Sliding Mode Control for Integrator Systems

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1 Linear feedback for single integrator and variants

1.1 Stabiliztion of Integrator Systems via linear feedback

From the linear control theory, we know the integrator system $x^{(n)} = u$ can be stabilized with linear feedback $u = -k_n x - k_{n-1}\dot{x} + ... - k_1 x^{(n-1)}$. then the closed loop system expressed in differential equation is

$$x^{(n)} + k_1 x^{(n-1)} + k_2 x^{(n-2)} + ... + k_{n-1}\dot{x} + k_n x = 0. \tag{1}$$

With Laplace transformation, the system can be expressed with

$$p(s) = s^n + k_1 s^{n-1} + k_2 s^{n-2} + ... + k_{n-1} s + k_n = 0. \tag{2}$$

To make sure the system is stable,

- all roots of $p(s) = 0$ must have negative real parts.
- the polynomial $p(s)$ must be a **Hurwitz** polynomial.
- The system must satisfy **Routh–Hurwitz stability criterion**
- The system matrix A associated with k_i must be a **Hurwitz matrix**: $\mathrm{Re}[\mathrm{eig}(A)] < 0$

Note: $k_i > 0$ is a necessary condition for the system to be stable.

First, we analyse the single integrator system:

$$\dot{x} = u \tag{3}$$

which can be stabilized with linear feedback $u = -kx$. Under linear feedback, the exponential stability will be observed.

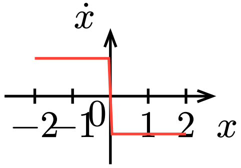
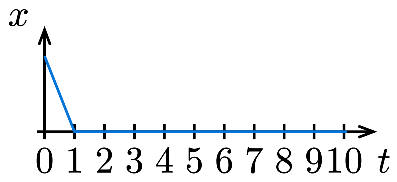
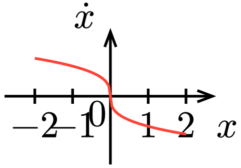
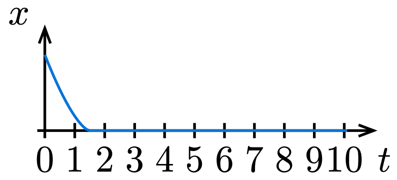
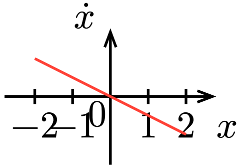
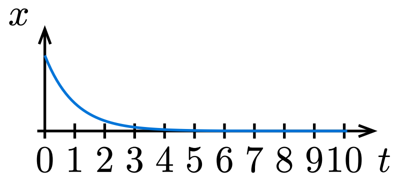
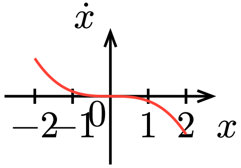
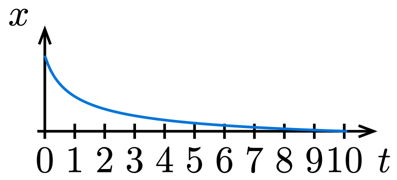
1.2 Stabiliztion of Single Integrator Systems via $-|x|^v \operatorname{sign}(x)$

From (POLYAKOV, 2020), we can find solutions of the system $\dot{x} = -x^v$ has some good properties. Due to the definition of power function, usually we extend the function's definition to the whole rational field $\mathbb{R}$ as $\dot{x} = -\operatorname{sign}(x)\ |x|^v, v \geq 0$. In some books like (POLYAKOV, 2020), a condition on v is used.[1] The general solution to this system is

$$x(t) = \left(x(0)^{-v+1} + (v-1)t\right)^{\frac{1}{-v+1}} \operatorname{sign}(x(0)) = \frac{x(0)}{\left(1 + (v-1)t|x(0)|^{v-1}\right)^{\frac{1}{v-1}}} \text{ if } v \neq 1. \tag{4}$$

[1]I haven't figure out this condition yet. The condition write v as $v = \frac{p}{q}$ and says p should be an odd integer and q is an even natual number. I think both p and q should be odd natural number and $q \neq 0$.

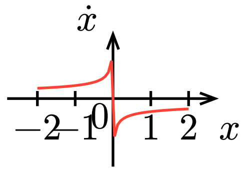
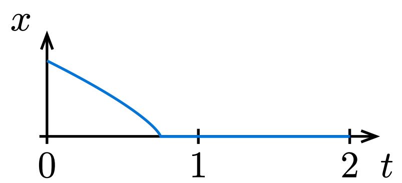
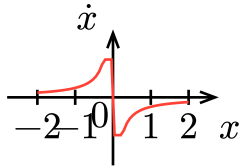
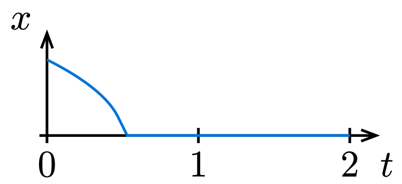
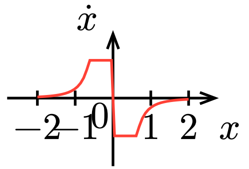
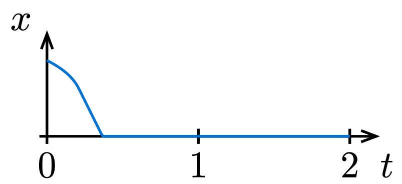
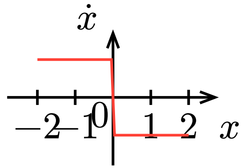
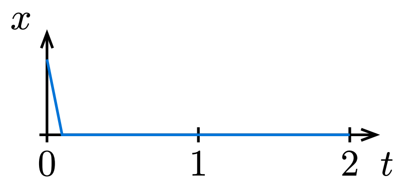
Table 1: fractional power feedback $\dot{x} = -\mathrm{sign}(x)|x|^v, v \geq 0$

system	$x - \dot{x}$ curve	numerical solution $x = x(t)$	analytical solution	stability
$\dot{x} = -x^0 = -\mathrm{sign}(x)$			when $x(0) > 0$ $x(t) = \begin{cases} x(0)-t & t\in(0,x(0)) \\ 0 & t\geq x(0) \end{cases}$	Finite time if $v < 1$
$\dot{x} = -x^{\frac{1}{3}}$			when $x(0) > 0$ $x(t) = \begin{cases} \left(\lvert x(0)\rvert^{\frac{2}{3}}-\frac{2}{3}t\right)^{\frac{3}{2}} & t\in\left(0,(\frac{2}{3})\lvert x(0)\rvert^{\frac{3}{2}}\right) \\ 0 & t\geq x(0) \end{cases}$	Finite time if $v < 1$
$\dot{x} = -x^1 = -x$			$x = x(0)e^{-t}$	Exponential if $= 1$
$\dot{x} = -x^3$			$x(t) = \left(x(0)^{-2} + 2t\right)^{-\frac{1}{2}}$	(practical) Fixed time if $v > 1$[2]

[2]converges to a neighborhood of the origin in a fixed time independent of the initial condition.

