筆記

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1 離散分佈

1.1 Geometric Distribution 幾何分佈

1.1.1 密度函數

$$f(x) = q^{x-1}p$$

1.1.2 累積函數

$$F(x) = \sum_{x=1}^{\infty} q^{x-1}p = p\frac{1-q^x}{1-q} = p\frac{1-q^x}{p} = 1-q^x$$

1.1.3 期望值

$$\begin{aligned}
E[X] &= \sum_{x=1}^{\infty} xq^{x-1}p = p\sum_{x=1}^{\infty} xq^{x-1} = p\sum_{x=1}^{\infty}\frac{dq^x}{dq} \\
&= p\frac{d}{dq}(q\frac{1}{1-q}) = p\frac{q'(1-q)-q(1-q)'}{(1-q)^2} \\
&= p\frac{(1-q)+q}{p^2} = p\frac{1}{p^2} = \frac{1}{p}
\end{aligned}$$

1.1.4 變異數

$$
\begin{aligned}
Var(X) &= E[(X-\mu)^2] = E[(X^2-2\mu X+\mu^2] = E[X^2]-2\mu E[X]+\mu^2 \\
&= E[X^2]-2E[X]^2+E[X]^2 = E[X^2]-E[X]^2 \\
E[X^2] &= \sum_{x=1}^{\infty} x^2 q^{x-1} p = p\sum_{x=1}^{\infty} x^2 q^{x-1}
\end{aligned}
$$

由於我們無法計算出 x^2q^{x-1} 的積分，式子的 x^2q^{x-1} 無法代換成 $\frac{df(q)}{dq}$ 的形式。

藉由 $x(x-1)q^{x-2} = \frac{dq^x}{dq}$，我們可以改算 $E[X(X-1)]$。

$$
\begin{aligned}
E[X(X-1)] &= \sum_{x=1}^{\infty} x(x-1)q^{x-1}p = pq\sum_{x=1}^{\infty} x(x-1)q^{x-2} = pq\sum_{x=1}^{\infty}\frac{dq^x}{dq^2} \\
&= pq\frac{d}{dq^2}(\frac{1}{1-q}) = pq(\frac{0(1-q)-1(-1)}{(1-q)^2})' \\
&= pq(\frac{1}{(1-q)^2})' = pq(\frac{0(1-q)^2-1\cdot 2(1-q)(-1)}{(1-q)^4}) = pq\frac{2p}{p^4} = \frac{2q}{p^2} \\
E[X^2] &= E[X(X-1)]+E[X] = \frac{2q}{p^2}+\frac{1}{p} = \frac{2q+p}{p^2} = \frac{2(1-p)+p}{p^2} = \frac{2-p}{p^2} \\
Var(X) &= E[X^2]-E[X]^2 = \frac{2-p}{p^2}-\frac{1}{p^2} = \frac{1-p}{p^2} = \frac{q}{p^2}
\end{aligned}
$$

1.1.5 動差生成函數

$$
m_x(t) = E[e^{tX}] = \sum_{x=1}^{\infty} e^{tx}q^{x-1}p = p\sum_{x=1}^{\infty} e^{tx}q^{x-1}
$$

把 $x-1$ 提出來

$$
= p\sum_{x=1}^{\infty} e^{t(x-1)}e^t q^{x-1} = pe^t\sum_{x=1}^{\infty}(qe^t)^{x-1} = pe^t\frac{1}{1-qe^t} = \frac{pe^t}{1-qe^t}
$$

1.2 Binomial Distribution 二項分佈

1.2.1 密度函數

$$f(x) = \binom{n}{x} p^x q^{n-x}$$

1.2.2 累積函數

$$F(x) = \sum_{k=0}^{x} \binom{n}{k} p^k q^{n-k}$$

1.2.3 期望值

$$\begin{aligned}
E[X] &= \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} = \sum_{x=1}^{n} x \binom{n}{x} p^x q^{n-x} = \sum_{x=1}^{n} n \frac{(n-1)!}{(x-1)!\,(n-x)!}\, p^x q^{n-x} \\
&= \sum_{x=1}^{n} n \binom{n-1}{x-1} p^x q^{n-x} = \sum_{x=1}^{n} n \binom{n-1}{x-1} p^{x-1} p q^{(n-1)-(x-1)} \\
&= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{(n-1)-(x-1)}
\end{aligned}$$

Let $y = x-1, m = n-1$

$$= np \sum_{y=0}^{n} \binom{n-1}{y} p^y q^{(n-1)-y} = np \sum_{y=0}^{n} \binom{m}{y} p^y q^{m-y} = np$$

1.2.4 變異數

$$Var(X) = E[X^2] - E[X]^2$$

Let $y = x-1, m = n-1$

$$\begin{aligned}
E[X^2] &= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x q^{n-x} = np \sum_{y=0}^{m} x \binom{m}{y} p^y q^{m-y} = np \sum_{y=0}^{m} (y+1) \binom{m}{y} p^y q^{m-y} \\
&= np \left(\sum_{y=0}^{m} y \binom{m}{y} p^y q^{m-y} + \sum_{y=0}^{m} \binom{m}{y} p^y q^{m-y} \right) \\
&= np(mp+1) = np((n-1)p+1) = np(np-p+1) \\
Var(X) &= E[X^2] - E[X]^2 = np(np-p+1) - np \cdot np = np(1-p) = npq
\end{aligned}$$

1.2.5 動差生成函數

$$m_t(x) = E[e^{tx}] = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x} = (pe^t + q)^n$$

1.3 Negative Binomial Distribution 負二項分佈

1.3.1 密度函數

$$f(x) = \binom{x-1}{r-1} p^r q^{x-r}$$

1.3.2 累積函數

$$F(x) = \sum_{k=r}^{x} \binom{k-1}{r-1} p^r q^{k-r} = \sum_{l=0}^{x} \binom{l+r-1}{r-1} p^r q^l$$

1.3.3 期望值

$$\begin{aligned}
E[X] &= \sum_{x=0}^{\infty} x \binom{x+r-1}{r-1} p^r q^x = \sum_{x=1}^{\infty} x \binom{x+r-1}{r-1} p^r q^x = \sum_{x=1}^{\infty} \frac{(x+r-1)!}{(r-1)!\,(x-1)!}\, p^r q^x \\
&= \sum_{y=0}^{\infty} \frac{(y+r)!}{(r-1)!\,y!}\, p^r q^{y+1} = \sum_{y=0}^{\infty} \frac{(y+s-1)!}{(s-2)!\,y!}\, p^{s-1} q^{y+1} = \frac{(s-1)q}{p} \sum_{y=0}^{\infty} \frac{(y+s-1)!}{(s-1)!\,y!}\, p^s q^y \\
&= \frac{rq}{p} \sum_{y=0}^{\infty} \binom{y+s-1}{s-1} p^s q^y = \frac{rq}{p}
\end{aligned}$$

