UNIVERSITY OF NEW SOUTH WALES
DEPARTMENT OF STATISTICS

MATH5856 Introduction to Statistics and Statistical Computations
Tutorial Problems week 12 solutions

1. (a)

$$\begin{aligned} f(\theta|x) &= \theta^{\sum(x_i-1)}(1-\theta)^n \frac{\theta^{p-1}(1-\theta)^{q-1}}{Be(p,q)} \\ &= \frac{\theta^{\sum(x_i-1)+p-1}(1-\theta)^{n+q-1}}{Be(p,q)} \end{aligned}$$

   hence $f(\theta|x) \sim Be(\sum x_i - n + p, n+q)$.

   (b)

$$f(\theta|x) \propto e^{-n\theta}\theta^{\sum x_i} e^{-\theta} = e^{-(n+1)\theta}\theta^{\sum x_i}$$

   hence $f(\theta|x) \sim gamma(\sum x_i + 1, n+1)$.

2. $x|\theta \sim Bin(100, \theta)$,

$$\begin{aligned} f(\theta|x) &\propto \theta^3(1-\theta)^{100-3}\theta(1-\theta)^{199} \\ &= \theta^4(1-\theta)^{296} \end{aligned}$$

   then $f(\theta|x) \sim Be(5, 297)$.

   Posterior distributions will have parameter $p = 4$ and $q = 98$ of beta distribution, or solve $E(\theta) = p/(p+q) = 4/102$ and $var(\theta) = \frac{pq}{(p+q)^2(p+q+1)} = 0.003658$ for p. The prior must be from $Beta(p', q')$ distribution then $p' + x = 4$ and $q' + n - x = 98$, therefore $p' = 1$ and $q' = 1$.

3. prior has distribution $N(10, 0.25)$ and likelihood is $N(\theta, 1)$ then from the conjuate table gives that $p(\theta|x) \sim N(\frac{10*4+12*31/3}{4+12}, \frac{1}{4+12}) = N(10.25, 0.0625)$ thus

   $p(\theta|x > 10) = 0.84$.

4. ```
> plot(Year,Rate)
> # Suggestion of a nonlinear trend
> firearm.lm<-lm(Rate ~ Year, data=firearm)
> summary(firearm.lm)

Call: lm(formula = Rate ~ Year, data = firearm)
Residuals:
     Min   1Q   Median   3Q   Max
 -0.3814 -0.1682 -0.01667 0.2207 0.307

Coefficients:
                Value Std. Error   t value Pr(>|t|)
(Intercept) 306.3199   29.8444   10.2639   0.0000
       Year   -0.1521   0.0150   -10.1424   0.0000

Residual standard error: 0.251 on 13 degrees of freedom
Multiple R-Squared: 0.8878
F-statistic: 102.9 on 1 and 13 degrees of freedom, the p-value is 1.527e
-007

Correlation of Coefficients:
     (Intercept)
Year -1
> # The F-test and partial t-test for the slope indicate
> # strong evidence of a linear trend
> par(mfrow=c(2,3))
> # Split graphsheet into six, two rows and three
> # columns so that linear model diagnostic plots
> # can fit on one graphsheet.
> plot(firearm.lm)
```

5. ```
> height<-c(68,61,63,70,69,65,72)
> weight<-c(155,99,115,205,170,125,220)
> hw.lm<-lm(height ~ weight)
> summary(hw.lm)

Call: lm(formula = height ~ weight)
Residuals:
```

```
    1      2       3      4      5      6       7
 1.191 -1.085 -0.4345 -1.027 0.9257 0.7219 -0.2924

Coefficients:
              Value Std. Error t value Pr(>|t|)
(Intercept) 53.7330  1.4802    36.3007  0.0000
     weight  0.0844  0.0092     9.1897  0.0003

Residual standard error: 1.03 on 5 degrees of freedom
Multiple R-Squared: 0.9441
F-statistic: 84.45 on 1 and 5 degrees of freedom, the p-value is 0.00025
6

Correlation of Coefficients:
       (Intercept)
weight -0.9648
> hwnewdata.df<-data.frame(weight=100)
> hw.pred<-predict(hw.lm,newdata=hwnewdata.df,
+ se.fit=T)
> hw.pred
$fit:
       1
 62.1691

$se.fit:
         1
 0.6416656

$residual.scale:
[1] 1.029776

$df:
[1] 5

> pointwise(hw.pred,coverage=0.95)
$upper:
        1
 63.81855
```

```
$fit:
       1
 62.1691

$lower:
        1
 60.51964

>resid<-height-hw.lm$fitted
> plot(hw.lm$fitted,resid,xlab="Fitted values",
+ ylab="Residuals")
```