1.3 Discussion: negative power feedback

Faster covergence will be found if we use negative power. However, this will need infinite gain (infinite energy). So, it is impossible for physical implementation. Simulations are carried out with a satutated power function for the infinite gain is not possible to simulate.

system	$x - \dot{x}$ curve	numerical solution $x = x(t)$	step	stability
$\dot{x} = -\mathrm{sat}\left(x^{-\frac{1}{3}}\right)$			0.0012	Finite time
$\dot{x} = -\mathrm{sat}(x^{-1}) = -\mathrm{sat}\left(\frac{1}{x}\right)$			0.01	Finite Time
$\dot{x} = -\mathrm{sat}(x^{-3})$			0.01	Finite Time
$\dot{x} = -10 * \mathrm{sign}(x)$			0.1	Finite time

2 Time-varying Gain

2.1 Prescribed Time Stabiliztion of Single Integrator Systems by Time-varying Gain

Generally prescribed/preassigned/pre-appointed time stability is reached by time-varying gain (time-varying scaling function, time-base generator). Following table gives the basic example, we see that the solution for the first case is the same as $\dot{x} = -\mathrm{sign}(x)$. (SONG et al., 2023) .

The system is:

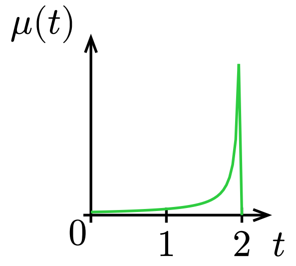
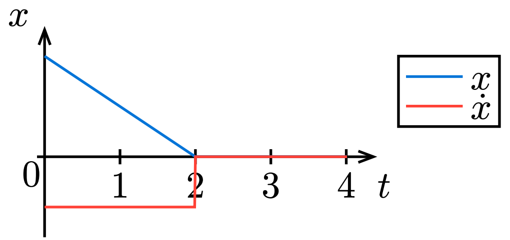
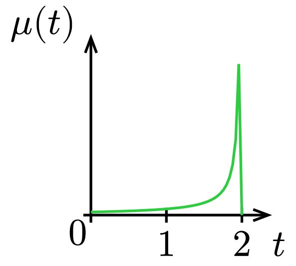
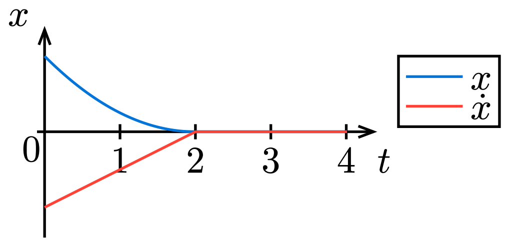
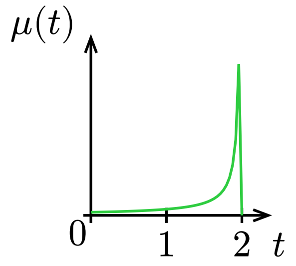
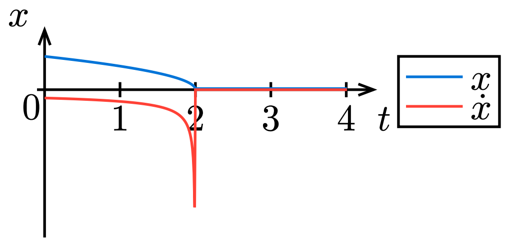
$$\dot{x} = -\mu(t)x, \mu(t) = \begin{cases} \frac{k_1}{(T-t)^h} & 0 < t < T \\ 0 \quad t \geq T \end{cases}, \tag{5}$$

with $T > 1$ to be prescribed and $k_1 > 0, k_2 > 0, h = 1$.
The analytical solution with $h = 1$ can be found easily as:

$$\begin{gathered} x(t) = x(0)\left(\frac{T-t}{T}\right)^{k_1}, t \in [0, T) \\ x(t) = 0, t \in [T, \infty) \end{gathered} \tag{6}$$

.

Table 3: time-varying gain control with prescribed time stability

system	$\mu(t)$	numerical solution	stability
$T = 2$, $h = 1$, $k_1 = 1$			Prescribed Time Stable with T
$T = 2$, $h = 1$, $k_1 = 2$			Prescribed Time Stable with T
$T = 2$, $h = 1$, $k_1 = 0.5$			Prescribed Time Stable with T

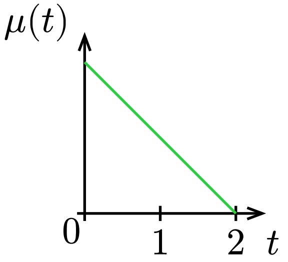
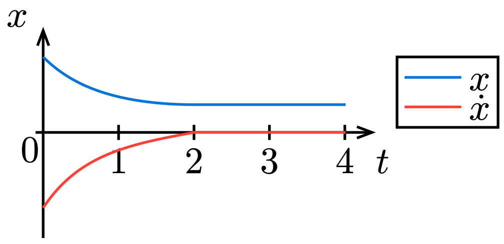
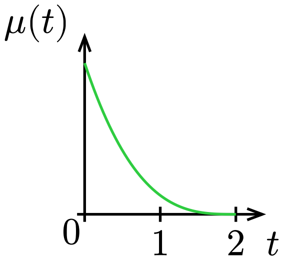
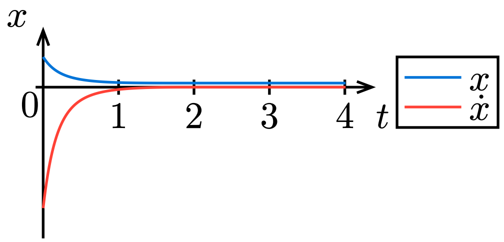
2.2 Discussion: Time-varying Gain with different power

The analytical solution with $h \neq 1$ can be found easily as:

$$
\begin{aligned}
x(t) &= x(0)\exp\left(-\frac{k_1}{-h+1}\left(T^{-h+1} - (T-t)^{-h+1}\right)\right) \\
x(t) &= x(0)\exp\left(-\frac{k_1}{-h+1}T^{-h+1}\right), t \in [T, \infty)
\end{aligned} \tag{7}
$$

. These system can be nearly stable but we can always observe some error.

