Supplement for Nonlinear Systems and Control

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Work In Progress

Contents

Some exercises are mentioned in the textbook's mainbody. So I organize some solutions here for reference. Only a little solutions is presented in this supplement. They are 1.1 3.24 5.6 .

$\boxed{\textbf{Math Review}}$

1 Order Linear Differential Equations

- “https://www.sfu.ca/math-coursenotes/Math%20158%20Course%20Notes/chap_DifferentialEquations.html

1.1 Homogeneity of a Linear DE

Given a linear differential equation

$$F_{n(x)}\frac{d^n y}{dx^n} + F_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + F_2(x)\frac{d^2 y}{dx^2} + F_1(x)\frac{dy}{dx} + F_0(x)y = G(x) \tag{1}$$

where $F_{i(x)}$ and $G(x)$ are functions of x, the differential equation is said to be **homogeneous** if $G(x) = 0$ and **non-homogeneous** otherwise.

Note: One implication of this definition is that $y = 0$ is a constant solution to a linear homogeneous differential equation, but not for the non-homogeneous case.

1.2 First Order Linear Differential Equations

Given a first order non-homogeneous linear differential equation

$$y' + p(t)y = f(t) \tag{2}$$

using variation of parameters the general solution is given by

$$y(t) = v(t)e^{P(t)} + Ae^{P(t)} \tag{3}$$

where $v'(t) = e^{-P(t)}f(t)$ and $P(t)$ is an antiderivative of $-p(t)$

1.3 Numerical methods for ode

The closed-loop control system is usually written as

$$\dot{x} = f(t, x). \tag{4}$$

To verify the control performance, several numerical method is important.

- https://www.math.hkust.edu.hk/~machas/numerical-methods-for-engineers.pdf

1.3.1 Euler method – First Order

$$x_{n+1} = x_n + \Delta t f(t_n, x_n) \tag{5}$$

For small enough Δt, the numerical solution should converge to the exact solution of the ode, when such a solution exists. The Euler Method has a local error, that is, the error incurred over a single time step, of $O(\Delta t^2)$. The global error, however, comes from integrating out to a time T. If this integration takes N time steps, then the global error is the sum of N local errors. Since $N = \frac{T}{\Delta t}$, the global error is given by $O(\Delta t)$, and it is customary to call the Euler Method a first-order method.

1.3.2 Modified Euler,Heun’s method,predictor-corrector method – Second Order

$$\begin{gathered} k_1 = \Delta t f(t_n, x_n) \quad k_2 = \Delta t f(t_n + \Delta t, x_n + k_1) \\ x_{n+1} = x_n + \frac{1}{2}(k_1 + k_2) \end{gathered} \tag{6}$$

1.3.3 Runge-Kutta methods

First, we compute the Taylor series for x_{n+1} directly:

$$x_{n+1} = x(t_n + \Delta t) = x(t_n) + \Delta t \dot{x}(t_n) + \frac{1}{2}(\Delta t)^2 \ddot{x}(t_n) + O(\Delta t^3) \tag{7}$$

Now, $\dot{x}(t_n) = f(t_n, x_n)$. The second derivative is more tricky and requires partial derivatives. We have

$$\ddot{x}(t_n) = \frac{d}{dt} f(t, x(t))\bigg|_{t=t_n} = f_t(t_n, x_n) + \dot{x}(t_n) f_x(t_n, x_n) = f_t(t_n, x_n) + f(t_n, x_n) f_x(t_n, x_n) \quad (8)$$

Putting all the terms together, we obtain

$$x_{n+1} = x_n + \Delta t f(t_n, x_n) + \frac{1}{2}(\Delta t)^2 \left(f_t(t_n, x_n) + f(t_n, x_n) + f(t_n, x_n) f_{x(t_n, x_n)} \right) + O(\Delta t^3) \quad (9)$$

