Posted-Price Retailing of Transactive Energy: An Optimal Online Mechanism without Prediction

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***Abstract*—In this paper, we study a general transactive energy (TE) retailing problem in smart grids: a TE retailer (e.g., a utility company) publishes the energy price, which may vary over time. TE customers arrive in an arbitrary manner and may choose to either purchase a certain amount of energy based on the posted price, or leave without buying. Typical examples of such a setup include a transactive electric vehicle charging platform, or a general market-based demand-side management program, etc. We consider the setting where the customer arrival information is unknown (i.e., without prediction), and focus on maximizing the social welfare of the TE system through a posted-price mechanism (PPM) that runs in an online fashion with causal information only. We quantify the performance of the proposed PPM in the competitive analysis framework, and show that our proposed PPM is optimal in the sense that no other online mechanisms can achieve a better competitive ratio. We evaluate our theoretic results for the case of transactive electric vehicle charging. Our extensive experimental results show that the proposed PPM is competitive and robust against system uncertainties, and outperforms several existing benchmarks.**

***Index Terms*—Pricing, Mechanism Design, Competitive Analysis, Transactive Energy, Smart Grid**

I. Introduction

Over the past decade, a growing attention has been devoted to the development of new economic models and control strategies to ensure market efficiency and grid reliability through the use of demand response techniques [1] and distributed energy resources (DERs) [2]. This has led to a focus on a new area of transactive energy (TE), defined by the Gridwise Architecture Council as "a system of economic and control mechanisms to balance the supply and demand using value as the key operational parameter" [3]. According to National Institute of Standards and Technology (NIST) [4], TE has a great potential for efficiency improvement through market-based transactive exchange between energy suppliers and energy consumers.

It is commonly believed that traditional flat-rate pricing models must be updated with the TE framework since they cannot handle the added complexity and cost to the energy suppliers due to the increasing intermittency of DERs [3]. Pricing schemes such as time-of-use (TOU) and real-time pricing (RTP) can better describe time-varying marginal costs of energy supply [5]. However, these pricing schemes are usually designed in heuristic ways [6], and thus it is difficult to achieve reliable and predictable performances in highly-uncertain environments. Therefore, a TE system with novel pricing schemes that can counteract the intermittency of DERs is a necessary step to achieve a welfare-maximized energy future (i.e., a win-win solution for both the supply and consumption sides). On the other hand, energy consumers are usually self-interested. Meanwhile, increasing availability of evolving technologies such as batteries also bring more choices and price transparency to energy consumers [4]. Therefore, without an appropriate market design, consumers may not be well-incentivized to behave in a desired collective manner. Worse yet is that energy consumers may strategically influence the market, leading to a poor system-wide performance. Towards this end, designing market mechanisms that can guarantee a competitive and robust system-wide performance under *uncertain* and *strategic* environments plays a pivotal role in the future of TE.

Motivated by these considerations, various market-based demand-side management (DSM) problems have been studied, e.g., pricing mechanism for electric vehicle (EV) charging [7], market-based coordination mechanism to manage thermalstatically-controlled loads (TCLs, e.g., air conditioners and heaters) in buildings [8], [9], and demand bidding and auction [10], [11]. However, all the above works have focused on market design in an *offline and static* setting, in which all the energy suppliers and energy customers have to participate in the mechanism at the beginning of the time horizon. However, in practice, customers often arrive sequentially in an *online and dynamic* manner, and are uncertain in multiple dimensions (e.g., arrival times, durations, power rates, energy demand, etc.). The uncertainty of customers is further complicated when they are equipped with DERs such as wind, solar and energy storage devices, which are key elements in the concept of TE. Therefore, a static and synchronized setup might not be amenable for a highly-uncertain and dynamic TE framework in future smart grid.

To address this concern, we borrow ideas from online mechanism design in multi-agent systems [12], [13] and economics [14], and aim to study the above market-based DSM problems in a unified framework of posted-price TE retailing[1]. Specifically, we consider a TE retailer (e.g., a utility company, an aggregator of EVs/TCLs, etc.) is selling energy to customers

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[1]Posted-price, auction, and bargaining are considered the three most commonly-used selling methods. We refer to [15] [16] for detailed discussions.

who arrive in a sequential and online manner. Upon the arrival of each customer (where ties are broken arbitrarily), the TE retailer publishes an energy price, which may vary over time. Customers can choose to either purchase a certain amount of energy based on the posted price, or leave without purchasing anything. In this context, customers are faced with *take-it-or-leave-it* offers, and therefore strategic behaviors naturally vanish [17], [18]. Meanwhile, such a sequential posted-price setting is also aligned with the trend of TE with a more transparent, customer-centric and dynamic exchange between energy suppliers and consumers [3]. For example, in market-based EV charging platforms, an EV can accept the posted price from one platform, or leave for alternatives. In our posted-price TE retailing framework, we do not assume any knowledge of future arrival information, i.e., arbitrary arrivals without prediction. Under such a highly-uncertain setting, we are interested in designing a posted-price mechanism (PPM) to achieve the following objectives: i) *individual rationality*, i.e., each customer suffers no loss from participating in the mechanism; ii) *incentive compatibility*, i.e., each customer can achieve the best outcome (e.g., the maximized utility) if she follows her true preference, regardless of actions taken by other customers; and iii) *tight competitive ratio*, i.e., the ratio between the social welfare achieved by the proposed PPM and its offline counterpart (assuming full knowledge of future information), is as close to 1 as possible.

We note that the above posted-price TE retailing framework is not entirely new in the smart grid literature. A similar setup has been studied in other settings such as RTP-based load control [19], [20], [21], auction-based real-time demand-side flexibility management [22], and auction-based online EV charging [23], [24], [25], [26], etc. However, the posted-price retailing framework differs from all these existing works in the following features. First, the prices in our TE retailing framework are updated for each individual customer (i.e., customer-driven), this differs from traditional RTP schemes which broadcast the prices for all the customers periodically (e.g., every one hour). Second, our proposed PPM does not require customers to reveal their private information (e.g., valuation). Instead, they make their own decisions based on the posted prices. This differs from existing auction-based market mechanisms such as [22], [23], [24], [25], [26], where customers need to reveal their private information to the retailer in order to maximize the social welfare. Eliminating the revelation of private information not only leads to a privacy-preserving mechanism, but also reduces the communication overhead between TE retailer and customers. Therefore, our proposed PPM is very suitable for a distributed implementation in large-scale TE systems[2].

(**Our Contributions**) We propose an optimal PPM for TE retailing without prediction. The proposed PPM is optimal in the sense that no other online mechanisms can achieve a better competitive ratio, and thus no other online mechanisms can achieve a better performance in expectation. To the best of our knowledge, the proposed PPM is the first online mechanism in TE retailing to achieve individual rationality, incentive compatibility[3] and optimal competitive ratio simultaneously. Specifically, the key to our online mechanism design is the proof of existence and uniqueness of optimal pricing functions that satisfy a group of first-order ordinary differential equations (ODE) with boundary conditions. We validate our design via case studies of market-based online EV charging. Extensive numerical results show that our proposed PPM is competitive and robust against system uncertainties, and outperforms several existing benchmarks.

The rest of this paper is organized as follows. We introduce the problem formulation, design objectives and assumptions in Section II. In Section III, we present our major results regarding the optimal competitive ratio and optimal pricing functions. We perform extensive experimental simulations in Section V and conclude this paper in Section VI.

II. Problem Formulation

In this section, we first present the problem formulations in the offline setting and online posted-price setting, and then describe the objectives and assumptions of our online mechanism design.

A. Problem Formulation in Offline Setting

We consider a general TE retailing problem as follows: a TE retailer is selling TE to customers who arrive in a sequential manner with a certain demand. We consider a group of customers $\mathcal{N} = \{1, \cdots, N\}$ over a slotted time horizon $\mathcal{T} = \{1, 2, \ldots, T\}$ with slot length Δ_T. Each customer is represented by a type vector $\boldsymbol{\theta}_n = (\boldsymbol{r}_n, v_n)$, where $\boldsymbol{r}_n = \{r_n^t\}_{\forall t \in \mathcal{T}_n}$ denotes the power demand profile during the consumption interval $\mathcal{T}_n$ and v_n denotes the monetary valuation, i.e., the maximum money customer n is willing to pay for consuming the power demand $\boldsymbol{r}_n$ over interval $\mathcal{T}_n$. Note that if we denote $\mathcal{T}_n$ by $\mathcal{T}_n = \{t_n^a, \cdots, t_n^d\}$, then t_n^a and t_n^d can be interpreted as the *arrival* and *departure* of customer n, respectively (e.g., an EV charging duration). We assume that $r_n^t \geq 0$ always hold during the consumption interval $\mathcal{T}_n$ (i.e., negative demand is not allowed in our TE retailing system). Meanwhile, for notational convenience, we denote the demand profile of customer n over the entire horizon by $\{\hat{r}_n^t\}_{\forall t \in \mathcal{T}}$, where $\hat{r}_n^t$ is given by

$$\hat{r}_n^t = \begin{cases} r_n^t & \text{if } t \in \mathcal{T}_n, \\ 0 & \text{if } t \in \mathcal{T} \backslash \mathcal{T}_n. \end{cases} \tag{1}$$

In the following we may use $\{\hat{r}_n^t\}_{\forall t \in \mathcal{T}}$ and $\{r_n^t\}_{\forall t \in \mathcal{T}_n}$ interchangeably.

For each time slot $t \in \mathcal{T}$, we consider the base load b_t is given and the TE retailer has a limited capacity of c_t. Here we allow c_t to be time-dependent to make it more general. When

[2]For instance, it is possible to implement a distributed and privacy-preserving transaction system based on our proposed PPM via blockchain [27].

[3]Posted-pricing is inherently incentive compatible [17], [18], [28].

all the future arrival information is known, the *offline social welfare maximization* problem can be formulated as follows:

$$\max_{\boldsymbol{x},\boldsymbol{y}} \quad \sum_{n\in\mathcal{N}} v_n x_n - \Big(\sum_{t\in\mathcal{T}} f_t(y_t) - \sum_{t\in\mathcal{T}} f_t(b_t)\Big)\Delta_T \tag{2a}$$

$$s.t. \quad y_t = b_t + \sum_{n\in\mathcal{N}} \hat{r}_n^t x_n, \forall t \in \mathcal{T}, \tag{2b}$$

$$b_t \le y_t \le c_t, t \in \mathcal{T}, \tag{2c}$$

$$x_n \in \{0,1\}, \forall n \in \mathcal{N}, \tag{2d}$$

where the binary variable $x_n \in \{0,1\}$ denotes the status of customer n, the auxiliary variable y_t denotes the total power consumption, and $f_t(\cdot)$ denotes the supply cost of the TE retailer. Specifically, $x_n = 1$ denotes that customer n purchases the demand and $x_n = 0$ otherwise. Following the convention in the literature (e.g., [1], [19], [29]), the cost function $f_t(\cdot)$ is assumed to take the following quadratic form

$$f_t(y) = a_{t,2}y^2 + a_{t,1}y + a_{t,0}, \text{(unit: \$/hour)}. \tag{3}$$

When the TE retailer is a utility company, then $f_t(y)$ represents the fuel cost of electricity generation per hour. In the following, we will also frequently use the derivative of $f_t(y)$, i.e., the marginal cost function $f_t'(y) = 2a_{t,2}y + a_{t,1}$ (unit: \$/kWh).

Problem (2) aims at optimizing the selection of customers (i.e., whom to sell/serve) such that the social welfare of the entire TE system can be maximized[4]. For example, for those customers who are in an urgent demand (e.g., an electric taxi who wants to be recharged as fast as possible to return to work), their valuations tend to be high and thus will have a higher chance to be selected, and vice versa. Note that in our formulated TE retailing problem, the demand vector $\boldsymbol{r}_n = \{r_n^t\}_{t\in\mathcal{T}_n}$ or $\hat{\boldsymbol{r}}_n = \{\hat{r}_n^t\}_{\forall t\in\mathcal{T}}$ is given by each customer $n \in \mathcal{N}$, and thus $\boldsymbol{r}_n$ or $\hat{\boldsymbol{r}}_n$ is fixed and not a decision variable in Problem (2). Therefore, we focus on TE customers with flexibility in consumption durations but without elasticity in power demand, e.g., EV charging control with flexible charging durations and given charging profiles.

B. Posted-Price Retailing in Online Setting

Directly solving Problem (2) is possible, provided that all the future information is given. However, we are interested in a more practical scenario when customers arrive in an online and dynamic fashion and there is no predictability in the sequence of customer arrivals as well as their demand information. Meanwhile, we consider the decisions of whether to make a purchase or not are made by the customers in a distributed manner so as to be more scalable and privacy-preserving.

Towards this end, we reformulate the above offline social welfare maximization problem into an online posted-price retailing problem as follows. We consider a group of $\mathcal{N} = \{1, \cdots, N\}$ customers whose arrival sequences $\{t_n^a\}_{\forall n}$ are, without loss of generality, in a non-descending order, i.e., $1 \le t_1^a \le t_2^a \le \cdots \le t_N^a \le T$. At each round when there is an arrival of customer $n \in \{1, 2, \cdots, N\}$, the TE retailer will offer him/her a price profile for the remaining horizon of interest $\mathcal{T}^{(n)} \triangleq \{t_n^a, \cdots, T\}$. Customer n may either leave without purchasing anything, or purchase the required power profile $\{r_n^t\}_{\forall t\in\mathcal{T}_n}$ by paying the TE retailer based on the current price. The same process will repeat upon the arrival of customer $n+1$. Our target is to design a sequence of price profiles at each round so that the social welfare of the whole TE system can be as close to the offline social welfare as possible. Below we present the design details of our PPM.

Let us denote the initial price by $\{\lambda_t^{(0)}\}_{\forall t\in\mathcal{T}}$, namely, the posted price before processing[5] the first customer, and denote the posted price by $\{\lambda_t^{(n)}\}_{\forall t\in\mathcal{T}}$ *after* processing customer n, where $n \in \{1, \cdots, N\}$. Following this notation, upon the arrival of customer n, the posted price for the remaining horizon $\mathcal{T}^{(n)}$ can be denoted by $\{\lambda_t^{(n-1)}\}_{\forall t\in\mathcal{T}^{(n)}}$. If customer n decides to purchase the power demand, she will pay

$$\pi_n = \sum_{t\in\mathcal{T}_n} \lambda_t^{(n-1)} r_n^t \Delta_T \tag{4}$$

to the TE retailer. We assume that customer n makes her own decision based on the following criteria:

$$x_n = \begin{cases} 1 & \text{if } v_n - \pi_n \ge 0, \\ 0 & \text{if } v_n - \pi_n < 0. \end{cases} \tag{5}$$

Therefore, we can quantify the utility of customer n by

$$U_n = (v_n - \pi_n)x_n. \tag{6}$$

The above decision-making model follows the conventional individual rationality principle in game theoretic literature [12] [28], namely, a customer will make a purchase only if her utility is non-negative.

Let us denote the total power consumption by $\{y_t^{(n)}\}_{\forall t\in\mathcal{T}}$ *after* processing customer n. For notational convenience, let us define $y_t^{(0)} \triangleq b_t, \forall t \in \mathcal{T}$. Hence, we have

$$y_t^{(n)} = y_t^{(n-1)} + r_n^t x_n, \forall t \in \mathcal{T}_n. \tag{7}$$

Intuitively, we have $y_t^{(n)} = y_t^{(n-1)}$ if customer n does not make any purchase, namely $x_n = 0$. Meanwhile, $b_t \le y_t^{(n)} \le c_t$ always holds, $\forall t \in \mathcal{T}, n \in \mathcal{N}$. After processing all the customers, the utility of the TE retailer is given by

$$U_r = \sum_{n\in\mathcal{N}} \pi_n x_n - \Big(\sum_{t\in\mathcal{T}} f_t\big(y_t^{(N)}\big) - \sum_{t\in\mathcal{T}} f_t(b_t)\Big)\Delta_T, \tag{8}$$

where $\{y_t^{(N)}\}_{\forall t}$ denotes the final total power consumption profile. Based on Eq. (6) and Eq. (8), summing over the utilities of all the customers and the TE retailer leads to the social welfare of the whole TE system, which is denoted by W_{ppm} as follows:

$$W_{\text{ppm}} = \sum_{n\in\mathcal{N}} v_n x_n - \Big(\sum_{t\in\mathcal{T}} f_t\big(y_t^{(N)}\big) - \sum_{t\in\mathcal{T}} f_t(b_t)\Big)\Delta_T. \tag{9}$$

[4]It is worth pointing out that social welfare maximization is not the only design objective of interest. For example, profit-maximization is also an important design objective if the TE retailer is profit-oriented, e.g., an aggregator of EVs. Meanwhile, other design objectives such as load shifting and/or peak-shaving are also common in literature, e.g., [19], [20].

[5]By *processing*, we mean the procedures of posting the price and then obtaining the decision results from customers. Note that this is different from *serving* energy to customers. In particular, customers can be served concurrently (e.g., multiple EVs are being charged simultaneously), but must be processed sequentially based on their arrival orders (where ties are broken arbitrarily). Meanwhile, it is possible that multiple customers are admitted in a single time slot.

Note that the payment terms cancel out in W_{ppm}. Meanwhile, W_{ppm} is in the same form as the objective of Problem (2) except that their final total power consumption profiles are calculated in different ways.

C. Objectives: Competitive Pricing Function Design

In the above posted-price retailing processes, any two different sequences of the posted prices $\{\lambda_t^{(n)}\}_{\forall n,t}$ will lead to a different welfare performance of the TE system. When there is no future information, our idea is to design the price $\lambda_t^{(n)}$ as a function of the current total power consumption $y_t^{(n)}$, i.e.,

$$\lambda_t^{(n)} \triangleq \Phi_t\big(y_t^{(n)}\big), \forall t, \tag{10}$$

where Φ_t is referred to as the *pricing function* and $\boldsymbol{\Phi} = \{\Phi_t\}_{\forall t}$ is called a *pricing scheme* hereinafter. We note that following the above definition, the initial price $\lambda_t^{(0)}$, namely the price posted for the first customer, is given by $\lambda_t^{(0)} = \Phi_t(y_t^{(0)}) = \Phi_t(b_t), \forall t$. Meanwhile, since the current power consumption is causal information, our design of $\{\Phi_t\}_{\forall t}$ is independent of future information and facilitates an online implementation. The detailed procedure of our proposed PPM is summarized in Algorithm 1.

