

Algebra II: Rings and Fields

MATH 340

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1 Foundational Definitions

Definition 1.1. A **group** is a pairing of a set and a binary operation such that the operation is associative, and each element of the set has both an inverse and an identity.

Remark 1.1.1. Considering an identity element is defined both as a left and a right identity, it can be proven that it is unique for the group.

Remark 1.1.2. Similarly, we can prove that the inverse is unique for every element of the set.

Definition 1.1.1. A **commutative group** is a group where the binary operation is commutative.

Definition 1.2. A **ring** is a commutative group with another operation defined such that the two operations are similar to “addition” and “multiplication” of the integers. The multiplication operation must be associative and distributive.

Definition 1.2.1. A **commutative ring** is a ring where multiplication is commutative.

Definition 1.2.2. We say that a ring has **unity** if there is a multiplicative identity.

Remark 1.2.1. For any ring R with additive identity 0 , it can be proven that $0a = 0$ for all $a \in R$.

Remark 1.2.2. Using Remark 1.2.1, it can be proven that $-ab = (-a)b = a(-b)$ for all $a, b \in R$.

Definition 1.2.3. For some non-zero $a, b \in R$, we say a and b are **zero divisors** if $ab = 0$.

Remark 1.2.3. For some non-zero $a \in R$, it can be proven that a is a left zero divisor if and only if there exists non-zero $b, c \in R$ such that $b \neq c$ and $ab = ac$.

Remark 1.2.4. It follows from Remark 1.2.3, that if a ring R does *not* have any zero divisors, then $ab = ac \implies b = c$ for all $a, b, c \in R$ and $a \neq 0$.

Definition 1.2.4. A **unit** in a ring with unity is an element which has a multiplicative inverse.

Remark 1.2.5. A unit cannot be a zero-divisor.

Definition 1.3. An **integral domain** is a commutative ring with unity and no zero divisors.

Definition 1.4. A **field** is a commutative ring where every non-zero element is a unit, and the additive and multiplicative identities are not equal.

2 Exploration

Definition 2.1. The **characteristic** of a ring is the lowest integer c such that $\underbrace{1 + 1 + \cdots + 1}_{c \text{ times}} = 0$.

Theorem 2.1.1. *If the characteristic of a ring is composite, it must have zero divisors.*

Proof. Let c be the characteristic of some ring where there exists positive integers m, n such that $c = mn$ and $m, n < c$. Consider, using the distributivity of multiplication, that

$$\underbrace{(1 + 1 + \cdots + 1)}_{m \text{ times}} \underbrace{(1 + 1 + \cdots + 1)}_{n \text{ times}} = 0. \quad \square$$

Theorem 2.1.2 (Euler's Theorem). *Let R^* be the finite set of the units in a ring. For all $a \in R^*$, $a^{|R^*|} = 1$.*

Proof. We have $R^* = \{r_1, \dots, r_n\} = \{ar_1, \dots, ar_n\}$ since a is not a zero-divisor (it is a unit) so $ar_i \neq ar_j$. Then, $r_1 \cdots r_n = (ar_1) \cdots (ar_n) = a^n(r_1 \cdots r_n) \implies a^n = 1$. \square

Theorem 2.1.3. *For a finite ring with unity, any element is either 0, a zero divisor, or a unit.*

Proof. For an element r that is not zero or a zero divisor, we have the following set of non-zero elements $\{r, r^2, \dots\}$. Since the ring is finite, we have $r^{e_1} = r^{e_2}$ for some $e_1 < e_2$. Then, $r^{e_1} = r^{e_2} = r^{e_1} r^{e_2 - e_1} \implies r^{e_2 - e_1} = 1$. Therefore, $r \cdot r^{e_2 - e_1 - 1} = 1$. \square

Remark 2.1.1. It follows from Theorem 2.1.3 that every finite integral domain is a field.

