FPI Observer with State Disturbances - Git Test

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Abstract

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1. Introduction

2. Problem Formulation

Consider a Linear Time-Invariant LTI) plant described by the state-space equations

$$\dot{x} = Ax + Bu + N\delta, \quad x(0) = x_o \tag{1a}$$

$$y = Cx + Du \tag{1b}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^p, y \in \mathbb{R}^m$, $\delta \in \mathbb{R}^k$ and without loss of generality N full column rank.

Let the input disturbance be

$$d = N\delta, \quad N \in \mathbb{R}^{n\times k}. \tag{2}$$

and assume it can be modeled as

$$\dot{z} = A_d z, \quad z(0) = z_o \in \mathbb{R}^r \tag{3a}$$

$$d = Wz + d_{\overline{o}}, \quad \mathcal{R}(W) \subseteq \mathcal{R}(N) \tag{3b}$$

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where $z \in \mathbb{R}^r$, $d \in \mathbb{R}^n$, $W \in \mathbb{R}^{n \times r}$ is full column rank and

$$d_{\overline{o}} = P_{W^\perp}\, d \tag{4}$$

where $P_{W^\perp}$ is the projection onto the orthogonal complement of $\mathcal{R}(W)$, *i.e.*, onto $\mathcal{N}(W^T)$.

The combined dynamic equations of the plant and perturbation are

$$\begin{bmatrix} \dot{x} \\ \dot{z} \\ y \end{bmatrix} = \left[\begin{array}{cc|c} A & W & B \\ O_{r\times n} & A_d & O_{r\times p} \\ \hline C & O_{m\times r} & D \end{array}\right] \begin{bmatrix} x \\ z \\ u \end{bmatrix} + \begin{bmatrix} I_n \\ O_{r\times n} \\ O_{m\times n} \end{bmatrix} d_{\overline{o}} \tag{5}$$

Now consider a state observer of the form

$$\dot{\hat{x}} = A\hat{x} + Bu + W\hat{z} + L(y - \hat{y}) \tag{6a}$$

$$\hat{y} = C\hat{x} + Du \tag{6b}$$

driven by the input u and the output error

$$\tilde{y} = y - \hat{y} = C\tilde{x} \tag{7}$$

where the measurement perturbations are estimated using

$$\dot{\hat{z}} = A_d\hat{z} + T(y - \hat{y}), \quad \hat{z}(0) = \hat{z}_o \tag{8a}$$

$$\hat{d} = W\hat{z} \tag{8b}$$

[**JC] Why is the disturbance estimate $\hat{d} = W\hat{z}$ instead of $\hat{d} = W\hat{z} + d_{\overline{o}}$?

The observer parameters are $L \in \mathbb{R}^{n\times m}$, $T \in \mathbb{R}^{r\times m}$ and $W \in \mathbb{R}^{n\times r}$.

Subtracting (6) and (8) from (5)

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{z}} \\ \tilde{y} \end{bmatrix} = \left[\begin{array}{cc|c} A & W & B \\ O_{r\times n} & A_d & O_{r\times p} \\ \hline C & O_{m\times r} & O_{m\times p} \end{array}\right] \begin{bmatrix} \tilde{x} \\ \tilde{z} \\ u \end{bmatrix} + \begin{bmatrix} I_n \\ O_{r\times n} \\ O_{m\times n} \end{bmatrix} d_{\overline{o}} - \begin{bmatrix} L \\ T \\ O_{m\times m} \end{bmatrix} \tilde{y} \tag{9}$$

Eliminating $\tilde{y}$

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{z}} \end{bmatrix} = \left(\begin{bmatrix} A & O_{n\times r} \\ O_{r\times n} & A_d \end{bmatrix} - \begin{bmatrix} L \\ T \end{bmatrix} \begin{bmatrix} C & V \end{bmatrix} \right) \begin{bmatrix} \tilde{x} \\ \tilde{z} \end{bmatrix} + \begin{bmatrix} I_n \\ O_{r\times n} \end{bmatrix} d_{\overline{o}} \tag{10}$$

$$\dot{\hat{\beta}} = (\overline{A} - \overline{L}\,\overline{C})\,\tilde{\beta} + \overline{B}\, d_{\overline{o}}, \quad \hat{\beta}(0) = \hat{\beta}_o \tag{11}$$

where

$$\tilde{\beta} = \begin{bmatrix} \tilde{x} \\ \tilde{z} \end{bmatrix}, \quad \overline{A} = \begin{bmatrix} A & W \\ O_{r\times n} & A_d \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} C & O_{m\times r} \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} I_n \\ O_{r\times n} \end{bmatrix} \tag{12}$$

Remark 1. *Since $W \in \mathbb{R}^{n\times r}$ is full column rank $\dim \mathcal{R}(W) = r$ and $\dim \mathcal{N}(W^T) = n - r$.*

Note that the estimation error dynamics (11) are driven by the component of the perturbation, $d_{\overline{o}} \in \mathcal{N}(W^T)$, that cannot be "observed".

