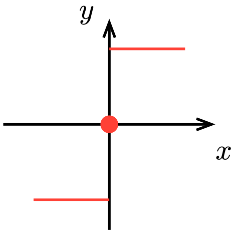
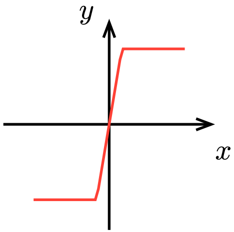
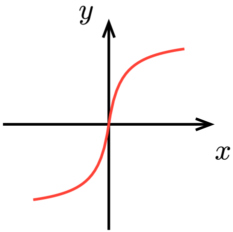
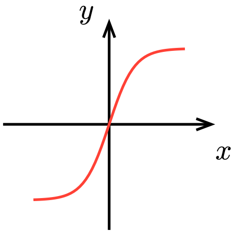
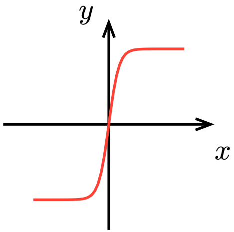
Illustrations for Control Theory

Contents

SMC Chattering Elimination: Quasi-Sliding Mode

In many practical control systems, including DC motors and aircraft control, it is important to avoid control chattering by providing continuous/smooth signals. One obvious solution to make the control function continuous/smooth is to approximate the discontinuous function $v(\sigma) = -\rho \text{ sign } (\sigma)$ by some continuous/smooth function. For instance, it could be replaced by a “sigmoid function”.

Table 1: replaced sign by a “sigmoid function”

	$\text{sign}(x)$	$\text{sat}\left(\frac{x}{\varepsilon}\right)$	$\frac{x}{\lvert x\rvert+\varepsilon}$	$\tanh(x)$	$\frac{1-e^{-Tx}}{1+e^{-Tx}}$
continuity	discontinuous	continuous	smooth[1]	smooth	smooth
		 $\varepsilon = 0.5$	 $\varepsilon = 0.5$		 $T = 5$

[1] I am not sure about this

Ternary Differential Equations' Solutions

Table 2: solutions to ternary differential equations

differetial equation	**differetial inclusion**	**classical solution**	**caratheodory solution**	**Filippov solution**
$\dot{x} = \begin{cases} 1 & \text{if } x<0 \\ a & \text{if } x=0 \\ -1 & \text{if } x>0 \end{cases}$	$\dot{x} \in \mathcal{F}(x) = \begin{cases} 1 & \text{if } x<0 \\ [-1,1] & \text{if } x=0 \\ -1 & \text{if } x>0 \end{cases}$	Only when $a = 0$, classical solution exists. The maximal classical solution is 1. if $x(0) > 0$, $x_1(t) = x(0) - t, t < x(0)$ 2. if $x(0) < 0$, $x_2(t) = x(0) + t, t < -x(0)$ 3. if $x(0) = 0$, $x_3(t) = 0, t \in [0, \infty)$	Only when $a = 0$, caratheodory solution exists. The maximal classical solution is 1. if $x(0) > 0$, $x_1(t) = \max(x(0) - t, 0), t \in [0, \infty)$ 2. if $x(0) < 0$, $x_2(t) = \min(x(0) + t, 0), t \in [0, \infty)$ 3. if $x(0) = 0$, $x_3(t) = 0, t \in [0, \infty)$ **Note**:These only absolutely continuous (not continuously differentiable)	Whatever the value of a is, the Filippov solution is 1. if $x(0) > 0$, $x_1(t) = \max(x(0) - t, 0), t \in [0, \infty)$ 2. if $x(0) < 0$, $x_2(t) = \min(x(0) + t, 0), t \in [0, \infty)$ 3. if $x(0) = 0$, $x_3(t) = 0, t \in [0, \infty)$
$\dot{x} = \begin{cases} -1 & \text{if } x<0 \\ a & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$	$\dot{x} \in \mathcal{F}(x) = \begin{cases} -1 & \text{if } x<0 \\ [-1,1] & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$	From $x = x(0) \neq 0$,classical solution exists as 1. $x_1(t) = x(0) + t$ if $x(0) > 0$ 2. $x_2(t) = x(0) - t$ if $x(0) < 0$ From $x = x(0) = 0$, classical solution exists when $a = 1$ or $a = -1$ 1. when $a = 1$, $x_1(t) = t, t \in [0, \infty)$ 2. when $a = -1$,$x_2(t) = -t, t \in [0, \infty)$	From $x = x(0) \neq 0$,classical solution exists as 1. $x_1(t) = x(0) + t$ if $x(0) > 0$ 2. $x_2(t) = x(0) - t$ if $x(0) < 0$. From $x = x(0) = 0$, two caratheodory solutions exist for **all** $a \in \mathbb{R}$ 1. $x_1(t) = t, t \in [0, \infty)$ 2. $x_2(t) = -t, t \in [0, \infty)$ These two solutions only violate the vector field in $t = 0$	Filippov solution exists for all $a \in \mathbb{R}$ and $x(0) \in \mathbb{R}$. 1. if $x(0) \geq 0$, $x_1(t) = x(0) + t, t \in [0, \infty)$ 2. if $x(0) \leq 0$,, $x_2(t) = x(0) - t, t \in [0, \infty)$ **Note**: When $x(0) = 0$, exists two Filippov solutions.
$\dot{x} = \begin{cases} 1 \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$	$\dot{x} \in \{1\}$	$x = 0, t \in [0, \infty)$	two caratheodory solutions: 1. $x(t) = 0, t \in [0, \infty)$ 2. $x(t) = t, t \in [0, \infty)$	one unique solution: 1. $x(t) = t, t \in [0, \infty)$

