筆記

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1　離散分佈

1.1　Geometric Distribution 幾何分佈

1.1.1　密度函數

$$f(x) = q^{x-1}p$$

1.1.2　累積函數

$$F(x) = \sum_{x=1}^{\infty} q^{x-1}p = p\frac{1-q^x}{1-q} = p\frac{1-q^x}{p} = 1-q^x$$

1.1.3　期望值

$$E[X] = \sum_{x=1}^{\infty} xq^{x-1}p = p\sum_{x=1}^{\infty} xq^{x-1} = p\sum_{x=1}^{\infty} \frac{dq^x}{dq}$$
$$= p\frac{d}{dq}(q\frac{1}{1-q}) = p\frac{q'(1-q)-q(1-q)'}{(1-q)^2}$$
$$= p\frac{(1-q)+q}{p^2} = p\frac{1}{p^2} = \frac{1}{p}$$

1.1.4 變異數

$$Var(X) = E[(X - \mu)^2] = E[(X^2 - 2\mu X + \mu^2] = E[X^2] - 2\mu E[X] + \mu^2$$
$$= E[X^2] - 2E[X]^2 + E[X]^2 = E[X^2] - E[X]^2$$
$$E[X^2] = \sum_{x=1}^{\infty} x^2 q^{x-1} p = p \sum_{x=1}^{\infty} x^2 q^{x-1}$$

由於我們無法計算出 $x^2 q^{x-1}$ 的積分，式子的 $x^2 q^{x-1}$ 無法代換成 $\frac{df(q)}{dq}$ 的形式。

藉由 $x(x-1)q^{x-2} = \frac{dq^x}{dq}$，我們可以改算 $E[X(X-1)]$。

$$E[X(X-1)] = \sum_{x=1}^{\infty} x(x-1)q^{x-1} p = pq \sum_{x=1}^{\infty} x(x-1)q^{x-2} = pq \sum_{x=1}^{\infty} \frac{dq^x}{dq^2}$$
$$= pq \frac{d}{dq^2} (\frac{1}{1-q}) = pq (\frac{0(1-q) - 1(-1)}{(1-q)^2})'$$
$$= pq (\frac{1}{(1-q)^2})' = pq (\frac{0(1-q)^2 - 1 \cdot 2(1-q)(-1)}{(1-q)^4}) = pq \frac{2p}{p^4} = \frac{2q}{p^2}$$
$$E[X^2] = E[X(X-1)] + E[X] = \frac{2q}{p^2} + \frac{1}{p} = \frac{2q+p}{p^2} = \frac{2(1-p)+p}{p^2} = \frac{2-p}{p^2}$$
$$Var(X) = E[X^2] - E[X]^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} = \frac{q}{p^2}$$

1.1.5 動差生成函數

$$m_x(t) = E[e^{tx}] = \sum_{x=1}^{\infty} e^{tx} q^{x-1} p = p \sum_{x=1}^{\infty} e^{tx} q^{x-1}$$

把 $x-1$ 提出來

$$= p \sum_{x=1}^{\infty} e^{t(x-1)} e^t q^{x-1} = pe^t \sum_{x=1}^{\infty} (qe^t)^{x-1} = pe^t \frac{1}{1-qe^t} = \frac{pe^t}{1-qe^t}$$

1.2 Binomial Distribution 二項分佈

1.2.1 密度函數

$$f(x) = \binom{n}{x} p^x q^{n-x}$$

1.2.2 累積函數

$$F(x) = \sum_{k=0}^{x} \binom{n}{k} p^k q^{n-k}$$

1.2.3 期望值

$$E[X] = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} = \sum_{x=1}^{n} x \binom{n}{x} p^x q^{n-x} = \sum_{x=1}^{n} n \frac{(n-1)!}{(x-1)!\,(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} n \binom{n-1}{x-1} p^x q^{n-x} = \sum_{x=1}^{n} n \binom{n-1}{x-1} p^{x-1} p q^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{(n-1)-(x-1)}$$

Let $y = x - 1, m = n - 1$

$$= np \sum_{y=0}^{n} \binom{n-1}{y} p^y q^{(n-1)-y} = np \sum_{y=0}^{n} \binom{m}{y} p^y q^{m-y} = np$$

1.2.4 變異數

$$Var(X) = E[X^2] - E[X]^2$$

Let $y = x - 1, m = n - 1$

$$E[X^2] = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x q^{n-x} = np \sum_{y=0}^{m} x \binom{m}{y} p^y q^{m-y} = np \sum_{y=0}^{m} (y+1) \binom{m}{y} p^y q^{m-y}$$

$$= np \left(\sum_{y=0}^{m} y \binom{m}{y} p^y q^{m-y} + \sum_{y=0}^{m} \binom{m}{y} p^y q^{m-y} \right)$$

$$= np(mp + 1) = np((n-1)p + 1) = np(np - p + 1)$$

$$Var(X) = E[X^2] - E[X]^2 = np(np - p + 1) - np \cdot np = np(1 - p) = npq$$

1.2.5 動差生成函數

$$m_t(x) = E[e^{tx}] = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x} = (pe^t + q)^n$$

1.3 Negative Binomial Distribution 負二項分佈

1.3.1 密度函數

$$f(x) = \binom{x-1}{r-1} p^r q^{x-r}$$

1.3.2 累積函數

$$F(x) = \sum_{k=r}^{x} \binom{k-1}{r-1} p^r q^{k-r} = \sum_{l=0}^{x} \binom{l+r-1}{r-1} p^r q^l$$

1.3.3 期望值

$$E[X] = \sum_{x=0}^{\infty} x \binom{x+r-1}{r-1} p^r q^x = \sum_{x=1}^{\infty} x \binom{x+r-1}{r-1} p^r q^x = \sum_{x=1}^{\infty} \frac{(x+r-1)!}{(r-1)!(x-1)!} p^r q^x$$

$$= \sum_{y=0}^{\infty} \frac{(y+r)!}{(r-1)! \, y!} p^r q^{y+1} = \sum_{y=0}^{\infty} \frac{(y+s-1)!}{(s-2)! \, y!} p^{s-1} q^{y+1} = \frac{(s-1)q}{p} \sum_{y=0}^{\infty} \frac{(y+s-1)!}{(s-1)! \, y!} p^s q^y$$

$$= \frac{rq}{p} \sum_{y=0}^{\infty} \binom{y+s-1}{s-1} p^s q^y = \frac{rq}{p}$$

1.3.4 變異數

$$Var(X) = E[X^2] - E[X]^2$$

$$E[X^2] = \sum_{x=0}^{\infty} x^2 \binom{x+r-1}{r-1} p^r q^x = \sum_{x=1}^{\infty} x^2 \binom{x+r-1}{r-1} p^r q^x = \sum_{x=1}^{\infty} x \frac{(x+r-1!)}{(r-1)!\,(x-1)!} p^r q^x$$

$$= \sum_{y=0}^{\infty} (y+1) \frac{(y+r!)}{(r-1)!\,y!} p^r q^{y+1} = \sum_{y=0}^{\infty} (y+1) \frac{(y+s-1!)}{(s-2)!\,y!} p^{s-1} q^{y+1}$$

$$= \frac{(s-1)q}{p} \sum_{y=0}^{\infty} (y+1) \frac{(y+s-1!)}{(s-1)!\,y!} p^s q^y = \frac{rq}{p} \sum_{y=0}^{\infty} (y+1) \frac{(y+s-1!)}{(s-1)!\,y!} p^s q^y$$

$$= \frac{rq}{p} \sum_{y=0}^{\infty} (y+1) \binom{y+s-1}{s-1} p^s q^y$$

$$= \frac{rq}{p} \left(\sum_{y=0}^{\infty} y \binom{y+s-1}{s-1} p^s q^y + \sum_{y=0}^{\infty} \binom{y+s-1}{s-1} p^s q^y \right)$$

$$= \frac{rq}{p} (\frac{sq}{p} + 1) = \frac{rq}{p} (\frac{(r+1)q}{p} + 1) = \frac{rq}{p} (\frac{rq}{p} + \frac{q}{p} + 1)$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{rq}{p} (\frac{rq}{p} + \frac{q}{p} + 1) - \frac{rq}{p} \frac{rq}{p} = \frac{rq}{p} (\frac{q}{p} + 1) = \frac{rq}{p} (\frac{q+p}{p}) = \frac{rq}{p^2}$$

1.3.5 動差生成函數

泰勒展開式

$$f(x) = \frac{f^{(0)}(a)}{0!}(x-a)^0 + \frac{f^{(1)}(a)}{1!}(x-a)^1 + ... + \frac{f^{(k)}(a)}{k!}(x-a)^k + ...\Big|_{a=0}$$

$$= \frac{f^{(0)}(0)}{0!}x^0 + \frac{f^{(1)}(0)}{1!}x^1 + ... + \frac{f^{(k)}(0)}{k!}x^k + ...$$

