Assignment 1: MATH3911

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Assignment 1: Version from June 2, 2014

This assignment is my own work. I have read and understood the University Rules in respect to Student Academic Misconduct.

1 Question 1

(a) $W = I_{\{X_1=2\}}(\mathbf{X})$ is a simple unbiased estimator of $\tau(\lambda)$ since $E(W) = 1 \times Pr(X_1 = 2) = \frac{\lambda^2 e^{-\lambda}}{2!}$

(b) By rearranging the probability density function of Poisson Distribution:

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-\lambda}}{x!}e^{xlog\lambda} = a(\lambda)b(x)exp\{c(\lambda)d(x)\}$$

It is clear that $a(\lambda) = e^{-\lambda}, b(x) = \frac{1}{x!}, c(\lambda) = log\lambda$ and $d(x) = x$. Therefore, the distribution belongs to 1-parameter-exponential family. So the statistic $T = \sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i$ is complete and minimal sufficient for λ.
Then in order to derive the UMVUE of $\tau(\lambda)$, we use the Lehmann-Scheffe theorem which states that $E(W|T)$ is the unique UMVUE of $\tau(\lambda)$
Hence we can obtain the UMVUE by finding $E(W|T)$,
Note that $X \sim Poi(\lambda), T = \sum_{i=1}^{n} X_i \sim Poi(n\lambda), \sum_{i=2}^{n-1} X_i \sim Poi((n-1)\lambda)$

$$\begin{aligned}
E(W|T=t) &= E(X_1 = 2|\sum_{i=1}^{n} X_i = t) \\
&= \frac{P(X_1 = 2 \cap \sum_{i=1}^{n-1} X_i = t-2)}{P(T=t)} \\
&= \frac{P(X_1 = 2)P(\sum_{i=1}^{n-1} X_i = t-2)}{P(T=t)} \\
&= \frac{\frac{\lambda^2 e^{-\lambda}}{2!} \times \frac{[(n-1)\lambda]^{t-2}e^{-(n-1)\lambda}}{(t-2)!}}{\frac{(n\lambda)^t e^{n\lambda}}{t!}} \\
&= \frac{t!(n-1)^{t-2}}{2(t-2)!n^t} \\
&= \frac{t(t-1)(n-1)^{t-2}}{2n^t} \\
&= \frac{t(t-1)}{2n^2}\left(1 - \frac{1}{n}\right)^{t-2}
\end{aligned}$$

Hence, by Lehmann-Scheffe theorem , $E(W|T=t) = \frac{t(t-1)}{2n^2}\left(1 - \frac{1}{n}\right)^{t-2}$ is the UMVUE for $\tau(\lambda)$

(c) First obtain the MLE of λ:

$$\begin{aligned}
0 &= V(\mathbf{X}, \hat{\lambda}) \\
&= -n + \frac{\sum_{i=1}^{n} X_i}{\hat{\lambda}} \\
\hat{\lambda} &= \bar{X}
\end{aligned}$$

Then using the transformation invariance property of the MLE, we can obtain the MLE $\hat{\tau}$ of $\tau(\lambda)$ which is,

$$\hat{\tau}_{\mathbf{MLE}} = \tau(\hat{\lambda}) = \frac{\bar{X}^2 e^{-\bar{X}}}{2!}$$

Now as MLE is asymptotically normal, unbiased and efficient, so by delta method, the asymptotic distribution is

$$\sqrt{n}(\hat{\tau} - \tau(\lambda)) \xrightarrow{d} N\left(0, \left[\frac{\partial \tau(\lambda)}{\partial \lambda}\right]^2 I_{X_1}^{-1}(\lambda)\right)$$

where

$$\frac{\partial \tau(\lambda)}{\partial \lambda} = \frac{\partial}{\partial \lambda}\left(\frac{\lambda^2 e^{-\lambda}}{2!}\right) = \frac{2\lambda e^{-\lambda} - \lambda^2 e^{-\lambda}}{2} = \lambda(2-\lambda)\frac{e^{-\lambda}}{2}$$

and

$$I_X(\lambda) = E\left(-\frac{\partial}{\partial \lambda} V(\mathbf{X}, \lambda)\right) = \mathbf{E}\left(-\frac{\partial}{\partial \lambda}\left(-\mathbf{n} + \frac{\sum_{\mathbf{i}=1}^{\mathbf{n}} \mathbf{X_i}}{\hat{\lambda}}\right)\right) = \mathbf{E}\left(\frac{\sum_{\mathbf{i}=1}^{\mathbf{n}} \mathbf{X_i}}{\hat{\lambda}^{\mathbf{2}}}\right) = \frac{\mathbf{n}}{\lambda}$$

therefore

$$I_{X_1}(\lambda) = \frac{1}{\lambda}$$

Substitute all the calculation back we obtain the distribution as follow:

$$\sqrt{n}(\hat{\tau} - \tau(\lambda)) \xrightarrow{d} N\left(0, \left[\lambda(2-\lambda)\frac{e^{-\lambda}}{2}\right]^2 \lambda\right) = N\left(0, \frac{\lambda^2(2-\lambda)e^{-2\lambda}}{4}\lambda\right) = N\left(0, \frac{\lambda^3(2-\lambda)e^{-2\lambda}}{4}\right)$$

