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Confidence Intervals

- frequently it is desirable to report an interval of plausible values of a parameter; e.g., in estimating the mean, $\bar{X} \pm$ small quantity

- the plausibility of an interval can be measured by the probability that the parameter lies in the interval

- Model: $X_1, X_2, \ldots, X_n$ iid $f_X(x;\theta)$, θ a scalar parameter

- we aim to find functions $\underline{\theta}(X_1, \ldots, X_n)$ and $\bar{\theta}(X_1, \ldots, X_n)$ such that, for a given α,

$$P(\underline{\theta} \leq \theta \leq \bar{\theta}) = 1 - \alpha, \text{ for all } \theta$$

then $(\underline{\theta}, \bar{\theta})$ is a $(1-\alpha)$ (or $(100(1-\alpha)\%)$ *confidence interval* for θ.

Confidence Intervals

- if one of $\underline{\theta}, \bar{\theta}$ is not random (usually 0 or $\pm\infty$), the confidence interval is one-sided

- construction of confidence intervals often relies on **pivotal functions**. These are functions of the data, the parameter of interest and possibly other parameters which are completely specified distribution (nothing unknown)

- **Confidence intervals for parameters of Normal distributions**

$$X_1, X_2, \ldots, X_n \quad iid \quad N(\mu, \sigma^2)$$

Confidence Intervals

$(1-\alpha)$-confidence interval for μ when σ is known:

- A confidence interval should be based on $\bar{X}$, the usual point estimator of μ.

- now,

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

so $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ is a pivotal function - it is a function of the parameter of interest (μ), the observations (through $\bar{X}$) and some known constants (σ, n) whose distribution is completely specified (no unknown parameters).

Confidence Intervals

- if $Z_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of the $N(0,1)$ distribution, whatever the value of μ,

$$\begin{aligned} 1-\alpha &= P(-z_{1-\alpha/2} < \tfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} < Z_{1-\alpha/2}) \\ &= P(\bar{X} - \tfrac{\sigma}{\sqrt{n}} z_{1-\alpha/2} < \mu < \bar{X} + \tfrac{\sigma}{\sqrt{n}} z_{1-\alpha/2}) \end{aligned}$$

- thus $\bar{X} \pm \frac{\sigma}{\sqrt{n}} z_{1-\alpha/2}$ is a $(1-\alpha)$-confidence interval for μ

- intervals like this are called *central* or *equal tailed*

Confidence Intervals

- for an interval to have length no more than l units we require

$$\frac{2\sigma}{\sqrt{n}} z_{1-\alpha/2} \leq l$$

  so the required sample size n is the smallest integer larger than $\frac{4\sigma^2 z^2_{1-\alpha/2}}{l^2}$ (n increases as α decreases)

- one-sided intervals:

$$\begin{aligned} P(-z_{1-\alpha} < \tfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}) &= 1-\alpha \\ &= P(\mu < \bar{X} + \tfrac{\sigma}{\sqrt{n}} z_{1-\alpha}) \end{aligned}$$

so $(-\infty, \bar{X} + \frac{\sigma}{\sqrt{n}} z_{1-\alpha})$ is a one sided $(1-\alpha)$-confidence interval for μ

- similarly, $P(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < z_{1-\alpha}) = 1-\alpha$ gives the one-sided interval $(\bar{X} - \frac{\sigma}{\sqrt{n}} z_{1-\alpha}, \infty)$

Confidence Intervals

$(1-\alpha)$-confidence interval for μ when σ is unknown

- a confidence interval should be based on $\bar{X}$
- pivotal function: $\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$
- if $t_{n-1,1-\alpha/2}$ is the $(1-\alpha/2)$th quantile of the t_{n-1} distribution,

$$\begin{aligned}
&1-\alpha \\
&= P(-t_{n-1,1-\alpha/2} < \tfrac{\bar{X}-\mu}{S/\sqrt{n}} < t_{n-1,1-\alpha/2}) \\
&= P(\bar{X} - \tfrac{S}{\sqrt{n}} t_{n-1,1-\alpha/2} < \mu < \bar{X} + \tfrac{S}{\sqrt{n}} t_{n-1,1-\alpha/2})
\end{aligned}$$

