Solutions to MST - Math3811/3911

2016

Q1.) a) $a(\theta) = \frac{1}{\theta}$, $b(x) = x$, $c(\theta) = -\frac{1}{2\theta}$, $d(x) = x^2$ imply that $f(x,\theta)$ is one-parameter exponential. Hence $T = \sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i^2$ is minimal sufficient for θ.

b) $E(X_1^2) = \int_0^\infty \frac{x^3}{\theta} e^{-\frac{x^2}{2\theta}} dx = -\int_0^\infty x^2 d e^{-\frac{x^2}{2\theta}} \overset{\text{by parts}}{=}$

$$= 2\theta \int_0^\infty e^{-\frac{x^2}{2\theta}} d\frac{x^2}{2\theta} = 2\theta \int_0^\infty e^{-y} dy = 2\theta$$

c) $E(T) = 2n\theta$ Therefore $U = \frac{1}{2n}\sum_{i=1}^{n} X_i^2$ is unbiased for θ. Since it is also a function of sufficient and complete statistic, it must be the UMVUE (by Lehmann-Scheffe theorem).

For CR bound: $\ln f(x,\theta) = \ln x - \ln\theta - \frac{x^2}{2\theta}$

$$\frac{\partial}{\partial\theta}\ln f(x,\theta) = -\frac{1}{\theta} + \frac{x^2}{2\theta^2}$$

$$\frac{\partial^2}{\partial\theta^2}\ln f(x,\theta) = \frac{1}{\theta^2} - \frac{x^2}{\theta^3} \to E\left(-\frac{\partial^2}{\partial\theta^2}\ln f(X_1,\theta)\right) =$$

$$= -\frac{1}{\theta^2} + \frac{2*\theta}{\theta^3} = \frac{1}{\theta^2} = I_{X_1}(\theta)$$ Hence CR bound is

$$\frac{1}{nI_{X_1}(\theta)} = \boxed{\frac{\theta^2}{n}}$$

d) Since $f(x,\theta)$ is a one-parameter exponential, a uniformly most powerful unbiased α-size test φ^* of $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$ exists. Its structure is:

$$\varphi^* = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i^2 < C_1 \text{ or } \sum_{i=1}^{n} X_i^2 > C_2 \\ 0 & \text{else} \end{cases}$$

Note: $Var(X_1^2) = 8\theta^2 - 4\theta^2 = 4\theta^2$

Hence $\dfrac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - 2\theta_0\right)}{2\theta_0} \xrightarrow{d} N(0,1)$ by CLT, or,

equivalently $\dfrac{T - 2n\theta_0}{2\sqrt{n}\,\theta_0} \xrightarrow{d} N(0,1)$

So, a monotone transformation of T is asymptotically standard normal : $T \approx N(2n\theta_0, 4n\theta_0^2)$ Hence the ump unbiased α-test is asymptotically equivalent to a z-test $\tilde{\varphi}^* = \begin{cases} 1 & \text{if } \frac{|T - 2n\theta_0|}{2\sqrt{n}\,\theta_0} > z_{\alpha/2} \\ 0 & \text{else} \end{cases}$

e) $f(X|\theta)\tau(\theta) = \dfrac{\prod_{i=1}^{n} x_i}{\theta^n} e^{-\frac{\sum_{i=1}^{n} x_i^2}{2\theta}} \dfrac{1}{\theta^3} e^{-\frac{1}{\theta}}$

$\propto \theta^{-n-3} e^{-\left(\frac{\sum x_i^2}{2}+1\right)/\theta}$

which means that $h(\theta/X)$ must be Inverse gamma with parameters $n+2$ and $\frac{\sum_{i=1}^{n} x_i^2}{2} + 1$. Hence $\hat{\theta}_{Bayes} = \dfrac{\frac{\sum x_i^2}{2}+1}{n+1} = \boxed{\dfrac{\sum_{i=1}^{n} x_i^2 + 2}{2(n+1)}}$. (Notice that this is close to the UMVUE $U = \sum_{i=1}^{n} x_i^2/(2n)$ when n is large.

f) First, note that CR bound $\frac{\theta^2}{n}$ is attained by U since $L(X,\theta) = -\frac{n}{\theta} + \frac{\sum x_i^2}{2\theta^2} = \frac{n}{\theta^2}(U - \theta)$

This implies that U is MLE. By transformation invariance, $\hat{h} = \sum_{i=1}^{n} x_i^2 / \sum_{i=1}^{n} Y_i^2$ is then MLE of $h(\theta_1, \theta_2) = \theta_1/\theta_2$.

