筆記

Lien Chiang

1 離散分佈

1.1 Geometric Distribution 幾何分佈

1.1.1 密度函數

$$f(x) = q^{x-1}p$$

1.1.2 累積函數

$$F(x) = p\frac{1-q^x}{1-q} = p\frac{1-q^x}{p} = 1 - q^x$$

1.1.3 期望值

$$\begin{aligned} E[X] &= \sum_{x=1}^{\infty} xq^{x-1}p = p\sum_{x=1}^{\infty} xq^{x-1} = p\sum_{x=1}^{\infty}\frac{dq^x}{dq} \\ &= p\frac{d}{dq}(q\frac{1}{1-q}) = p\frac{q'(1-q)-q(1-q)'}{(1-q)^2} \\ &= p\frac{(1-q)+q}{p^2} = p\frac{1}{p^2} = \frac{1}{p} \end{aligned}$$

1.1.4 變異數

先算 $Var(X)$

$$\begin{aligned} Var(X) &= E[(X-\mu)^2] = E[(X^2-2\mu X+\mu^2] = E[X^2]-2\mu E[X]+\mu^2 \\ &= E[X^2]-2E[X]^2+E[X]^2 = E[X^2]-E[X]^2 \end{aligned}$$

再算其中的 $E[X^2]$

$$E[X^2] = \sum_{x=1}^{\infty} x^2 q^{x-1} p = p\sum_{x=1}^{\infty} x^2 q^{x-1}$$

不過由於我們無法計算出 x^2q^{x-1} 的積分，式子的 x^2q^{x-1} 無法代換成 $\frac{df(q)}{dq}$ 的形式。相反的 $x(x-1)q^{x-2} = \frac{dq^x}{dq}$ 卻成立，我們可以利用該式對 $E[X]$ 做些手腳：算 $E[X(X-1)]$ 作為替代。

$$\begin{aligned} E[X(X-1)] &= \sum_{x=1}^{\infty} x(x-1)q^{x-1}p = pq\sum_{x=1}^{\infty} x(x-1)q^{x-2} = pq\sum_{x=1}^{\infty}\frac{dq^x}{dq^2} = pq\frac{d}{dq^2}(\frac{1}{1-q}) \\ &= pq(\frac{0(1-q)-1(-1)}{(1-q)^2})' = pq(\frac{1}{(1-q)^2})' = pq(\frac{0(1-q)^2-1\cdot 2(1-q)(-1)}{(1-q)^4}) \\ &= pq\frac{2p}{p^4} = \frac{2q}{p^2} \end{aligned}$$

$$\begin{aligned} E[X^2] &= E[X(X-1)] + E[X] \\ &= \frac{2q}{p^2} + \frac{1}{p} = \frac{2q+p}{p^2} = \frac{2(1-p)+p}{p^2} = \frac{2-p}{p^2} \end{aligned}$$

$$\begin{aligned} Var(X) &= E[X^2] - E[X]^2 \\ &= \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} = \frac{q}{p^2} \end{aligned}$$

1.1.5 動差生成函數

技巧：把 $x-1$ 提出來

$$\begin{aligned} m_x(t) &= E[e^{tX}] = \sum_{x=1}^{\infty} e^{tx}q^{x-1}p = p\sum_{x=1}^{\infty} e^{tx}q^{x-1} = p\sum_{x=1}^{\infty} e^{t(x-1)}e^t q^{x-1} = pe^t\sum_{x=1}^{\infty}(qe^t)^{x-1} \\ &= pe^t\frac{1}{1-qe^t} = \frac{pe^t}{1-qe^t} \end{aligned}$$