Second, we compute the Taylor series for x_{n+1} from the Runge-Kutta formula. We start with

$$x_{n+1} = x_n + a \Delta t f(t_n, x_n) + b \Delta t f(t_n + \alpha \Delta t, x_n + \beta \Delta t f(t_n, x_n)) + O(\Delta t^3) \quad (10)$$

and the Taylor series that we need is

$$\begin{aligned} & f(t_n + \alpha \Delta t, x_n + \beta \Delta t f(t_n, x_n)) \\ &= f(t_n, x_n) + \alpha \Delta t f_{t(t_n, x_n)} + \beta \Delta t f(t_n, x_n) f_{x(t_n, x_n)} + O(\Delta t^2) \end{aligned} \quad (11)$$

The Taylor-series for x_{n+1} from the Runge-Kutta method is therefore given by

$$x_{n+1} = x_n + (a + b) \Delta t f(t_n, x_n) + (\Delta t)^2 \left(\alpha b f_{t(t_n, x_n)} + \beta b f(t_n, x_n) f_{x(t_n, x_n)} \right) + O(\Delta t^3) \quad (12)$$

Comparing (9) and (12), we find three constraints for the four constants.

$$a + b = 1, \alpha b = 1/2, \beta b = 1/2 \quad (13)$$

1.3.4 Second-order Runge-Kutta methods

The family of second-order Runge-Kutta methods that solve $\dot{x} = f(t, x)$ is given by

$$\begin{gathered} k_1 = \Delta t f(t_n, x_n), \quad k_2 = \Delta t (f_n + \alpha \Delta t, x_n + \beta k_1), \\ x_{n+1} = x_n + a k_1 + b k_2 \end{gathered} \quad (14)$$

where we have derived three constraints for the four constants α,β,a and b:

$$a + b = 1, \alpha b = \frac{1}{2}, \beta b = \frac{1}{2} \quad (15)$$

The modified Euler method corresponds to $\alpha = \beta = 1$ and $a = b = \frac{1}{2}$. The function $f(t, x)$ is evaluated at the times $t = t_n$ and $t = t_n + \Delta t$.

The midpoint method corresponds to $\alpha = \beta = \frac{1}{2}$, $a = 0$ and $b = 1$. In this method, the function $f(t, x)$ is evaluated at the times $t = t_n$ and $t = t_n + \Delta t/2$ and we have

$$\begin{gathered} k_1 = \Delta t f(t_n, x_n) \quad k_2 = \Delta t f \left(t_n + \frac{1}{2} \Delta t, x_n + \frac{1}{2} k_1 \right), \\ x_{n+1} = x_n + k_2 \end{gathered} \quad (16)$$

1.3.5 Higher Order Runge-Kutta methods

Higher-order Runge-Kutta methods can also be derived, but require substantially more algebra. For example, the general form of the third-order method is given by

$$\begin{aligned}k_1 &= \Delta t f(t_n, x_n),\\ k_2 &= \Delta t f(t_n + \alpha\Delta t, x_n + \beta k_1),\\ k_3 &= \Delta t f(t_n + \gamma\Delta t, x_n + \delta k_1 + \varepsilon k_2),\\ x_{n+1} &= x_n + a k_1 + b k_2 + c k_3\end{aligned} \tag{17}$$

with constraints α,β,γ,δ,ϵ,a,b and c. The foutth-order method has stages k_1,k_2,k_3 and k_4. The fifth-order methood requires at least six stages. The table below gives the order of the method and the minimum number of stages required.

