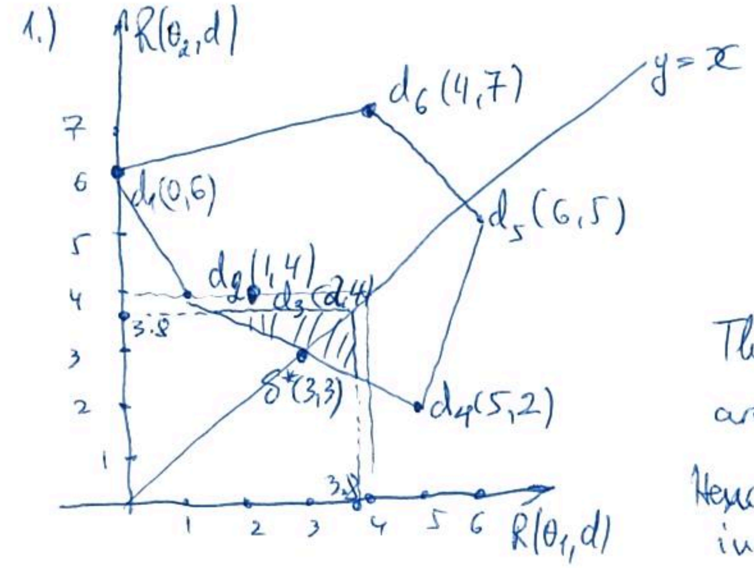
Solutions to Assignment 1, 2014
MATH5905

1.)



a) $d_{minimax}$ delivers

$$\min_{d \in D} \max_{\theta_1, \theta_2} \{R(\theta_1, d), R(\theta_2, d)\}$$

The minimax risks of the 6 rules are (6, 4, 4, 5, 6, 7), respectively. Hence there are ~~two~~ minimax rules in D: namely, d_2 and d_3.

b) See the convex hull plotted above

c) We need to find the intersection of the line $y = x$ with the line $y - 4 = \frac{4-2}{1-5}(x-1)$, i.e. solve $\left| \begin{array}{l} y = x \\ y - 4 = -\frac{1}{2}(x-1) \end{array} \right\} \Rightarrow y = x = 3$

The solution gives $\delta^*(3,3)$ as the minimax rule in $\mathcal{D}$, with a minimax risk of 3.

d) Apparently $\delta^* = \begin{cases} \text{choose } d_2 \text{ with probability } \alpha \\ \text{choose } d_4 \text{ with probability } 1-\alpha \end{cases}$

But $R(\theta_1, \delta^*) = \alpha R(\theta_1, d_2) + (1-\alpha) R(\theta_2, d_4)$

$$3 = \alpha * 1 + (1-\alpha) * 5 \Longrightarrow \alpha = \frac{1}{2}$$

that is: δ^* chooses d_2 with probability $\frac{1}{2}$ and d_4 with probability $\frac{1}{2}$.

e) The line $\overline{d_2 d_4}$ has a slope of $-\frac{1}{2}$. The Bayes risk w.r. to any prior $(p, 1-p)$ is calculated as $px + (1-p)y$ for a risk point (X, Y).

All rules with risk points (x, y) satisfying
$px + (1-p)y = c$, are equivalent w.r. to the Bayes criterion and have the same Bayes risk of c.
We want δ^* to be Bayes $\rightarrow$ $c = 3$
So $px + (1-p)y = 3$ and the slope $-\frac{p}{1-p}$ must match the slope of d_2d_4 (which is $-\frac{1}{2}$)
Hence $-\frac{p}{1-p} = -\frac{1}{2} \Rightarrow p = \frac{1}{3}$
Hence δ^* turns out to be minimax and Bayes w.r. to the prior $(1/3, 2/3) = (p^*, 1-p^*)$ (this is also the least favourable prior)

f) All rules with risk points (x, y) satisfying

$\frac{4}{5}x + \frac{1}{5}y = \text{const}$ are equivalent w.r. to their Bayes risk.
The slope of this line is -4. We want to minimize the Bayes risk when looking for the Bayes rule. Hence we "move" the lines with slopes of (-4) south-west while still keeping intersected with the risk set. The extreme point is $(0, 6)$. Hence d_1 is the Bayes rule w.r. to the prior $(\frac{4}{5}, \frac{1}{5})$. Its Bayes risk is:
$0 * \frac{4}{5} + 6 * \frac{1}{5} = \underline{1.2}$

g) See the shaded area on the graph of the risk set. It extends north-east up to the point $(3.8, 3.8)$.