Let $f(x) = (x+1)^r$

$$(x+1)^r = \frac{(a+1)^r}{0!}x^0 + \frac{r(a+1)^{r-1}}{1!}x^1 + \frac{r(r-1)(a+1)^{r-2}}{2!}x^2 + ...$$

$$+ \frac{r!\,(a+1)^{r-k}}{k!\,(r-k)!}x^k + ...\Big|_{a=0}$$

$$= \frac{1}{0!}x^0 + \frac{r}{1!}x^1 + \frac{r(r-1)}{2!}x^2 + ... + \frac{r!}{k!\,(r-k)!}x^k + ...$$

$$= \binom{r}{0}x^0 + \binom{r}{1}x^1 + \binom{r}{2}x^2 + ... + \binom{r}{k}x^k + ... = \sum_{k=0}^{\infty}\binom{r}{k}x^k$$

$$(x+1)^{-r} = \sum_{k=0}^{\infty}\binom{-r}{k}x^k = \sum_{k=0}^{\infty}\frac{(-r)(-r-1)...(-r-k+1)}{k!}x^k$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k\,r(r+1)...(r+k-1)}{k!}x^k = \sum_{k=0}^{\infty}\frac{r(r+1)...(r+k-1)}{k!}(-x)^k$$

$$= \sum_{k=0}^{\infty}\frac{(r+k-1)!}{k!\,(r-1)!}(-x)^k = \sum_{k=0}^{\infty}\binom{k+r-1}{r-1}(-x)^k$$

$$(-x+1)^{-r} = (1-x)^{-r} = \sum_{k=0}^{\infty}\binom{k+r-1}{r-1}x^k \tag{1}$$

$$m_t(x) = E[e^{tx}]$$

$$= \sum_{x=0}^{\infty}e^{tx}\binom{x+r-1}{r-1}p^r q^x = \sum_{x=0}^{\infty}\binom{x+r-1}{r-1}p^r(qe^t)^x = p^r\sum_{x=0}^{\infty}\binom{x+r-1}{r-1}(qe^t)^x$$

代入 (1) 式

$$= p^r(1-qe^t)^{-r} = \left(\frac{p}{1-qe^t}\right)^r$$

1.4 Hypergeometric Distribution 超幾何分佈

1.4.1 密度函數

$$f(x) = \frac{\binom{r}{x}\binom{N-r}{N-x}}{\binom{N}{n}}$$

1.4.2 累積函數

$$F(x) = \sum_{k=0}^{x} \frac{\binom{r}{k}\binom{N-r}{n-k}}{\binom{N}{n}}$$

1.4.3 期望值

$$E[X] = \sum_{x=0}^{r} x\frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}} = \sum_{x=1}^{r} x\frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}} = \sum_{x=1}^{r} r\frac{\binom{r-1}{x-1}\binom{N-r}{n-x}}{\binom{N}{n}} = \sum_{y=0}^{r} (s+1)\frac{\binom{s}{y}\binom{N-s-1}{n-y-1}}{\binom{N}{n}}$$

$$= \sum_{y=0}^{r} \frac{(s+1)n}{N}\frac{\binom{s}{y}\binom{N-s-1}{n-y-1}}{\binom{N-1}{n-1}} = \frac{rn}{N}\sum_{y=0}^{r} \frac{\binom{s}{y}\binom{N-s-1}{n-y-1}}{\binom{N-1}{n-1}} = \frac{rn}{N}\sum_{y=0}^{r} \frac{\binom{s}{y}\binom{M-s}{m-y}}{\binom{M}{m}} = \frac{rn}{N} = \frac{nr}{N}$$

1.4.4 變異數

$$Var(X) = E[X^2] - E[X]^2$$

$$E[X^2] = \sum_{x=0}^{r} x^2\frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}} = \frac{nr}{N}\sum_{y=0}^{s} x\frac{\binom{s}{y}\binom{N-s}{n-y}}{\binom{N}{n}} = \frac{nr}{N}\sum_{y=0}^{s} (s+1)\frac{\binom{s}{y}\binom{N-s}{n-y}}{\binom{N}{n}}$$

$$= \frac{nr}{N}\left(\sum_{y=0}^{s} s\frac{\binom{s}{y}\binom{N-s}{n-y}}{\binom{N}{n}} + \sum_{y=0}^{s} \frac{\binom{s}{y}\binom{N-s}{n-y}}{\binom{N}{n}}\right) = \frac{nr}{N}(\frac{ms}{M}+1)$$

$$= \frac{nr}{N}(\frac{(n-1)(r-1)}{N-1}+1) = \frac{nr}{N}(\frac{nr-n-r+1}{N-1}+1)$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{nr}{N}(\frac{nr-n-r+1}{N-1}+1) - \frac{nr}{N}\frac{nr}{N} = \frac{nr}{N}(\frac{nr-n-r+N}{N-1}) - \frac{nr}{N}\frac{nr}{N}$$

