THE TIME-SPACE SIMULATION OF A GUITAR: MODEL, METHOD, IMPLEMENTATION AND DETAILS

TECHNICAL REPORT*

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ABSTRACT

In this technical report, we present the time-space simulation of a guitar in details and provide every necessary detailed intermediate information for the reader to know the whole process of the simulation. We introduce the model of the guitar in Section 2, the finite element method to solve the model in Section 3, the details of derivation of the finite element method in Section 4, and the implementation detail and experiment in Section 5. Our main contrinutions are (1) providing a much more detailed derivation of the finite element method to solve the model of a guitar, (2) providing the implementation detail and experiment of the simulation of a guitar, increasing much reproducibility (3) providing the code in the GitHub repository for the reader to reproduce the result, which is at https://github.com/xsjk/SI114H-Project.

Keywords Finite Element Method · Acoustic Model · Reproducibility Report

1 Introduction

The acoustic simulation of an instrument is an active area of research in the field of computer music and physical simulation. The goal is to create a model (and the corresponding computing method) that can generate sound similar to areal instrument, to be used in music production, sound synthesis, and so on. However, the simulation of an instrument is a complex task, as it involves multiple physical phenomena and medium, such as different mediums, multiple physical phenomena, and so on.

Specifically, the guitar is composed of a *String*, a *Sound board*, an *Air Field* within the cabin. The palyer's perturbation on the string is the input of the system, and the air pressure at the sound hole is the output. For each part of the guitar, we use a mathematical model (ie. a differential equation taken from physics) to describe the physical phenomena. For the connective between each component, they are coupled together by the boundary conditions. In this report, we first present the detailed model of the guitar in section 2 and then give the numerical method to solve the model in section 3. Finally, we show the key indispensable derivation details in 4.

2 Mathematical Model of a Guitar

2.1 Definition

As is depicted in Fig. 1: The body of the guitar is delimited by a surface denoted G which is divided into two parts: $\Gamma = \omega \cup \Sigma$, where ω is the top plate (i.e. sound board) of the instrument and Σ is the rest of the surface (i.e. sides and back). The boundary of ω itself is divided into two parts: γ_0 is the outer boundary of the top plate and γ_f is the inner

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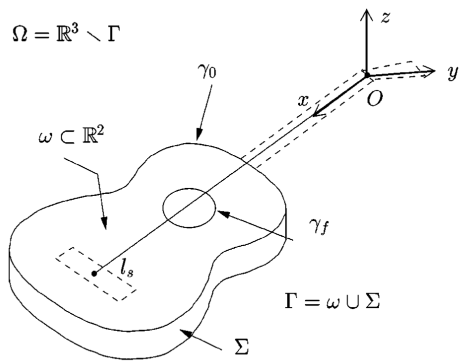

Figure 1: The Illutration of a Guitar with Symbols

boundary, along the hole. The surrounding air occupies the domain $\Omega = \mathbb{R}^3 - \Gamma$. Ω corresponds to the internal cavity and the external domain, which communicate via the sound hole. The string of length l_s is rigidly fixed to the neck at a point denoted O, chosen as the origin of the coordinate system. All above are depicted in Fig. 1. Also, we present the unknowns in the guitar model in Table 1.

Unknown	Physical Meaning	Domain Mapping	Governing Equation
u_s	Displacement of the string	$\mathbb{R} \to \mathbb{R}$	1D Damped Wave Equation
u_p	Displacement of the plate	$\omega \to \mathbb{R}$	Kirchhoff-Love Equation
$\mathbf{v}_a$	Air velocity in the sound field	$\Omega \to \mathbb{R}^3$	Linearized Euler's Equations
p	Sound pressure of the sound field	$\Omega \to \mathbb{R}$	Linearized Euler's Equations

Table 1: Summary of Main Unknowns in the Guitar Model

However, since the orginal model is

2.2 The Model of String

We assume that the string is uniform and inextensible (i.e. no stifness), with a uniform density ρ_s. The motion of the string is described by the vertical transverse displacement $u_s(x,t) \quad x \in (0, l_s)$. The string is subject equation is a classical 1D damped wave equation:

$$\rho_s \frac{\partial^2 u_s}{\partial t^2} - T\left(1 + \eta_s \frac{\partial}{\partial t}\right) \frac{\partial^2 u_s}{\partial x^2} + \rho_s R_s \frac{\partial u_s}{\partial t} = f_s(x,t) \tag{1}$$

where μ_s, R_s are constants with provided typical values. The function $f_s(x,t)$ is the known external force applied to the string. Notably, the only unknown in the equation is the displacement $u_s(x,t)$.

And the boundary conditions are:

$$u_s(0,t) = 0 \quad \forall t > 0 \quad \text{and} \quad u_s(l_s,t) = u_p(x_0, y_0, t) \quad \forall t > 0$$

where $u_p(x_0, y_0, t)$ is the displacement of the plate at the connecting bridge (x_0, y_0).

Also, the load function $f_s(x,t)$ is given by:

$$f_s(x,t) = g(x)h(t)$$

and $g(x)$ and $h(t)$ are known functions.

Key Takeaways

- **Point 1:** The String is Governed by a 1D Damped Wave Equation.
- **Point 2:** The String is Subject to External Forces (pluck the string)
- **Point 3:** The String is associated with the rest of the model by the boundary conditions.

