 1\\1 & 1 & 0\end{pmatrix}$$

2×3

2. Determine the matrix A so that

$$A\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} \quad \text{and} \quad A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

$L_A(e_0) = a_0$

$L_A(e_1) = a_1$

$$2A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = A\begin{pmatrix} 1 \\ 1 \end{pmatrix} - A\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$$

3. For each of the following functions, indicate whether it is a linear transformation (circle TRUE) or not (circle FALSE). Justify your answer.

(a) $f\left(\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) = \begin{pmatrix} \chi_1\chi_0 \\ \chi_2 \end{pmatrix}$ TRUE/FALSE

$f(\alpha x) = \alpha f(x)$

$\begin{pmatrix} \alpha x_1 \alpha x_0 \\ \alpha x_2 \end{pmatrix} \neq \begin{matrix} \alpha x_1 x_0 \\ x_2 \end{matrix}$

(b) $f\left(\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) = \begin{pmatrix} 3\chi_0 + \chi_1 \\ \chi_0 \\ \chi_2 \end{pmatrix}$ TRUE/FALSE

$f(x+y) = f(x) + f(y)$

$f\begin{pmatrix} x_0 + y_0 \\ x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} = f\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} + f\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$

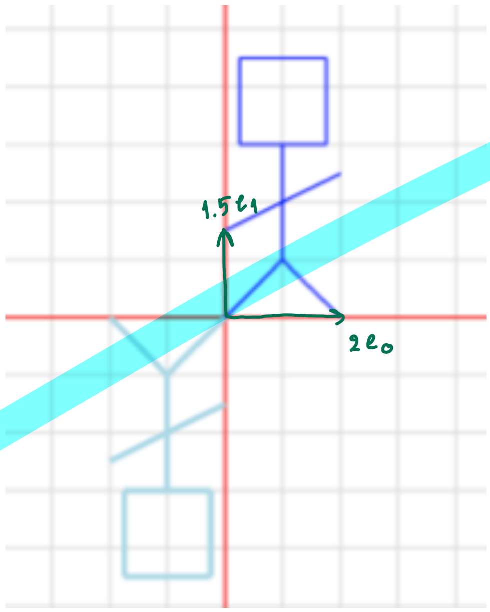
$\begin{pmatrix} 3x_0 + 3y_0 + x_1 + y_1 \\ x_0 + y_0 \\ x_2 + y_2 \end{pmatrix} = \begin{pmatrix} 3x_0 + x_1 \\ x_0 \\ x_2 \end{pmatrix} + \begin{pmatrix} 3y_0 + y_1 \\ y_0 \\ y_2 \end{pmatrix}$

$\sim = \begin{pmatrix} 3x_0 + 3y_0 + x_1 + y_1 \\ x_0 + y_0 \\ x_2 + y_2 \end{pmatrix}$

(c) $f\left(\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) = \begin{pmatrix} 3\chi_0 + \chi_1 + 1 \\ \chi_0 \\ \chi_2 \end{pmatrix}$ TRUE/FALSE

$\begin{matrix} 3x_0 + 3y_0 + x_1 + y_1 + 1 \\ x_0 + y_0 \\ x_2 + y_2 \end{matrix} = \begin{pmatrix} 3x_0 + x_1 + 1 \\ x_0 \\ x_2 \end{pmatrix} + \begin{pmatrix} 3y_0 + y_1 + 1 \\ y_0 \\ y_2 \end{pmatrix}$

4. Consider the following picture of "Timmy":



$$f(2e_0) = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$2(f(e_0)) = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$f(e_0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = a_0$$

$$f(1.5e_1) = \begin{pmatrix} 0 \\ -1.5 \end{pmatrix}$$

$$1.5 f(e_1) = \begin{pmatrix} 0 \\ -1.5 \end{pmatrix}$$

$$f(e_1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = a_1$$

What matrix, A, transforms Timmy (the blue figure drawn with thin lines that is in the top-right quadrant) into the target (the light blue figure drawn with thicker lines that is in the bottom-left quadrant)?

$$A = \begin{pmatrix} \boxed{-1} & \boxed{0} \\ \boxed{0} & \boxed{-1} \end{pmatrix}$$

$L(x) = \underbrace{\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{2\times 1}$ $\quad \begin{matrix} 1 \\ 0 \end{matrix} \quad \begin{matrix} 2 \\ 5 \end{matrix}$

5. Let $A = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$ and $L_A : \mathbb{R}^2 \to \mathbb{R}^2$, the linear transformation defined by $L(x) = Ax$.

(a) Find the vector x such that $L_A(x) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

$L_A(e_0)$

$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(b) Find the vector y such that $L_A(y) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$L_A(e_1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(c) Find the vector z such that $L_A(z) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

$L_A(z) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$= L_A\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + L_A\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$

$z = e_0 + e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

6. Consider the MATLAB (M-script) code

```
function [ y_out ] = foo( A, x, y )

n = size( A, 1 );
for j = 1:n
    for i = 1:j
        y( i ) = A( j,i ) * x( j ) + y( i );
    end
    for i = j+1:n
        y( i ) = A( i,j ) * x( j ) + y( i );
    end
end

y_out = y;

return
end
```

(If you have a hard time interpreting this algorithm, you may want to consider the algorithm typeset with FLAME notation at the end of this exam.)

