# Midsession test - MATH 3811/3911 S1 2014 - Solutions

1.) a) $a(\theta) = \frac{1}{\theta}$, $b(x) = 2x$, $c(\theta) = -\frac{1}{\theta}$, $d(x) = x^2$

imply that $f(x,\theta)$ is one-parameter exponential family.
Hence $T = \sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} x_i^2$ is minimal sufficient

b) $$EX_1 = \int_0^\infty \frac{2x^2}{\theta}\exp\left(-\frac{x^2}{\theta}\right)dx = \underbrace{\frac{\sqrt{2}}{\sqrt{2\pi}\sqrt{\theta}}\int_{-\infty}^{\infty} x^2\exp\left(-\frac{x^2}{2(\frac{\theta}{2})}\right)dx}_{=\frac{\theta}{2}} \frac{\sqrt{\pi\theta}}{\theta}$$

$\frac{\theta}{2}$ as the variance of $N(0, \frac{\theta}{2})$

Hence $EX_1 = \cancel{\frac{\theta}{2}}\frac{\sqrt{\pi\theta}}{\cancel{\theta}} = \boxed{\frac{\sqrt{\pi\theta}}{2}}$

$$EX_1^2 = \int_0^\infty \frac{2x^3}{\theta}\exp\left(-\frac{x^2}{\theta}\right)dx = -\int_0^\infty x^2\, d\exp\left(-\frac{x^2}{\theta}\right) = \theta\underbrace{\boxed{\int_0^\infty \exp\left(-\frac{x^2}{\theta}\right)d\frac{x^2}{\theta}}}_{=1} = \theta$$

c) Using (b), we have $E(T) = n\theta$. Therefore $\frac{T}{n}$ is unbiased for $\theta$ and is a function of complete and sufficient statistic. Hence, by Lehmann-Scheffe theorem, $\frac{T}{n}$ is the UMVUE

For CR bound: $\frac{\partial}{\partial\theta}\ln f(x,\theta) = -\frac{1}{\theta} + \frac{x^2}{\theta^2} \Rightarrow V = -\frac{n}{\theta} + \frac{T}{\theta^2}$

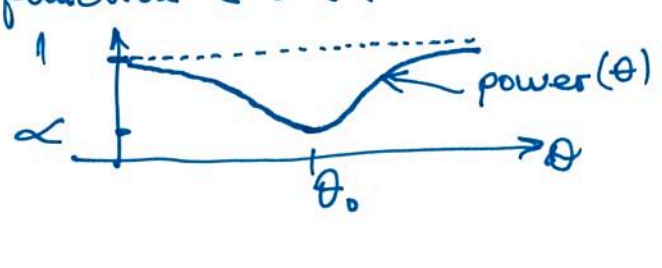
$$\frac{\partial^2}{\partial\theta^2}\ln f(x,\theta) = \frac{1}{\theta^2} - \frac{2x^2}{\theta^3} \Rightarrow E\left(-\frac{\partial^2}{\partial\theta^2}\ln f(X_1,\theta)\right) = \frac{1}{\theta^2}$$

Hence CR bound is $\frac{1}{nI_{X_1}(\theta)} = \boxed{\frac{\theta^2}{n}}$ and it is attainable since $V = \frac{n}{\theta^2}(\frac{T}{n} - \theta)$.

d) Since $f(x,\theta)$ is one-parameter exponential $\Rightarrow$ uniformly most powerful unbiased $\alpha$-size test $\varphi^*$ of $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$ exists. Its structure is:

$$\varphi^* = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i^2 < C_1 \text{ or } \sum_{i=1}^{n} x_i^2 > C_2 \\ 0 & \text{else} \end{cases}$$

The powerfunction should look like:



e) Note: $Var(X_1^2) = 2\theta^2 - (EX_1^2)^2 = 2\theta^2 - \theta^2 = \theta^2$

Hence $\sqrt{n}\,\frac{\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \theta_0\right)}{\theta_0} \xrightarrow{d} N(0,1)$ (CLT)

So, a monotone transformation of $T$ is asymptotically standard normal, or "equivalently" $T \approx N(n\theta_0, n\theta_0^2)$

