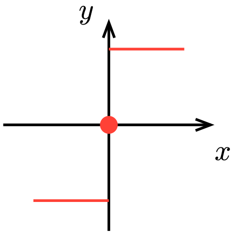
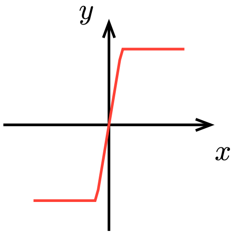
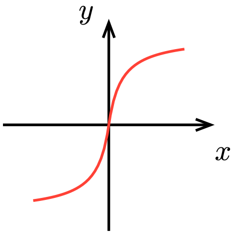
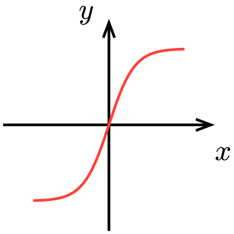
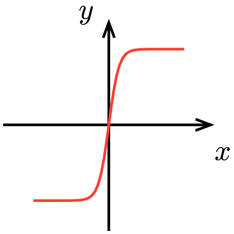
Illustrations for Control Theory

Contents

SMC Chattering Elimination: Quasi-Sliding Mode

In many practical control systems, including DC motors and aircraft control, it is important to avoid control chattering by providing continuous/smooth signals. One obvious solution to make the control function continuous/smooth is to approximate the discontinuous function $v(\sigma) = -\rho \text{ sign } (\sigma)$ by some continuous/smooth function. For instance, it could be replaced by a "sigmoid function".

Table 1: replaced sign by a "sigmoid function"

	$\text{sign}(x)$	$\text{sat}\left(\frac{x}{\varepsilon}\right)$	$\frac{x}{\lvert x\rvert+\varepsilon}$	$\tanh(x)$	$\frac{1-e^{-Tx}}{1+e^{-Tx}}$
continuity	discontinuous	continuous	smooth[1]	smooth	smooth
	 	 $\varepsilon = 0.5$	 $\varepsilon = 0.5$	 	 $T = 5$

[1] I am not sure about this

Ternary Differential Equations' Solutions

Table 2: solutions to ternary differential equations

differetial equation	differetial inclusion	classical solution	caratheodory solution	Filippov solution
$\dot{x} = \begin{cases} 1 & \text{if } x<0 \\ a & \text{if } x=0 \\ -1 & \text{if } x>0 \end{cases}$	$\dot{x} \in \mathcal{F}(x) = \begin{cases} 1 & \text{if } x<0 \\ [-1,1] & \text{if } x=0 \\ -1 & \text{if } x>0 \end{cases}$	Only when $a = 0$, classical solution exists. The maximal classical solution is 1. if $x(0) > 0$, $x_1(t) = x(0) - t, t < x(0)$ 2. if $x(0) < 0$, $x_2(t) = x(0) + t, t < -x(0)$ 3. if $x(0) = 0$, $x_3(t) = 0, t \in [0, \infty)$	Only when $a = 0$, caratheodory solution exists. The maximal classical solution is 1. if $x(0) > 0$, $x_1(t) = \max(x(0) - t, 0), t \in [0, \infty)$ 2. if $x(0) < 0$, $x_2(t) = \min(x(0) + t, 0), t \in [0, \infty)$ 3. if $x(0) = 0$, $x_3(t) = 0, t \in [0, \infty)$ **Note**:These only absolutely continuous (not continuously differentiable)	Whatever the value of a is, the Filippov solution is 1. if $x(0) > 0$, $x_1(t) = \max(x(0) - t, 0), t \in [0, \infty)$ 2. if $x(0) < 0$, $x_2(t) = \min(x(0) + t, 0), t \in [0, \infty)$ 3. if $x(0) = 0$, $x_3(t) = 0, t \in [0, \infty)$
$\dot{x} = \begin{cases} -1 & \text{if } x<0 \\ a & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$	$\dot{x} \in \mathcal{F}(x) = \begin{cases} -1 & \text{if } x<0 \\ [-1,1] & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$	From $x = x(0) \neq 0$,classical solution exists as 1. $x_1(t) = x(0) + t$ if $x(0) > 0$ 2. $x_2(t) = x(0) - t$ if $x(0) < 0$ From $x = x(0) = 0$, classical solution exists when $a = 1$ or $a = -1$ 1. when $a = 1$, $x_1(t) = t, t \in [0, \infty)$ 2. when $a = -1$,$x_2(t) = -t, t \in [0, \infty)$	From $x = x(0) \neq 0$,classical solution exists as 1. $x_1(t) = x(0) + t$ if $x(0) > 0$ 2. $x_2(t) = x(0) - t$ if $x(0) < 0$. From $x = x(0) = 0$, two caratheodory solutions exist for **all** $a \in \mathbb{R}$ 1. $x_1(t) = t, t \in [0, \infty)$ 2. $x_2(t) = -t, t \in [0, \infty)$ These two solutions only violate the vector field in $t = 0$	Filippov solution exists for all $a \in \mathbb{R}$ and $x(0) \in \mathbb{R}$. 1. if $x(0) \geq 0$, $x_1(t) = x(0) + t, t \in [0, \infty)$ 2. if $x(0) \leq 0$,, $x_2(t) = x(0) - t, t \in [0, \infty)$ **Note**: When $x(0) = 0$, exists two Filippov solutions.
$\dot{x} = \begin{cases} 1 \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$	$\dot{x} \in \{1\}$	$x = 0, t \in [0, \infty)$	two caratheodory solutions: 1. $x(t) = 0, t \in [0, \infty)$ 2. $x(t) = t, t \in [0, \infty)$	one unique solution: 1. $x(t) = t, t \in [0, \infty)$

Conditions for Existence and Uniqueness of Classical, Caratheodory, Filippov Solutions

Table 3: conditions of solutions to $\dot{x} = X(x(t))$

	solution	existence	uniqueness
classical	continuously differentiable	$X : \mathbb{R}^d \to \mathbb{R}^d$ is continuous	essentially one-sided Lipschitz on $B(x, \varepsilon)$,[2]
Filippov	absolutely continuous	$X : \mathbb{R}^d \to \mathbb{R}^d$ is measurable and locally essentially bounded	essentially one-sided Lipschitz on $B(x, \varepsilon)$

[2] Every vector field that is locally Lipschitz at x satisfies the one-sided Lipschitz condition on a neighborhood of x, but the converse is not true.

Sliding Mode Control Algorithms

Consider the system $\ddot{y} = u + f$, design u to regulate y to track y_c. define error variable $e = y_c - y$, then $\ddot{e} = \ddot{y}_c - f - u$. Convergence requirements and analysis can be found at (SHTESSEL et al., 2014) for SMC , (XIAN et al., 2004) for RISE

Table 4: regulate the system $\ddot{e} = \ddot{y}_c - f - u$

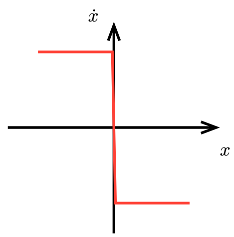
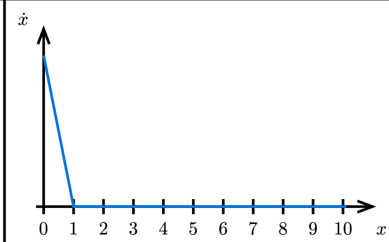
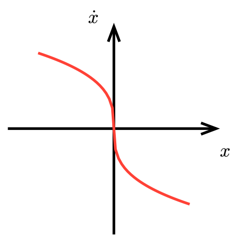
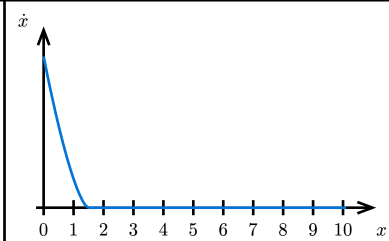
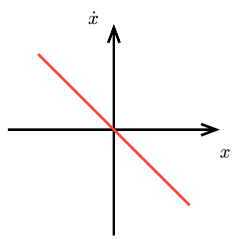
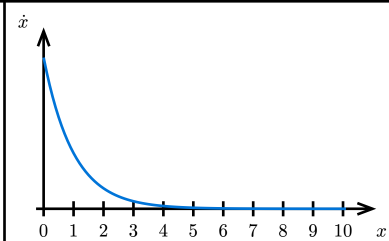
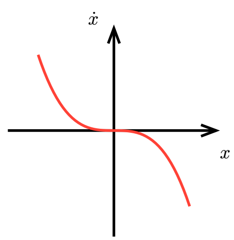
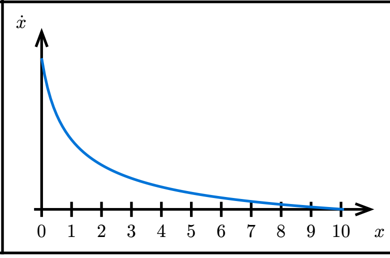
Sliding Mode Controller	name	Sliding variable convergence	Output Tracking convergence	Type
$u = \rho \text{ sign } (\sigma)$ $\sigma = \dot{e} + ce$	Traditional	Finite time	Asymptotic	Discontinuous
$u = u_1 + u_2,$ $u_1 = \rho_1 \text{ sign}(s), u_2 = -\int \sigma dt$ $\sigma = \dot{e} + ce$ $s = \sigma - z, \dot{z} = -u_2$	integral	s: Finite time σ: Asysmptotic	Asymptotic	Discontinuous
$\dot{u} = v,$ $v = c\overline{c}\dot{e} + (c + \overline{c})u + \rho \text{ sign}(s)$ $s = \dot{\sigma} + \overline{c}\sigma, \sigma = \dot{e} + ce$	integral	s: Finite time σ: Asysmptotic	Asymptotic	Continuous
$u = c \ \lvert\sigma\rvert^{\frac{1}{2}} \text{ sign}(\sigma) + w$ $\dot{w} = b \text{ sign}(\sigma)$ $\sigma = \dot{e} + ce$	Super-twisting (Second Order SMC)	Finite time	Asymptotic	Continuous
$u = -\rho \text{ sign}(\sigma)$ $\sigma = \dot{e} + c\lvert e\rvert^{\frac{1}{2}}\text{sign}(e)$	prescribed convergence law	Finite time	Finite time	Discontinuous
$u = (k_s + 1)e_2(t) - (k_s + 1)e_2(0) + w$ $\dot{w} = (k_s + 1)\alpha e_2 + \beta \text{ sign } e_2$ $e_1 = e, e_2 = \dot{e}_1 + e_1$	RISE		Asysmptotic	Continuous

Fractional power feedback

From (POLYAKOV, 2020), we can find solutions of the system $\dot{x} = -x^v$ has some good properties. Due to the definition of power function, usually we extend the function's definition to the whole rational field $\mathbb{R}$ as $\dot{x} = -\operatorname{sign}(x)\ |x|^v, v \geq 0$. In some books like (POLYAKOV, 2020), a condition on v is used.[3] The general solution to this system is

$$x(t) = \left(x(0)^{-v+1} + (v-1)t\right)^{\frac{1}{-v+1}} \operatorname{sign}(x(0)) = \frac{x(0)}{\left(1 + (v-1)t|x(0)|^{v-1}\right)^{\frac{1}{v-1}}} \text{ if } v \neq 1.$$

Table 5: fractional power feedback $\dot{x} = -\operatorname{sign}(x)|x|^v, v \geq 0$

system	$x - \dot{x}$ curve	numerical solution $x = x(t)$	analytical solution	stability
$\dot{x} = -x^0 = -\operatorname{sign}(x)$			when $x(0) > 0$ $x(t) = \begin{cases} x(0)-t & t\in(0,x(0)) \\ 0 & t\geq x(0) \end{cases}$	Finite time if $v < 1$
$\dot{x} = -x^{\frac{1}{3}}$			when $x(0) > 0$ $x(t) = \begin{cases} \left(\lvert x(0)\rvert^{\frac{2}{3}} - \frac{2}{3}t\right)^{\frac{3}{2}} & t\in\left(0, \left(\frac{2}{3}\right)\lvert x(0)\rvert^{\frac{3}{2}}\right) \\ 0 & t\geq x(0) \end{cases}$	Finite time if $v < 1$
$\dot{x} = -x^1 = -x$			$x = x(0)e^{-t}$	Exponential if $= 1$
$\dot{x} = -x^3$			$x(t) = \left(x(0)^{-2} + 2t\right)^{-\frac{1}{2}}$	(practical) Fixed time if $v > 1$[4]

[3] I haven't figure out this condition yet. The condition write v as $v = \frac{p}{q}$ and says p should be an odd integer and q is an even natual number. I think both p and q should be odd natural number and $q \neq 0$.

[4] converges to a neighborhood of the origin in a fixed time independent of the initial condition.

Bibliography

POLYAKOV A, 2020. Generalized homogeneity in systems and control[M/OL]. Cham: Springer International Publishing[2024-01-18]. http://link.springer.com/10.1007/978-3-030-38449-4. DOI:10.1007/978-3-030-38449-4.

SHTESSEL Y, EDWARDS C, FRIDMAN L, et al., 2014. Sliding Mode Control and Observation[M/OL]. New York, NY: Springer New York. DOI:10.1007/978-0-8176-4893-0.

XIAN B, DAWSON D, DEQUEIROZ M, et al., 2004. A Continuous Asymptotic Tracking Control Strategy for Uncertain Nonlinear Systems[J/OL]. IEEE Transactions on Automatic Control, 49(7): 1206. DOI:10.1109/TAC.2004.831148.