A Simulation of Playing a Guitar

吉他的物理仿真

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The Model of the Guitar

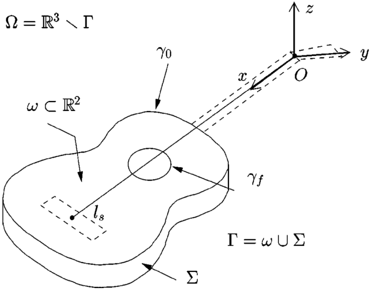

图 1: The Model of the Guitar

- The Initial Impulse
- The Movement of the String
- The Movement of the Soundboard
- The Mixture Acoustic Field

Combine the Submodels

The Impulse $f_s(x,t) = g(x)f(t)$

$$\downarrow$$

The String's wave equation $F_s(u_s(x,t)) = -f_s(x,t)$

$$\Bigg\downarrow \quad \begin{aligned} u_p(l_s,t) &= u_p(x_0,y_0,t) \\ \mathcal{F}(x,y,t) &= -T\partial_x u_s(l_s,t)\partial_{x_0,y_0}(x,y) \end{aligned}$$

The Soundboard's wave equation $F_b(u_b(x,y,t)) = \mathcal{F} - [p]_\omega$

$$\Bigg\downarrow \quad \begin{aligned} \mathbf{v}_a(x,y,0,t)\cdot\mathbf{e}_z &= \partial_t u_p(x,y,z) \text{ in } \omega \\ \mathbf{v}_a(x,y,z,t)\cdot\mathbf{N}_\Gamma &= 0 \text{ on } \Gamma \end{aligned}$$

The Acoustic Field equations

$$\begin{aligned} \partial p/\partial t &= -c_a^2\rho_a \text{div } \mathbf{v}_a \\ \rho_a \partial\mathbf{v}_a/\partial t &= -\nabla p \end{aligned}$$

Discrete Time Domain

Explicit centered finite difference is used to iterate over time Δt, taking the sound field as an example

$$\mathbf{v}_a(x, y, 0, t) \cdot \mathbf{e}_z = \partial_t u_p(x, y, z) \text{ in } \omega$$

$$\mathbf{v}_a(x, y, z, t) \cdot \mathbf{N}_\Gamma = 0 \text{ on } \Gamma$$

Is discretized as

$$\frac{p^{n+1} - 2p^n + p^{n-1}}{\Delta t^2} = -c_a^2 \rho_a \nabla \cdot \mathbf{v}_a^n$$

$$\rho_a \frac{\mathbf{v}_a^{n+1} - \mathbf{v}_a^{n-1}}{2\Delta t} = -\nabla p^n$$

We will iterate through the discrete time point to solve the system of equations.

Discrete Space Domain(1)

For the space domain, we plan to use Finite Element Method.

未知量	物理意义	域映射	定义	有限元	系数向量
v_s	弦的法向速度	$\mathbb{R} \to \mathbb{R}$	$\partial_t u_s$	P_0 (piecewise constant)	v_{s_h}
q	弦的张力	$\mathbb{R} \to \mathbb{R}$	$T\partial_x u_s$	P_1 (piecewise linear)	q_h
v_p	板的法向速度	$\omega \to \mathbb{R}$	$\partial_t u_p$	P_2 (lagrange polynomial)	v_{p_h}
M	板的弯曲矩	$\omega \to \mathbb{R}^3$	$a^3\mathbf{C}\varepsilon(\nabla u_p)$	P_2 (lagrange polynomial)	M_h
λ	接触面压力差	$\Gamma \to \mathbb{R}$	$p_e - p_i$	P_0 (lagrange polynomial)	λ_h
p	声场的声压	$\Omega \to \mathbb{R}$		P_0 (piecewise constant in cube)	p_h
$\mathbf{v}_a$	声场的空气速度	$\Omega \to \mathbb{R}^3$		P_0 (Raviart − Thomas in cube)	$\mathbf{v}_{a_h}$

表 1: 吉他模型中的未知量

Discrete Space Domain(2)

