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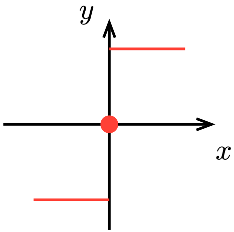
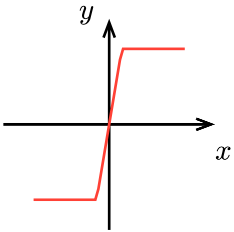
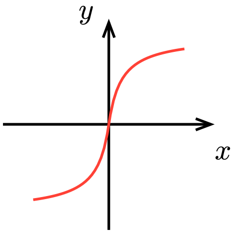
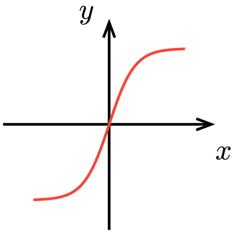
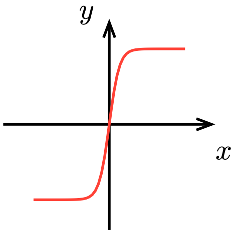
Illustrations for Control Theory

Contents

SMC Chattering Elimination: Quasi-Sliding Mode

In many practical control systems, including DC motors and aircraft control, it is important to avoid control chattering by providing continuous/smooth signals. One obvious solution to make the control function continuous/smooth is to approximate the discontinuous function $v(\sigma) = -\rho \operatorname{sign}(\sigma)$ by some continuous/smooth function. For instance, it could be replaced by a "sigmoid function".

Table 1: replaced sign by a "sigmoid function"

	$\operatorname{sign}(x)$	$\operatorname{sat}\left(\frac{x}{\varepsilon}\right)$	$\frac{x}{\lvert x\rvert+\varepsilon}$	$\tanh(x)$	$\frac{1-e^{-Tx}}{1+e^{-Tx}}$
continuity	discontinuous	continuous	smooth[1]	smooth	smooth
		 $\varepsilon = 0.5$	 $\varepsilon = 0.5$		 $T = 5$

[1] I am not sure about this

Ternary Differential Equations' Solutions

Table 2: solutions to ternary differential equations

differetial equation	**differetial inclusion**	**classical solution**	**caratheodory solution**	**Filippov solution**
$\dot{x} = \begin{cases} 1 & \text{if } x<0 \\ a & \text{if } x=0 \\ -1 & \text{if } x>0 \end{cases}$	$\dot{x} \in \mathcal{F}(x) = \begin{cases} 1 & \text{if } x<0 \\ [-1,1] & \text{if } x=0 \\ -1 & \text{if } x>0 \end{cases}$	Only when $a = 0$, classical solution exists. The maximal classical solution is 1. if $x(0) > 0$, $x_1(t) = x(0) - t, t < x(0)$ 2. if $x(0) < 0$, $x_2(t) = x(0) + t, t < -x(0)$ 3. if $x(0) = 0$, $x_3(t) = 0, t \in [0, \infty)$	Only when $a = 0$, caratheodory solution exists. The maximal classical solution is 1. if $x(0) > 0$, $x_1(t) = \max(x(0) - t, 0), t \in [0, \infty)$ 2. if $x(0) < 0$, $x_2(t) = \min(x(0) + t, 0), t \in [0, \infty)$ 3. if $x(0) = 0$, $x_3(t) = 0, t \in [0, \infty)$ **Note**:These only absolutely continuous (not continuously differentiable)	Whatever the value of a is, the Filippov solution is 1. if $x(0) > 0$, $x_1(t) = \max(x(0) - t, 0), t \in [0, \infty)$ 2. if $x(0) < 0$, $x_2(t) = \min(x(0) + t, 0), t \in [0, \infty)$ 3. if $x(0) = 0$, $x_3(t) = 0, t \in [0, \infty)$
$\dot{x} = \begin{cases} -1 & \text{if } x<0 \\ a & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$	$\dot{x} \in \mathcal{F}(x) = \begin{cases} -1 & \text{if } x<0 \\ [-1,1] & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$	From $x = x(0) \neq 0$,classical solution exists as 1. $x_1(t) = x(0) + t$ if $x(0) > 0$ 2. $x_2(t) = x(0) - t$ if $x(0) < 0$ From $x = x(0) = 0$, classical solution exists when $a = 1$ or $a = -1$ 1. when $a = 1$, $x_1(t) = t, t \in [0, \infty)$ 2. when $a = -1$,$x_2(t) = -t, t \in [0, \infty)$	From $x = x(0) \neq 0$,classical solution exists as 1. $x_1(t) = x(0) + t$ if $x(0) > 0$ 2. $x_2(t) = x(0) - t$ if $x(0) < 0$. From $x = x(0) = 0$, two caratheodory solutions exist for **all** $a \in \mathbb{R}$ 1. $x_1(t) = t, t \in [0, \infty)$ 2. $x_2(t) = -t, t \in [0, \infty)$ These two solutions only violate the vector field in $t = 0$	Filippov solution exists for all $a \in \mathbb{R}$ and $x(0) \in \mathbb{R}$. 1. if $x(0) \geq 0$, $x_1(t) = x(0) + t, t \in [0, \infty)$ 2. if $x(0) \leq 0$,, $x_2(t) = x(0) - t, t \in [0, \infty)$ **Note**: When $x(0) = 0$, exists two Filippov solutions.
$\dot{x} = \begin{cases} 1 \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$	$\dot{x} \in \{1\}$	$x = 0, t \in [0, \infty)$	two caratheodory solutions: 1. $x(t) = 0, t \in [0, \infty)$ 2. $x(t) = t, t \in [0, \infty)$	one unique solution: 1. $x(t) = t, t \in [0, \infty)$