order	2	3	4	5	6	7	8
minimum #stage	2	3	4	6	7	9	11

Because the fifth-order method requires two more stages than the fourth-order method, the fourth-order method has found some popularity. The general fouth-order method with four stages has 13 constants and 11 constraints. A particularly simple fourth-order method that has been widely used in the past by physicsts ig given by

$$\begin{aligned}k_1 = \Delta t f(t_n, x_n), \qquad\qquad & k_2 = \Delta t\left(t_n + \frac{1}{2}\Delta t, x_n + \frac{1}{2}k_1\right),\\ k_3 = \Delta t f\left(t_n + \frac{1}{2}\Delta t, x_n + \frac{1}{2}k_2\right), & k_4 = \Delta t f(t_n + \Delta t, x_n + k_3);\end{aligned} \tag{18}$$

$$x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{19}$$

1.3.6 Adaptive Runge-Kutta methods

An adaptive ode solver automatically finds the best integration step-size Δt at each time step. The Dormand-Prince method, which is implemented in MATLAB's most widely used solver, `[t,y,te,ye,ie] = ode45(odefun,tspan,y0,options)`, determines the step size by comparing the results of fourth- and fifth- order Runge-Kutta methods. This solver requires six function evaluations per time step, and saves computational time by constructing both fourth- and fifth-order methods using the same function evaluation's.

1.3.7 stiff ODE

- https://ww2.mathworks.cn/help/matlab/math/solve-stiff-odes.html?lang=en

For some ODE problems, the step size taken by the solver is forced down to an unreasonably small level in comparison to the interval of integration, even in a region where the solution curve is smooth. **These step sizes can be so small that traversing a short time interval might require millions of evaluations**. This can lead to the solver failing the integration, but even if it succeeds it will take a very long time to do so.

Equations that cause this behavior in ODE solvers are said to be stiff. The problem that stiff ODEs pose is that explicit solvers (such as ode45) are untenably slow in achieving a solution. This is why ode45 is classified as a nonstiff solver along with ode23, ode78, ode89, and ode113.

Solvers that are designed for stiff ODEs, known as stiff solvers, typically do more work per step. The pay-off is that they are able to take much larger steps, and have improved numerical stability compared to the nonstiff solvers.

2 Topological Space

- 拓扑学 http://staff.ustc.edu.cn/~wangzuoq/Courses/22S-Topology/

2.1 Metic Space

A metric space (M, d) consists of a set M and a mapping, called distance, $d : M \times M \to \mathbb{R}$, which satisfies the following:

1. $0 \leq d(x, y) < \infty, \forall x, y \in M$
2. $d(x, y) = 0$, if and only if $x = y$
3. $d(x, y) = d(y, x)$
4. Triangle Inequality: $d(x, z) \leq d(x, y) + d(x, z), x, y, z \in M$

2.2 Topological Spaces

2.3 Continuous Mapping

Definition 2.3.1: Let M, N be two topological spaces. A mapping $\pi : M \to N$ is continuous, if one of the following two equivalent conditions holds:

- For any $U \subset N$ open, its inverse image

$$\pi^{-1}(U) := \{x \in M \mid \pi(x) \in U\} \tag{20}$$

  is open
- For any $C \subset N$ closed, its inverse image $\pi^{-1}(C)$ is closed

Note: the two conditions are equivalent.

Definition 2.3.2: Let M,N be two topological spaces. M and N are said to be homeomorphic(同胚 pei 胚胎) if there exists a mapping $\pi : M \to N$, which is

1. one-to-one(单射 injective),
2. onto(满射 surjective)
3. and continuous (both π and π^{-1} are continuous).

π is called a homeomorphism.

Note: If a mapping is both injective and surjective it is said to bijective(满射).

Definition 2.3.3: Given a topological space M.

1. A set $U \subset M$ is said to be clopen if it is both closed and open. A topological space(度量空间), M, is said to be **connected** if the only two clopen sets are M and $\emptyset$
2. A continuous mapping $\pi : I = [0, 1] \to M$ is called a path on M. M is said to be **pathwise(or arcwise) connected** if for any two points $x, y \in M$ there exists a path, π, such that $\pi(0) = x$ and $\pi(1) = y$

Tip on open and closed set: As described by topologist James Munkres, unlike a door, "a set can be open, or closed, or both, or neither!" https://en.wikipedia.org/wiki/Clopen_set A set is closed if its complement is open. But A set can be closed or open if its complement is closed.
https://en.wikipedia.org/wiki/Open_set
A subset U of a metric space (M, d) is called open if, for any point x in U, there exists a real number ε such that any point $y \in M$ satisfying $d(x, y) < \varepsilon$ belongs to U. Equivalently, U is open if every point in U has a neighborhood contained in U.