有限元系数向量	物理意义	域映射	定义	有限元
v_{s_h}	弦的法向速度	$\mathbb{R} \to \mathbb{R}$	$\partial_t u_s$	P_0 (piecewise constant)
q_h	弦的张力	$\mathbb{R} \to \mathbb{R}$	$T \partial_x u_s$	P_1 (piecewise linear)
v_{p_h}	板的法向速度	$\omega \to \mathbb{R}$	$\partial_t u_p$	P_2 (lagrange polynomial)
M_h	板的弯曲矩	$\omega \to \mathbb{R}^3$	$a^3 \mathbf{C} \varepsilon(\nabla u_p)$	P_2 (lagrange polynomial)
λ_h	接触面压力差	$\Gamma \to \mathbb{R}$	$p_e - p_i$	P_0 (lagrange polynomial)
p_h	声场的声压	$\Omega \to \mathbb{R}$		P_0 (piecewise constant in cube)
$\mathbf{v}_{a_h}$	声场的空气速度	$\Omega \to \mathbb{R}^3$		P_0 (Raviart $-$ Thomas in cube)

表 2: 吉他模型中有限元系数向量

We will use Finite Element Method to solve for the unknowns in the space domain.

String's Equation and FEM

$u_s(x,t)$: the displacement along the normal direction of the string at x and t

$$\rho_s \frac{\partial^2 u_s}{\partial t^2} - T\left(1 + \eta_s \frac{\partial}{\partial t}\right)\frac{\partial^2 u_s}{\partial x^2} + \rho_s R_s \frac{\partial u_s}{\partial t} = f_s(x,t)$$

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$$\Rightarrow \begin{cases} \rho_s \partial_t v_s - \partial_x q = f_s \\ \partial_t q - T \partial_x v_s = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{d}{dt} \int_0^{l_s} \rho_s v_s v_s^* \, dx - \int_0^{l_s} \partial_x q v_s^* = \int_0^{l_s} f_s v_s^* \, dx \\ \frac{d}{dt} \int_0^{l_s} \frac{1}{T} q q^* + \int_0^{l_s} \partial_x q^* v_s - q^*(l_s, t) v_p(x_0, y_0) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} M_h^q \frac{dq_h}{dt} - D_h q_h = f_{s_h} \\ M_h^v \frac{dv_{sh}}{dt} - D_h^T v_{sh} + J_h^T v_{ph} = 0 \end{cases}$$

The Sound Board Model

$$a\rho_p \frac{\partial^2 u_p}{\partial t^2} + \left(1 + h_p \frac{\partial}{\partial t}\right) \nabla_{x,y} \cdot \nabla_{x,y} a^3 \mathbf{C}\varepsilon(\nabla_{x,y} u_p) + a r_p R_p \frac{\partial u_p}{\partial t} = F - [p]_\omega, \quad \text{in} \quad \omega$$

- $u_p(x, y, t)$: transverse displacement of the sound board at position (x, y) and time t.
- a: thickness of the sound board
- r_p: density of the sound board
- h_p, R_p: damping coefficients
- F: force exerted by the strings on the sound board
- $[p]_\omega$: pressure difference of the sound field on the sound board

The Soundboard Model FEM

$$\int_\Omega a\rho_p v_p \frac{\partial v_p^*}{\partial t}\, d\Omega - \int_\Omega (\mathrm{Div} M)\cdot(\nabla v_p^*)\, d\Omega + \int_\Omega a\rho_p R_p v_p v_p^*\, d\Omega = \int_\Omega (F - [p]_\omega) v_p^*\, d\Omega$$

其中

$$M = \begin{bmatrix} M_{xx} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix}$$

$$\mathrm{Div} M = \left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y}, \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y}\right)^T$$

$$\int_\Omega (\mathrm{Div} M)\cdot(\nabla v_p^*)\, d\Omega = \int_\Omega \left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y}\right)\frac{\partial v_p^*}{\partial x} + \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y}\right)\frac{\partial v_p^*}{\partial y}\, d\Omega$$