Algorithm 1: PPM with Pricing Scheme $\boldsymbol{\Phi}$ (PPM$_{\boldsymbol{\Phi}}$)

1: **Initialization**: $y_t^{(0)} = b_t, \forall t$.
2: **while** a new customer n arrives **do**
3: Customer n calculates π_n by Eq. (4).
4: **if** $v_n \geq \pi_n$ and $y_t^{(n-1)} + r_n^t \leq c_t$ for all $t \in \mathcal{T}_n$, **then**
5: Customer n purchases the power demand $\boldsymbol{r}_n$ and pays π_n to the TE retailer (i.e., $x_n = 1$).
6: **else**
7: Customer n leaves without purchasing anything (i.e., $x_n = 0$).
8: **end if**
9: Update the total power consumption by Eq. (7).
10: Update the current power price by Eq. (10).
11: **end while**

(**Competitive Analysis**) Recall that our target is to make W_{ppm} as close to its offline counterpart as possible. For online settings without future information, the performance of PPM$_{\boldsymbol{\Phi}}$ can be measured via the competitive analysis framework [30]. Let W_{opt} denote the offline optimal objective value of Problem (2). We say PPM$_{\boldsymbol{\Phi}}$ is α-competitive if there exists a constant α such that

$$W_{\text{ppm}} \geq \frac{1}{\alpha} W_{\text{opt}}$$

holds for all possible arrival instances, meaning that PPM$_{\boldsymbol{\Phi}}$ achieves at least $1/\alpha$ of the offline optimal social welfare when there is no future information. Note that α is at least 1, and the closer to 1 the better. Meanwhile, we say the competitive ratio α of PPM$_{\boldsymbol{\Phi}}$ is *optimal* if no other online algorithms can outperform PPM$_{\boldsymbol{\Phi}}$ with a smaller competitive ratio of $\alpha - \epsilon$, $\forall \epsilon > 0$. If the competitive ratio of PPM$_{\boldsymbol{\Phi}}$ is optimal, we denote it by α^*.

The main objective of this paper is to strategically design a pricing scheme $\boldsymbol{\Phi} = \{\Phi_t\}_{\forall t}$ so that PPM$_{\boldsymbol{\Phi}}$ can achieve a competitive performance. In particular, we will characterize under what conditions PPM$_{\boldsymbol{\Phi}}$ can achieve a bounded competitive ratio, and whether the bounded competitive ratio is optimal or not.

D. Assumption of VERs and Setup

To help our pricing function design, in this subsection we make a few definitions and assumptions. For each customer n, let us define the *valuation-to-energy ratio* (VER) by

$$\xi_n \triangleq \frac{v_n}{\sum_{t\in\mathcal{T}_n} r_n^t \Delta_T}, \forall n \in \mathcal{N}, \tag{11}$$

where ξ_n has the same unit as the energy price, i.e., \$/kWh. Therefore, we can interpret ξ_n as the *maximum average energy price* that customer n is willing to accept.

Based on the above definitions, we make the following assumption throughout the paper.

Assumption 1 (**Upper Bound**). *The VERs of all the customers are upper bounded, namely,*

$$\xi_n \leq \overline{p}, \forall n \in \mathcal{N}, \tag{12}$$

where $\overline{p}$ is referred to as the upper bound hereinafter.

Since the demand profiles $\{r_n^t\}_{\forall t,n}$ are all finite, Assumption 1 basically states that all the customers are rational and will not have exceptionally-high valuations, namely, the valuations of all the customers are upper bounded. We emphasize that Assumption 1 is a mild assumption that naturally holds in practice. It should be note that $\overline{p}$ not only provides a natural upper bound for the maximum average energy prices that customers are willing to accept, but also indicates the uncertainty level of VERs. In particular, a larger $\overline{p}$ indicates that the uncertainty of VERs is higher since ξ_n is totally random in $[0, \overline{p}]$, $\forall n \in \mathcal{N}$. In the extreme case when $\overline{p}$ is arbitrarily large, the customers are arbitrarily heterogeneous in the sense that the next customer may accept an extremely-high energy price.

(**Setup**) Based on Assumption 1, we define all the information known by the TE retailer by $\mathcal{S}$ as follows:

$$\mathcal{S} \triangleq \left\{\left\{b_t, c_t, f_t\right\}_{\forall t}, \overline{p}\right\}, \tag{13}$$

which includes the base load profile $\{b_t\}_{\forall t}$, the capacity limit $\{c_t\}_{\forall t}$, the cost coefficients in function $\{f_t\}_{\forall t}$, and the upper bound $\overline{p}$. In the following, $\mathcal{S}$ is referred to as a *setup*. Other than the setup $\mathcal{S}$, we consider all other information is unknown. Therefore, for a given setup $\mathcal{S}$, the competitive ratio of PPM$_{\boldsymbol{\Phi}}$ can only depend on $\mathcal{S}$, and must be independent of other factors such as the number of customers, the valuation and the demand of customers, etc.

III. Optimal Design

In this section, we present the major results of this paper. We start by introducing two heuristic pricing schemes, and then describe the principles of our optimal pricing function design. After that, we characterize the optimal competitive ratio $\alpha^*(\mathcal{S})$ and present the optimal pricing scheme $\boldsymbol{\Phi} = \{\Phi_t^*\}_{\forall t}$.

A. Two Heuristic Pricing Schemes

To reveal the intuition of our optimal pricing function design, in this subsection we introduce two heuristic pricing schemes that are designed purely based on the initial marginal cost $f_t'(b_t)$, the maximum marginal cost $f_t'(c_t)$, and the upper bound $\overline{p}$. For notational convenience, let us define p_t^b and p_t^c as follows:

$$p_t^b \triangleq f_t'(b_t), p_t^c \triangleq f_t'(c_t), \forall t \in \mathcal{T}. \tag{14}$$

For simplicity of exposition, in the following we assume that $\overline{p} > \max_t\{p_t^c\}$ always holds[6], namely, the upper bound $\overline{p}$ is larger than the maximum marginal cost of all time slots. In particular, we define the complete set of $\overline{p}$'s by $\mathcal{P}$ as follows:

$$\overline{p} \in \mathcal{P} \triangleq \big(\max_t\{p_t^c\}, +\infty\big). \tag{15}$$

Based on p_t^b, p_t^c, and $\overline{p}$, we give the following two pricing schemes:

- **Linear**. For each time slot $t \in \mathcal{T}$, the pricing function of this scheme is given by

$$\Phi_t^{\text{linear}}(y) = \frac{\overline{p} - p_t^b}{c_t - b_t}(y - b_t) + p_t^b, y \in [b_t, c_t], \tag{16}$$

  which is linear w.r.t. the total power consumption by directly connecting p_t^b and $\overline{p}$. The implementation of $\text{PPM}_{\boldsymbol{\Phi}}$ with this pricing scheme will be referred to as **Linear** hereinafter.
- **Greedy**. For each time slot $t \in \mathcal{T}$, the pricing function of this scheme is given by

$$\Phi_t^{\text{greedy}}(y) = f_t'(y), y \in [b_t, c_t], \tag{17}$$

  which simply uses the marginal cost function as the pricing function when $y \in [b_t, c_t]$. This is a greedy pricing scheme and the available capacity will be sold without any reservation for possible future high-VER customers. The implementation of $\text{PPM}_{\boldsymbol{\Phi}}$ with this pricing scheme will be referred to as **Greedy** hereinafter.

Fig. 1 illustrates $\Phi_t^{\text{linear}}(y)$, $\Phi_t^{\text{greedy}}(y)$, and other two pricing functions $\hat{\Phi}_t(y)$ and $\Phi_t(y)$. It can be seen that **Greedy** always has a cheaper selling price than **Linear** except the initial price p_t^b. In particular, **Greedy** does not depend on $\overline{p}$, meaning that **Greedy** always depletes the available capacity myopically, regardless of the valuations of future customers. In contrast, **Linear** sets the price based on $\overline{p}$ in a linear manner, and thus **Linear** attempts to reserve some capacity for future arrivals with high-VERs. In the following, we will say pricing scheme "A" is more *aggressive* than pricing scheme "B" if "A" always sets the price cheaper than "B" for the same power consumption level. In this regard, **Greedy** is more aggressive than **Linear**, and the aggressiveness of the four pricing strategies in Fig. 1 in ascending order is given as follows: $\Phi_t^{\text{linear}}(y), \hat{\Phi}_t(y), \Phi_t(y), \Phi_t^{\text{greedy}}(y)$. Intuitively, a less aggressive pricing scheme tends to believe the usefulness of $\overline{p}$ and will try to reserve some available capacity for potential future high-VER customers. In economics, an aggressive pricing strategy[7] is also known as a predatory pricing strategy [14].

[6]It is possible to relax this assumption but will complicate the exposition. Please refer to Appendix I for a detailed discussion of how to extend our current design without this assumption.

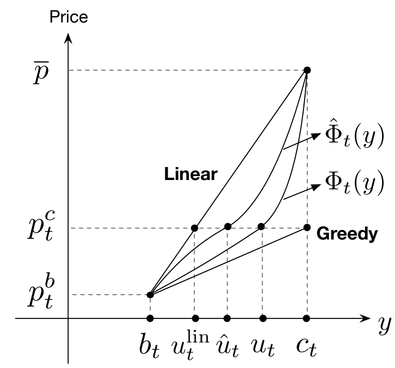


Fig. 1. Illustration of different pricing functions with dividing thresholds.

(**Dividing Threshold**) Given a pricing function Φ_t, we define its *dividing threshold* as the power consumption level u_t so that $\Phi_t(u_t) = p_t^c$, as shown in Fig. 1. Based on this definition, the dividing threshold of $\Phi_t^{\text{linear}}(y)$, denoted by u_t^{lin}, is given by

$$u_t^{\text{lin}} \triangleq b_t + \frac{p_t^c - p_t^b}{\overline{p} - p_t^b} \cdot (c_t - b_t), \tag{18}$$

namely, $\Phi_t^{\text{linear}}(u_t^{\text{lin}}) = p_t^c$, $\forall t \in \mathcal{T}$. Similarly, the dividing threshold of $\Phi_t^{\text{greedy}}(y)$ is c_t since $\Phi_t^{\text{greedy}}(c_t) = p_t^c$, $\forall t \in \mathcal{T}$. Intuitively, for any $u_t \in (b_t, c_t)$, the whole interval $[b_t, c_t]$ is divided into two segments by u_t, i.e., $[b_t, u_t]$ and $[u_t, c_t]$. We name these two stages as the *high-risk segment* and *low-risk segment*, and describe some key features of them as follows:

- *High-Risk Segment*: $[b_t, u_t]$. At this segment, the VER of any served customer may be smaller than the final marginal cost. Therefore, the pricing function design at this segment is risky in the sense that a customer may appear to have a reasonably-high VER now, but ultimately lead to negative social welfare contribution in the end. Thus, we refer to this segment as the high-risk segment. As illustrated by $\Phi_t(y)$ in Fig. 1, a smaller dividing threshold u_t indicates a shorter high-risk segment, and the pricing function tends to be less aggressive, and vice versa.
- *Low-Risk Segment*: $[u_t, c_t]$. At this segment, all the served customers will lead to positive social welfare contribution since their VERs are always larger than or equal to the maximum marginal cost p_t^c. Therefore, we refer to this stage as the low-risk segment.

The above descriptions suggest that the pricing function design for the high-risk segment should be inherently different from the low-risk segment. From this perspective, **Linear** fails to distinguish the pricing function design between these two segments. For example, when $\overline{p}$ is very large, a simple linear pricing function will set the prices too high at the high-risk segment (i.e., overreact to $\overline{p}$), leading to excessive refusal of

[7] Usually, an aggressive pricing strategy undergoes a short-term pain for long-term gain by deliberately increasing the market share, and is considered anti-competitive in many jurisdictions [14]. However, since we consider only one TE retailer in our setting, the competition of market share among different TE retailers is beyond the scope of this paper.

customers. In comparison, **Greedy** completely overlooks the usage of $\overline{p}$. As a consequence, **Greedy** will fail to reserve some capacity for potential high-VER customers in the future. In other words, **Linear** is over optimistic for the usefulness of $\overline{p}$ while **Greedy** is over pessimistic. For both pricing schemes, we can expect that their performances are sensitive towards the changes of $\overline{p}$.

B. Design Principles: Conditions for PPM$_{\mathbf{\Phi}}$ Being Competitive

Motivated by the discussions of the two heuristic pricing schemes in the previous subsection, we aim at designing a pricing scheme that takes into account the useful information of $\overline{p}$, but will not overreact to $\overline{p}$ to avoid excessive customer refusal at the high-risk segment. In particular, given a setup $\mathcal{S}$, we are interested in understanding whether the curvature of $\Phi_t(y)$ can be designed in a strategic way so as to achieve a 'smart' balance between **Linear** and **Greedy**, namely, a balance between optimism and pessimism. This subsections shows that such nonlinear pricing functions indeed exist, provided that they satisfy a certain conditions.

Below we first give Theorem 1 to summarize the sufficient conditions for $\mathbf{\Phi} = \{\Phi_t(y)\}_{\forall t \in \mathcal{T}}$ to guarantee that PPM$_{\mathbf{\Phi}}$ has a bounded competitive ratio.

Theorem 1 (**Sufficiency**). *Given a setup $\mathcal{S}$ with $\overline{p} \in \mathcal{P}$,* PPM$_{\mathbf{\Phi}}$ *is $\max_t\{\alpha_t\}$-competitive and incentive compatible if for each $t \in \mathcal{T}$, $\Phi_t(y)$ satisfies the following conditions:*

- ***C1**): $\Phi_t(b_t) = p_t^b$ and $\Phi_t(c_t) \geq \overline{p}$.*
- ***C2**): $\Phi_t(y)$ is strictly increasing in $y \in [b_t, c_t]$.*
- ***C3**): $\Phi_t(y)$ satisfies the following ODEs with dividing threshold $u_t \in (b_t, c_t)$:*

$$\begin{cases} \Phi_t(y) - f_t'(y) = \frac{1}{\alpha_t}\Phi_t'(y)\left(f_t'^{-1}(\Phi_t(y)) - b_t\right), \forall y \in [b_t, u_t]; \\ \Phi_t(y) - f_t'(y) = \frac{1}{\alpha_t}\Phi_t'(y)\big(c_t - b_t\big), \forall y \in [u_t, c_t], \end{cases} \tag{19}$$

where $f_t'^{-1}$ denotes the inverse of the marginal cost function f_t', and $\alpha_t \geq 1$ is a competitive ratio parameter that depends on $\mathcal{S}$ only.

We also give Theorem 2 below to show that the existence of pricing schemes to satisfy the three conditions in Theorem 1 is necessary for the existence of any general α-competitive online algorithm.

Theorem 2 (**Necessity**). *Given a setup $\mathcal{S}$ with $\overline{p} \in \mathcal{P}$, if there exists an α-competitive online algorithm, then for each $t \in \mathcal{T}$, there must exist a pricing function $\Phi_t(y)$ that satisfies **C1**, **C2**, and **C3** with some dividing threshold $u_t \in (b_t, c_t)$ and competitive ratio parameter $\alpha_t \leq \alpha$.*

Proof. We note that the three sufficient conditions in Theorem 1 are derived based on the online primal-dual analysis [32] of Problem (2), and the proof of the necessity in Theorem 2 is based on constructing a special arrival instance such that any α-competitive online algorithm must satisfy the above three conditions in order to achieve at least $1/\alpha$ of the offline optimal social welfare. The detailed proofs of the above two theorems are given in Appendix A. □

(**Rationality and Intuition**) The three conditions in Theorem 1 are critical in our following pricing function design. Below we briefly explain the rationality and intuition behind Theorem 1 and Theorem 2.

- Rationality of $\Phi_t(b_t) = p_t^b$ in **C1**. We can prove that if $\Phi_t(b_t) = p_t^b$ does not hold for all $t \in \mathcal{T}$, then it is always possible to construct an arrival instance such that $W_{\text{opt}} \neq 0$ but $W_{\text{ppm}} = 0$, leading to an unbounded competitive ratio. For example, if $\Phi_t(b_t) > p_t^b$ holds for $t = t_0$, then we can construct a sequence of customers who intend to purchase some power for time slot t_0 only with VERs drawn from $(p_{t_0}^b, \Phi_{t_0}(b_{t_0}))$. For such an arrival instance, no customer will make a purchase under PPM$_{\mathbf{\Phi}}$ (i.e., $W_{\text{ppm}} = 0$) while $W_{\text{opt}} \neq 0$. Obviously, all the four pricing functions illustrated in Fig. 1 satisfy this necessary condition.
- Intuition of **C2** and **C3**. The monotonicity condition **C2** is because a higher power consumption indicates a higher supply cost, and consequently leads to a higher selling price. The ODEs in Eq. (19) are derived based on a principled primal-dual analysis of Problem (2). Note that for both ODEs in Eq. (19), the left-hand-side parts are given by $\Phi_t(y) - f_t'(y)$, which is the difference between the selling price $\Phi_t(y)$ and the marginal cost $f_t'(y)$. Therefore, **C3** provides an analytical way to design the curvatures of $\{\Phi_t(y)\}_{\forall t \in \mathcal{T}}$ so that a smart balance between **Linear** and **Greedy** can be achieved. Moreover, Eq. (19) explicitly shows that the curvature of Φ_t is directly related to the dividing threshold u_t and the competitive ratio parameter α_t, which follows our intuition.

Before leaving this subsection, it is worth pointing out that Theorem 2 does not necessarily require that all α-competitive online algorithms must be PPMs (online algorithms/mechanisms have various types and structures). Instead, Theorem 2 argues that if there exists any general α-competitive online algorithm/mechanism for a given setup $\mathcal{S}$, then there must exist a pricing function $\Phi_t(y)$ for each time slot $t \in \mathcal{T}$ that satisfies the three conditions in Theorem 1. In the next subsection we show that the necessity of Theorem 2 plays a critical role in our optimal pricing function design.

C. Existence, Uniqueness, and Optimality

Below in Theorem 3, we show the major results of this paper, namely, the existence of a unique dividing threshold $u_t^* \in (b_t, c_t)$ for each $t \in \mathcal{T}$ such that the optimal competitive ratio achievable by all online mechanisms can be characterized.