2.3 The Model of the Sound Board

It is assumed that the only vibrating part of the guitar is the soundboard. The other parts of the guitar body (back, sides, neck) are assumed to be perfectly rigid. The motion of the strutted soundboard is thus completely described by the transverse displacement of the top plate, denoted $u_p(x, y, t), (x, y) \in \omega$. Here we present the model for the soundboard without further details. The model is based on the Kirchhoff-Love. For details, please see Derveaux et al. [2003]. We first present a compacted representation of the equation for intuition.

$$a\rho_p \frac{\partial^2 u_p}{\partial t^2} + \left(1 + h_p \frac{\partial}{\partial t}\right) \nabla_{x,y} \cdot \nabla_{x,y} a^3 \mathbf{C} \underline{\varepsilon}(\nabla_{x,y} u_p) + a r_p R_p \frac{\partial u_p}{\partial t} = F - [p]_\omega, \quad \text{in} \quad \omega \tag{2}$$

$\underline{\varepsilon}$ is the plane linearized strain tensor $\left[\varepsilon_{\alpha\beta}(\theta) = \frac{1}{2}(\partial_\beta \theta_\alpha + \partial_\alpha \theta_\beta)\right]$, and ∇ denote the usual divergence and gradiant. The above euqation is to provide intuition on the model. Notably, the second term is called *bending moment*. The more explicit way to represent the above equation is as follows:

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{xx} \\ \mathcal{M}_{yy} \\ \mathcal{M}_{xy} \end{pmatrix} = a^3 \begin{pmatrix} D_1 & D_2/2 & 0 \\ D_2/2 & D_3 & 0 \\ 0 & 0 & D_4/2 \end{pmatrix} \begin{pmatrix} \partial_{xx} u_p \\ \partial_{yy} u_p \\ \partial_{xy} u_p \end{pmatrix} \tag{3}$$

Where D_1, D_2, D_3 are constants. And the equation 2 can be written as:

$$a\rho_p \frac{\partial^2 u_p}{\partial t^2} + \left(1 + h_p \frac{\partial}{\partial t}\right) \left(\frac{\partial^2 \mathcal{M}_{xx}}{\partial x^2} + \frac{\partial^2 \mathcal{M}_{yy}}{\partial y^2} + 2\frac{\partial^2 \mathcal{M}_{xy}}{\partial x \partial y}\right) + a r_p R_p \frac{\partial u_p}{\partial t} = F - [p]_\omega, \quad \text{in} \quad \omega \tag{4}$$

where

$$\mathcal{F}(x, y, t) \approx -T \partial_x u_S(l_S, t) \delta_{x_0, y_0}(x, y).$$

And $[p]_\omega$ is the pressure applied on the soundboard. Please note that except for the u_p and $[p]_\omega$, all the other parameters are constants.

And the Boundary conditions are as follows:

$$u_p(x, y, t) = 0 \quad \text{and} \quad \partial_n u_p(x, y, t) = 0 \quad \text{on} \quad \gamma_0 \tag{5}$$

$$(\mathcal{M}n) \cdot n = 0 \quad \text{on} \quad \gamma_f \tag{6}$$

$$(\nabla_{x,y} \cdot \mathcal{M}) \cdot n + \partial_\tau[(\mathcal{M}n) \cdot \tau] = 0 \quad \text{on} \quad \gamma_f \tag{7}$$

where n is the outer normal and τ is the tangent to the boundary.

If you are lost, no worries. We will provide a summary of all variables and the correpsonding derivations later.

Key Takeaways

- **Point 1:** The Sound Board is Governed by the Kirchhoff-Love Equation.
- **Point 2:** The String is associated with the Sound Board model by the load function.
- **Point 3:** The String is associated with the conditions and the load function.

2.4 The Model of the Air Field

The acoustic field is governed by the linearized Euler's equations

$$\frac{\partial p}{\partial t} = c_a^2 \rho_a \mathrm{div}(\mathbf{v}_a) \quad \text{in } \Omega, \tag{8}$$

$$\rho_a \frac{\partial \mathbf{v}_a}{\partial t} = -\nabla p \quad \text{in } \Omega, \tag{9}$$

where c_a is the speed of the sound in air, ρ_a is the density of air, p is the sound pressure in Ω, and $\mathbf{v}_a$ the acoustic velocity in Ω. These equations are complemented by a condition of continuity for the normal component of the velocity at the surface ω of the plate

$$\mathbf{v}_a(x, y, 0, t) \cdot \mathbf{e}_z = \partial_t u_p(x, y, t), \quad \forall (x, y) \in \omega, \quad \forall t > 0. \tag{10}$$

In addition, as the body of the guitar is assumed to be perfectly rigid, one has

$$\mathbf{v}_a(x, y, z, t) \cdot \mathbf{N}_\Gamma = 0, \quad \forall (x, y, z) \in \Gamma, \quad \forall t > 0. \tag{11}$$

where $\mathbf{N}_\Gamma$ denotes the outer normal to the boundary Σ.

Key Takeaways

- **Point 1:** The Air Field is Governed by linearized Euler's equations
- **Point 2:** The Air Field is associated with the String Model by the Boundary Conditions.

3 Finite Element Method to Solve the Model

To find the numerical solution to the our simulation, we follow a standard FEM convention to solve the problem. We first derive the variational formulation by muplication-and-integrate in section 3.2 and then discretize them in space domain and in the domain in section 3.5. Before we start, we first present the result and the whole picture of our derivation in section 3.1.

3.1 The Whole Picture

We first present the whole picture (without detailed definition) of the Finite Element Method (FEM) for this problem. Later, we will provide key steps of derivation in section 3.2 and 3.5. The whole picture is shown in the following equations 12, 13, 14, 15, 16, and 17. And the definition of the variables are summarized in the Table. 3.2, also the discretized variables are summarized in the Table. 3.5.

The Brown colored lowercase letters represent the unknowns (is in the form of vector). And the uppercase letters represent the cofficient matrices produced by the Finite Element Method. And the non-Brown lowercase variables are the known variables. The upper-righer superscript n means the variables at time time point n.