Note: $\mathbb{R}$ is connected and A pathwise connected space M is connected while the converse is incorrect.

Definition 2.3.4: A topological space M is said to be *locally connected* at $x \in M$ if every neighborhood N_x of x contains a connected neighborhood U_x, i.e., $x \in U_x \subset N_x$. M is said to be locally connected if it is locally connected at each $x \in M$

local connectedness does not imply connectedness (hence no pathwise connectedness); conversely, pathwise connectedness (connectedness) does not imply local connectedness.

Definition 2.3.5: Let $\{U_\lambda | \lambda \in \Lambda\}$ be a set of open sets in M. The set is called an open covering of M if

$$\bigcup_{\lambda \in \Lambda} U_\lambda \supset M. \tag{21}$$

M is said to be a compact space if every open covering has a finite sub-covering, i.e., there exists a finite subset $\left\{U_{\lambda_i} |\ i = 1, 2, ..., k\right\}$ such that

$$\bigcup_{i=1}^{k} U_{\lambda_i} \supset M \tag{22}$$

From Calculus we know that with the conventional topology, a set, $U \subset \mathbb{R}^n$ is compact, if and only if it is bounded and closed. Unfortunately, it is not true for general metric spaces.

Definition 2.3.6: In a topological space, M, a sequence $\{x_k\}$ is said to converge to x, if for any neighborhood $U \ni x$ there exists a positive integer $N > 0$ such that when $n > N$, $x_n \in U$.

1. Impose the discrete topology on $\mathbb{R}^1$. i.e., each point is an open set. Then x_k converges to nowhere. Because it can never get into any $\{r\}$, which is a neighborood of r

Proposition 2.3.1: Let M, N be two topological spaces. M is first countable. $f : M \to N$ is continuous, if and only if for each $x_k \to x$, $f(x_k) \to f(x)$

Definition 2.3.7: A topological space is called sequentially compact if every sequence contains a convergent subsequence.

Definition 2.3.8: (**Bolzano-Weierstrass**) Let M be a first countable topological space. if M is compact, it is sequentially compact.

2.4 Quotient Spaces

3 Differentiable Manifold

Definition 3.1: Let $(M, \mathcal{T})$ be a second coutable, T_2(Hausdorff) topological space. M is called an n dimensional topological manifold if there exists a subset $\mathcal{A} = \{A_\lambda \mid \lambda \in \Lambda\} \subset \mathcal{T}$, such that

1. $\bigcup_{\lambda \in \Lambda} A_\lambda \supset M$;
2. For each $U \in \mathcal{A}$ there exists a homeomorphism $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$, which is called a coordinate chart, denoted by (U, φ).
3. Moreover, if for two coordinate charts:

(U, φ) and (V, Ψ), if $U \cap V$ is not empty, then both $\Psi \circ \varphi^{-1} : \varphi(U \cap V) \to \Psi(U \cap V)$ and $\varphi \circ \Psi^{-1} : \Psi(U \cap V) \to \varphi()$

3.1 Structure of Manifolds

3.2 Fiber Bundle

3.3 Vector Field

3.4 One Parameter Group

3.5 Lie Algebra of Vector Fields

3.6 Co-tangent Space

3.7 Lie Derivatives

3.8 Frobenius' Theory

3.9 Lie Series, Chow's Theorem

3.10 Tensor Field

3.11 Riemannian Geometry

3.12 Symplectic Geometry

Solutions

chapter 1 Introduction

Exercise 1.1: A mathematical model that describes a wide variety of physical nonlinear systems is the nth-order differential equation

$$y^{(\text{n})} = g\left(t, y, \dot{y}, ..., y^{(\text{n-1})}, u\right) \tag{23}$$

where u and y are scalar variables. With u as input and y as output, find a state model.