Theorem 3 (**Existence, Uniqueness, and Optimality**). *Given a setup $\mathcal{S}$ with $\overline{p} \in \mathcal{P}$, for each $t \in \mathcal{T}$, there exists a unique pricing function $\Phi_t^*(y)$ that satisfies*

- ***OC1**): $\Phi_t^*(b_t) = p_t^b$ and $\Phi_t^*(c_t) = \overline{p}$.*
- ***OC2**): $\Phi_t^*(y)$ is strictly increasing in $y \in [b_t, c_t]$.*
- ***OC3**): $\Phi_t^*(y)$ satisfies the ODEs in Eq. (19) with $u = u_t^*$ and $\alpha_t = \Gamma_t(u_t^*)$, where $\Gamma_t(u_t^*)$ is a function of $u_t^* \in (b_t, c_t)$ given as follows:*

$$\Gamma_t(u_t^*) = \begin{cases} \frac{(c_t - b_t)^2}{(u_t^* - b_t)(c_t - u_t^*)} & \text{if } u_t^* \in \left(b_t, \frac{b_t + c_t}{2}\right), \\ 4 & \text{if } u_t^* \in \left[\frac{b_t + c_t}{2}, c_t\right), \end{cases} \tag{20}$$

and the dividing threshold $u_t^* \in (b_t, c_t)$ is the unique root to the following equation

$$\frac{c_t - u_t^* - \frac{c_t - b_t}{\Gamma_t(u_t^*)}}{\exp\left(\frac{u_t^*}{c_t - b_t} \cdot \Gamma_t(u_t^*)\right)} = \frac{f_t'^{-1}(\overline{p}) - c_t - \frac{c_t - b_t}{\Gamma_t(u_t^*)}}{\exp\left(\frac{c_t}{c_t - b_t} \cdot \Gamma_t(u_t^*)\right)}. \quad (21)$$

Meanwhile, the implementation of PPM$_{\mathbf{\Phi}}$ with $\mathbf{\Phi} = \{\Phi_t^*(y)\}_{\forall t}$ achieves an optimal competitive ratio of $\alpha^*(\mathcal{S})$, where $\alpha^*(\mathcal{S})$ is given by

$$\alpha^*(\mathcal{S}) = \max_{t \in \mathcal{T}} \{\Gamma_t(u_t^*)\}. \quad (22)$$

Proof. The proof of the above existence, uniqueness, and optimality is non-trivial, and the details are given in Appendix B. Our proof heavily relies on the uniqueness and existence properties of first-order ODEs with boundary conditions [33], [34]. Note that the three conditions **OC1-OC3** in Theorem 3 correspond to the three conditions **C1-C3** in Theorem 1, respectively, where 'OC' is short for 'optimal condition'. □

Theorem 3 shows that there exists an optimal pricing scheme $\mathbf{\Phi} = \{\Phi_t^*(y)\}_{\forall t}$, with optimal dividing thresholds $\{u_t^*\}_{\forall t}$, such that PPM$_{\mathbf{\Phi}}$ achieves the optimal competitive ratio of $\alpha^*(\mathcal{S})$, namely, no other online mechanisms can achieve a better competitive ratio than PPM$_{\mathbf{\Phi}}$ if $\mathbf{\Phi} = \{\Phi_t^*\}_{\forall t}$. Note that both $\{u_t^*\}_{\forall t}$ and $\{\Gamma_t(\cdot)\}_{\forall t}$ depend on $\mathcal{S}$ only, and thus the optimal competitive ratio also depends on $\mathcal{S}$ only. We note that the optimality in Theorem 3 is based on the necessity in Theorem 2. Specifically, given a setup $\mathcal{S}$ and for any $t \in \mathcal{T}$, we can prove that when $\alpha_t < \alpha^*(\mathcal{S})$, there exists no such a strictly-increasing pricing function that satisfies the two ODEs in Eq. (19) with the two boundary conditions given by **C1**, and thus it is impossible to have an online algorithm that is $(\alpha^*(\mathcal{S}) - \epsilon)$-competitive, $\forall \epsilon > 0$.

Based on the above analysis, the following two corollaries directly follow Theorem 3.

Corollary 4. *For any $\epsilon > 0$, there exists no $(\alpha^*(\mathcal{S}) - \epsilon)$-competitive online algorithms/mechanisms.*

Corollary 5. *For any $\epsilon \geq 0$, there exists a pricing scheme $\mathbf{\Phi} = \{\Phi_t\}_{\forall t}$ so that PPM$_{\mathbf{\Phi}}$ is $(\alpha^*(\mathcal{S}) + \epsilon)$-competitive and incentive compatible.*

Proof. Note that Corollary 5 guarantees the existence of competitive and incentive compatible pricing schemes for PPM$_{\mathbf{\Phi}}$, while Corollary 4 holds for general online algorithms. The detailed proofs for these two corollaries are also given in Appendix B. □

Lemma 6. *If we define the unique root of Eq. (21) in variable $u_t^* \in (b_t, c_t)$ as a function of $\overline{p} \in \mathcal{P}$ as follows:*

$$u_t^* \triangleq \Lambda_t(\overline{p}), \quad (23)$$

then $\Lambda_t(\overline{p})$ is strictly decreasing in $\overline{p} \in \mathcal{P}$.

Proof. Proving the monotonicity of $\Lambda_t(\overline{p})$ is elementary, and the detailed proof is given in Appendix C. □

The intuition of Lemma 6 is as follows: when $\overline{p}$ is larger, to achieve the optimal competitive ratio of $\alpha^*(\mathcal{S})$, $\{\Phi_t^*\}_{\forall t}$ tends to become less aggressive by having a smaller $\{u_t^*\}_{\forall t}$. In Fig. 1, the monotonicity of $\Lambda_t(\overline{p})$ means that a larger $\overline{p}$ will lift the pricing curve higher (but the lower bound p_t^b will remain unchanged) and thus the optimal dividing threshold u_t^* shifts towards the left, and vice versa.

If we assume $\overline{p}_t^{\text{cut}}$ satisfies $\Lambda_t(\overline{p}_t^{\text{cut}}) = \frac{b_t + c_t}{2}$, then $\overline{p}_t^{\text{cut}}$ can be calculated as follows:

$$\overline{p}_t^{\text{cut}} \triangleq \Lambda_t^{-1}\big(\frac{b_t + c_t}{2}\big) = p_t^c + \frac{1 + e^2}{4} \cdot \left(p_t^c - p_t^b\right), \quad (24)$$

where $\overline{p}_t^{\text{cut}}$ is defined to be a *cut-off* point for $\overline{p} \in \mathcal{P}$. Since $\Lambda_t(\overline{p})$ is strictly decreasing in $\overline{p} \in \mathcal{P}$, we have

- when $\overline{p} = \overline{p}_t^{\text{cut}}$, $u_t^* = \Lambda_t(\overline{p}) = \frac{b_t + c_t}{2}$;
- when $\overline{p} \geq \overline{p}_t^{\text{cut}}$, $u_t^* = \Lambda_t(\overline{p}) \in (b_t, \frac{b_t + c_t}{2}]$,
- when $\overline{p} < \overline{p}_t^{\text{cut}}$, $u_t^* = \Lambda_t(\overline{p}) \in (\frac{b_t + c_t}{2}, c_t)$.

Note that based on Eq. (24), $\overline{p}_t^{\text{cut}} > p_t^c$ always holds since $c_t > b_t$, or equivalently, $p_t^c > p_t^b$.

Based on Theorem 3 and the above analysis, when $\overline{p} \leq \overline{p}_t^{\text{cut}}$ holds for all $t \in \mathcal{T}$, i.e., when $\max_{t \in \mathcal{T}}\{\overline{p}_t^c\} \leq \overline{p} \leq \min_{t \in \mathcal{T}} \{\overline{p}_t^{\text{cut}}\}$, the optimal competitive ratio $\alpha^*(\mathcal{S}) = 4$ regardless of having exact knowledge of $\overline{p}$. This result is, in our opinion, very counter-intuitive since it has been argued in many literature (e.g., [32], [35]) that the exact knowledge of such upper bound information is necessary in order to achieve a bounded competitive ratio. Here our results show that it is actually not necessary in this scenario. However, when $\overline{p} \leq \min_{t \in \mathcal{T}} \{\overline{p}_t^{\text{cut}}\}$ does not hold, the optimal competitive ratio $\alpha^*(\mathcal{S}) > 4$, and it indeed depends on $\overline{p}$.

We note that when $\overline{p} \neq \overline{p}_t^{\text{cut}}$, calculating the optimal dividing thresholds $\{u_t^* = \Lambda_t(\overline{p})\}_{\forall t}$ requires solving the nonlinear equation (21) for each time slot $t \in \mathcal{T}$. However, this is a lightweight task and can be computed via various numerical methods such as bisection searching. Moreover, the computation of $\{u_t^*\}_{\forall t}$ can be performed offline (before running PPM$_{\mathbf{\Phi}}$).

D. Optimal Pricing Functions

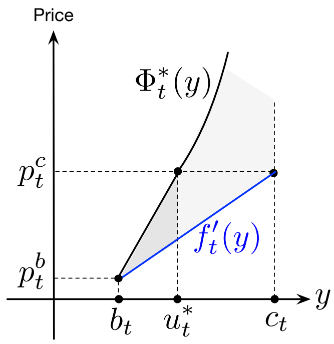
In this subsection, we present the optimal pricing functions $\{\Phi_t^*(y)\}_{\forall t}$ that achieve the optimal competitive ratio of $\alpha^*(\mathcal{S})$. Our optimal pricing function design is based on Theorem 3 and consists of the following two steps. First, for each given setup $\mathcal{S}$ with $\overline{p} \in \mathcal{P}$, we solve Eq. (21) to get the optimal dividing threshold u_t^* for each time slot $t \in \mathcal{T}$, and then obtain the optimal competitive ratio parameter $\alpha_t = \Gamma_t(u_t^*)$ based on Eq. (20). Second, we substitute $u_t = u_t^*$ and $\alpha_t = \Gamma_t(u_t^*)$ into Eq. (19), and then get the optimal pricing function Φ_t^* by solving the ODEs in Eq. (19) with the boundary conditions **OC1**. Based on Theorem 3, the resulting optimal pricing functions $\{\Phi_t^*\}_{\forall t \in \mathcal{T}}$ are guaranteed to satisfy the monotonicity condition **OC2**.

Before presenting the specific forms of $\{\Phi_t^*(y)\}_{\forall t}$, we first give the following lemma and definition.

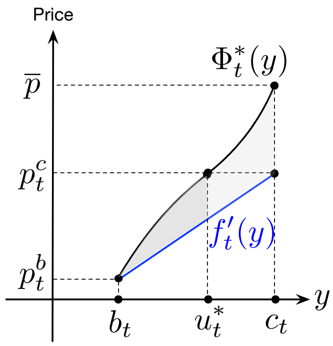
Lemma 7. *Suppose the optimal dividing threshold $u_t^* \in (\frac{b_t + c_t}{2}, c_t)$, then for any $y \in (b_t, u_t^*)$, the following equation has a unique root in $H_t \in (y - b_t, 2(y - b_t))$:*

$$\frac{2(y - b_t)}{H_t - 2(y - b_t)} - \frac{2(u_t^* - b_t)}{c_t + b_t - 2u_t^*} = \ln\big(\frac{H_t - 2(y - b_t)}{c_t + b_t - 2u_t^*}\big). \quad (25)$$

Proof. The proof is related to Theorem 8 below, and the details are given in Appendix E. □



(a) Case-1: $\overline{p} \geq \overline{p}_t^{\text{cut}}$



(b) Case-2: $\overline{p} < \overline{p}_t^{\text{cut}}$

Fig. 2. Illustration of the optimal pricing function. Subfigure (a): $\overline{p} \geq \overline{p}_t^{\text{cut}}$ and $u_t^* \in (b_t, \frac{b_t+c_t}{2}]$; Subfigure (b): $\overline{p} < \overline{p}_t^{\text{cut}}$ and $u_t^* \in (\frac{b_t+c_t}{2}, c_t)$.

We define the above one-to-one mapping by $H_t(y; u_t^*)$, where $y \in (b_t, u_t^*)$ and $u_t^* \in (\frac{b_t+c_t}{2}, c_t)$. Below in Definition 1, we define another similar function $L_t(y; u_t^*)$ as a function of $y \in (u_t^*, c_t]$ for any given optimal dividing threshold $u_t^* \in (b_t, c_t)$.

Definition 1. *Given the optimal dividing threshold $u_t^* \in (b_t, c_t)$, let us define $L_t(y; u_t^*)$ as a function of $y \in [u_t^*, c_t]$ as follows:*

$$L_t(y; u_t^*) \triangleq \frac{p_t^c - f_t'(u_t^*) - \frac{p_t^c - p_t^b}{\Gamma_t(u_t^*)}}{\exp\left(\frac{\Gamma_t(u_t^*) \cdot u_t^*}{c_t - b_t}\right)} \cdot \exp\left(\frac{\Gamma_t(u_t^*) \cdot y}{c_t - b_t}\right) + \frac{p_t^c - p_t^b}{\Gamma_t(u_t^*)}. \tag{26}$$

The above two functions $H_t(y; u_t^*)$ and $L_t(y; u_t^*)$ are defined for the high-risk segment and the low-risk segment, respectively. We emphasize that both $H_t(y; u_t^*)$ and $L_t(y; u_t^*)$ are derived from solving the two ODEs in Eq. (19). Based on the these two functions, below we present the optimal pricing scheme $\boldsymbol{\Phi} = \{\Phi_t^*(y)\}_{\forall t}$ that achieves the optimal competitive ratio of $\alpha^*(\mathcal{S})$.

Theorem 8 (Optimal Pricing Functions). *Given a setup $\mathcal{S}$ with $\overline{p} \in \mathcal{P}$, PPM$_{\boldsymbol{\Phi}}$ is $\alpha^*(\mathcal{S})$-competitive if $\boldsymbol{\Phi} = \{\Phi_t^*(y)\}_{\forall t}$, where $\Phi_t^*(y)$ is given as follows:*

- *Case-1: $\overline{p} \geq \overline{p}_t^{\text{cut}}$. In this case, $\Phi_t^*(y)$ is given by:*

$$\Phi_t^*(y) = \begin{cases} \frac{p_t^c - p_t^b}{u_t^* - b_t}(y - b_t) + p_t^b, & \text{if } y \in [b_t, u_t^*), \\ f_t'(y) + L_t(y; u_t^*), & \text{if } y \in [u_t^*, c_t]. \end{cases} \tag{27}$$

- *Case-2: $\overline{p} < \overline{p}_t^{\text{cut}}$. In this case, $\Phi_t^*(y)$ is given by:*

$$\Phi_t^*(y) = \begin{cases} p_t^b, & \text{if } y = b_t, \\ f_t'\Big(b_t + H_t\big(y; u_t^*\big)\Big), & \text{if } y \in (b_t, u_t^*), \\ f_t'(y) + L_t\big(y; u_t^*\big), & \text{if } y \in [u_t^*, c_t]. \end{cases} \tag{28}$$

Proof. The functions given by Eq. (27) and Eq. (28) are derived by solving the two ODEs in Eq. (19) with the two boundary conditions in **OC1**. The detailed proof is given in Appendix E. □

We illustrate the optimal pricing functions of both cases in Fig. 2. As can be seen, Fig. 2(a) shows the optimal pricing function in Case-1, where $u_t^* \in (b_t, \frac{b_t+c_t}{2}]$, and the optimal pricing function is always linear at the high-risk segment and grows exponentially fast at the low-risk segment. In this case, the TE retailer tends to believe that there will be high-VER customers in the future (since $\overline{p}$ is large) and thus sets the price less aggressively. Fig. 2(b) illustrates the optimal pricing function in Case-2, where $u_t^* \in (\frac{b_t+c_t}{2}, c_t)$. We can see from Fig. 2(b) that the optimal pricing function grows slowly at the high-risk segment (a concave pricing function) since the TE retailer tends to believe that there is no high-VER customer in the future (since $\overline{p}$ is small).

Note that in Case-1, the optimal pricing function is analytically given by Eq. (27) except the numerical computation of the optimal dividing threshold u_t^* (which can be computed offline, as mentioned previously). In comparison, the optimal pricing function in Case-2 needs some extra computation of $H_t(y; u_t^*)$ by solving Eq. (25) numerically for each updated power consumption level $y \in (b_t, u_t^*)$, and moreover, this must be performed 'on-the-fly' (during the running of PPM$_{\boldsymbol{\Phi}}$). However, we argue that this should not be a concern for the online implementation of PPM$_{\boldsymbol{\Phi}}$ since the computation is lightweight. Moreover, for each customer n, we just need to solve Eq. (25) for the time slots whose total power consumption levels have been updated (i.e., the time slots within $\mathcal{T}_n$).