With notation as above, the unbiased observer problem can be stated as follows:

Problem 1. *Find $\overline{L}$ and W such that $\tilde{\beta} \to 0$ asymptotically for all $\tilde{\beta}(0)$ and for all constant perturbations $d_{\overline{o}} \in \mathcal{N}(W^T)$.*

When $d_{\overline{o}} = 0$ problem 1 reduces to a standard observer problem that is solvable if and only if $(\overline{A}, \overline{C})$ detectable [1]. Furthermore, if $(\overline{A}, \overline{C})$ observable the rate of convergence of the estimation error can be made arbitrarily fast.

In this paper the focus is on the analysis and design of observers subject to constant state perturbations. This enables a simple characterization of all allowable perturbations and corresponds to the case when $A_d = O_{n\times n}$.

3. Observer Error Dynamics Stability

As mentioned above a necessary condition for problem 1 to have a solution is that $(\overline{A}, \overline{C})$ be detectable since that is it necessary and sufficient for the existence of a stabilizing gain for the error dynamics.

In this section we show that detectability of $(\overline{A}, \overline{C})$ is sufficient for detectability of (A, C). However, for it to be necessary additional requirements on the output matrix of the perturbation model, W must be satisfied. The proof uses standard linear algebra results and block Gaussian elimination arguments.

Theorem 1. *If $(\overline{A}, \overline{C})$ detectable then (A, C) detectable.*

Proof. By the PBH rank test

$$\mathbf{rank} \begin{bmatrix} sI_{n+r} - \overline{A} \\ \overline{C} \end{bmatrix} = \mathbf{rank} \begin{bmatrix} sI_n - A & -W \\ O_{r\times n} & sI_r \\ C & O_{m\times r} \end{bmatrix} = n + r, \forall\ s \in \overline{\mathbb{C}}_+ \tag{13}$$

where $\overline{\mathbb{C}}_+ = \{s \in \mathbb{C} : \mathrm{Re}\{s\} \geq 0\}$ denotes the closed right half plane.

If $s = 0$, the PBH rank condition becomes

$$\mathbf{rank}\begin{bmatrix} -A & -W \\ O_{r\times n} & O_{r\times r} \\ C & O_{n\times r} \end{bmatrix} = \mathbf{rank}\begin{bmatrix} A & W \\ C & O_{n\times r} \end{bmatrix} = n + r. \tag{14}$$

Hence, it follows that

$$\mathbf{rank}\, W = r \quad \text{and} \quad \mathbf{rank}\begin{bmatrix} A \\ C \end{bmatrix} = n, \tag{15}$$

If $s \in \overline{\mathbb{C}}_+ \setminus 0$, by block Gaussian elimination

$$\begin{bmatrix} sI_n - A & -W \\ O_{r\times n} & sI_r \\ C & O_{m\times r} \end{bmatrix} \overset{\text{row}}{\sim} \begin{bmatrix} sI_n - A & -W \\ C & O_{m\times r} \\ O_{r\times n} & sI_r \end{bmatrix} \overset{\text{row}}{\sim} \left[\begin{array}{c|c} sI_n - A & O_{n\times r} \\ C & O_{m\times r} \\ \hline O_{r\times n} & sI_r \end{array}\right] \tag{16}$$

Therefore

$$\mathbf{rank}\begin{bmatrix} sI_{n+r} - \overline{A} \\ \overline{C} \end{bmatrix} = \mathbf{rank}\begin{bmatrix} sI_n - A \\ C \end{bmatrix} + \mathbf{rank}\, sI_r = n + r \Rightarrow \mathbf{rank}\begin{bmatrix} sI_n - A \\ C \end{bmatrix} = n. \tag{17}$$

which together with (15) implies (A, C) detectable. □

The next theorem establishes the conditions that the plant and perturbation model matrices must satisfy to guarantee the existence of an asymptotic stabilizing gain for the observer error dynamics.