- $\bar{X} \pm \frac{S}{\sqrt{n}} t_{n-1,1-\alpha/2}$ is a $(1-\alpha)$-confidence interval for μ

Confidence Intervals

- the length of the interval is a random variable, length $= \frac{2S}{\sqrt{n}} t_{n-1,1-\alpha/2}$

- one-sided $(1-\alpha)$-confidence intervals are

$$(-\infty, \bar{X} + \frac{S}{\sqrt{n}} t_{n-1,1-\alpha})$$

and

$$(\bar{X} - \frac{S}{\sqrt{n}} t_{n-1,1-\alpha}, \infty)$$

Confidence Intervals

$(1-\alpha)$-confidence interval for σ^2 (or σ) when μ is unknown

- an interval should be based on S^2, the usual point estimator of σ^2

- pivotal function:

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$\begin{aligned} 1-\alpha &= P(\chi^2_{n-1,\alpha/2} < \tfrac{(n-1)S^2}{\sigma^2} < \chi^2_{n-1,1-\alpha/2}) \\ &= P(\tfrac{(n-1)S^2}{\chi^2_{n-1,1-\alpha/2}} < \sigma^2 < \tfrac{(n-1)S^2}{\chi^2_{n-1,\alpha/2}}) \end{aligned}$$

Confidence Intervals

- $(\frac{(n-1)S^2}{\chi^2_{n-1,1-\alpha/2}}, \frac{(n-1)S^2}{\chi^2_{n-1,\alpha/2}})$ is a $(1-\alpha)$-confidence interval for σ^2 and

$$(\sqrt{\frac{(n-1)S^2}{\chi^2_{n-1,1-\alpha/2}}}, \sqrt{\frac{(n-1)S^2}{\chi^2_{n-1,\alpha/2}}})$$

is a $(1-\alpha)$-confidence interval for σ

Confidence Intervals

- suppose now we have random samples from two (possibly different) Normal distributions, thus assume $X_1, X_2, \ldots, X_{n_1}$ iid $N(\mu_1, \sigma_1^2)$ and $Y_1, Y_2, \ldots, Y_{n_2}$ iid $N(\mu_2, \sigma_2^2)$ and assume that the samples are independent (i.e., the Xs are independent of the Ys)

$(1-\alpha)$**-confidence interval for** $\mu_1 - \mu_2$ **when** σ_1**,**σ_2 **are unknown**

- we first look at the common variance case $\sigma_1 = \sigma_2 = \sigma$ (unknown)

- Let $S_1^2 = \frac{1}{n_1-1}\sum_{i=1}^{n_1}(X_i - \bar{X})^2$, $S_2^2 = \frac{1}{n_2-1}\sum_{i=1}^{n_2}(Y_i - \bar{Y})^2$ and $S_p^2 = \frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}$

Confidence Intervals

- S_p^2 is the *pooled sample variance* since it is an estimator of σ^2 based on the combined sample of $X_1, X_2, \ldots, X_{n_1}, Y_1, Y_2, \ldots, Y_{n_2}$

- now $\frac{(n_1-1)S_1^2}{\sigma^2} \sim \chi^2_{n_1-1}$, $\frac{(n_2-1)S_2^2}{\sigma^2} \sim \chi^2_{n_2-1}$ and S_1^2 and S_2^2 are independent, so

$$\begin{aligned} \frac{(n_1+n_2-2)S_p^2}{\sigma^2} &= \frac{(n_1-1)S_1^2}{\sigma^2} + \frac{(n_2-1)S_2^2}{\sigma^2} \\ &\sim \chi^2_{n_1+n_2-2} \end{aligned}$$

- hence $E\{\frac{(n_1+n_2-2)S_p^2}{\sigma^2}\} = (n_1 + n_2 - 2)$ and $E(S_p^2) = \sigma^2$

- now $\bar{X} - \bar{Y} \sim N(\mu_1 - \mu_2, \sigma^2(\frac{1}{n_1} + \frac{1}{n_2}))$ and so

$$\frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim N(0, 1)$$

Confidence Intervals

- therefore

$$\frac{\dfrac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}}{\sqrt{\dfrac{S_p^2}{\sigma^2}}} = \frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{S_p\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}} \sim t_{n_1+n_2-2}$$

