Solutions to MST'2013 - Math 3811/3911

1a) $\frac{L(X,\theta)}{L(Y,\theta)} = (1-\theta)^{\sum_{i=1}^n x_i - \sum_{i=1}^n y_i}$ which does not depend on θ iff $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Hence $T=\sum_{i=1}^n x_i$ is minimal sufficient.

It is also complete (see one-par. expon. argument below)

1b) $EX_1 = \sum_{x=1}^{\infty} x\theta(1-\theta)^{x-1} = \theta \sum_{x=1}^{\infty} x(1-\theta)^{x-1} = \theta \frac{d}{d\theta}\left(-\sum_{x=1}^{\infty}(1-\theta)^x\right) =$

$= -\theta \frac{d}{d\theta}\left[(1-\theta)(1+(1-\theta)+(1-\theta)^2+\cdots)\right] = -\theta \frac{d}{d\theta}\left(\frac{1-\theta}{\cancel{1}+\theta-\cancel{1}}\right) =$

$= -\theta\left(\frac{-\cancel{\theta}-1+\cancel{\theta}}{\theta^2}\right) = \frac{1}{\theta}$ (Note: $\theta\in(0,1)$)

1c) $\ln L(\underline{X},\theta) = n\ln\theta + \left(\sum_{i=1}^n x_i - n\right)\ln(1-\theta)$

$\frac{\partial}{\partial\theta}\ln L(\underline{X},\theta) = \frac{n}{\theta} - \frac{\sum_{i=1}^n x_i - n}{1-\theta}$; $\frac{\partial^2}{\partial\theta^2}\ln L(\underline{X},\theta) = -\frac{n}{\theta^2} - \frac{\sum_{i=1}^n x_i - n}{(1-\theta)^2}$

$E\left(-\frac{\partial^2}{\partial\theta^2}\ln L\right) = \frac{n}{\theta^2} + \frac{n/\theta - n}{(1-\theta)^2} = \frac{n}{\theta^2} + \frac{n\cancel{(1-\theta)}}{\theta(1-\theta)^{\cancel{2}}} = \frac{n}{\theta^2(1-\theta)} = I_{\underline{X}}(\theta)$

BUT: $I_T(\theta) = I_{\underline{X}}(\theta)$ because T is sufficient! Hence $I_T(\theta) = \frac{n}{\theta^2(1-\theta)}$

(1d) We want the nth success to happen at trial t.

Immediately before the t-th trial we must have had $(n-1)$ successes (with a probability for this being $\binom{t-1}{n-1}\theta^{n-1}(1-\theta)^{t-n}$) and, independently of this event, the last trial must be a success (with a probability θ). Multiply the two expressions due to independence.

We have an unbiased estimator of θ: $W = I_{\{X_1=1\}}(\underline{X})$ is unbiased since $EW = P(X_1=1) = \theta$

T is a function of sufficient, but also a complete statistic since $f(x;\theta) = \theta(1-\theta)^{x-1}$ belongs to a one-parameter exponential family with (for example) $a(\theta) = \frac{\theta}{\theta-1}$, $b(x)=1$, $c(\theta) = \ln(1-\theta)$, $d(x) = x$

Hence UMVUE: $E(I_{\{X_1=1\}}(\underline{X}) \mid T=t) = \frac{P(X_1=1 \cap \sum_{i=1}^n x_i = t)}{P(\sum_{i=1}^n x_i = t)}$

$= \frac{P(X_1=1 \cap \sum_{i=1}^n x_i = t-1)}{P(\sum_{i=1}^n x_i = t)} = \frac{\cancel{\theta}\binom{t-2}{n-2}\cancel{\theta^{n-1}}\cancel{(1-\theta)^{t-n}}}{\binom{t-1}{n-1}\cancel{\theta^n}\cancel{(1-\theta)^{t-n}}} = \frac{(t-2)!\,(n-1)!}{(n-2)!\,(t-1)!} = \boxed{\frac{n-1}{t-1}}$

e) $f(X|\theta)\pi(\theta) = \theta^n(1-\theta)^{\sum_{i=1}^n x_i - n}\,\theta^2(1-\theta) = \theta^{n+2}(1-\theta)^{\sum_{i=1}^n x_i - n+1}$

This implies that $h(\theta|X) \propto f(X|\theta)\pi(\theta)$ must be a Beta density with parameters $(n+3, \sum_{i=1}^n x_i - n+2)$

Since $\hat{\theta}_{Bayes} = E(\theta|X)$, $\hat{\theta}_{Bayes}$ must be the conditional expected value of $h(\theta|X)$, i.e. $\frac{n+3}{\cancel{n}+3+\sum_{i=1}^n x_i - \cancel{n}+2} = \boxed{\frac{n+3}{\sum_{i=1}^n x_i + 5}}$

f) We use again the fact that $f(x;\theta)$ belongs to one-parameter exponential but denote: $a(\theta) = \frac{\theta}{\theta-1}$, $b(x)=1$, $c(\theta) = -\ln(1-\theta)$, $d(x) = -x$ (since we need $c(\theta)$ to be monotonically increasing in θ). Then, using the BG theorem, we know that an UMP α test for $H_0: \theta \le \theta_0$ vs $H_1: \theta > \theta_0$ exists and has the structure

$$\varphi^* = \begin{cases} 1 & -\sum x_i > \tilde{K} \\ \gamma & -\sum x_i = \tilde{K} \\ 0 & -\sum x_i < \tilde{K} \end{cases}$$

This is the same as $\varphi^* = \begin{cases} 1 & \text{if } \sum_{i=1}^n x_i < K \\ \gamma & \text{if } \sum_{i=1}^n x_i = K \\ 0 & \text{if } \sum_{i=1}^n x_i < K \end{cases}$

(by renaming the constant).

