Solutions to MST 3811/3911 (2011) p. 1

Q1) a) The density is $(1+\theta) * 1 * e^{\theta \ln x} = a(\theta) b(x) e^{c(\theta) d(x)}$ with $d(x) = \ln x$. Hence $T = \sum_{i=1}^{n} d(x_i) = \boxed{\sum_{i=1}^{n} \ln x_i}$ is a minimal sufficient and complete statistics for θ.

b) $\ln f(x_i;\theta) = \ln(1+\theta) + \theta \ln x$

$\frac{\partial}{\partial\theta} \ln f(x_i;\theta) = \frac{1}{1+\theta} + \ln x$; $\frac{\partial^2}{\partial\theta^2} \ln f(x_i;\theta) = -\frac{1}{(1+\theta)^2}$

Hence $I_{X_1}(\theta) = -E\left(-\frac{1}{(1+\theta)^2}\right) = \frac{1}{(1+\theta)^2}$ is the information in one observation. The $I_{\underline{X}}(\theta) = \frac{n}{(1+\theta)^2}$ is the information in the whole sample and, since T is sufficient, $I_T(\theta) = I_{\underline{X}}(\theta) = \boxed{\frac{n}{(1+\theta)^2}}$

c) $\log L(X;\theta) = n \log(1+\theta) + \theta \sum_{i=1}^{n} \ln x_i$

$V(X;\theta) = \frac{\partial}{\partial\theta} \log L(X;\theta) = \frac{n}{1+\theta} + \sum_{i=1}^{n} \ln X_i$

Hence $V(X;\hat\theta) = 0 \implies \frac{n}{1+\hat\theta} = -\sum_{i=1}^{n} \ln X_i$ and $\boxed{\hat\theta = -1 - \frac{n}{\sum_{i=1}^{n} \ln X_i}}$

By invariance property of MLE:

$\widehat{\tau(\theta)} = \tau(\hat\theta) = \frac{1}{1+\hat\theta} = \boxed{\frac{-\sum_{i=1}^{n} \ln x_i}{n}}$

Now since $E_\theta V(X;\theta) = 0 = n E_\theta\left(\frac{1}{1+\theta} + \sum_{i=1}^{n} \frac{\ln X_i}{n}\right) = n E_\theta(\tau(\theta) - \widehat{\tau(\theta)})$ and $n \neq 0$ we see that $\underline{E\widehat{\tau(\theta)} = E\tau(\theta) = \tau(\theta)}$ holds that is, $\widehat{\tau(\theta)}$ is unbiased for $\tau(\theta)$. Since $\hat\theta$ is a nonlinear transformation of $\widehat{\tau(\theta)}$ then $E\hat\theta \neq E\theta$ (however, $\hat\theta$ is still asymptotically unbiased for θ)

d) $CRLB = \frac{\left[\frac{\partial}{\partial\theta}\left(\frac{1}{1+\theta}\right)\right]^2}{I_{\underline{X}}(\theta)} = \frac{1}{(1+\theta)^4 \cdot n} (1+\theta)^2 = \boxed{\frac{1}{n(1+\theta)^2}}$ and since $V = -n(\hat\tau - \tau(\theta))$ we once again see that $\hat\tau$ is unbiased for $\tau(\theta)$ and is UMVUE that attains the bound

e) $\hat{\tau} \approx N\left(\tau(\theta), \frac{1}{n(1+\theta)^2}\right) \approx N\left(\tau(\theta), \frac{1}{n(1+\hat{\theta})^2}\right) \approx N\left(\tau(\theta), \frac{\hat{\tau}^2}{n}\right)$

The we can pretend that we have one observation ($\hat{\tau}$) from $N\left(\tau(\theta), \frac{\hat{\tau}^2}{n}\right)$ and we want to construct CI for the mean of this normal distribution. Hence, 95% CI will be

$$\hat{\tau} \pm 1.96 \frac{\hat{\tau}}{\sqrt{n}} = .55 \pm 1.96 \frac{0.55}{\sqrt{40}} = (0.3795, 0.72045)$$

f) $\tau(\theta) = 3e^{-3(\theta+1)}, \theta > -1$

$$h(\theta|\underline{X}) \propto (1+\theta)^n \left(\prod_{i=1}^n x_i\right)^\theta e^{-3(\theta+1)} \propto (1+\theta)^n e^{\left(\sum_{i=1}^n \ln X_i - 3\right)(1+\theta)}$$

Setting $\theta+1 = \eta > 0$, this is $\propto \eta^n e^{-\left(-\sum_{i=1}^n \ln X_i + 3\right)\eta}$

which is a Gamma density with $\alpha = n+1$; $\beta = \frac{1}{3 - \sum_{i=1}^n \ln X_i}$

Now $E(\theta/\underline{X}) = E(\eta - 1/\underline{X}) = E(\eta/\underline{X}) - 1$