- since the numerator and denominator above are independent as S_1^2 and S_2^2 are independent of $\bar{X}$ and $\bar{Y}$. Thus $\frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$ is a pivotal function and

$$P(-t_{n_1+n_2-2,1-\alpha/2} < \frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} < t_{n_1+n_2-2,1-\alpha/2}) = 1-\alpha$$

- thus $\bar{X} - \bar{Y} \pm S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} t_{n_1+n_2-2, 1-\alpha/2}$ is a $(1-\alpha)$-confidence interval for $\mu_1 - \mu_2$

Confidence Intervals

the paired observations case

- now $n_1 = n_2 = n$, say, and in most applications, when the observations are in pairs, the paired observations are usually dependent.

$$X_1, X_2, \ldots, X_n \quad iid N(\mu_1, \sigma_1^2)$$

$$Y_1, Y_2, \ldots, Y_n \quad iid N(\mu_2, \sigma_2^2)$$

  Thus we assume X_i is independent of Y_j only for $j \neq i$

- if we let $Z_i = X_i - Y_i, i = 1, 2, \ldots, n$, then $Z_1, Z_2, \ldots, Z_n$ are iid $N(\mu_1 - \mu_2, \sigma_Z^2)$, where σ_Z^2 is unknown ($\sigma_1^2 + \sigma_2^2$ if X_i and Y_i are independent)

Confidence Intervals

- a confidence interval should be based on the point estimator $\bar{Z} = \bar{X} - \bar{Y}$

- if $S_Z^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Z_i - \bar{Z})^2 =$ sample variance of the differences within pairs, then

$$\frac{\bar{Z} - (\mu_1 - \mu_2)}{S_Z/\sqrt{n}} \sim t_{n-1}$$

- thus $\frac{\bar{Z} - (\mu_1 - \mu_2)}{S_Z/\sqrt{n}}$ is a pivotal function and

$$P(-t_{n-1,1-\alpha/2} < \frac{\bar{Z} - (\mu_1 - \mu_2)}{S_Z/\sqrt{n}} < t_{n-1,1-\alpha/2}) = 1 - \alpha$$

- $\bar{Z} \pm \frac{S_Z}{\sqrt{n}} t_{n-1,1-\alpha/2}$ is a $(1-\alpha)$-confidence interval for $\mu_1 - \mu_2$

Confidence Intervals

- if we do not have either common variance or observations in pairs, we cannot find an exact $(1-\alpha)$-confidence interval for $\mu_1 - \mu_2$

- assume again that the samples are independent

$(1-\alpha)$-**confidence interval for** $\frac{\sigma_1^2}{\sigma_2^2}$ **(or** $\frac{\sigma_2^2}{\sigma_1^2}$ **or** $\frac{\sigma_1}{\sigma_2}$ **or** $\frac{\sigma_2}{\sigma_1}$**) when** μ_1, μ_2 **are unknown**

$$\frac{(n_1-1)S_1^2}{\sigma_1^2} \sim \chi^2_{n_1-1}, \quad \frac{(n_2-1)S_2^2}{\sigma_2^2} \sim \chi^2_{n_2-1}$$

and S_1^2 and S_2^2 are independent, so

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{n_1-1,n_2-1}$$

Confidence Intervals

- thus $\dfrac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$ is a pivotal function and

$$\begin{aligned}
& P(F_{n_1-1,n_2-1,\alpha/2} < \tfrac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} < F_{n_1-1,n_2-1,1-\alpha/2}) = 1-\alpha \\
\Rightarrow\quad & P(F_{n_1-1,n_2-1,\alpha/2} < \tfrac{\sigma_2^2}{\sigma_1^2} \cdot \tfrac{S_1^2}{S_2^2} < F_{n_1-1,n_2-1,1-\alpha/2}) = 1-\alpha \\
\Rightarrow\quad & P(\tfrac{1}{F_{n_1-1,n_2-1,\alpha/2}} > \tfrac{\sigma_1^2}{\sigma_2^2} \cdot \tfrac{S_2^2}{S_1^2} > \tfrac{1}{F_{n_1-1,n_2-1,1-\alpha/2}}) = 1-\alpha \\
\Rightarrow\quad & P(\tfrac{1}{F_{n_1-1,n_2-1,1-\alpha/2}} < \tfrac{\sigma_1^2}{\sigma_2^2} \cdot \tfrac{S_2^2}{S_1^2} < \tfrac{1}{F_{n_1-1,n_2-1,\alpha/2}}) = 1-\alpha \\
\Rightarrow\quad & P(\tfrac{1}{F_{n_1-1,n_2-1,1-\alpha/2}} < \tfrac{\sigma_1^2}{\sigma_2^2} \cdot \tfrac{S_2^2}{S_1^2} < F_{n_2-1,n_1-1,1-\alpha/2}) = 1-\alpha \\
\Rightarrow\quad & P(\tfrac{S_1^2}{S_2^2} \cdot \tfrac{1}{F_{n_1-1,n_2-1,1-\alpha/2}} < \tfrac{\sigma_1^2}{\sigma_2^2} < \tfrac{S_1^2}{S_2^2} \cdot F_{n_2-1,n_1-1,1-\alpha/2}) = 1-\alpha
\end{aligned}$$