(d) From the data we can obtain $\bar{X} = \frac{T}{n} = \frac{\sum_{i=1}^{25} X_i}{25} = 75/25 = 3$
Using (b), the point estimate is:

$$\hat{\tau}_{UMVUE} = \frac{T(T-1)}{2n^2}\left(1 - \frac{1}{n}\right)^{T-2} = \frac{75(75-1)}{2 \times 25^2}\left(1 - \frac{1}{25}\right)^{75-2} = 0.2255189$$

Using (c), the point estimate is:

$$\hat{\tau}_{MLE} = \frac{\bar{X}^2 e^{-\bar{X}}}{2!} = \frac{3^2 e^{-3}}{2!} = 0.22404$$

Both values are very close to each other, which is expected as UMVUE approaches MLE asymptotically when the sample size is sufficiently large. i.e.

$$\begin{aligned}
\lim_{n\to\infty} \frac{T(T-1)}{2n^2}\left(1 - \frac{1}{n}\right)^{T-2} &= \lim_{n\to\infty} \frac{n\bar{X}(n\bar{X}-1)}{2(n-1)^2}\left(1 - \frac{1}{n}\right)^{n\bar{X}-2} \\
&= \lim_{n\to\infty} \frac{n^2\bar{X}^2 - n\bar{X}}{2n^2}\left(1 - \frac{1}{n}\right)^{n\bar{X}}\left(1 - \frac{1}{n}\right)^{-2} \\
&= \frac{\bar{X}^2}{2}\left[\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^{n\bar{X}}\right]\left[\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^{-2}\right] \\
&= \frac{\bar{X}^2}{2} e^{-\bar{X}} = \hat{\tau}_{MLE}
\end{aligned}$$

(e) The variance given by the Cramer-Rao lower bound:

$$\frac{(\tau'(\lambda))^2}{nI_{X_1}(\lambda)} = \frac{\lambda^2(2-\lambda)^2e^{-2\lambda}}{4} \times \frac{\lambda}{n} = \frac{\lambda^3(2-\lambda)^2e^{-2\lambda}}{4n}$$

To find out if the variance of the proposed estimator in (b) is equal to the Cramer-Rao lower bound, we can check if the Cramer-Rao lower bound is attainable or not.
SInce If the Cramer-Rao lower bound is attainable by T($\mathbf{X}$), an unbiased estimator of $\tau(\lambda)$. Then the score $V(\mathbf{X}, \lambda)$must have a representation of the form $k_n(\lambda)[W(\mathbf{X}) - \tau(\lambda)]$.
Since

$$\begin{aligned} V(\mathbf{X}, \lambda) &= -n + \frac{1}{\lambda}\sum_{i=1}^{n} X_i \\ &= -n + \frac{n\bar{X}}{\lambda} \\ &= \frac{n}{\lambda}(\bar{X} - \lambda) \\ &= \frac{n}{\lambda} \times \frac{2}{\lambda e^{-\lambda}}\left(\frac{\lambda e^{-\lambda}\bar{X}}{2} - \frac{\lambda^2 e^{-\lambda}}{2}\right) \\ &= \frac{n}{\lambda} \times \frac{2}{\lambda e^{-\lambda}}\left(\frac{\lambda e^{-\lambda}\bar{X}}{2} - \tau(\lambda)\right) \end{aligned}$$

and $\frac{\lambda e^{-\lambda}\bar{X}}{2}$ is clearly not a statistic as it depends on parameter λ so the Cramer-Rao lower bound is not attainable. Thus the variance of the UMVUE does not have the same variance as that implied by the Cramer-Rao lower bound.

(f) Using part (c), we know that the asymptotic distribution is $N\left(\tau(\lambda), \frac{\lambda^3(2-\lambda)e^{-2\lambda}}{4n}\right)$ and using the available data ($\hat{\lambda} = \bar{X} = 3, n = 25$), we can calculate that

$$\hat{se}(\hat{\tau}) = \sqrt{\frac{\hat{\lambda}^3(2-\hat{\lambda})e^{-2\hat{\lambda}}}{4n}} = 0.025870119$$

Therefore, the asymptotic 95% confidence interval for $\tau(\lambda)$ is

$$(\hat{\tau} \pm \Phi^{-1}(0.0975)\hat{se}(\hat{\tau})) = (0.173334566, 0.274745433)$$

2 Question 2

(a) First look at Distribution I:
Suppose that $E_\theta[g(x)] = 0$, we then have:

$$\begin{aligned}
0 &= E_\theta[g(x)] \\
&= \sum_{x=1}^{4} g(x)P(X = x) \\
&= \theta g(1) + 2\theta g(2) + (2\theta^3 - \theta^4)g(3) + (1 + \theta^4 - 3\theta - 2\theta^3)g(4) \\
0 &= g(4) + \theta(g(1) + 2g(2) - 3g(4)) + \theta^3(2g(3) - 2g(4)) + \theta^4(g(4) - g(3))
\end{aligned}$$

By equating the coefficient, we can obtain the following result:

$$g(4) = 0, g(4) - g(3) = 0 \Rightarrow g(4) = g(3) = 0, g(1) + 2g(2) = 0$$

But $g(1) + 2g(2) = 0$ doesn't imply that $g(1) + g(2) = 0$. Therefore, the family of distribution $\{P_\theta\}$ is not complete, since it doesn't imply $P_\theta(g(T) = 0) = 1$.