1.3.4 變異數

$$
\begin{aligned}
Var(X) &= E[X^2] - E[X]^2 \\
E[X^2] &= \sum_{x=0}^{\infty} x^2 \binom{x+r-1}{r-1} p^r q^x = \sum_{x=1}^{\infty} x^2 \binom{x+r-1}{r-1} p^r q^x = \sum_{x=1}^{\infty} x \frac{(x+r-1!)}{(r-1)!\,(x-1)!} p^r q^x \\
&= \sum_{y=0}^{\infty} (y+1) \frac{(y+r!)}{(r-1)!\,y!}\, p^r q^{y+1} = \sum_{y=0}^{\infty} (y+1) \frac{(y+s-1!)}{(s-2)!\,y!}\, p^{s-1} q^{y+1} \\
&= \frac{(s-1)q}{p} \sum_{y=0}^{\infty} (y+1) \frac{(y+s-1!)}{(s-1)!\,y!}\, p^s q^y = \frac{rq}{p} \sum_{y=0}^{\infty} (y+1) \frac{(y+s-1!)}{(s-1)!\,y!}\, p^s q^y \\
&= \frac{rq}{p} \sum_{y=0}^{\infty} (y+1) \binom{y+s-1}{s-1} p^s q^y \\
&= \frac{rq}{p} \left(\sum_{y=0}^{\infty} y \binom{y+s-1}{s-1} p^s q^y + \sum_{y=0}^{\infty} \binom{y+s-1}{s-1} p^s q^y \right) \\
&= \frac{rq}{p} (\frac{sq}{p} + 1) = \frac{rq}{p} (\frac{(r+1)q}{p} + 1) = \frac{rq}{p} (\frac{rq}{p} + \frac{q}{p} + 1) \\
Var(X) &= E[X^2] - E[X]^2 = \frac{rq}{p} (\frac{rq}{p} + \frac{q}{p} + 1) - \frac{rq}{p} \frac{rq}{p} = \frac{rq}{p} (\frac{q}{p} + 1) = \frac{rq}{p} (\frac{q+p}{p}) = \frac{rq}{p^2}
\end{aligned}
$$

1.3.5 動差生成函數

泰勒展開式

$$
\begin{aligned}
f(x) &= \frac{f^{(0)}(a)}{0!}(x-a)^0 + \frac{f^{(1)}(a)}{1!}(x-a)^1 + ... + \frac{f^{(k)}(a)}{k!}(x-a)^k + ... \Big|_{a=0} \\
&= \frac{f^{(0)}(0)}{0!}x^0 + \frac{f^{(1)}(0)}{1!}x^1 + ... + \frac{f^{(k)}(0)}{k!}x^k + ...
\end{aligned}
$$

Let $f(x) = (x+1)^r$

$$
\begin{aligned}
(x+1)^r &= \frac{(a+1)^r}{0!}x^0 + \frac{r(a+1)^{r-1}}{1!}x^1 + \frac{r(r-1)(a+1)^{r-2}}{2!}x^2 + ... \\
&+ \frac{r!\,(a+1)^{r-k}}{k!\,(r-k)!}x^k + ... \Big|_{a=0} \\
&= \frac{1}{0!}x^0 + \frac{r}{1!}x^1 + \frac{r(r-1)}{2!}x^2 + ... + \frac{r!}{k!\,(r-k)!}x^k + ... \\
&= \binom{r}{0}x^0 + \binom{r}{1}x^1 + \binom{r}{2}x^2 + ... + \binom{r}{k}x^k + ... = \sum_{k=0}^{\infty}\binom{r}{k}x^k \\
(x+1)^{-r} &= \sum_{k=0}^{\infty}\binom{-r}{k}x^k = \sum_{k=0}^{\infty}\frac{(-r)(-r-1)...(-r-k+1)}{k!}x^k \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k\,r(r+1)...(r+k-1)}{k!}x^k = \sum_{k=0}^{\infty}\frac{r(r+1)...(r+k-1)}{k!}(-x)^k \\
&= \sum_{k=0}^{\infty}\frac{(r+k-1)!}{k!\,(r-1)!}(-x)^k = \sum_{k=0}^{\infty}\binom{k+r-1}{r-1}(-x)^k \\
(-x+1)^{-r} &= (1-x)^{-r} = \sum_{k=0}^{\infty}\binom{k+r-1}{r-1}x^k \qquad (1) \\
m_t(x) &= E[e^{tx}] \\
&= \sum_{x=0}^{\infty}e^{tx}\binom{x+r-1}{r-1}p^r q^x = \sum_{x=0}^{\infty}\binom{x+r-1}{r-1}p^r(qe^t)^x = p^r\sum_{x=0}^{\infty}\binom{x+r-1}{r-1}(qe^t)^x
\end{aligned}
$$

代入 (1) 式

$$
= p^r(1-qe^t)^{-r} = (\frac{p}{1-qe^t})^r
$$