Learning Guide Module

Subject Code	Math 3	Mathematics 3
Module Code	4.0	*Graphs of Polynomial and Rational Functions*
Lesson Code	4.2	*Rational Root Theorem*
Time Frame		30 minutes

After completing this learning guide, you should be able to:

1. factor completely a polynomial function by expressing it as a product of linear and quadratic factors;
2. list all possible rational zeroes of a polynomial function using the rational root theorem; and
3. find the rational and complex zeroes of a rational function.

If our thoughts cannot be expressed as real and rational in a way that we can easily visualize and understand them, what kind of thoughts should that be?

In this section, we will determine the rational and complex zeroes of the polynomial function.

10 MINUTES

TIP (The Important Point)

Theorem on Conjugate Zeros of a Polynomial

If $z = a + bi$ is a complex zero of a polynomial function of degree more than 1 then the conjugate $\bar{z} = a - bi$ is also a zero of $f(x)$ with $b \neq 0$.

Example 1.

What polynomial $f(x)$ with $n = 4$ has real coefficients and zeroes $3 - i$ and $2i$?

Solution.

By the Theorem on Conjugate Zeros of a polynomial,
$3 + i$ and $-2i$ are also zeroes of $f(x)$.
By factor theorem, $f(x)$ has the following factors:
$x - (3 - i)$, $x - (3 + i)$, $x - 2i$, $x - (-2i)$

Multiplying these factors gives us

$$f(x) = [x - (3 - i)][x - (3 + i)][x - 2i][x - (-2i)]$$
$$= (x^2 - 6x + 10)(x^2 + 4)$$
$$f(x) = x^4 - 6x^3 + 14x^2 - 24x + 40$$

Note that $[x - (a + bi)][x - (a - bi)] = x^2 - 2ax + a^2 + b^2$.

Example 2.

Express $f(x) = x^5 + 3x^4 + 2x^3 - x^2 - 3x - 2$ as a product of

a) linear and quadratic polynomials with real coefficients that are irreducible over $\mathbb{R}$

b) linear polynomials

Solution:

a) $$\begin{aligned} f(x) &= x^5 + 3x^4 + 2x^3 - x^2 - 3x - 2 \\ &= (2x^3 + 2x^2 - x - 2)(x^2 + x + 1) \\ &= (x+2)(x^2-1)(x^2+x+1) \\ &= (x+2)(x+1)(x-1)(x^2+x+1) \end{aligned}$$

The resulting expression is the desired factorization that is irreducible over $\mathbb{R}$ because using the quadratic formula, we see that the polynomial $x^2 + x + 1$ has the complex zeros

$$\frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{3}i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

b) From part (a), $f(x)$ has factors

$$x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \quad \text{and} \quad x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

Thus, the linear factors of $f(x)$ are

$$(x+2)(x+1)(x-1)\left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

TIP (The Important Point)

Every polynomial function can be expressed as a product of linear and quadratic polynomials such that the quadratic factors are irreducible over $\mathbb{R}$.

TIP (The Important Point)

Rational Root Theorem

- If the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has *integer* coefficients and if p/q is a rational zero of $f(x)$ such that p and q have no common prime factor, then
  (1) p is a factor of the constant term a_0
  (2) q is a factor of the leading coefficient a_n

Example 3. *Finding the rational zeros of an equation*

Find all the rational zeroes of $f(x) = 6x^4 + 11x^3 + 5x + 2$.

Solution.
$p = \pm 1, \pm 2$
$q = \pm 1, \pm 2, \pm 3, \pm 6$
The possible rational zeros of $f(x)$ are: $\frac{p}{q} = \pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{2}{3}$

We can either use synthetic division or factor theorem to see which among the possible rational zeros is the solution to $f(x) = 0$.

By synthetic division,

$$
\begin{array}{r|rrrrr}
-2 & 6 & 11 & 0 & 5 & 2 \\
+ & & -12 & 2 & -4 & -2 \\
\hline
 & \mathbf{6} & \mathbf{-1} & \mathbf{2} & \mathbf{1} & \mathbf{0}
\end{array}
$$

By factor theorem,
$f\left(-\frac{1}{3}\right) = 6\left(-\frac{1}{3}\right)^4 + 11\left(-\frac{1}{3}\right)^3 + 5\left(-\frac{1}{3}\right) + 2 = 0$

This implies that -2 and $-\frac{1}{3}$ are zeros.

$f(x) = (x+2)(3x+1)(2x^2 - x + 1)$
The remaining zeros are the solutions of the equation $2x^2 - x + 1 = 0$

By the quadratic formula,

$$\frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(1)}}{2(2)} = \frac{1 \pm \sqrt{7}i}{4} = \frac{1}{4} \pm \frac{\sqrt{7}}{4}i$$

Thus, $f(x)$ has two rational roots, -2 and $-\frac{1}{3}$, and two imaginary roots $\frac{1}{4} + \frac{\sqrt{7}}{4}i$ and $\frac{1}{4} - \frac{\sqrt{7}}{4}i$.

Example 4. *Showing that a polynomial function has no rational roots*

Show that $f(x) = 2x^4 - 3x^3 + x^2 - x + 3$ has no rational roots.

Solution.

The factors p of the last term a_0 are ± 1 and ± 3.
The factors q of the leading coefficient a_n are ± 1 and ± 2.

The possible rational roots $\frac{p}{q}$ of $f(x)$ are $\pm\frac{1}{2}, \pm 1$, $\pm\frac{3}{2}$, and ± 3.
Substituting each of these numbers for x, we obtain

$f\left(\frac{1}{2}\right) = \frac{5}{2}$	$f\left(-\frac{1}{2}\right) = \frac{17}{4}$	$f\left(\frac{3}{2}\right) = \frac{15}{4}$	$f\left(-\frac{3}{2}\right) = 27$
$f(1) = 2$	$f(-1) = 2$	$f(3) = 90$	$f(-3) = 10$

Since $f\left(\pm\frac{1}{2}\right) \neq 0$, $f(\pm 1) \neq 0$, $f\left(\pm\frac{3}{2}\right) \neq 0$, and $f(\pm 3) \neq 0$, it follows that $f(x)$ has no rational roots.

Example 5. *Finding a polynomial with specific graph*

How many rational roots does $f(x) = x^4 - x^2 - 4x - 4$ have? How many non-real zeros does it have? The graph of $f(x)$ is shown below.

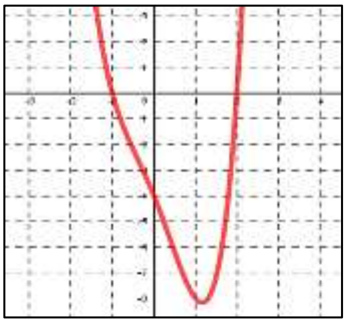

Figure 1: [x:y=1:1]

Solution.
The graph has x-intercepts -1 and 2. This implies that $f(x)$ has 2 real zeroes which are rational.

The degree of $f(x)$ is 4 which also indicates that it has 4 zeros according to the theorem on the number of zeros of a polynomial function.

Thus, there are 2 non-real zeros for $f(x)$.

Real Life Application

A storage is to be constructed in the shape of cube with a triangular prism with a total height of $6ft$ and a volume of $V = x^3 + \frac{1}{2}x^2(6 - x)$. What is the value of x so that the volume is $80ft^3$?

Solution.

$V = x^3 + \frac{1}{2}x^2(6 - x)$
$x^3 + \frac{1}{2}x^2(6 - x) = 80$
$x^3 + 6x^2 - 160 = 0$

The possible rational roots are:
$\pm1, \pm2, \pm4, \pm5, \pm8, \pm10, \pm16, \pm20, \pm32, \pm40, \pm80, \pm160$

By synthetic division,

$$\begin{array}{r|rrrr} \underline{4} & 1 & 6 & 0 & -160 \\ & + & 4 & 40 & 160 \\ \hline & \mathbf{1} & \mathbf{10} & \mathbf{40} & \mathbf{0} \end{array}$$

Thus, the value of $x = 4\ ft$.