Hence the ump $\alpha$ test (unbiased) is asymptotically equivalent to a Z test $\varphi^* = \begin{cases} 1 & \text{if } \left|\frac{T - n\theta_0}{\sqrt{n}\theta_0}\right| > z_{\frac{\alpha}{2}} \\ 0 & \text{else} \end{cases}$

f) We see that $\tau(\theta)$ represents Invgamma$(4,2)$

Also: $f(\mathbf{X},\theta)\tau(\theta) \propto \frac{1}{\theta^n}\exp\left(-\frac{\Sigma X_i^2}{\theta}\right)\frac{1}{\theta^5}\exp\left(-\frac{2}{\theta}\right)$

i.e., $f(\mathbf{X},\theta)\tau(\theta) \propto \theta^{-n-5} e^{-\left(\sum_{i=1}^{n} X_i^2 + 2\right)/\theta}$ which implies that

$h(\theta|\mathbf{X})$ is Invgamma $\left(n+4, \sum_{i=1}^{n} X_i^2 + 2\right)$

In our case: $h(\theta|\mathbf{X})$ is Invgamma$(14, 18)$

Hence $\boxed{0.1426} = \frac{18^{14}}{\Gamma(14)} \int_0^1 x^{-15} \exp\left(-\frac{18}{x}\right) dx$

The Bayes estimate is $\frac{18}{14-1} \approx 1.385$

The Frequentist estimate (the UMVUE) is $16/10 = 1.6$

<u>Comment</u>: For the prior, the expected value is $\frac{2}{3}$, i.e. using b)): $(2EX_1)^2 = \theta\pi \to \theta \approx \frac{16}{9 * \pi} \approx 0.566$ is indicated as the value of $\theta$ under the prior. This drags down the data-supported 1.6 to just 1.385 but 1.385 is still much larger than the borderline value $\theta_0 = 1$ and we reject $H_0$ as a consequence.

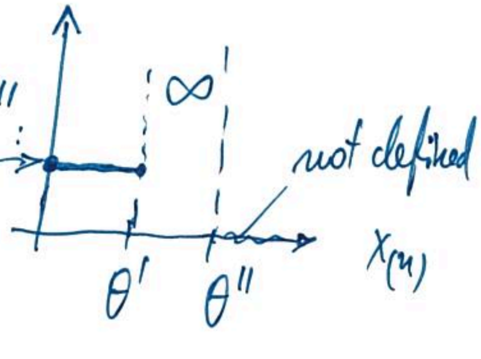
(That is, even the largest $\theta$-value under $H_0$ $(\theta_0 = 1)$ is not large enough to be supported by the combined information in data $\oplus$ prior)

**Q2** a) $L(\underline{X};\theta) = \frac{3^n}{\theta^{3n}} \prod_{i=1}^{n} x_i^2 \prod_{i=1}^{n} I_{(0,\theta)}(x_i) = \left(\frac{3}{\theta^3}\right)^n \prod_{i=1}^{n} x_i^2 \, I_{(0,\theta)}(X_{(n)})$

Then the ratio

$$\frac{L(X;\theta'')}{L(X;\theta')} = \left(\frac{\theta'}{\theta''}\right)^{3n} \frac{I_{(0,\theta'')}(X_{(n)})}{I_{(0,\theta')}(X_{(n)})}$$

is monotone non-decreasing in $X_{(n)}$ for fixed $0<\theta'<\theta''$.




Hence we have a family with monotone LR in $X_{(n)}$. $\left(\frac{\theta'}{\theta''}\right)^{3n}$

b) $F_{X_1}(x;\theta) = \begin{cases} 0 & x<0 \\ x^3/\theta^3 & 0<x<\theta \\ 1 & x>\theta \end{cases}$

Hence $F_{X_{(n)}}(x) = P(X_1 \le x \cap X_2 \le x \cap \dots \cap X_n \le x) = \begin{cases} 0 & x<0 \\ \left(\frac{x}{\theta}\right)^{3n} & 0<x<\theta \\ 1 & x>\theta \end{cases}$

Then density: $f_{X_{(n)}}(x) = \begin{cases} 3n x^{3n-1}/\theta^{3n}, & 0<x<\theta \\ 0 & \text{else} \end{cases}$

c) $\theta_0 = 2$ BG theorem says that a UMP $\alpha$-test $\varphi^*$ exists with $\varphi^* = \begin{cases} 1 & X_{(n)} > c \\ 0 & X_{(n)} \le c \end{cases}$

To find the threshold $c$, we "exhaust the level":

$$\alpha = E_{\theta=2}\varphi^* = P_{\theta=2}(X_{(n)}>c) = 1 - \left(\frac{c}{2}\right)^{3n} = \alpha$$

Hence $\left(\frac{c}{2}\right)^{3n} = 1-\alpha \Rightarrow c = \boxed{2(1-\alpha)^{\frac{1}{3n}}}$

and the test is completely determined