- thus

$$(\frac{S_1^2}{S_2^2} \cdot \frac{1}{F_{n_1-1,n_2-1,1-\alpha/2}}, \frac{S_1^2}{S_2^2} \cdot F_{n_2-1,n_1-1,1-\alpha/2})$$

is a $(1-\alpha)$-confidence interval for σ_1^2/σ_2^2

- similarly,

$$(\frac{S_2^2}{S_1^2} \cdot \frac{1}{F_{n_2-1,n_1-1,1-\alpha/2}}, \frac{S_2^2}{S_1^2} \cdot F_{n_1-1,n_2-1,1-\alpha/2})$$

is a $(1-\alpha)$-confidence interval for σ_2^2/σ_1^2 and by taking square roots we obtain intervals for σ_1/σ_2 and σ_2/σ_1

Constructing Confidence Intervals

- $X_1, X_2, \ldots, X_n$ iid $f_X(x;\theta)$. If the distribution of $q(X_1, X_2, \ldots, X_n;$ does not depend on θ, then q is a pivotal function

- if q is a pivotal function, then we can find numbers q_1, q_2 such that

$$P(q_1 < q(X_1, X_2, \ldots, X_n; \theta) < q_2) = 1 - \alpha$$

q_1, q_2 will depend on α, e.g., if q_1, q_2 are the lower and upper $\alpha/2$ points of the distribution of $q(X_1, X_2, \ldots, X_n; \theta)$.

Constructing Confidence Intervals

- suppose the inequalities

$$q_1 < q(X_1, X_2, \ldots, X_n; \theta) < q_2$$

are equivalent to the inequalities

$$t_1(X_1, X_2, \ldots, X_n) < \theta < t_2(X_1, X_2, \ldots, X_n)$$

where t_1, t_2 are functions only of $X_1, X_2, \ldots, X_n$ and not functions of θ. Then

$$(t_1(X_1, X_2, \ldots, X_n), t_2(X_1, X_2, \ldots, X_n))$$

is a $(1-\alpha)$-confidence interval for θ since

$$\begin{aligned} & 1-\alpha \\ = \; & P(q_1 < q(X_1, X_2, \ldots, X_n; \theta) < q_2) \\ = \; & P(t_1(X_1, X_2, \ldots, X_n) < \theta < t_2(X_1, X_2, \ldots, X_n)) \end{aligned}$$

- note that t_1, t_2 are not unique, so we might select the pair q_1, q_2 which minimizes the length of the interval

Constructing Confidence Intervals

- **Example 1:** $X_1, X_2, \ldots, X_n$ iid $N(\mu, \sigma^2)$, σ known. $q(X_1, X_2, \ldots,$ $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ is a pivotal function

$$q_1 < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < q_2 \Leftrightarrow \bar{X} - \frac{\sigma}{\sqrt{n}} q_2 < \mu < \bar{X} - \frac{\sigma}{\sqrt{n}} q_1$$

- the interval $(\bar{X} - \frac{\sigma}{\sqrt{n}} q_2, \bar{X} - \frac{\sigma}{\sqrt{n}} q_1)$ has length $L = \frac{\sigma}{\sqrt{n}}(q_2 - q_1)$ and is a $(1-\alpha)$-confidence interval if $\int_{q_1}^{q_2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1 - \alpha$.

