Learning Guide Module

Subject Code	Math 3	Mathematics 3
Module Code	4.0	*Graphs of Polynomial and Rational Functions*
Lesson Code	4.1	*Remainder Theorem, Factor Theorem, and Descartes' Rule of Signs*
Time Frame		30 minutes

After completing this learning guide, you should be able to

1. find the remainder of a polynomial function using synthetic division, long division, and remainder theorem;
2. identify if an expression is a factor of a polynomial function using synthetic division, long division, and factor theorem;
3. identify the polynomial function from a given graph; and
4. enumerate the number of possible combinations of real and imaginary zeros of a polynomial function using Descartes' Rule of Signs.

With the COVID-19 crisis these past months, how did you overcome all the challenges? How did your perception in life change? What type of ideas and actions resulted to zero worries? Were they real or imaginary? Positive or negative? Rational or Irrational?

In this section, we will determine the zeroes of the polynomial which are the solutions of the equation $f(x) = 0$ and each real zero is an x-intercept of the graph of $f(x)$.

10 MINUTES

TIP (The Important Point)

Fundamental Theorem of Algebra

If a polynomial $f(x)$ has a positive degree and complex coefficients, then $f(x)$ has at least one complex zero.

Complete Factorization Theorem for Polynomials

If $f(x)$ is a polynomial degree $n > 0$, then there exist n complex numbers $c_1, c_2, \ldots, c_n$ such that $f(x) = a(x - c_1)(x - c_2) \cdots (x - c_n)$, where a is the leading coefficient of $f(x)$. Each number c_k is a zero of $f(x)$.

Remainder Theorem	A polynomial function has a remainder $f(c)$ when divided by $x - c$

Example 1. *Using synthetic division & remainder theorem*

When $f(x) = x^3 + 9x^2 + 17x + 5$ is divided by $x + 3$, what is the remainder?

Solutions.

By synthetic division,

$$
\begin{array}{r|rrrr}
-3 & 1 & 9 & 17 & 5 \\
+ & & -3 & -18 & 3 \\
\hline
 & \mathbf{1} & \mathbf{6} & \mathbf{-1} & \mathbf{8}
\end{array}
$$

coefficients of quotient *remainder*

By remainder theorem,

$$
\begin{aligned}
f(-3) &= (-3)^3 + 9(-3)^2 + 17(-3) + 5 \\
&= -27 + 81 - 51 + 5 \\
&= 8
\end{aligned}
$$

Example 2. *Using long division & remainder theorem*

What is the remainder if $f(x) = 4x^3 - 12x^2 + 11x + 10$ is divided by $2x - 5$?

Solutions.

By long division,

$$
\begin{array}{r|l}
 & 2x^2 - x + 3 \ (quotient) \\
2x - 5 & 4x^3 - 12x^2 + 11x + 10 \\
 & \underline{4x^3 - 10x^2} \\
 & \quad -2x^2 + 11x \\
 & \quad \underline{-2x^2 + \ 5x} \\
 & \qquad\qquad 6x + 10 \\
 & \qquad\qquad \underline{6x - 15} \\
 & \qquad\qquad\qquad 25 \ (remainder)
\end{array}
$$

By remainder theorem,

$$
\begin{aligned}
f\left(\tfrac{5}{2}\right) &= 4\left(\tfrac{5}{2}\right)^3 - 12\left(\tfrac{5}{2}\right)^2 + 11\left(\tfrac{5}{2}\right) + 10 \\
&= 4\left(\tfrac{125}{8}\right) - 12\left(\tfrac{25}{4}\right) + \tfrac{55}{2} + 10 \\
&= 25
\end{aligned}
$$

Factor Theorem	$x - c$ is a factor of a polynomial function if and only if $f(c) = 0$.

Example 3. *Using synthetic division & factor theorem*

Is $x - 4$ a factor of $f(x) = x^4 - 4x^3 + 4x^2 - 18x + 8$?

Solutions.

