Supplement for
Nonlinear Systems and Control

Contents

Some exercises are mentioned in the textbook's mainbody. So I organize some solutions here for reference. Only a little solutions is presented in this supplement. They are 1.1 3.24 5.6 .

Math Review

Order Linear Differential Equations

Following content is from https://www.sfu.ca/math-coursenotes/Math%20158%20Course%20Notes/chap_DifferentialEquations.html

Homogeneity of a Linear DE

Given a linear differential equation

$$F_{n(x)}\frac{d^n y}{dx^n} + F_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + F_2(x)\frac{d^2 y}{dx^2} + F_1(x)\frac{dy}{dx} + F_0(x)y = G(x)$$

where $F_{i(x)}$ and $G(x)$ are functions of x, the differential equation is said to be **homogeneous** if $G(x) = 0$ and **non-homogeneous** otherwise.

Note: One implication of this definition is that $y = 0$ is a constant solution to a linear homogeneous differential equation, but not for the non-homogeneous case.

First Order Linear Differential Equations

Given a first order non-homogeneous linear differential equation

$$y' + p(t)y = f(t)$$

using variation of parameters the general solution is given by

$$y(t) = v(t)e^{P(t)} + Ae^{P(t)}$$

where $v'(t) = e^{-P(t)}f(t)$ and $P(t)$ is an antiderivative of $-p(t)$

Topological Space

Metic Space

A metric space (M, d) consists of a set M and a mapping, called distance, $d : M \times M \to \mathbb{R}$, which satisfies the following:

1. $0 \leq d(x, y) < \infty, \forall x, y \in M$
2. $d(x, y) = 0$, if and only if $x = y$
3. $d(x, y) = d(y, x)$
4. Triangle Inequality: $d(x, z) \leq d(x, y) + d(x, z), x, y, z \in M$

Topological Spaces

Continuous Mapping

Definition 0.1: Let M, N be two topological spaces. A mapping $\pi : M \to N$ is continuous, if one of the following two equivalent conditions holds:

- For any $U \subset N$ open, its inverse image

  $$\pi^{-1}(U) := \{x \in M \mid \pi(x) \in U\}$$

  is open
- For any $C \subset N$ closed, its inverse image $\pi^{-1}(C)$ is closed

Note: the two conditions are equivalent.

Definition 0.2: Let M,N be two topological spaces. M and N are said to be homeomorphic(同胚 pei 胚胎) if there exists a mapping $\pi : M \to N$, which is

1. one-to-one(单射 injective),
2. onto(满射 surjective)
3. and continuous (both π and π^{-1} are continuous).

π is called a homeomorphism.

Note: If a mapping is both injective and surjective it is said to bijective(满射).

Definition 0.3: Given a topological space M.

1. A set $U \subset M$ is said to be clopen if it is both closed and open. A topological space(度量空间), M, is said to be **connected** if the only two clopen sets are M and $\emptyset$
2. A continuous mapping $\pi : I = [0, 1] \to M$ is called a path on M. M is said to be **pathwise(or arcwise) connected** if for any two points $x, y \in M$ there exists a path, π, such that $\pi(0) = x$ and $\pi(1) = y$

Tip on open and closed set: As described by topologist James Munkres, unlike a door, "a set can be open, or closed, or both, or neither!" https://en.wikipedia.org/wiki/Clopen_set A set is closed if its complement is open. But A set can be closed or open if its complement is closed.
https://en.wikipedia.org/wiki/Open_set
A subset U of a metric space (M, d) is called open if, for any point x in U, there exists a real number ε such that any point $y \in M$ satisfying $d(x, y) < \varepsilon$ belongs to U. Equivalently, U is open if every point in U has a neighborhood contained in U.

Note: $\mathbb{R}$ is connected and A pathwise connected space M is connected while the converse is incorrect.

Definition 0.4: A topological space M is said to be *locally connected* at $x \in M$ if every neighborhood N_x of x contains a connected neighborhood U_x, i.e., $x \in U_x \subset N_x$. M is said to be locally connected if it is locally connected at each $x \in M$

local connectedness does not imply connectedness (hence no pathwise connectedness); conversely, pathwise connectedness (connectedness) does not imply local connectedness.

Definition 0.5: Let $\{U_\lambda | \lambda \in \Lambda\}$ be a set of open sets in M. The set is called an open covering of M if

$$\cup_{\lambda \in \Lambda} U_\lambda \supset M.$$

M is said to be a compact space if every open covering has a finite sub-covering, i.e., there exists a finite subset $\left\{U_{\lambda_i} \middle| \, i = 1, 2, \ldots, k\right\}$ such that

$$\cup_{i=1}^{k} U_{\lambda_i} \supset M$$

From Calculus we know that with the conventional topology, a set, $U \subset \mathbb{R}^n$ is compact, if and only if it is bounded and closed. Unfortunately, it is not true for general metric spaces.