2.) It is given that $f(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \dots$

and the prior is $\pi(\lambda) = \frac{\lambda^{a-1} e^{-\lambda/b}}{\Gamma(a) b^a}$, $\lambda > 0$

a) For $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_T \end{pmatrix}$ we get $f(\mathbf{x}|\lambda) = \frac{e^{-T\lambda}\lambda^{\sum_{i=1}^T x_i}}{\prod_{i=1}^T (x_i!)}$

Then

$$h(\lambda|\mathbf{X}) \propto e^{-T\lambda - \frac{\lambda}{b}} \lambda^{\sum_{i=1}^T x_i + a - 1} = e^{-\lambda(T + \frac{1}{b})} \lambda^{\sum_{i=1}^T x_i + a - 1}$$

Hence $h(\lambda|\mathbf{X})$ <u>must</u> be the Gamma density with parameters $\sum_{i=1}^T X_i + a$ and $\frac{b}{Tb+1}$, i.e.

$$h(\lambda|\mathbf{X}) \sim \text{Gamma}\left(\sum_{i=1}^T X_i + a, \frac{b}{Tb+1}\right)$$

The Bayes estimator of λ w.r. to quadratic loss is the expected values of this (conditional) density. This is known to be the product of the parameters: $\hat{\lambda}_{Bayes} = \frac{(\sum_{i=1}^T X_i + a) b}{Tb + 1}$

b) Given the data, we have $T = 6$, $b = 2$, $a = 2$
$X = (0, 2, 3, 3, 2, 2)$ so $\sum_{i=1}^T x_i = \underline{12}$

and $\hat{\lambda}_{Bayes} = 2\frac{2}{13}$ (which is just a bit higher than the hypothetical borderline value of 2 so the role of the prior in testing $H_0: \lambda \le 2$ vs $H_1: \lambda > 2$ becomes very important. To test in Bayesian setting we need $P(\lambda \le 2|\mathbf{X})$ and the posterior is Gamma $(14, \frac{2}{13})$

Now $P(\lambda \le 2|\mathbf{X}) = \int_0^2 \frac{e^{-\frac{13\lambda}{2}} \lambda^{13}}{\Gamma(14)(\frac{2}{13})^{14}} d\lambda = \frac{1}{\Gamma(14)(\frac{2}{13})^{14}} \int_0^2 e^{-\frac{13}{2}\lambda} \lambda^{13} d\lambda$

$= 0.427$ (using integration program for example). The posterior being $< \frac{1}{2}$ implies to <u>reject</u> the Bank's claim that $\lambda < 2$.

Q3

$X_1, X_2, \ldots, X_n$ are i.i.d. uniform in $(0, \theta)$

$\Rightarrow f(x_i|\theta) = \frac{1}{\theta} I_{(x_i,\infty)}(\theta)$

Hence $\prod_{i=1}^{n} f(x_i|\theta) = \frac{1}{\theta^n} I_{(x_{(n)},\infty)}(\theta)$

The prior $\pi(\theta) = \begin{cases} \beta\alpha^\beta \theta^{-(\beta+1)}, & \theta > \alpha > 0 \\ 0 \end{cases}$ can also be written

via indicators on "one line" as $\beta\alpha^\beta\theta^{-(\beta+1)} I_{(\alpha,\infty)}(\theta)$

The joint is a product; the product of indicators is an indicator

so we end up with

$$\prod_{i=1}^{n} f(x_i|\theta)\pi(\theta) = \frac{\beta\alpha^\beta\theta^{-(\beta+1)}}{\theta^n} I_{(\max(x_{(n)},\alpha),\infty)}(\theta) =$$

$$= \beta\alpha^\beta\theta^{-(n+\beta+1)} I_{(\max(x_{(n)},\alpha),\infty)}(\theta)$$

Hence the Bayes estimator w.r. quadratic loss is

$$E(\theta|X) = \frac{\cancel{\beta\alpha^\beta}\int_{\max(x_{(n)},\alpha)}^{\infty} \theta^{-(n+\beta)}\,d\theta}{\cancel{\beta\alpha^\beta}\int_{\max(x_{(n)},\alpha)}^{\infty} \theta^{-(n+\beta+1)}\,d\theta} =$$

$$= \frac{(n+\beta)}{(n+\beta-1)} \max(x_{(n)},\alpha)$$

Q4 The observation scheme: we have $n=1$ observation only from a geometric distribution with

$$f(x/\theta) = (1-\theta)^{x-1}\theta$$

where $\theta \in (0,1)$ is the probability of success in a single trial (since our data expresses the total number of trials until the first success).

The two priors are $\pi_1(\theta) = 6\theta(1-\theta)$ of the aviation minister and $\pi_2(\theta) = 4\theta^3$ of the prime minister.

The two corresponding posteriors are:

$h_1(\theta/x) \propto \theta^2(1-\theta)^x$ and $h_2(\theta/x) \propto \theta^4(1-\theta)^{x-1}$

These can easily be identified as

$h_1(\theta/x) \sim \text{Beta}(3, x+1)$

$h_2(\theta/x) \sim \text{Beta}(5, x)$

We have 2 actions available: $a_0 \equiv$ continue
$a_1 \equiv$ abandon

The Losses related to these actions are:-

$$L(\theta, a_0) = \begin{cases} \frac{1}{2} - \theta & \text{if } \theta < \frac{1}{2} \\ 0 & \text{if } \theta \geq \frac{1}{2} \end{cases}$$

and

$$L(\theta, a_1) = \begin{cases} 0 & \text{if } \theta < \frac{1}{2} \\ \theta - \frac{1}{2} & \text{if } \theta \geq \frac{1}{2} \end{cases}$$

For an optimal Bayes decision we need to compare: $Q(x, a_0) = \int_0^{\frac{1}{2}} (\frac{1}{2} - \theta) h(\theta) \, d\theta = \frac{1}{2} \int_0^{\frac{1}{2}} h(\theta|x) d\theta - \int_0^{1/2} \theta h(\theta|x) d\theta$ with

$$Q(x, a_1) = \int_{\frac{1}{2}}^{1} \left(\theta - \frac{1}{2}\right) h(\theta|x) \, d\theta = \int_{\frac{1}{2}}^{1} \theta h(\theta|x) \, dx - \frac{1}{2} \int_{1/2}^{1} h(\theta|x) d\theta.$$

Then a_0 would be preferred to a_1 if $Q(x, a_0) < Q(x, a_1)$ (alternatively if $Q(x, a_0) > Q(x, a_1)$ then a_1 would be preferred (and there is hesitation if $Q(x, a_0) = Q(x, a_1)$)

From the inequality

$$\frac{1}{2} \int_0^{\frac{1}{2}} h(\theta|x) \, d\theta - \int_0^{1/2} \theta h(\theta|x) \, dx < \int_{1/2}^{1} \theta h(\theta|x) \, d\theta - \frac{1}{2} \int_{1/2}^{1} h(\theta|x) \, d\theta$$

we see that adding $\pm \frac{1}{2} \int_0^{\frac{1}{2}} h(\theta|x) d\theta$, noting that $\frac{1}{2} \int_0^1 h(\theta|x) d\theta = \frac{1}{2}$ and re-arranging we get

$$\cancel{\frac{1}{2} \int_0^{1/2} h(\theta|x) \, d\theta} - \int_0^{1/2} \theta h(\theta|x) \, d\theta < \int_{\frac{1}{2}}^{1} \theta h(\theta|x) \, d\theta - \frac{1}{2} + \cancel{\frac{1}{2} \int_0^{1/2} h(\theta|x) \, d\theta}$$

Hence $\quad \frac{1}{2} < \int_0^1 \theta h(\theta|x) \, d\theta = E(\theta|x)$

In other words, we choose a_0 = continue if $E(\theta|x) > \frac{1}{2}$ (and, of course, this decision is also intuitively appealing).