Table 4: time-varying gain control with similar form without stability

system	$\mu(t)$	numerical solution	stability
$T = 2$, $h = -1$, $k_1 = 0.5$			unstable x(t)=0.3678794411816698 $t \geq T$
$T = 2$, $h = -3$, $k_1 = 0.5$			unstable x(t)=0.13533528470886394 $t \geq T$

3 Control Single Integrator System

3.1 Robustness of $\dot{x} = -k\ \mathrm{sign}(x) + \delta$

Consider the fisrt order system $\dot{x} = u + \delta$, δ is the bounded disturbance $|\delta| < C$. The first control law is

$$u = -k\mathrm{sign}x \tag{8}$$

where $k > C$. Select

$$V = \frac{1}{2}x^2. \tag{9}$$

Calculate the derivative along the system trajectory:

$$\begin{aligned} \dot{V} &= x\dot{x} = x(-k\mathrm{sign}x + \delta) \\ &= -k|x| + x\delta \leq -k|x| + |x|\ |\delta| \\ &\leq -(k - C)|x| \leq 0 \end{aligned} \tag{10}$$

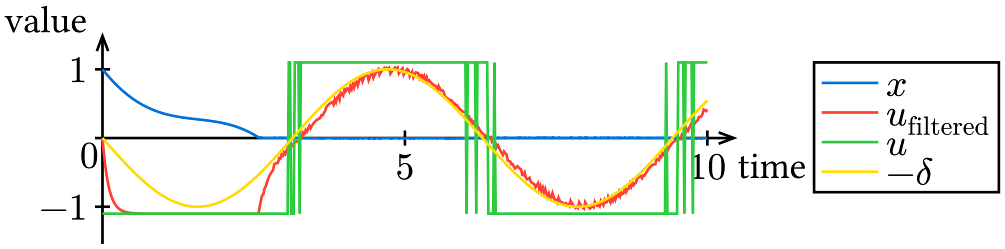
This implies $x \to 0$(Lyapunov direct method) and $\dot{x} \to 0$ (Barbalat's lemma). If so, $-k\ \mathrm{sign}(x) + \delta \to 0$?

This differential equation should be understood in **Pilippov sense**. The solution here is not a classical(continuously differentiable) solution but a absolutely continuous solution. the solution satisfy the **Pilippov DI** (Pilippov Differential Inclusion) of the differential equation.

We use the concept of **Equivalent Control** describes this feature, that is ,$\mathrm{sign}x = \delta$. Use a low pass filter, we can say $\mathrm{LPF}(\mathrm{sign}(x)) \approx \delta$.

For example, the following low pass filter is used in simulation,

$$\begin{gathered} \frac{U_{\mathrm{filtered}(s)}}{U(s)} = \frac{1}{Ts + 1} \\ T\dot{u}_{\mathrm{filtered}} + u_{\mathrm{filtered}} = u \end{gathered} \tag{11}$$



3.2 Finite-time convergence of $\dot{x} = -k\mathrm{sign}(x) + \delta$

Another important feature of this system is finite-time stability. From (9) and (10), we have

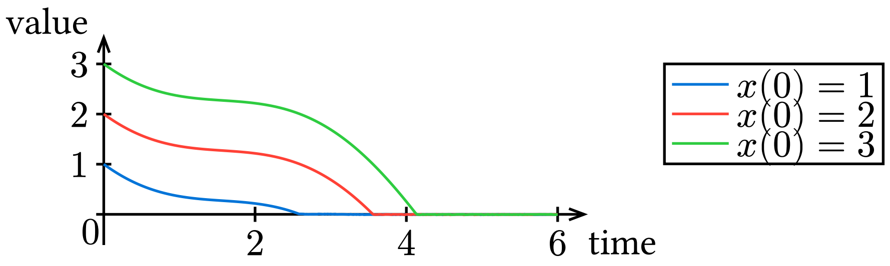
$$\dot{V}(t) \leq -(k - C)\sqrt{2V(t)} \tag{12}$$

Let $a = (k - C)\sqrt{2}$ and integrate it. we get

$$\begin{aligned} \frac{dV(t)}{\sqrt{V(t)}} &\leq -adt \\ 2\sqrt{V(t)} - 2\sqrt{V(0)} &\leq -at \\ \sqrt{V(t)} &\leq \sqrt{V(0)} - \frac{a}{2}t \end{aligned} \tag{13}$$

Consequently, $V(t)$ reaches zero in a finite time t_r that is bounded by

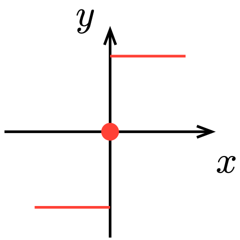
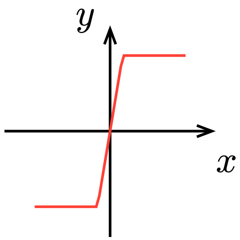
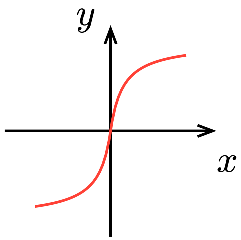
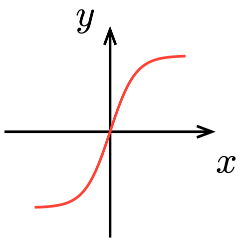
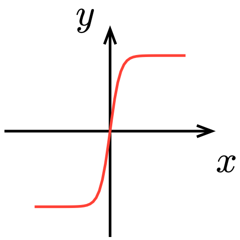
$$t_r \leq 2\frac{\sqrt{V(0)}}{a} = \frac{|x(0)|}{k - C} \tag{14}$$



3.3 SMC Chattering Elimination: Quasi-Sliding Mode

In many practical control systems, including DC motors and aircraft control, it is important to avoid control chattering by providing continuous/smooth signals. One obvious solution to make the control function continuous/smooth is to approximate the discontinuous function $v(\sigma) = -\rho \text{ sign } (\sigma)$ by some continuous/smooth function. For instance, it could be replaced by a "sigmoid function".

