Table 1: solutions to ternary differential equations

| differetial equation | differetial inclusion | classical solution | caratheodory solution | Filippov solution |
|---|---|---|---|---|
| $\dot{x} = \begin{cases} 1 & \text{if } x<0 \\ a & \text{if } x=0 \\ -1 & \text{if } x>0 \end{cases}$ | $\dot{x} \in \mathcal{F}(x) = \begin{cases} 1 & \text{if } x<0 \\ [-1,1] & \text{if } x=0 \\ -1 & \text{if } x>0 \end{cases}$ | Only when $a = 0$, classical solution exists.<br>The maximal classical solution is<br>1. if $x(0) > 0$, $x_1(t) = x(0) - t, t < x(0)$<br>2. if $x(0) < 0$, $x_2(t) = x(0) + t, t < -x(0)$<br>3. if $x(0) = 0$, $x_3(t) = 0, t \in [0, \infty)$ | Only when $a = 0$, caratheodory solution exists.<br>The maximal classical solution is<br>1. if $x(0) > 0$, $x_1(t) = \max(x(0) - t, 0), t \in [0, \infty)$<br>2. if $x(0) < 0$, $x_2(t) = \min(x(0) + t, 0), t \in [0, \infty)$<br>3. if $x(0) = 0$, $x_3(t) = 0, t \in [0, \infty)$<br>**Note**:These only absolutely continuous (not continuously differentiable) | Whatever the value of $a$ is, the Filippov solution is<br>1. if $x(0) > 0$, $x_1(t) = \max(x(0) - t, 0), t \in [0, \infty)$<br>2. if $x(0) < 0$, $x_2(t) = \min(x(0) + t, 0), t \in [0, \infty)$<br>3. if $x(0) = 0$, $x_3(t) = 0, t \in [0, \infty)$ |
| $\dot{x} = \begin{cases} -1 & \text{if } x<0 \\ a & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$ | $\dot{x} \in \mathcal{F}(x) = \begin{cases} -1 & \text{if } x<0 \\ [-1,1] & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$ | From $x = x(0) \neq 0$,classical solution exists as<br>1. $x_1(t) = x(0) + t$ if $x(0) > 0$<br>2. $x_2(t) = x(0) - t$ if $x(0) < 0$<br>From $x = x(0) = 0$, classical solution exists when $a = 1$ or $a = -1$<br>1. when $a = 1$, $x_1(t) = t, t \in [0, \infty)$<br>2. when $a = -1$,$x_2(t) = -t, t \in [0, \infty)$ | From $x = x(0) \neq 0$,classical solution exists as<br>1. $x_1(t) = x(0) + t$ if $x(0) > 0$<br>2. $x_2(t) = x(0) - t$ if $x(0) < 0$.<br>From $x = x(0) = 0$, two caratheodory solutions exist for **all** $a \in \mathbb{R}$<br>1. $x_1(t) = t, t \in [0, \infty)$<br>2. $x_2(t) = -t, t \in [0, \infty)$<br>These two solutions only violate the vector field in $t = 0$ | Filippov solution exists for all $a \in \mathbb{R}$ and $x(0) \in \mathbb{R}$.<br>1. if $x(0) \geq 0$, $x_1(t) = x(0) + t, t \in [0, \infty)$<br>2. if $x(0) \leq 0$,, $x_2(t) = x(0) - t, t \in [0, \infty)$<br>**Note**: When $x(0) = 0$, exists two Filippov solutions. |
| $\dot{x} = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ | $\dot{x} \in \{1\}$ | $x = 0, t \in [0, \infty)$ | two caratheodory solutions:<br>1. $x(t) = 0, t \in [0, \infty)$<br>2. $x(t) = t, t \in [0, \infty)$ | one unique solution:<br>1. $x(t) = t, t \in [0, \infty)$ |

Table 2: conditions of solutions to $\dot{x} = X(x(t))$

| | solution | existence | uniqueness |
|---|---|---|---|
| classical | continuously differentiable | $X : \mathbb{R}^d \to \mathbb{R}^d$ is continuous | essentially one-sided Lipschitz on $B(x, \varepsilon)$,[1] |
| Filippov | absolutely continuous | $X : \mathbb{R}^d \to \mathbb{R}^d$ is measurable and locally essentially bounded | essentially one-sided Lipschitz on $B(x, \varepsilon)$ |

[1]Every vector field that is locally Lipschitz at $x$ satisfies the one-sided Lipschitz condition on a neighborhood of $x$, but the converse is not true.