$$\begin{aligned} v_{p_h}^{n+(1/2)} = & \left[\cos(\sqrt{K_h}\Delta t) v_{p_h}^{n-(1/2)} + \frac{\sin(\sqrt{K_h}\Delta t)}{\sqrt{K_h}} \frac{\mathrm{d} v_{p_h}^{n-(1/2)}}{\mathrm{d}t} \right] \\ & + \frac{I - \cos(\sqrt{K_h}\Delta t)}{\sqrt{K_h}} \Big(- J_h \frac{q_h^{n+1} - q_h^{n-1}}{2\Delta t} \\ & - B_\omega^T \frac{\lambda_h^{n+(1/2)} - \lambda_h^{n-(1/2)}}{\Delta t} \Big). \end{aligned} \tag{12}$$

$$M_h^s \frac{v_{s_h}^{n+1/2} - v_{s_h}^{n-1/2}}{\Delta t} - D_h q_h^n = f_{s_h}^n. \tag{13}$$

$$M_h^q \frac{q_h^{n+1} - q_h^n}{\Delta t} + D_h^T v_{s_h}^{n+1/2} - J_h^T v_{p_h}^{n+1/2} = 0, \tag{14}$$

$$M_h^a \frac{v_{a_h}^{n+1} - v_{a_h}^n}{\Delta t} - G_h p_h^{n+1/2} - B_\Gamma^T \lambda_h^{n+1/2} = 0, \tag{15}$$

$$M_h^{p_a} \frac{p_h^{n+1/2} - p_h^{n-1/2}}{\Delta t} + G_h v_{a_h}^n = 0, \tag{16}$$

$$B_{\omega_h} \frac{v_{p_h}^{n+1/2} - v_{ph}^{n-1/2}}{\Delta t} - B_{\Gamma_h} \frac{v_{a_h}^{n+1} - v_{a_h}^{n-1}}{2\Delta t} = 0. \tag{17}$$

There are six unknowns in total, $v_{s_h}^{n+1/2}$, q_h^{n+1}, $v_{a_h}^{n+1}$, $p_h^{n+1/2}$, $v_{p_h}^{n+1/2}$, and $\lambda_h^{n+1/2}$ and accordingly six equations. Solving these equations simultaneously, finally we can get the unknowns by equations 18, 19, 20:

$$v_{s_h}^{n+\frac{1}{2}} = v_{s_h}^{n-\frac{1}{2}} + (M_h^s)^{-1}(f_{s_h}^n + D_h q_h^n)\,\Delta t \tag{18}$$

$$p_h^{n+\frac{1}{2}} = p_h^{n-(\frac{1}{2})} - (M_h^{p_a})^{-1} G_h \mathbf{v}_{a_h}^n\,\Delta t \tag{19}$$

$$\begin{pmatrix} I & 0 & \frac{1-\cos(\sqrt{K_h}\Delta t)}{\sqrt{K_h}} \frac{B_{\omega_h}^T}{\Delta t} & \frac{I-\cos(\sqrt{K_h}\Delta t)}{\sqrt{K_h}} \frac{J_h}{2\Delta t} \\ 0 & \frac{M_h^a}{\Delta t} & -B_{\Gamma_h}^T & 0 \\ 2B_{\omega_h} & -B_{\Gamma_h} & 0 & 0 \\ -J_h{}^T & 0 & 0 & \frac{M_h^q}{\Delta t} \end{pmatrix} \begin{pmatrix} v_{p_h}^{n+\frac{1}{2}} \\ \mathbf{v}_{a_h}^{n+1} \\ \lambda_h^{n+\frac{1}{2}} \\ q_h^{n+1} \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\sqrt{K_h}\Delta t) v_{p_h}^{n-\frac{1}{2}} + \frac{\sin(\sqrt{K_h}\Delta t)}{\sqrt{K_h}} \dot{v}_{p_h}^{n-\frac{1}{2}} + \frac{1-\cos(\sqrt{K_h}\Delta t)}{\sqrt{K_h}} \left(\frac{B_{\omega_h}^T}{\Delta t} \lambda_h^{n-\frac{1}{2}} + \frac{J_h}{2\Delta t} q_h^n \right) \\ \frac{M_h^a}{\Delta t} \mathbf{v}_{a_h}^n + G_h p_h^{n+\frac{1}{2}} \\ 2B_{\omega_h} v_{p_h}^{n-\frac{1}{2}} - B_{\Gamma_h} \mathbf{v}_{a_h}^{n-1} v_{p_h}^{n+\frac{1}{2}} \\ M_h^q v_{s_h}^{n+\frac{1}{2}} + D_h^T v_{s_h}^{n+\frac{1}{2}} \end{pmatrix} \tag{20}$$

Later in section 3.2 and 3.5 , we will give the derivation of the above equations. The derivation steps are typical in the Finite Element Method, that is to find the variational formulation (section 3.2) (multiplies a test function) of the problem and then discretize (section 3.5) the variational formulation (breaks down into linear combination of trial function).

Also, we will provide mathematical details of some of the derivation in section 4.

Key Takeaways

- **Point 1:** By FEM, we can solve out unknowns at time-point $n + \frac{1}{2}$ by six equations with the variables at time-point $n-1, n-\frac{1}{2}$,and n are known.
- **Point 2:** The equation provides a method to compute the unknowns in time-evolving order.
- **Point 3:** Later in section 3.2 and 3.5, we will provide the derivation of the above equations.

3.2 Variational Formulation

We follow the standard way of Finite Element Method. Generally, we get the variational form by multiplying the equation by a test function v and integrating over the space domain Ω with integrate-by-part technique.

we define auxiliary variational unknowns in 3.2 to enable the variational form of the equation.

3.2.1 Variational Form of the String Equation

According the main governing equation 1, we get the variational form of the string equation by multiplying the equation by the P_0, P_1 test function v_s^* and integrating over the space domain Ω_s with integrate-by-part technique.

$$\frac{d}{dt}\int_0^{l_s} \rho_s v_s^* v_s^*\,dx - \int_0^{l_s} \partial_x q v_s^*\,dx = \int_0^{l_s} f_s v_s^*\,dx, \quad \forall v_s^* \tag{21}$$

Unknown	Physical Meaning	Domain Mapping	Definition	Finite Element
v_s	Normal velocity of the string	$\mathbb{R} \to \mathbb{R}$	$\partial_t u_s$	P_0 (piecewise constant)
q	Tension of the string	$\mathbb{R} \to \mathbb{R}$	$T\partial_x u_s$	P_1 (piecewise linear)
v_p	Normal velocity of the plate	$\omega \to \mathbb{R}$	$\partial_t u_p$	P_2 (Lagrange polynomial)
$\mathcal{M}$	Bending moment of the plate	$\omega \to \mathbb{R}^3$	$a^3\mathbf{C}\underline{\varepsilon}(\nabla u_p)$	P_2 (Lagrange polynomial)
λ	Pressure difference on the contact surface	$\Gamma \to \mathbb{R}$	$p_e - p_i$	P_0 (Lagrange polynomial)
p	Sound pressure of the sound field	$\Omega \to \mathbb{R}$		P_0 (piecewise constant in cube)
$\mathbf{v}_a$	Air velocity in the sound field	$\Omega \to \mathbb{R}^3$		P_1 (piecewise linear)

