UNIVERSITY OF NEW SOUTH WALES
DEPARTMENT OF STATISTICS

MATH5856 Introduction to Statistics and Statistical Computations
Tutorial Problems week 10 (ESTIMATION)

1. Let $X_1, \ldots, X_5$ be a random sample from a distribution with mean μ and variance σ^2. Two estimators of μ are

$$\widehat{\mu}_1 = \frac{X_1 + X_2 + X_3 + X_4 + X_5}{5} \quad \text{and} \quad \widehat{\mu}_2 = \frac{X_1 + 2X_2 + 3X_3 + 4X_4 + 5X_5}{15}.$$

Which estimator is the better one? Provide mathematical support for your answer. Intuitively, why is one better than the other?

2. Let $X_1, \ldots, X_n$ be a random sample from the Poisson(λ) distribution. Let

$$\widehat{\lambda} = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

be an estimator of λ. Find the bias, standard error (se) and mean squared error (MSE) of $\widehat{\lambda}$ (i.e. find $\mathsf{bias}(\widehat{\lambda})$, $\mathsf{se}(\widehat{\lambda})$ and $\text{MSE}(\widehat{\lambda})$).

3. Let $X_1, \ldots, X_n$ be a random sample from the $N(\mu, \sigma^2)$ distribution.

   (a) Let

$$\widehat{\mu} = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

   be an estimator of μ. Find $\mathsf{bias}(\widehat{\mu})$, $\mathsf{se}(\widehat{\mu})$ and $\text{MSE}(\widehat{\mu})$.

   (b) Let

$$\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

   be an estimator of σ^2. Find $\mathsf{bias}(\widehat{\sigma}^2)$, $\mathsf{se}(\widehat{\sigma}^2)$ and $\text{MSE}(\widehat{\sigma}^2)$.

4. Let $X_1, \ldots, X_n$ be a random sample the Uniform$(0, \theta)$ distribution.

   (a) Obtain the maximum likelihood estimator $\widehat{\theta}$.

   (b) Show that $\widehat{\theta}$ is consistent.

5. Find the maximum likelihood estimate of θ based on a random sample of size n from the density
$f_X(x; \theta) = 2\theta x e^{-\theta x^2}, \ x \geq 0; \ \theta > 0.$

6. Determine the maximum likelihood estimates of μ and σ^2 when $X_1, X_2, \ldots, X_n$ are independent $N(\mu, \sigma^2)$ variables.

7. Suppose $X_1, X_2, \ldots, X_n$ are independent random variables each with density $f_X(x; \theta) = e^{\theta - x}, x \geq \theta$. Find the maximum likelihood estimator of θ based on $X_1, X_2, \ldots, X_n$.

8. Suppose that an epidemiologist wishes to compare the number of emergency asthma admissions per day for two different regions. One hundred independent observations on the numbers of admissions are to be measured for each region. Let X_i denote the ith observation for the first region and Y_i denote the ith observation for the second region. For the first region, assume that the number of admissions per day has a Poisson distribution with mean λ_X. For the second region, assume

that the number of admissions per day has a Poisson distribution with mean λ_Y.

(a) Write down the joint probability function of

$$X_1, \ldots, X_{100}, Y_1, \ldots, Y_{100}.$$

(b) Derive the maximum likelihood estimators for each of λ_X and λ_Y.

(c) Write down the likelihood function for λ under the hypothesis

$$\lambda = \lambda_X = \lambda_Y.$$

(d) Show that, under the hypothesis of (iii), the maximum likelihood estimator for λ is

$$\widehat{\lambda} = \tfrac{1}{2}(\overline{X} + \overline{Y}).$$