Conditions for Existence and Uniqueness of Classical, Caratheodory, Filippov Solutions

Table 3: conditions of solutions to $\dot{x} = X(x(t))$

	solution	existence	uniqueness
classical	continuously differentiable	$X : \mathbb{R}^d \to \mathbb{R}^d$ is continuous	essentially one-sided Lipschitz on $B(x, \varepsilon)$,[2]
Filippov	absolutely continuous	$X : \mathbb{R}^d \to \mathbb{R}^d$ is measurable and locally essentially bounded	essentially one-sided Lipschitz on $B(x, \varepsilon)$

[2]Every vector field that is locally Lipschitz at x satisfies the one-sided Lipschitz condition on a neighborhood of x, but the converse is not true.

Sliding Mode Control Algorithms

Consider the system $\ddot{y} = u + f$, design u to regulate y to track y_c. define error variable $e = y_c - y$, then $\ddot{e} = \ddot{y}_c - f - u$. Convergence requirements and analysis can be found at (SHTESSEL et al., 2014) for SMC , (XIAN et al., 2004) for RISE

Table 4: regulate the system $\ddot{e} = \ddot{y}_c - f - u$

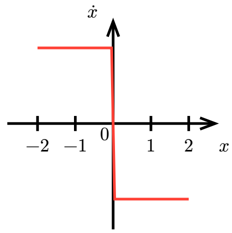
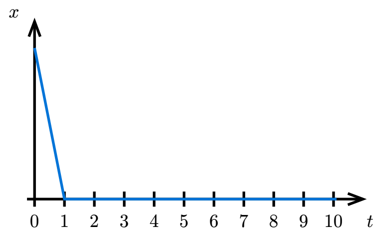
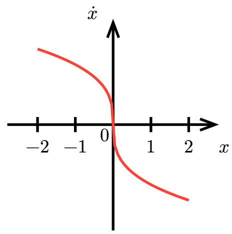
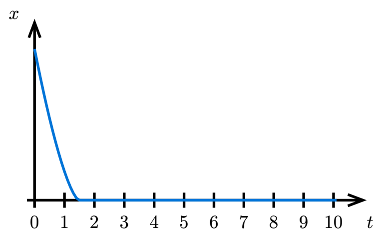
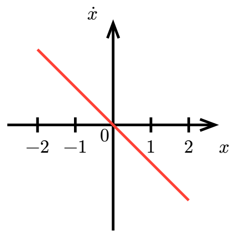
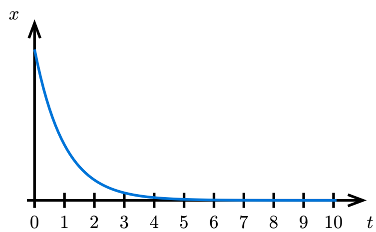
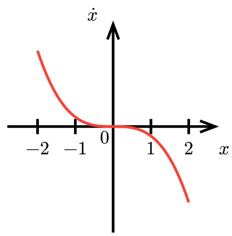
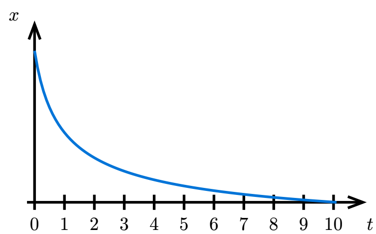
Sliding Mode Controller	name	Sliding variable convergence	Output Tracking convergence	Type
$u = \rho \text{ sign } (\sigma)$ $\sigma = \dot{e} + ce$	Traditional	Finite time	Asymptotic	Discontinuous
$u = u_1 + u_2,$ $u_1 = \rho_1 \text{ sign}(s), u_2 = -\int \sigma dt$ $\sigma = \dot{e} + ce$ $s = \sigma - z, \dot{z} = -u_2$	integral	s: Finite time σ: Asysmptotic	Asymptotic	Discontinuous
$\dot{u} = v,$ $v = c\bar{c}\dot{e} + (c + \bar{c})u + \rho \text{ sign}(s)$ $s = \dot{\sigma} + \bar{c}\sigma, \sigma = \dot{e} + ce$	integral	s: Finite time σ: Asysmptotic	Asymptotic	Continuous
