1 Systems of Linear Equations

1.1 Definitions

- A **linear equation** in variables $x_1, \ldots, x_n$ is of the form $a_1x_1 + x_2x_2 + \ldots a_nx_n = b$ where $a_1, \ldots, a_n, b$ are numbers.
- A **system** of linear equations in variables $x_1, \ldots, x_n$ is a set of one or more linear equations.
- A **solution** to a system of linear equations is a set of numbers such that when plugged in for the variables, the equations are satisfited.
- Two systems of linear equations sharing the same solution(s) are considered to be **equivalent**.
- A system with any nonzero number of solutions is **consistent**.
- A system of linear equations with no solutions is **inconsistent**.
- A **coefficient matrix** is a matrix holding the coefficients of a system of linear equations.
- An **augmented matrix** is a coefficient matrix with an additional column on the right holding the constants of each linear equation.

1.2 Lecture Material

There are three basic operations which can be done on a system of linear equations called **elementary row operations**:

1. **Interchange** the order of two rows in the system.
2. **Scale** all terms in a row by the same nonzero constant.
3. **Replace** a row by adding a multiple of another row to it.

2 Row Reduction and Echelon Forms

2.1 Definitions

- The **leading entry** of a row is the first nonzero term in that linear equation.
- A matrix is in **echelon form** if it has the following properties:
    1. Any rows of all zero are at the bottom.
    2. The leading entry of each row is to the right of the leading entry to each row above it.

- A matrix is in **reduced echelon form** if it is in echelon form and has the following additional property:

    1. The leading entry in each nonzero row is 1 and also the only nonzero entry in that column.

- A **pivot position** in matrix M is a position in M that holds a leading 1 of the corresponding reduced echelon matrix of M.
- A **pivot column** is a column that contains a pivot position
- A **basic variable** is a variable in a pivot column.
- A **free variable** is a variable not in a pivot column. This means that the variable can be set to any value and still be part of a solution.
- A **parametric description** of a solution set is one in which the basic variables are described in terms of the free variables and the free variables are listed as such. An empty solution set has no parametric description.

2.2 Lecture Material

Theorem 1 *There is only one reduced echelon matrix corresponding to each matrix.*

The **row reduction algorithm** is a five-step plan to acing your linear algebra course:[1]

1. The leftmost nonzero column is a pivot column with the pivot position in the first row.
2. Interchange rows such that a non-zero entry is in the pivot position.
3. Use replacement operations to set all entries below the pivot to 0.
4. Ignore everything at or above the row that was just operated on. If the matrix is not in echelon form, go back to step one.
5. Starting from the bottommost row, use replacement operations to set all entries above the pivot to 0. Repeat this until the matrix is in reduced echelon form.

Theorem 2 *A linear system is consistent if and only if none of the equations (in echelon form) describe zero to be equal to a nonzero constant. A consistent linear system contains one unique solution if and only if there are no free variables, otherwise, it has an infinite number of solutions.*

[1] If you don't think about this too much it'll make you feel better.

The following steps can be done to find all the solutions of a linear system with row reduction:

1. Use row reduction to obtain an equivalent matrix in echelon form. If there is no solution (this can be determined with Theorem 2), stop here, do not pass go, and do not collet $200.

2. Continue row reduction to obtain the reduced echelon form.

3. Write the system of equations that corresponding to the reduced echelon matrix.

4. Solve each basic variable in terms of the free variables.