Now, for a Beta (α, β) distribution, the expected value is $\frac{\alpha}{\alpha + \beta}$ which implies in our case:

$E(\theta|x) = 3/(x+4)$ for aviation minister

$E(\theta|x) = 5/(x+5)$ for prime minister

Hence the aviation minister wants the project to continue when $x = 1$, hesitates when $x = 2$ and wants to stop when $x = 3, 4, 5 \ldots$. The prime minister wants to continue when $x = 1, 2, 3, 4$; hesitates when $x = \mathbf{5}$ and wants to stop when $x = 6, 7, 8, \ldots$. Obviously, for $\underline{x = 3}$ and $\underline{x = 4}$ we have the most serious disagreement

Q5

a) Let $Z_i = \frac{X_i}{\sigma}, i=1,2,\dots,n, \ \sigma > 0$

$$P(Z_i < z) = P(X_i < \sigma z) = F_\sigma(\sigma z) = F\left(\frac{\cancel{\sigma} z}{\cancel{\sigma}}\right) = F(z)$$

i.e. the distribution of Z_i does not depend on σ.

$$F_{X_1/X_n, X_2/X_n, \dots, X_{n-1}/X_n}(y_1, y_2, \dots, y_{n-1}) = P\left(\frac{X_1}{X_n} \le y_1, \dots, \frac{X_{n-1}}{X_n} \le y_{n-1}\right)$$

$$= P\left(\frac{X_1/\sigma}{X_n/\sigma} \le y_1, \dots, \frac{X_{n-1}/\sigma}{X_n/\sigma} \le y_{n-1}\right) = P\left(\frac{Z_1}{Z_n} \le y_1, \dots, \frac{Z_{n-1}}{Z_n} \le y_{n-1}\right)$$

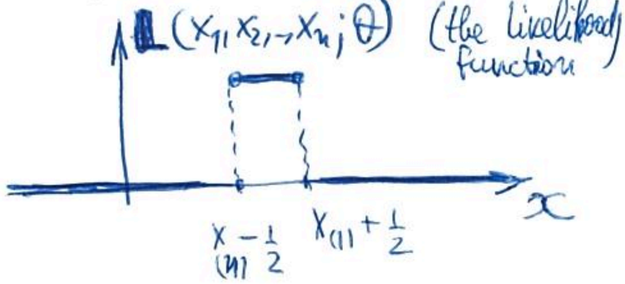
which also does not depend on σ. Then for any statistic $T = g\left(\frac{X_1}{X_n}, \dots, \frac{X_{n-1}}{X_n}\right)$, we arrive at the conclusion, that its distribution would not depend on σ.

b) Since $X_1, \dots, X_n$ are i.i.d. uniform in $[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$ their density is $f(x_i, \theta) = 1 * I_{[\theta-\frac{1}{2}, \theta+\frac{1}{2}]}(x_i) = f(x_i - \theta)$

Then $L(X_1, X_2, \dots, X_n; \theta) = \prod_{i=1}^{n} I_{[\theta-\frac{1}{2}, \theta+\frac{1}{2}]}(x_i) = \prod_{i=1}^{n} f(x_i - \theta)$

$$= I_{[\theta-\frac{1}{2}, \infty)}(X_{(1)}) \, I_{(-\infty, X_{(n)}]}(\theta + \tfrac{1}{2})$$

$$= I_{(-\infty, X_{(1)}+\frac{1}{2})}(\theta) \, I_{(X_{(n)}-\frac{1}{2}, \infty)}(\theta)$$



Hence the Pitman estimator is

$$\hat{\theta}_P = \frac{\int_{-\infty}^{\infty} \theta \prod_{i=1}^{n} f(X_i - \theta) d\theta}{\int_{-\infty}^{\infty} \prod_{i=1}^{n} f(X_i - \theta) d\theta} = \frac{\int_{X_{(n)}-\frac{1}{2}}^{X_{(1)}+\frac{1}{2}} \theta \, d\theta}{\int_{X_{(n)}-\frac{1}{2}}^{X_{(1)}+\frac{1}{2}} 1 * d\theta} = \frac{\frac{1}{2}\left[(X_{(1)}+\frac{1}{2})^2 - (X_{(n)}-\frac{1}{2})^2\right]}{X_{(1)}+\frac{1}{2} - X_{(n)} + \frac{1}{2}}$$

$$= \boxed{\frac{X_{(1)} + X_{(n)}}{2}}$$