Now, Consider Distribution II:
Suppose that $E_\theta[g(x)] = 0$, we then have:

$$\begin{aligned}
0 &= E_\theta[g(x)] \\
&= \sum_{x=1}^{4} g(x)P(X = x) \\
&= \theta g(1) + (\theta - \theta^3)g(2) + 3\theta^2 g(3) + (1 - 2\theta - 3\theta^2 + \theta^3)g(4) \\
0 &= g(4) + \theta(g(1) + g(2) - 2g(4)) + \theta^2(3g(3) - 3g(4)) + \theta^3(g(4) - g(2))
\end{aligned}$$

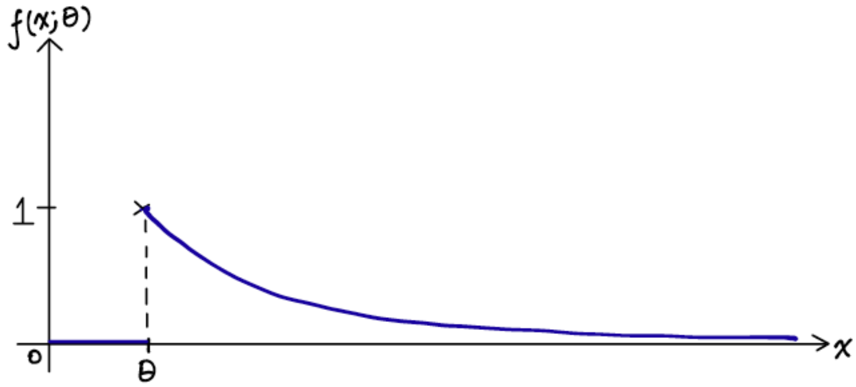
By equating the coefficient, we can obtain the following result:

$$g(4) = 0, g(4) - g(2) = 0 \Rightarrow g(4) = g(2) = 0, 3g(3) - 3g(4) = 0 \Rightarrow g(4) = g(3) = 0$$

This implies that $P_\theta(g(T) = 0) = 1$ and therefore, the family of distribution $\{P_\theta\}$ is complete.

3 Question 3

(a) Graph of a density from the family for a fixed θ:



(b) For $x \geq \theta$ the cumulative distribution function $F(x;\theta)$ of X_1 is

$$\begin{aligned} F(x;\theta) &= \int_{-\infty}^{x} f(t;\theta)dt \\ &= \int_{\theta}^{x} e^{(\theta-t)}dt \\ &= -e^{\theta-t}\Big|_{\theta}^{x} \\ &= 1-e^{\theta-x} \end{aligned}$$

Then

$$\begin{aligned} F_{X_{(1)}}(y;\theta) &= P(X_{(1)} \leq y) \\ &= 1-P(X_{(1)} \geq y) \\ &= 1-P(X_1 \geq y, ..., X_n \geq y) \\ &= 1-\Big[P(X_1 \geq y)\Big]^n && (i.i.d.) \\ &= 1-\Big[1-(1-e^{\theta-x})\Big]^n \\ &= 1-e^{n(\theta-x)} \\ &= 1-e^{n\theta}e^{-nx} \end{aligned}$$

Therefore

$$\begin{aligned} f_{X_{(1)}}(x;\theta) &= \frac{d}{dx}F_{X_{(1)}}(y;\theta) \\ &= -e^{n\theta}(-n)e^{(}-nx) \\ &= ne^{n(\theta-x)} \end{aligned}$$

Hence

$$f(z) = \begin{cases} ne^{n(\theta-x)} & \text{for} \quad x \geq \theta \\ 0 & \text{for} \quad x < \theta \end{cases}$$

(c) Since $X_1, ..., X_n$ are i.i.d. random variable with $f(x;\theta) = e^{\theta - x} I_{(-\infty,x]}(\theta) = \begin{cases} ne^{n(\theta-x)} & \text{for} \quad x \geq \theta \\ 0 & \text{for} \quad x < \theta \end{cases}$

Then the likelihood function is

$$\begin{aligned} L(x;\theta) &= \prod_{i=1}^{n} f(X_i;\theta) && (i.i.d) \\ &= \prod_{i=1}^{n} e^{\theta - x_i} I_{(-\infty, x_{(1)}]}(\theta) \\ &= exp\left\{ n\theta - \sum_{i=1}^{n} x_t \right\} I_{(-\infty, x_1]}(\theta) \end{aligned}$$

Let $\psi(\theta)$ be the ratio of the likelihood between two sample points $\mathbf{x}$, $\mathbf{y}$

$$\psi(\theta) = \frac{L(x;\theta)}{L(y;\theta)} = \frac{exp\{n\theta - \sum_{i=1}^{n} x_i\} I_{(-\infty,x_1]}(\theta)}{exp\{n\theta - \sum_{i=1}^{n} y_i\} I_{(-\infty,y_1]}(\theta)} = exp\left\{ \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \right\} \frac{I_{(-\infty,x_1]}(\theta)}{I_{(-\infty,y_1]}(\theta)}$$

For $x_{(1)} < y_{(1)}$,

$$\psi(\theta) = \begin{cases} exp\left[\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \right] & \text{for} \quad \theta \leq x_{(1)} \\ 0 & \text{for} \quad x_{(1)} < \theta \leq y_{(1)} \\ \text{undefined} & \text{for} \quad \theta > y_{(1)} \end{cases}$$

