Homework for Chapter 1 (Theoretical Questions)

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Problem I. & Problem II.

These two problems can be extended to general cases.

Suppose we are working on interval $[a_0, b_0]$, with initial length $l_0 := b_0 - a_0$. By definition of bisection method, The length of interval at the nth step is

$$l_n = \frac{b_0 - a_0}{2^n},$$

and $\dfrac{l_n}{2} = \dfrac{b_0 - a_0}{2^{n+1}}$ is the supremum of the distance between the root r and the midpoint of the interval.

Take $[a_0, b_0] = [1.5, 3.5]$, we have $l_n = 2^{1-n}$, $\dfrac{l_n}{2} = 2^{-n}$.

Let α be the root. To ensure the **relative** error $\dfrac{l_n}{2}\Big/\alpha$ no more than ε, we need to have

$$\frac{l_n}{2\alpha} \leqslant \frac{l_n}{2a_0} = \frac{b_0 - a_0}{2^{n+1}a_0} < \varepsilon,$$

which is equivalent to

$$n \geqslant \frac{\log(b_0 - a_0) - \log\varepsilon - \log a_0}{\log 2}.$$

Problem III.

Since $p'(x) = 12x^2 - 4x$, with the iteration formula $x_{n+1} = x_n - \dfrac{p(x_n)}{p'(x_n)}$, we have

$$\begin{aligned}
x_1 &= x_0 - \frac{p(x_0)}{p'(x_0)} = -1 - \frac{-3}{16} = -\frac{21}{16} = -0.8125,\\
x_2 &= x_1 - \frac{p(x_1)}{p'(x_1)} \approx -0.7708,\\
x_3 &= x_2 - \frac{p(x_2)}{p'(x_2)} \approx -0.7688,\\
x_4 &= x_3 - \frac{p(x_3)}{p'(x_3)} \approx -0.7688.
\end{aligned}$$

Problem IV.

Let the root be α, and $e_n = x_n - \alpha$. