UNIVERSITY OF NEW SOUTH WALES
DEPARTMENT OF STATISTICS

MATH5856 Introduction to Statistics and Statistical Computations
Tutorial Problems week 11 solutions

1.

x	1	2	3	4	5	6
$f_1(x)/f_0(x)$	$\frac{20}{3}$	5	$\frac{1}{10}$	$\frac{1}{8}$	$\frac{40}{3}$	10

(a)

$$\text{size} = P_{f_0}(X \leq 2) = 0.05$$
$$\text{power} = P_{f_1}(X \leq 2) = 0.3$$

(b) If the rejection region is $x = 5$ then the size is 0.03.
If the rejection region is $x \geq 5$ then the size is 0.05.

2. (a) According to the Neyman-Pearson lemma, the most powerful test rejects H_0 if

$$\frac{\prod_{i=1}^n f(X_i;4)}{\prod_{i=1}^n f(X_i;1)} \geq k$$

for some $k > 0$ (determined by the significance level of the test). The left-hand side of this expression equals

$$\begin{aligned}
\frac{\prod_{i=1}^n \frac{1}{8}e^{-|X_i|/4}}{\prod_{i=1}^n \frac{1}{2}e^{-|X_i|}} &= \frac{1}{4^n}e^{-\frac{1}{4}\sum_{i=1}^n |X_i| + \sum_{i=1}^n |X_i|} \\
&= \frac{1}{4^n}e^{\frac{3}{4}\sum_{i=1}^n |X_i|}
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\prod_{i=1}^n f(X_i;4)}{\prod_{i=1}^n f(X_i;1)} \geq k &\iff \frac{1}{4^n}e^{\frac{3}{4}\sum_{i=1}^n |X_i|} \geq k \\
&\iff e^{\frac{3}{4}\sum_{i=1}^n |X_i|} \geq k' \\
&\iff \frac{3}{4}\sum_{i=1}^n |X_i| \geq k'' \\
&\iff \frac{1}{n}\sum_{i=1}^n |X_i| \geq c
\end{aligned}$$

(b)

$$\begin{aligned}
\alpha &= P(\text{reject } H_0|H_0 \text{ true}) = P(|X_1| \geq c|\theta = 1) \\
&= 2P(X_1 > c|\theta = 1) = 2\int_c^\infty \tfrac{1}{2}e^{-|x|}\,dx \\
&= \int_c^\infty e^{-x}\,dx = [-e^{-x}]_c^\infty = e^{-c}
\end{aligned}$$

Therefore,

$$c = -\ln(\alpha).$$

3. (a) The transformation from Y to W is given by

$$w = e^{y-\theta}$$

and so the inverse transformation is given by

$$y = \ln(w) + \theta \implies \frac{dy}{dw} = (1/w).$$

Hence

$$f_W(w) = f_Y(\ln(w)+\theta)\left|\frac{dy}{dw}\right| = ne^{-n(\ln(w)+\theta-\theta)}|1/w| = \frac{n}{w^{n+1}}.$$

Also $y > \theta \iff \ln(w) + \theta > \theta \iff w > 1$ so

$$f_W(w) = \frac{n}{w^{n+1}}, \quad w > 1.$$

Since this not depend on any unknown parameters it is a pivotal function.

(b)

$$0.95 = P(W < b) = 1 - \frac{1}{b^n} \iff \frac{1}{b^n} = 0.05 \iff b^n = \frac{1}{0.05}$$

$$\iff b = \frac{1}{(0.05)^{1/n}} = 20^{1/n}.$$

(c) A one-sided 95% interval is derived as follows:

$$
\begin{aligned}
0.95 &= P(\exp(Y-c) < 20^{1/n}) \\
&= P(Y - c < \ln(20)/n) \\
&= P(c > Y - \ln(20)/n)
\end{aligned}
$$

When Y is observed to be $y = 5$ and $n = 10$ the lower confidence bound is

$$5 - \ln(20)/10 \simeq 4.70.$$

4. The sample sizes, sample means and sample standard deviations in each group are $n_1 = n_2 = 21$, $\overline{x} = 17.5$, $\overline{y} = 16.9$, $s_1 = 0.55$, $s_2 = 0.49$

(a)

$$\frac{\overline{X} - \overline{Y} - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}$$

A 95% confidence interval for $\mu_1 - \mu_2$ is:

$$\overline{X} - \overline{Y} \pm S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\, t_{n_1+n_2-2,0.975}.$$