For $y_{(1)} < x_{(1)}$,

$$\psi(\theta) = \begin{cases} exp\left[\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \right] & \text{for} \quad \theta \leq y_{(1)} \\ 0 & \text{for} \quad y_{(1)} < \theta \leq x_{(1)} \\ \text{undefined} & \text{for} \quad \theta > x_{(1)} \end{cases}$$

If $x_{(1)} \neq y_{(1)}, \psi(\theta)$ is not a constant with respect to θ.
If $x_{(1)} = y_{(1)}$,

$$\psi(\theta) = \begin{cases} exp\left[\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \right] & \text{for} \quad \theta \leq x_{(1)} = y_{(1)} \\ \text{undefined} & \text{for} \quad \theta > x_{(1)} = y_{(1)} \end{cases}$$

$\psi(\theta)$ is constant with respect to θ. Hence, by Lehmann-Schette's method, $X_{(1)}$ is a minimal sufficient statistic for θ.

(d)

$$\begin{aligned} E_\theta[X_{(1)}] &= \int_\theta^\infty x f_{X_{(1)}}(x;\theta) dx \\ &= \int_\theta^\infty x n e^{n(\theta - x)} dx \\ &= n e^{n\theta} \int_\theta^\infty x e^{-nx} dx \\ &= e^{n\theta} \left(-x e^{-nx} \Big|_\theta^\infty + \int_\theta^\infty e^{-nx} dx \right) && \text{(by parts)} \\ &= e^{n\theta} \left(\theta e^{-n\theta} + \frac{1}{n} e^{-n\theta} \right) \\ &= e^{n\theta} e^{-n\theta} \left(\theta + \frac{1}{n} \right) \\ &= \theta + \frac{1}{n} \end{aligned}$$

(e) Let g be a function from $[\theta, \infty)$ to $\mathbb{R}$ and suppose that $E_\theta[g(X_{(1)}] = 0$ for all $\theta \in \mathbb{R}$, which means:

$$\begin{aligned}
0 &= \int_\theta^\infty g(t) f_{X_{(1)}}(t;\theta)dt \\
&= \int_\theta^\infty g(t) n e^{n(\theta - t)} dt \\
&= n e^{n\theta} \int_\theta^\infty g(t) e^{-nt} dt \\
0 &= \int_\theta^\infty g(t) e^{-nt} dt
\end{aligned}$$

Differentiate both sides and apply the fundamental theorem of calculus,

$$0 = -g(\theta) e^{-n\theta}$$

Therefore $g(\theta) = 0$ for all $\theta \in \mathbb{R}$ since $e^{-\theta} > 0$ for all $\theta \in \mathbb{R}$.
Then $g(x) = 0$ for $x \in [\theta, \infty)$, so $Pr_\theta(g(X_{(1)} = 0) = 1$ for all $\theta \in \mathbb{R}$ and hence $X_{(1)}$ is complete.
From part (c), we have justified that $X_{(1)}$ is also minimal sufficient for θ.
Therefore, considering part (d) where $E_\theta[X_{(1)}] = \theta + \frac{1}{n}$ which means that in order to obtain UMVUE of θ, we will need $W = X_{(1)} - \frac{1}{n}$ which will give us $E(W) = E(X_{(1)} - \frac{1}{n}) = E(X_{(1)}) - \frac{1}{n} = \theta + \frac{1}{n} - \frac{1}{n} = \theta$ for all θ.
Then by the Lehmann-Scheffe Theorem, the UMVUE of θ is

$$E[W|X_{(1)}] = E[X_{(1)} - \frac{1}{n}|X_{(1)}] = E[X_{(1)}|X_{(1)}] - \frac{1}{n} = X_{(1)} - \frac{1}{n}$$

4 Question 4

(a) Given that $\mathbf{X} = (X_1, X_2, ..., X_n)$ is a sample of n i.i.d observations from Bernoulli distribution with probability of success θ.
Suppose that an unbiased estimator of $\tau(\theta) = \frac{\theta}{1-\theta}$ did exist, then by Rao- Blackwell Theorem, an unbiased estimator $\mathbf{T}$ of $\tau(\theta) = \frac{\theta}{1-\theta}$ will exist, where $\mathbf{T}$ is a function of sufficient statistic.
Considering the statistic $\tilde{X} = \sum_{i=1}^n X_i \sim Bin(n, \theta)$ since each X is a $Bern(\theta)$.
Then

$$\begin{aligned}
E_\theta[\mathbf{T}(\tilde{X})] &= \tau(\theta) && \text{(using Rao- Blackwell Theorem)} \\
\sum_{x=0}^n T(x) \binom{n}{x} \theta^x (1-\theta)^{n-x} &= \frac{\theta}{1-\theta} && \forall \theta \in (0, \infty)
\end{aligned}$$

However, this is impossible, since as $\theta \to 1$ the L.H.S of the above equation will be equal to $\mathbf{T}(1)$ which is a finite constant (M)

$$\lim_{\theta \to 1} \sum_{x=0}^n T(x) \binom{n}{x} \theta^x (1-\theta)^{n-x} = T(1) = M$$

While the R.H.S (odd ratio) can be larger than any finite constant M (given from the hints in the question).
This is a contradiction as L.H.S $\neq$ R.H.S hence no unbiased estimator of $\tau(\theta) = \frac{\theta}{1-\theta}$ exists.