$u = c \ \lvert\sigma\rvert^{\frac{1}{2}} \text{ sign}(\sigma) + w$ $\dot{w} = b \text{ sign}(\sigma)$ $\sigma = \dot{e} + ce$	Super-twisting (Second Order SMC)	Finite time	Asymptotic	Continuous
$u = -\rho \text{ sign}(\sigma)$ $\sigma = \dot{e} + c\lvert e\rvert^{\frac{1}{2}}\text{sign}(e)$	prescribed convergence law	Finite time	Finite time	Discontinuous
$u = (k_s + 1)e_2(t) - (k_s + 1)e_2(0) + w$ $\dot{w} = (k_s + 1)\alpha e_2 + \beta \text{ sign } e_2$ $e_1 = e, e_2 = \dot{e}_1 + e_1$	RISE		Asysmptotic	Continuous

Fractional Power Feedback

From (POLYAKOV, 2020), we can find solutions of the system $\dot{x} = -x^v$ has some good properties. Due to the definition of power function, usually we extend the function's definition to the whole rational field $\mathbb{R}$ as $\dot{x} = -\operatorname{sign}(x)\ |x|^v, v \geq 0$. In some books like (POLYAKOV, 2020), a condition on v is used.[3] The general solution to this system is

$$x(t) = \left(x(0)^{-v+1} + (v-1)t\right)^{\frac{1}{-v+1}} \operatorname{sign}(x(0)) = \frac{x(0)}{\left(1 + (v-1)t|x(0)|^{v-1}\right)^{\frac{1}{v-1}}} \text{ if } v \neq 1.$$

Table 5: fractional power feedback $\dot{x} = -\operatorname{sign}(x)|x|^v, v \geq 0$

system	$x - \dot{x}$ curve	numerical solution $x = x(t)$	analytical solution	stability
$\dot{x} = -x^0 = -\operatorname{sign}(x)$			when $x(0) > 0$ $x(t) = \begin{cases} x(0)-t & t\in(0,x(0)) \\ 0 & t \geq x(0) \end{cases}$	Finite time if $v < 1$
$\dot{x} = -x^{\frac{1}{3}}$			when $x(0) > 0$ $x(t) = \begin{cases} \left(\lvert x(0)\rvert^{\frac{2}{3}} - \frac{2}{3}t\right)^{\frac{3}{2}} & t \in \left(0, \left(\frac{2}{3}\right)\lvert x(0)\rvert^{\frac{3}{2}}\right) \\ 0 & t \geq x(0) \end{cases}$	Finite time if $v < 1$
$\dot{x} = -x^1 = -x$			$x = x(0)e^{-t}$	Exponential if $= 1$
$\dot{x} = -x^3$			$x(t) = \left(x(0)^{-2} + 2t\right)^{-\frac{1}{2}}$	(practical) Fixed time if $v > 1$[4]

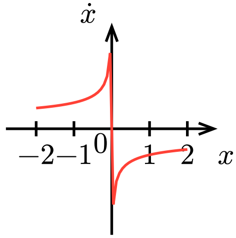
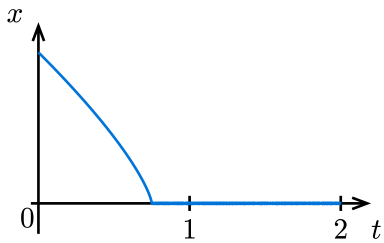
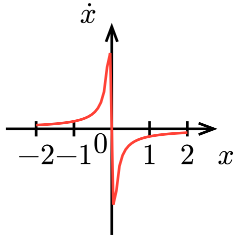
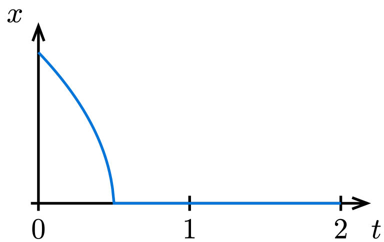
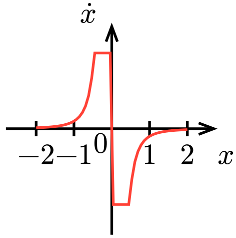
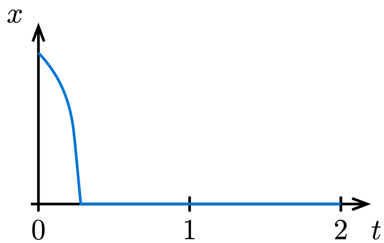
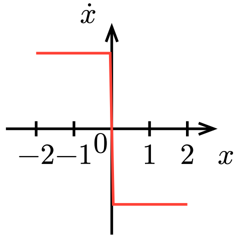
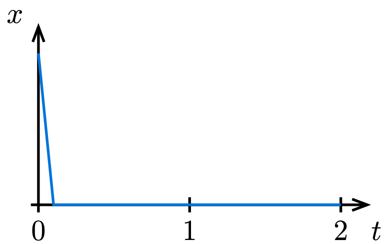
[3] I haven't figure out this condition yet. The condition write v as $v = \frac{p}{q}$ and says p should be an odd integer and q is an even natual number. I think both p and q should be odd natural number and $q \neq 0$.

[4] converges to a neighborhood of the origin in a fixed time independent of the initial condition.

Fractional (negative) Power Feedback

Faster covergence will be found if we use negative power. However, this will need infinite gain (infinite energy). So, it is impossible for physical implementation. Simulations are carried out with a satutated power function for the infinite gain is not possible to simulate.

Table 6: fractional power feedback $\dot{x} = -\operatorname{sign}(x)|x|^v, v < 0$, the right-hand side function is satutated with threhold 10

system	$x - \dot{x}$ curve	numerical solution $x = x(t)$	numerical simulation step	stability
$\dot{x} = -\operatorname{sat}\left(x^{-\frac{1}{3}}\right)$			0.0012	Finite time
$\dot{x} = -\operatorname{sat}(x^{-1}) = -\operatorname{sat}\left(\frac{1}{x}\right)$			0.01	Finite Time
$\dot{x} = -\operatorname{sat}(x^{-3})$			0.01	Finite Time
$\dot{x} = -10 * \operatorname{sign}(x)$			0.1	Finite time

Prescribed Time Stability by Time-varying Gain

Generally prescribed/preassigned/pre-appointed time stability is reached by time-varying gain (time-varying scaling function, time-base generator). Following table gives the basic example, we see that the solution for the first case is the same as $\dot{x} = -\text{sign}(x)$. (SONG et al., 2023) .