- suppose we wish to minimise L subject to $\int_{q_1}^{q_2} \phi(x)dx = 1-\alpha$, where $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

- if we let $\Phi(x) = \int_{-\infty}^{\phi}(u)du$ then the constraint $\int_{q_1}^{q_2} \phi(x)dx = 1 - \alpha$ is $\Phi(q_2) - \Phi(q_1) = 1 - \alpha$ or $\Phi(q_2) = \Phi(q_1) + 1 - \alpha$

Constructing Confidence Intervals: Example 1

- thus q_2 is a function of q_1, so that L may be treated as a function of a single variable q_1

- so

$$\frac{dL}{dq_1} = \frac{\sigma}{\sqrt{n}}(\frac{dq_2}{dq_1} - 1)$$

- we have $\phi(q_2) \cdot \frac{dq_2}{dq_1} = \phi(q_1)$ or $\frac{dq_2}{dq_1} = \frac{\phi(q_1)}{\phi(q_2)}$

- then

$$
\begin{aligned}
\frac{dL}{dq_1} &= \frac{\sigma}{\sqrt{n}}\left(\frac{\phi(q_1)}{\phi(q_2)} - 1\right) = 0 \\
\Leftrightarrow \quad & \phi(q_1) = \phi(q_2) \\
\Rightarrow \quad & q_1 = -q_2 (q_1 < q_2) \\
\Rightarrow \quad & q_2 = z_{1-\alpha/2} \text{ and } q_1 = -z_{1-\alpha/2}
\end{aligned}
$$

Constructing Confidence Intervals:

- therefore the usual interval $\bar{X} \pm \frac{\sigma}{\sqrt{n}} z_{1-\alpha/2}$ is the $(1-\alpha)$-confidence interval with minimum length.

- **Example 2:** $X_1, X_2, \ldots, X_n$ iid $N(\mu, \sigma^2)$, σ unknown, $q(X_1, X_2, \ldots$ $\frac{\bar{X}-\mu}{S/\sqrt{n}}$ is a pivotal function.

  $q_1 < \frac{\bar{X}-\mu}{S/\sqrt{n}} < q_2$ gives an interval $(\bar{X} - \frac{S}{\sqrt{n}} q_2, \bar{X} - \frac{S}{\sqrt{n}} q_1)$ with length $L = \frac{S}{\sqrt{n}}(q_2 - q_1)$, The interval is a $(1-\alpha)$-confidence interval if $\int_{q_1}^{q_2} t_{n-1}$ density$=1-\alpha$.

  Similar calculations to those in Example 1 show that L is minimised when $q_2 = t_{n-1,1-\alpha/2}$ and $q_1 = t_{n-1,\alpha/2}$

Constructing Confidence Intervals:

- **Example 3:** $X_1, X_2, \ldots, X_n$ iid $f_X(x;\theta) = \frac{1}{\theta}e^{-x/\theta}, x > 0, \theta > 0. Y = \frac{X}{\theta}$ has density

$$f_Y(Y) = f_X(x;\theta)|\frac{dx}{dy}| = \frac{1}{\theta}e^{-y}|\theta| = e^{-y}, y > 0$$

Thus $Y \sim Gamma(1)$ If $Y_i = \frac{X_i}{\theta}$, then $Y_1, Y_2, \ldots, Y_n$ are iid Gamma(1) and so $\sum_{i=1}^n Y_i = \frac{1}{\theta}\sum_{i=1}^n X_i \sim Gamma(n)$.

Therefore $\frac{1}{\theta}\sum_{i=1}^n X_i$ is a pivotal function.

if $\gamma_{n,\alpha/2}$ is the upper $\alpha/2$ point of the Gamma(n) density $(\int_0^{\gamma_{n,\alpha/2}} \frac{e^{-x}x^{n-1}}{\Gamma(n)}dx = 1-\alpha/2)$ and $\gamma_{n,1-\alpha/2}$ is the lower $\alpha/2$

point, then

$$P(\gamma_{n,1-\alpha/2} < \frac{1}{\theta}\sum X_i < \gamma_{n,\alpha/2}) = 1 - \alpha$$

so $(\frac{\sum X_i}{\gamma_{n,\alpha/2}}, \frac{\sum X_i}{\gamma_{n,1-\alpha/2}})$ is a $(1-\alpha)$-confidence interval for θ