3 Unique Factorization

Definition 3.1. A **quadratic ring extension** $R[\gamma]$ of some ring R is created by adding an element γ to R such that $\gamma^2 = c$ for some $c \in R$ and $\gamma \notin R$.

Remark 3.1.1. Elements in $R[\gamma]$ are denoted $a + \gamma b$ for $a, b \in R$. This means elements in $R[\gamma]$ can be seen as elements in $R \times R$.

Theorem 3.1.1. *The norm map¹ $N : R[\gamma] \rightarrow R$ is defined as $N(a + \gamma b) = a^2 - cb^2$ and has the property that $N(a + \gamma b)$ is a unit in R if and only if $a + \gamma b$ is a unit in $R[\gamma]$.*

Proof. We see that $N(a + \gamma b)^{-1}$ exists if and only if $N(a + \gamma b)$ is a unit. Then, $(a + \gamma b)(a - \gamma b) = N(a + \gamma b)$ so $(a + \gamma b) [(a - \gamma b)N(a + \gamma b)^{-1}] = 1$. □

Remark 3.1.2. Theorem 3.1.1 shows the quadratic ring extension of any field or integral domain maintains that status.

Definition 3.2. An element in an integral domain is called **irreducible** if it cannot be written as a product of two non-units.

Definition 3.3. Elements a, b in an integral domain R are called **associates** if there exists a unit $u \in R$ such that $a = ub$.

Definition 3.4. An integral domain has **unique factorization** if every element can be written as a product of irreducibles which are unique up to order and associates.

Theorem 3.4.1. *An integral domain R has unique factorization if all irreducible elements are prime.*

¹We have not yet formally defined a *norm map*.

Proof. Let $x = a_1 \cdots a_n = b_1 \cdots b_m$. Since a_1 is prime, we know that it divides one of b_i . Without loss of generality, let $b_1 = ca_1$. However, since b_1 is irreducible and $a_1 \neq 1$, we have $c = 1$. Then, we can repeat this process on $a_2 \cdots a_n = b_2 \cdots b_m$. \square

Definition 3.5. A **polynomial ring** $R[x]$ of some ring R is created by using polynomials of the variable x using coefficients from R .

Remark 3.5.1. For some field F , Euclidean division works on $F[x]$ because all non-zero coefficients are units. It then follows that irreducible elements are prime, so unique factorization exists in $F[x]$.

Theorem 3.5.1 (Fundamental Theorem of Algebra). *The only irreducible polynomials in $\mathbb{C}[x]$ are linear.*

Remark 3.5.2. It follows from Theorem 3.5.1 that the only irreducible polynomials in $\mathbb{R}[x]$ are linear or quadratic. This can be proven using $\mathbb{R}[x] \subset \mathbb{C}[x]$ and that multiplying some linear $f(x) \in \mathbb{C}[x]$ with its conjugate results in some $g(x) \in \mathbb{R}[x]$ with $\deg(g(x)) = 2$.

Definition 3.6. A subring I of ring R is called an **ideal** if for all $a \in I$ and $r \in R$, $ra, ar \in I$.

Definition 3.7. For a commutative ring R and $a \in R$, a **principal ideal** generated by a is defined as $aR = \{ar : r \in R\}$. For $a, b \in R$, we can also generate $(a, b)R = \{xa + yb : x, y \in R\}$.

Remark 3.7.1. For $a \in R$ with integral domain R , $aR = 1R = R$ if and only if a is a unit.

Remark 3.7.2. For $a, b \in R$, $b \mid a \implies aR \subseteq bR$. Furthermore, $aR = bR$ if and only if a and b are associates.

Remark 3.7.3. For $a, b \in R$, if a is irreducible and $b \mid a$, then $aR \subseteq bR \subseteq R$ so either $aR = bR$ or $bR = R$. Therefore, aR is not properly contained in any other principal ideal. Also, if a is not irreducible and b is not a unit, then $aR \subset bR \subset R$.