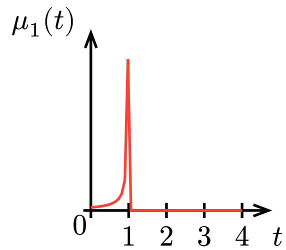
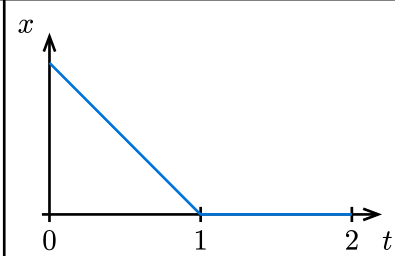
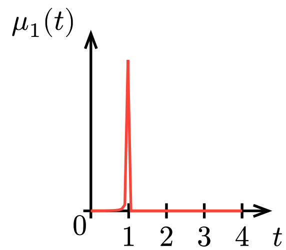
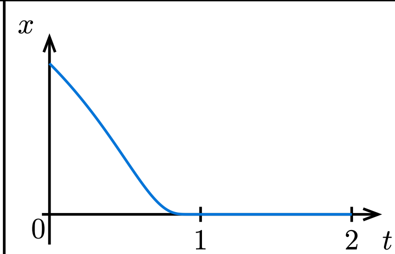
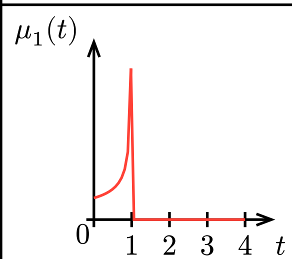
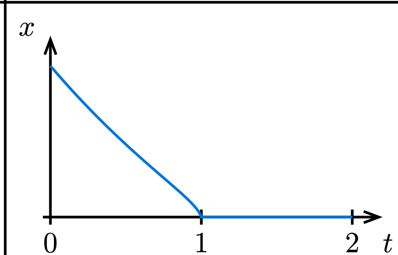
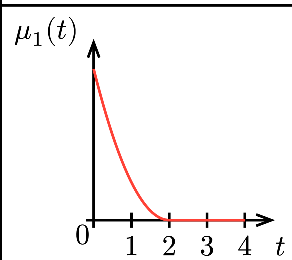
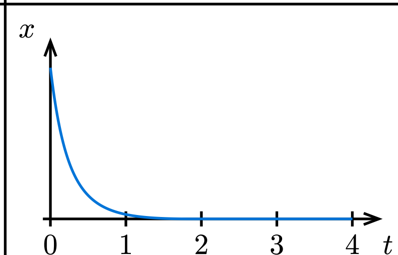
The system is:

$$\dot{x} = -\mu_2(t)x$$

$$\mu_1(t) = \begin{cases} \frac{k_1}{(T-t)^h} + k_2 & 0 < t < T \\ 0 \quad t \geq T \end{cases},$$

with $T > 1$ to be prescribed and $k_1 > 0, k_2 > 0, h > 0$.

Table 7: time-varying gain for single integrator, $\dot{x} = -\mu(t)x$, $\mu(t) = \frac{k_1}{(T-t)^h} + k_2, t \in (0, T]$

system	$\mu(t)$	numerical solution	stability
$T = 1, h = 1, k_1 = 1, k_2 = 0$			prescribed time stability convergence time is T
$T = 1, h = 2, k_1 = 1, k_2 = 0$ and the control signal $\mu_2(t)x$ is saturated with 2			prescribed time stability convergence time is T
$T = 1, h = \frac{1}{2}, k_1 = 1, k_2 = 0$ and the control signal $\mu_2(t)x$ is saturated with 2			prescribed time stability convergence time is T
$T = 2, h = -2, k_1 = 1, k_2 = 0$			prescribed time stability convergence time is T

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SHTESSEL Y, EDWARDS C, FRIDMAN L, et al., 2014. Sliding Mode Control and Observation[M/OL]. New York, NY: Springer New York. DOI:10.1007/978-0-8176-4893-0.

SONG Y, YE H, LEWIS F L, 2023. Prescribed-Time Control and Its Latest Developments[J/OL]. IEEE Transactions on Systems, Man, and Cybernetics: Systems: 1-15[2023-06-02]. https://ieeexplore.ieee.org/document/10044666/. DOI: 10.1109/TSMC.2023.3240751.

XIAN B, DAWSON D, DEQUEIROZ M, et al., 2004. A Continuous Asymptotic Tracking Control Strategy for Uncertain Nonlinear Systems[J/OL]. IEEE Transactions on Automatic Control, 49(7): 1206. DOI:10.1109/TAC.2004.831148.