Mark which operation this implements (check all correct answers):

- ☐ $y := Lx + y$, where L is a lower triangular matrix, stored only in the lower triangular part of array `A` .
- ☐ $y := Ux + y$, where U is a upper triangular matrix, stored only in the lower triangular part of array `A` .
- ☐ $y := Ax + y$, where A is symmetric, stored only in the lower triangular part of array `A` .
- ☐ $y := Ax + y$, where A is symmetric, stored only in the upper triangular part of array `A` .
- ☐ The equivalent of `y = ( tril( A ) + tril( A, -1 )' ) * x + y` in MATLAB's M-script.
- ☐ The equivalent of `y = ( triu( A ) + triu( A, 1 )' ) * x + y` in MATLAB's M-script.

2 ANS

7. Compute

(a) $2\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix}$

(b) $\begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix} + (-1)\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix}$

(d) $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}^T \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} =$ 2

(e) $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}^T \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} =$ 5

(f) $\begin{pmatrix} 3 & -1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}^T \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

(g) $\left\| \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\|_2 =$ 5

(h) $\begin{pmatrix} 3 & 2 & 1 \\ 4 & 4 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 4 & 4 \\ -1 & 1 \end{pmatrix} =$ $\begin{pmatrix} 16 & 15 \\ 29 & 23 \end{pmatrix}$

(i) $\begin{pmatrix} 3 & 2 & 1 \\ 4 & 4 & -1 \end{pmatrix}^T \begin{pmatrix} 3 & 2 \\ 4 & 4 \\ -1 & 1 \end{pmatrix}^T =$ $\begin{pmatrix} 3 & 4 \\ 2 & 4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 4 & -1 \\ 2 & 4 & 1 \end{pmatrix}$

$= \begin{pmatrix} 17 & 28 & 1 \\ 14 & 24 & 2 \\ 1 & 0 & -2 \end{pmatrix}$

8. For an $m \times n$ matrix A and an $n \times m$ matrix B,

$$(AB)^T = A^T B^T.$$

Always/Sometimes/Never

Answer:

Sometimes

- When confronted with this kind of question, it pays to first think to yourself "What if I pick somethings really simple?"

  The simplest thing one can pick is $m = n = 1$ and then you can see what happens if you take $A = \left(\begin{array}{c} 0 \end{array} \right)$ and $B = \left(\begin{array}{c} 0 \end{array} \right)$.

  You quickly conclude that for this example $(AB)^T = A^T B^T$ (both sides evaluate to $\left(\begin{array}{c} 0 \end{array} \right)$.

- Now think a little further: What if $m = n = 1$ and $A = \left(\begin{array}{c} \alpha \end{array} \right)$ and $B = \left(\begin{array}{c} \beta \end{array} \right)$, where α and β are arbitrary scalars. You notice that $AB = \left(\begin{array}{c} \alpha \end{array} \right) \left(\begin{array}{c} \beta \end{array} \right) = \left(\begin{array}{c} \alpha\beta \end{array} \right)$ and $A^T B^T = \left(\begin{array}{c} \alpha \end{array} \right)^T \left(\begin{array}{c} \beta \end{array} \right)^T = \left(\begin{array}{c} \alpha \end{array} \right) \left(\begin{array}{c} \beta \end{array} \right) = \left(\begin{array}{c} \alpha\beta \end{array} \right)$. So now you start thinking that perhaps the answer may be "always".
- But then you think to yourself: AB is a matrix that is $m \times m$ and $A^T B^T$ is a matrix that is $n \times n$ (because A^T is $n \times m$ and B^T is $m \times n$. So, if you pick $m \neq n$ and you are guaranteed to have an example where $AB \neq A^T B^T$.
- But then you think to yourself: What if $m = n$? If you pick almost any matrix where $A \neq B$ you will find that $(AB)^T \neq A^T B^T$.

Algorithm: $y := \textsc{foo}(A, x, y)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$,

$x \rightarrow \left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right)$, $y \rightarrow \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right)$

where A_{TL} is 0×0, x_T, y_T are 0×1

while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right),$$

$$\left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \rightarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \rightarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

where α_{11}, χ_1, and ψ_1 are scalars

$y_0 := \chi_1 (a_{10}^T)^T + y_0$

$\psi_1 := \chi_1 \alpha_{11} + \psi_1$

$y_2 := \chi_1 a_{21} + y_2$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right),$$

$$\left(\begin{array}{c} x_T \\ \hline x_B \end{array}\right) \leftarrow \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right), \left(\begin{array}{c} y_T \\ \hline y_B \end{array}\right) \leftarrow \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline y_2 \end{array}\right)$$

endwhile