IV. Extensions

In this section, we extend our previous model to account for scenarios when customers have multiple options to choose from instead of a binary option of take-it-or-leave-it. We consider the same setup of TE retailing as our previous model in Section II, but assume each customer $n \in \mathcal{N}$ has a set of options $\mathcal{O}_n$. Each option or selection $s \in \mathcal{O}_n$ represents a potential demand service for customer n. Similar to our previous model, we assume that each customer is represented by a type vector $\boldsymbol{\theta}_n = (\{\boldsymbol{r}_n^s, v_n^s\}_{\forall s \in \mathcal{O}_n})$, where $\boldsymbol{r}_n^s = \{r_n^{s,t}\}_{\forall t \in \mathcal{T}_n^s}$ denotes the power demand profile during the consumption interval $\mathcal{T}_n^s$ and v_n^s denotes the monetary valuation, i.e., the maximum money customer n is willing to pay for consuming the power demand $\boldsymbol{r}_n^s$ over interval $\mathcal{T}_n^s$. We still assume that $r_n^{s,t} \geq 0$ always hold during the consumption interval $\mathcal{T}_n^s$. Meanwhile, for notational convenience, we also denote the demand profile of customer n over the entire horizon by $\{\hat{r}_n^{s,t}\}_{\forall t \in \mathcal{T}}$, where $\hat{r}_n^{s,t}$ is given by

$$\hat{r}_n^{s,t} = \begin{cases} r_n^{s,t} & \text{if } t \in \mathcal{T}_n^s, \\ 0 & \text{if } t \in \mathcal{T} \backslash \mathcal{T}_n^s. \end{cases} \tag{29}$$

Based on the above setup, the new social welfare maximization problem can be formulated as follows:

$$\begin{aligned} \max_{\boldsymbol{x}, \boldsymbol{y}} \quad & \sum_{n \in \mathcal{N}} \sum_{s \in \mathcal{O}_n} v_n^s x_n^s - \Big(\sum_{t \in \mathcal{T}} f_t(y_t) - \sum_{t \in \mathcal{T}} f_t(b_t)\Big)\Delta_T \\ s.t. \quad & y_t = b_t + \sum_{n \in \mathcal{N}} \sum_{s \in \mathcal{O}_n} \hat{r}_n^{s,t} x_n^s, \forall t \in \mathcal{T}, && \text{(30a)} \\ & b_t \leq y_t \leq c_t, t \in \mathcal{T}, && \text{(30b)} \\ & \sum_{s \in \mathcal{O}_n} x_n^s \leq 1, \forall n \in \mathcal{N}, && \text{(30c)} \\ & x_n^s \in \{0, 1\}, \forall s \in \mathcal{O}_n, n \in \mathcal{N}. && \text{(30d)} \end{aligned}$$

We can see that Problem (30) differs from Problem (2) in the following aspects. First, in Problem (30), each customer n has a set of options, which is denoted by $\mathcal{O}_n$, while Problem (2) has only a binary option. In Problem (30), for each possible selection or option $s \in \mathcal{O}_n$, customer n has a valuation v_n^s on the demand vector $\{r_n^{s,t}\}_{\forall t \in \mathcal{T}_n^s}$, where $r_n^{s,t}$ denotes the power demand of customer n at t, and $\mathcal{T}_n^s$ denotes the selected consumption interval. In this regard, the multi-option model in Problem (30) can capture more flexibility in the demand side since customers are flexible in choosing not only consumption intervals, but also power profiles. Second, similar to Problem (2), at most one of the options in $\mathcal{O}_n$ will be selected by customer n, namely, $\sum_{s\in\mathcal{O}_n} x_n^s \leq 1$. Meanwhile, $x_n^s = 1$ denotes that option s of customer n is selected and $x_n^s = 0$ otherwise. Intuitively, when each customer has only one option, i.e., $|\mathcal{O}_n| = 1$ holds for all $n \in \mathcal{N}$, Problem (30) reduces to Problem (2), and thus Problem (2) is a special case of Problem (30). In the general case when $|\mathcal{O}_n| > 1$, say $|\mathcal{O}_n| = 3$ and the three options respectively represent 100%, 80%, and 50% of the total demand of customer n, then Problem (30) allows customer n to select not just 0 and 100% of her demand, but a predetermined proportion within 0 and 1, namely, 100%, 80%, 50% or 0%.

Extending our previous theoretic results to account for the above multi-option TE retailing problem is straightforward. To be more specific, we can still use the same technique to design a pricing function $\lambda_t^{(n)} = \Phi_t(y_t^{(n)})$ for each time slot $t \in \mathcal{T}$, and let customer $n \in \mathcal{N}$ make her own decision by solving the following utility-maximization problem

$$s_n^* = \arg\max_{s\in\mathcal{O}_n} \ v_n^s - \sum\nolimits_{t\in\mathcal{T}_n^s} \lambda_t^{(n-1)} r_n^{s,t}\Delta_T. \tag{31}$$

Where s_n^* denotes the utility-maximizing selection or option for customer n. After that, customer n chooses to purchase option s_n^* if the following inequality holds:

$$v_n^{s_n^*} - \sum\nolimits_{t\in\mathcal{T}_n^{s_n^*}} \lambda_t^{(n-1)} r_n^{s_n^*,t}\Delta_T \geq 0. \tag{32}$$

In this case, the TE retailer collects the following payment from customer n

$$\pi_n = \sum\nolimits_{t\in\mathcal{T}_n^{s_n^*}} \lambda_t^{(n-1)} r_n^{s_n^*,t}\Delta_T, \tag{33}$$

and update the total power consumption in the same way as before. In contrast, if the inequality in Eq. (32) does not hold, then customer n leaves without purchasing anything.

V. Case Studies

In this section, we evaluate $\text{PPM}_{\mathbf{\Phi}}$ based on extensive experimental simulations. We start by introducing the setup of our experiments, and then describe the detailed numerical results and insights.

A. Experimental Setup

We validate the performance of $\text{PPM}_{\mathbf{\Phi}}$ through the case of EV charging. We use a set of driving traces for GPS-equipped taxi vehicles from [36] to construct a market-based online EV charging setting. The detailed experimental setup is as follows. We simulate over a horizon of 24 hours with $\Delta_T = 1/2$ hour per time slot. Therefore, we have $T = 48$ time slots in total. The base load $\{b_t\}_{\forall t}$ in our simulation is the real-world load data from NYISO[8] and varies within 1300 kW and 1650 kW. The capacity limit is set to be $c_t = 1.7 \times 10^3$ kW for all time slots unless otherwise specified, and thus we may refer to the capacity limit by c without the time index. The cost coefficients of $f_t(\cdot)$, namely $a_{t,2}, a_{t,1}$ and $a_{t,0}$, are also assumed to be time-invariant, and thus we drop the time index and assume $a_2 = 10^{-4}$ \$/kWh/kW, $a_1 = 10^{-4}$ \$/kWh, and $a_0 = 0$. By this setup, the marginal cost is around $0.26 \sim 0.34$ \$/kWh within the available capacity.

(**Real-world Trace Setup**) We consider a group of $N = 1000$ EVs, and for each EV n, we consider its arrival and departure times are randomly drawn from the real-world traces of GPS-equipped taxi vehicles [36]. The charging power r_n^t for customer n is assumed to be time invariant (i.e., the ideal charging process), and thus we drop the time index and assume r_n is randomly drawn from [3.7, 7, 22] kW, which are typical charging rates of Tesla Model S[9]. The total energy demand of customer n is given by $e_n = |\mathcal{T}_n| \cdot r_n\Delta_T$, where $|\mathcal{T}_n|$ is the charging duration of customer n. Based on the energy demand e_n of customer n, the valuation v_n is given by $v_n = \xi_n e_n$, where ξ_n is the VER of EV n. In our simulation, ξ_n is randomly drawn from a truncated Gaussian distribution with mean μ, variance σ, lower bound lb and upper bound ub, as follows:

$$\xi_n \sim \text{TruncGaussian}(\mu, \sigma, lb, ub).$$

Since the average marginal cost is around 0.3, a reasonable setup should have $\mu \geq 0.3$. Therefore, throughout our simulation, μ is drawn from $[0.3, 0.7]$ and σ varies within $[0.01, 2]$. Meanwhile, we set $lb = 0.2$ and $ub = 1$ for all the time unless otherwise specified. For this case, we assume the upper bound $\overline{p} = \max_n\{\xi_n\} = ub = 1$.

(**Extreme-Case Setup**) Other than generating the VERs of all the EVs by the aforementioned truncated Gaussian distribution, we also artificially construct the following extreme cases to test the performance and robustness of our proposed online mechanism. Moreover, we are interested in understanding whether the value of $\overline{p}$ influences the performance and robustness of $\text{PPM}_{\mathbf{\Phi}}$ since it plays a critical role in our pricing function design. The extreme cases are constructed as follows: First, we consider the same setting of EVs as the above Real-World Trace Setup in terms of N and $\{e_n, r_n, \mathcal{T}_n\}_{\forall n}$, but the VERs are generated in the following three extreme cases:

- High-Low Case. This is the case where the VERs $\{\xi_n\}_{\forall n}$ of the first 500 customers are generated with $(\mu, \sigma, lb, ub) = (0.7, 0.1, 0.6, 1)$, and the second 500 customers are generated with $(\mu, \sigma, lb, ub) = (0.3, 0.1, 0.2, 0.5)$. Therefore, the first-half customers have high VERs while the second-half have low VERs.
- Constant Case. This is the case where the VERs $\{\xi_n\}_{\forall n}$ are generated with $(\mu, \sigma, lb, ub) = (0.5, 0, 0.5, 0.5)$ for all 1000 customers, namely, $\xi_n = 0.5$ for all $n \in \mathcal{N}$.
- Low-High Case. This is the case where the VERs of all the customers are constructed in the opposite way to the above High-Low Case.

[8]https://www.nyiso.com/load-data
[9]https://pod-point.com/landing-pages/tesla-charging

Third, for all the three extreme cases, we assume $\overline{p}$ is drawn from $\{3,4,5,6,7\}$. Note that in the High-Low Case and Low-High Case, we have $\max_n\{\xi_n\} = 1$, and in the Constant Case, we have $\max_n\{\xi_n\} = 0.5$. Thus, our setup of $\overline{p} = \{3,4,5,6,7\}$ is feasible since $\max_n\{\xi_n\} \leq \overline{p}$ always holds. The purpose of setting $\overline{p}$ and ξ_n in such a way is to evaluate the performances of $\text{PPM}_{\boldsymbol{\Phi}}$ when $\overline{p}$ deviates from the exact value of $\max_n\{\xi_n\}$. In reality, this can simulate the robustness of our proposed pricing mechanism when there exist estimation errors of $\overline{p}$.

B. Benchmarks

We benchmark $\text{PPM}_{\boldsymbol{\Phi}}$ based on the offline optimal result by assuming complete knowledge of future information. The offline problem (2) is solved by Gurobi 8.1 via its Python API[10]. To compare the performance of different pricing schemes, we define the empirical ratio of an online mechanism by

$$\text{Empirical Ratio} \triangleq W_{\text{opt}}/W_{\text{online}}.$$

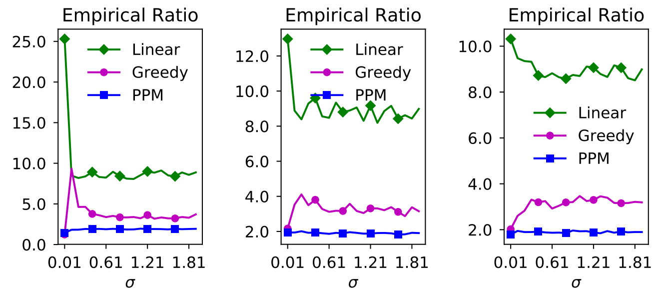
In the following figures, the results of our proposed $\text{PPM}_{\boldsymbol{\Phi}}$ with the optimal pricing scheme $\boldsymbol{\Phi} = \{\Phi_t^*\}_{\forall t}$ are simply marked as PPM, to differentiate it from the results by **Linear**, **Greedy** and **Offline**. Next we present our numerical results based on the above setup.

C. Numerical Results

1) Results based on Real-world Trace Setup. We first show the performances of $\text{PPM}_{\boldsymbol{\Phi}}$ based on the real-world trace setup and present our results in Fig. 3-Fig. 5. As shown in Fig. 3, $\text{PPM}_{\boldsymbol{\Phi}}$ shows a very competitive performance w.r.t. the changes of μ and σ. In most cases, the empirical ratios of $\text{PPM}_{\boldsymbol{\Phi}}$ are below 2, and are very robust w.r.t. the changes of μ and σ. Among the three online mechanisms, **Linear** performs the worst, and even the best empirical ratio of **Linear** is still larger than 8. Note that although the pricing function of **Linear** shares the same lower and upper bounds of our proposed pricing function in $\text{PPM}_{\boldsymbol{\Phi}}$, these two mechanisms result in totally different empirical ratios. The big performance difference between $\text{PPM}_{\boldsymbol{\Phi}}$ and **Linear** demonstrates the effectiveness of our optimal pricing function design.

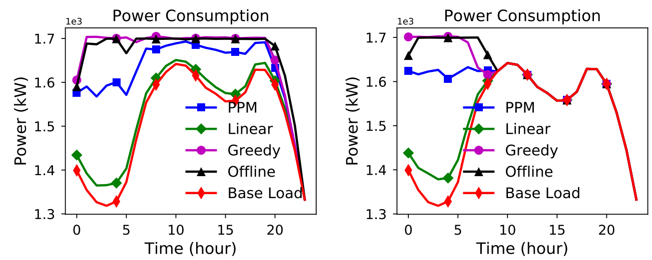
Fig. 3 also reveals some interesting results regarding **Greedy** and **Linear**. When the VERs of customers are less volatile, i.e., a smaller σ, **Greedy** depicts approximately the same performance as $\text{PPM}_{\boldsymbol{\Phi}}$, as shown in all the three subfigures in Fig. 3 when $\sigma = 0.01$. This is not a surprising result since when σ is extremely small, all the customers are similar in terms of VERs, and thus a myopic pricing scheme performs good enough since the future arrivals will be almost the same as the current one. When σ increases (i.e., the valuations become more volatile), the empirical ratio of **Linear** always improves until becomes stable around 8, regardless of the values of μ. However, **Greedy** in general performs worse w.r.t. the increasing of σ. In particular, Fig. 3(a) shows that when μ is small, **Greedy** first starts to perform worse and then performs better until its empirical ratios become stable around 3. However, Fig. 3(c) shows that when μ is large, **Greedy**

[10]http://www.gurobi.com/index



(a) $\mu = 0.3$ (b) $\mu = 0.5$ (c) $\mu = 0.7$

Fig. 3. Empirical ratios of different online mechanisms w.r.t. $\sigma \in [0.1, 2]$ under different settings of $\mu = 0.3, 0.5, 0.7$. Each point in the figure is an average of 1000 evaluations.



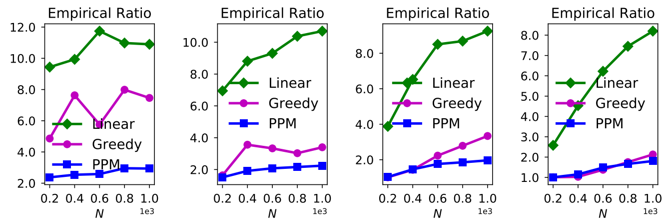
(a) Arrivals within $[0, 20]$ (b) Arrivals within $[0, 6]$

Fig. 4. Total power consumption profiles under different mechanisms with arrivals occur within [0,20] and [0,6]. In both figures, $(\mu, \sigma) = (0.7, 1)$.

always performs worse when σ increases and then becomes stable. Therefore, both **Linear** and **Greedy** are sensitive to the mean and variance of VERs, while $\text{PPM}_{\boldsymbol{\Phi}}$ are not.

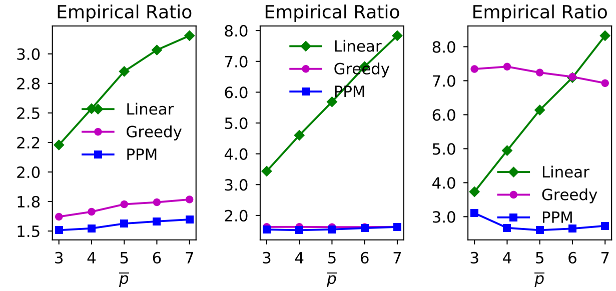
To better illustrate the difference among all the mechanisms, we plot one instance of the total power consumption profiles in Fig. 4 when EV arrivals occur within $[0, 20]$ and $[0, 6]$. Among all the mechanisms, **Greedy** always has the quickest depletion of the available capacity. In contrast, **Linear** consumes the least power since most customers cannot afford the price posted by **Linear**. In between, $\text{PPM}_{\boldsymbol{\Phi}}$ achieves a good balance between aggressiveness and conservativeness, leading to a good performance in empirical ratios. Note that for both figures in Fig. 4, **Greedy** and **Offline** both deplete the whole available capacity, while $\text{PPM}_{\boldsymbol{\Phi}}$ tends to reserve some capacity for future use. This is the key smartness of $\text{PPM}_{\boldsymbol{\Phi}}$ since it avoids the cases when high-VER customers who come at a latter stage cannot consume any energy due to the capacity limit.

Fig. 5 shows our results regarding whether the system scale (i.e., the number of customers N) and the available capacity (i.e., $c - b_t$) will influence the performance of our proposed online mechanisms. We vary N from 200 to 1000 and plot the empirical ratios of three online mechanisms when the capacity limit is $c = 1650$ (scarce), 2000, 2400, 2800 (sufficient). We can see that the empirical ratios of all the three mechanisms become smaller when the available capacity becomes more sufficient. Meanwhile, **Greedy** tends to have a similar performance as $\text{PPM}_{\boldsymbol{\Phi}}$ when the available capacity is sufficient, but both **Greedy** and **Linear** have a poor performance when the available capacity is very limited, as can be seen in Fig. 5(a).



(a) $c = 1650$ (b) $c = 2000$ (c) $c = 2400$ (d) $c = 2800$

Fig. 5. Impact of the number of customers N and the total capacity c on the performances of online mechanisms. Each point in the figure is an average of 1000 evaluations. For all the three figures, $(\beta, \sigma) = (0.5, 1)$.



(a) High-Low Case (b) Constant Case (c) Low-High Case

Fig. 6. Empirical ratios of different online mechanisms under three extreme cases. Each point in the figure is an average of 1000 evaluations.

The results in Fig. 5 is intuitive since a more limited available capacity indicates a more difficult and risky online decision-making, as every inappropriate decision made now (e.g., prices are set too cheap) might have no remedy in the future. Fig. 5 demonstrates that our proposed $\text{PPM}_{\mathbf{\Phi}}$ has a very competitive performance even if the available capacity is very limited (Fig. 5(a)). Moreover, the performance of $\text{PPM}_{\mathbf{\Phi}}$ is very robust when the number of customers increases, and thus is amenable for large-scale systems.

