`MATH5905Inference` `inference]` `umvue`

Week5 CR Inequality and UMVUE

4.1 Cramer-Rao Inequality

Mean squared error: $MSE_\theta(T_n) = Var_\theta T_n + (b_n(\theta))^2$
Our aim is to minimize MSE, by imposing the crterion of unbiasedness, we only have to minimize the Variance

Condition for attain CR bound: $V(X, \theta) = k_n(\theta)[W(X) - \tau(\theta)]$

CR-Bound:

$$Var_\theta(W(X)) \geq \frac{\frac{\partial}{\partial\theta}\tau(\theta)^2}{I_X(\theta)}$$

Where $I_X(\theta) = -E(\frac{\partial^2}{\partial\theta^2} InL(X, \theta))$ is fisher information.

4.5 Rao-Blackwell Theorem

$W: any\ unbiased\ estimator\ of\ \tau(\theta)$
$T: sufficient\ statistics\ for\ \theta$
What we need to find:

$$\hat{\tau}(T) = E(W|T)$$

and Rao-Blackwell theorem tells us that $E_\theta\hat{\tau}(T) = \tau(\theta)$ and $Var_\theta\hat{\tau}(T) \leq Var_\theta(W)$

4.6 Uniqueness

If an esitimator W is UMVUE for $\tau(\theta)$, then W is unique. Moreover, **W is UMVUE iff W is uncorrelated with all unbiased estimators of zero.**

4.7 Completeness

Let's say have a statistic T

$$E_\theta g(T) = 0,\ for\ all\ \theta \in \Theta\ implies\ P_\theta(g(T) = 0) = 1$$

> Meaning of complete statistic

Practice:

1. $T = \sum_{i=1}^{n} X_i\ for\ Bernoulli\ distribution$

4.8 Lehmann-Scheffe

It seems the only difference between Lehmann-Scheffe and Rao-Blackwell theorem is just "Completeness"
What we can do with it: find UMVUE even in situtations when the CR bound is not achievable
Once again, what we need to calculate is:

$$\hat{\tau}(T) = E(W|T)$$

Note: If W is a function of T, we can immediately come to the conclusion that W is UMVUE.

Conclusions

$Mean\ squared\ error: MSE_\theta(T_n) = Var_\theta T_n + (b_n(\theta))^2$

condition for attaining CR bound: $V(X, \theta) = k_n(\theta)[W(X) - \tau(\theta)]$

Confusion

1. During your note, example 4.9 - Poisson distribution
   page3: what's the meaning of $Var(X_1)$? X_1 only has one value, how do we describe the extent of data spreading out?
2. Exampl 4.9 - Uniform distribution
   Unbiased estimator $Y = \frac{n+1}{n} X_{(n)}$ Explain?

$$f_{\mathcal{X}(n)}(t,\theta) = \begin{cases} \frac{nt^{n-1}}{\theta^n} & 0<t<\theta \\ 0 & \text{else} \end{cases} \quad (1)$$

   Does it mean the probability density function of $X_{(n)}$?
   Also, in the additional notes, you mentioned if W is a function of T then W is UMVUE.
3. **4.6** uniqueness black bold part, explain the meaning?