(b) First obtain the MLE of θ:

$$\begin{aligned}
0 &= V(\mathbf{X}, \hat{\theta}) \\
&= \frac{\sum_{i=1}^{n} X_i}{\hat{\theta}} - \frac{n - \sum_{i=1}^{n} X_i}{1-\hat{\theta}} \\
\frac{\sum_{i=1}^{n} X_i}{\hat{\theta}} &= \frac{n - \sum_{i=1}^{n} X_i}{1-\hat{\theta}} \\
\sum_{i=1}^{n} X_i - \hat{\theta}\sum_{i=1}^{n} X_i &= n\hat{\theta} - \hat{\theta}\sum_{i=1}^{n} X_i \\
\hat{\theta} &= \frac{\sum_{i=1}^{n} X_i}{n} \\
\hat{\theta} &= \bar{X}
\end{aligned}$$

Then using the transformation invariance property of the MLE, we can obtain the MLE $\hat{\tau}$ of $\tau(\theta)$ which is,

$$\hat{\tau}_{\mathbf{MLE}} = \tau(\hat{\theta}) = \frac{\hat{\theta}}{1-\hat{\theta}} = \frac{\bar{X}}{1-\bar{X}}$$

Now as MLE is asymptotically normal, unbiased and efficient, so by delta method, the asymptotic distribution is

$$\sqrt{n}(\hat{\tau} - \tau(\theta)) \xrightarrow{d} N\left(0, \left[\frac{\partial\tau(\theta)}{\partial\theta}\right]^2 I_{X_1}^{-1}(\theta)\right)$$

where

$$\frac{\partial\tau(\theta)}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\frac{\theta}{1-\theta}\right) = \frac{1-\theta-(-1)\theta}{(1-\theta)^2} = \frac{1}{(1-\theta)^2}$$

and

$$\begin{aligned}
I_X(\theta) &= E\left[-\frac{\partial}{\partial\theta}V(\mathbf{X},\theta)\right] = \mathbf{E}\left[-\frac{\partial}{\partial\theta}\left(\frac{\sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{X_i}}{\hat{\theta}} - \frac{\mathbf{n} - \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{X_i}}{\mathbf{1}-\hat{\theta}}\right)\right] \\
&= E\left[\frac{\sum_{i=1}^{n} X_i}{\hat{\theta}^2} + \frac{n - \sum_{i=1}^{n} X_i}{(1-\theta)^2}\right] = \frac{n\theta}{\theta^2} + \frac{n-n\theta}{(1-\theta)^2} \qquad \text{(we know that } n\hat{\theta} = \sum_{i=1}^{n} X_i\text{)} \\
&= n\left(\frac{1}{\theta} + \frac{1}{1-\theta}\right) = \frac{n}{\theta(1-\theta)}
\end{aligned}$$

therefore

$$I_{X_1}(\theta) = \frac{1}{\theta(1-\theta)}$$

Substitute all the calculation back we obtain the distribution as follow:

$$\sqrt{n}(\hat{\tau} - \tau(\theta)) \xrightarrow{d} N\left(0, \left[\frac{1}{(1-\theta)^2}\right]^2 \theta(1-\theta)\right) = N\left(0, \frac{\theta}{(1-\theta)^3}\right)$$

Therefore, the asymptotic 90% confidence interval for $\tau(\theta)$ is

$$(\hat{\tau} \pm \Phi^{-1}(0.95)\hat{se}(\hat{\tau})) = \left(\frac{\theta}{1-\theta} \pm \Phi^{-1}(0.95)\sqrt{\frac{\theta}{(1-\theta)^3}}\right)$$

5 Question 5

(a) (i) Since it is given that the common density function is $f(x;\theta) = \theta x^{(\theta-1)}$ for $0 < x < 1$ and $\theta > 0$, then it can be easily be represented in the form of exponential family

$$\begin{aligned}
f(x;\theta) &= \theta x^{(\theta-1)} \\
&= exp[log(\theta x^{(\theta-1)})] \\
&= exp[log(\theta) + (\theta-1)log(x)] \\
&= \theta \times e^{(\theta-1)log(x)} \\
&= a(\theta)b(x)exp[c(\theta)d(X)]
\end{aligned}$$

It is clear that $a(\theta) = \theta,\ b(x) = 1,\ c(\theta) = \theta - 1,\ d(X) = log(x)$. Therefore, the distribution belongs to the exponential family. So the statistic will be $T = \sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} log(X_i)$ and it is complete and minimal sufficient. To double check for minimal sufficient we can look at the likelihood function:

$$\begin{aligned}
L(\mathbf{x},\theta) &= \prod_{i=1}^{n} f(x_i;\theta) && \text{as } X_1, ..., X_n \text{ are i.i.d.)} \\
&= \prod_{i=1}^{n} \theta e^{(\theta-1)log(x_i)} \\
&= \theta^n exp\left\{(\theta-1)\sum_{i=1}^{n} log x_i\right\}
\end{aligned}$$

Then the ratio of the likelihoods for data vectors $\mathbf{x}$ and $\mathbf{y}$ is

$$\begin{aligned}
\frac{L(\mathbf{x};\theta)}{L(\mathbf{y};\theta)} &= \frac{\theta^n exp\left\{(\theta-1)\sum_{i=1}^{n} log x_i\right\}}{\theta^n exp\left\{(\theta-1)\sum_{i=1}^{n} log y_i\right\}} \\
&= exp\left\{\sum_{i=1}^{n} log x_i - \sum_{i=1}^{n} log y_i\right\}
\end{aligned}$$

as it is independent of θ if and only if $\sum_{i=1}^{n} log x_i = \sum_{i=1}^{n} log y_i$. Hence , by the Lehmann and Scheffe theorem, $T = \sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} log(X_i)$ is a minimal sufficient statistic for θ.