But $E(\eta|\underline{X}) = \frac{n+1}{3 - \sum \ln X_i}$ So $\hat{\theta}_{Bayes} = E(\theta/\underline{X}) = \boxed{-1 - \frac{n+1}{\sum_{i=1}^n \ln X_i - 3}}$

when n is large, we see that $\hat{\theta}_{Bayes} \approx -1 - \frac{n}{\sum \ln X_i} = \hat{\theta}_{MLE}$

g) Since $h(\theta_1, \theta_2) = \frac{\theta_1+1}{\theta_2+1} \Rightarrow \nabla h(\theta_1, \theta_2) = \left(\frac{1}{\theta_2+1}, -\frac{(1+\theta_1)}{(1+\theta_2)^2}\right)$

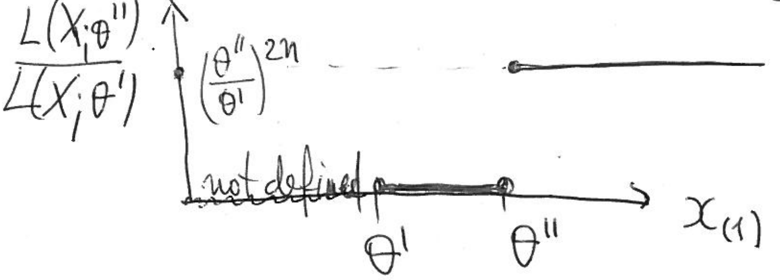
Because of the independence of the two samples, and using b), we have $I_{\binom{X_1}{Y_1}}(\theta_1, \theta_2) = \begin{pmatrix} \frac{1}{(1+\theta_1)^2} & 0 \\ 0 & \frac{1}{(1+\theta_2)^2} \end{pmatrix}$

The delta method says that the asymptotic distribution of $\sqrt{n}(\hat{h} - h(\theta_1, \theta_2))$ is e zero-mean normal and the variance is

$$\nabla h' \, I_{\binom{X_1}{Y_1}}(\theta_1, \theta_2)^{-1} \nabla h = \left(\frac{1}{\theta_2+1}, -\frac{(1+\theta_1)}{(1+\theta_2)^2}\right) \begin{pmatrix} (1+\theta_1)^2 & 0 \\ 0 & (1+\theta_2)^2 \end{pmatrix} \begin{pmatrix} \frac{1}{\theta_2+1} \\ -\frac{1+\theta_1}{(1+\theta_2)^2} \end{pmatrix} = \frac{2(\theta_1+1)^2}{(\theta_2+1)^2}$$

Q2) a) Obviously $L(x;\theta) = \dfrac{(2\theta^2)^n}{\left(\prod_{i=1}^n x_i\right)^3} I_{[\theta,\infty)}(x_{(1)})$

Hence for two fixed values of θ such that $\theta'' > \theta' > 0$ we have $\dfrac{L(X;\theta'')}{L(X;\theta')} = \left(\dfrac{\theta''}{\theta'}\right)^{2n} \dfrac{I_{[\theta'',\infty)}(X_{(1)})}{I_{[\theta',\infty)}(X_{(1)})}$ which is



obviously a non-decreasing function in the statistic $X_{(1)}$. This shows the MLR property

b) Integrating $f(x;\theta)$ we get the cdf of a single observation: $F_X(x;\theta) = \begin{cases} 0 & x < \theta \\ 1 - \dfrac{\theta^2}{x^2}, & x \ge \theta > 0 \end{cases}$

Now $F_Z(z;\theta) = 1 - P(\text{all } X_i \ge z) = 1 - [1 - F_X(z;\theta)]^n$

$= \begin{cases} 0 & \text{if } z < \theta \\ 1 - \left(\dfrac{\theta}{z}\right)^{2n} & \text{if } z \ge \theta \end{cases}$ Hence $f_Z(z;\theta) = \begin{cases} 0 & z < \theta \\ \dfrac{2n\theta^{2n}}{z^{2n+1}}, & z \ge \theta > 0 \end{cases}$

c) The BG theorem specifies the structure of UMP α test as:

$\varphi^*(X) = \begin{cases} 1 & \text{if } X_{(1)} > c \\ 0 & \text{if } X_{(1)} \le c \end{cases}$

To determine c, we "exhaust the level": $\alpha = E_{\theta=4}\varphi^* = P_{\theta=4}(X_{(1)} > c) =$

$= 1 - F_Z(c;4) = \begin{cases} 1 & c < 4 \\ \left(\dfrac{4}{c}\right)^{2n} & c \ge 4 \end{cases} \longrightarrow \alpha^{\frac{1}{2n}} = \dfrac{4}{c} \to \boxed{c = 4\alpha^{-\frac{1}{2n}}}$

Sketch of power: