THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS

FINAL EXAMINATION

JUNE 2006

MATH3911
HIGHER STATISTICAL INFERENCE

(1) TIME ALLOWED – 2 $\frac{1}{2}$ Hours

(2) TOTAL NUMBER OF QUESTIONS – 5

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE **NOT** OF EQUAL VALUE

(5) TOTAL NUMBER OF MARKS – 125

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

1. **[28 marks]** Suppose a sample $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ from the density

$$f(x,\theta) = \begin{cases} (1+\theta)x^\theta, x \in (0,1), \\ 0 \text{ elsewhere} \end{cases}$$

is given where $\theta > -1$ is the unknown parameter.

a) Find a minimal sufficient statistic T for θ. Give reasons for your answer.

b) Calculate the Fisher information about θ contained in the statistic T that you found in a). Show that it is equal to $\frac{n}{(1+\theta)^2}$.

c) Find the MLE of θ and also the MLE of $h(\theta) = \frac{1}{1+\theta}$.

d) Find the Cramer-Rao lower bound for the variance of an unbiased estimator of $h(\theta) = \frac{1}{1+\theta}$.

e) Checking out the form of the score function, find out if an UMVUE of $h(\theta) = \frac{1}{1+\theta}$ exists and if it exists, write it down.

f) Show that the UMVUE of θ is $U = -1 - \frac{n-1}{\sum_{i=1}^n \ln X_i}$.
Hint: Using the density transformation formula, show that $Y_i = -\ln X_i$ are exponentially distributed and hence the density of $\sum_{i=1}^n -\ln X_i$ can be found; this helps to show that U is unbiased for θ.

g) State the asymptotic distribution of $\sqrt{n}(\hat{h}_{mle} - h(\theta))$ for $h(\theta) = \frac{1}{1+\theta}$.

Hint: The "delta method" implies that for any smooth function $h(\theta)$:

$$\sqrt{n}(h(\hat{\theta}_{mle}) - h(\theta_0)) \xrightarrow{d} N(0, (\frac{\partial h}{\partial \theta}(\theta_0))^2 I^{-1}(\theta_0)),$$

$$I(\theta) = E\{\frac{\partial}{\partial \theta}[lnf(x,\theta)]\}^2 = E\{-\frac{\partial^2}{\partial \theta^2}[lnf(x,\theta)]\}.$$

2. **[26 marks]** Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be i.i.d. observations, each from a Poisson distribution:

$$f(y|\lambda) = e^{(-\lambda)} \cdot \lambda^y/(y!), \; y = 0, 1, 2, \ldots, \; \lambda > 0.$$

The prior on λ is believed to be Gamma(2,3).
(Note that, for any values of $\alpha > 0, \beta > 0$, the density of the Gamma(α, β) distribution is given by

$$\tau(\lambda) = \begin{cases} \frac{1}{\beta^\alpha \cdot \Gamma(\alpha)} \cdot \lambda^{\alpha-1} \exp(-\lambda/\beta) & ,\lambda > 0; \\ 0 & \text{else} \end{cases}$$

and $\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx$ is the Gamma function with the property $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.)

a) Find the posterior density $h(\lambda|\mathbf{X})$ of λ given $\mathbf{X} = (X_1, X_2, \ldots, X_n)$. Show that, like the prior, the posterior density is also a member of the Gamma family.

b) Find the Bayesian estimator of λ for quadratic error-loss with respect to the prior $\tau(\lambda)$.

c) For a sample size $n = 10$, you observed $\sum_{i=1}^{10} X_i = 18$. You want to test $H_0 : \lambda \leq 2$ versus $H_1 : \lambda > 2$ using the Bayesian approach and a 0-1 loss. Do you accept H_0? Give a reason for your answer.
(**Note:** the cumulative distribution function $F(x)$ of the Gamma(20, $\frac{3}{31}$) distribution is equal to 0.588 when the argument is equal to 2).

