# Mechanism Design for Online Resource Allocation: A Unified Approach

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# Joint Work with


- **Alberto Leon-Garcia**, University of Toronto, Canada.
- **Bo Sun, Danny H.K. Tsang**, HKUST, Hong Kong SAR, China.
- **Yuan Wu**, University of Macau, Macau SAR, China.

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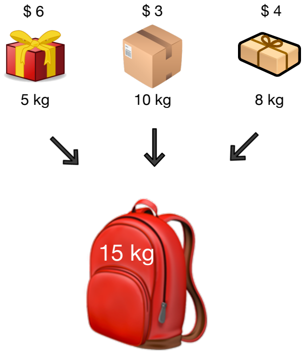

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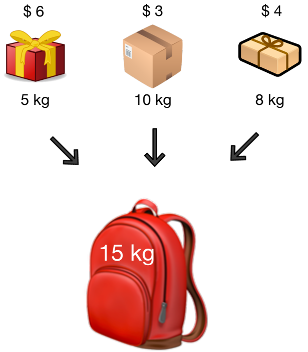

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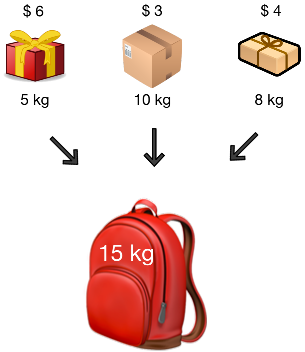

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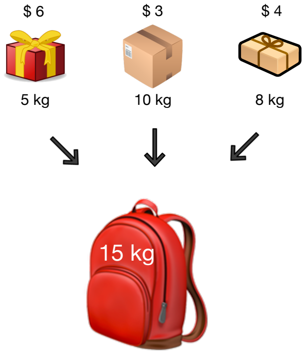
- Welfare maximization: $\sum_n v_n x_n - f(\sum_n r_n x_n)$.



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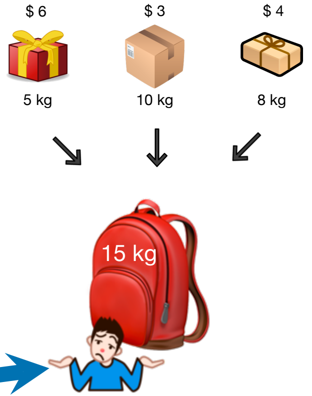
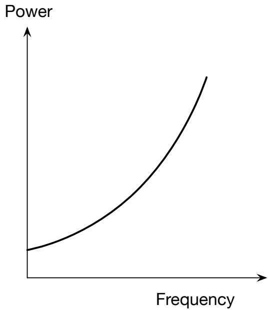


# Supply Costs in Different Forms

- Cloud resource allocation with **operating costs**, e.g., power v.s. CPU utilization.



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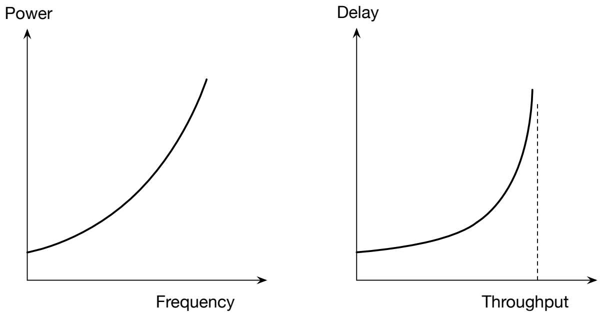


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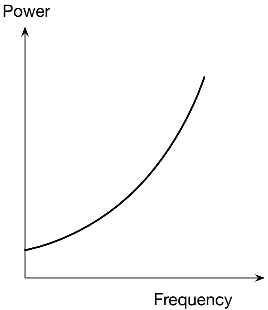


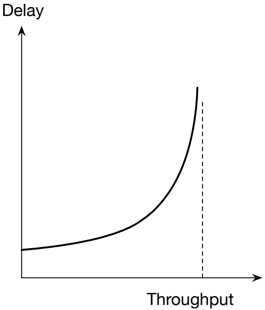


**Strictly-convex $f$**

**Increasing Marginal Costs $f'$**

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## Competitive Ratio

- **Offline Setting**: knows all arrival information:

$$S_{\text{offline}}(\mathcal{A}) = \sum_{n \in \{1,2,\cdots\}} v_n x_n^* - f\left(\sum_{n \in \{1,2,\cdots\}} r_n x_n^*\right),$$

where $\mathcal{A} = \{(v_1, r_1), (v_2, r_2), \cdots\}$ denotes an arrival instance.