Theorem 3.7.1. *If an element $a \in R$ cannot be written as a finite product of irreducibles, then R has an infinite ascending chain of principal ideals.*

Proof. Assume that a cannot be written as a finite product of irreducibles. Then, $a = r_1 a_1 = r_1 r_2 a_2 = \dots$ for non-units r_i, a_i and reducible a_i . This implies $aR \subset a_1 R \subset a_2 R \subset \dots$. \square

Remark 3.7.4. This tells us that every element in \mathbb{N} has a factorization into irreducibles since every proper divisor is “smaller” so there cannot be an infinite chain.

Definition 3.8. An integral domain R is a **principal ideal domain** if every ideal in R is a principal ideal.

Proposition 3.8.1. *The ring \mathbb{Z} is a principal ideal domain.*

Proof. If $I = \{0\}$, then $I = 0\mathbb{Z}$. Therefore, we prove with $I \neq \{0\}$. Then, there exists a positive element in I . Let a be the least positive element in I and we claim that $I = a\mathbb{Z}$.

Let $b \in I$ be some other element in I . Then we have $b = qa + r$ for $0 \leq r < a$. This also means $b - qa = r$ so $r \in I$. However, by the minimality of a , this implies $r = 0$ so b is a multiple of a and $b \in a\mathbb{Z}$. \square

Corollary 3.8.1. *For field F , $F[x]$ is a principal ideal domain.*

Theorem 3.8.1. *For a principal ideal domain, every ascending chain of ideals stabilizes.*

Proof. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of ideals in a principal ideal domain R . Then, $\bigcup_{i=1}^{\infty} I_i$ is a principal ideal aR . For some j , $a \in I_j$ so $aR = I_j = I_{j+1} = \cdots$. \square

Definition 3.9. Let I, J be ideals of R . Then, $I + J$ is the smallest ideal which contains both I and J . Therefore, $I + J = \{a + b : a \in I \text{ and } b \in J\}$.

Remark 3.9.1. Since \mathbb{Z} is a principal ideal domain, $a\mathbb{Z} + b\mathbb{Z} = d\mathbb{Z}$. Then, $d\mathbb{Z} = \{xa + yb : x, y \in \mathbb{Z}\}$. Therefore, $d = \gcd(a, b)$ since it is the least positive element (by proof of Theorem 3.8.1).

Definition 3.10. An ideal I of ring R is a **prime ideal** if $ab \in I$ implies $a \in I$ or $b \in I$ for all $a, b \in R$.

Remark 3.10.1. An element $p \in R$ is prime if and only if pR is prime. Why?

Remark 3.10.2. Not all prime ideals are principal (eg. $(x, y) \subset \mathbb{Q}[x, y]$).

Definition 3.11. An ideal I in ring R is called **maximal** if for any ideal $J \subseteq R$ where $I \subseteq J \subseteq R$, it follows that $I = J$ or $J = R$.

Remark 3.11.1. In a principal ideal domain, the principal ideal generated by an irreducible element is maximal.

Theorem 3.11.1. In an integral domain, maximal ideals are prime.

Proof. Let I be a maximal ideal of ring R with $bc \in I$. Suppose $b \notin I$. Then, we have $I \subsetneq I + bR \subseteq R$ so, by the maximality of I , $I + bR = R$. This also means that $1 \in I + bR$ so $1 = a + br$ for $a \in I$ and $r \in R$. Multiplying through by c , this gives us $c = ac + bcr \in I$ since $a, bc \in I$. \square

Remark 3.11.2. For a principal ideal domain R , this gives us that $a \in R$ is irreducible implies aR is maximal implies aR is prime implies a is prime. Therefore, by Theorem 3.4.1, every principal ideal domain has unique factorization.

Definition 3.12. A ring is a unique factorization domain if every non-zero non-unit can be written uniquely as a product of irreducible elements, up to order and associates. Duplicate of Definition 3.4; don't ask why.

