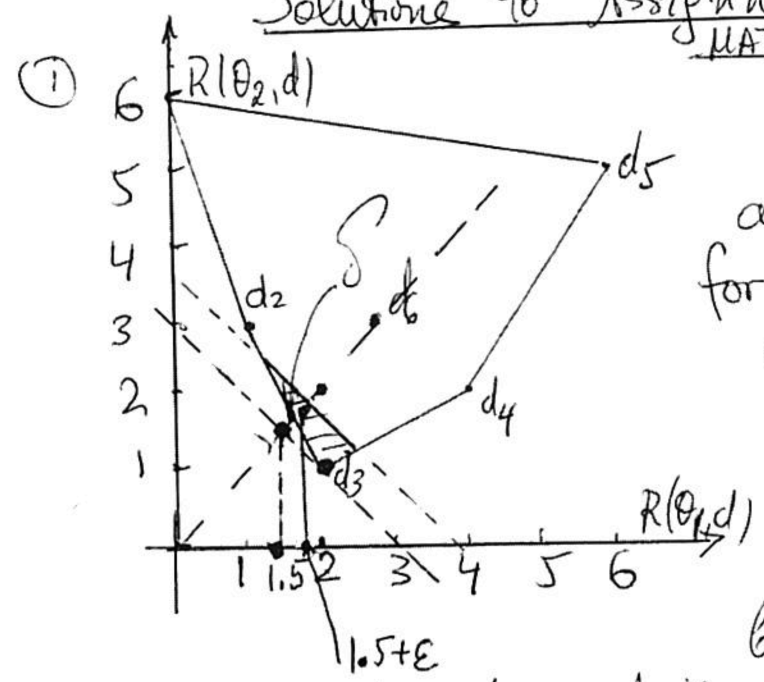
Solutions to Assignment 1, 2017
MATH5905

(1)



a) The individual minimax risks for the 6 rules $d_1 \div d_6$ are 6, 3, 2, 4, 6, 3 respectively. Hence d_3 is the minimax rule in the set D.

b) see graph

c) Taking intersection of the line $y=x$ with the line connecting $\overline{d_2 d_3}$ we end up with the minimax rule δ in the set $\mathcal{D}$.

Now: $\overline{d_2 d_3}$: $y-1 = \frac{3-1}{1-2}(x-2) = -2(x-2)$. Solving together with $y=x$ gives for δ: $3x=5 \Rightarrow x=y=5/3$

Hence $(5/3, 5/3)$ represents the risk point corresponding to the minimax decision rule in $\mathcal{D}$ and its minimax risk is $\frac{5}{3}$.

d) δ chooses d_3 with probability α and d_2 with probability $(1-\alpha)$. Hence for the θ_1-component of the risks we have: $\frac{5}{3} = \alpha * 2 + (1-\alpha) * 1 \Rightarrow \alpha = \frac{2}{3}$

i.e., δ chooses d_3 with probability $\frac{2}{3}$ and d_2 with probability $1/3$.

e) Writing the $\overline{d_2 d_3}$ line in the form $2x+y-5=0$ we see that the vector $(2,1)$ is $\perp$ to this line. We norm this vector so that the sum of the components is 1. This gives us $(p_1, p_2) = (\frac{2}{3}, \frac{1}{3})$ as the prior for which δ is Bayes (this is also the least favourable prior).

f) the line $\frac{1}{3}x + \frac{2}{3}y = c$ has a slope $k = -\frac{1}{2}$. By moving lines with slopes $(-\frac{1}{2})$ as south-west as possible we end up with $d_3 \Rightarrow d_3$ is the Bayes rule w.r.to the prior $(\frac{1}{3}, \frac{2}{3})$. Its Bayes risk is

$$\frac{1}{3} * 2 + \frac{2}{3} * 1 = 1\frac{1}{3}$$

g) See the shaded area. The line bounding the area from above (North-East) has a slope equal to -1.

