MATH3911 Assignment One

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This assignment is my own work. I have read and understood the University Rules with respect to Student Academic Misconduct.

Question 1

a) A simple unbiased estimator is given by $W = I_{\{X_1=3\}}(\mathbf{X})$ since $E_\lambda(W) = Pr(X_1 = 3) = \frac{\lambda^3 e^{-\lambda}}{6}$.

b) We see that the given density can be written in the following form

$$f(x;\theta) = \frac{e^{-\lambda}}{x!} e^{x \log \lambda} = a(\lambda)b(x)exp(c(\lambda)d(x))$$

where $a(\lambda) = e^{-\lambda}$, $b(x) = 1/x!$, $c(\lambda) = \log \lambda$ and $d(x) = x$, noting that $c(\lambda)$ is strictly monotone. Thus, the density is a one parameter exponential family density so the statistic $T = \sum_{i=1}^n d(X_i) = \sum_{i=1}^n X_i$ is complete and sufficient.

By noting that $T \sim Po(n\lambda)$ and $\sum_{i=2}^n X_i \sim Po((n-1)\lambda)$, we have

$$\begin{aligned}
E(W|\sum_{i=1}^n X_i = t) &= P(W=1|\sum_{i=1}^n X_i = t) \\
&= \frac{P(X_1 = 3 \cap \sum_{i=1}^n X_i = t)}{P(\sum_{i=1}^n X_i = t)} \\
&= \frac{P(X_1 = 3 \cap \sum_{i=2}^n X_i = t-3)}{P(\sum_{i=1}^n X_i = t)} \\
&= \frac{P(X_1 = 3)P(\sum_{i=2}^n X_i = t-3)}{P(\sum_{i=1}^n X_i = t)} \\
&= \frac{[\lambda^3 e^{-\lambda}/6][e^{-\lambda(n-1)}(\lambda(n-1))^{t-3}((t-3)!)^{-1}]}{e^{-n\lambda}(n\lambda)^t(t!)^{-1}} \\
&= \frac{t(t-1)(t-2)}{6}\frac{(n-1)^{t-3}}{n^t}
\end{aligned}$$

By Lehmann-Scheffe's theorem, since T is complete and sufficient and W is unbiased for τ, we conclude $\frac{T(T-1)(T-2)}{6}\frac{(n-1)^{t-3}}{n^t}$ is the UMVUE.

c) The MLE of λ is $\bar{X}$. Hence, the MLE of $\tau(\lambda) = \frac{\lambda^3 e^{-\lambda}}{6}$ is $\hat{\tau} = \frac{\bar{X}^3 e^{-\bar{X}}}{6}$ by the transformation invariance property of the MLE. We know that the MLE is asymptotically normal, unbiased and efficent, so using the delta method, the asymptoic distribution is given by:

$$\sqrt{n}(\hat{\tau} - \tau(\lambda)) \xrightarrow{d} N\left(0, \left(\frac{3\lambda^2 e^{-\lambda} - \lambda^3 e^{-\lambda}}{6}\right)^2 \lambda\right) = N\left(0, \frac{\lambda e^{-2\lambda}(3\lambda^2 - \lambda^3)^2}{36}\right).$$

d) To find a point estimate of $\tau(\lambda)$, we note that $T = \sum_{i=1}^{15} X_i = 45$ and $\bar{X} = \frac{T}{15} = 3$. By applying the method in part b), we obtain the estimate

$$\frac{T(T-1)(T-2)}{6}\frac{(n-1)^{t-3}}{n^t} = \frac{45(44)(43)}{6}\frac{15^{42}}{15^{45}} = 0.231875$$