Solution:

Let $x_1 = y$,$x_2 = y^{(1)}$,...,$x_n = y^{(\text{n-1})}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_{\text{n-1}} &= x_n \\ \dot{x}_n &= g(t, x_1, ..., x_n, u) \\ y &= x_1 \end{aligned} \tag{24}$$

chapter 2 Second Order Systems

chapter 3 Fundamental Properties

Exercise 3.24 : Let $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Suppose that $V(t, 0) = 0$ for all $t \geq 0$ and

$$V(t, x) \geq c_1 \|x\|^2; \left\| \frac{\partial V}{\partial x}(t, x) \right\| \leq c_4 \|x\|, \forall (t, x) \in [0, \infty) \times D \tag{25}$$

where c_1 and c_4 are positive constants and $D \subset \mathbb{R}^n$ is a convex domain that contains the origin $x = 0$

1. Show that $V(t, x) \leq \frac{1}{2} c_4 \|x\|^2$ for all $x \in D$.
   Hint: Use the representation $V(t, x) = \int_0^1 \frac{\partial V}{\partial x}(t, \sigma x) d\sigma x$
2. Show that the constants c_1 and x_4 must satisfy $2c_1 \leq c_4$
3. Show that $W(t, x) = \sqrt{V(t, x)}$ satisfies the Lipschitz condition

$$|W(t, x_2) - W(t, x_1)| \leq \frac{c_4}{2\sqrt{c_1}} \|x_2 - x_1\|, \forall t \geq 0, \forall x_1, x_2 \in D \tag{26}$$

Solution to 1

$$V(t, x) = \int_0^1 \frac{\partial V}{\partial V}(t, \sigma x) dx \leq \int_0^1 \left\| \frac{\partial V}{\partial x}(t, \sigma x) \right\| \|x\| d\sigma \leq \int_0^1 c_4 \sigma d\sigma \|x\|^2 \leq \frac{1}{2} c_4 \|x\|^2 \tag{27}$$

Solution to 2

Since

$$c_1 \|x\|^2 \leq V(t, x) \leq \frac{1}{2} c_4 \|x\|^2, \forall x \in D \tag{28}$$

we must have $c_1 \leq \frac{1}{2} c_4$

Solution to 3

Consider two ponts x_1 and x_2 such that $\alpha x_1 + (1-\alpha)x_2 \neq 0$ for all $0 \leq \alpha \leq 1$; that is, the orgin does not lie on the line connecting x_1 and x_2. The Jacobian $[\partial W/\partial x]$ is defined for every $x = \alpha x_1 + (1-\alpha)x_2$ and given by

$$\frac{\partial W}{\partial x}(t,x) = \frac{1}{2\sqrt{V(t,x)}}\frac{\partial V}{\partial x}(t,x) \tag{29}$$

By the mean value theorem, there is $\alpha^* \in (0,1)$ such that, with $z = \alpha^* x_1 + (1-\alpha^*)x_2$

$$W(t,x_2) - W(t,x_1) = \frac{\partial W}{\partial x}(t,z)(x_2 - x_1) = \frac{1}{2\sqrt{V(t,z)}}\frac{\partial V}{\partial x}(t,z)(x_2 - x_1) \tag{30}$$

Hence

$$|W(t,x_2) - W(t,x_1)| \leq \frac{1}{2\sqrt{c_1}\|z\|} \tag{31}$$

Consider now the case when the origin lies on the line connecting x_1 and x_2; that is , $0 = \alpha_0 x_1 + (1-\alpha_0)x_2$ for some $\alpha_0 \in [0,1]$. We have

$$\begin{aligned} |W(t,x_2) - W(t,0)| = |W(t,x_2)| = \sqrt{V(t,x_2)} \leq \sqrt{\frac{c_4}{2}}\|x_2\| \\ |W(t,x_1) - W(t,0)| = |W(t,x_1)| = \sqrt{V(t,x_1)} \leq \sqrt{\frac{c_4}{2}}\|x_1\| \end{aligned} \tag{32}$$

$$|W(t,x_2) - W(t,x_1)| = |W(t,x_2) - W(t,0) + W(t,0) - W(t,x_1)| \leq \sqrt{\frac{c_4}{2}}(\|x_1\| + \|x_2\|)$$

