Algorithm Design and Analysis
Assignment 6
Deadline: Jan 15, 2025

Choose two of the four questions.

1. (50 points) Given an undirected graph $G = (V, E)$ and an integer k, decide if G has a spanning tree with maximum degree at most k.
   (a) Prove that this problem is NP-complete.
   (b) Prove that this problem is NP-complete for $k = 3$. (Notice that, in the first part, an instance of this problem is (G, k) which consists of both the graph G and the parameter k; in this part, k becomes a fixed constant, and an instance of this problem is just a graph G.)

2. (50 points) Let's reconsider problem 2(b) in our midterm exam, but now with a *directed* and *weighted* graph. That is, the inputs include a directed weighted graph $G = (V, E, w)$ where edges have non-negative *integer* weights, two vertices $s, t \in V$, and a non-negative integer k. You are to decide if there is an s-t path with a length of exactly k. Again, in this question, a path can visit a vertex more than once.
   (a) Prove that this problem is NP-complete.
   (b) (Not for credit, just for fun) Does this problem still NP-complete for undirected (but still positively weighted) graphs?

3. (50 points) In Lecture 8, we have seen that the K-Centers problem does not admit a polynomial-time $(2 - \varepsilon)$-approximation for any $\varepsilon > 0$, unless P = NP, and we have proved this by a reduction from the *dominating set problem*. We have mentioned that the dominating set problem is NP-complete. Now it's your turn to prove it!

   Recall that, given an undirected graph $G = (V, E)$, a dominating set $S \subseteq V$ is a set of vertices such that each $u \in V$ is either in S or has a neighbor in S. The dominating set problem takes a graph $G = (V, E)$ and a positive integer k as inputs, and asks if G contains a dominating set with size k.

4. (50 points) Given a ground set $U = \{1, \ldots, n\}$ and a collection $\mathcal{T} = \{S_1, \ldots, S_m\}$ of subsets of U such that $\bigcup_{i=1}^{m} S_i = U$, the *set cover problem* asks for a minimum number of subsets from $\mathcal{T}$ whose union is U, and the *max-k-coverage problem* asks for k subsets from $\mathcal{T}$ with the maximum union size. Prove that both problems are NP-hard.

5. How long does it take you to finish the assignment (including thinking and discussion)? Give a score (1,2,3,4,5) to the difficulty. Do you have any collaborators? Please write down their names here.