The observed value of s_p is:

$$s_p^2 = \tfrac{1}{2}(0.3025 + 0.2401) = 0.2713 \implies s_p = 0.52$$

Also, $t_{40,0.975} = 2.02$. Hence, the confidence interval is

$$0.6 \pm 0.52 \times 0.31 \times 2.02 = (0.27, 0.93).$$

(b) We use the result:

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{n_1-1,n_2-1}.$$

$$
\begin{aligned}
0.99 &= P\left(F_{n_1-1,n_2-1,0.005} < \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} < F_{n_1-1,n_2-1,0.995}\right) \\
&= P\left(F_{n_1-1,n_2-1,0.005} < \frac{\sigma_2^2}{\sigma_1^2}\frac{S_1^2}{S_2^2} < F_{n_1-1,n_2-1,0.995}\right) \\
&= P\left(\frac{S_2/S_1}{\sqrt{F_{n_2-1,n_1-1,0.005}}} < \frac{\sigma_2}{\sigma_1} < (S_2/S_1)\sqrt{F_{n_1-1,n_2-1,0.995}}\right)
\end{aligned}
$$

Therefore a 99% confidence interval for σ_2/σ_1 is

$$\left(\frac{S_2/S_1}{\sqrt{F_{n_2-1,n_1-1,0.995}}}, (S_2/S_1)\sqrt{F_{n_1-1,n_2-1,0.995}} \right).$$

Using $F_{20,20,0.995} = 3.32$ we obtain the interval

$$(0.49, 1.62).$$

5. Let the two samples be distinguished with subscripts of 1 and 2. Let σ_1 and σ_2 denote the respective (population) standard deviations. The hypotheses to be tested are:

$$H_0 : \sigma_1 = \sigma_2 \text{ versus } \quad H_1 : \sigma_1 < \sigma_2.$$

Under H_0,

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{S_1^2}{S_2^2} \sim F_{n_1-1,n_2-1}.$$

Therefore we should reject H_0 at the 5% level of significance if

$$\frac{S_1^2}{S_2^2} < F_{24,20,0.05} = 1/F_{20,24,0.95} = 1/2.03 = 0.49.$$

We observe

$$s_1 = 32,\ s_2 = 54,\ n_1 = 25,\ n_2 = 21$$

Since the observed value of $\frac{S_1^2}{S_2^2}$ is

$$32^2/54^2 = 0.35$$

H_0 is rejected at the 5% level of significance.

6. (a) A $100(1-\alpha)\%$ confidence interval for the mean ultimate tensile strength is

$$\overline{X} \pm \frac{S}{\sqrt{n}} t_{n-1,1-\alpha/2}$$

$n = 25,\ \overline{x} = 152.7,\ s = 4.6,\ \alpha = 0.05,\ t_{24,0.975} = 2.06$ so the confidence interval is

$$152.7 \pm (4.6/5) \times 2.06 = (150.8, 154.6).$$

(b) A $100(1-\alpha)\%$ confidence interval for the variance of the ultimate tensile strength is

$$\left(\frac{(n-1)S^2}{\chi^2_{n-1,1-\alpha/2}}, \frac{(n-1)S^2}{\chi^2_{n-1,\alpha/2}} \right)$$

$n = 25, s = 4.6,\ \alpha = 0.01,\ \chi^2_{24,0.005} = 9.89,\ \chi^2_{24,0.995} = 45.56$ so the confidence interval is

$$(11.15, 51.35).$$

7. The test statistic is

$$T = \frac{\overline{X} - \overline{Y}}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}.$$

The observed values are $n_1 = 15$, $n_2 = 76$, $\overline{x} = 110$, $\overline{y} = 101$, $s_1 = 24$, $s_2 = 22$ which gives $s_p = 22.3$. The observed value of T is 1.4. Since $t_{89,0.025} \simeq 2$ we are nowhere near the tail of the distribution of T under H_0 so we should accept H_0.

8. Since $\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2 \sim F_{12,12}}$,

$$\begin{aligned}
0.9 &= P\left(F_{12,12,0.05} < \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} < F_{12,12,0.95} \right) \\
&= P\left((S_2^2/S_1^2)F_{12,12,0.05} < \frac{\sigma_2^2}{\sigma_1^2} < (S_2^2/S_1^2)F_{12,12,0.95} \right) \\
&= P\left((S_2^2/S_1^2)\frac{1}{F_{12,12,0.95}} < \frac{\sigma_2^2}{\sigma_1^2} < (S_2^2/S_1^2)F_{12,12,0.95} \right)
\end{aligned}$$

From tables, $F_{12,12,0.95} = 2.69$. Also, the observed values of S_1^2 and S_2^2 are

$$s_1^2 = 237 \quad \text{and} \quad s_2^2 = 389.$$

Hence, the required 90% confidence interval is

$$((389/237)/2.69, (389/237) \times 2.69) = (0.610, 4.42).$$

9. Refer to the normal infants by the random variable X and the SGA infants using the random variable Y.

(a) Since the true variance of SGA infants is not known need to use the t-statistic

$$\frac{\bar{Y} - \mu_2}{S_2/\sqrt{n}} \sim t_{n-1}$$

from which the 95% confidence interval for μ_2 (the true mean for SGA infants) is derived as

$$\bar{y} \pm \frac{s_2}{\sqrt{10}} t_{9,0.975} = 43.5 \pm \frac{12.03}{3.162} 2.262 = (34.89, 52.11).$$

(b) Recognise this as a two independent samples situation. Assume that the true variances of the normal infants and SGA infants are equal in order to use the two-sample pooled variance t-statistic. Null hypothesis $H_0 : \mu_1 = \mu_2$ versus alternative $H_1 : \mu_2 < \mu_1$. Test statistic uses the pooled estimate of common standard deviation

$$S_p = \sqrt{\frac{4(63.7) + 9(144.72)}{13}} = 10.94.$$

The value of the t-statistic is

$$\frac{\bar{x} - \bar{y} - 0}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{57.2 - 43.5}{10.94\sqrt{\frac{1}{5} + \frac{1}{10}}} = 2.2863$$

The 95% quantile for the t_{13} distribution is 1.771 and since the value of the test statistic of 2.2863 exceeds this we reject H_0 in favour of H_1 and conclude that mean glucose level is lower for the SGA infants.

10. The observed mean and standard deviation are

$$\overline{x} = 60.174 \quad \text{and} \quad s = 6.807$$

and the sample size is $n = 23$.

(a) Since the normality assumption is valid we can use the t-distribution based confidence interval (with $\alpha = 0.10$ for a 90% confidence interval):

$$
\begin{aligned}
\overline{x} \pm t_{n-1,1-\alpha/2}\frac{s}{\sqrt{n}} &= 60.174 \pm t_{22,0.95}\frac{6.807}{\sqrt{23}} \\
&= 60.174 \pm 1.717 \times 1.419358 \\
&= (57.7, 62.6).
\end{aligned}
$$

We can be 90% confident that the mean speed along Anzac Parade is between 57.7km/hour and 62.6km/hour.

(b) The formula for a $100(1-\alpha)\%$ confidence interval for the standard deviation is (with $\alpha = 0.05$):

$$
\left(\sqrt{\frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha/2}}}, \sqrt{\frac{(n-1)s^2}{\chi^2_{n-1,\alpha/2}}}\right)
$$

For a 95% confidence interval we set $\alpha = 0.05$ to get

$$
\begin{aligned}
\left(s\sqrt{\frac{(n-1)}{\chi^2_{n-1,1-\alpha/2}}}, s\sqrt{\frac{(n-1)}{\chi^2_{n-1,\alpha/2}}}\right) &= \left(6.807\sqrt{\frac{22}{\chi^2_{22,0.975}}}, 6.807\sqrt{\frac{22}{\chi^2_{22,0.025}}}\right) \\
&= \left(6.807\sqrt{\frac{22}{36.78}}, 6.807\sqrt{\frac{22}{10.98}}\right) \\
&= (5.26, 9.63)
\end{aligned}
$$

We can be 95% confident that the standard deviation is between 5.26km/hour and 9.63km/hour.

11. By the normality assumption

$$
\frac{24S^2}{\sigma^2} \sim \chi^2_{24}.
$$

Hence,

$$
0.99 = P\left(\chi^2_{24,0.005} < \frac{24S^2}{\sigma^2} < \chi^2_{24,0.995}\right)
$$

Some straightforward algebraic manipulation leads to

$$0.99 = P\left(\frac{24S^2}{\chi^2_{24,0.995}} < \sigma^2 < \frac{24S^2}{\chi^2_{24,0.005}}\right).$$

From tables, $\chi^2_{24,0.005} = 9.89$ and $\chi^2_{24,0.995} = 45.56$ so the 99% confidence in interval is

$$\left(\frac{24 \times 4.6^2}{45.56}, \frac{24 \times 4.6^2}{9.89}\right) = (11.147, 51.349).$$