By synthetic division,

$$\begin{array}{r|rrrrr} \underline{4} & 1 & -4 & 4 & -18 & 8 \\ + & & 4 & 0 & 16 & -8 \\ \hline & \mathbf{1} & \mathbf{0} & \mathbf{4} & \mathbf{-2} & \mathbf{0} \end{array}$$

coefficients of quotient *remainder*

By factor theorem,

$f(4) = (4)^4 - 4(4)^3 + 4(4)^2 - 18(4) + 8$

$= 256 - 256 + 64 - 72 + 8 = 0$

The remainder is 0. Thus, $x - 4$ a factor of $f(x)$.

Example 4. *Using long division & factor theorem*

Is $3x - 1$ a factor of $f(x) = 27x^4 - 9x^3 + 6x^2 - 11x + 4$?

Solutions.

By long division,

$$\begin{array}{r l} & 9x^3 + 2x - 3 \\ 3x - 1 & \overline{)\,27x^4 - 9x^3 + 6x^2 - 11x + 4} \\ & \underline{27x^4 - 9x^3} \\ & \qquad\qquad 6x^2 - 11x \\ & \qquad\qquad \underline{6x^2 - \;\, 2x} \\ & \qquad\qquad\qquad -9x + 4 \\ & \qquad\qquad\qquad \underline{-9x + 3} \\ & \qquad\qquad\qquad\qquad 1 \end{array}$$

By factor theorem,

$$f\left(\tfrac{1}{3}\right) = 27\left(\tfrac{1}{3}\right)^4 - 9\left(\tfrac{1}{3}\right)^3 + 6\left(\tfrac{1}{3}\right)^2 - 11\left(\tfrac{1}{3}\right) + 4$$

$$= \frac{3^4}{3^3} - \frac{3^2}{3^1} - \frac{2(3)}{3^2} - \frac{11}{3} + 4$$

$$= \frac{1}{3} - \frac{1}{3} + \frac{2}{3} - \frac{11}{3} + \frac{12}{3}$$

$$= \frac{1-1+2-11+12}{3} = \frac{3}{3} = 1$$

There is a remainder. Thus, $3x - 1$ is NOT a factor of $f(x)$.

TIP (The Important Point)

Theorems on the Number of Zeros of Polynomial

- A polynomial of degree $n > 0$ has at most n different complex zeros.
- If $f(x)$ is a polynomial of degree $n > 0$ and if a zero of multiplicity m is counted m times, then $f(x)$ has precisely n zeros.

Example 5. *Finding a polynomial with specific graph*

What function $f(x)$ of degree 3 whose graph is shown below?

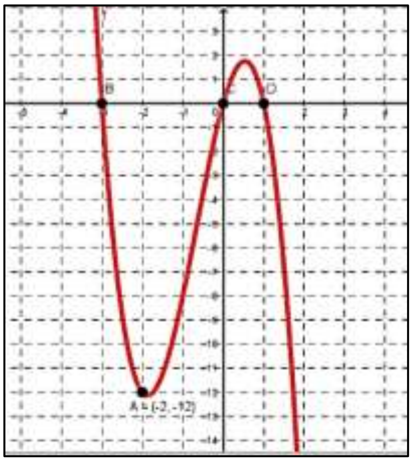

Figure 1: [x:y=1:1]

Solution.

The graph of the function has zeroes $\{-3, 0, 1\}$ and $f(-2) = -12$.

By factor theorem, $x + 3, x,$ and $x - 1$ are factors of $f(x)$. Thus,

$$f(x) = a(x+3)(x)(x-1)$$

Since $f(-2) = -12$,

$$f(-2) = a(-2+3)(-2)(-2-1)$$
$$-12 = a(1)(-2)(-3)$$
$$-12 = a(6)$$
$$a = -2$$

Thus, $f(x) = -2(x+3)(x)(x-1)$

$$f(x) = -2x^3 - 4x^2 + 6x$$

TIP (The Important Point)

Multiplicity

If c is a real zero of $f(x)$ of multiplicity m, the following behaviors apply in the graph:

- If m is odd
  - The graph of $f(x)$ passes thru and crosses the x – axis at $(c,0)$.
  - The behavior of the graph as to the trend does not change whether it is increasing or decreasing.
- If m is even
  - The graph of $f(x)$ passes thru the x – axis at $(c,0)$ but does not cross, instead, it either curves up or down
  - The behavior of the graph as to the trend changes either from increasing to decreasing or vice versa.