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- **Online Setting**: develop posted prices, $\{p_n\}_{\forall n}$, whose competitive ratio:

$$\alpha \triangleq \max_{\text{all possible } \mathcal{A}} \frac{S_{\text{offline}}(\mathcal{A})}{S_{\text{online}}(\mathcal{A})}$$

is bounded by a constant independent of the number of agents.

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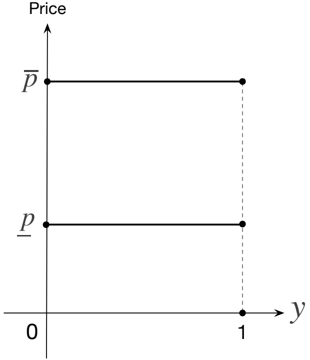


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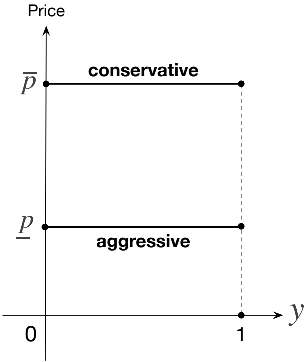


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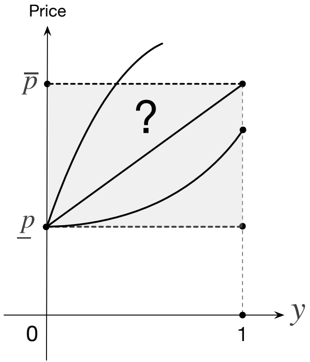
$$\boxed{\text{competitive ratio } \alpha \longrightarrow 1}$$

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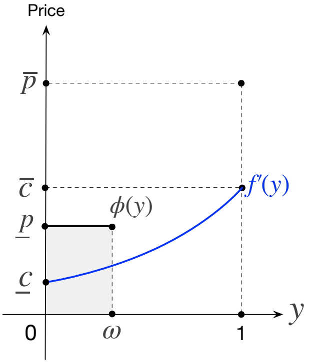

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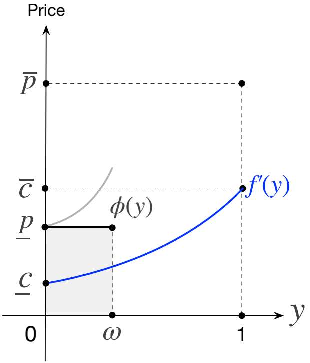


$$\boxed{\alpha \geq \frac{S_{\text{offline}}(\mathcal{A}_{\underline{p}})}{S_{\text{online}}(\mathcal{A}_{\underline{p}})} = \frac{\underline{p}\underline{\rho} - f(\underline{\rho})}{\underline{p}r_1 - f(r_1)} \xrightarrow{r_1 \to 0} +\infty}$$

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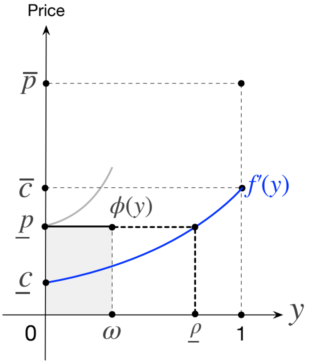

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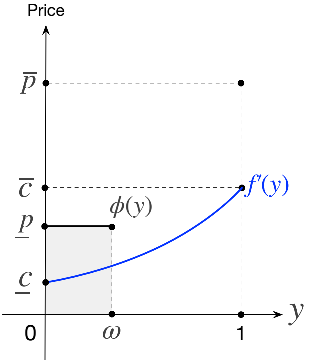

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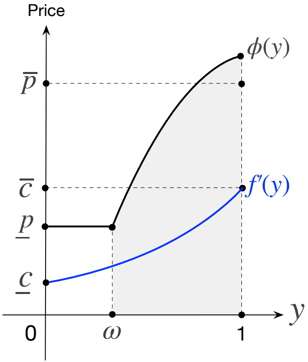

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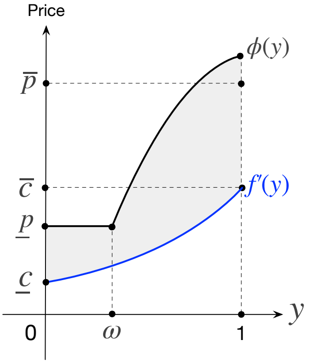
- **P3**: sell the resource at a profitable price.
  - $\phi(y) > f'(y)$ must hold for all $y \in [0, 1]$.



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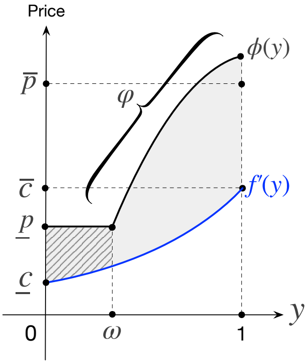

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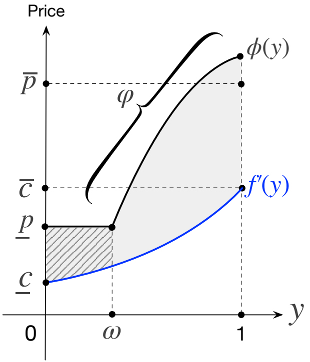


The **flat segment** $\omega$ and the **increasing segment** $\varphi(y)$.

# Main Results: Sufficient Conditions

- **Flat-Segment:** $\omega$ should satisfy

$$\boxed{\underline{p}\omega - f(\omega) \geq \frac{1}{\alpha} h(\underline{p}) \text{ and } f(\omega) \leq \underline{p},}$$

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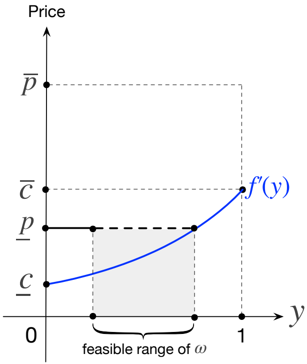

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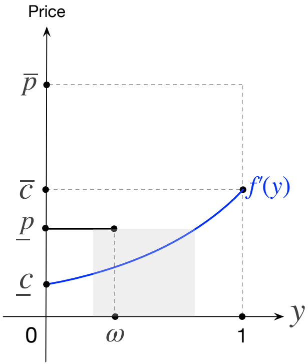

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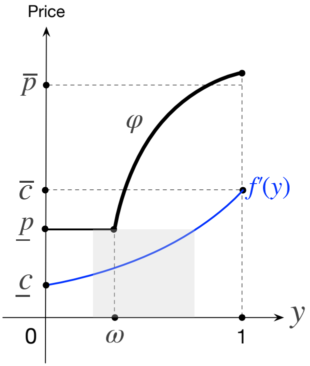

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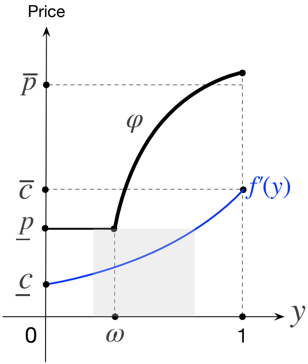


**Online Primal-Dual Approach**

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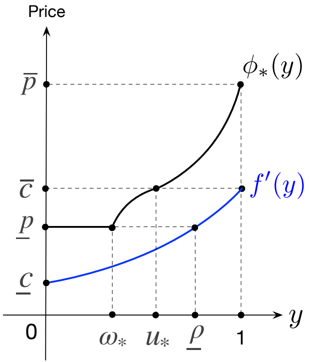

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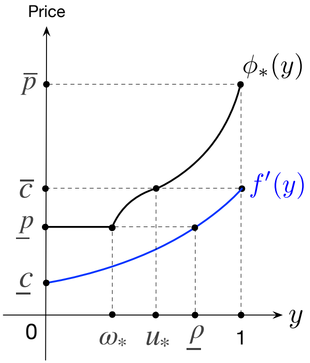

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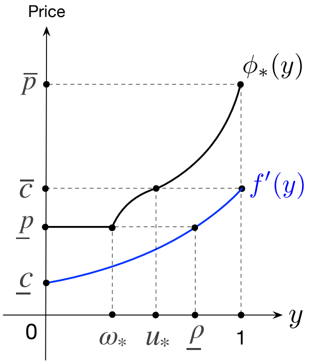
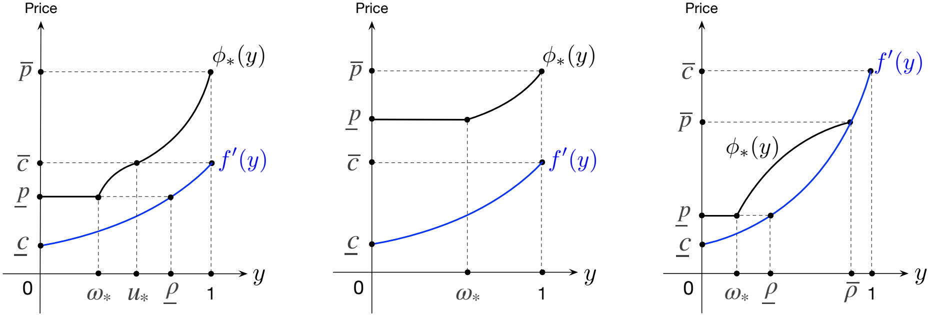



**Remarks**: i) $\omega_*$ depends on $f, \underline{p},$ and $\bar{p}$ and ii) $\boxed{\varphi_*'(y) = \alpha_* \cdot \Phi(\varphi_*, y)}$.

# General Cases: Strictly-Convex Supply Costs




**Case-1**: $\underline{c} < \underline{p} < \bar{c} < \bar{p}$ **Case-2**: $\underline{c} < \bar{c} \le \underline{p} \le \bar{p}$ **Case-3**: $\underline{c} < \underline{p} \le \bar{p} \le \bar{c}$

# Special Cases: Zero and Linear Supply Costs

- Given $\mathcal{S} = \{f, \underline{p}, \bar{p}\}$ with $f(y) = qy$, where $q \geq 0$, there exists a unique $\phi_*$:

$$\phi_*(y) = \begin{cases} \underline{p} & \text{if } y \in [0, \omega_*), \\ (\underline{p} - q) \cdot \exp\left(\frac{y}{\omega_*} - 1\right) + q & \text{if } y \in [\omega_*, 1], \\ +\infty & \text{if } y \in (1, +\infty), \end{cases}$$

such that $\text{PM}_{\phi_*}$ is $\alpha_*$-competitive, where $\alpha_*$ and $\omega_*$ are given by

$$\alpha_* = 1 + \ln\left(\frac{\bar{p} - q}{\underline{p} - q}\right), \quad \omega_* = \frac{1}{\alpha_*}.$$

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**Remark**: the logarithmic competitive ratio is not new, see [42] for details.

# A Unified Approach

- zero supply costs: $\alpha_* = 1 + \ln\left(\frac{\bar{p}}{\underline{p}}\right)$ when $f(y) = 0$ (e.g., [41], [42]).
- linear supply costs: $\alpha_* = 1 + \ln\left(\frac{\bar{p}-q}{\underline{p}-q}\right)$ when $f(y) = qy$ with $q > 0$ (e.g., [41]).
- **strictly-convex supply costs:** $\alpha_* = \frac{h(\underline{p})}{\underline{p}\omega_* - f(\omega_*)}$.

# A Case Study: Quadratic Supply Costs

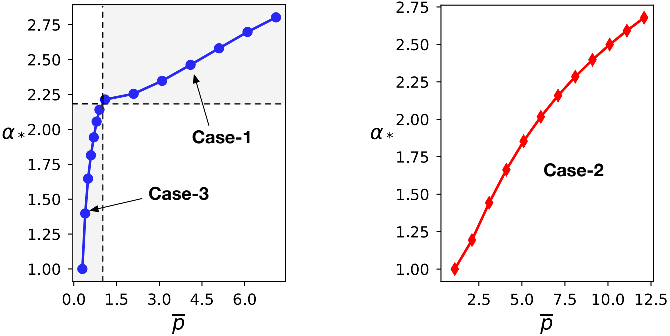


Figure: Illustration of $\alpha_*$ when $f(y) = \frac{1}{2}y^2$. Left: $\underline{p} = 0.3$. Right: $\underline{p} = 1.1$.

# Conclusions

- A **unified** approach for online resource allocation.
  - characterization of optimal competitive ratios.
  - computation of optimal pricing functions.

# Conclusions

- A **unified** approach for online resource allocation.
  - characterization of optimal competitive ratios.
  - computation of optimal pricing functions.
- A **general** model that can be extended to more complex settings.
  - multi-knapsack problems.
  - multi-unit auctions and combinatorial auctions.
  - . . .

# Thank You

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