Conditions for Existence and Uniqueness of Classical, Caratheodory, Filippov Solutions

Table 3: conditions of solutions to $\dot{x} = X(x(t))$

	solution	existence	uniqueness
classical	continuously differentiable	$X : \mathbb{R}^d \to \mathbb{R}^d$ is continuous	essentially one-sided Lipschitz on $B(x, \varepsilon)$,[2]
Filippov	absolutely continuous	$X : \mathbb{R}^d \to \mathbb{R}^d$ is measurable and locally essentially bounded	essentially one-sided Lipschitz on $B(x, \varepsilon)$

[2]Every vector field that is locally Lipschitz at x satisfies the one-sided Lipschitz condition on a neighborhood of x, but the converse is not true.

Sliding Mode Control Algorithms

Consider the system $\ddot{y} = u + f$, design u to regulate y to track y_c. define error variable $e = y_c - y$, then $\ddot{e} = \ddot{y}_c - f - u$. see [1] for convergence requirements and analysis.

Sliding Mode Controller	name	Sliding variable convergence	Output Tracking convergence	Type
$u = \rho \text{ sign } (\sigma)$ $\sigma = \dot{e} + ce$	Traditional	Finite time	Asymptotic	Discontinuous
$u = u_1 + u_2,$ $u_1 = \rho_1 \text{ sign}(s), u_2 = -\int \sigma dt$ $\sigma = \dot{e} + ce$ $s = \sigma - z, \dot{z} = -u_2$	integral	Asysmptotic	Asymptotic	Discontinuous
$u = c\ \lvert\sigma\rvert^{\frac{1}{2}} \text{ sign}(\sigma) + w$ $\dot{w} = b \text{ sign}(\sigma)$ $\sigma = \dot{e} + ce$	Super-twisting	Finite time	Asymptotic	Continuous
$u = -\rho \text{ sign}(\sigma)$ $\sigma = \dot{e} + c\lvert e\rvert^{\frac{1}{2}}\text{sign}(e)$	prescribed convergence law	Finite time	Finite time	Discontinuous

Bibliography

[1] Y. Shtessel, C. Edwards, L. Fridman, and A. Levant, *Sliding Mode Control and Observation*. in Control Engineering. New York, NY: Springer New York, 2014. doi: 10.1007/978-0-8176-4893-0.