Table 2: Variational Unknowns in the Guitar Model

$$\frac{d}{dt}\left(\int_0^{l_s} \frac{1}{T} q q^* \, dx\right) + \int_0^{l_s} \partial_x q^* v_s \, dx - q^*(l_s, t) v_p(x_0, y_0) = 0, \quad \forall q^*. \tag{22}$$

where v_s^* and q^* are the test functions for the string equation, defined in the *Element* column of Table 3.2. For detailed derivation, we write some in section 4.

3.3 Variational Form of the Plate Equation

Coming from the main governing equation 4, we get the variational form of the plate equation by multiplying the equation by the P_2 test function v_p^* and integrating over the space domain ω with integrate-by-part technique.

Where : operator is the double-dot product of two tensors, and $\mathcal{M}^*$ is the test bending moment function. The actual derivation is written in section 4.5.

$$\begin{aligned}\frac{d}{dt}\int_\omega a_p p v_p v_p^* \, d\omega - \int_\omega \mathrm{Div}\mathcal{M} \cdot \nabla v_p^* \, d\omega - \int_{\gamma_f} \partial_\tau[(\mathcal{M}\mathbf{n}) \cdot \tau] v_p^* \, d\gamma_f \\ + \int_\omega a_p p R_p v_p v_p^* \, d\omega \\ = -T\partial_x u_s(l_s, t) v_p^*(x_0, y_0) - \int_\omega [p]\omega v_p^* \, d\omega, \quad \forall v_p^*,\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{d}{dt}\int_\omega a^{-3} C^{-1} \mathcal{M} : \mathcal{M}^* \, d\omega + \int_\omega \mathrm{Div}\mathcal{M}^* \cdot \nabla v_p \, d\omega \\ + \int_{\gamma_f} \partial_\tau[(\mathcal{M}^*\mathbf{n}) \cdot \tau] v_p \, d\gamma_f = 0, \quad \forall \mathcal{M}^*.\end{aligned} \tag{24}$$

The detail of the derivation is provided in section 4.5.

3.4 Variational Form of the Air Field Equation

$$\frac{d}{dt}\int_{R^3} \frac{1}{\rho_a c_a^2} p p^* + \int_{R^3} p^* \,\mathrm{div}\, \mathbf{v}_a = 0, \quad \forall p^*, \tag{25}$$

$$\begin{aligned}\frac{d}{dt}\int_{R^3} \rho_a \mathbf{v}_a \cdot \mathbf{v}_a^* \, dR^3 - \int_{R^3} p \,\mathrm{div}\, \mathbf{v}_a^* \, dR^3 \\ - \int_\Gamma (\mathbf{v}_a^* \cdot \mathbf{N}) \lambda \, d\Gamma = 0, \quad \forall \mathbf{v}_a^*,\end{aligned} \tag{26}$$

$$\int_\omega v_p \lambda^* \, d\omega - \int_\Gamma (\mathbf{v}_a \cdot \mathbf{N}) \lambda^* \, d\Gamma = 0, \quad \forall \lambda^*. \tag{27}$$

The definition of the test functions, please see 3.2.

Key Takeaways

- **Point 1:** By variational forms, we can get the weak form of the governing equations.
- **Point 2:** By weak form, we can get the finite element discretization of the governing equations, by plugging the test functions into the weak form.
- **Point 3:** By Finite Difference in the time-domain, we build computing equation from time-points $n-1, n-\frac{1}{2}, n$ to $n+\frac{1}{2}$.
- **Point 4:** We provide the expressions of the finite element and the detailed derivation in the section 4.
- **Point 5:** We provide code implementation at `https://github.com/xsjk/SI114H-Project` and provide some information of our code in the section 5.

3.5 Discretization

FinEle Coeff Vector	Physical Meaning	Domain Mapping	Finite Element
v_{s_h}	Normal velocity of the string	$\mathbb{R} \to \mathbb{R}$	P_0 (piecewise constant)
q_h	Tension of the string	$\mathbb{R} \to \mathbb{R}$	P_1 (piecewise linear)
v_{p_h}	Normal velocity of the plate	$\omega \to \mathbb{R}$	P_2 (Lagrange polynomial)
M_h	Bending moment of the plate	$\omega \to \mathbb{R}^3$	P_2 (Lagrange polynomial)
λ_h	Pressure difference on the contact surface	$\Gamma \to \mathbb{R}$	P_0 (Lagrange polynomial)
p_h	Sound pressure of the sound field	$\Omega \to \mathbb{R}$	P_0 (piecewise constant in cube)
$\mathbf{v}_{a_h}$	Air velocity in the sound field	$\Omega \to \mathbb{R}^3$	P_1 (piecewise linear)

Table 3: Finite Element Coefficient Vectors in the Guitar Model

3.5.1 Space Discretization

By the Variational Forms 21, 22 23, 24, 25 , 26, 27 provided in section 3.2, we can now discretize the space domain of the guitar model by simply replacing the continuous functions with a combination of the finite element test functions (or basis function in this context). To understand the following results, the reader is required with some basic knowledge of finite element methods Zienkiewicz et al. [2005]. Back to this report, plugging all unknown variables with linear combination of test(basis/trial) functions in the variational equations, will produce a system of ordinary linear equations:

$$M_s^h \frac{dv_{sh}}{dt} - D_h q_h = f_{sh}, \tag{28}$$

$$M_q^h \frac{dq_h}{dt} + D_h^T v_{sh} - J_h^T v_{ph} = 0, \tag{29}$$

$$M_p^h \frac{dv_{ph}}{dt} - H_h^T \mathcal{M}_h + R_p M_p^h v_{ph} = -J_h q_h - (B_\omega^h)^T \lambda_h, \tag{30}$$

$$M_{\mathcal{M}}^h \frac{d\mathcal{M}_h}{dt} + H_h v_{ph} = 0, \tag{31}$$

$$M_a^h \frac{dv_{ah}}{dt} - G_h p_h - (B_\Gamma^h)^T \lambda_h = 0, \tag{32}$$

$$M_p^a \frac{dp_h}{dt} + G_h^T v_{ah} = 0, \tag{33}$$

$$B_\omega^h v_{ph} - B_\Gamma^h v_{ah} = 0, \tag{34}$$

where $M_{\dots}^h, J, H, D$ matrices are coefficient matrices of the test functions, $B_{\dots}^h$ matrices are related to the boundary conditions.

