THE UNIVERSITY OF NEW SOUTH WALES

DEPARTMENT OF STATISTICS

MID SESSION TEST - 2017 -Tuesday, 5th September (Week 7)

MATH5905

Time allowed: 75 minutes

1. Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be i.i.d. Bernoulli (θ) random variables (that is,

$$f(x, \theta) = \theta^x (1-\theta)^{1-x}, x = \{0, 1\}; \theta \in (0, 1)).$$

   a) Justify that that $T = \sum_{i=1}^n X_i$ is sufficient and complete for θ.

   b) Derive the UMVUE of $h(\theta) = \theta^2$. Justify each step in your answer.

   c) Calculate the Cramer-Rao bound for the minimal variance of an unbiased estimator of $h(\theta) = \theta^2$. Does the variance of the UMVUE of $h(\theta)$ attain this bound? Give reasons.

   d) Find the MLE $\hat{h}$ of $h(\theta)$. Justify your answer.

   e) When testing $H_0 : \theta \le 0.6$ versus $H_1 : \theta > 0.6$ with a 0-1 loss in Bayesian setting with the prior $\tau(\theta) = 12\theta^2(1-\theta)$, what is your decision when $T = 5$ and $n = 8$. (You may use: $\int_0^{0.6} x^7(1-x)^4 dx = 0.00011$.)
   Note: The continuous random variable X has a beta density f with parameters $\alpha > 0$ and $\beta > 0$ if $f(x; \alpha, \beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, x \in (0, 1)$. It holds: $E(X) = \frac{\alpha}{\alpha+\beta}$. Here

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx, B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \Gamma(\alpha+1) = \alpha\Gamma(\alpha).$$

2. Let $X_1, X_2, \ldots, X_n$ be independent random variables, with a density

$$f(x; \theta) = \begin{cases} \frac{\theta}{x^2}, x > \theta, \\ 0 \text{ else} \end{cases}$$

   where $\theta > 0$ is an unknown parameter. If $Z_n = X_{(1)}$, (i.e., the minimal of the n observations) then

   a) Argue that Z_n is a sufficient statistic for θ.

   b) Show that the density of Z_n is

$$f_{Z_n}(z) = \begin{cases} \frac{n\theta^n}{z^{n+1}}, z > \theta, \\ 0 \text{ else} \end{cases}$$

   (**Hint:** You may find the cdf first by using $P(X_{(1)} > x) = P(X_1 > x \cap X_2 > x \ldots \cap X_n > x)$.)

   c) Find the MLE of θ. Justify your answer.

   d) (*) Given that $X_{(1)}$ is complete for θ, find the UMVUE of θ.