Galois Theory

MATH 440

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Galois Theory is the study of symmetries among roots of polynomials.

— Professor (Spring 2023)

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1 Introduction

Definition 1.1. The **degree** of K over L is written $[K : L]$.

Proposition 1.1. *Let $F \subseteq K$ and $K \subseteq L$ be field extensions. Then, $[L : F] = [L : K][K : F]$.*

Proof. We may assume $[L : K]$ and $[K : F]$ are finite. Then, L has K-basis $\{a_1, \ldots, a_k\}$ and K has F-basis $\{b_1, \ldots, b_f\}$. We can show $\{a_i b_j\}$ is an F-basis of L. $\square$

Definition 1.2. A field extension $F \subseteq K$ is **finite** if the degree of K over F is finite.

Definition 1.3. A field extension $F \subseteq K$ is **finitely generated** if there exists a finite set S such that $F(S) = K$.

Proposition 1.2. *A $F \subseteq K$ is finitely generated if it is finite.*

Definition 1.4. For $F \subseteq K$, some $\alpha \in K$ is **algebraic** over F if there exists a non-constant $f \in F[x]$ such that $f(\alpha) = 0$.

Definition 1.5. The minimal polynoimial of α over F is written $m_{\alpha,F}(\alpha) = 0$. Then, the degree of α over F is $\deg(m_{\alpha,F})$.

Definition 1.6. A $F \subseteq K$ is an algebraic field extension if every $\alpha \in K$ is algebraic.

Theorem 1.1. *A $F \subseteq K$ is finite if and only if it is finitely generated and algebraic.*

Proof. Suppose $F \subseteq K$ is finite. We will show K is algebraic over F (finitely generated follows from Proposition 1.2). Let $\alpha \in K$ be nonzero and see that $\alpha^0, \ldots, \alpha^m \in K$ is linearly dependent if $m \geq \deg(m_{\alpha,F}) = [K : F]$.

Now, suppose $K = F(\alpha_1, \ldots, \alpha_m)$ is algebraic and define $K_i = K_{i-1}(\alpha_i)$ with $K_0 = F$. By an implicit induction on i, we see that $K_m = K$ is finite. $\square$