The Soundboard Model FEM

$$M_p \frac{dv_{ph}}{dt} - H^T M_h + R_p M_p v_{ph} = -J_h q_h - (B_v)^T \lambda_h$$

$$M_p : N_p \times N_p$$

$$v_{ph} : N_p \times 1$$

$$H^T : N_p \times (3N_m)$$

$$M_h : 3N_m \times 1$$

$$R_p : N_p \times N_p$$

$$J_h : N_p \times N_s$$

$$q_h : N_s \times 1$$

$$B_v : N_p \times N_\lambda$$

$$\lambda_h : N_\lambda \times 1$$

The Soundboard Model FEM

$$\int_\Omega a^{-3} C_e^{-1} M \frac{\partial M^*}{\partial t}\, d\Omega + \int_\Omega (\mathrm{Div} M^*) \cdot (\nabla v_p)\, d\Omega = 0$$

$$M_M \frac{dM_h}{dt} + H v_{ph} = 0$$

$$M_M : 3N_m \times 3N_m,\ \tfrac{dM_h}{dt} : 3N_m \times 1,\ H : 3N_m \times N_p,\ v_{ph} : N_p \times 1$$

The Sound Filed Model

$$\frac{\partial p}{\partial t} = -c_a^2 \rho_a \nabla \cdot \mathbf{v}_a$$

$$\rho_a \frac{\partial \mathbf{v}_a}{\partial t} = -\nabla p$$

其中： $p(x, y, z, t)$： 声压 $\mathbf{v}_a(x, y, z, t)$： 声速 c_a： 空气中的声速 ρ_a： 空气密度边界条件

$$\mathbf{v}_a \cdot \mathbf{e}_z = \frac{\partial u_p}{\partial t} \quad \text{in} \quad \omega$$

初始条件

$$p(x, y, z, 0) = p^0(x, y, z), \quad \mathbf{v}_a(x, y, z, 0) = \mathbf{v}_a^0(x, y, z)$$

The Sound Field Model FEM

$$\int_V \frac{1}{\rho_a c_a^2} p \frac{\partial p^*}{\partial t}\, dV + \int_V p^* (\mathrm{div} v_a)\, dV = 0$$

$$M_{pa} \frac{dp_h}{dt} + G^T v_{ah} = 0$$

$M_{pa} : N_p \times N_p,\ \frac{dp_h}{dt} : N_p \times 1,\ G^T : N_p \times N_a,\ v_{ah} : N_a \times 1$

$$\int_V \rho_a v_a \frac{\partial v_a^*}{\partial t}\, dV - \int_V p(\nabla \cdot v_a^*)\, dV = \int_\Gamma \lambda v_a^* \cdot \mathbf{n}\, d\Gamma$$

$$M_a \frac{dv_{ah}}{dt} - G_h p_h - (B_G)^T \lambda_h = 0$$

$M_a : N_a \times N_a,\ \frac{dv_{ah}}{dt} : N_a \times 1,\ G_h : N_a \times N_p,\ p_h : N_p \times 1,\ B_G^T : N_a \times N_\lambda,\ \lambda_h : N_\lambda \times 1$
$B_v v_{ph} - B_G v_{ah} = 0\ B_v v_{ph} - B_G v_{ah} = 0\ B_v : N_\lambda \times N_p,\ v_{ph} : N_p \times 1,\ B_G : N_\lambda \times N_a,$
$v_{ah} : N_a \times 1$

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表 3: 吉他模型中的未知量

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表 4: 吉他模型中有限元系数向量

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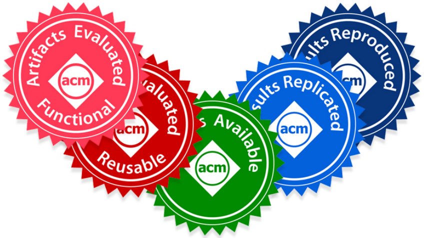
How to evaluate our Implementation



- The string's movement.
- The string and the soundboard in the air.
- The string, the soundboard, and the sound field.
- The string, the soundboard, the sound filed, and the air.
- CUDA Acceleration/PyTorch/TensorFlow

Related Work

ANSYS、ADINA、ABAQUS、MSC、ASFEM

Thanks!

[DCJB03]

[DCJB03] Grégoire Derveaux, Antoine Chaigne, Patrick Joly, and Eliane Bécache. Time-domain simulation of a guitar: Model and method. *The Journal of the Acoustical Society of America*, 114(6):3368–3383, 12 2003.