3.5.2 Time Discretization

The basic idea of Time Discretization is to replace $\frac{d}{dt}$ with a centered-finite difference, using a time step Δt. First, we will eliminate the bending moment which is complex and hinders our replacement of $\frac{d}{dt}$ with finite difference. So, the Equation 30 and 31 take us to

$$M_p^h \frac{d^2 v_{ph}}{dt^2} + R_p M_p^h \frac{dv_{ph}}{dt} + K_h v_{ph} = -J_h \frac{dq_h}{dt} - (B_\omega^h)^T \frac{d\lambda_h}{dt}, \quad \forall t \geq 0, \tag{35}$$

where K_h is the matrix defined by

$$K_h = H_h^T (M_{\mathcal{M}}^h)^{-1} H_h. \tag{36}$$

Plus, by the centered finite difference, we have equations like

$$\frac{\mathrm{d}v_{s_h}}{\mathrm{d}t} \approx \frac{v_{s_h}^{n+1} - v_{s_h}^{n-1}}{2\Delta t}, \tag{37}$$

Directly replacing the time derivative with the finite difference, we can get the final system of linear equations 12 13 14 15 16 17 . The system is then solved by a time-stepping method. Finally, the computational algorithm is clear.

$$v_{s_h}^{n+\frac{1}{2}} = v_{s_h}^{n-\frac{1}{2}} + (M_h^s)^{-1}(f_{s_h}^n + D_h q_h^n)\,\Delta t$$

$$p_h^{n+\frac{1}{2}} = p_h^{n-(\frac{1}{2})} - (M_h^{p_a})^{-1} G_h \mathbf{v}_{a_h}^n\,\Delta t$$

$$\begin{pmatrix} I & 0 & \frac{1-\cos(\sqrt{K_h}\Delta t)}{\sqrt{K_h}} \frac{B_{\omega_h}^T}{\Delta t} & \frac{I-\cos(\sqrt{K_h}\Delta t)}{\sqrt{K_h}} \frac{J_h}{2\Delta t} \\ 0 & \frac{M_h^a}{\Delta t} & -B_{\Gamma_h}^T & 0 \\ 2B_{\omega_h} & -B_{\Gamma_h} & 0 & 0 \\ -{J_h}^T & 0 & 0 & \frac{M_h^q}{\Delta t} \end{pmatrix} \begin{pmatrix} v_{p_h}^{n+\frac{1}{2}} \\ \mathbf{v}_{a_h}^{n+1} \\ \lambda_h^{n+\frac{1}{2}} \\ q_h^{n+1} \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\sqrt{K_h}\Delta t) v_{p_h}^{n-\frac{1}{2}} + \frac{\sin(\sqrt{K_h}\Delta t)}{\sqrt{K_h}} \dot{v}_{p_h}^{n-\frac{1}{2}} + \frac{1-\cos(\sqrt{K_h}\Delta t)}{\sqrt{K_h}} \left(\frac{B_{\omega_h}^T}{\Delta t} \lambda_h^{n-\frac{1}{2}} + \frac{J_h}{2\Delta t} q_h^n \right) \\ \frac{M_h^a}{\Delta t} \mathbf{v}_{a_h}^n + G_h p_h^{n+\frac{1}{2}} \\ 2B_{\omega_h} v_{p_h}^{n-\frac{1}{2}} - B_{\Gamma_h} \mathbf{v}_{a_h}^{n-1} v_{p_h}^{n+\frac{1}{2}} \\ M_h^q v_{s_h}^{n+\frac{1}{2}} + D_h^T v_{s_h}^{n+\frac{1}{2}} \end{pmatrix}$$

(reappearance of 18, 19 and 20)

In section 5, we will show the results of the numerical simulation of the guitar model by the computational algorithm described above.

4 Details of Derivation of the Finite Element Method

4.1 String Base Functions

Suppose we have N_s simulate points in the string, the length of the string is l_s. For a piece of segment segment domain in the string $\omega_i = \{x | \frac{l_s \cdot i}{N_s - 1} \leq x < \frac{l_s \cdot (i+1)}{N_s - 1}\}$, where $i \in [0, N_s - 2] \cap \mathbb{Z}$, we have a position position mapping $M_i : \omega_i \to [0, 1]$.

We list the affects produced by base functions on this domain.

$$v_s^i(v) = 1, v \in [0, 1]$$

$$q^i(v) = -v + 1, v \in [0, 1]$$

$$q^{i+1}(v) = v, v \in [0, 1]$$

4.2 Plate Base Functions

Suppose we have N_p^{point} points, N_p^{tri} triangles and N_p^{edge} edges in the plate. For a piece of triangle domain in the plate tri, we have a position mapping $M_{tri}^{pos} : tri \to u : [0,1] \times v : [0, 1-u]$ and a index mapping $M_{tri}^{idx} : idx_{local} \in [0,6] \cap \mathbb{Z} \to idx_{global} \in [0, N_p^{point} + N_p^{edge} + N_p^{tri} - 1] \cap \mathbb{Z}$.

We list the affects produced by base functions in this domain.