(ii) Suppose $W = -log X_1$ is a an unbiased estimator of $\tau(\theta) = \frac{1}{\theta}$, which means we must have $E[W] = \frac{1}{\theta}$. Now check the condition:

$$\begin{aligned}
E[W] &= E[-log X_1] \\
&= \int_0^1 (-log x)\theta x^{(\theta-1)} dx \\
&= \int_0^\infty w\theta e^{-w\theta} dw && \text{by substitution, } w = -log x \\
&= we^{-w\theta}\Big|_0^\infty + \int_0^\infty e^{-w\theta} dw && \text{by parts, } u = w, v' = e^{-w\theta} \\
&= \frac{e^- w\theta}{\theta}\Big|_0^\infty \\
&= \frac{1}{\theta}
\end{aligned}$$

Therefore, $W = -logX_1$ is an unbiased estimator of $\frac{1}{\theta}$.
Now, we first look at Variance of $\sqrt{W}$

$$
\begin{aligned}
Var[\sqrt{W}] &> 0 \qquad \text{(since variance is positive and is only equal to zero if its not random)}\\
E[\sqrt{W}^2] - E[\sqrt{W}]^2 &> 0\\
E[\sqrt{W}^2] &> E[\sqrt{W}]^2\\
\frac{1}{\sqrt{\theta}} &> E[\sqrt{W}]
\end{aligned}
$$

This means that $\sqrt{W}$ is a biased estimator of $\xi(\theta) = \frac{1}{\sqrt{\theta}}$ since $E[\sqrt{W}] < \frac{1}{\sqrt{\theta}}$.

(iii) Suppose $X_1 \sim U[0,1]$ and $W = -log(X_1)$, then consider the cumulative distribution function of W will be as follow

$$
\begin{aligned}
F_W(w) &= Pr(W < w)\\
&= Pr(-logX < w) \qquad \text{(substitute } W = -logX\text{)}\\
&= 1 - Pr(X < e^{-w})\\
&= 1 - F_X[e^{-w}]\\
\frac{\partial}{\partial W}F_W(w) &= \frac{\partial}{\partial W}\Big[1 - F_X[e^{-w}]\Big]
\end{aligned}
$$

To obtain $F_X[e^{-w}]$ we can do the follow:

$$
\begin{aligned}
F_X(x) &= \int_0^x \theta t^{(\theta-1)}dt\\
&= \theta\int_0^x t^{(\theta-1)}dt\\
&= x^\theta
\end{aligned}
$$

Then applied that into the previous equation for cdf of W:

$$
\begin{aligned}
f_W(w) &= \frac{\partial}{\partial W}\Big[1 - [e^{-w\theta}]\Big]\\
&= \theta e^{-w\theta}
\end{aligned}
$$

which is the probability density of a Exponential distribution with parameter θ.

(iv) From previous part we found that T is a complete sufficient statistic for θ and W is an unbiased estimator of $\tau(\theta)$. Note that since $W \sim exp(rate = \theta)$ and therefore,

$$T = \sum_{i=1}^{n} -log(X_i) \sim \Gamma(n, \theta).$$

Then By Lehmann-Scheffe theorem, if we can find a function of T whose expected value is $\frac{1}{\theta}$, we have an UMVUE for $\tau(\theta) = \frac{1}{\theta}$. First look at $E(T)$:

$$E[T] = E\Bigg[\sum_{i=1}^{n} -log(X_i)\Bigg] = \sum_{i=1}^{n} E\Bigg[-log(X_i)\Bigg] \stackrel{iid}{=} nE[-log(X_1)] = nE[W]$$

since we know that $W = -logX_1$ is an unbiased estimator of $\frac{1}{\theta}$, that means

$$E[T] = nE[W] = \frac{n}{\theta}$$

Therefore, using the Lehmann-Scheffe Theorem, we have $\frac{T}{n} = \frac{-\sum_{i=1}^{n} logX_i}{n}$ as the UMVUE for $\frac{1}{\theta}$. It can be checked by looking at the expected value of $\frac{T}{n}$ as follow:

$$E\left[\frac{T}{n}\right] = \frac{E[T]}{n} = \frac{n/\theta}{n} = \frac{1}{\theta}$$

(v) For the UMVUE of $\bar{\tau} = \theta$, we first consider $E[\frac{1}{T}]$
Note that $W = -logX_1 \sim \Gamma(1,\theta)$ and $T = -\sum_{i=1}^{n} logX_i \sim \Gamma(n,\theta)$ since they are i.i.d distributed.

$$\begin{aligned}
E\left[\frac{1}{T}\right] &= \int_0^\infty \frac{1}{t} f_T(t)dt \\
&= \int_0^\infty \frac{1}{t}\frac{1}{\Gamma(n)}\theta^n t^{n-1}e^{-\theta t}dt \\
&= \frac{\Gamma(n-1)}{\Gamma(n)}\theta\int_0^\infty \frac{1}{\Gamma(n-1)}\theta^{n-1}t^{n-2}e^{-\theta t}dt \\
&= \frac{\Gamma(n-1)}{\Gamma(n)}\theta \times 1 && \left(\text{pdf of } \Gamma(n-1,\theta)\right) \\
&= \frac{(n-2)!}{(n-1)!}\theta \\
&= \frac{\theta}{n-1}
\end{aligned}$$

Therefore,

$$(n-1)\frac{1}{T} = \frac{n-1}{\sum_{i=1}^{n} -logX_i}$$

is a function of the complete and sufficient statistic T that is unbiased for θ. So, by the Lehmann-Scheffe Theorem, we have

$$\hat{\theta}_{UMVUE} = \frac{n-1}{\sum_{i=1}^{n} -logX_i}$$

(b) Since it is given that n i.i.d. samples are taken from $N(\mu,\theta^2)$ with known μ. Then the probability density function for X_i is

$$f(x;\theta) = \frac{1}{\theta\sqrt{2\pi}}exp\left\{-\frac{(x-\mu)^2}{2\theta^2}\right\} \qquad \text{for } x \in \mathbb{R} \text{ and } \theta > 0$$

Then it can be easily be represented in the form of exponential family

$$\begin{aligned}
f(x;\theta) &= \frac{1}{\theta\sqrt{2\pi}}exp\left\{-\frac{(x-\mu)^2}{2\theta^2}\right\} \\
&= a(\theta)b(x)exp[c(\theta)d(X)]
\end{aligned}$$

It is clear that $a(\theta) = \frac{1}{\theta}$, $b(x) = 1\frac{1}{\sqrt{2\pi}}$, $c(\theta) = -\frac{1}{2\theta^2}$, $d(X) = (x-\mu^2)$. Therefore, the distribution belongs to the exponential family. So the statistic will be $T = \sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n}(X_i-\mu)^2$ and it is a complete and minimal sufficient statistic.

Now consider S^2

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2 = \frac{T}{n}$$

as it is a function of T, it is also a complete and minimal sufficient statistic.
Now we want to find the distribution of S^2, but first note that $X_i \sim N(\mu, \theta^2)$ where $i = 1, ..., n$. Then we can standardised the distribution of X_i which is

$$\begin{aligned} X_i &\sim N(\mu, \theta^2) \\ \frac{X_i-\mu}{\theta} &\sim N(0,1) \end{aligned}$$

Then we can square it to form something similar to S^2

$$\frac{(X_i-\mu)^2}{\theta^2} \sim \chi^2_{(1)}$$

Since square of a Normal random variable will give a 1 degree Chi-Square random variable. Then we are able to deduce that the sum of n Chi-Square random variable will be a n degree Chi-Square random variable which is

$$\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\theta^2} \sim \chi^2_{(n)}$$

Therefore,

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2 \sim \frac{\theta^2}{n}\chi^2_{(n)}$$

Hence

$$\frac{n}{\theta^2}S^2 \sim \chi^2_n.$$

Now by the Lehmann-Scheffe theorem, if we can find a function of S whose expected value is θ, then we have an UMVUE for θ.
It is not clear which function to choose. So we can first find $E[\frac{n}{\theta^2}S^2]$. Then try to multiply or divide to make a function of S that is unbiased for θ.
Note that in order to find $E[\frac{n}{\theta^2}S^2]$, it is essential to obtain the moment generating function of Chi-Square distribution with n degrees of freedom first. Let the m.g.f of Chi-Square distribution be $m_Y(t)$

$$\begin{aligned}
m_Y(t) &= E[e^{tY}] \\
&= \int_{-\infty}^{\infty} e^{tY} f_Y(y)dy \\
&= \frac{1}{2^{n/2}\Gamma(\frac{n}{2})}\int_0^{\infty} e^{tY} y^{\frac{n}{2}-1}e^{-\frac{1}{2}y}dy \\
&= \frac{1}{2^{n/2}\Gamma(\frac{n}{2})}\int_0^{\infty} y^{\frac{n}{2}-1}exp\bigg\{-\Big(\frac{1}{2}-t\Big)y\bigg\}dy \\
&= \frac{1}{2^{n/2}\Gamma(\frac{n}{2})}\int_0^{\infty} \bigg(\frac{2}{1-2t}u\bigg)^{\frac{n}{2}-1} e^{-u}\frac{2}{1-2t}du & \bigg(\text{changing variable : } u = (\frac{1}{2}-t)y\bigg) \\
&= \frac{1}{2^{n/2}\Gamma(\frac{n}{2})}\int_0^{\infty} \bigg(\frac{2}{1-2t}\bigg)^{\frac{n}{2}} u^{\frac{n}{2}-1} e^{-u}du \\
&= \frac{1}{2^{n/2}\Gamma(\frac{n}{2})}\bigg(\frac{2}{1-2t}\bigg)^{\frac{n}{2}}\int_0^{\infty} u^{\frac{n}{2}-1} e^{-u}du \\
&= \frac{1}{2^{n/2}\Gamma(\frac{n}{2})}\bigg(\frac{2}{1-2t}\bigg)^{\frac{n}{2}}\Gamma\bigg(\frac{n}{2}\bigg) & \bigg(\text{by definition of Gamma function}\bigg) \\
&= (1-2t)^{-\frac{n}{2}}
\end{aligned}$$

Hence we can now find the m^{th} non-central moment using the m.g.f $(1-2t)^{-\frac{n}{2}}$ of Chi-Square distribution with n degree of freedoms

$$E[(\frac{n}{\theta^2}S^2)^m] = n(n+2)(n+4)...(n+2m-2) = 2^m\frac{\Gamma(m+\frac{n}{2})}{\Gamma(\frac{n}{2})}$$

Apply the above formula, we can obtain

$$E[\frac{n}{\theta^2}S^2] = 2\frac{\Gamma(1+\frac{n}{2})}{\Gamma(\frac{n}{2})}$$
$$E[S^2] = 2\frac{\Gamma(1+\frac{n}{2})}{\Gamma(\frac{n}{2})}\frac{\theta^2}{n}$$

However, we are looking for UMVUE for θ therefore we should find $E[S]$ instead:

$$E\left[\left(\frac{n}{\theta^2}S^2\right)^{1/2}\right] = 2^{\frac{1}{2}}\frac{\Gamma(\frac{1}{2}+\frac{n}{2})}{\Gamma(\frac{n}{2})}$$
$$E[S] = \sqrt{\frac{2}{n}}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}\theta$$

Therefore,

$$\sqrt{\frac{n}{2}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}S = \sqrt{\frac{n}{2}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\sqrt{\frac{T}{n}} = \sqrt{\frac{n}{2}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2}$$

is a function of the complete and sufficient statistic S that is unbiased for θ. So by the Lehmann-Scheffe Theorem, we have

$$\hat{\theta}_{UMVUE} = \sqrt{\frac{n}{2}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}S$$

where $S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$ as given in the question.

(c) From the previous part, we have $S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$, however, since μ is unknown. We have to consider the sample variance and sample mean instead of the population variance and population mean. To check if the sample variance is unbiased:

$$
\begin{aligned}
s^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \\
E[s^2] &= \frac{1}{n-1}\sum_{i=1}^{n} E[(X_i - \bar{X})^2] \\
&= \frac{1}{n-1}\sum_{i=1}^{n} E[X_i^2 - 2X_i\bar{X} + \bar{X}^2] \\
&= \frac{1}{n-1}\sum_{i=1}^{n} E[X_i^2] - 2E[X_i\bar{X}] + E[\bar{X^2}] \\
&\left(\text{using the identity:} E[X_i^2] = Var[X_i] + E[X_i]^2\right) \\
&= \frac{1}{n-1}\sum_{i=1}^{n} Var[X_i] + E[X_i]^2 - 2E[X_i\frac{1}{n}\sum_{j=1}^{n} X_j] + E[\bar{X^2}] \\
&= \frac{1}{n-1}\sum_{i=1}^{n} Var[X_i] + E[X_i]^2 - \frac{2}{n}\sum_{j=1}^{n} E[X_iX_j] + E[\bar{X^2}]
\end{aligned}
$$

$$
\left(E[X_iX_j] = Cov(X_i, X_j) + E[X_i]E[X_j] = \begin{cases} \theta^2 + \mu^2 & \text{for} \quad i = j \\ \mu^2 & \text{for} \quad i \neq j \end{cases}\right)
$$

$$
\begin{aligned}
E[s^2] &= \frac{1}{n-1}\sum_{i=1}^{n} Var[X_i] + E[X_i]^2 - \frac{2}{n}\sum_{j=1}^{n} E[X_iX_j] + Var[\bar{X}] + E[\bar{X}]^2 \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(\theta^2 + \mu^2 - \frac{2}{n}\theta^2 + 2\mu^2 + \frac{\theta^2}{n} + \mu^2\right) \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(\theta^2 - \frac{\theta^2}{n}\right) \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(\theta^2\left(1 - \frac{1}{n}\right)\right) \\
&= \frac{1}{n-1}\sum_{i=1}^{n}\left(\theta^2\left(1 - \frac{1}{n}\right)\right) \\
&= \frac{\theta^2}{n-1}\sum_{i=1}^{n}\left(\frac{n-1}{n}\right) \\
&= \frac{\theta^2 n}{n-1}\left(\frac{n-1}{n}\right) \\
&= \theta^2
\end{aligned}
$$

Now obtain the distribution of s^2

$$
\begin{aligned}
X_i &\sim N(\mu, \theta^2) \\
\frac{X_i - \bar{X}}{\theta} &\sim N(0, 1)
\end{aligned}
$$

Then we can square it to form something similar to s^2

$$\frac{(X_i-\mu)^2}{\theta^2} \sim \chi^2_{(1)}$$

Since square of a Normal random variable will give a 1 degree Chi-Square random variable. Then we are able to deduce that the sum of n Chi-Square random variable will be a n degree Chi-Square random variable which is

$$\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\theta^2} \sim \chi^2_{(n)}$$

Therefore,

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\mu)^2 \sim \frac{\theta^2}{n-1}\chi^2_{(n)}$$

Hence

$$\frac{n}{\theta^2}S^2 \sim \chi^2_n.$$

Now by the Lehmann-Scheffe theorem, if we can find a function of S whose expected value is θ, then we have an UMVUE for θ.