$$= \frac{nr}{N}(\frac{nr-n-r+N}{N-1} - \frac{nr}{N}) = \frac{nr}{N}(\frac{nrN-nN-rN+N^2-nr(N-1)}{N(N-1)})$$

$$= \frac{nr}{N}(\frac{nrN-nN-rN+N^2-nrN-nr}{N(N-1)}) = \frac{nr}{N}(\frac{nN-rN+N^2-nr}{N(N-1)})$$

$$= \frac{nr}{N}(\frac{(N-n)(N-r)}{N(N-1)}) = \frac{nr(N-n)(N-r)}{N^2(N-1)}$$

1.4.5 動差生成函數

$$m_t(x) = E[e^{tx}] = \sum_{x=0}^{r} e^{tx}\frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

1.5 Poisson Distribution 步瓦松分佈

1.5.1 密度函數

$$f(x) = \lim_{n \to \infty} \binom{n}{x} p^x q^{n-x} = \lim_{n \to \infty} \frac{n!}{x!\,(n-x)!} (\frac{\lambda}{n})^x (1 - \frac{\lambda}{n})^{n-x}$$

$$= \lim_{n \to \infty} \frac{n!}{(n-x)!\,n^x} \frac{\lambda^x}{x!} (1 - \frac{\lambda}{n})^n (1 - \frac{\lambda}{n})^{-x} = \lim_{n \to \infty} \frac{n!}{(n-x)!\,n^x} \frac{\lambda^x}{x!} (1 - \frac{\lambda}{n})^n$$

$$\lim_{n \to \infty} (1 + \frac{x}{n})^n = \lim_{n \to \infty} exp(\ln((1 + \frac{x}{n})^n)) = exp(\lim_{n \to \infty} \ln((1 + \frac{x}{n})^n)) = exp(\lim_{n \to \infty} n \ln(1 + \frac{x}{n}))$$

$$= exp(\lim_{n \to \infty} \frac{x \ln(1 + \frac{x}{n})}{\frac{x}{n}}) = exp(\lim_{\frac{x}{n} \to 0} \frac{x \ln(1 + \frac{x}{n})}{\frac{x}{n}})$$

$$= exp(\lim_{\frac{x}{n} \to 0} \frac{d(x \ln(1 + \frac{x}{n}))/d(\frac{x}{n})}{d(\frac{x}{n})/d(\frac{x}{n})}) = exp(\lim_{\frac{x}{n} \to 0} \frac{x \frac{1}{1 + \frac{x}{n}}}{1}) = exp(\frac{x \frac{1}{1+0}}{1}) = e^x$$

$$f(x) = e^{-\lambda} \lim_{n \to \infty} \frac{n!}{(n-x)!\,n^x} \frac{\lambda^x}{x!} = \frac{\lambda^x e^{-\lambda}}{x!} \lim_{n \to \infty} \frac{n!}{(n-x)!\,n^x}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \lim_{n \to \infty} \frac{n(n-1)...(n-(x-1))}{n^x} = \frac{\lambda^x e^{-\lambda}}{x!} \lim_{n \to \infty} 1(1 - \frac{1}{n})...(1 - \frac{x-1}{n})$$

$$= \frac{\lambda^x e^{-\lambda}}{x!}$$

1.5.2 累積函數

$$F(x) = \sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{x} \frac{\lambda^k}{k!}$$

1.5.3 期望值

$$E[X] = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=1}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1} e^{-\lambda}}{(x-1)!} = \lambda \sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} = \lambda$$

1.5.4 變異數

$$Var(X) = E[X^2] - E[X]^2$$

$$E[X^2] = \sum_{x=0}^{\infty} x^2 \frac{\lambda^x e^{-\lambda}}{x!} = \lambda \sum_{y=0}^{\infty} x \frac{\lambda^y e^{-\lambda}}{y!} = \lambda \sum_{y=0}^{\infty} (y+1) \frac{\lambda^y e^{-\lambda}}{y!} = \lambda \left(\sum_{y=0}^{\infty} y \frac{\lambda^y e^{-\lambda}}{y!} + \sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} \right)$$

$$= \lambda(\lambda + 1)$$

$$Var(X) = E[X^2] - E[X]^2 = \lambda(\lambda + 1) - \lambda\lambda = \lambda$$

1.5.5 動差生成函數

$$m_t(x) = E[e^{tx}] = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$