Algebra III: Groups

MATH 341

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Contents

1 Assorted Introductions

Author's Remark

This section is weird because most of the material was already covered either in MATH 340 or MATH 440. That it to say, there is material that I am too lazy to add.

Theorem 1.1 (Internal Characterization). *For $G_1, G_2 \subseteq G$ groups, $G \cong G_1 \times G_2$ if and only if all the following apply:*

(i) $G = \{g_1 g_2 : g_1 \in G_1, g_2 \in G_2\}$,

(ii) $G_1 \cap G_2 = \{e_G\}$, *and*

(iii) $g_1 g_2 = g_2 g_1$ *for all* $g_1 \in G_1$ *and* $g_2 \in G_2$.

Proof. Tedious rule checking. □

Theorem 1.2 (Cayley). *Every finite group of order n is isomorphic to some subgroup of S_n.*

Proof. Let G be a finite group of order n. Define $\phi : G \to S_n$ as $\phi(g) = \sigma_g$ where $\sigma_g(h) = gh$, an isomorphism. □

Theorem 1.3 (Lagrange). *Let $H \subseteq G$ be groups (not necessarily finite). Then, $|G| = [G : H]|H|$ (with cardinality if infinite).*

Proof. We prove only for the finite case, by seeing that cosets partition the group, and that all cosets are of the same size. □

Theorem 1.4 (Cauchy). *Let G be a finite group of order n. If a prime p divides n, there exists an element of order p.*

Proof. Define $X = \{(x_1, \dots, x_p) \in G^p : x_1 \cdots x_p = e\}$ and see that x_p is determined entirely by the choices of $x_1, \dots, x_{p-1}$. Since $x_1, \dots, x_{p-1}$ can be chosen arbitrarily, $|X| = n^{p-1}$.

Let $\mathbb{Z}_p$ act on X by cyclic permutation of the p-tuple. Since stabilizers are subgroups, the orbit-stabilizer theorem says that all orbits of X are size either 1 or p. We note an orbit of some $(x_1, \dots, x_p) \in X$ is size 1 if and only if $x_1 = \cdots = x_p$, id est, x_1 is of order p or $x_1 = e$. Finally, since $|X|$ is a multiple of p, the class equation says there must be at least p elements of with an orbit of size 1, hence $p - 1$ elements of order p. □

Theorem 1.5. *Let $H \subseteq G$ be a normal subgroup. Then, the quotient set G/H has a group structure.*

Proof. Tedious rule checking. □

Theorem 1.6. *A group $H \subseteq G$ is normal if and only if $gHg^{-1} \subseteq H$ for all $g \in G$.*

Proof. The "only if" is trivial, so we prove the "if" direction by showing $gHg^{-1} \supseteq H$. We see $gHg^{-1} \subseteq H \implies g^{-1}Hg \subseteq H$. For some $ghg^{-1} \in H$, we have $ghg^{-1} = h' \iff h = g^{-1}h'g$ for some $h' \in H$. But then $g^{-1}h'g \in g^{-1}Hg$ so $h = gh'g \in H$. $\square$

Theorem 1.7. *Subgroups of index 2 are normal.*

Proof. Let $H \subseteq G$ be a subgroup of index 2. Then, the left cosets of H are H, gH and the right cosets of H are H, Hg, so $gH = Hg$. $\square$

Theorem 1.8 (First Isomorphism)**.** *Let $\phi : G \to H$ be a surjective homomorphism. Then, $H \cong G/\ker(\phi)$.*

Proof. Tedious rule checking. $\square$

Corollary 1.8.1. *Let $\phi : G \to H$ be a homomorphism. Cosets of* $\ker(\phi)$ *contain all values mapping to some element in the codomain.*

Remark 1.1. Normal subgroups of a group are exactly all possible kernels of homomorphisms from that group.

2 Alternating Groups

Definition 2.1. A **transposition** in a symmetric group is a cycle with support size 2.

Remark 2.1. Every permutation can be written (not uniquely) as a product of transpositions.

Theorem 2.1. *For some permutation σ, all writings of σ as a product of transpositions has the same parity of the number of factors.*

Proof. Let S_n act on $f = \prod_{i<j}(x_i - x_j) \in F[x_1, \dots, x_n]$ by permuting the x's, and see that every transposition negates f. $\square$

Remark 2.2. The parity of a permutation σ is the number of transpositions in a 2-cycle factorization of σ.

Definition 2.2. The **alternating group** on n elements, A_n, is the set of even permutations.

Definition 2.3. The **cycle structure** of a permutation σ is the list of cycle lengths in σ sorted in non-increasing order.

Theorem 2.2 (Conjugation). *Every conjugate of a permutation has the same cycle structure.*

Proof. Let $\alpha \in S_n$ and see $\alpha(a_1 \cdots a_k)\alpha^{-1} = (\alpha(a_1) \cdots \alpha(a_k))$. $\square$

Theorem 2.3. *Every pair of permutations of the same cycle structure is conjugates in S_n.*

Definition 2.4. A group is **simple** if the only normal subgroups are the trivial group or itself.

Theorem 2.4. *The group A_n is simple if and only if $n \neq 4$.*

Proof sketch. Let $H \subseteq A_n$ be proper normal, and choose $\alpha \in H$ of minimal support. Claim α must be a cycle of length 3, so normality forces H to have all cycles of length 3, which generates A_4. $\square$

3 Sylow Theorems

Definition 3.1. For prime p, a p-**group** is one which only contains elements of order a power of p.

Theorem 3.1. *A finite group is a p-group if and only if it is of order a power of p.*

Theorem 3.2. *Let G be an abelian group with $|G| = p^n m$ where $\gcd(p, m) = 1$. Defining P as all elements of order a power of p and M as all other elements, we see $G \cong P \times M$.*

Corollary 3.2.1. *Suppose G is finite abelian of order $p_1^{a_1} \cdots p_n^{a_n}$ with $p^i \neq p_j$ for $i \neq j$, then $G \cong P_1 \times \cdots \times P_n$.*

Definition 3.2. Let G act on X. We say $x, y \in X$ are **G-equivalent** if there exists $g \in G$ such that $g(x) = y$.

Definition 3.3. Let G act on X, the subgroups of G, by conjugation. The **normalizer** of $H \in G$ is $N(H) = \{g \in G : g(H) = H\}$, and is the largest subgroup of G in which H is normal.