2) Results based on Extreme-Case Setup. We next present the performance of $\text{PPM}_{\mathbf{\Phi}}$ based on the extreme-case setup in Fig. 6. As mentioned earlier in this section, we plot the empirical ratios of three online mechanisms w.r.t. the values of $\overline{p}$ in the following three extreme cases: High-Low Case, Constant Case, and Low-High Case. Fig. 6(a) shows that $\text{PPM}_{\mathbf{\Phi}}$ achieves an empirical ratio around $1.5 \sim 1.7$ in the High-Low Case. Meanwhile, the other two online mechanisms also achieve a decent performance. For example, the empirical ratios of **Greedy** are smaller than 2. This should be expected since the High-Low Case is easy in terms of online decision-making. Fig. 6(b) shows the performance of three online mechanisms in a relatively more difficult Constant Case. As shown in Fig. 6(b), when the VERs of all customers are constant, both **Greedy** and $\text{PPM}_{\mathbf{\Phi}}$ have a similar performance, while **Linear** performs much worse than **Greedy** and $\text{PPM}_{\mathbf{\Phi}}$. A similar performance between **Greedy** and $\text{PPM}_{\mathbf{\Phi}}$ in the Constant Case is consistent with our results in Fig. 3 when σ is extremely small (e.g., $\sigma = 0.01$). Among the three extreme cases, the Low-High Case is the most difficult case since it is likely to deplete too much available capacity at the earlier stage and thus latter high-VER customers cannot make any purchase due to the capacity limit. However, kind of surprisingly, as shown in Fig. 6(c), $\text{PPM}_{\mathbf{\Phi}}$ still demonstrates a relatively competitive results (the empirical ratios are below 3 for most of the cases). As can be expected, **Greedy** fails to reserve enough capacity for future high-VER customers, leading to a very poor performance in the Low-High Case. Fig. 6(c) also shows that **Linear** performs better than **Greedy** when $\overline{p}$ is small. This follows the intuition since it is reasonable to price as conservative as **Linear** in the Low-High Case for the purpose of future high-VER customers. However, a larger $\overline{p}$ will make **Linear** excessively conservative, and thus **Linear** performs even worse than **Greedy** when $\overline{p}$ is larger than 6.

In summary, from Fig. 6(a) to Fig. 6(c), the difficulty of online decision-making is increasing, and thus all the three online mechanisms shows a certain degree of performance degradation. However, Fig. 6 shows that $\text{PPM}_{\mathbf{\Phi}}$ achieves a very stable performance w.r.t. $\overline{p}$. We argue that this is kind of counter-intuitive, since $\overline{p}$ is the upper bound of $\Phi_t^*(y)$ and directly influences the calculation of the optimal dividing threshold u_t^*. Thus, $\overline{p}$ is supposed to play a critical role in determining the final curvature of $\Phi_t^*(y)$. However, unlike **Linear**, the performance of $\text{PPM}_{\mathbf{\Phi}}$ is not sensitive to the changes of $\overline{p}$. Note that our previous theoretic analyses provide no such guarantee, and thus we consider this is another advantage of our pricing function design.

VI. Conclusion

In this paper, we proposed a theoretic framework to study a general TE retailing problem in smart grid. In our studied problem, the TE retailer sells energy to customers that arrive in an arbitrary manner and may choose to purchase a certain amount of TE based on the current posted prices, or leave without buying anything. We proposed an optimal posted-pricing mechanism (PPM) for TE retailing without assuming any knowledge of future arrival information. Our proposed PPM is optimal in the sense that no other online mechanisms can achieve a better competitive ratio, and consequently, no other online algorithms can achieve a better performance in expectation. We evaluated the proposed online mechanism in the setting of online EV charging. Extensive experimental results show that our proposed PPM achieves a very competitive empirical result compared to its offline counterpart. Meanwhile, our proposed PPM is robust against system uncertainties and outperforms several existing benchmarks.

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APPENDIX A
PROOF OF THEOREM 1 AND THEOREM 2

The proof of this two theorems are based on the online primal-dual analysis. In the following, we first give the primal and dual versions of the offline social welfare maximization problem in Eq. (2), and then discuss how the online version of the primal and dual analysis can help design competitive pricing functions. After that, we prove the necessary and sufficient conditions for any pricing function that has a bounded competitive ratio.

A. *Offline Primal-Dual Analysis*

For the offline problem (2), let us relax $\{x_n\}_{\forall n}$ to be continuous, relax the equality constraint (2b) to an inequality one, and further introduce a barrier cost function $\hat{f}_t$ to eliminate the capacity limit constraint $y_t \leq c_t$. Then, the resulting relaxed problem is given as follows:

$$\max_{x_n, y_t} \quad \sum_{n=1}^{N} v_n x_n - \Big(\sum_{t=1}^{T} \hat{f}_t(y_t) - \sum_{t=1}^{T} \hat{f}_t(b_t)\Big)\Delta_T \tag{34a}$$

$$s.t. \quad y_t \geq \sum_{n=1}^{N} r_n^t x_n + b_t, \forall t, \qquad (\lambda_t) \tag{34b}$$

$$x_n \leq 1, \forall n, \qquad (\gamma_n) \tag{34c}$$

$$x_n \geq 0, y_t \geq b_t, \tag{34d}$$

where λ_t and γ_n denote the corresponding dual variables of each constraint[11], and the barrier cost function $\hat{f}_t(y)$ is given by

$$\hat{f}(y) = \begin{cases} f_t(y), & \text{if } y \in [b_t, c_t], \\ +\infty, & \text{if } y \in (c_t, +\infty). \end{cases} \tag{35}$$

Since cost function $f_t(\cdot)$ is non-decreasing, constraints (34b) will always be binding. Meanwhile, the introduction of barrier cost function $\hat{f}(y)$ is also known to be an equivalent transformation [37]. Therefore, the only difference between Problem (2) and Problem (34) is the relaxation of $\{x_n\}_{\forall n}$.

[11] Here the shadow price λ_t is being used by abusing the notation of $\lambda_t^{(n)}$ introduced in Section II-B. We consider λ_t is for the offline case while $\lambda_t^{(n)}$ is for the online case.

Given the relaxed offline problem (34), we can derive its dual as follows:

$$\min \quad \sum_{n=1}^{N} \gamma_n + \Big(\sum_{t=1}^{T} g_t\left(\lambda_t\right)\Big)\Delta_T \tag{36a}$$

$$s.t. \quad \gamma_n \geq v_n - \sum_{t=1}^{T} \lambda_t r_n^t \Delta_T, \forall n, \tag{36b}$$

$$\text{variables:} \quad \lambda_t \geq 0, \forall t; \gamma_n \geq 0, \forall n, \tag{36c}$$

where function $g_t(\lambda_t)$ is defined by

$$g_t(\lambda_t) = \max_{y_t \geq b_t} \ \lambda_t(y_t - b_t) - \big(\hat{f}_t(y_t) - \hat{f}_t(b_t)\big). \tag{37}$$

Recall that in Eq. (14), we define the initial marginal cost p_t^b and the maximum marginal cost p_t^c as follows

$$p_t^b \triangleq f_t'(b_t), p_t^c \triangleq f_t'(c_t), \forall t \in \mathcal{T}, \tag{38}$$

then $g_t(\cdot)$ can be analytically given by

$$g_t(\lambda_t) = \begin{cases} 0, & \text{if } \lambda_t \in [0, p_t^b] \\ q_{t,2}\lambda_t^2 + q_{t,1}\lambda_t + q_{t,0}, & \text{if } \lambda_t \in [p_t^b, p_t^c], \\ \ell_{t,1}\lambda_t + \ell_{t,0}, & \text{if } \lambda_t \geq p_t^c, \end{cases} \tag{39}$$

where the coefficients are given as follows: $q_{t,2} = \frac{1}{4a_{t,2}}$, $q_{t,1} = -\frac{a_{t,1}}{2a_{t,2}} - b_t$, $q_{t,0} = \frac{a_{t,1}^2}{4a_{t,2}} + a_{t,2}b_t^2 + a_{t,1}b_t$, $\ell_{t,1} = c_t - b_t$, and $\ell_{t,0} = a_{t,2}b_t^2 + a_{t,1}b_t - a_{t,2}c_t^2 - a_{t,1}c_t$. It can be observed that function $g_t(\lambda_t)$ is strictly increasing in $\lambda_t \geq p_t^b$. Meanwhile, $g_t(\lambda_t)$ is quadratic when $\lambda_t \in [p_t^b, p_t^c]$ and is linear when $\lambda_t \geq p_t^c$. Note that once the setup $\mathcal{S}$ is given, all the coefficients of $g_t(\lambda_t)$ can be calculated. We will provide the detailed derivation of $g_t(\cdot)$ in Appendix F.

If we denote the optimal objective of the relaxed primal problem (34) and its dual (36) by $W_{\text{r-primal}}$ and $W_{\text{r-dual}}$, respectively, then we have

$$W_{\text{opt}} \leq W_{\text{r-primal}} \leq W_{\text{r-dual}}, \tag{40}$$

where W_{opt} is the optimal objective of the original offline problem (2). In particular, the first inequality in Eq. (40) is due to the relaxation of $\{x_n\}_{\forall n}$ and the second inequality comes form weak duality [37].

B. Online Primal-Dual Analysis

The key to the design of $\text{PPM}_{\mathbf{\Phi}}$ is to link the pricing function $\lambda_t^{(n)} = \Phi_t(y_t^{(n-1)})$ to the offline shadow price λ_t. Specifically, when there is no future information, it is impossible to know the exact value of λ_t. Our idea is to design the posted price $\lambda_t^{(n)}$ as a function of the current total power consumption $y_t^{(n)}$, and using $\lambda_t^{(n)}$ to approximate the exact shadow price at each round.

Following this idea, let us denote the primal and dual objective by P_n and D_n after processing customer n, respectively. Intuitively, P_0 and D_0 denote the initial values (i.e., before processing the first customer), and P_N and D_N represent the terminal values (i.e., after processing the last customer of interest). Obviously, $P_0 = 0$ and D_0 is given by

$$D_0 = \sum_{t\in\mathcal{T}} g_t\big(\lambda_t^{(0)}\big) = \sum_{t\in\mathcal{T}} g_t\big(\Phi_t(b_t)\big). \tag{41}$$

where $\Phi_t(b_t)$ represents the initial price at time t.

The key design idea is that, if the pricing function $\Phi_t(\cdot)$ is constructed in a certain way so that i) $D_0 = 0$ and the solutions found by $\text{PPM}_{\mathbf{\Phi}}$ are feasible, and ii) the following *incremental inequality* $P_n - P_{n-1} \geq \frac{1}{\alpha}(D_n - D_{n-1})$ holds for each round with a constant α, then $P_N = \sum_{n=1}^{N}(P_n - P_{n-1}) \geq \frac{1}{\alpha}\sum_{n=1}^{N}(D_n - D_{n-1}) = \frac{1}{\alpha}D_N$. Note that P_N denotes the social welfare achieved by $\text{PPM}_{\mathbf{\Phi}}$, i.e., $W_{\text{ppm}} = P_N$. Based on (40), we have

$$W_{\text{ppm}} = P_N \geq \frac{1}{\alpha}D_N \geq \frac{1}{\alpha}W_{\text{r-dual}} \geq \frac{1}{\alpha}W_{\text{opt}},$$

which thus indicates that $\text{PPM}_{\mathbf{\Phi}}$ is α-competitive.

The above design principle follows the online primal-dual approach [32]. Typically, one needs to directly construct a feasible pricing scheme $\mathbf{\Phi} = \{\Phi_t\}_{\forall t}$ so that the above incremental inequality holds at each round, e.g., [38], [28], [17], and our recent work [22], [26]. However, unlike all of these existing approaches, this paper adopts a different design philosophy and proposes a systematic framework for designing competitive pricing functions. Our design framework is based on the uniqueness and existence theories of first-order ODE with boundary conditions, i.e., boundary value problems (BVPs). Below we give the necessary and sufficient condition for any pricing function that has a bounded competitive ratio.

C. Sufficient and Necessary Conditions

Below we give Theorem 9 which summarizes the sufficient conditions to guarantee a bounded competitive ratio for $\text{PPM}_{\mathbf{\Phi}}$.

Theorem 9 (Sufficiency)**.** $\text{PPM}_{\mathbf{\Phi}}$ *is α-competitive and incentive compatible if each $t \in \mathcal{T}$, pricing function $\Phi_t(y)$ satisfies:*

- ***S1**): $\Phi_t(b_t) = p_t^b$ and $\Phi_t(c_t) \geq \overline{p}$.*
- ***S2**): $\Phi_t(y)$ is monotonically increasing in $y \in [b_t, c_t]$.*
- ***S3**): $\Phi_t(y)$ satisfies the following inequality:*

$$\Phi_t(y) - f_t'(y) \geq \frac{\Phi_t'(y) \cdot g_t'\big(\Phi_t(y)\big)}{\alpha}, \forall y \in [b_t, c_t], \tag{42}$$

where α is a constant that depends on the setup $\mathcal{S}$ only.

Proof. The above theorem gives three sufficient conditions, namely, the boundary condition **S1**, the monotonicity condition **S2**, and the differential inequality condition **S3**. These three sufficient conditions are derived based on the online primal-dual analysis and the detailed proof is deferred to Appendix G. □

In addition to the sufficiency in Theorem 1, we also have the following theorem which indicates the necessity of any online algorithm that is α-competitive.

Theorem 10 (Necessity)**.** *If there exists an α-competitive online algorithm for our setup, then for each $t \in \mathcal{T}$, there exists a function $\Psi_t(\lambda)$ that satisfies:*

- ***N1**): $\Psi_t(p_t^b) = b_t$ and $\Psi_t(\overline{p}) \leq c_t$.*
- ***N2**): $\Psi_t(\lambda)$ is monotonically increasing in $\lambda \in [p_t^b, \overline{p}]$.*
- ***N3**): $\Psi_t(\lambda)$ satisfies the following inequality:*

$$\int_{p_t^b}^{\lambda} \eta\Psi_t'(\eta)d\eta - f_t(\Psi_t(\lambda)) + f_t(b_t) \geq \frac{g_t(\lambda)}{\alpha}, \forall \lambda \in [p_t^b, \overline{p}]. \tag{43}$$

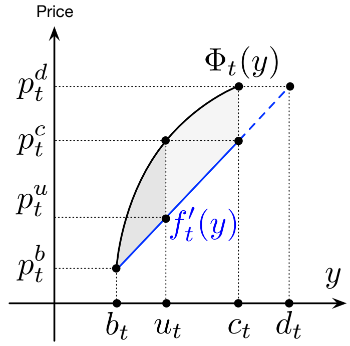

Fig. 7. The pricing function $\Phi_t(y)$ and the marginal cost function $f_t'(y)$ with respect to the total power consumption $y \in [b_t, c_t]$.

Proof. The proof of the above necessity is based on constructing a special arrival instance such that any α-competitive online algorithm must satisfy the above three conditions in order to achieve at least $1/\alpha$ of the offline optimal social welfare. The detailed proof is deferred to Appendix H. □

We emphasize that Theorem 1 provides three sufficient conditions for $\text{PPM}_{\mathbf{\Phi}}$ to be competitive and incentive compatible, while Theorem 2 holds for general online algorithms. Notice that, if we assume $\lambda = \Phi_t(y)$ and $y = \Psi_t(\lambda)$, then Ψ_t and Φ_t are inverse to each other since $\Phi_t(\cdot)$ and $\Psi_t(\cdot)$ are both monotonic. Therefore, Φ_t is the pricing function, and Ψ_t is the inverse pricing function.

We illustrate some key features of a feasible pricing function in Fig. 7. For instance, $\Phi_t(y)$ is strictly increasing in $y \in [b_t, c_t]$ (condition **S1**), and satisfies the following two boundary conditions: $\Phi(b_t) = f_t'(b_t) = p_t^b$ and $\Phi_t(c_t) = f_t'(d_t) \triangleq p_t^d$, where d_t is any constant such that $f_t'(d_t) \geq \overline{p}$, namely

$$d_t \in \mathcal{D}_t \triangleq [f_t'^{-1}(\overline{p}), +\infty), \forall t \in \mathcal{T}. \tag{44}$$

Therefore, condition **S2** is satisfied as long as $d_t \in \mathcal{D}_t$. Geometrically, the grey area in Fig. 7 represents the left-hand-side of (42), i.e., $\Phi_t(y) - f_t'(y)$, and the increasing rate of Φ_t (i.e., $\Phi_t'(y)$) should satisfy Eq. (42) (condition **S3**).

As illustrated in Fig. 7, the pricing function $\Phi_t(y)$ passes a unique point at (u_t, p_t^c), namely, $\Phi_t(u_t) = p_t^c$. Recall that in Section III-A, u_t is defined as the dividing threshold for pricing function $\Phi_t(y)$. For notational convenience, let us define the set of all possible dividing thresholds by

$$\mathcal{U}_t \triangleq (b_t, c_t). \tag{45}$$

Based on the above two theorems and the definition of dividing threshold, we give the following proposition.

Proposition 11. $\text{PPM}_{\mathbf{\Phi}}$ *is* $\max_t\{\alpha_t\}$*-competitive only if for all* $t \in \mathcal{T}$*, there exist a pair of monotonically increasing functions* $\Phi_{t,1}(y)$ *and* $\Phi_{t,2}(y)$ *that are solutions to the following two boundary value problems (BVPs), respectively:*

$$\begin{cases} \Phi_{t,1}'(y) = \alpha_t \cdot \frac{\Phi_{t,1}(y) - f_t'(y)}{f_t'^{-1}(\Phi_{t,1}(y)) - b_t}, y \in (b_t, u_t), \\ \Phi_{t,1}(b_t) = f_t'(b_t), \Phi_{t,1}(u_t) = f_t'(c_t), \end{cases} \tag{46a}$$

$$\begin{cases} \Phi_{t,2}'(y) = \alpha_t \cdot \frac{\Phi_{t,2}(y) - f_t'(y)}{c_t - b_t}, y \in (u_t, c_t), \\ \Phi_{t,2}(u_t) = f_t'(c_t), \Phi_{t,2}(c_t) = f_t'(d_t), \end{cases} \tag{46b}$$

where $(d_t, u_t) \in \mathcal{D}_t \times \mathcal{U}_t$ *and* $\alpha_t \geq 1$ *is a constant that depends on the setup* $\mathcal{S}$ *only. On the other hand, if we denote the solutions to the above two BVPs by* $\Phi_{t,1}(y)$ *and* $\Phi_{t,2}(y)$*, respectively, then* $\text{PPM}_{\mathbf{\Phi}}$ *is* α*-competitive if the pricing function* Φ_t *is given by*

$$\Phi_t(y) = \begin{cases} \Phi_{t,1}(y) & \text{if } y \in [b_t, u_t), \\ \Phi_{t,2}(y) & \text{if } y \in [u_t, c_t]. \end{cases} \tag{47}$$

Proof. This proposition directly follows Theorem 1 and Theorem 2. On the one hand, compared to the three sufficient conditions in Theorem 1, we replace the differential inequality (42) by an equality one, and then substitute $g_t'(\cdot)$ from Eq. (39) into the resulting differential equations to get the two ODEs in the above two BVPs (condition **S3**). Meanwhile, we perform an equivalent transformation to change the condition of $\Phi_t(c_t) \geq \overline{p}$ (condition **S1**) to $\Phi_t(c_t) = f_t'(d_t)$ with $d_t \in \mathcal{D}_t$ (as illustrated in Fig. 7). Condition **S2** remains unchanged since both $\Phi_{t,1}$ and $\Phi_{t,2}$ are monotonically increasing. On the other hand, if there exists a $\max_t\{\alpha_t\}$-competitive online algorithm, we can find a monotonically-increasing function Ψ_t that satisfies (43) by equality (condition **N3**) and the other two necessary conditions **N1** and **N2**. The inverse of Ψ_t is the monotonically increasing solution Φ_t that satisfies the above two BVPs with a feasible pair of constants $(d_t, u_t) \in \mathcal{D}_t \times \mathcal{U}_t$. □

Proposition 11 provides two first-order two-point BVPs for the design of $\{\Phi_t\}_{\forall t}$ in two segments. Basically, if we can find a pair of constants (d_t, u_t) and α_t for each time slot $t \in \mathcal{T}$, and solve Problem (46a) and Problem (46b) to get two monotonically-increasing pricing functions, then $\text{PPM}_{\mathbf{\Phi}}$ achieves a competitive ratio of $\max_t\{\alpha_t\}$. It is worth pointing out that for a given triple (d_t, u_t, α_t), Problem (46a) is only indexed by (u_t, α_t), while Problem (46b) is indexed by all of these three parameters.