Table 5: replaced sign by a "sigmoid function"

	$\text{sign}(x)$	$\text{sat}\left(\frac{x}{\varepsilon}\right)$	$\frac{x}{\lvert x\rvert+\varepsilon}$	$\tanh(x)$	$\frac{1-e^{-Tx}}{1+e^{-Tx}}$
continuity	discontinuous	continuous	smooth[3]	smooth	smooth
		 $\varepsilon = 0.5$	 $\varepsilon = 0.5$		 $T = 5$

[3] I am not sure about this

3.4 SMC Chattering Attenuation: Asymptotic Sliding Mode

Assume $\dot{\delta}$ is also bounded and $\dot{x}$ is measurable. Let $|\dot{\delta}| \leq C_1$. Define $s = x + c\dot{x}$. The control law is given by

$$\begin{aligned} \dot{x} &= u + \delta \\ \dot{u} &= v \\ v &= -\rho \ \mathrm{sign}(s) - \frac{1}{c} u \end{aligned} \tag{15}$$

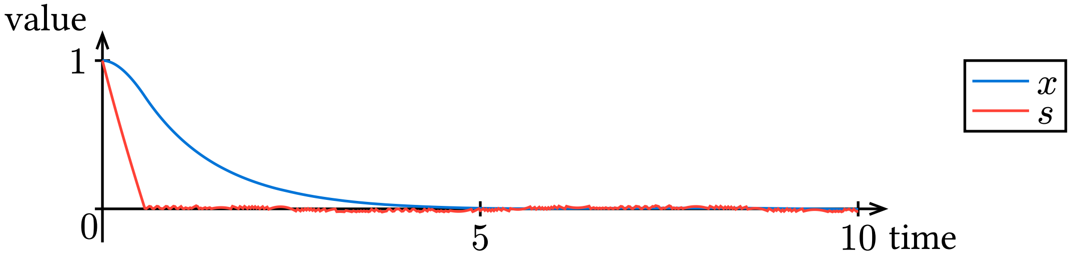
Select

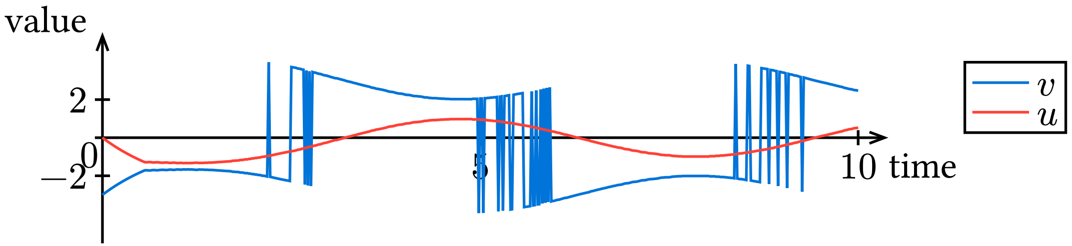
$$V = \frac{1}{2} s^2 \tag{16}$$

Calculate the derivative along the system trajectory:

$$\begin{aligned} \dot{V} &= s\dot{s} = s(\dot{x} + c\ddot{x}) \\ &= s\left(u + \delta + c\left(v + \dot{\delta}\right)\right) \\ &= s\left(-c\rho \ \mathrm{sign}(s) + \delta + c\dot{\delta}\right) \\ &\leq -(c\rho - C - cC_1)|s| \end{aligned} \tag{17}$$

s converges to zero in finite time while x converges to zero asymptotically.





3.5 Integral Sliding Mode Control

Assuming the initial conditions are known, we can split the control with

$$\begin{gathered} \dot{x} = u + \delta \\ u = u_1 + u_2 = -\rho_1 \operatorname{sign}(s) - kx \end{gathered} \tag{18}$$

The auxiliary sliding variable is designed as

$$\begin{cases} s = x - z \\ \dot{z} = u_2 = -kx \end{cases} \tag{19}$$

then

$$\dot{s} = \dot{x} - \dot{z} = u + \delta - u_2 = u_1 - \delta \tag{20}$$

Select

$$u_1 = -\rho_1 \operatorname{sign}(s) \tag{21}$$

Then

$$s = 0 \Rightarrow x = z \Rightarrow u_2 = \dot{z} = \dot{x} \tag{22}$$

So design $u_2 = -kx$ such that $\dot{x} = -kx$

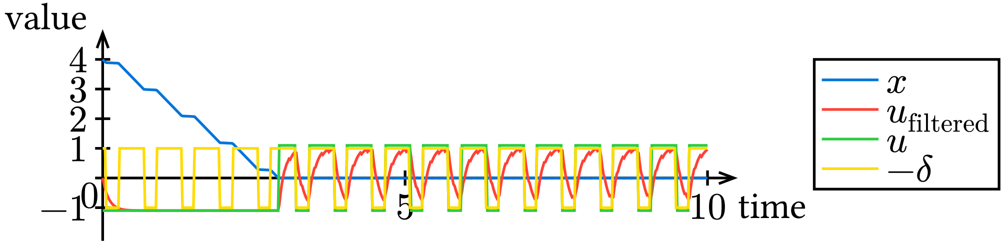
Now we will address the issue of starting the auxiliary sliding mode from the very beginning without any reaching phase. In order to achieve it we have to enforce the initial condition $s(0) = 0$

$$s(0) = 0 \Rightarrow z(0) = x(0) \tag{23}$$

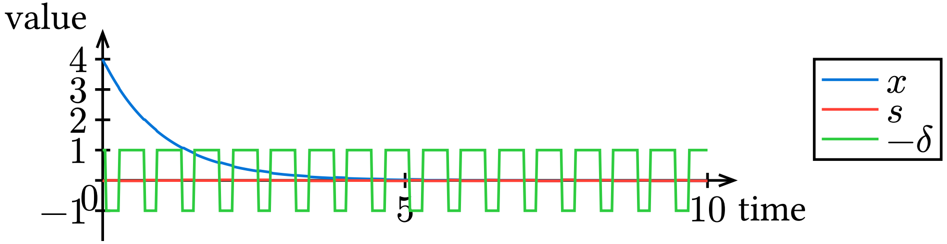
Advantage: We can set $z(0)$ to keep $s(0) = 0$, The system is starting from auxiliary sliding surface.

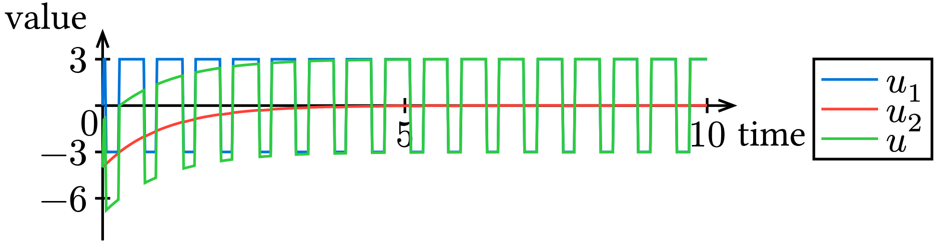
- Elimination of Reaching Phase: The system state always starts on the sliding surface, simplifying control design.
- Improved Robustness: ISMC extends this robustness to the entire state space, making the system less sensitive to uncertainties.
- Guaranteed Stability: Once the sliding mode is achieved, ISMC guarantees the system's stability. This provides a strong theoretical foundation for the control performance.

The first simulation demonstrates the traditional SMC is sensitive to disturbance in reaching phase.



The second one uses integral SMC is





3.6 Super Twist Algorithm (STA)

$$
\begin{aligned}
\dot{x} &= u + \delta \\
u &= -c\ |x|^{\frac{1}{2}}\ \mathrm{sign}(x) - w \\
\dot{w} &= b\ \mathrm{sign}(x)
\end{aligned} \tag{24}
$$

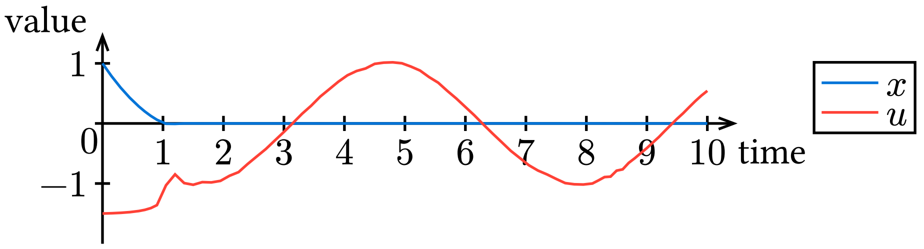
- The super-twisting control is a **second-order** sliding mode control, since it drives both $\sigma \to 0$ and $\dot{\sigma} \to 0$ in finite time. (Second-Order Sliding Mode or 2-SM means the control law drives the sliding variable and its derivative to zero in **finite time**)
- The super-twisting control is **continuous**.

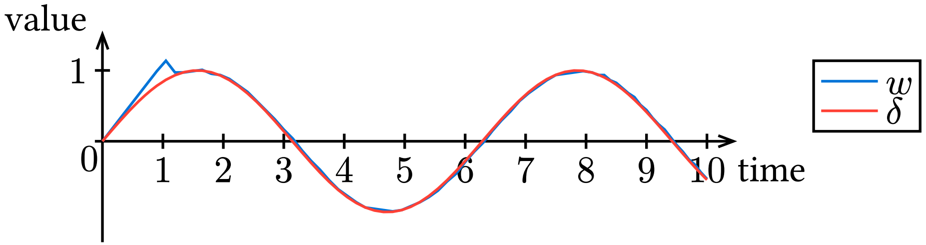
The parameters c and b are quite difficult to select. The parameters given by (SHTESSEL et al., 2014) are

$$
c = 1.5\sqrt{C}, b = 1.1C \tag{25}
$$

(SEEBER et al., 2017) gives a condition

$$
b > C, c > \sqrt{b + C} \tag{26}
$$





3.7 RISE for single integrator

Here we follow the design of RISE (XIAN et al., 2004) and apply them to single integrator.

$$\dot{x} = u + \delta \tag{27}$$

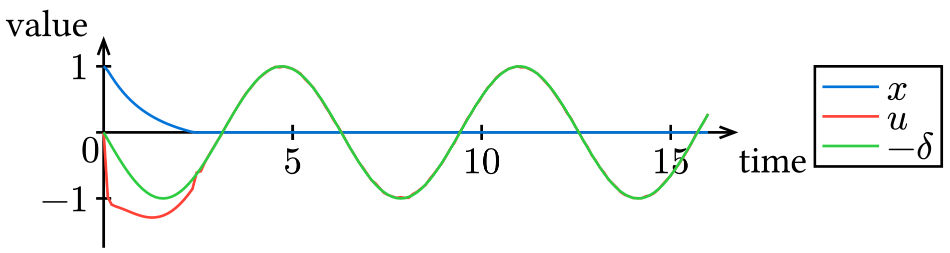
where $\delta \in \mathcal{C}^2$ (both $|\delta|$ and $\left|\dot{\delta}\right|$ is bounded).

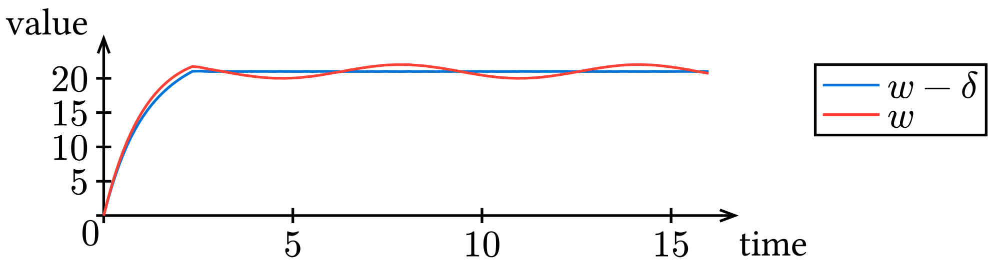
The main idea of RISE is using a Lyapunov function contains δ. Let $L = (\alpha x + \dot{x})\left(\dot{\delta} - \beta \operatorname{sign}(x)\right)$, we calculate

$$\begin{aligned} V &= \frac{1}{2}(\alpha x + \dot{x})^2 + \xi_b - P \\ \dot{V} &= (\alpha x + \dot{x})\left(\alpha \dot{x} + \dot{u} + \dot{\delta}\right) - L \\ &= (\alpha x + \dot{x})\left(\alpha \dot{x} + \dot{u} + \dot{\delta} - \dot{\delta} + \beta \operatorname{sign}(x)\right) \\ &= (\alpha x + \dot{x})(\alpha \dot{x} + \dot{u} + \beta \operatorname{sign}(x)) \end{aligned} \tag{28}$$

so we design

$$\begin{aligned} \dot{u} &= -\alpha \dot{x} - \beta \operatorname{sign}(x) - k_s \alpha(\alpha x + \dot{x}) \\ u &= -(k_s + 1)\alpha x(t) + (k_s + 1)\alpha x(0) - w \\ \dot{w} &= k_s \alpha^2 x + \beta \operatorname{sign}(x) \end{aligned} \tag{29}$$





$$\begin{aligned} P &= \int_0^t L(\tau) d\tau \\ &= \int_0^t \alpha x\left(\dot{\delta} - \beta \operatorname{sign}(x)\right) d\tau + \int_0^t \dot{x}\left(\dot{\delta} - \beta \operatorname{sign}(x)\right) d\tau \\ &= \int_0^t \alpha x\left(\dot{\delta} - \beta \operatorname{sign}(x)\right) d\tau + x\dot{\delta}|_0^t - \int_0^t x\ddot{\delta} d\tau - \int_0^t \dot{x}\beta \operatorname{sign}(x) d\tau \\ &= \int_0^t \alpha x\left(\dot{\delta} - \frac{1}{\alpha}\ddot{\delta}\right) - \alpha\beta\ |x|\ d\tau + x\dot{\delta}\ |_0^t - \beta|x||_0^t \\ &\le \xi_b := \int_0^t \alpha|x|\left(\left|\dot{\delta}\right| + \frac{1}{\alpha}\left|\ddot{\delta}\right| - \beta\right) d\tau + |x(t)|\left(\left|\dot{\delta}(t)\right| - \beta\right) - x(0)\dot{\delta}(0) + \beta|x(0)| \end{aligned} \tag{30}$$

4 Control Algorithm for Double Integrator

4.1 Conventional Sliding Mode Control

The basic idea of Sliding Mode Control is reduce the order of system. Take the double integrator system for example,

$$\ddot{x} = u + d \tag{31}$$

The disturbance d here is called **matched disturbance** which can be eliminated by SMC.

Define a **sliding variable**

$$\sigma = \dot{x} + cx. \tag{32}$$

Calculate the derivative

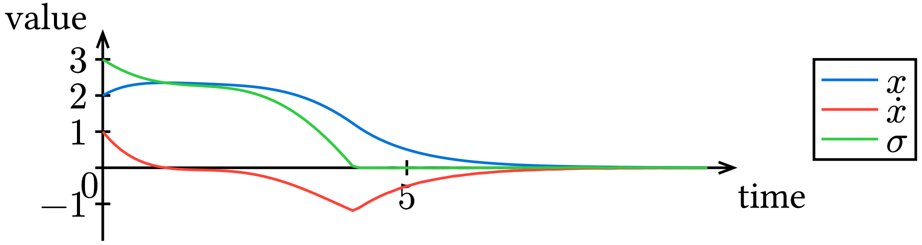
$$\dot{\sigma} = \ddot{x} + c\dot{x} = u + d + cv \tag{33}$$

Design the control input as $u = -cv - k \text{ sign } (\sigma)$. The dynamic of the sliding variable is

$$\dot{\sigma} = -k \text{ sign } (\sigma) + \delta. \tag{34}$$

This means the system state **reaches** the surface $\sigma = 0$ in finite time. Then the system state **slides** in the surface.

The first simulation shows x converges asymptotically.



4.2 Terminal SMC

If we need the system converges in finite time, we can desgin the sliding surface (33) as

$$\begin{gathered}\sigma = \dot{x} + c\lfloor x \rceil^{q} \\ \dot{\sigma} = \ddot{x} + qc\lfloor x \rceil^{q-1}) = u + \delta + qc\lfloor x \rceil^{q-1})\end{gathered} \tag{35}$$

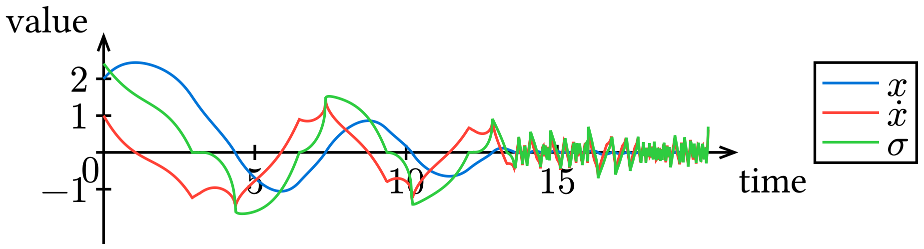
and the corresponding control law is

$$u = -\rho \ \mathrm{sign}\ (\sigma) - qc\lfloor x \rceil^{q-1}) \tag{36}$$

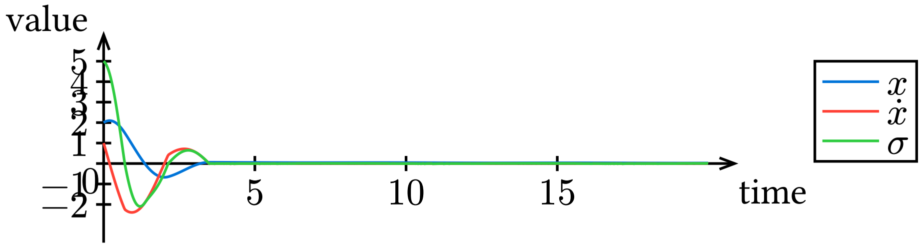
We call this 2-SM(Second Order Sliding Mode).