---

② We have $f(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, ---$ and a prior $\tau(\lambda) = \frac{\lambda^{a-1}e^{-\lambda/b}}{\Gamma(a)b^a}$, $\lambda > 0$

a) For sample $\mathbb{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ the joint conditional distribution of the Sample given λ is $f(\mathbb{X}|\lambda) = \frac{e^{-T\lambda}\lambda^{\sum_{i=1}^T x_i}}{\prod_{i=1}^T x_i!}$

Hence $h(\lambda|\mathbb{X}) \propto e^{-T\lambda - \frac{\lambda}{b}} \lambda^{\sum_{i=1}^T x_i + a - 1}$

which implies that $h(\lambda|\mathbb{X})$ must be the Gamma density with parameters $\sum_{i=1}^T x_i + a$ and $\frac{b}{Tb+1}$, that is,

$$h(\lambda|\mathbb{X}) \sim \text{Gamma}\left(\sum_{i=1}^T x_i + a, \frac{b}{Tb+1}\right)$$

The Bayes estimator of λ w.r. to quadratic loss is the expected value of the posterior distribution of λ given $\mathbb{X}$. This expected value is known to be the product of the parameters $\Longrightarrow$ $\hat{\lambda}_{Bayes} = \frac{(\sum_{i=1}^T x_i + a)\, b}{Tb+1}$

b) For the given data: $T = 7$, $b = 2$, $a = 2$
$\mathbb{X} = (0, 2, 3, 3, 2, 2, 4)$. Hence $\sum_{i=1}^7 x_i = 16$, $\hat{\lambda}_{Bayes} = \frac{36}{15}$

This is a bit higher than the hypothetical borderline value of 2 so the role of the prior becomes critical. We need $P(\lambda \le 2|\mathbb{X})$ for the posterior which is $\text{Gamma}(18, \frac{2}{15})$

Now $P(\lambda \le 2|\mathbb{X}) = \int_0^2 \frac{e^{-\frac{15\lambda}{2}}\lambda^{17}}{\Gamma(18)\left(\frac{2}{15}\right)^{18}} d\lambda = \frac{1}{\Gamma(18)\left(\frac{2}{15}\right)^{18}} \int_0^2 e^{-\frac{15\lambda}{2}}\lambda^{17} d\lambda =$

$= \frac{15^{18}}{\Gamma(18)2^{18}} * 0.0158447 = 0.2511$ hence we reject H_0 and decide that the claim of the Bank is false

3

Q3 i) $ET = E\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} EX_i = \sum_{i=1}^{n} E_{\theta_i}\left(E(X_i|\theta_i)\right) =$

$= \sum_{i=1}^{n} E_{\theta_i}(\theta_i) = n\frac{\alpha}{\alpha+\beta}$

Since $X_i|\theta_i \sim Bernoulli(\theta_i)$

ii) $Var\,T = Var\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} Var\,X_i$

since X_i are independent

Now $Var\,X_i = EX_i^2 - (EX_i)^2 = E_{\theta_i}\left(E(X_i^2|\theta_i)\right) - \left(E_{\theta_i}E(X_i|\theta_i)\right)^2$

$= E_{\theta_i}\left(E(X_i|\theta_i)\right) - \left(E_{\theta_i}(E(X_i|\theta_i))\right)^2 = \frac{\alpha}{\alpha+\beta} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 = \frac{\alpha\beta}{(\alpha+\beta)^2}$

since $X_i^2 = X_i$ for 0,1-valued variables

Hence $Var\,T = n\frac{\alpha\beta}{(\alpha+\beta)^2}$

Q4 The observation scheme: we have $n=1$ observation only from a geometric distribution with

$$f(x/\theta) = (1-\theta)^{x-1}\theta$$

where $\theta \in (0,1)$ is the probability of success in a single trial (since our data expresses the total number of trials until the first success).

