Solutions to MST questions
MATH 3811/3911 , 2015

1.) a) $f(x;\theta) = \frac{\theta}{1-\theta} e^{x \ln(1-\theta)}$ is a one-parameter exponential family with $d(x) = x$. Hence $T = \sum_{i=1}^{n} X_i$ is minimal sufficient.

b) $E\,W = 1 * P(X_1 = 1) = \theta(1-\theta)^0 = \theta \to$ unbiased

c) T is complete and minimal sufficient, hence $E(W|T=t)$ will be the UMVUE (by Lehmann-Scheffe's theorem)

Now $E(W|T=t) = P(W=1|T=t) = \frac{P(X_1 = 1 \cap \sum_{i=1}^{n} X_i = t)}{P(\sum_{i=1}^{n} X_i = t)} =$

$= \frac{P(X_1 = 1 \cap \sum_{i=2}^{n} X_i = t-1)}{P(\sum_{i=1}^{n} X_i = t)} = \frac{\cancel{\theta} \binom{t-2}{n-2} \cancel{\theta^{n-1}} \cancel{(1-\theta)^{t-n}}}{\binom{t-1}{n-1} \cancel{\theta^{n}} \cancel{(1-\theta)^{t-n}}} =$

$= \frac{(t-2)!}{(n-2)!\cancel{(n-t)}!} \frac{(n-1)!\cancel{(n-t)}!}{(t-1)!} = \boxed{\frac{n-1}{t-1}}$ which is the UMVUE of θ

d) $L(\mathbf{X},\theta) = \theta^n (1-\theta)^{\sum_{i=1}^{n} X_i - n}$; $\ln L(\mathbf{X},\theta) = n \ln\theta + (\sum_{i=1}^{n} X_i - n)\ln(1-\theta)$

$\frac{\partial}{\partial\theta} \ln L = \frac{n}{\theta} - \frac{\sum_{i=1}^{n} X_i - n}{1-\theta} = 0 \Rightarrow \frac{n}{\theta} = \frac{\sum_{i=1}^{n} X_i - n}{1-\theta} \Rightarrow \hat{\theta}_{MLE} = \frac{1}{\bar{X}}$

$\frac{\partial^2}{\partial\theta^2} \ln L = -\frac{n}{\theta^2} - \frac{\sum_{i=1}^{n} X_i - n}{(1-\theta)^2} \Rightarrow E\left(-\frac{\partial^2}{\partial\theta^2}\ln L\right) = \frac{n}{\theta^2} + \frac{\frac{n}{\theta} - n}{(1-\theta)^2} =$

$\frac{n - 2\theta n + \cancel{n\theta^2} + n\theta - \cancel{n\theta^2}}{\theta^2(1-\theta)^2} = \frac{n - \theta n}{\theta^2(1-\theta)^2} = \frac{n}{\theta^2(1-\theta)} = I_{\mathbf{X}}(\theta)$

The CR bound is $\frac{\theta^2(1-\theta)}{n}$ and the MLE is different from the UMVUE as seen above. Hence the MLE does not attain the bound (in fact the MLE is biased so it is outside the class)

e) $f(x;\theta) = \frac{\theta}{1-\theta} e^{-x\ln\left(\frac{1}{1-\theta}\right)}$, with $\ln\left(\frac{1}{1-\theta}\right)$ monotonically increasing in θ

Hence we have MLR in $Z = -\sum_{i=1}^{n} X_i = -T$ and BG theorem tells us that a UMP α test exists with a structure

$$\varphi^* = \begin{cases} 1 & \text{if } Z > K \\ \gamma & Z = K \\ 0 & Z < K \end{cases}, \quad E_{\theta_0}\varphi^* = \alpha$$

f) The posterior satisfies

$$h(\theta/\mathbf{X}) \propto f(\mathbf{X}/\theta)\tau(\theta) = \theta^n(1-\theta)^{T-n} * \theta(1-\theta)^2 = \theta^{n+1}(1-\theta)^{T-n+2}$$

this implies that necessarily $h(\theta/\mathbf{X})$ is Beta$(n+2, T-n+3)$ and the Bayes estimator is $\frac{n+2}{\cancel{n}+2+T-\cancel{n}+3} = \boxed{\frac{n+2}{T+5}}$

To solve the hypothesis testing problem, we note that

$$P(\theta \in \Theta_0 | X) = \int_0^{.37} h(\theta/X)\,d\theta \overset{\text{for } n=3,\ T=9}{=} \frac{\Gamma(14)}{\Gamma(5)\Gamma(9)} \int_0^{0.37} x^4(1-x)^8 dx =$$

$= \frac{13!}{4!8!} * 0.00008707 = 0.56 > \frac{1}{2}$ Hence we accept $H_0 : \theta \le .37$

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(2) a) Take $0 < \theta' < \theta''$ and fix them. Consider

$$\frac{L(X, \theta'')}{L(X, \theta')} = \left(\frac{\theta''}{\theta'}\right)^{2n} \frac{I_{(\theta'', \infty)}(X_{(1)})}{I_{(\theta', \infty)}(X_{(1)})}$$

which is not defined in $(0, \theta')$, equals to 0 in (θ', θ''), and is $\left(\frac{\theta''}{\theta'}\right)^{2n} > 0$ when $X_{(1)} > \theta''$. Hence we have MLR in $\underline{Z = X_{(1)}}$.

b) $F_Z(z) = P(X_{(1)} \le z) = 1 - P(X_{(1)} > z) = 1 - (P(X_1 > z))^n$

Now $F_{X_1}(x) = \begin{cases} 0 & \text{if } x \le \theta \\ \int_\theta^x \frac{2\theta^2}{t^3} dt = 1 - \frac{\theta^2}{x^2}, & x \ge \theta \end{cases}$

Hence $F_Z(z) = 1 - (1 - F_{X_1}(z))^n = \begin{cases} 0 & z < \theta \\ 1 - \frac{\theta^{2n}}{z^{2n}}, & z \ge \theta \end{cases}$

Then $f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} \frac{2n\theta^{2n}}{z^{2n+1}}, & z \ge \theta > 0 \\ 0 & \text{else} \end{cases}$

c) Thanks to a), UMP α-test exists and its structure is:

$$\varphi^*(\mathbf{X}) = \begin{cases} 1 & \text{if } X_{(1)} \geq K \\ 0 & \text{if } X_{(1)} < K. \end{cases}$$

To find K we need:

$$E_{\theta_0} \varphi^*(\mathbf{X}) = \alpha \quad \text{where } \theta_0 = 4$$

Now $E_{\theta_0} \varphi^*(\mathbf{X}) = P_{\theta_0}(X_{(1)} \geq K) = 1 - F_{X_{(1)}}(K) = \frac{\theta_0^{2n}}{K^{2n}} = \alpha$ must hold.

Hence $K = \alpha^{-\frac{1}{2n}} \cdot \theta_0 = 4\alpha^{-\frac{1}{2n}}$

Note that $E_\theta \varphi^*(X) = \begin{cases} \left(\frac{\theta}{4}\right)^{2n} \cdot \alpha, & \theta < K \\ 1, & \theta \geq K \end{cases}$

so the graph of the power function: