THE UNIVERSITY OF NEW SOUTH WALES

DEPARTMENT OF STATISTICS

MID SESSION TEST - 2016 -wednesday, 27th April

MATH3811/MATH3911

Time allowed: 50 minutes; (*) question is for 3911 students only

1. Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ $(n \geq 2)$ be i.i.d. random variables each with Rayleigh distribution density:

$$f(x, \theta) = \frac{x}{\theta} e^{(-\frac{x^2}{2\theta})}, x > 0, \theta > 0.$$

Here θ is unknown parameter.

a) Argue that $T(\mathbf{X}) = \sum_{i=1}^{n} X_i^2$ is a complete and minimal sufficient statistic for θ.

b) Show that $E(X_1^2) = 2\theta$ holds.

c) Using a) and b), or otherwise, derive the UMVUE of θ. What is the value of the Cramer-Rao bound for an unbiased estimator of θ?

d) Argue that a uniformly most powerful unbiased α-size test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ exists. Given that $E(X_1^4) = 8\theta^2$, use the Central Limit Theorem to show that the rejection region of the test can be approximated by $S = \{\mathbf{X} : \frac{|T - 2n\theta_0|}{2\sqrt{n}\theta_0} > z_{\alpha/2}\}$.

e) If the prior for θ is $\tau(\theta) = \theta^{-3} e^{-1/\theta}, \theta > 0$, find the Bayesian estimator of θ with respect to quadratic loss.

You may use: The Inverse Gamma distribution with parameters $\alpha > 1$ and $\beta > 0$ has a density

$$g(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta/x}, x > 0$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp(-x) dx$. Its expected value is equal to $\frac{\beta}{\alpha-1}$.

f) (*) Suppose that besides the sample $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ from the Rayleigh distribution, with a parameter θ_1, another independent sample $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ is available but with a parameter θ_2. Show that for the MLE $\hat{h}$ of the ratio $h(\theta_1, \theta_2) = \frac{\theta_1}{\theta_2}$ it holds

$$\sqrt{n}(\hat{h} - h(\theta_1, \theta_2)) \xrightarrow{d} N(0, 2\frac{\theta_1^2}{\theta_2^2}).$$

2. Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a sample of n observations each with a uniform in $[0, \theta)$ density

$$f(x, \theta) = \begin{cases} \frac{1}{\theta} & 0 \leq x < \theta \\ 0 & \text{else} \end{cases}$$

where $\theta > 0$ is an unknown parameter. Denote the joint density by $L(X, \theta)$.

a) Find the density of $T = X_{(n)}$.

b) Show that the family $\{L(X, \theta)\}$, $\theta > 0$ has a monotone likelihood ratio in $T = X_{(n)}$.

c) Show that the uniformly most powerful α-size test of $H_0 : \theta \leq 3$ versus $H_1 : \theta > 3$ is given by

$$\varphi^*(\mathbf{X}) = \begin{cases} 1 \text{ if } X_{(n)} > 3(1-\alpha)^{\frac{1}{n}} \\ 0 \text{ if } X_{(n)} \leq 3(1-\alpha)^{\frac{1}{n}} \end{cases}$$

and sketch the graph of $E_\theta \varphi^*$ as accurately as possible.