$I_{\binom{X_1}{Y_1}}(\theta_1, \theta_2) = \begin{pmatrix} 1/\theta_1^2 & 0 \\ 0 & 1/\theta_2^2 \end{pmatrix}$ holds and then

$$\sqrt{n}\left(\hat{h} - \frac{\theta_1}{\theta_2}\right) \xrightarrow{d} N\left(0, \left(\frac{\partial h}{\partial \theta_1} \ \frac{\partial h}{\partial \theta_2}\right)\begin{pmatrix} \theta_1^2 & 0 \\ 0 & \theta_2^2 \end{pmatrix}\begin{pmatrix} \frac{\partial h}{\partial \theta_1} \\ \frac{\partial h}{\partial \theta_2} \end{pmatrix}\right)$$

So the asymptotic variance is

$$\left(\frac{1}{\theta_2} \ -\frac{\theta_1}{\theta_2^2}\right)\begin{pmatrix} \theta_1^2 & 0 \\ 0 & \theta_2^2 \end{pmatrix}\begin{pmatrix} \frac{1}{\theta_2} \\ -\frac{\theta_1}{\theta_2^2} \end{pmatrix} = \left(\frac{1}{\theta_2} \ -\frac{\theta_1}{\theta_2^2}\right)\begin{pmatrix} \theta_1^2/\theta_2 \\ -\theta_1 \end{pmatrix} = 2\frac{\theta_1^2}{\theta_2^2}$$

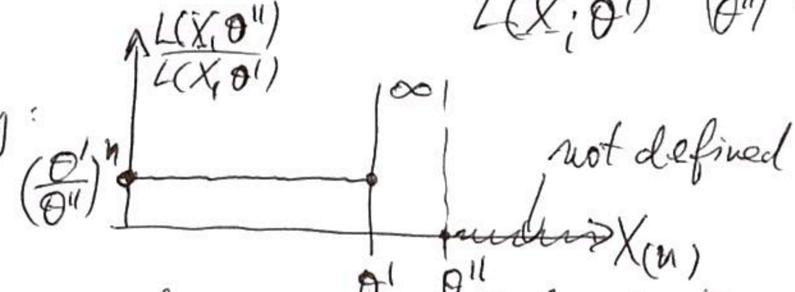
Q2 a) the density is $f_{X_{(n)}}(x;\theta) = \frac{n x^{n-1}}{\theta^n}, \ \theta > x > 0$

You get this by differentiating the cdf $F_{X_{(n)}} = \begin{cases} 0 & x<0 \\ \left(\frac{x}{\theta}\right)^n & 0<x<\theta \\ 1 & x>\theta \end{cases}$

b) $L(\underline{X};\theta) = \frac{1}{\theta^n}\prod_{i=1}^{n} I_{(0,\theta)}(X_i) = \frac{1}{\theta^n} I_{(0,\theta)}(X_{(n)})$

Hence for $0<\theta'<\theta''$ we have $\frac{L(\underline{X};\theta'')}{L(\underline{X};\theta')} = \left(\frac{\theta'}{\theta''}\right)^n \frac{I_{(0,\theta'')}(X_{(n)})}{I_{(0,\theta')}(X_{(n)})}$

Graphically:



Over the region where it is defined, the ratio is non-decreasing in the statistic $T = X_{(n)}$ → hence we have MLR in this statistic.

c) BG theorem tells us that there exists a UMP α-test φ^*

$$\varphi^* = \begin{cases} 1 & \text{if } X_{(n)} > C \\ 0 & \text{if } X_{(n)} \le C \end{cases}$$

To find the threshold c, we must "exhaust" the level:

$$\alpha = E_{\theta=3}\varphi^* = P_{\theta=3}(X_{(n)}>C) = 1-\left(\frac{c}{3}\right)^n = \alpha$$

Hence $c = 3(1-\alpha)^{\frac{1}{n}} < 3$

Now since $F_{X_{(n)}}(x) = \begin{cases} 0 & x<0 \\ \left(\frac{x}{\theta}\right)^n & 0<x<\theta \\ 1 & x>\theta \end{cases} \Rightarrow P_\theta(X_{(n)}>c) = \begin{cases} 1-\left(\frac{c}{\theta}\right)^n \\ 0, \ \theta<c \end{cases}$

i.e. $P_\theta(X_{(n)}>c) = \begin{cases} 0, & 0<\theta<c \\ 1-(1-\alpha)\left(\frac{3}{\theta}\right)^n, & \theta>c \end{cases}$ so the sketch of the graph is