Theorem 1 and Theorem 2 directly follow Theorem 9, Theorem 10, and Proposition 11.

Appendix B
Proof of Theorem 3

Based on the above analysis in Appendix A, below we prove Theorem 3 in two parts, namely, i) the existence and uniqueness, and ii) the optimality.

*1) **Part-1: Existence and Uniqueness**:* Before proving the existence and uniqueness properties in Theorem 3, we first give the following two theorems regarding the existing and uniqueness of solutions to Problem (46a) and Problem (46b).

Theorem 12. *For any* $u_t \in \mathcal{U}_t$*, there exists a unique monotonically-increasing pricing function* $\Phi_{t,1}(y)$ *that satisfies Problem* (46a) *if and only if* $\alpha_t \geq \Gamma_t(u_t)$*, where* $\Gamma_t(u_t)$ *is given as follows:*

$$\Gamma_t(u_t) = \begin{cases} \frac{(c_t - b_t)^2}{(u_t - b_t)(c_t - u_t)} & \text{if } u_t \in \left(b_t, \frac{b_t + c_t}{2}\right), \\ 4 & \text{if } u_t \in \left[\frac{b_t + c_t}{2}, c_t\right). \end{cases} \tag{48}$$

Proof. Proving the above necessary and sufficient condition is deferred to Appendix D. Here we briefly describe the intuition of this theorem. Note that $\frac{(c_t - b_t)^2}{(u_t - b_t)(c_t - u_t)}$ achieves its minimum of 4 when $u_t = \frac{b_t + c_t}{2}$. Therefore, $\Gamma_t(u_t)$ is continuous and

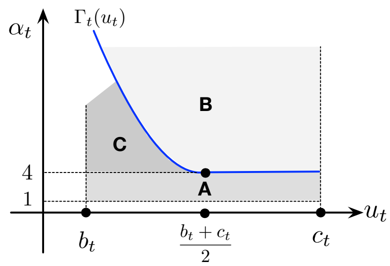

Fig. 8. Illustration of $\alpha_t = \Gamma_t(u)$ w.r.t. $u_t \in \mathcal{U}_t = (b_t, c_t)$. The horizontal axis represents the dividing threshold u_t and the vertical axis denotes the competitive ratio parameter α_t. The blue real curve $\alpha_t = \Gamma_t(u_t)$ separates the whole area of interest into three regions.

monotonically non-increasing in $u_t \in (b_t, c_t)$. As illustrated in Fig. 8, the blue curve represents $\alpha_t = \Gamma_t(u_t)$ and separates the whole area of interest (i.e., $(u_t, \alpha_t) \in (b_t, c_t) \times [1, +\infty)$) into three regions, namely Region A, B, and C. The definition of these three regions are as follows:

- Region A: $u_t \in (b_t, c_t)$ and $1 \leq \alpha_t < 4$.
- Region B: $u_t \in (b_t, c_t)$ and $\alpha_t \geq \Gamma_t(u_t)$.
- Region C: $u_t \in (b_t, \frac{b_t+c_t}{2})$ and $4 \leq \alpha_t < \Gamma_t(u_t)$.

Theorem 12 basically argues that for any (u_t, α_t) in Region B, Problem (46a) has a unique solution which is monotonically increasing, while for any (u_t, α_t) in Region A and Region C, there exists no solution for Problem (46a). For the detailed discussion of existence and uniqueness properties in these three regions, please refer to Appendix D. □

The analytical lower bound $\Gamma_t(u_t)$ in each time slot provides a necessary and sufficient condition for the unique existence of monotonically-increasing pricing functions to Problem (46a). Below we also give a proposition regarding the unique existence of monotonically-increasing pricing functions to Problem (46b).

Theorem 13. *For any $\alpha_t \geq 1$, there exists a unique monotonically increasing pricing function $\Phi_{t,2}(y)$ that satisfies Problem* (46b) *if and only if $(d_t, u_t) \in \mathcal{D}_t \times \mathcal{U}_t$ and satisfies*

$$\frac{c_t - u_t - \frac{c_t - b_t}{\alpha_t}}{\exp\left(\frac{u_t}{c_t - b_t} \cdot \alpha_t\right)} = \frac{d_t - c_t - \frac{c_t - b_t}{\alpha_t}}{\exp\left(\frac{c_t}{c_t - b_t} \cdot \alpha_t\right)}. \tag{49}$$

Proof. Note that the ODE of Problem (46b) can be analytically solved and the solution is given as follows:

$$\Phi_{t,2}(y) = f_t'(y) + k_t \exp\left(\frac{\alpha_t y}{c_t - b_t}\right) + \frac{p_t^c - p_t^b}{\alpha_t}. \tag{50}$$

where the coefficient k_t depends on the two boundary conditions: $\Phi_{t,2}(u_t) = f_t'(c_t), \Phi_{t,2}(c_t) = f_t'(d_t)$. Substituting these two boundary conditions into the above formula and eliminating the coefficient k_t leads to Eq. (49). □

Below we give Proposition 14 which shows the unique existence of $u_t \in \mathcal{U}_t$ for each given $d_t \in \mathcal{D}_t$ when $\alpha_t = \Gamma_t(u_t)$.

Proposition 14. *For each given $d_t \in \mathcal{D}_t$, there exists a unique $u_t \in \mathcal{U}_t$ that satisfies the following equation:*

$$\frac{c_t - u_t - \frac{c_t - b_t}{\Gamma_t(u_t)}}{\exp\left(\frac{u_t}{c_t - b_t} \cdot \Gamma_t(u_t)\right)} = \frac{d_t - c_t - \frac{c_t - b_t}{\Gamma_t(u_t)}}{\exp\left(\frac{c_t}{c_t - b_t} \cdot \Gamma_t(u_t)\right)}. \tag{51}$$

Meanwhile, $u_t(d_t)$ is strictly decreasing in $d_t \in \mathcal{D}_t = [f_t'^{-1}(\overline{p}), +\infty)$.

Proof. To help our derivation, we revisit the Eq. (51) as follows:

$$\underbrace{\frac{c_t - u_t - \frac{c_t - b_t}{\Gamma_t(u_t)}}{\exp\left(\frac{u_t}{c_t - b_t} \cdot \Gamma_t(u_t)\right)}}_{\text{LHS}(u_t)} = \underbrace{\frac{d_t - c_t - \frac{c_t - b_t}{\Gamma_t(u_t)}}{\exp\left(\frac{c_t}{c_t - b_t} \cdot \Gamma_t(u_t)\right)}}_{\text{RHS}(u_t)}. \tag{52}$$

Note that to prove the unique existence of $u_t \in \mathcal{U}_t$ for a given $d_t \in \mathcal{D}_t$, we can analyse the property of function $\text{LHS}(u_t) - \text{RHS}(u_t)$ and then show that there exists a unique root when $\text{LHS}(u_t) - \text{RHS}(u_t) = 0$. However, this approach is tedious since we need to evaluate the derivative of $\text{LHS}(u_t) - \text{RHS}(u_t)$. Below we provide a much easier proof.

First, if $d_t - c_t - \frac{c_t - b_t}{\Gamma_t(u_t)} = 0$, namely,

$$\Gamma_t(u_t) = \frac{c_t - b_t}{d_t - c_t}, \tag{53}$$

then, substituting $\Gamma_t(u_t)$ into Eq. (51) we can get the unique u_t as follows:

$$u_t = 2c_t - d_t. \tag{54}$$

In this case, if $u_t = 2c_t - d_t \leq \frac{b_t + c_t}{2}$, then we have

$$\Gamma_t(u_t) = \frac{c_t - b_t}{d_t - c_t} = \frac{(c_t - b_t)^2}{(u_t - b_t)(c_t - u_t)}, \tag{55}$$

which leads to the solution of $d_t = c_t$ and $u_t = c_t$, and thus infeasible. Therefore, $u_t = 2c_t - d_t > \frac{b_t + c_t}{2}$ must hold, which leads to the following equation

$$\Gamma_t(u_t) = \frac{c_t - b_t}{d_t - c_t} = 4 \Rightarrow d_t = c_t + \frac{c_t - b_t}{4} \tag{56}$$

Therefore, in this case, Eq. (51) has a unique u_t, given by

$$u_t = 2c_t - d_t = c_t - \frac{c_t - b_t}{4} = \frac{3c_t + b_t}{4}. \tag{57}$$

In addition to this special case, we have the following two general cases:

- If $d_t > c_t + \frac{c_t - b_t}{4}$, Eq. (51) is equivalent to the following function
$$\exp\left(\frac{c_t - u_t}{c_t - b_t} \cdot \Gamma_t(u_t)\right) = \frac{d_t - c_t - \frac{c_t - b_t}{\Gamma_t(u_t)}}{c_t - u_t - \frac{c_t - b_t}{\Gamma_t(u_t)}}, \tag{58}$$
where the left-hand-side is monotonically-decreasing in $u_t \in (b_t, \frac{3c_t + b_t}{4})$, and the right-hand-side is monotonically-increasing in $u_t \in (b_t, \frac{3c_t + b_t}{4})$. By evaluating both sides at $u_t = b_t$ and $u_t = \frac{3c_t + b_t}{4}$, it is easy to prove that the above equation has a unique root u_t in $(b_t, \frac{3c_t + b_t}{4})$.
- if $c_t < d_t < \frac{c_t - b_t}{4} + c_t$, we can prove similarly that the above equation has a unique root u_t in $(\frac{3c_t + b_t}{4}, c_t)$.

In summary, for any $d_t \in \mathcal{D}_t$, there exists a unique $u_t \in \mathcal{U}_t$

The monotonicity of $u_t(d_t)$ can be proved as follows. First, when $d_t > c_t + \frac{c_t - b_t}{4}$, the right-hand-side of Eq. (58) is increasing in d_t. Therefore, when d_t increases, the intersection point between the left-hand-side and right-hand-side of Eq. (58) moves to the left, i.e., decreasing. Therefore, in this case $u_t(d_t)$ is strictly decreasing in $d_t > c_t + \frac{c_t - b_t}{4}$. We can prove similarly that $u_t(d_t)$ is strictly decreasing when $c_t < d_t < c_t + \frac{c_t - b_t}{4}$. In summary, $u_t(d_t)$ is strictly decreasing in $d_t \in (c_t, +\infty)$. Therefore, $u_t(d_t)$ is also strictly decreasing in $d_t \in [f_t'^{-1}(\overline{p}), +\infty) \subset (c_t, +\infty)$. $\square$

Therefore, when $\alpha_t = \Gamma_t(u_t)$ and $d_t = f_t^{-1}(\overline{p})$, there must exist a unique dividing threshold $u_t \in (0,1)$ such that both Problem (46a) and Problem (46b) have unique monotonically-increasing solutions. *We thus complete the proof of the uniqueness and existence in Theorem 3.*

*2) **Part-2: Optimality**:* Proposition 14 shows that Eq. (51) has a unique solution in $u_t \in \mathcal{U}_t$. Since the lower bound $\Gamma_t(u_t)$ is continuous and monotonically non-increasing in $u_t \in (b_t, c_t)$, to get the minimum competitive ratio parameter α_t^*, we just need to find the maximum dividing threshold u_t^* so that $\alpha_t^* = \Gamma_t(u_t^*)$ is minimized, leading to the minimized competitive ratio of $\max_t\{\alpha_t^*\}$ with the unique existence of monotonically-increasing solutions to Problem (46a) and Problem (46b).

Based on the monotonicity of $\Gamma_t(u_t)$ and $u_t(d_t)$, to minimize α_t for each time slot, we set d_t to its minimum, namely

$$d_t^* = f_t'^{-1}(\overline{p}), \forall t \in \mathcal{T}, \tag{59}$$

and thus the dividing threshold $u_t^*(d_t^*)$ achieves its maximum for each time slot $t \in \mathcal{T}$. As a result, α_t achieves its minimum α_t^* as follows

$$\alpha_t^* = \Gamma_t(u_t^*(f_t'^{-1}(\overline{p}))), \forall t \in \mathcal{T}. \tag{60}$$

Therefore, when $(d_t, u_t, \alpha_t) = (d_t^*, u_t^*, \alpha_t^*)$, PPM$_{\boldsymbol{\Phi}}$ achieves the optimal competitive ratio of $\max_t\{\alpha_t^*\}$, i.e.,

$$\alpha^*(\mathcal{S}) = \max_{t\in\mathcal{T}} \left\{\Gamma_t(u_t^*)\right\}. \tag{61}$$

(**Proof of Corollary 4**) Now we prove that given the setup $\mathcal{S}$, for any $\epsilon > 0$, there exist no $(\alpha^*(\mathcal{S}) - \epsilon)$-competitive online mechanisms. Based on the definition of α_t^*, it is impossible for Problem (46a) and Problem (46b) to have monotonically increasing solutions when $\alpha_t = \alpha_t^* - \epsilon$. Consequently, there exists no such an inverse pricing function $\Psi_t(\lambda)$ that satisfies inequality (43) (i.e., the necessary condition **N3**). As a result, no online algorithm can achieve a competitive ratio of $\alpha^*(\mathcal{S}) - \epsilon$.

(**Proof of Corollary 5**) Meanwhile, we can prove that for any point in Region B in Fig. 8, we can find the corresponding pricing functions to satisfy Problem (46a) and Problem (46b). Therefore, for any $\epsilon \geq 0$, there exists a pricing scheme $\boldsymbol{\Phi} = \{\Phi_t\}_{\forall t}$ so that PPM$_{\boldsymbol{\Phi}}$ is $(\alpha^*(\mathcal{S}) + \epsilon)$-competitive. The details are deferred to our proof of Theorem 12 in Appendix D.

We thus complete the proof of Theorem 3.

APPENDIX C
PROOF OF LEMMA 6

According to Proposition 14, $u_t(d_t)$ is strictly decreasing in $d_t \in \mathcal{D}_t = [f_t'^{-1}(\overline{p}), +\infty)$. Since $f_t'^{-1}(\overline{p})$ is strictly increasing in $\overline{p} \in \mathcal{P}$, Lemma 6 directly follows.

APPENDIX D
PROOF OF THEOREM 12

In this section we present the proofs for Theorem 12. We first discuss some preliminaries regarding the BVP in Problem (46a), and then present the detailed proof for Theorem 12.

A. Preliminaries of BVPs

To help present the proof of Theorem 12 in this section, we revisit Problem (46a) as follows

$$\begin{cases} \Phi_{t,1}'(y) = \alpha_t \frac{\Phi_{t,1}(y) - f_t'(y)}{f_t'^{-1}(\Phi_{t,1}(y)) - b_t}, y \in (b_t, u_t), \\ \Phi_{t,1}(b_t) = f_t'(b_t), \Phi_{t,1}(u_t) = f_t'(c_t), \end{cases} \tag{62}$$

Our target is to study the existence and uniqueness of solutions to the above BVP for each given $u_t \in (b_t, c_t)$. In particular, given $u_t \in (b_t, c_t)$, we want to know what is the minimum α_t such that there exists a monotonically increasing solution for Problem (62).

Our following proof relies on the existence and uniqueness theories in the field of differential equations [33] [34]. In particular, let us define an initial value problem (IVP) as follows:

$$\begin{cases} \Phi_{t,1}'(y) = \alpha_t \frac{\Phi_{t,1}(y) - f_t'(y)}{f_t'^{-1}(\Phi_{t,1}(y)) - b_t}, y \in (b_t, u_t), \\ \Phi_{t,1}(u_t) = f_t'(c_t). \end{cases} \tag{63}$$

We denote the solution to the above IVP by $\Phi_{t,1}^{\text{ivp}}(y)$ (if it exists). Note that the only difference between Problem (63) and Problem (62) is the removal of $\Phi_{t,1}(b_t) = f_t'(b_t)$.

Let us denote $w = y - b_t$ and $\chi_t = f_t'^{-1}(\Phi_{t,1}) - b_t = \frac{\Phi_{t,1} - a_{t,1}}{2a_{t,2}} - b_t$. Therefore, we can get to the following BVP:

$$\text{BVP}_\chi \begin{cases} \chi_t'(w) = \alpha_t \frac{\chi_t(w) - w}{\chi_t(w)}, w \in (0, u_t - b_t), \\ \chi_t(0) = 0, \chi_t(u_t - b_t) = c_t - b_t, \end{cases} \tag{64}$$

Similarly, we define the following first-order IVP based on the above BVP:

$$\text{IVP}_\chi \begin{cases} \chi_t'(w) = \alpha_t \frac{\chi_t(w) - w}{\chi_t(w)}, w \in (0, u_t - b_t), \\ \chi_t(u_t - b_t) = c_t - b_t. \end{cases} \tag{65}$$

We denote the unique solution to the above IVP by $\chi_t(w; u_t, \alpha_t)$ (if it exists) to explicitly denote the dependency on u_t and α_t.

Based on standard differential equation theories [34], we have the following lemma.

Lemma 15. *For each given $u_t \in \mathcal{U}_t$, Problem (63) has a unique solution $\Phi_{t,1}^{ivp}(y)$. Meanwhile, $\Phi_{t,1}^{ivp}(y)$ is monotonically increasing in $y \in (b_t, u_t]$.*

Proof. The existence of a unique solution to standard first-order IVPs directly follows the Picard–Lindelöf theorem. We refer the details to [33], [34]. $\square$

To study the existence conditions for Problem (62), we give the following lemma.