Since the origin lies on the line connecting x_1 and x_2, we have $\|x_2\| + \|x_1\| = \|x_2 - x_1\|$. We also have $1 \leq \sqrt{c_4/2c_1}$. Therefore,

$$|W(t,x_2) - W(t,x_1)| \leq \frac{c_4}{2\sqrt{c_1}}\|x_2 - x_1\| \tag{33}$$

chapter 4 Lyapunov Stability

chapter 5 Input-Output Stability

Exercise 5.6: Verify that $D_+W(t)$ satisfies (34)(5.12 in textbook) when $V(t,x(t)) = 0$.

$$D_+W \leq \frac{c_4 L}{2\sqrt{c_1}}\|u(t)\| \tag{34}$$

Hint: Using Exercise 3.24, show that

$$V(t+h, x(t+h)) \leq c_4 h^2 L^2 \|u\|^2/2 + ho(h) \tag{35}$$

where $\frac{o(h)}{h} \to 0$ as $h \to 0$. Then apply $c_4 \geq 2c_1$

From textbook, the system is

$$\begin{aligned} \dot{x} = f(t,x,u), x(0) = x_0 \\ y = h(t,x,u) \end{aligned} \tag{36}$$

From $V(t, x(t)) = 0$ and (28), we have $x(t) = 0$

Let $V(t, x(t)) = 0$.

$$\begin{aligned} D_+W &= \limsup_{h\to 0^+} \frac{1}{h}[W(t+h, x(t+h)) - W(t, x(t))] \\ &= \limsup_{h\to 0^+} \frac{1}{h}\sqrt{V(t+h, x(t+h))} \end{aligned} \tag{37}$$

From (28), We have

$$V(t+h, x(t+h)) \le \frac{c_4}{2}\|x(t+h)\|^2 \tag{38}$$

From textbook 5.9, we have

$$\|f(t, x(t), u) - f(t, x(t), 0)\| \le L\|u\| \tag{39}$$

Use Taylor Series:

$$\begin{aligned} x(t+h) &= f(t, x, u)h + o(h) \\ \Rightarrow \|x(t+h)\|^2 &\le (\|f(t, x, u)\|h + \|o(h)\|)^2 \end{aligned} \tag{40}$$

$$\frac{1}{h^2}V(t+h, x(t+h)) \le \frac{c_4}{2}\left(\frac{\|x(t+h)\|}{h}\right)^2 \le \frac{c_4}{2}\left(\|f(t, x, u)\| + \frac{\|o(h)\|}{h}\right)^2 \tag{41}$$

$$\limsup_{h\to 0^+} \frac{1}{h}\sqrt{V(t+h, x(t+h))} \le \sqrt{\frac{c_4}{2}}\|f(t, x, u)\| \le \sqrt{\frac{c_4}{2}}L\|u\| \tag{42}$$

since $\sqrt{c_4/(2c_1)} \ge 1$. Thus

$$D_+W \le \sqrt{\frac{c_4}{2}}L\|u\|\sqrt{c_4/(2c_1)} = \frac{c_4 L}{2\sqrt{c_1}}\|u(t)\| \tag{43}$$

which agrees with the right hand side of (34)

chapter 6 Passivity

chapter 7 Frequency Domain analysis of Feedback Systems

chapter 8 Advanced Stability Analysis

chapter 9 Stability of Perturbed Systems

chapter 10 Perturbation Theory and Averaging

chapter 11 Singular Perturbations

chapter 12 Feedback Control

chapter 13 Feedback Linearization

chapter 14 Nonlinear Design Tools