Remark 3.12.1. Not all unique factorization domains are principal ideal domains. (eg. $\mathbb{Z}[x]$ with $2\mathbb{Z}[x] + x\mathbb{Z}[x]$).

4 Ring Homomorphisms

Definition 4.1. A **ring homomorphism** for rings R, S is a map $\varphi : R \rightarrow S$ satisfying $\varphi(a + b) = \varphi(a) + \varphi(b)$ and $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in R$.

Definition 4.2. If there exists a $\varphi^{-1} : S \rightarrow R$ for a ring homomorphism $\varphi : R \rightarrow S$ such that $\varphi^{-1}(\varphi(a)) = a$ for all $a \in R$, we call R and S **isomorphic**.

Remark 4.2.1. We can show the inverse of a bijective ring homomorphism $\varphi : R \rightarrow S$ is also a ring homomorphism, therefore, R and S are isomorphic.

Remark 4.2.2. For ring homomorphism $\varphi : R \rightarrow S$, we have $\varphi(0_R) = 0_S$, $\varphi(-a) = -\varphi(a)$ for all elements $a \in R$, and $\varphi(a^{-1}) = \varphi(a)^{-1}$ for all units $a \in R$. We can also show $\varphi(R) = \{\varphi(r) : r \in R\}$ is a subring of S .

Definition 4.3. The **kernel** of $\varphi : R \rightarrow S$ is defined as $\ker(\varphi) = \{\varphi(r) = 0_S : r \in R\}$.

Remark 4.3.1. We can show that $\ker(\varphi)$ is an ideal.

Theorem 4.3.1. *A ring homomorphism $\varphi : R \rightarrow S$ is injective if and only if $\ker(\varphi) = \{0_R\}$.*

Proof. Suppose, towards a contradiction, that φ is not injective, that is, there exists $a, b \in R$ satisfying $\varphi(a) = \varphi(b)$ and $a \neq b$. Then, $a - b \in \ker(\varphi)$ but $a - b \neq 0_R$. \square

Theorem 4.3.2. *Let R be a commutative ring with unity and $\varphi : R \rightarrow S$ be a surjective ring homomorphism. Then, S is an integral domain if and only if $\ker(\varphi)$ is a prime ideal.*

Proof. Let $a, b \in S$ with $\varphi(x) = a$ and $\varphi(y) = b$. Suppose $\ker(\varphi)$ is a prime ideal. Then, $ab = 0 \implies \phi(x)\phi(y) = 0 \implies \phi(xy) = 0$ so either $x \in \ker(\varphi)$ or $y \in \ker(\varphi)$. Without loss of generality, let $x \in \ker(\varphi)$ so $\varphi(x) = a = 0$.

Suppose $\ker(\varphi)$ is not a prime ideal. There exists $xy \in \ker(\varphi)$ such that $x \notin \ker(\varphi)$ and $y \notin \ker(\varphi)$. Then, $0 = \varphi(xy) = \varphi(x)\varphi(y) = ab = 0$ but $\varphi(x) \neq 0$ and $\varphi(y) \neq 0$. \square

Theorem 4.3.3. *Let R be a commutative ring with unity, S a non-zero ring, and $\varphi : R \rightarrow S$ a surjective ring homomorphism. Then, S is a field if and only if $\ker(\varphi)$ is a maximal ideal.*

Proof. Suppose $\ker(\varphi)$ is maximal. Let $b \in S$ be non-zero and define $J = \varphi^{-1}(bS) = \{\varphi(r) \in bS : r \in R\}$. We can show that J is an ideal satisfying $\ker(\varphi) \subseteq J \subseteq R$. Then, by maximality of J , $\ker(\varphi) = J$ or $J = R$. However, since $b \neq 0$, $J = R$ so b is a unit. It then follows that S is a field.