For three vertices of this triangle domain:

$$\begin{cases} v_p^{M_{tri}^{idx}(0)} = (1 - \frac{u+v}{2}) \cdot (1-u-v) \\ v_p^{M_{tri}^{idx}(1)} = \frac{1+u}{2} \cdot u \\ v_p^{M_{tri}^{idx}(2)} = \frac{1+v}{2} \cdot v \end{cases}$$

For three edges of this triangle domain:

$$\begin{cases} v_p^{M_{tri}^{idx}(3)} = 4 \cdot u \cdot (1-u-v) \\ v_p^{M_{tri}^{idx}(4)} = 4 \cdot u \cdot v \\ v_p^{M_{tri}^{idx}(5)} = 4 \cdot v \cdot (1-u-v) \end{cases}$$

For the center of this triangle domain:

$$v_p^{M_{tri}^{idx}(6)} = 27 \cdot u \cdot v \cdot (1-u-v)$$

And for the simulate point $i \in [0,6] \cap \mathbb{Z}$,

$$\mathcal{M}^{M_{tri}^{idx}(i)} = \{\mathcal{M}_{xx}^{M_{tri}^{idx}(i)} = \begin{bmatrix} v_p^{M_{tri}^{idx}(i)} & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{M}_{xy}^{M_{tri}^{idx}(i)} = \begin{bmatrix} 0 & v_p^{M_{tri}^{idx}(i)} \\ v_p^{M_{tri}^{idx}(i)} & 0 \end{bmatrix}, \mathcal{M}_{yy}^{M_{tri}^{idx}(i)} = \begin{bmatrix} 0 & 0 \\ 0 & v_p^{M_{tri}^{idx}(i)} \end{bmatrix}\}$$

4.3 Surfaces Base Functions

Suppose we have $N_{surface}^{point}$ points and $N_{surface}^{tri}$ triangles in the surface. For a piece of triangle domain in the surface tri, we have a position mapping $M_{tri}^{pos} : tri \to u : [0,1] \times v : [0, 1-u]$ and a index mapping $M_{tri}^{idx} : idx_{local} \in [0,2] \cap \mathbb{Z} \to idx_{global} \in [0, N_{surface}^{point} - 1] \cap \mathbb{Z}$.

We list the affects produced by base function in this domain.

$$\begin{cases} \lambda_{surface}^{M_{tri}^{idx}(0)} = 1-u-v \\ \lambda_{surface}^{M_{tri}^{idx}(1)} = u \\ \lambda_{surface}^{M_{tri}^{idx}(2)} = v \end{cases}$$

4.4 Sound Field Base Functions

Suppose we have $N_{cube} = N_X \times N_Y \times N_Z$ grids in the space, thus we have $N_{faces} = 3 \times N_X \times N_Y \times N_Z + N_X \times N_Y + N_X \times N_Z + N_Y \times N_Z$ faces. For a piece of cube domain c in the space Ω, we have a position mapping $M_c^{pos} : c \to x : [0,1] \times y : [0,1] \times z : [0,1]$ and an index mapping $M_{cube}^{idx} : idx_{local} \in [0,5] \cap \mathbb{Z} \to [0, N_{faces} - 1] \cap \mathbb{Z}$.

We list the affects produced by base functions in this domain.

$$p^{cube}(x,y,z) = 1$$

$$\begin{cases} \mathbf{v}_a^{M_c^{\text{idx}}(0)}(x,y,z) = \begin{bmatrix} 1-x & 0 & 0 \end{bmatrix}^T \\ \mathbf{v}_a^{M_c^{\text{idx}}(1)}(x,y,z) = \begin{bmatrix} x & 0 & 0 \end{bmatrix}^T \\ \mathbf{v}_a^{M_c^{\text{idx}}(2)}(x,y,z) = \begin{bmatrix} 0 & 1-y & 0 \end{bmatrix}^T \\ \mathbf{v}_a^{M_c^{\text{idx}}(3)}(x,y,z) = \begin{bmatrix} 0 & y & 0 \end{bmatrix}^T \\ \mathbf{v}_a^{M_c^{\text{idx}}(4)}(x,y,z) = \begin{bmatrix} 0 & 0 & 1-z \end{bmatrix}^T \\ \mathbf{v}_a^{M_c^{\text{idx}}(5)}(x,y,z) = \begin{bmatrix} 0 & 0 & z \end{bmatrix}^T \end{cases}$$

4.5 Detail of Derivation of Equation 24

We provide the hardest derivation for you reference.

$$\mathcal{M} = a^3\mathbf{C}\varepsilon(\nabla u_p)$$

$$a^{-3}\mathbf{C}^{-1}\mathcal{M} - \varepsilon(\nabla u_p) = 0$$

$$a^{-3}\mathbf{C}^{-1}\partial_t\mathcal{M} - \epsilon(\nabla v_p) = 0$$

$$\int_\omega \left(a^{-3}\mathbf{C}^{-1}\partial_t\mathcal{M} - \epsilon(\nabla v_p)\right) : \mathcal{M}^*\, d\omega = 0 \quad \forall\, \mathcal{M}^*$$

$$\int_\omega a^{-3}\mathbf{C}^{-1}\partial_t\mathcal{M} : \mathcal{M}^*\, d\omega - \int_\omega \epsilon(\nabla v_p) : \mathcal{M}^*\, d\omega = 0 \quad \forall\, \mathcal{M}^*$$

$$\frac{d}{dt}\int_\omega a^{-3}\mathbf{C}^{-1}\mathcal{M} : \mathcal{M}^*\, d\omega + \int_\omega (\nabla\cdot\mathcal{M}^*)\cdot\nabla v_p\, d\omega - \int_\omega \nabla\cdot(\mathcal{M}^*\,\nabla v_p) d\omega = 0 \quad \forall\, \mathcal{M}^*$$

$$\frac{d}{dt}\int_\omega a^{-3}\mathbf{C}^{-1}\mathcal{M} : \mathcal{M}^*\, d\omega + \int_\omega (\nabla\cdot\mathcal{M}^*)\cdot\nabla v_p\, d\omega - \int_{\gamma_f} (\mathcal{M}^*\,\nabla v_p)\cdot\mathbf{n}\, d\gamma = 0 \quad \forall\, \mathcal{M}^*$$