When $q < 1$, the term $\lfloor x \rceil^{q-1})$ is singular.

q=0.5



q=2



4.3 Second Order Sliding Mode Control

The n-th Order Sliding Mode Control means the relative degree of the sliging variable system is n and the controller drives the system variable to zero in **finite time**.

Consider the system

$$\ddot{x} = \delta + g(x,t)u \tag{37}$$

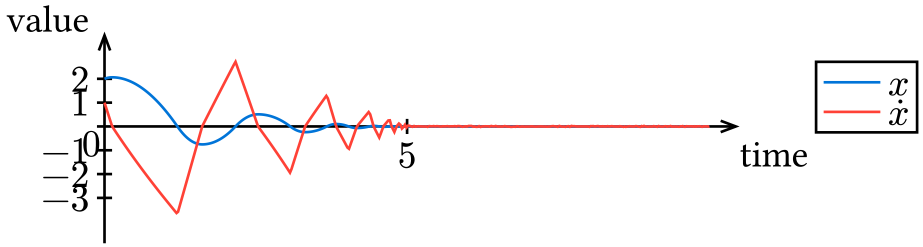
where $|\delta| \leq C$ and $|g(x,t)| \in [K_m, K_M]$.

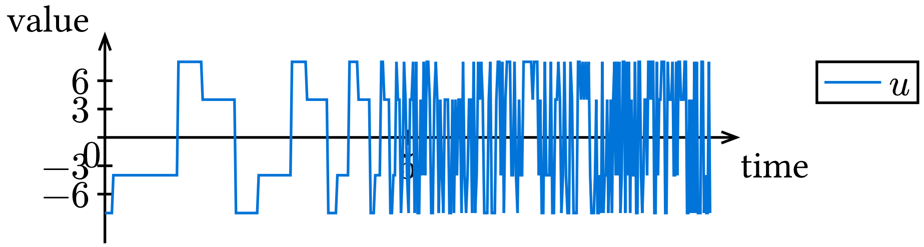
Twisting Control is the typical controller charaterized by

$$u = -k_1 \text{ sign } (x) - k_2 \text{ sign } (v) \tag{38}$$

where $(k_1 + k_2)K_m - C > (k_1 - k_2)K_M + C$, $(k_1 - k_2)K_m > C$.

The first simulation shows x converges asymptotically.





4.4 Robust Integral Sign Error for Double Integrator

(XIAN DAWSON DEQUEIROZ CHEN, 2004) gives the RISE control for high-order integrator. Consider the system

$$\ddot{x} = u - f \tag{39}$$

where

- $f(x)$ are uncertain nonlinear C^2 functions.

Define

$$\begin{aligned} e_1 &= -x \\ e_2 &= \dot{e}_1 + e_1 = -\dot{x} - x \end{aligned} \tag{40}$$

The final control law is

$$\begin{aligned} u = {} & (k_s + 1)e_2(t) - (k_s + 1)e_2(0) \\ & + \int_0^t (k_s + 1)\alpha e_2(\tau) + \beta \text{ sign } e_2(\tau) d\tau \end{aligned} \tag{41}$$

where $\beta > \|N_d(t)\| + \frac{1}{\alpha}\left\|\dot{N}_d(t)\right\|$ and $\alpha > \frac{1}{2}$ and the control gain k_s is selected sufficiently large relative to the system initial conditions.

Take the derivative of u:

$$\begin{aligned} \dot{u} = {} & (k_s + 1)\dot{e}_2(t) \\ & + (k_s + 1)\alpha e_2(t) + \beta \text{ sign } e_2(t) \\ = {} & (k_s + 1)r + \beta \text{ sign } (e_2) \end{aligned} \tag{42}$$

where $r = \dot{e}_2 + \alpha e_2 = \ddot{e}_1 + (1 + \alpha)\dot{e}_1 + \alpha e_1 =$.

Then

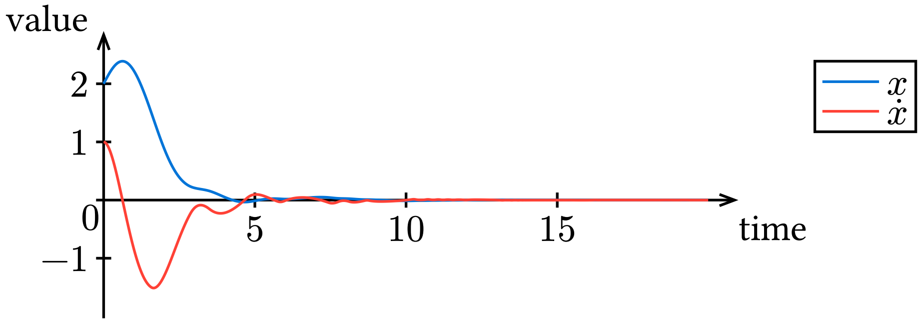
$$\begin{aligned} \dot{r} &= -e_2 - \dot{u} + N \\ N &= \ddot{e}_1 + \alpha \dot{e}_2 + e_2 + \dot{f} \\ \tilde{N} &= \ddot{e}_1 + \alpha \dot{e}_2 + e_2 \\ N_d &= \dot{f} \end{aligned} \tag{43}$$