3. [**22 marks**] Let $\mathbf{X} = (\mathbf{X_1}, \mathbf{X_2}, \ldots, \mathbf{X_n})$ be i.i.d. random variables, each with a density

$$f(x, \theta) = \begin{cases} \frac{1}{\sqrt{2\pi}x\theta} e^{\{-\frac{1}{2}[\frac{\ln(x)}{\theta}]^2\}}, x > 0 \\ 0 \text{ elsewhere} \end{cases}$$

where $\theta > 0$ is a parameter. (This is called the log-normal density.)

a) Prove that the family $L(\mathbf{X}, \theta)$ has a monotone likelihood ratio in $T = \sum_{i=1}^{n} (\ln X_i)^2$.

b) Argue that there is a uniformly most powerful (UMP) $\alpha-$size test of the hypothesis $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$ and exhibit its structure.

c) Using the density transformation formula (or otherwise) show that

$$Y_i = \ln X_i$$

has a $N(0, \theta^2)$ distribution.

d) Using c) (or otherwise), find the threshold constant in the test and hence determine completely the uniformly most powerful $\alpha-$ size test φ^* of

$$H_0 : \theta \leq \theta_0 \text{ versus } H_1 : \theta > \theta_0.$$

Calculate the power function $E_\theta \varphi^*$ and sketch a graph of it.

4. [**22 marks**]

a) In an agricultural experiment, three blocks of similar size and location have been selected. On each block, four varieties (A, B, C and D) of potatoes have been grown on plots of equal size. The yields of potato from the 12 plots have been registered in the table below:

	Variety A	Variety B	Variety C	Variety D
Block 1	68	67	71	77
Block 2	82	83	86	89
Block 3	56	59	64	60

Analyze these data using the Friedman test. At 10% level of significance, decide if the hypothesis of equal yields for the four varieties can be accepted.

Please see over ...

b) Two groups of students attempt a test with the following results:
Group 1: 91, 83, 76, 81
Group 2: 72, 61, 74, 63, 82
Using a (large sample) Wilcoxon test and $\alpha = 0.05$, test the hypothesis of no difference between the two groups.

5. [27 marks]

a) Suppose $X_{(1)} < X_{(2)} < X_{(3)}$ are the order statistics based on a random sample of size 3 from the standard exponential density $f(x) = e^{-x}, x > 0$. Show that $g(m) = 12e^{-2m}(1-e^{-m})^2, m > 0$ is the density of the midrange $M = \frac{1}{2}(X_{(1)} + X_{(3)})$ and that $P(M > 1) = 0.4687$.

b) For the following sample of eight measurements:

$$7.3, 12.7, 6.8, 8.3, 9.1, 10.6, 8.7, 11.6$$

find the sample median and construct a confidence interval for the true unknown median at a confidence level as close as possible to 95%. Determine the exact level of your confidence interval.

Some useful formulae

1. Friedman's Test:

$$F = \frac{12l}{K(K+1)} \sum_{i=1}^{K} (\bar{R}_{i.} - \frac{K+1}{2})^2 = \frac{12}{lK(K+1)} \sum_{i=1}^{K} R_{i.}^2 - 3l(K+1)$$

has χ^2_{K-1} as a limiting distribution under the null hypothesis.

2. rth order statistic $(r = 1, 2, \ldots, n)$ of the sample of size n from a distribution with a density $f_X(.)$ and a cdf $F_X(.)$:

$$f_{X_{(r)}}(y_r) = \frac{n!}{(r-1)!(n-r)!} [F_X(y_r)]^{r-1} [1 - F_X(y_r)]^{n-r} f_X(y_r)$$

Joint density of the couple $(X_{(i)}, X_{(j)}), 1 \le i < j \le n$:

$$f_{X_{(i)}, X_{(j)}}(x, y) =$$

$$\frac{n!}{(i-1)!(j-1-i)!(n-j)!} f_X(x) f_X(y) [F_X(x)]^{i-1} [F_X(y) - F_X(x)]^{j-1-i} [1 - F_X(y)]^{n-j}$$

for $-\infty < x < y < \infty$.

3. Wilcoxon: Two independent samples $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots Y_n$, $W_{m+n} = \sum_{i=1}^{m} R(X_i)$. Then $\frac{W_{m+n} - m(m+n+1)/2}{\sqrt{mn(m+n+1)/12}} \xrightarrow{d} N(0, 1)$.

4. Bayesian inference:

$$h(\theta|\mathbf{X}) = \frac{f(\mathbf{X}|\theta)\tau(\theta)}{g(\mathbf{X})}, g(\mathbf{X}) = \int_{\Theta} f(\mathbf{X}|\theta)\tau(\theta) d\theta.$$

5. Density transformation: $Y = W(X) \longrightarrow f_Y(y) = f_X(W^{-1}(y)) |\frac{d(W^{-1}(y))}{dy}|$.