Definition 0.6: In a topological space, M, a sequence $\{x_k\}$ is said to converge to x, if for any neighborhood $U \ni x$ there exists a positive integer $N > 0$ such that when $n > N$, $x_n \in U$.

Quotient Spaces

Differentiable Manifold

Structure of Manifolds

Fiber Bundle

Vector Field

One Parameter Group

Lie Algebra of Vector Fields

Co-tangent Space

Lie Derivatives

Frobenius' Theory

Lie Series, Chow's Theorem

Tensor Field

Riemannian Geometry

Symplectic Geometry

Some Solutions to Nonlinear Systems(3rd edition)

chapter 1 Introduction

Exercise 1.1: A mathematical model that describes a wide variety of physical nonlinear systems is the nth-order differential equation

$$y^{(\mathrm{n})} = g\left(t, y, \dot{y}, ..., y^{(\mathrm{n\text{-}1})}, u\right) \tag{1}$$

where u and y are scalar variables. With u as input and y as output, find a state model.

Solution:

Let $x_1 = y$,$x_2 = y^{(1)}$,...,$x_n = y^{(\mathrm{n\text{-}1})}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_{\mathrm{n\text{-}1}} &= x_n \\ \dot{x}_n &= g(t, x_1, ..., x_n, u) \\ y &= x_1 \end{aligned} \tag{2}$$

chapter 2 Second Order Systems

chapter 3 Fundamental Properties

Exercise 3.24 : Let $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Suppose that $V(t, 0) = 0$ for all $t \geq 0$ and

$$V(t, x) \geq c_1 \|x\|^2; \left\| \frac{\partial V}{\partial x}(t, x) \right\| \leq c_4 \|x\|, \forall (t, x) \in [0, \infty) \times D \tag{3}$$

where c_1 and c_4 are positive constants and $D \subset \mathbb{R}^n$ is a convex domain that contains the origin $x = 0$

1. Show that $V(t, x) \leq \frac{1}{2} c_4 \|x\|^2$ for all $x \in D$.
   Hint: Use the representation $V(t, x) = \int_0^1 \frac{\partial V}{\partial x}(t, \sigma x) d\sigma x$
2. Show that the constants c_1 and x_4 must satisfy $2c_1 \leq c_4$
3. Show that $W(t, x) = \sqrt{V(t, x)}$ satisfies the Lipschitz condition

$$|W(t, x_2) - W(t, x_1)| \leq \frac{c_4}{2\sqrt{c_1}} \|x_2 - x_1\|, \forall t \geq 0, \forall x_1, x_2 \in D \tag{4}$$

Solution to 1

$$V(t, x) = \int_0^1 \frac{\partial V}{\partial V}(t, \sigma x) dx \leq \int_0^1 \left\| \frac{\partial V}{\partial x}(t, \sigma x) \right\| \|x\| d\sigma \leq \int_0^1 c_4 \sigma d\sigma \|x\|^2 \leq \frac{1}{2} c_4 \|x\|^2 \tag{5}$$

Solution to 2

Since

$$c_1 \|x\|^2 \leq V(t, x) \leq \frac{1}{2} c_4 \|x\|^2, \forall x \in D \tag{6}$$

we must have $c_1 \leq \frac{1}{2} c_4$

Solution to 3

Consider two ponts x_1 and x_2 such that $\alpha x_1 + (1-\alpha)x_2 \neq 0$ for all $0 \leq \alpha \leq 1$; that is, the orgin does not lie on the line connecting x_1 and x_2. The Jacobian $[\partial W / \partial x]$ is defined for every $x = \alpha x_1 + (1-\alpha)x_2$ and given by

$$\frac{\partial W}{\partial x}(t,x) = \frac{1}{2\sqrt{V(t,x)}} \frac{\partial V}{\partial x}(t,x) \tag{7}$$

By the mean value theorem, there is $\alpha^* \in (0,1)$ such that, with $z = \alpha^* x_1 + (1-\alpha^*)x_2$

$$W(t,x_2) - W(t,x_1) = \frac{\partial W}{\partial x}(t,z)(x_2 - x_1) = \frac{1}{2\sqrt{V(t,z)}} \frac{\partial V}{\partial x}(t,z)(x_2 - x_1) \tag{8}$$

Hence

$$|W(t,x_2) - W(t,x_1)| \leq \frac{1}{2\sqrt{c_1}\|z\|} \tag{9}$$

Consider now the case when the origin lies on the line connecting x_1 and x_2; that is , $0 = \alpha_0 x_1 + (1-\alpha_0)x_2$ for some $\alpha_0 \in [0,1]$. We have

$$\begin{aligned} |W(t,x_2) - W(t,0)| = |W(t,x_2)| = \sqrt{V(t,x_2)} \leq \sqrt{\frac{c_4}{2}}\|x_2\| \\ |W(t,x_1) - W(t,0)| = |W(t,x_1)| = \sqrt{V(t,x_1)} \leq \sqrt{\frac{c_4}{2}}\|x_1\| \\ |W(t,x_2) - W(t,x_1)| = |W(t,x_2) - W(t,0) + W(t,0) - W(t,x_1)| \leq \sqrt{\frac{c_4}{2}}(\|x_1\| + \|x_2\|) \end{aligned} \tag{10}$$

Since the origin lies on the line connecting x_1 and x_2, we have $\|x_2\| + \|x_1\| = \|x_2 - x_1\|$. We also have $1 \leq \sqrt{c_4/2c_1}$. Therefore,

$$|W(t,x_2) - W(t,x_1)| \leq \frac{c_4}{2\sqrt{c_1}}\|x_2 - x_1\| \tag{11}$$

chapter 4 Lyapunov Stability

chapter 5 Input-Output Stability

Exercise 5.6: Verify that $D_+W(t)$ satisfies Equation 12(5.12 in textbook) when $V(t,x(t)) = 0$.

$$D_+W \leq \frac{c_4 L}{2\sqrt{c_1}}\|u(t)\| \tag{12}$$

Hint: Using Exercise 3.24, show that

$$V(t+h, x(t+h)) \leq c_4 h^2 L^2 \|u\|^2/2 + ho(h) \tag{13}$$

where $\frac{o(h)}{h} \to 0$ as $h \to 0$. Then apply $c_4 \geq 2c_1$

From textbook, the system is

$$\begin{aligned} \dot{x} = f(t,x,u), x(0) = x_0 \\ y = h(t,x,u) \end{aligned} \tag{14}$$

From $V(t,x(t))=0$ and Equation 6, we have $x(t)=0$

Let $V(t,x(t))=0$.

$$
\begin{aligned}
D_+W &= \lim\sup_{h\to 0^+}\frac{1}{h}[W(t+h,x(t+h))-W(t,x(t))] \\
&= \lim\sup_{h\to 0^+}\frac{1}{h}\sqrt{V(t+h,x(t+h))}
\end{aligned}
\tag{15}
$$

From Equation 6, We have

$$
V(t+h,x(t+h)) \le \frac{c_4}{2}\|x(t+h)\|^2 \tag{16}
$$

From textbook 5.9, we have

$$
\|f(t,x(t),u)-f(t,x(t),0)\| \le L\|u\| \tag{17}
$$

Use Taylor Series:

$$
\begin{aligned}
& x(t+h) = f(t,x,u)h + o(h) \\
\Rightarrow\ & \|x(t+h)\|^2 \le (\|f(t,x,u)\|h + \|o(h)\|)^2
\end{aligned}
\tag{18}
$$

$$
\frac{1}{h^2}V(t+h,x(t+h)) \le \frac{c_4}{2}\left(\frac{\|x(t+h)\|}{h}\right)^2 \le \frac{c_4}{2}\left(\|f(t,x,u)\| + \frac{\|o(h)\|}{h}\right)^2 \tag{19}
$$

$$
\lim\sup_{h\to 0^+}\frac{1}{h}\sqrt{V(t+h,x(t+h))} \le \sqrt{\frac{c_4}{2}}\|f(t,x,u)\| \le \sqrt{\frac{c_4}{2}}L\|u\| \tag{20}
$$

since $\sqrt{c_4/(2c_1)} \ge 1$. Thus

$$
D_+W \le \sqrt{\frac{c_4}{2}}L\|u\|\sqrt{c_4/(2c_1)} = \frac{c_4 L}{2\sqrt{c_1}}\|u(t)\| \tag{21}
$$

which agrees with the right hand side of Equation 12

chapter 6 Passivity

chapter 7 Frequency Domain analysis of Feedback Systems

chapter 8 Advanced Stability Analysis

chapter 9 Stability of Perturbed Systems

chapter 10 Perturbation Theory and Averaging

chapter 11 Singular Perturbations

chapter 12 Feedback Control

chapter 13 Feedback Linearization

chapter 14 Nonlinear Design Tools