Lemma 16. *$\Phi_{t,1}^{ivp}(y)$ is the unique solution to Problem (62) if and only if the unique solution $\Phi_{t,1}^{ivp}(y)$ satisfies*

$$\lim_{y \to b_t^+} \Phi_{t,1}^{ivp}(y) = f_t'(b_t) = p_t^b. \tag{66}$$

Proof. It is easy to prove that $\Phi_{t,1}^{\text{ivp}}(y)$ is also a solution to Problem (62) if the above limit holds. We focus on proving the uniqueness property. We can prove the uniqueness by contradiction. Suppose $\Phi_{t,1}^{\text{ivp}}(y) \neq \Phi_{t,1}(y)$, then $\Phi_t^{\text{bvp}}(y)$ is another solution to Problem (63) with different evaluations at

$y = b_t$. However, this contradicts with the uniqueness property of $\Phi_t^{\text{ivp}}(y)$. Therefore, as long as $\lim_{y\to b_t^+} \Phi_t^{\text{ivp}}(y) = p_t^b$, $\Phi_t^{\text{ivp}}(y)$ must be the unique solution to Problem (62). Note that we check the limit of $\Phi_t^{\text{ivp}}(y)$ at $y = b_t$ since $\Phi_t^{\text{ivp}}(y)$ may not be defined at $y = b_t$. □

Based on the above two lemmas, we can study Problem (63) instead of Problem (62). In particular, if we can find the condition for α_t such that the unique solution to Problem (63) always satisfies the limiting behavior in (66), then the unique solution to Problem (62) can be obtained. Below in Proposition 17 we can see that, a subgroup of such solutions can be found in a special case when α_t and u_t satisfy a certain relationship.

Proposition 17 (Linear Solution)**.** *For any $u_t \in \mathcal{U}_t$, when $\alpha_t = \frac{(c_t-b_t)^2}{(u_t-b_t)(c_t-u_t)}$, the unique solution to Problem* (62) *is*

$$\Phi_{t,1}(y) = \frac{p_t^c - p_t^b}{u_t - b_t}(y - b_t) + f_t'(b_t). \tag{67}$$

Proof. Let us rewrite the ODE of IVP_χ as follows:

$$\chi_t(w) - w = \frac{1}{\alpha_t}\chi_t(w)\chi_t'(w) \tag{68}$$

If we take derivative w.r.t. w in both sides, and after some manipulation, we have the following equation:

$$\chi_t''(w) = -\frac{(\chi_t'(w))^2 - \alpha_t\chi'(w) + \alpha_t}{\chi_t(w)}. \tag{69}$$

Substituting $\chi_t'(w)$ into the numerator of the above equation leads to the following

$$\begin{aligned}
&(\chi_t'(w))^2 - \alpha_t\chi'(w) + \alpha_t \\
&= \Big(\alpha_t - \alpha_t\frac{w}{\chi_t(w)}\Big)^2 - \alpha_t\Big(\alpha_t - \alpha_t\frac{w}{\chi_t(w)}\Big) + \alpha_t \\
&= \alpha_t^2\Big(\frac{w}{\chi_t(w)}\Big)^2 - \alpha_t^2\frac{w}{\chi_t(w)} + \alpha_t \\
&= \alpha_t\Big(\frac{w}{\chi_t(w)}\Big)^2 \cdot \Big[\Big(\frac{\chi_t(w)}{w}\Big)^2 - \alpha_t\Big(\frac{\chi_t(w)}{w}\Big) + \alpha_t\Big]
\end{aligned} \tag{70}$$

Therefore, if $\frac{\chi_t(w)}{w}$ is the root to the following quadratic equation

$$\left(\frac{\chi_t(w)}{w}\right)^2 - \alpha_t\left(\frac{\chi_t(w)}{w}\right) + \alpha_t = 0, \tag{71}$$

then $\chi_t''(w) = 0$, vice versa. Since $\chi_t(0) = 0$ and $\chi_t(u_t - b_t) = c_t - b_t$, if we draw a line between $(0, 0)$ and $(u_t - b_t, c_t - b_t)$ and denote it by $\hat{\chi}_t(w)$, then $\hat{\chi}_t(w) = \frac{c_t-b_t}{u_t-b_t}\cdot w$ and $\frac{\hat{\chi}_t(w)}{w} = \frac{c_t-b_t}{u_t-b_t}$, substituting it to the above formula leads to

$$\left(\frac{c_t - b_t}{u_t - b_t}\right)^2 - \alpha_t \cdot \frac{c_t - b_t}{u_t - b_t} + \alpha_t = 0, \tag{72}$$

which leads to

$$\alpha_t = \frac{(c_t - b_t)^2}{(u_t - b_t)(c_t - u_t)}. \tag{73}$$

Therefore, when α_t is given by the above equation, there exists a linear solution for $\chi_t(w)$, which is given by

$$\chi_t(w; \alpha_t, u_t) = \frac{c_t - b_t}{u_t - b_t}w. \tag{74}$$

Since $\chi_t = f_t'^{-1}(\Phi_{t,1}) - b_t$, after some simple algebra we obtain the linear solution in Eq. (67). It is easy to check that the limiting behaviour in Eq. (66) is satisfied. Therefore, when $\alpha_t = \frac{(c_t-b_t)^2}{(u_t-b_t)(c_t-u_t)}$, the unique solution to Problem (62) and Problem (63) is linear and is given by (67). □

Proposition 18. *Given $u_t \in \mathcal{U}_t$, there exists no solution for Problem* (62) *if $\alpha_t < 4$.*

Proof. According to the definition of limit, we have

$$\lim_{w\to 0^+} \frac{\chi_t(w)}{w} = \lim_{w\to 0^+} \chi_t'(w) = \lim_{w\to 0^+} \alpha_t\left(1 - \frac{w}{\chi_t(w)}\right)$$

which indicates that $\lim_{w\to 0^+} \chi_t'(w)$ must be finite. Let us denote $\lim_{w\to 0^+} \chi_t'(w) = \chi_t'(0^+)$, then we have

$$\left(\chi_t'(0^+)\right)^2 - \alpha_t \cdot \chi_t'(0^+) + \alpha_t = 0. \tag{75}$$

Therefore, we have $\alpha_t^2 - 4\alpha_t \geq 0$, i.e., $\alpha_t \geq 4$. Otherwise, the above equation has no real root, i.e., $\chi_t'(0^+)$ does not exist. □

B. Proof of Theorem 12

Based on the above lemmas and propositions, below we give the proof of Theorem 12. Our proof is organized in the following three parts based on the three regions in Fig. 8.

Region A. In this region, based on Proposition 18, we can directly conclude that there exists no solution to Problem (46a).

Region B. In this region, we need to prove the following subregions:

- *Subregion B-1*: $\alpha_t = \Gamma_t(u_t)$ and $u_t \in (b_t, \frac{b_t+c_t}{2}]$, which corresponds to the blue line between Region A and Region C in Fig. 8. Based on Proposition 17, there exists a unique linear solution to Problem (46a).
- *Subregion B-2*: $\alpha_t = 4$ and $u_t \in (\frac{b_t+c_t}{2}, c_t)$, which corresponds to the horizontal blue line between Region A and Region B in Fig. 8. We can prove that
$$w \leq \chi_t(w; \alpha_t, u_t) \leq \chi_t\left(w; \frac{b_t + c_t}{2}, 4\right) = 2w, \tag{76}$$
where $\chi_t(w; \frac{b_t+c_t}{2}, 4) = 2w$ denotes the unique solution to Problem (65) when $u_t = \frac{b_t+c_t}{2}$ and $\alpha_t = 4$. Therefore, when w approaches 0 from the right, based on the squeeze theorem, we have
$$\lim_{w\to 0^+} \chi_t(w; u_t, \alpha_t) = 0. \tag{77}$$
Therefore, for all $(u_t, 4)$ in Subregion B-2, the unique solution to Problem (65) is also the unique solution to Problem (64), which thus means that Problem (46a) has a unique solution which is monotonically increasing.
- *Subregion B-3*: $\alpha_t > \Gamma_t(u_t)$ and $u_t \in (b_t, c_t)$. Based on Lemma we have
$$w \leq \chi_t(w; u_t, \alpha_t) \leq \chi_t(w; u_t, \Gamma_t(u_t)). \tag{78}$$
When w approaches 0 from the right, based on the squeeze theorem, we have
$$0 \leq \lim_{w\to 0^+} \chi_t(w; u_t, \alpha_t) \leq 0. \tag{79}$$

Therefore, for all (u_t, α_t) in subregion B-3, the unique solution to Problem (65) is also the unique solution to Problem (64), which thus means that Problem (46a) has a unique solution which is monotonically increasing.

Region C. We prove in this region, it is impossible to have

$$\lim_{w\to 0^+} \chi_t(w; u_t, \alpha_t) = 0. \tag{80}$$

We prove this by contradiction. Suppose the above limit holds, then we have

$$\big(\chi_t'(0^+; u_t, \alpha_t)\big)^2 - \alpha_t \cdot \chi_t'(0^+; u_t, \alpha_t) + \alpha_t = 0. \tag{81}$$

where $\chi_t'(0^+; u_t, \alpha_t) = \lim_{w\to 0^+} \chi_t'(w; u_t, \alpha_t)$. When $4 \le \alpha_t < \Gamma_t(u_t)$, let us denote the two roots of Eq. (81) by $R^+(\alpha_t)$ and $R^-(\alpha_t)$ as follows:

$$R^-(\alpha_t) = \frac{\alpha_t - \sqrt{\alpha_t^2 - 4\alpha_t}}{2}, \tag{82}$$

$$R^+(\alpha_t) = \frac{\alpha_t + \sqrt{\alpha_t^2 - 4\alpha_t}}{2}. \tag{83}$$

where $R^+(\alpha_t) \ge R^-(\alpha_t)$ and the equality is reached if $\alpha_t = 4$. Therefore, when the limit in Eq. (80) holds, $\chi_t'(0^+; u_t, \alpha_t)$ equals either $R^-(\alpha_t)$ or $R^+(\alpha_t)$.

On the other hand, since $4 \le \alpha_t < \Gamma_t(u_t)$ implies that $\chi_t(w; u_t, \alpha_t)$ is strictly concave in $w \in (0, u_t - b_t)$. The concavity of $\chi_t(w; u_t, \alpha_t)$ means that $\chi_t'(w; u_t, \alpha_t)$ is monotonically decreasing in $w \in (0, u_t - b_t)$, namely,

$$\chi_t'(0^+; u_t, \alpha_t) > \chi_t'(u_t - b_t; u_t, \alpha_t) = \alpha_t \cdot \frac{c_t - u_t}{c_t - b_t}. \tag{84}$$

When we substitute $\alpha_t \cdot \frac{c_t - u_t}{c_t - b_t}$ into the left-hand-side of Eq. (81), we have

$$\left(\alpha_t \cdot \frac{c_t - u_t}{c_t - b_t}\right)^2 - \alpha_t^2 \cdot \frac{c_t - u_t}{c_t - b_t} + \alpha_t \tag{85}$$

$$= \alpha_t \left(1 - \alpha_t \cdot \frac{(c_t - u_t)(u_t - b_t)}{(c_t - b_t)^2}\right) > 0, \tag{86}$$

where the last inequality is because in Region-C we have $4 \le \alpha_t < \Gamma_t(u_t) = \frac{(c_t - b_t)^2}{(c_t - u_t)(u_t - b_t)}$ and $u_t \in (b_t, \frac{b_t + c_t}{2})$. Therefore, we have

$$\chi_t'(0^+; u_t, \alpha_t) > \alpha_t \cdot \frac{c_t - u_t}{c_t - b_t} > R^+(\alpha_t) \ge R^-(\alpha_t), \tag{87}$$

which leads to a contradiction as $\chi_t'(0^+; u_t, \alpha_t)$ must be equal to either $R^+(\alpha_t)$ or $R^-(\alpha_t)$ according to Eq. (81).

Therefore, the limit in Eq. (80) does not hold, and thus in Region C, there exists no solution to Problem (64), or equivalently, Problem (62).

We thus complete the proof of Theorem 12.

Appendix E
Proofs of Lemma 7 and Theorem 8

Our optimal pricing functions given by Theorem 8 is obtained by solving the two BVPs in Proposition 11 when $u_t = u_t^*(f_t'^{-1}(\overline{p}))$, $\alpha_t = \Gamma_t(u_t^*)$ and $d_t = f_t'^{-1}(\overline{p})$. To solve the original BVP in Problem (46a), we just need to solve the BVP in Problem (64). Let us revisit the ODE of Problem (64) as follows:

$$\chi_t'(w) = \alpha_t \frac{\chi_t(w) - w}{\chi_t(w)} \tag{88}$$

Let us denote $\phi_t = \frac{\chi_t}{w}$, then the above ODE can be transformed into the following one

$$\phi_t' w + \phi_t = \alpha_t \frac{\phi_t - 1}{\phi_t}, \tag{89}$$

which leads to the following ODE with separable variables:

$$\frac{d\phi_t}{\alpha_t \frac{\phi_t - 1}{\phi_t} - \phi_t} = \frac{dw}{w}, \tag{90}$$

and thus we have the following equation:

$$\int^{\phi_t} \frac{-\eta}{\eta^2 - \alpha_t \eta + \alpha_t} d\eta = \ln|w| + C_t, \tag{91}$$

where $C_t > 0$ is a coefficient that depends on the two boundary conditions. When $\alpha_t = 4$, the left-hand-side of the Eq. (91) can be written as follows:

$$\int^{\phi_t} \frac{-\eta}{(\eta - 2)^2} d\eta = \int^{\phi_t} \left(-\frac{1}{\eta - 2} + \frac{-2}{(\eta - 2)^2} \right) d\eta$$
$$= -\ln|\phi_t - 2| + \frac{2}{\phi_t - 2}.$$

Therefore, we have

$$-\ln|\phi_t - 2| + \frac{2}{\phi_t - 2} = \ln|w| + C_t, \tag{92}$$

where $\phi_t = \frac{\chi_t}{w}$ and $\phi_t(u_t^* - b_t) = \frac{c_t - b_t}{u_t^* - b_t}$. Therefore, we have

$$\frac{2w}{\chi_t - 2w} - \ln|\chi_t - 2w| = C_t \tag{93}$$

Substituting the boundary condition into the above equation leads to the following

$$\frac{2(y - b_t)}{\chi_t - 2(y - b_t)} - \frac{2(u_t^* - b_t)}{c_t + b_t - 2u_t^*} = \ln\left|\frac{\chi_t - 2(y - b_t)}{c_t + b_t - 2u_t^*}\right| \tag{94}$$

Based on our proof of Theorem 12, for each given $y \in (b_t, u_t^*)$, the above nonlinear equation always has a unique solution χ_t and $\chi_t \in (y - b_t, 2(y - b_t))$. We thus complete the proof of Lemma 7 by replacing χ_t by $H_t(y; u_t^*)$. In this case, the

$$\Phi_{t,1}(y) = f_t'\Big(b_t + H_t(y; u_t^*)\Big). \tag{95}$$

Solving the ODE of Problem (46b) leads to the following general solution

$$\Phi_{t,2}(y) = f_t'(y) + k_t \exp\left(\frac{\alpha_t y}{c_t - b_t}\right) + \frac{p_t^c - p_t^b}{\alpha_t}. \tag{96}$$

After substituting $\alpha_t = \Gamma_t(u_t)$ and the two boundary conditions, we have:

$$k_t \exp\left(\frac{u_t^* \cdot \Gamma_t(u_t^*)}{c_t - b_t}\right) + f_t'(u_t^*) + \frac{p_t^c - p_t^b}{\Gamma_t(u_t)} = p_t^c, \tag{97a}$$

$$k_t \exp\left(\frac{c_t \cdot \Gamma_t(u_t^*)}{c_t - b_t}\right) + p_t^c + \frac{p_t^c - p_t^b}{\Gamma_t(u_t^*)} = \overline{p}. \tag{97b}$$

Therefore, $\Phi_{t,2}(y)$ can be written as

$$\Phi_{t,2}(y) = f_t'(y) + L_t(y; u_t^*), \tag{98}$$

where according to Eq. (96) and Eq. (97), $L_t(y; u_t^*)$ is given by

$$L_t(y; u_t^*) = \frac{p_t^c - f_t'(u_t^*) - \frac{p_t^c - p_t^b}{\Gamma_t(u_t^*)}}{\exp\left(\frac{\Gamma_t(u_t^*)\cdot u_t^*}{c_t - b_t}\right)} \cdot \exp\left(\frac{\Gamma_t(u_t^*)\cdot y}{c_t - b_t}\right) + \frac{p_t^c - p_t^b}{\Gamma_t(u_t^*)},$$

which is the same as our definition of $L(y; u_t)$ in Definition 1.

Therefore, when we substitute the optimal dividing threshold u_t^* into Eq. (95) and Eq. (98), we obtain the optimal pricing functions in Eq. (28). In particular, when $\overline{p} \geq \overline{p}_t^{\text{cut}}$, the optimal pricing function at the high-risk segment is linear, as given by Proposition 17. Therefore, we have the optimal pricing functions in Eq. (27).

We thus complete the proof of Theorem 8.