Since $n=5$, possible realisations of $T = \sum_{i=1}^5 x_i$ are $5, 6, 7, \ldots$ Using the formula we have for $\theta_0 = .3$:

$P_{\theta_0}(T=5) = .3^5 = 0.00243$; $P_{\theta_0}(T=6) = \binom{5}{4} \cdot .3^5 .7 = 0.0085$

$P_{\theta_0}(T=7) = \binom{6}{4} .3^5 (.7)^2 = 0.01786$ and we see that

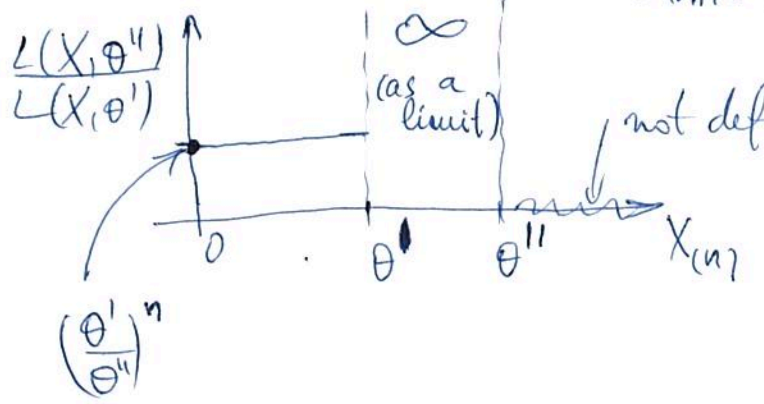
$P_{\theta_0}(T<7) < 0.02$ but $P_{\theta_0}(T \le 7) > 0.02$. Hence

$K = 7$. $\gamma = \frac{0.02 - P(T<7)}{P(T=7)} = \frac{0.02 - (0.00243 + .0085)}{0.01786}$

Hence $\gamma = .5078 \approx \underline{0.5}$

2 a) $L(X,\theta) = \frac{1}{\theta^n} I_{(X_{(n)},\infty)}(\theta)$. Fix $0<\theta'<\theta''$:

$$\frac{L(X,\theta'')}{L(X,\theta')} = \left(\frac{\theta'}{\theta''}\right)^n \frac{I_{(X_{(n)},\infty)}(\theta'')}{I_{(X_{(n)},\infty)}(\theta')}$$



Hence we have MLR in $T = X_{(n)}$

b) We know that $F_{X_{(n)}}(y) = P(X_{(n)} \le y) = P(X_1 \le y \cap X_2 \le y \cap \dots, X_n \le y)$
$= (F_{X_1}(y))^n = \left(\frac{y}{\theta}\right)^n$ when $0<y<\theta$
(and, of course, $F_{X_{(n)}}(y) = 0$ for $y<0$, $F_{X_{(n)}}(y) = 1$ for $y>\theta$.

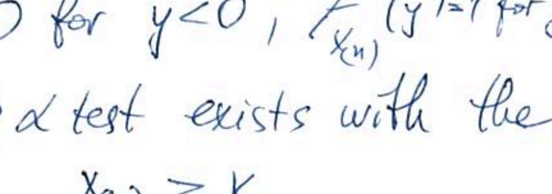
BG theorem tells us that a UMP α test exists with the structure

$$\varphi^* = \begin{cases} 1 & X_{(n)} > K \\ 0 & X_{(n)} \le K . \end{cases}$$

To find K, we need $E_{\theta=3}\varphi^* = \alpha = P_{\theta=3}(X_{(n)} > K) =$
$= 1 - \left(\frac{K}{3}\right)^n$. Hence $K = 3(1-\alpha)^{\frac{1}{n}}$

The powerfunction is $P_\theta(X_{(n)} > K) = \begin{cases} 1 - \left(\frac{K}{\theta}\right)^n, & 0<K<\theta \\ 0 & K>\theta \end{cases}$

i.e. $P_\theta(X_{(n)}>K) = \begin{cases} 1 - (1-\alpha)\left(\frac{3}{\theta}\right)^n \\ 1, \quad K>\theta \end{cases} \Rightarrow$



c) $P_\theta(Y_n < y) = P_\theta\left(\frac{Y_n}{n} < \frac{y}{n}\right) = P_\theta\left(1 - \frac{X_{(n)}}{\theta} < \frac{y}{n}\right)$
$= P_\theta\left(\theta\left(1-\frac{y}{n}\right) < X_{(n)}\right) = 1 - P_\theta\left(X_{(n)} \le \theta\left(1-\frac{y}{n}\right)\right) = 1 - \left(1-\frac{y}{n}\right)^n \to 1 - e^{-y}$

This means that $P_\theta\left(\frac{Y_n}{n} \to 0\right) \xrightarrow[n\to\infty]{} 1$ i.e. $P\left(1 - \frac{X_{(n)}}{\theta} \to 0\right) \to 0 \Rightarrow X_{(n)} \xrightarrow{P} \theta$