Example 6. *Finding a polynomial with specific zeroes and multiplicities*

What polynomial function of degree 6 has zeroes 2, -3 (multiplicity 2), -1 (multiplicity 3), and $f(-2) = 8$?

Solution.

By factor theorem, $x-2, x+3, x+1$ are factors of $f(x)$. Thus,

$$f(x) = a(x-2)(x+3)^2(x+1)^3$$

Since $f(-2) = 8$,

$$f(-2) = a(-2-2)(-2+3)^2(-2+1)^3$$
$$8 = a(-4)(1)^2(-1)^3$$
$$8 = a(4)$$
$$a = 2$$

Thus, $f(x) = 2(x-2)(x+3)^2(x+1)^3$.

The graph of $f(x)$ is shown below.

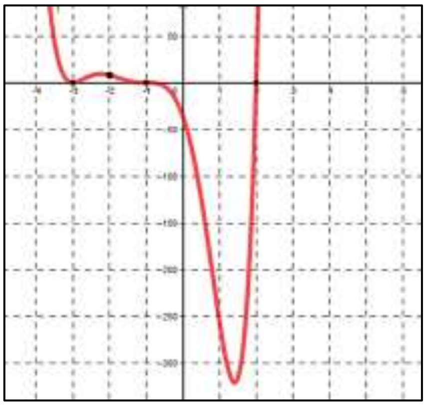

Figure 2: [x:y=1:50]

2 and -1 have odd multiplicities where the graph maintained its behavior at these zeroes while -3 has even multiplicity where the graph changes behavior from decreasing to increasing as it passes through $(-3,0)$.

TIP (The Important Point)

Descartes' Rule of Signs

Let $f(x)$ be a polynomial with real coefficients and a nonzero constant term.

- The number of *positive* real zeros of $f(x)$ either is equal to the number of variations of sign in $f(x)$ or is less than that number by an even integer.
- The number of *negative* real zeros of $f(-x)$ either is equal to the number of variations of sign in $f(x)$ or is less than that number by an even integer.

There is variation of sign in $f(x)$ if two consecutive coefficients have opposite signs

Example 7. *Using Descartes' Rule of Signs*

What are the number of possible and negative real solutions and imaginary solutions of the equation $f(x) = 0$, where $f(x) = 2x^5 + x^4 - 7x^3 + 7x - 3$?

Solution.

$$f(x) = 2x^5 + x^4 - 7x^3 + 7x - 3$$

has 3 variations of signs

$$f(-x) = -2x^5 + x^4 + 7x^3 - 7x - 3$$

has 2 variations of signs

The following table summarizes the various possibilities that can occur for solutions of the equation. Because $f(x)$ has degree 5, there are a total of 5 solutions.

No. of positive real solutions	3	1	3	1
No. of negative real solutions	2	2	0	0
No. of imaginary solutions	0	2	2	4
Total number of solutions	5	5	5	5

Real –Life Application

A storage tank is to be constructed where the volume is determined by the function $V(x) = \frac{4}{3}x^3 + 5x^2$. Determine the radius x so that the resulting volume is $\frac{92}{3}ft^3$?

Solution:

$V(x) = \frac{4}{3}x^3 + 5x^2$

$\frac{4}{3}x^3 + 5x^2 = \frac{92}{3}$

$4x^3 + 15x^2 = 92$

$4x^3 + 15x^2 - 92 = 0$

$4x^3 + 15x^2 - 92 = 0$

$(x-2)(4x^2 + 23x + 46) = 0$

By Factor Theorem,

$V(2) = 4(2)^3 + 15(2)^2 - 92 = 0$

Thus, the radius is $2ft$.

Answer the following problems below.
Believe in yourself that you can do it.

Do It Yourself

15 MINUTES

Note: Items marked with an asterisk (*) are graded.