Constructing Confidence Intervals:

- **Example 4:** $X_1, X_2, \ldots, X_n$ iid $N(\theta, \theta^2)$

  $q(X_1, X_2, \ldots, X_n; \theta) = \frac{\bar{X} - \theta}{\theta/\sqrt{n}}$ is a pivotal function and $q(X_1, X_2, \ldots, X_n; \theta) \sim N(0, 1)$

$$\begin{aligned} & q_1 < \frac{\bar{X} - \theta}{\theta/\sqrt{n}} < q_2 \\ \Leftrightarrow \; & \bar{X}(1 + \tfrac{q_2}{\sqrt{n}})^{-1} < \theta < \bar{X}(1 + \tfrac{q_1}{\sqrt{n}})^{-1} \end{aligned}$$

  if we choose $q_1 = -z_{1-\alpha/2}$ and $q_2 = z_{1-\alpha/2}$, then $(\bar{X}(1 + \frac{z_{1-\alpha/2}}{\sqrt{n}})^{-1}, \bar{X}(1 - \frac{z_{1-\alpha/2}}{\sqrt{n}})^{-1})$ is a $(1 - \alpha)$-confidence interval for θ

Large sample CI for Bernoulli parameters

- **One sample case** $X_1, X_2, \ldots, X_n$ iid Bernoulli(p). The mle of p is $\hat{p} = \bar{X}, E(\hat{p}) = p, Var(\hat{p}) = \frac{p(1-p)}{n}$.

- central limit theorem: when n is large, $\hat{p}$ is approximately $N(p, \frac{p(1-p)}{n})$ and so

$$P(-z_{1-\alpha/2} < \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} < z_{1-\alpha/2}) \approx 1 - \alpha$$

now

$$
\begin{aligned}
& -z_{1-\alpha/2} < \frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} < z_{1-\alpha/2} \\
\Leftrightarrow \quad & (\hat{p}-p)^2 \leq \frac{p(1-p)}{n} z^2_{1-\alpha/2} \\
\Leftrightarrow \quad & (1+\frac{z^2_{1-\alpha/2}}{n})p^2 - (2\hat{p}+\frac{z^2_{1-\alpha/2}}{n})p + \hat{p}^2 \leq 0
\end{aligned}
$$

Large sample CI for Bernoulli parameters

- a large sample (or approximate) $(1-\alpha)$-confidence interval for p is the set of values of p which satisfy the above inequalities. Thus the roots of the quadratic in p determine the interval; i.e.,

$$\frac{\hat{p} + \frac{z^2_{1-\alpha/2}}{2n} \pm \frac{z_{1-\alpha/2}}{\sqrt{n}} \sqrt{\hat{p}(1-\hat{p}) + \frac{z^2_{1-\alpha/2}}{4n}}}{1 + \frac{z^2_{1-\alpha/2}}{n}}$$

- if we omit the $O(\frac{1}{n})$ terms, the interval becomes $\hat{p} \pm \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} z_{1-\alpha/2}$, which is precisely the interval we would have obtained by replacing $Var(\hat{p})$ by $\frac{\hat{p}(1-\hat{p})}{n}$ and using the "pivotal function" $\frac{\hat{p}-p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$ which is also approximately $N(0,1)$

Large sample CI for Bernoulli parameters

- **Two sample case** $X_1, \ldots, X_m$ iid Bernoulli(p_1), mle of p_1 is $\hat{p}_1 = \bar{X}$. $Y_1, Y_2, \ldots, Y_n$ iid Bernoulli(p_2), mle of p_2 is $\hat{p}_2 = \bar{Y}$.

- if the samples are independent and m and n are large, by the Central Limit theorem, $\hat{p}_1$ is approximately $N(p_1, \frac{p_1(1-p_1)}{m})$, $\hat{p}_2$ is approximately $N(p_2, \frac{p_2(1-p_2)}{n})$ and so $\hat{p}_1 - \hat{p}_2$ is approximately $N(p_1 - p_2, \frac{p_1(1-p_1)}{m} + \frac{p_2(1-p_2)}{n})$ and, approximating variances as in case 1,

$$\frac{\hat{p}_1 - \hat{p}_2 - (p_1 - p_2)}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{m} + \frac{\hat{p}_2(1-\hat{p}_2)}{n}}}$$

is approximately $N(0, 1)$

Large sample CI for Bernoulli parameters

- and so a large sample (or approximate) $(1-\alpha)$-confidence interval for $p_1 - p_2$ is

$$\hat{p}_1 - \hat{p}_2 \pm \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{m} + \frac{\hat{p}_2(1-\hat{p}_2)}{n}}\, z_{1-\alpha/2}$$