6.
```
> cheese.dat<-read.table("cheese.dat", header=T)
> pairs(cheese.dat)

#all predictors seem to have a linear relationship with taste

>cheese.lm<-lm(taste~., data=cheese.dat)
>summary(cheese.lm)

Call:
lm(formula = taste ~ ., data = cheese.dat)

Residuals:
    Min      1Q  Median      3Q     Max
-17.391  -6.612  -1.009   4.908  25.449

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -28.8768    19.7354  -1.463  0.15540
acetic        0.3277     4.4598   0.073  0.94198
H2S           3.9118     1.2484   3.133  0.00425 **
lactic       19.6705     8.6291   2.280  0.03108 *
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 10.13 on 26 degrees of freedom
```

```
Multiple R-Squared: 0.6518,Adjusted R-squared: 0.6116
F-statistic: 16.22 on 3 and 26 DF,  p-value: 3.81e-06

#F test is the test for the hypothesis that at least
#one of the predictor should be included in the
# model, this has a p-value of almost 0, suggesting
#strong evidence that at least one predictor is important

#T-test suggest the exclusion of acetic first, so we fit
#the model without this predictor


>cheese1.lm<-lm(taste~.-acetic, data=cheese.dat)
> summary(cheese1.lm)

Call:
lm(formula = taste ~ . - acetic, data = cheese.dat)

Residuals:
    Min      1Q  Median      3Q     Max
-17.343  -6.530  -1.164   4.844  25.618

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -27.592      8.982  -3.072  0.00481 **
H2S            3.946      1.136   3.475  0.00174 **
lactic        19.887      7.959   2.499  0.01885 *
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 9.942 on 27 degrees of freedom
Multiple R-Squared: 0.6517,Adjusted R-squared: 0.6259
F-statistic: 25.26 on 2 and 27 DF,  p-value: 6.551e-07

#all remaining predictors are significant at 0.05 level,
#so the best model is the one without acetic
```

7. Proof of a) is obvious.

b) $a^T Y = \sum_{i=1}^{k} a_i Y_i$ and hence

$$\begin{aligned} E(a^T Y) &= E(\sum_{i=1}^{k} a_i Y_i) \\ &= \sum_{i=1}^{k} a_i E(Y_i) \\ &= \sum_{i=1}^{k} a_i \mu_i \\ &= a^T \mu \end{aligned}$$

c) $(AY)_i = \sum_{m=1}^{k} A_{im} Y_m$ and so

$$\begin{aligned} E((AY)_i) &= E(\sum_{m=1}^{k} A_{im} Y_m) \\ &= \sum_{m=1}^{k} A_{im} E(Y_m) \\ &= \sum_{m=1}^{k} A_{im} \mu_m \\ &= (A\mu)_i. \end{aligned}$$

8. a)

$$
\begin{aligned}
Var(a^TY) &= Var(\sum_{i=1}^{k} a_iY_i) \\
&= E\left(\left(\sum_{i=1}^{k} a_iY_i - \sum_{i=1}^{k} a_i\mu_i\right)^2\right) \\
&= E\left(\left(\sum_{i=1}^{k} a_i(Y_i - \mu_i)\right)^2\right) \\
&= E\left(\sum_{i=1}^{k}\sum_{j=1}^{k} a_ia_j(Y_i - \mu_i)(Y_j - \mu_j)\right) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} a_ia_jE((Y_i - \mu_i)(Y_j - \mu_j)) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} a_ia_jV_{ij} \\
&= a^TVa.
\end{aligned}
$$

b)

$$
\begin{aligned}
Cov(Z_i, Z_j) &= Cov\left(\sum_{q=1}^{k} A_{iq}Y_q, \sum_{r=1}^{k} A_{jr}Y_r\right) \\
&= E\left(\left(\sum_{q=1}^{k} A_{iq}(Y_q - \mu_q)\right)\right. \\
&\qquad \left.\left(\sum_{r=1}^{k} A_{jr}(Y_r - \mu_r)\right)\right) \\
&= E\left(\sum_{q=1}^{k}\sum_{r=1}^{k} A_{iq}A_{jr}(Y_r - \mu_r)(Y_q - \mu_q)\right) \\
&= \sum_{q=1}^{k}\sum_{r=1}^{k} A_{iq}A_{jr}V_{qr} \\
&= \sum_{q=1}^{k} A_{iq}\sum_{r=1}^{k} V_{qr}A_{jr} \\
&= \sum_{q=1}^{k} A_{iq}(VA^T)_{qj} \\
&= (AVA^T)_{ij}
\end{aligned}
$$