By applying the method in part c), we obtain the estimate

$$\frac{\bar{X}^3 e^{-\bar{X}}}{6} = \frac{3^3 e^{-3}}{6} = 0.2240418$$

Both values are relatively similar to each other. This is to be expected since we know that the UMVUE approaches the MLE asymptotically. More explicity, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{T(T-1)(T-2)}{6}\frac{(n-1)^{t-3}}{n^t} &= \lim_{n\to\infty} \frac{n\bar{X}(n\bar{X}-1)(n\bar{X}-2)}{6}\frac{(n-1)^{n\bar{X}-3}}{n^{n\bar{X}}} \\
&= \lim_{n\to\infty} \frac{n^3\bar{X}^3}{6}\frac{(n-1)^{n\bar{X}-3}}{n^{n\bar{X}}} \\
&= \lim_{n\to\infty} \frac{\bar{X}^3}{6}\left(1-\frac{1}{n}\right)^{n\bar{X}-3} \\
&= \frac{\bar{X}^3}{6}\left[\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n\bar{X}}\right]\left[\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-3}\right] \\
&= \frac{\bar{X}^3 e^{-\bar{X}}}{6} \\
&= \hat{\tau}.
\end{aligned}$$

e) The variance according to the Cramer-Rao lower bound is given by:

$$\frac{(r'(\lambda))^2}{nI_{X_1}(\lambda)} = \frac{((3\lambda^2 e^{-\lambda} - \lambda^3 e^{-\lambda})/6)^2}{n/\lambda} = \frac{\lambda e^{-2\lambda}(3\lambda^2-\lambda^3)^2}{36n}.$$

Suppose that the Cramer-Rao lower bound is attainable by $T(\mathbf{X})$, an unbiased estimator of $\tau(\lambda)$. Then the score $V(\mathbf{X},\lambda)$ can be expressed as $k_n(\lambda)[W(\mathbf{X}) - \tau(\lambda)]$. Now,

$$\begin{aligned}
V(\mathbf{X},\lambda) &= \frac{1}{\lambda}\sum_{i=1}^{n} X_i - n \\
&= \frac{n}{\lambda}(\bar{X}-\lambda) \\
&= \frac{n\lambda^2 e^{-\lambda}}{6}\left(\frac{6\bar{X}}{n\lambda^2 e^{-\lambda}} - \frac{\lambda^3 e^{-\lambda}}{6}\right) \\
&= k_n(\lambda)\left(\frac{6\bar{X}}{n\lambda^2 e^{-\lambda}} - \tau(\lambda)\right)
\end{aligned}$$

This implies that $T(\mathbf{X}) = \frac{\mathbf{6\bar{X}}}{\mathbf{n\lambda^2 e^{-\lambda}}}$. However, this is not an estimator since it depends on the parameter λ so the Cramer-Rao lower bound is not attainable. Thus the variance of the UMVUE does not have the same variance as that implied by the Cramer-Rao lower bound.

f) Note that the asymptotic distribution is $N\left(\tau(\lambda), \frac{\lambda e^{-2\lambda}(3\lambda^2-\lambda^3)^2}{36n}\right)$. Thus, the standard error of $\hat{\tau}$ is given by

$$
\begin{aligned}
\hat{se}[\hat{\tau}] &\approx \sqrt{\frac{1}{36n}\hat{\lambda}e^{-2\hat{\lambda}}(3\hat{\lambda}^2-\hat{\lambda}^3)^2} \\
&= \sqrt{\frac{1}{36n}\bar{x}e^{-2\bar{x}}(3\bar{x}^2-\bar{x}^3)^2} \\
&= \sqrt{\frac{1}{36\times 15}3\times e^{-2\times 3}(3\times 3^2-3^3)^2} \\
&= 0
\end{aligned}
$$

so an asymptotic 90% confidence interval for $\tau(\lambda)$ is given by:

$$(\hat{\tau}-\Phi^{-1}(0.95)\hat{se}[\hat{\tau}]\,,\,\hat{\tau}+\Phi^{-1}(0.95)\hat{se}[\hat{\tau}]) = (0.2240418, 0.2240418).$$

The reason why we only get a point interval is that coincidentally, the sample mean was exactly equal to 3 so the standard error was 0.

Question 2

Firstly, consider Distribution 1. Suppose that $E_\theta g(T) = 0$ for all $\theta \in \Theta$. That is, for all $\theta \in \Theta$, we have,

$$g(1)3\theta + g(2)2\theta + g(3)(2\theta^3-\theta^4) + g(4)(1+\theta^4-5\theta-2\theta^3) = 0$$

$$[g(4)-g(3)]\theta^4 + [2g(3)-2g(4)]\theta^3 + [3g(1)+2g(2)-5g(4)]\theta + g(4) = 0.$$

By the fundamental theorem of algebra, this implies each coefficient is equal to 0 so, we have

$$g(4) = 0 \Rightarrow g(4)-g(3) = 0 \Rightarrow g(3) = 0 \Rightarrow 3g(1)+2g(2) = 0,$$

but the latter relationship does not necessarily imply both $g(1)$ and $g(2)$ are equal to 0. Hence Distribution 1 is not complete.

Now, consider Distribution 2. Suppose that $E_\theta g(T) = 0$ for all $\theta \in \Theta$. That is, for all $\theta \in \Theta$, we have,

$$g(1)\theta + g(2)(\theta-\theta^3) + g(3)2\theta^2 + g(4)(1-\theta-2\theta^2+\theta^3) = 0$$

$$[g(4)-g(2)]\theta^3 + [2g(3)-2g(4)]\theta^2 + [g(1)+g(2)-2g(4)]\theta + g(4) = 0$$

By the fundamental theorem of algebra, this implies each coefficient is equal to 0 so, we have

$$g(4) = 0 \Rightarrow 2g(3)-2g(4) = 0 \Rightarrow g(3) = 0$$

$$\Rightarrow g(4)-g(2) = 0 \Rightarrow g(2) = 0 \Rightarrow g(1)+g(2)-2g(4) \Rightarrow g(1) = 0$$

Indeed, this implies $P_\theta(g(T) = 0) = 1$ for all $\theta \in \Theta$. Hence, Distribution 2 is complete.

Question 3

a) See Appendix

b) The cumulative distribution function $F(x;\theta)$ of X_1 is given by

$$
\begin{aligned}
F(x;\theta) &= \int_{-\infty}^{x} f(t;\theta)dt \\
&= \int_{\theta}^{x} \frac{3\theta^3}{t^4}dt \\
&= 1-\left(\frac{\theta}{x}\right)^3
\end{aligned}
$$

c) We note that $X_1, ..., X_n$ are i.i.d random variables with common density $f(x;\theta) = \frac{3\theta^3}{x^4} I_{(-\infty,x]}(\theta)$. Then the likelihood function is

$$
\begin{aligned}
L(x,\theta) &= \prod_{i=1}^{n} f(x_i;\theta) \\
&= \prod_{i=1}^{n} \frac{3\theta^3}{x_i^4} I_{(-\infty,x_i]}(\theta) \\
&= \frac{(3\theta^3)^n}{\prod_{i=1}^{n} x_i^4} I_{(-\infty,x_{(1)}]}(\theta) \qquad (1)
\end{aligned}
$$

To show that $X_{(1)}$ is a minimal sufficient statistic for θ, we consider the ratio

$$
\psi(\theta) = \frac{L(x,\theta)}{L(y,\theta)} = \frac{(3\theta^3)^n / \prod_{i=1}^{n} x_i^4 I_{(-\infty,x_{(1)}]}}{(3\theta^3)^n / \prod_{i=1}^{n} y_i^4 I_{(-\infty,y_{(1)}]}} = \prod_{i=1}^{n}\left(\frac{y_i}{x_i}\right)^4 \frac{I_{(-\infty,x_{(1)}]}}{I_{(-\infty,y_{(1)}]}}.
$$

When $x_{(1)} < y_{(1)}$, we have

$$
\psi(\theta) = \begin{cases} \prod_{i=1}^{n}\left(\frac{y_i}{x_i}\right)^4 & ,\theta \le x_{(1)} \\ 0 & ,x_{(1)} < \theta < y_{(1)} \\ \text{undefined} & ,\theta > y_{(1)} \end{cases}
$$

When $y_{(1)} < x_{(1)}$, we have

$$
\psi(\theta) = \begin{cases} \prod_{i=1}^{n}\left(\frac{y_i}{x_i}\right)^4 & ,\theta \le y_{(1)} \\ \infty & ,y_{(1)} < \theta < x_{(1)} \\ \text{undefined} & ,\theta > x_{(1)} \end{cases}
$$

When $x_{(1)} = y_{(1)}$,we have

$$
\psi(\theta) = \begin{cases} \prod_{i=1}^{n}\left(\frac{y_i}{x_i}\right)^4 & ,\theta \le x_{(1)} \\ \text{undefined} & ,\theta > x_{(1)} \end{cases}
$$

So we see that $\psi(\theta)$ is constant with respect to θ, where it is defined. Hence, by the Lehmann and Scheffe's method, $X_{(1)}$ is a minimal sufficient statistic for θ.

d) Using the given fact, we have

$$\begin{aligned}
F(y;\theta) &= P(X_{(1)} \leq y) \\
&= 1 - [P(X_1 \geq y)]^n \\
&= 1 - [1 - P(X_1 \leq y)]^n \\
&= 1 - \left(\frac{\theta}{y}\right)^{3n}
\end{aligned}$$

To find the density of $X_{(1)}$, we differentiate the cumulative distribution function,

$$\begin{aligned}
f(y;\theta) &= \frac{\partial}{\partial y} F(y;\theta) \\
&= \frac{\partial}{\partial y}\left[1 - \left(\frac{\theta}{y}\right)^{3n}\right] \\
&= \frac{3n\theta^{3n}}{y^{3n+1}}
\end{aligned}$$

if $y \geq \theta$ and 0 otherwise.

Now, we calculate the expectation of $X_{(1)}$ as

$$\begin{aligned}
EX_{(1)} &= \int_{-\infty}^{\infty} y f(y;\theta) dy \\
&= \int_{\theta}^{\infty} y \frac{3n\theta^{3n}}{y^{3n+1}} dy \\
&= \frac{3n}{3n-1}\theta.
\end{aligned}$$

e) Let g be a function from $[\theta, \infty)$ to $\mathbb{R}$. Suppose that $E_\theta g(X_{(1)}) = 0$ for all $\theta \in \mathbb{R}$. That is, for all $\theta \in \Theta$, we have,

$$\int_{\theta}^{\infty} g(x) f_{X_{(1)}}(x;\theta) dx = 0$$

$$3n\theta^{3n} \int_{\theta}^{\infty} g(x) \frac{1}{x^{3n+1}} dx = 0$$

By differentiating both sides with respect to x and using the fundamental theorem of calculus, we obtain $-g(\theta)\frac{1}{\theta^{3n+1}} = 0$ and so $g(\theta) = 0$ for all $\theta \in \mathbb{R}$. Thus, for all $\theta \in \mathbb{R}$, we have $P_\theta(g(X_{(1)}) = 0) = 1$ and as a result, $X_{(1)}$ is complete.

Consider the estimator $W = \frac{3n-1}{3n} X_{(1)}$. We have $E_\theta(W) = \frac{3n-1}{3n} E_\theta(X_{(1)}) = \frac{3n-1}{3n}\frac{3n}{3n-1}\theta = \theta$, so clearly W is an unbiased estimator of θ. Now, we know that $X_{(1)}$ is complete and sufficient from the previous parts and it is clear from part d) that $\frac{3n-1}{3n} X_{(1)}$ is an unbiased estimator of θ. Thus, by the Theorem of Lehmann-Scheffe, $\frac{3n-1}{3n} X_{(1)}$ is the UMVUE of θ is $E[W|X_{(1)}] = \frac{3n-1}{3n} X_{(1)}$.

Question 4

a) We note that $\sum_{i=1}^{n} X_i \sim Bin(n, \theta)$ since it is made of n i.i.d. Bernoulli observations with probability of success θ. Suppose that an unbiased estimator of $\tau(\theta) = 1/\theta$ did exist and let this estimator be $T(X)$. Then we must have $E_\theta[T(X)] = 1/\theta$. That is,

$$\sum_{x=0}^{n} T(X)\binom{n}{x}\theta^x(1-\theta)^{n-x} = \frac{1}{\theta}.$$

However, this is impossible since no polynomial in θ is equal to $1/\theta$. Furthermore, we note that as $\theta \to 0$, the left hand side tends to $T(0)$ which is finite but the right hand side tends to ∞, again this is not possible. Therefore, there is a contraction and so there exists no unbiased estimator of $1/\theta$.