$$\frac{d}{dt}\int_\omega a^{-3}\mathbf{C}^{-1}\mathcal{M} : \mathcal{M}^*\, d\omega + \int_\omega (\nabla\cdot\mathcal{M}^*)\cdot\nabla v_p\, d\omega - \int_{\gamma_f} (\mathcal{M}^*\,\mathbf{n})\cdot\nabla v_p\, d\gamma = 0 \quad \forall\, \mathcal{M}^*$$

$$\frac{d}{dt}\int_\omega a^{-3}\mathbf{C}^{-1}\mathcal{M} : \mathcal{M}^*\, d\omega + \int_\omega (\nabla\cdot\mathcal{M}^*)\cdot\nabla v_p\, d\omega - \int_{\gamma_f} (\mathcal{M}^*\,\mathbf{n})\cdot(\partial_\tau v_p \tau + \partial_{\mathbf{n}} v_p \mathbf{n})\, d\gamma = 0 \quad \forall\, \mathcal{M}^*$$

Since $\mathcal{M}\mathbf{n}\cdot\mathbf{n} = 0$ on γ_f

$$\frac{d}{dt}\int_\omega a^{-3}\mathbf{C}^{-1}\mathcal{M} : \mathcal{M}^*\, d\omega + \int_\omega (\nabla\cdot\mathcal{M}^*)\cdot\nabla v_p\, d\omega - \int_{\gamma_f} [(\mathcal{M}^*\,\mathbf{n})\cdot\tau]\,\partial_\tau v_p\, d\gamma = 0 \quad \forall\, \mathcal{M}^*$$

$$\frac{d}{dt}\int_\omega a^{-3}\mathbf{C}^{-1}\mathcal{M} : \mathcal{M}^*\, d\omega + \int_\omega (\nabla\cdot\mathcal{M}^*)\cdot\nabla v_p\, d\omega + \int_{\gamma_f} \partial_\tau\left[(\mathcal{M}^*\mathbf{n})\cdot\tau\right]\, v_p\, d\gamma = 0 \quad \forall\, \mathcal{M}^*$$

$$\frac{d}{dt}\int_\omega a^{-3}\mathbf{C}^{-1}\mathcal{M} : \mathcal{M}^*\, d\omega + \int_\omega (\nabla\cdot\mathcal{M}^*)\cdot\nabla v_p\, d\omega - \int_{\gamma_f} (\nabla\cdot\mathcal{M}^*)\cdot\mathbf{n}\, v_p\, d\gamma = 0 \quad \forall\, \mathcal{M}^*$$

5 Implementation and Experiment

After we discretize the domain and define the finite element function. We look into the formulas (28) to (34) and derive the coefficient matries from its variation form. Most of coefficient matries with can be calculate easily with an integration of the product of two functions in the small unit domain defined by the discretization. For example.

The local kernel products of first-order Raviart–Thomas basis functions in cubes

6 basis functions in a cube (1 on each face)

$$\int_{\Omega_0} \phi[i]\cdot\phi[j]\, dudvdw = \frac{h^3}{6}\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\int_{\Omega_0} \nabla \cdot \phi[i] \;\, dudvdw = h^2 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

And local kernel products of the extended Second-order Lagrange basis functions in triangles:

7 basis functions in a triangle (3 vertices, 3 edge center and 1 mass center)

$$\int_{\omega_0} \phi[i] \cdot \phi[j] \, dudv = \begin{pmatrix} \frac{13}{240} & \frac{29}{1440} & \frac{29}{1440} & \frac{1}{20} & \frac{1}{45} & \frac{1}{20} & \frac{3}{56} \\ \frac{29}{1440} & \frac{13}{240} & \frac{29}{1440} & \frac{1}{20} & \frac{1}{20} & \frac{1}{45} & \frac{3}{56} \\ \frac{29}{1440} & \frac{29}{1440} & \frac{13}{240} & \frac{1}{45} & \frac{1}{20} & \frac{1}{20} & \frac{3}{56} \\ \frac{1}{20} & \frac{1}{20} & \frac{1}{45} & \frac{4}{45} & \frac{2}{45} & \frac{2}{45} & \frac{3}{35} \\ \frac{1}{45} & \frac{1}{20} & \frac{1}{20} & \frac{2}{45} & \frac{4}{45} & \frac{2}{45} & \frac{3}{35} \\ \frac{1}{20} & \frac{1}{45} & \frac{1}{20} & \frac{2}{45} & \frac{2}{45} & \frac{4}{45} & \frac{3}{35} \\ \frac{3}{56} & \frac{3}{56} & \frac{3}{56} & \frac{3}{35} & \frac{3}{35} & \frac{3}{35} & \frac{81}{560} \end{pmatrix}$$

$$\int_{\omega_0} \partial_u \phi[i] \cdot \partial_u \, \phi[j] dudv = \begin{pmatrix} \frac{3}{8} & -\frac{1}{3} & 0 & -\frac{1}{6} & -\frac{1}{2} & \frac{1}{2} & -\frac{9}{40} \\ -\frac{1}{3} & \frac{3}{8} & 0 & -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & -\frac{9}{40} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & -\frac{1}{6} & 0 & \frac{4}{3} & 0 & 0 & \frac{9}{5} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{4}{3} & -\frac{4}{3} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{4}{3} & \frac{4}{3} & 0 \\ -\frac{9}{40} & -\frac{9}{40} & 0 & \frac{9}{5} & 0 & 0 & \frac{81}{20} \end{pmatrix}$$

$$\int_{\omega_0} \partial_u \phi[i] \cdot \partial_v \phi[j] \, dudv = \begin{pmatrix} \frac{3}{8} & 0 & -\frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{6} & -\frac{9}{40} \\ -\frac{1}{3} & 0 & \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & 0 & 0 & \frac{2}{3} & -\frac{2}{3} & \frac{2}{3} & \frac{9}{10} \\ -\frac{1}{2} & 0 & \frac{2}{3} & -\frac{2}{3} & \frac{2}{3} & -\frac{2}{3} & -\frac{9}{10} \\ \frac{1}{2} & 0 & -\frac{2}{3} & \frac{2}{3} & -\frac{2}{3} & \frac{2}{3} & \frac{9}{10} \\ -\frac{9}{40} & 0 & 0 & \frac{9}{10} & -\frac{9}{10} & \frac{9}{10} & \frac{81}{40} \end{pmatrix}$$