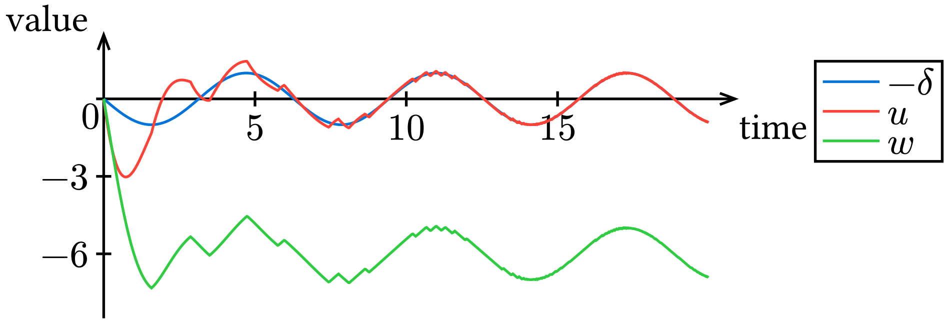
select

$$V = \frac{1}{2}e_2^2 + \frac{1}{2}e_1^2 + \frac{1}{2}r^2 + P \tag{44}$$

where $L := r(N_d - \beta \text{ sgn } (e_2))$ and $P = \xi_b - \int_0^t L(\tau) d\tau$

$$
\begin{aligned}
\dot{V} &= e_2\dot{e}_2 + e_1\dot{e}_1 + r\dot{r} - L \\
&= e_2(r - \alpha e_2) - e_1(e_2 - e_1) \\
&\quad +r(-e_2 - ((k_s+1)r + \beta\ \mathrm{sign}(e_2)) + N) \\
&\quad -r(N_d - \beta\ \mathrm{sign}(e_2)) \\
&= -e_1^2 - \alpha e_2^2 + e_1e_2 + r\tilde{N} - (k_s+1)r^2 \\
&= -e_1^2 - \alpha e_2^2 + \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 + |r|\ \rho(\|z\|)\|z\| - (k_s+1)r^2 \\
&\le -\lambda_3\|z\|^2 + |r|\rho(\|z\|)\|z\| - k_s r^2 \\
&\le -\left(\lambda_3 - \frac{\rho^2(\|z\|)}{4k_s}\right)\|z\|^2
\end{aligned}
\tag{45}
$$





5 Discontinuous System Theory

see (CORTES, 2008)

5.1 Ternary Differential Equations' Solutions

Table 6: solutions to ternary differential equations

differetial equation	**differetial inclusion**	**classical solution**	**caratheodory solution**	**Filippov solution**
$\dot{x} = \begin{cases} 1 & \text{if } x<0 \\ a & \text{if } x=0 \\ -1 & \text{if } x>0 \end{cases}$	$\dot{x} \in \mathcal{F}(x) = \begin{cases} 1 & \text{if } x<0 \\ [-1,1] & \text{if } x=0 \\ -1 & \text{if } x>0 \end{cases}$	Only when $a = 0$, classical solution exists. The maximal classical solution is 1. if $x(0) > 0$, $x_1(t) = x(0) - t, t < x(0)$ 2. if $x(0) < 0$, $x_2(t) = x(0) + t, t < -x(0)$ 3. if $x(0) = 0$, $x_3(t) = 0, t \in [0, \infty)$	Only when $a = 0$, caratheodory solution exists. The maximal classical solution is 1. if $x(0) > 0$, $x_1(t) = \max(x(0) - t, 0), t \in [0, \infty)$ 2. if $x(0) < 0$, $x_2(t) = \min(x(0) + t, 0), t \in [0, \infty)$ 3. if $x(0) = 0$, $x_3(t) = 0, t \in [0, \infty)$ **Note**:These only absolutely continuous (not continuously differentiable)	Whatever the value of a is, the Filippov solution is 1. if $x(0) > 0$, $x_1(t) = \max(x(0) - t, 0), t \in [0, \infty)$ 2. if $x(0) < 0$, $x_2(t) = \min(x(0) + t, 0), t \in [0, \infty)$ 3. if $x(0) = 0$, $x_3(t) = 0, t \in [0, \infty)$
$\dot{x} = \begin{cases} -1 & \text{if } x<0 \\ a & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$	$\dot{x} \in \mathcal{F}(x) = \begin{cases} -1 & \text{if } x<0 \\ [-1,1] & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$	From $x = x(0) \neq 0$,classical solution exists as 1. $x_1(t) = x(0) + t$ if $x(0) > 0$ 2. $x_2(t) = x(0) - t$ if $x(0) < 0$ From $x = x(0) = 0$, classical solution exists when $a = 1$ or $a = -1$ 1. when $a = 1$, $x_1(t) = t, t \in [0, \infty)$ 2. when $a = -1$,$x_2(t) = -t, t \in [0, \infty)$	From $x = x(0) \neq 0$,classical solution exists as 1. $x_1(t) = x(0) + t$ if $x(0) > 0$ 2. $x_2(t) = x(0) - t$ if $x(0) < 0$. From $x = x(0) = 0$, two caratheodory solutions exist for **all** $a \in \mathbb{R}$ 1. $x_1(t) = t, t \in [0, \infty)$ 2. $x_2(t) = -t, t \in [0, \infty)$ These two solutions only violate the vector field in $t = 0$	Filippov solution exists for all $a \in \mathbb{R}$ and $x(0) \in \mathbb{R}$. 1. if $x(0) \geq 0$, $x_1(t) = x(0) + t, t \in [0, \infty)$ 2. if $x(0) \leq 0$,, $x_2(t) = x(0) - t, t \in [0, \infty)$ **Note**: When $x(0) = 0$, exists two Filippov solutions.
$\dot{x} = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$	$\dot{x} \in \{1\}$	$x = 0, t \in [0, \infty)$	two caratheodory solutions: 1. $x(t) = 0, t \in [0, \infty)$ 2. $x(t) = t, t \in [0, \infty)$	one unique solution: 1. $x(t) = t, t \in [0, \infty)$

5.2 Conditions for Existence and Uniqueness of Classical, Caratheodory, Filippov Solutions

Table 7: conditions of solutions to $\dot{x} = X(x(t))$

	solution	existence	uniqueness
classical	continuously differentiable	$X : \mathbb{R}^d \to \mathbb{R}^d$ is continuous	essentially one-sided Lipschitz on $B(x, \varepsilon)$,[4]
Filippov	absolutely continuous	$X : \mathbb{R}^d \to \mathbb{R}^d$ is measurable and locally essentially bounded	essentially one-sided Lipschitz on $B(x, \varepsilon)$

[4] Every vector field that is locally Lipschitz at x satisfies the one-sided Lipschitz condition on a neighborhood of x, but the converse is not true.

THANKS