b) Since $\mathbf{X}$ is made up of i.i.d observations, we have $I_{\mathbf{X}}(\theta) = nI_{X_1}(\theta)$. Now,

$$V(X_1, \theta) = \frac{\partial}{\partial\theta}\log f(X_1; \theta) = \frac{x}{\theta} - \frac{1-x}{1-\theta}$$

$$\frac{\partial}{\partial\theta}V(X_1, \theta) = -\frac{x}{\theta^2} - \frac{1-x}{(1-\theta)^2}.$$

By noting that $X_1 \sim Bern(\theta)$ and that $E(X_1) = \theta$, we then have

$$I_{X_1}(\theta) = E_\theta(-\frac{\partial}{\partial\theta}V(X_1, \theta)) = \frac{E(x)}{\theta^2} + \frac{1-E(x)}{(1-\theta)^2} = \frac{\theta}{\theta^2} + \frac{1-\theta}{(1-\theta)^2} = \frac{1}{\theta(1-\theta)}$$

Thus,

$$I_{\mathbf{X}}(\theta) = nI_{X_1}(\theta) = \frac{n}{\theta(1-\theta)}.$$

Since $\mathbf{Y}$ is made up of i.i.d observations, we have $I_{\mathbf{Y}}(\theta) = nI_{Y_1}(\theta)$. Now,

$$V(Y_1, \theta) = \frac{\partial}{\partial\theta}\log f(Y_1; \theta) = \frac{1}{\theta} - \frac{y-1}{1-\theta}.$$

$$\frac{\partial}{\partial\theta}V(Y_1, \theta) = -\frac{1}{\theta^2} - \frac{y-1}{(1-\theta)^2}.$$

By noting that $Y_1 \sim Geom(\theta)$ and that $E(Y_1) = \frac{1}{\theta}$, we then have

$$I_{Y_1}(\theta) = E_\theta(-\frac{\partial}{\partial\theta}V(Y_1, \theta)) = \frac{1}{\theta^2} + \frac{E(y)-1}{(1-\theta)^2} = \frac{1}{\theta^2} + \frac{1/\theta-1}{(1-\theta)^2} = \frac{1}{\theta^2(1-\theta)}$$

Thus,

$$I_{\mathbf{Y}}(\theta) = nI_{Y_1}(\theta) = \frac{n}{\theta^2(1-\theta)}.$$

Now since $\theta \in (0, 1)$, we have $\theta^2 < \theta$ and so $\frac{n}{\theta^2(1-\theta)} > \frac{n}{\theta(1-\theta)}$, which implies $I_{\mathbf{Y}}(\theta) > I_{\mathbf{X}}(\theta)$. Since we know the variance of the estimate is inversely related to the Fisher information for n large enough, the variance of the sampling scheme $\mathbf{Y}$ would be lower than the variance of the sampling scheme $\mathbf{X}$ and so will lead to more precise inference about the parameter $\tau(\theta) = \theta$.

c) If we were to reparameterize the geometric observations scheme with η instead of θ, we would have

$$f(Y_1;\eta) = \frac{1}{\eta}\left(1-\frac{1}{\eta}\right)^{y-1} = \theta(1-\theta)^{y-1} = f(Y_1;\theta).$$

Now,

$$\log f(Y_1;\eta) = -\log\eta + (y-1)[\log(\eta-1)-\log\eta]$$
$$V(Y_1,\eta) = \frac{\partial}{\partial\theta}\log f(Y_1;\eta) = -\frac{1}{\eta} + (y-1)\left[\frac{1}{\eta-1}-\frac{1}{\eta}\right]$$
$$\frac{\partial}{\partial\theta}V(Y_1,\eta) = \frac{1}{\eta^2} + (y-1)\left[-\frac{1}{(\eta-1)^2}+\frac{1}{\eta^2}\right] = \frac{1}{\eta^2} + (y-1)\left[\frac{1-2\eta}{\eta^2(\eta-1)^2}\right]$$

By noting that $Y_1 \sim Geom(1/\eta)$ and that $E(Y_1) = \frac{1}{1/\eta} = \eta$, we then have

$$I_{Y_1}(\eta) = E_\theta(-\frac{\partial}{\partial\theta}V(Y_1,\eta)) = -\frac{1}{\eta^2} - (\eta-1)\left[\frac{1-2\eta}{\eta^2(\eta-1)^2}\right] = \frac{1}{\eta(\eta-1)}$$

Thus,

$$\begin{aligned}
I_{\mathbf{Y}}(\eta) &= nI_{Y_1}(\eta) \\
&= \frac{n}{\eta(\eta-1)} \\
&= \frac{1}{\frac{1}{\eta^2}\left(1-\frac{1}{\eta}\right)}\frac{1}{\eta^4} \\
&= \frac{1}{\frac{1}{\eta^2}\left(1-\frac{1}{\eta}\right)}\left(-\frac{1}{\eta^2}\right)^2 \\
&= I_{\mathbf{Y}}(\theta(\eta))\left(\frac{\partial\theta}{\partial\eta}\right)^2.
\end{aligned}$$

In general, when a smooth parameter transformation is performed on the fischer information such that $\theta = h(\eta)$, then we would expect the result

$$I_{\mathbf{Y}}(\eta) = I_{\mathbf{Y}}(\theta(\eta))\left(\frac{\partial\theta}{\partial\eta}\right)^2.$$

d) For convenience, we will represent the n i.i.d. observations $\mathbf{Y} = (Y_1, Y_2, ..., Y_n)$ each with geometric distribution rewritten as

$$f_{Y_i}(y) = \theta(1-\theta)^y, y = 0, 1, 2, ...$$

Firstly, we will argue that $T = \sum_{i=1}^n Y_i$ is complete. Since the sum of n i.i.d $Geom(p)$ random variables is a $NegativeBinomial(n,\theta)$, we have $T \sim NegativeBinomial(n,\theta)$. Suppose that $E_\theta g(T) = 0$ for all $\theta \in \Theta$. That is, for all $\theta \in \Theta$, we have,

$$\sum_{t=0}^{n}\binom{n-1+t}{t}\theta^n(1-\theta)^t g(t) = 0$$
$$\theta^n\sum_{i=0}^{n}\binom{n-1+t}{t}(1-\theta)^t g(t) = 0,$$

where the left hand side is a power series in $(1-\theta)$. Then for all possible values of t, we have

$$\binom{n-1+t}{t} g(t) = 0$$

which implies that $g(t) = 0$. Thus, for all $\theta \in \Theta$, we have $P_\theta(g(T) = 0) = 1$ and as a result, $T = \sum_{i=1}^n Y_i$ is complete.

Now, consider the estimator $W = I_{\{Y_1=0\}}(\mathbf{Y})$. We have $E_\theta(W) = Pr(Y_1 = 0) = \theta$, so W is an unbiased estimator of θ. Now,

$$\begin{aligned}
E(W|T=t) &= Pr(Y_1 = 0|T = t) \\
&= \frac{Pr(Y_1 = 0, \sum_{i=1}^n Y_i = t)}{Pr(T = t)} \\
&= \frac{Pr(Y_1 = 0, \sum_{i=2}^n Y_i = t)}{\binom{n-1+t}{t}\theta^n(1-\theta)^t} \\
&= \frac{Pr(Y_1 = 0)P(\sum_{i=2}^n Y_i = t)}{\binom{n-1+t}{t}\theta^n(1-\theta)^t} \\
&= \frac{\theta\binom{n-2+t}{t}\theta^{n-1}(1-\theta)^t}{\binom{n-1+t}{t}\theta^n(1-\theta)^t} \\
&= \frac{\binom{n-2+t}{t}}{\binom{n-1+t}{t}} \\
&= \frac{[(n-2+t)!]/[t!(n-2)!]}{[(n-1+t)!]/[t!(n-1)!]} \\
&= \frac{n-1}{n-1+t}
\end{aligned}$$

so by the Lehmann-Scheffe Theorem, the UMVUE of θ is $\frac{n-1}{n-1+T} = \frac{n-1}{n-1+n\bar{Y}}$.