Appendix F
Derivation of $g_t(\cdot)$ in the Dual

Function $g_t(\lambda_t)$ is defined by

$$g_t(\lambda_t) = \max_{y_t \geq b_t} \ \lambda_t(y_t - b_t) - \left(\hat{f}_t(y_t) - \hat{f}_t(b_t)\right). \tag{99}$$

Let us define the objective of the right-hand-side of the above equation by $G_t(y_t)$. After substituting $\hat{f}_t(y)$, we have

$$\begin{aligned} G_t(y_t) &= \lambda_t(y_t - b_t) - (a_{t,2}y^2 + a_{t,1}y + a_{t,0}) + f_t(b_t) \\ &= -a_{t,2}y_t^2 + (\lambda_t - a_{t,1})y_t - \lambda_t b_t + f_t(b_t) - a_{t,0}, \end{aligned}$$

which achieves its maximum when

$$y_t = \frac{\lambda_t - a_{t,1}}{2a_{t,2}}. \tag{100}$$

Therefore, we have the following three cases:

- $\frac{\lambda_t - a_{t,1}}{2a_{t,2}} \leq b_t$, i.e., $\lambda_t \leq 2a_{t,2}b_t + a_{t,1} = p_t^b$. In this case, when $y_t \in [b_t, c_t]$, $G_t(y_t)$ achieves its maximum when $y_t = b_t$. Thus, we have

$$g_t(\lambda_t) = G_t(b_t) = 0. \tag{101}$$

- $\frac{\lambda_t - a_{t,1}}{2a_{t,2}} \in (b_t, c_t)$, i.e., $2a_{t,2}b_t + a_{t,1} < \lambda_t < 2a_{t,2}c_t + a_{t,1}$, i.e., $\lambda_t \in (p_t^b, p_t^c)$. In this case, when $y_t \in [b_t, c_t]$, $G_t(y_t)$ achieves its maximum when $y_t = \frac{\lambda_t - a_{t,1}}{2a_{t,2}}$. Thus, we have

$$\begin{aligned} g_t(\lambda_t) &= G_t\left(\frac{\lambda_t - a_{t,1}}{2a_{t,2}}\right) \\ &= \frac{(\lambda_t - a_{t,1})^2}{4a_{t,2}} - \lambda_t b_t + a_{t,2}b_t^2 + a_{t,1}b_t \\ &= q_{t,2}\lambda_t^2 + q_{t,1}\lambda_t + q_{t,0}, \end{aligned} \tag{102}$$

where $q_{t,2}$, $q_{t,1}$, and $q_{t,0}$ are given by

$$q_{t,2} = \frac{1}{4a_{t,2}}, \tag{103}$$

$$q_{t,1} = -\frac{a_{t,1}}{2a_{t,2}} - b_t, \tag{104}$$

$$q_{t,0} = \frac{a_{t,1}^2}{4a_{t,2}} + a_{t,2}b_t^2 + a_{t,1}b_t. \tag{105}$$

- $\frac{\lambda_t - a_{t,1}}{2a_{t,2}} \geq c_t$, i.e., $\lambda_t \geq 2a_{t,2}c_t + a_{t,1} = p_t^c$. In this case, when $y_t \in [b_t, c_t]$, $G_t(y_t)$ achieves its maximum when $y_t = c_t$. In this case, we have

$$g_t(\lambda_t) = G_t(c_t) = \ell_{t,1}\lambda_t + \ell_{t,0}, \tag{106}$$

where $\ell_{t,1}$ and $\ell_{t,0}$ are given by

$$\ell_{t,1} = c_t - b_t, \tag{107}$$

$$\ell_{t,0} = a_{t,2}b_t^2 + a_{t,1}b_t - a_{t,2}c_t^2 - a_{t,1}c_t. \tag{108}$$

Summarizing the above three cases leads to Eq. (39).

Appendix G
Proof of Theorem 9

The proof is based on the online primal-dual analysis. In particular, we can prove that if condition **S1** is satisfied, then $P_0 = D_0 = 0$ and the solutions obtained by Algorithm 1 are feasible since no constraint will be violated. Condition **S2** is obvious since the price cannot be cheaper when more energy have been consumed. In fact, a monotonic pricing function is also key to the incentive compatibility since there is no chance for customers to strategically influence the price for their own benefits in the future. Below we focus on proving **S3**. Let us revisit the primal as follows:

$$\max \quad \sum_{n=1}^{N} v_n x_n - \left(\sum_{t=1}^{T} \hat{f}_t(y_t) - \sum_{t=1}^{T} \hat{f}_t(b_t)\right)\Delta_T \tag{109a}$$

$$s.t. \quad y_t \geq \sum_{n=1}^{N} r_n^t x_n + b_t, \forall t, \quad (\lambda_t) \tag{109b}$$

$$x_n \leq 1, \forall n, \quad (\gamma_n) \tag{109c}$$

$$\text{variables:} \quad x_n \geq 0, y_t \geq b_t, \tag{109d}$$

and the dual:

$$\min \quad \sum_{n=1}^{N} \gamma_n + \left(\sum_{t=1}^{T} g_t(\lambda_t)\right)\Delta_T \tag{110a}$$

$$s.t. \quad \gamma_n \geq v_n - \sum_{t=1}^{T} \lambda_t r_n^t \Delta_T, \forall n, \tag{110b}$$

$$\text{variables:} \quad \lambda_t \geq 0, \forall t; \gamma_n \geq 0, \forall n, \tag{110c}$$

We first calculate the change of the primal objective after processing customer n. Based on the above primal problem, we can calculate the difference between P_n and P_{n-1} as follows:

$$\begin{aligned} & P_n - P_{n-1} \\ &= v_n - \Delta_T \sum_{t=1}^{T} \left(\hat{f}_t(y_t^{(n)}) - \hat{f}_t(y_t^{(n-1)})\right) \\ &\overset{\text{(A)}}{=} \gamma_n + \sum_{t=1}^{T} \lambda_t^{(n-1)} r_n^t \Delta_T - \Delta_T \sum_{t=1}^{T} \left(f_t(y_t^{(n)}) - f_t(y_t^{(n-1)})\right) \\ &\overset{\text{(B)}}{=} \gamma_n + \Delta_T \sum_{t=1}^{T} \Phi_t\left(y_t^{(n-1)}\right)\left(y_t^{(n)} - y_t^{(n-1)}\right) \\ & -\Delta_T \sum_{t=1}^{T} \left(f_t\left(y_t^{(n)}\right) - f_t\left(y_t^{(n-1)}\right)\right), \end{aligned}$$

where (A) comes from constraint (110b) in the dual problem, (B) is because $r_n^t = y_k^{(n)} - y_k^{(n-1)}$ and $\Phi(y_t^{(n-1)})$ denotes the posted-price for customer n, which is calculated based on the power consumption $y_t^{(n-1)}$ after processing the previous customer $n-1$.

Similarly, calculating the change of the dual objective after processing customer n leads to

$$\begin{aligned} & D_n - D_{n-1} \\ &= \gamma_n + \Delta_T \left(\sum_{t=1}^{T} \left(g_t\left(\lambda_t^{(n)}\right) - g_t\left(\lambda_t^{(n-1)}\right)\right)\right) \\ &= \gamma_n + \Delta_T \left(\sum_{t=1}^{T} \left[g_t\left(\Phi_t\left(y_t^{(n)}\right)\right) - g_t\left(\Phi_t\left(y_t^{(n-1)}\right)\right)\right]\right), \end{aligned}$$

To guarantee that the incremental inequality $P_n - P_{n-1} \geq \frac{1}{\alpha}\left(D_n - D_{n-1}\right)$ holds at each round, the following inequality must be satisfied:

$$\begin{aligned} & \sum_{t\in\mathcal{T}} \Phi_t\left(y_t^{(n-1)}\right)\left(y_t^{(n)} - y_t^{(n-1)}\right) - \\ & \sum_{t\in\mathcal{T}} \left(f_t\left(y_t^{(n)}\right) - f_t\left(y_t^{(n-1)}\right)\right) \\ \geq & \frac{1}{\alpha} \sum_{t\in\mathcal{T}} \left[g_t\left(\Phi_t\left(y_t^{(n)}\right)\right) - g_t\left(\Phi_t\left(y_t^{(n-1)}\right)\right)\right]. \end{aligned} \tag{111}$$

Note that $y_t^{(n)} - y_t^{(n-1)} = 0$ for all the time slots with $r_n^t = 0$, we thus have

$$\begin{aligned} & \sum_{t\in\mathcal{T}_n} \Phi_t\left(y_t^{(n-1)}\right)\left(y_t^{(n)} - y_t^{(n-1)}\right) - \\ & \sum_{t\in\mathcal{T}_n} \left[f_t\left(y_t^{(n)}\right) - f_t\left(y_t^{(n-1)}\right)\right] \\ \geq & \frac{1}{\alpha} \sum_{t\in\mathcal{T}_n} \left[g_t\left(\Phi_t\left(y_t^{(n)}\right)\right) - g_t\left(\Phi_t\left(y_t^{(n-1)}\right)\right)\right]. \end{aligned} \tag{112}$$

Our target is to guarantee that the above inequality holds for all possible arrival instances, which thus also includes the cases of all possible $\mathcal{T}_n$'s. Note that in the extreme case, we may have $|\mathcal{T}_n| = 1$, meaning that customer n only purchases the demanded power r_n^t for a single time slot. Therefore, to guarantee the incremental inequality holds for all possible arrival instances, we must have

$$\begin{aligned} & \Phi_t\left(y_t^{(n-1)}\right)\left(y_t^{(n)} - y_t^{(n-1)}\right) - \left(f_t\left(y_t^{(n)}\right) - f_t\left(y_t^{(n-1)}\right)\right) \\ \geq & \frac{1}{\alpha}\left[g_t\left(\Phi_t\left(y_t^{(n)}\right)\right) - g_t\left(\Phi_t\left(y_t^{(n-1)}\right)\right)\right], \end{aligned} \tag{113}$$

which can be equivalently written as follows:

$$\begin{aligned} & \Phi_t(y_t^{(n-1)}) - \frac{f_t(y_t^{(n-1)} + r_n^t) - f_t(y_t^{(n-1)})}{y_t^{(n-1)} + r_n^t - y_t^{(n-1)}} \\ \geq & \frac{1}{\alpha} \cdot \frac{\Phi_t(y_t^{(n)}) - \Phi_t(y_t^{(n-1)})}{y_t^{(n-1)} + r_n^t - y_t^{(n-1)}} \cdot \frac{g_t(\Phi_t(y_t^{(n)})) - g_t(\Phi(y_t^{(n-1)}))}{\Phi_t(y_t^{(n)}) - \Phi_t(y_t^{(n-1)})}. \end{aligned} \tag{114}$$

Since r_n^t is very small compared to the base load b_t, the above equality can thus be written as follows:

$$\Phi_t\left(y_t^{(n-1)}\right) - f_t'\left(y_t^{(n-1)}\right) \geq \frac{1}{\alpha}\Phi_t'\left(y_t^{(n-1)}\right) \cdot g_t'\left(\Phi_t(y_t^{(n-1)})\right)$$

Therefore, if the above inequality holds for any realization of y, namely,

$$\Phi_t\left(y\right) - f_t'\left(y\right) \geq \frac{1}{\alpha}\Phi_t'\left(y\right) \cdot g_t'\left(\Phi_t(y)\right), \forall y \in [b^t, c^t], \tag{115}$$

then the incremental inequality $P_n - P_{n-1} \geq \frac{1}{\alpha}\left(D_n - D_{n-1}\right)$ holds at each round for all possible arrival instances, we thus complete the proof of condition **S3**.

In summary, if the three sufficient conditions are satisfied, then PPM$_{\mathbf{\Phi}}$ is α-competitive and incentive compatible. **We thus complete the proof of the sufficiency in Theorem 1**.

APPENDIX H
PROOF OF THEOREM 10

An online algorithm is α-competitive indicates that the social welfare achieved by this online algorithm is at least $1/\alpha$ of the optimal offline social welfare for all possible arrival instances. In the following we are going to prove that, If there exists an α-competitive online algorithm, then there must exist a monotonically increasing function $y = \Psi_t(p)$ such that

$$\int_{p_t^b}^{\lambda} \eta d(\Psi_t(\eta) - b_t) - f_t\big(\Psi_t(\lambda)\big) + f_t(b_t) \geq \frac{1}{\alpha} g_t(\lambda) \tag{116}$$

holds for all $\lambda \in [p_t^b, \overline{p})$ and $\Psi_t(p_t^b) = b_t$.

Let us construct a special arrival instance $\mathcal{A}_\lambda$ as follows. Suppose all customers arrive before t and want to purchase power for this single time slot. For any $\lambda \geq p_t^b$, let us assume for each $\eta \in [p_t^b, \lambda]$, there is a continuum of customers indexed by η, where each customer η has a total power demand of $g_t'(\eta)$ and the VER of this customer is η, i.e., $v_\eta = \eta \cdot g_t'(\eta)\Delta_T$.

Note that $g_t'(\eta) + b_t$ is the maximum power that can be provided on when the marginal cost is η per unit, and thus $g_t'(\eta)$ is the maximum power that can be sold for the customers in instance $\mathcal{A}_\lambda$. When $\eta \in [p_t^b, p_t^c]$, $g_t'(\eta) = \frac{\eta - a_{t,1}}{2a_{t,2}} - b_t = f_t'^{-1}(\eta) - b_t$; when $p \in (p_t^c, +\infty)$, $g_t'(\eta) = c_t - b_t$.

For the arrival instance $\mathcal{A}_\lambda$, the optimal offline result in hindsight is to allocate $g_t'(\lambda)$ unit of power to the last customer λ and none to all the previous continuum of customers. The optimal social welfare is thus

$$W_{\text{opt}} = \lambda \cdot g_t'(\lambda) \cdot \Delta_T - f_t\left(g_t'(\lambda) + b_t\right) \cdot \Delta_T = g_t(\lambda) \cdot \Delta_T. \tag{117}$$

Note that the output of g_t means the cost per hour, i.e., \$/hour, and thus W_{opt} means the total monetary welfare.

For the online algorithm, let $y = \Psi_t(\eta)$ denote the total power consumption after processing customer η, and thus $\Psi_t(\eta) - b_t$ represents the TE sold to the continuum of customers in $[p_t^b, \eta]$. Intuitively, $\Psi_t(\eta) \geq b_t$ always holds and $\Psi_t(\eta)$ is monotonically non-decreasing in $\eta \in [p_t^b, \lambda]$. The social welfare achieved by this online algorithm is thus the total valuation minus the total cost, namely,

$$\begin{aligned} & W_{\text{alg}} \\ &= \int_0^{\Psi_t(\lambda) - b_t} \eta \Delta_T d(\Psi_t(\eta) - b_t) - \Delta_T\left(f_t\big(\Psi_t(\lambda)\big) - f_t(b_t)\right) \\ &= \Delta_T\left(\int_{p_t^b}^{\lambda} \eta \Psi_t'(\eta) d\eta - f_t\big(\Psi_t(\lambda)\big) + f_t(b_t)\right) \end{aligned} \tag{118}$$

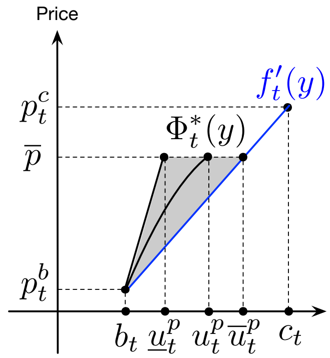

Fig. 9. Optimal pricing functions for the case when $\overline{p} \le p_t^c$. In this case, the optimal pricing function is not unique. In particular, for any $u_t^p \in [\underline{u}_t^p, \overline{u}_t^p]$, we can find a unique optimal pricing function that achieves an optimal competitive ratio of 4.

The online algorithm is α-competitive means that

$$\int_{p_t^b}^{\lambda} \eta \Psi_t'(\eta) d\eta - f_t\big(\Psi_t(\lambda)\big) + f_t(b_t) \ge \frac{1}{\alpha} g_t(\lambda) \tag{119}$$

holds for all $\lambda \ge p_t^b$. Therefore, if there exists an α-competitive online algorithm, then the above integral inequality must hold for all $\lambda \in [p_t^b, \overline{p}]$. According to the definition of Ψ_t, we have $\Psi_t(p_t^b) = b_t$ and $\Psi_t(\lambda) \le c_t$ always holds, namely, $\Psi_t(\overline{p}) \le c_t$. We thus complete the proof of Theorem 10.

Note that the difference between the above integral inequality (119) and the differential inequality (115) is that one is in the form of $\lambda = \Phi_t(y)$, while the other is in the form of $y = \Psi_t(\lambda)$. Therefore, if we can find a monotonically increasing function $\lambda = \Phi_t(y)$ based on the sufficient conditions in Theorem 9, then $y = \Phi_t^{-1}(\lambda) = \Psi_t(\lambda)$ is the monotonically increasing function that satisfies the conditions in Theorem 10.

APPENDIX I
RELAXATION OF THE ASSUMPTION OF $\overline{p} > \max_t\{p_t^c\}$

Recall that in the paper we assume $\overline{p} > \max_t\{p_t^c\}$. Here in this section we discuss how to relax this assumption by adding one more case to our optimal pricing function design.

For any time slot $t \in \mathcal{T}$, when $\overline{p} \le p_t^c$, there exists no such a threshold $u_t \in (b_t, c_t)$ such that $\Phi_t(u_t) = p_t^c$. Therefore, in this case the optimal pricing function reduces to only one segment, as illustrated in Fig. 9.

In particular, when $\overline{p} \le p_t^c$, the optimal pricing function is not unique. Specifically, when $u_t = \frac{b_t + c_t}{2}$, the linear solution in Eq. (67) is given by

$$\Phi_{t,1}(y) = \frac{p_t^c - p_t^b}{u_t - b_t}(y - b_t) + f_t'(b_t) = 4a_{t,2}(y - b_t) + p_t^b. \tag{120}$$

Therefore, when we define

$$\underline{u}_t^p \triangleq \frac{\overline{p} - p_t^b}{4a_{t,2}} + b_t \tag{121}$$

Then, the following pricing function

$$\Phi_t^*(y) = 4a_{t,2}(y - b_t) + p_t^b, y \in [b_t, \underline{u}_t^p] \tag{122}$$

achieves the optimal competitive ratio of 4. Moreover, when we define

$$\overline{u}_t^p \triangleq \frac{\overline{p} - a_{t,1}}{2a_{t,2}}, \tag{123}$$

then for any $u_t^p \in [\underline{u}_t^p, \overline{u}_t^p]$, we can find a unique pricing function that satisfies the following BVP

$$\begin{cases} \Phi_t'(y) = 4 \cdot \frac{\Phi_t(y) - f_t'(y)}{f_t'^{-1}(\Phi_t(y)) - b_t}, y \in (b_t, u_t^p), \\ \Phi_t(b_t) = f_t'(b_t), \Phi_t(u_t^p) = \overline{p}, \end{cases} \tag{124}$$

and the unique solution to the above BVP is 4-competitive. We can solve the above BVP in the similar way as Case-2 in Theorem 8, and thus the details are eliminated for brevity.

In summary, when $\overline{p} > \max_t\{p_t^c\}$ does not hold, then for any time slot $t \in \mathcal{T}$ with $\overline{p} \le p_t^c$, the optimal pricing function $\Phi_t^*(y)$ can be obtained by solving the above BVP in Eq. (124). In particular, the optimal pricing function in this case is not unique.