Large sample CI in general

- large sample confidence intervals in general rely on the general result

  Let $X_1, X_2, \ldots, X_n$ be iid from $f_X(x; \theta)$ and let $\hat{\theta}_{mle}$ be the maximum likelihood estimator of θ. Then, under certain regularity conditions

$$\frac{\hat{\theta}_{mle} - \theta}{\sqrt{CRLB(\hat{\theta}_{mle})}} \overset{approx}{\sim} N(0, 1)$$

where

$$CRLB(\theta) = \frac{-1}{nE\{\frac{\partial^2}{\partial\theta^2}\log f_X(X;\theta)\}}$$

is the Cramer-Rao lower bound for unbiased estimators of θ

Large sample CI in general

- from this it is easily shown that Let $X_1, X_2, \ldots, X_n$ be iid from $f_X(x;\theta)$ and let $\hat{\theta}_{mle}$ be the maximum likelihood estimator of θ. Then the large sample $(1-\alpha)$-confidence interval for θ is

$$\hat{\theta}_{mle} \pm z_{1-\alpha/2}\sqrt{CRLB(\hat{\theta}_{mle})}$$

- **Example 5:**

$X_1, X_2, \ldots, X_n$ iid $f_X(x;\theta) = \theta e^{-\theta x}, x > 0; \theta > 0$.

$$\begin{aligned}\hat{\theta}_{mle} = \; & \frac{1}{\bar{X}} \text{ is the mle of } \theta \\ & \log f_X(X;\theta) = \log\theta - \theta X \\ & \frac{\partial}{\partial\theta}\log f_X(X;\theta) = \frac{1}{\theta} - X\end{aligned}$$

$$E\{\frac{\partial^2}{\partial\theta^2}\log f_X(X;\theta)\} = E\{\frac{-1}{\theta^2}\} = \frac{-1}{\theta^2}$$

$$\therefore CRLB(\theta) = \frac{-1}{n(\frac{-1}{\theta^2})} = \frac{\theta^2}{n}$$

hence

$$\sqrt{CRLB(\theta)} = \frac{\theta}{\sqrt{n}}$$

and

$$\sqrt{CRLB(\hat{\theta}_{mle})} = \frac{1}{\bar{X}\sqrt{n}}$$

thus a large sample $(1-\alpha)$-confidence interval for θ is

$$\frac{1}{\bar{X}} \pm \frac{z_{1-\alpha/2}}{\bar{X}\sqrt{n}}$$

Large sample CI in general

- **Example 6:** $X_1, \ldots, X_n$ iid Poisson(λ), $\hat{\lambda}_{mle} = \bar{X}$ is the mle of λ.

$$f_X(X;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \ldots; \lambda > 0$$

$$\ln f_X(X;\lambda) = -\lambda + X \ln \lambda - \ln X!$$

$$\frac{\partial}{\partial \lambda} \ln f_X(X;\lambda) = -1 + \frac{X}{\lambda}$$

$$E\{\frac{\partial^2}{\partial \lambda^2} \ln f_X(X;\lambda)\} = E(\frac{-X}{\lambda^2}) = \frac{-1}{\lambda^2} E(X) = \frac{-1}{\lambda}$$

so

$$CRLB(\lambda) = \frac{-1}{nE\{\frac{\partial^2}{\partial\lambda^2}\ln f_X(X;\lambda)\}} = \frac{-1}{\frac{-n}{\lambda}} = \frac{\lambda}{n}$$

$$\sqrt{CRLB(\lambda)} = \sqrt{\frac{\lambda}{n}}$$

$$\sqrt{CRLB(\hat{\lambda}_{mle})} = \sqrt{\frac{\bar{X}}{n}}$$

so a large sample $(1-\alpha)$-confidence interval for λ is

$$\bar{X} \pm z_{1-\alpha/2}\sqrt{\frac{\bar{X}}{n}}.$$