Appendix

Sketch of density (used in 3.a)

Proof that sum of n i.i.d $Geom(\theta)$ random variables is $NegativeBinomial(n, \theta)$ (used in 4.d)

Let the common m.g.f of $Y_1, ..., Y_n$ be $m_Y(t)$. We then have

$$\begin{aligned} m_Y(t) &= E[e^{Yt}] \\ &= \sum_{i=0}^{\infty} e^{yt} f(y; \theta) \\ &= \sum_{i=0}^{\infty} e^{yt} \theta (1-\theta)^y \\ &= \theta \sum_{i=0}^{\infty} (e^t (1-\theta))^y \\ &= \frac{\theta}{1 - e^t(1-\theta)} \end{aligned}$$

Let $T = \sum_{i=1}^{n} Y_i$. Then the m.g.f. of T is given by

$$\begin{aligned} m_T(t) &= \prod_{i=1}^{n} m_X(t) \\ &= \prod_{i=1}^{n} \left(\frac{\theta}{1 - e^t(1-\theta)} \right) \\ &= \left(\frac{\theta}{1 - e^t(1-\theta)} \right)^n \end{aligned}$$

But this is the m.g.f. of a negative binomial distribution with parameters n and θ so $T \sim NegativeBinomial(n, \theta)$

Alternative proof that T is complete and minimal sufficient (in 1.b)

Firstly, we will show that $T = \sum_{i=1}^{n} X_i$ is complete. It is clear that $T \sim Po(-n\lambda)$. Suppose that $E_\lambda g(T) = 0$ for all $\lambda \in \Theta$. That is, for all $\lambda \in \Theta$, we have,

$$\sum_{i=1}^{n} \frac{e^{-n\lambda}(n\lambda)^t}{t!} g(t) = 0$$
$$e^{-n\lambda} \sum_{i=1}^{n} \frac{(n\lambda)^t}{t!} g(t) = 0,$$

which is a polynomial in terms of θ, so by the fundamental theorem of algebra, we have,

$$\frac{n^t}{t!} g(t) = 0$$

which implies that $g(t) = 0$. Thus, for all $\lambda \in \Theta$, we have $P_\lambda(g(T) = 0) = 1$ and as a result, $T = \sum_{i=1}^{n} X_i$ is complete.

Now we will show that $T = \sum_{i=1}^{n} X_i$ is minimal sufficient. Consider the following expression:

$$\frac{L(x,\theta)}{L(y,\theta)} = \frac{\prod_{i=1}^{n} y_i!}{\prod_{i=1}^{n} x_i!} \lambda^{\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i},$$

which is not a function of θ if and only if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$. Hence, $T = \sum_{i=1}^{n} X_i$ is minimal sufficient.

Explicit calculation of Variance of proposed estimator (in 1.e)

The variance of the proposed estimator in b) is given by:

$$\begin{aligned}
V(b(T)) &= \sum_{t=3}^{\infty} \frac{t(t-1)(t-2)}{6} \frac{(n-1)^{t-3}}{n^t} \frac{e^{-n\lambda}(n\lambda)^t}{t!} - \left(\frac{\lambda^3 e^{-\lambda}}{6}\right)^2 \\
&= \frac{\lambda^3 e^{-n\lambda}}{6} \sum_{t=3}^{\infty} \frac{(\lambda(n-1))^{t-3}}{(t-3)!} - \left(\frac{\lambda^3 e^{-\lambda}}{6}\right)^2 \\
&= \frac{\lambda^3 e^{-n\lambda}}{6} \sum_{t=0}^{\infty} \frac{(\lambda(n-1))^{t}}{t!} - \frac{\lambda^6 e^{-2\lambda}}{36} \\
&= \frac{\lambda^3 e^{-n\lambda}}{6} e^{\lambda(n-1)} - \frac{\lambda^6 e^{-2\lambda}}{36} \\
&= \frac{\lambda^3 e^{-\lambda}}{6} \left(1 - \frac{\lambda^3 e^{-\lambda}}{6}\right)
\end{aligned}$$