Now suppose $\ker(\varphi)$ is not maximal, that is, there exists an ideal I such that $\ker(\varphi) \subset I \subset R$. Pick $y \in I \setminus \ker(\varphi)$ and assume, for a contradiction, that S is a field. Then, there exists $x \in R$ such that $\varphi(x)\varphi(y) = 1_S$ so $\varphi(xy) = \varphi(1_R)$. It follows

that $xy - 1_R = r \in \ker(\varphi)$ so $1_R = xy - r \in I$ since $xy, r \in I$. Therefore, $I = R$, a contradiction. \square

Theorem 4.3.4 (First Isomorphism Theorem). *If $\varphi : R \rightarrow S_1$ and $\psi : R \rightarrow S_2$ are surjective ring homomorphisms with $\ker(\varphi) = \ker(\psi)$, there is a ring isomorphism $\sigma : S_1 \rightarrow S_2$ where $\sigma(\varphi(x)) = \psi(x)$ for all $x \in R$.*

Proof. Since σ is a composition of ring homomorphisms, it itself is a ring homomorphism. Therefore, it suffices to show σ is both injective and surjective.

Let $x, y \in R$ such that $\sigma(\varphi(x)) = \sigma(\varphi(y))$. Then, $\psi(x) = \psi(y)$ so $\psi(x - y) = 0_{S_2}$ and $x - y \in \ker(\psi)$. Since $\ker(\psi) = \ker(\varphi)$, $x - y \in \ker(\varphi)$ so $\varphi(x) = \varphi(y)$. Therefore, σ is injective.

For any $b \in S_2$, we have $\psi(x) = b$ for some $x \in R$. Then, $\sigma(\varphi(x)) = b$ so σ is surjective. \square

Remark 4.3.2. It follows from Theorem 4.3.4 that, for a surjective homomorphism $\varphi : R \rightarrow S$, S is entirely determined (up to isomorphism) by R and $\ker(\varphi)$.

Definition 4.4. For ring R , ideal $I \subseteq R$, and $a \in R$, the **coset** of a modulo I is defined $a + I = \{a + x : x \in I\} = [a]_I$.

Theorem 4.4.1. *Let $a + I$ and $b + I$ be cosets. Then, either $a + I = b + I$ or $(a + I) \cap (b + I) = \emptyset$.*

Proof. Suppose $(a + I) \cap (b + I) \neq \emptyset$ and let $c \in (a + I) \cap (b + I)$. Then, $c = a + x = b + y \implies a - b = y - x \in I$. Now, for all $a + z \in a + I$, $a + z - (a - b) = z + b \in b + I$. \square

Definition 4.5. The **quotient ring** for ring R by an ideal $I \subseteq R$ is denoted $R/I = \{a + I : a \in R\}$ and represents the collection of cosets modulo I .

Remark 4.5.1. It follows from Theorem 4.4.1 that the map for $\varphi : R \rightarrow R/I$ where $a \mapsto a + I$ is well defined. Furthermore, we can show φ is a surjective ring homomorphism with $\ker(\varphi) = I$.

Theorem 4.5.1 (Chinese Remainder Theorem). *Let R be a commutative ring with unity and $I, J \subset R$ be ideals satisfying $I + J = R$. Then, there is an isomorphism between $R/(I + J)$ and $R/I \times R/J$ defined $a + (I \cap J) \mapsto (a + I, a + J)$.*

Proof. We explain the isomorphism via Theorem 4.3.4. Define $\varphi : R \rightarrow R/(I \cap J)$ as $\varphi(a) \mapsto a + (I \cap J)$ and $\psi : R \rightarrow R/I \times R/J$ as $\psi(a) \mapsto (a + I, a + J)$. Then, $\ker(\varphi) = \ker(\psi)$ since $c \in (I \cap J) \iff c \in I$ and $c \in J$. Also, φ is surjective by definition. Therefore, it suffices to show ψ is surjective.

INCOMPLETE!! :D

□