The two priors are $\pi_1(\theta) = 6\theta(1-\theta)$ of the transport minister and $\pi_2(\theta) = 3\theta^2$ of the prime minister.

The two corresponding posteriors are:

$h_1(\theta/x) \propto \theta^2(1-\theta)^x$ and $h_2(\theta/x) \propto \theta^3(1-\theta)^{x-1}$

These can easily be identified as

$h_1(\theta/x) \sim \text{Beta}(3, x+1)$

$h_2(\theta/x) \sim \text{Beta}(4, x)$

We have 2 actions available: $a_0 \equiv$ continue
$a_1 \equiv$ abandon

The losses related to these actions are:

$$L(\theta, a_0) = \begin{cases} \frac{1}{2} - \theta & \text{if } \theta < \frac{1}{2} \\ 0 & \text{if } \theta \geq \frac{1}{2} \end{cases}$$ and

$$L(\theta, a_1) = \begin{cases} 0 & \text{if } \theta < \frac{1}{2} \\ \theta - \frac{1}{2} & \text{if } \theta \geq \frac{1}{2} \end{cases}$$

5

For an optimal Bayes decision we need to compare: $Q(x, a_0) = \int_0^{\frac{1}{2}} (\frac{1}{2} - \theta) h(\theta) d\theta = \frac{1}{2} \int_0^{\frac{1}{2}} h(\theta|x) d\theta - \int_0^{1/2} \theta h(\theta|x) d\theta$ with

$$Q(x, a_1) = \int_{\frac{1}{2}}^{1} (\theta - \tfrac{1}{2}) h(\theta|x) d\theta = \int_{\frac{1}{2}}^{1} \theta h(\theta|x) dx - \frac{1}{2} \int_{1/2}^{1} h(\theta|x) d\theta.$$

Then a_0 would be preferred to a_1 if $Q(x, a_0) < Q(x, a_1)$ (alternatively if $Q(x, a_0) > Q(x, a_1)$ then a_1 would be preferred (and there is hesitation if $Q(x, a_0) = Q(x, a_1)$)

From the inequality

$$\frac{1}{2} \int_0^{\frac{1}{2}} h(\theta|x) d\theta - \int_0^{1/2} \theta h(\theta|x) dx < \int_{1/2}^{1} \theta h(\theta|x) d\theta - \frac{1}{2} \int_{1/2}^{1} h(\theta|x) d\theta$$

we see that adding $\pm \frac{1}{2} \int_0^{\frac{1}{2}} h(\theta|x) d\theta$, noting that $\frac{1}{2} \int_0^1 h(\theta|x) d\theta = \frac{1}{2}$ and re-arranging we get

$$\cancel{\frac{1}{2} \int_0^{1/2} h(\theta|x) d\theta} - \int_0^{1/2} \theta h(\theta|x) d\theta < \int_{\frac{1}{2}}^{1} \theta h(\theta|x) d\theta - \frac{1}{2} + \cancel{\frac{1}{2} \int_0^{1/2} h(\theta|x) d\theta}$$

Hence $\quad \frac{1}{2} < \int_0^1 \theta h(\theta|x) d\theta = E(\theta|x)$

In other words, we choose $a_0 \equiv$ continue if $E(\theta|x) > \frac{1}{2}$ (and, of course, this decision is also intuitively appealing).

Now, for a Beta (α, β) distribution, the expected value is $\frac{\alpha}{\alpha+\beta}$ which implies in our case:

$E(\theta|x) = 3/(x+4)$ for transport minister

$E(\theta|x) = 4/(x+4)$ for prime minister Hatoyama

Hence the transport minister wants the project to continue when $x = 1$, hesitates when $x = 2$ and wants to stop when $x = 3, 4, 5, \ldots$. Hatoyama in turn wants to continue when $x = 1, 2, 3$; hesitates when $x = 4$ for wants to stop when $x = 5, 6, 7, \ldots$
Hence if $x = 3$, the most serious disagreement will occur.