# Solutions to MST - Math3811/3911

2012

Q1.) a) $a(\theta) = \frac{1}{\theta}$, $b(x) = x$, $c(\theta) = -\frac{1}{2\theta}$, $d(x) = x^2$ imply that $f(x,\theta)$ is one-parameter exponential. Hence $T = \sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i^2$ is minimal sufficient for $\theta$.

b) $E(X_1^2) = \int_0^\infty \frac{x^3}{\theta} e^{-\frac{x^2}{2\theta}} dx = -\int_0^\infty x^2 d e^{-\frac{x^2}{2\theta}} \overset{\downarrow \text{ by parts}}{=}$

$= 2\theta \int_0^\infty e^{-\frac{x^2}{2\theta}} d\frac{x^2}{2\theta} = 2\theta \int_0^\infty e^{-y} dy = 2\theta$

c) $E(T) = 2n\theta$. Therefore $U = \frac{1}{2n}\sum_{i=1}^{n} X_i^2$ is unbiased for $\theta$. Since it is also a function of sufficient and complete statistic, it must be the UMVUE (by Lehmann-Scheffe theorem)

For CR bound: $\ln f(x,\theta) = \ln x - \ln\theta - \frac{x^2}{2\theta}$

$$\frac{\partial}{\partial\theta}\ln f(x,\theta) = -\frac{1}{\theta} + \frac{x^2}{2\theta^2}$$

$$\frac{\partial^2}{\partial\theta^2}\ln f(x,\theta) = \frac{1}{\theta^2} - \frac{x^2}{\theta^3} \to E\left(-\frac{\partial^2}{\partial\theta^2}\ln f(X,\theta)\right) =$$

$= -\frac{1}{\theta^2} + \frac{2*\theta}{\theta^3} = \frac{1}{\theta^2} = I_{X_1}(\theta)$ Hence CR bound is

$$\frac{1}{n I_{X_1}(\theta)} = \boxed{\frac{\theta^2}{n}}$$

d) Since $f(x,\theta)$ is a one-parameter exponential, a uniformly most powerful unbiased $\alpha$-size test $\varphi^*$ of $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$ exists. Its structure is:

$$\varphi^* = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i^2 < C_1 \text{ or } \sum_{i=1}^{n} X_i^2 > C_2 \\ 0 & \text{else} \end{cases}$$

Note: $Var(X_1^2) = 8\theta^2 - 4\theta^2 = 4\theta^2$

Hence $\dfrac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - 2\theta_0\right)}{2\theta_0} \xrightarrow{d} N(0,1)$ by CLT, or,

equivalently $\dfrac{T - 2n\theta_0}{2\sqrt{n}\,\theta_0} \xrightarrow{d} N(0,1)$

So, a monotone transformation of $T$ is asymptotically standard normal : $T \approx N(2n\theta_0, 4n\theta_0^2)$. Hence the ump unbiased $\alpha$-test is asymptotically equivalent to a z-test $\tilde{\varphi}^* = \begin{cases} 1 & \text{if } \dfrac{|T - 2n\theta_0|}{2\sqrt{n}\,\theta_0} > z_{\alpha/2} \\ 0 & \text{else} \end{cases}$

e) $f(X|\theta)\tau(\theta) = \dfrac{\prod_{i=1}^{n} x_i}{\theta^n} e^{-\frac{\sum_{i=1}^{n} X_i^2}{2\theta}} \dfrac{1}{\theta^3} e^{-\frac{1}{\theta}}$

$\propto \theta^{-n-3} e^{-\left(\frac{\sum X_i^2}{2}+1\right)/\theta}$

which means that $h(\theta|X)$ must be Inverse gamma with parameters $n+2$ and $\sum_{i=1}^{n}\frac{X_i^2}{2} + 1$. Hence $\hat{\theta}_{Bayes} = \dfrac{\frac{\sum X_i^2}{2}+1}{n+1} =$

$= \boxed{\dfrac{\sum_{i=1}^{n} X_i^2 + 2}{2(n+1)}}$. (Notice that this is close to the UMVUE $U = \sum_{i=1}^{n} X_i^2/(2n)$ when $n$ is large.

f) First, note that CR bound $\frac{\theta^2}{n}$ is attained by $U$ since $L(X,\theta) = -\frac{n}{\theta} + \frac{\sum X_i^2}{2\theta^2} = \frac{n}{\theta^2}(U-\theta)$

This implies that $U$ is MLE. By transformation invariance, $\hat{h} = \sum_{i=1}^{n} X_i^2 / \sum_{i=1}^{n} Y_i^2$ is then MLE of $h(\theta_1,\theta_2) = \theta_1/\theta_2$.

$I_{\binom{X_1}{Y_1}}(\theta_1,\theta_2) = \begin{pmatrix} 1/\theta_1^2 & 0 \\ 0 & 1/\theta_2^2 \end{pmatrix}$ holds and then

$\sqrt{n}\left(\hat{h} - \frac{\theta_1}{\theta_2}\right) \xrightarrow{d} N\left(0, \left(\frac{\partial h}{\partial \theta_1}\ \frac{\partial h}{\partial \theta_2}\right)\begin{pmatrix} \theta_1^2 & 0 \\ 0 & \theta_2^2 \end{pmatrix}\begin{pmatrix} \frac{\partial h}{\partial \theta_1} \\ \frac{\partial h}{\partial \theta_2} \end{pmatrix}\right)$

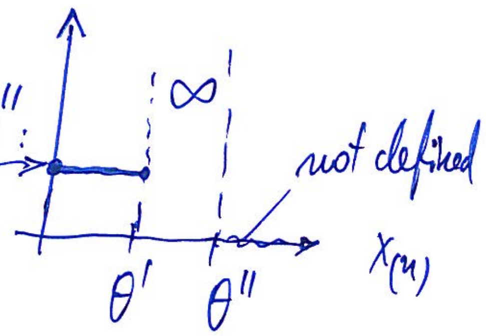
So the asymptotic variance is

$$\left(\frac{1}{\theta_2}\ \ -\frac{\theta_1}{\theta_2^2}\right)\begin{pmatrix} \theta_1^2 & 0 \\ 0 & \theta_2^2 \end{pmatrix}\begin{pmatrix} \frac{1}{\theta_2} \\ -\frac{\theta_1}{\theta_2^2} \end{pmatrix} = \left(\frac{1}{\theta_2}\ \ -\frac{\theta_1}{\theta_2^2}\right)\begin{pmatrix} \theta_1^2/\theta_2 \\ -\theta_1 \end{pmatrix} = 2\frac{\theta_1^2}{\theta_2^2}$$

**Q2** a) $L(\underline{X};\theta) = \frac{3^n}{\theta^{3n}} \prod_{i=1}^{n} x_i^2 \prod_{i=1}^{n} I_{(0,\theta)}(x_i) = \left(\frac{3}{\theta^3}\right)^n \prod_{i=1}^{n} x_i^2 \, I_{(0,\theta)}(X_{(n)})$

Then the ratio

$$\frac{L(\underline{X};\theta'')}{L(\underline{X};\theta')} = \left(\frac{\theta'}{\theta''}\right)^{3n} \frac{I_{(0,\theta'')}(X_{(n)})}{I_{(0,\theta')}(X_{(n)})}$$ is monotone non-decreasing in $X_{(n)}$ for fixed $0<\theta'<\theta''$:




Hence we have a family with monotone LR in $X_{(n)}$. $\left(\frac{\theta'}{\theta''}\right)^{3n}$

b) $F_{X_1}(x;\theta) = \begin{cases} 0 & x<0 \\ x^3/\theta^3 & 0<x<\theta \\ 1 & x>\theta \end{cases}$

Hence $F_{X_{(n)}}(x) = P(X_1 \le x \cap X_2 \le x \cap \dots \cap X_n \le x) = \begin{cases} 0 & x<0 \\ \left(\frac{x}{\theta}\right)^{3n} & 0<x<\theta \\ 1 & x>\theta \end{cases}$

Then density: $f_{X_{(n)}}(x) = \begin{cases} 3n x^{3n-1}/\theta^{3n}, & 0<x<\theta \\ 0 & \text{else} \end{cases}$

c) $\theta_0 = 4$ BG theorem says that a UMP $\alpha$-test $\varphi^*$ exists with $\varphi^* = \begin{cases} 1 & X_{(n)} > C \\ 0 & X_{(n)} \le C \end{cases}$

To find the threshold $C$, we "exhaust the level":

$$\alpha = E_{\theta=4}\varphi^* = P_{\theta=4}(X_{(n)} > c) = 1 - \left(\frac{C}{4}\right)^{3n} = \alpha$$

Hence $\left(\frac{C}{4}\right)^{3n} = 1-\alpha \Rightarrow C = \boxed{4(1-\alpha)^{\frac{1}{3n}}}$

and the test is completely determined