1. Use synthetic division and remainder theorem to find the remainder when $f(x)$ is divided by $x - 2$.
   Let $f(x) = x^3 - 3x^2 + 5x - 7$.

2. *Use long division and remainder theorem to find the remainder when $f(x)$ is divided by $x + 4$. Let $f(x) = x^3 + 2x^2 - 7x + 9$.

3. Use long division and factor theorem to determine whether $2x - 1$ is a factor of $f(x) = 2x^4 - x^3 + 10x - 5$.

4. *Use synthetic division and factor theorem to determine whether $2x - 1$ is a factor of $f(x) = 3x^3 - 5x^2 - x + 1$.

5. What function $f(x)$ of degree 4 whose graph is shown below?

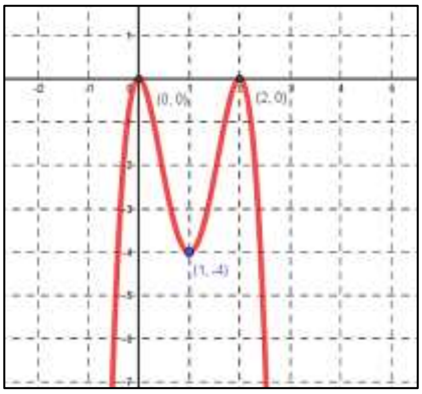

Figure 3: [x:y=1:1]

6. *What function $f(x)$ of degree 3 whose graph is shown below?

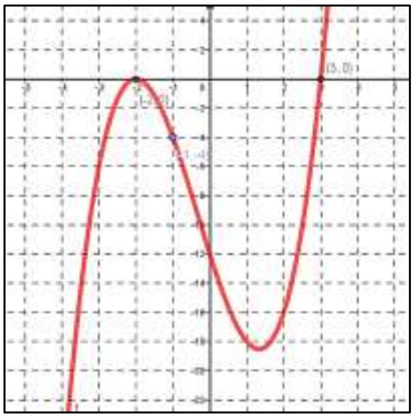

Figure 4: [x:y=1:2]

7. What are the number of possible positive and negative real solutions and imaginary solutions of the equation $f(x) = 0$, where

$$f(x) = 2x^5 + 4x^4 + x^3 + 2x^2 - 3x - 6?$$

Solution.

No. of positive real solutions				
No. of negative real solutions				
No. of imaginary solutions				
Total number of solutions				

8. *What are the number of possible positive and negative real solutions and imaginary solutions of the equation $f(x) = 0$, where

$$f(x) = x^4 - 2x^3 - 2x^2 - 2x - 3?$$

Solution.

No. of positive real solutions				
No. of negative real solutions				
No. of imaginary solutions				
Total number of solutions				

CRITICAL THINKING

9. Find all the possible values of k such that $f(x) = k^2x^3 - 4kx + 3$ is divisible by $x - 1$.

10. *Show that $x - c$ is not a factor of $f(x)$ for any real number c. Let $f(x) = -x^4 - 3x^2 - 2$.

KNOT

Before this lesson ends, *Keep* *Note* of these *Outstanding* *Thoughts*:

1. There are many ways in finding the zeroes of a polynomial function such as synthetic division, long division, remainder theorem, and factor theorem.
2. Descartes' Rule of Signs gives the various possibilities that can occur for solutions of the $f(x) = 0$.

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References:

[1] Albarico, J.M. (2013). THINK Framework. (Based on Ramos, E.G. and N. Apolinario. (n.d.) *Science LINKS*. Rex Bookstore, Inc.)
[2] International Geogebra Institute. (2020). *GeoGebra*. https://www.geogebra.org/
[3] Swokowski, E., & Cole, J. (2008). *Algebra and Trigonometry with Analytic Geometry, 12th Edition*. Thomson Learning, Inc.

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ANSWER KEY:

1. -1
3. Yes
5. $f(x) = -4x^2(x-2)^2$
7.

No. of positive real solutions	1	1	1	
No. of negative real solutions	4	2	0	
No. of imaginary solutions	0	2	4	
Total number of solutions	5	5	5	

9. $k = 1,3$