Theorem 2. *Let $[A, B, C, D]$ be a state-space realization of a system with m outputs such that $\dim \mathcal{N}(A) = q > 0$. Then if (A, C) detectable there exists a full column rank $W \in \mathbb{R}^{n\times r}$, with $0 < r \leq n - q$ such that $(\overline{A}, \overline{C})$ detectable.*

Proof. By the PBH rank test

$$\mathbf{rank}\begin{bmatrix} sI_n - A \\ C \end{bmatrix} = n, \forall\ s \in \overline{\mathbb{C}}_+ \tag{18}$$

Construct an invertible column compression $H = [H_1\ H_2]$ for A. Then

$$\begin{bmatrix} A \\ C \end{bmatrix} H = \begin{bmatrix} AH_1 & AH_2 \\ CH_1 & CH_2 \end{bmatrix} = \begin{bmatrix} A_1 & O_{n\times q} \\ C_1 & C_2 \end{bmatrix} \overset{\text{row}}{\sim} \begin{bmatrix} A_1 & O_{n\times q} \\ O_{m\times(n-q)} & C_2 \end{bmatrix} \tag{19}$$

where we used the fact that $A_1 \in \mathbb{R}^{n\times(n-q)}$ is full column rank.

For $s=0$ and from (19)

$$\mathbf{rank}\begin{bmatrix} sI_n - A \\ C \end{bmatrix} = \mathbf{rank}\begin{bmatrix} A \\ C \end{bmatrix} = \mathbf{rank}\, A_1 + \mathbf{rank}\, C_2 = n \Rightarrow \mathbf{rank}\, C_2 = q \tag{20}$$

By the Fundamental Theorem of Linear Algebra [**?**] it is always possible to construct a rank $r \le q$ extension to $A_1 = AH_1$ such that

$$\mathbf{rank}\begin{bmatrix} A_1 & W \end{bmatrix} = n - q + r. \tag{21}$$

Hence, from (19) and (21)

$$\mathbf{rank}\begin{bmatrix} A & W \\ C & O_{m\times r} \end{bmatrix} = n + r. \tag{22}$$

For $s \in \overline{\mathbb{C}}_+ \setminus 0$, extend the rank of the PBH test matrix (18) by adding r columns and rows and apply block elementary operations to obtain the final result

$$\mathbf{rank}\left[\begin{array}{c|c} sI_n - A & O_{n\times r} \\ O_{r\times n} & sI_r \\ C & O_{m\times r} \end{array}\right] = \mathbf{rank}\left[\begin{array}{c|c} sI_n - A & W \\ O_{r\times n} & sI_r \\ C & O_{m\times r} \end{array}\right] = \mathbf{rank}\begin{bmatrix} sI - \overline{A} \\ \overline{C} \end{bmatrix} = n + r, \tag{23}$$

wich together with (22) imply that $(\overline{A}, \overline{C})$ is detectable. □

Remark 2. *The number of columns of $W \in \mathbb{R}^{n\times r}$ determines the dimension of the state of the perturbation model and is constrained by the number of states and the nullity of the state matrix of the system being observed (see equation (21).)*

4. Unbiased Estimation

In this section we unveil necessary and sufficient conditions for the solution of the unbiased estimation problem 1.

Theorem 3. *Assume the conditions in Theorem 2 hold. Then $\lim_{t\to\infty} \tilde{\beta}(t) = 0$ for any constant $d_{\overline{o}}$ if and only if $\overline{A} - \overline{L}\,\overline{C}$ stable and $d_{\overline{o}} = 0$.*

Proof. Choose W to satisfy Theorem 2 and let $\overline{A}_f = \overline{A} - \overline{L}\,\overline{C}$. Using Laplace Transforms, the state trajectories of (11) for a step disturbance $d_{\overline{o}}$ are

$$\tilde{\beta}(s) = (sI - \overline{A}_f)^{-1}\tilde{\beta}(0) + \frac{1}{s}(sI - \overline{A}_f)^{-1}\overline{B}\, d_{\overline{o}} \tag{24}$$

By Theorem 2 and standard observer theory it is always possible to choose $\overline{L}$ such that $\overline{A}_f$ is stable and therefore invertible. For unbiased estimation $\tilde{\beta}_{ss} = \lim_{t\to\infty} \tilde{\beta}(t) = 0$. Applying the Final Value Theorem

$$\tilde{\beta}_{ss} = \lim_{s\to 0} s\tilde{\beta}(s) = (\overline{A}_f)^{-1}\overline{B}\, d_{\overline{o}} = 0 \Leftrightarrow \overline{B}\, d_{\overline{o}} = 0 \Leftrightarrow d_{\overline{o}} = 0 \tag{25}$$

where the last equivalence follows due to the fact that $\overline{B}$ in (12) is full column rank. □

5. References

[1] T. Kailath. *Linear Systems.* Prentice-Hall Inc., NJ, 1980.