# Algebra III: Groups

## MATH 341

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## Contents



# 1 Assorted Introductions

**Author's Remark**

This section is weird because most of the material was already covered either in MATH 340 or MATH 440. There is missing material, and I was simply too lazy to add it.

**Theorem 1.1** (Internal Characterization). *For $G_1, G_2 \subseteq G$ groups, $G \cong G_1 \times G_2$ if and only if all the following apply:*

*(i)* $G = \{g_1 g_2 : g_1 \in G_1, g_2 \in G_2\}$,
*(ii)* $G_1 \cap G_2 = \{e_G\}$, *and*
*(iii)* $g_1 g_2 = g_2 g_1$ *for all* $g_1 \in G_1$ *and* $g_2 \in G_2$.

*Proof.* Tedious rule checking. $\square$

**Theorem 1.2** (Cayley). *Every finite group of order $n$ is isomorphic to some subgroup of $S_n$.*

*Proof.* Let $G$ be a finite group of order $n$. Define $\phi : G \to S_n$ as $\phi(g) = \sigma_g$, where $\sigma_g(h) = gh$, an isomorphism. □

**Theorem 1.3** (Lagrange). *Let $H \subseteq G$ be groups (not necessarily finite). Then, $|G| = [G : H]|H|$.*

*Proof.* We prove only for the finite case, by seeing that cosets partition the group, and that all cosets are of the same size. □

**Theorem 1.4** (Cauchy). *Let $G$ be a finite group of order $n$. If a prime $p$ divides $n$, there exists an element of order $p$.*

*Proof.* Define $X = \{(x_1, \dots, x_p) \in G^p : x_1 \cdots x_p = e\}$ and see that $x_p$ is determined entirely by the choices of $x_1, \dots, x_{p-1}$. Since $x_1, \dots, x_{p-1}$ can be chosen arbitrarily, $|X| = n^{p-1}$.

Let $C_p$ act on $X$ by cyclic permutation of the $p$-tuple. Since stabilizers are subgroups, the orbit-stabilizer theorem says that all orbits of $X$ are size either 1 or $p$. We note an orbit of some $(x_1, \dots, x_p) \in X$ is size 1 if and only if $x_1 = \cdots = x_p$, id est, $x_1$ is of order $p$ or $x_1 = e$. Finally, since $|X|$ is a multiple of $p$, the class equation says there must be at least $p$ elements of with an orbit of size 1, hence $p - 1$ elements of order $p$. □

**Theorem 1.5.** *Let $H \subseteq G$ be a normal subgroup. Then, the quotient set $G/H$ has a group structure.*

*Proof.* Tedious rule checking. □