Answer to Hook Question: Complex ☺

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Answer the following problems below.
Believe in yourself that you can do it.

Do It Yourself

15 MINUTES

Note: Items marked with an asterisk (*) are graded

1. What polynomial $f(x)$ of degree 5 has real coefficients and zeroes 0, $-1+i$ and $3i$?

*2. Find all the rational zeroes of $f(x) = 2x^4 - 5x^3 + 10x^2 - 20x + 8$.

3. Express $f(x) = x^5 - 8x^3 + x^2 + 12x + 4$ as a product of
   a) linear and quadratic polynomials that are irreducible over $\mathbb{R}$
   b) linear polynomials

*4. Show that $f(x) = x^4 - x^3 + 5x^2 - 2x + 6$ has no rational roots

5. What polynomial function $f(x)$ of degree 4 has leading coefficient 1 whose graph is shown below? How many rational roots does it have? How many non-real zeros does it have?

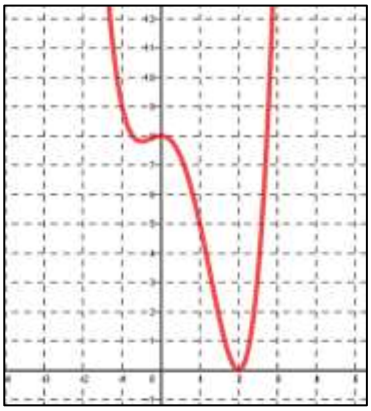

Figure 2: [x:y=1:1]

KNOT

Before this lesson ends, *K*eep *N*ote of these *O*utstanding *T*houghts:

1. If $z = a + bi$ is a complex zero of a polynomial function of degree more than 1 then the conjugate $\bar{z} = a - bi$ is also a zero of $f(x)$ with $b \neq 0$.
2. Every polynomial function can be expressed as a product of linear and quadratic polynomials such that the quadratic factors are irreducible over $\mathbb{R}$.
3. If the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ has *integer* coefficients and if p/q is a rational zero of $f(x)$ such that p and q have no common prime factor, then
   (1) p is a factor of the constant term a_0
   (2) q is a factor of the leading coefficient a_n

𝓜ΔTΠi∫ƒμN 𝓜ΔTΠi∫ƒμN 𝓜ΔTΠI ∫ƒμN 𝓜ΔTΠi∫ƒμN 𝓜ΔTΠi∫ƒμN 𝓜ΔTΠi∫ƒμN

References:

[1] Albarico, J.M. (2013). THINK Framework. (Based on Ramos, E.G. and N. Apolinario. (n.d.) *Science LINKS*. Rex Bookstore, Inc.)
[2] International Geogebra Institute. (2020). *GeoGebra*. https://www.geogebra.org/
[3] Swokowski, E., & Cole, J. (2008). *Algebra and Trigonometry with Analytic Geometry, 12th Edition*. Thomson Learning, Inc.

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ANSWER KEY:

1. $f(x) = x(x^2 + 2x + 2)(x^2 + 9) = x^5 + 2x^4 + 11x^3 + 18x^2 + 18x$

3. a) $f(x) = (x-1)(x-2)(x+1)(x^2 + 3x + 1)$
   b) $f(x) = (x-1)(x-2)(x+1)\left(x + \frac{3}{2} - \frac{\sqrt{5}}{2}\right)\left(x + \frac{3}{2} + \frac{\sqrt{5}}{2}\right)$

5. $f(x)$ has one real and rational zero with multiplicity 2 which is 2.
   $f(x) = (x-2)^2(x^2 + 2x + 2)$