$$\int_{\omega_0} \partial_v \phi[i] \cdot \partial_v \, \phi[j] dudv = \begin{pmatrix} \frac{3}{8} & 0 & -\frac{1}{3} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{6} & -\frac{9}{40} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{3}{8} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{6} & -\frac{9}{40} \\ \frac{1}{2} & 0 & -\frac{1}{2} & \frac{4}{3} & -\frac{4}{3} & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{4}{3} & \frac{4}{3} & 0 & 0 \\ -\frac{1}{6} & 0 & -\frac{1}{6} & 0 & 0 & \frac{4}{3} & \frac{9}{5} \\ -\frac{9}{40} & 0 & -\frac{9}{40} & 0 & 0 & \frac{9}{5} & \frac{81}{20} \end{pmatrix}$$

$$\int_{\omega_0} \partial_u \phi[i] \cdot \phi[j] \, dudv = \begin{pmatrix} -\frac{31}{240} & -\frac{11}{120} & -\frac{11}{120} & -\frac{3}{20} & -\frac{7}{60} & -\frac{3}{20} & \frac{-3}{16} \\ \frac{11}{120} & \frac{31}{240} & \frac{11}{120} & \frac{3}{20} & \frac{3}{20} & \frac{7}{60} & \frac{3}{16} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{20} & -\frac{3}{20} & 0 & 0 & -\frac{2}{15} & \frac{2}{15} & 0 \\ \frac{7}{60} & \frac{7}{60} & \frac{4}{15} & \frac{2}{15} & \frac{4}{15} & \frac{4}{15} & \frac{3}{10} \\ -\frac{7}{60} & -\frac{7}{60} & -\frac{4}{15} & -\frac{2}{15} & -\frac{4}{15} & -\frac{4}{15} & -\frac{3}{10} \\ \frac{3}{16} & -\frac{3}{16} & 0 & 0 & -\frac{3}{10} & \frac{3}{10} & 0 \end{pmatrix}$$

$$\int_{\omega_0} \partial_v \phi[i] \cdot \phi[j] \, dudv = \begin{pmatrix} -\frac{31}{240} & -\frac{11}{120} & -\frac{11}{120} & -\frac{3}{20} & -\frac{7}{60} & -\frac{3}{20} & -\frac{3}{16} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{11}{120} & \frac{11}{120} & \frac{31}{240} & \frac{7}{60} & \frac{3}{20} & \frac{3}{20} & \frac{3}{16} \\ -\frac{7}{60} & -\frac{4}{15} & -\frac{7}{60} & -\frac{4}{15} & -\frac{4}{15} & -\frac{2}{15} & -\frac{3}{10} \\ \frac{7}{60} & \frac{4}{15} & \frac{7}{60} & \frac{4}{15} & \frac{4}{15} & \frac{2}{15} & \frac{3}{10} \\ \frac{3}{20} & 0 & -\frac{3}{20} & \frac{2}{15} & -\frac{2}{15} & 0 & 0 \\ \frac{3}{16} & 0 & -\frac{3}{16} & \frac{3}{10} & -\frac{3}{10} & 0 & 0 \end{pmatrix}$$

5.1 Constants we Apply in our Code

See 5.1 for the typical values of the physical and numerical parameters used in the simulations.

Table 4: Typical values of physical and numerical values used for the simulations.

String:							
$\rho_s = 0.00525\,\mathrm{kg} \cdot \mathrm{m}^{-1}$ $R_s = 0.75\,\mathrm{s}^{-1}$ $l_s = 0.65\,\mathrm{m}$		$T = 60\,\mathrm{N}$ $\eta_s = 9 \cdot 10^{-8}\,\mathrm{s}$ $(x_0, y_0) = (65\,\mathrm{cm}, 4\,\mathrm{cm})$					
Soundboard:							
D_1 (MPa)	D_2 (MPa)	D_3 (MPa)	D_4 (MPa)	ρ_p (kg·m^{-3})	a (mm)	R_p (s^{-1})	η_p (s)
850	50	75	200	350	2.9	7	0.005
80	50	900	270	400	6	7	0.005
100	60	1250	300	400	14	7	0.005
Air:							
$c_a = 344\,\mathrm{m} \cdot \mathrm{s}^{-1}$		$\rho_a = 1.21\,\mathrm{kg} \cdot \mathrm{m}^{-3}$					
Numerical information:							
String: 99 nodes Plate: 20400 nodes Pressure jump: 44761 nodes		Time step: $\Delta t = 2 \times 10^{-5}$ s Sampling frequency: $f_e = 50\,000\,\mathrm{Hz}$ Acoustic field: $\approx 125000, 382500$ cubes (for pressure and velocity)					
Plucking force: $f_s(x,t) = g(x)h(t)$							
$g(x) = \frac{\exp\left(-(x-x_0)^2/\delta_s^2\right)}{\int_0^{l_s} \exp(-(x-x_0)^2/\delta_s^2)dx}$		$x_0 = 55\,\mathrm{cm}$, $\delta_s = 0.006\,\mathrm{m}$, $t_1 = 0.015\,\mathrm{s}$, $t_2 = 0.004\,\mathrm{s}$					
$h(t) = \begin{cases} (1-\cos(\pi t/t_1)), & 0 \le t \le t_1, \\ (1+\cos(\pi(t-t_1)/t_2)), & t_1 \le t \le t_2, \\ 0, & t > t_2 \end{cases}$							

5.2 Reproducibility Issue

However, even strictly following the guidelines in the paper, and concretizing the formulas to the best of our ability and leveraging to the best of our High Performance Computing (HPC) asset provided by the school, we were not able to reproduce the results. We found that the results of the simulations were not reproducible. Further investigation revealed that the issue is due to the lack of information in the paper, such as techniques to retain the stability of the simulations, and the reduction of the space model to reduce computation.

References

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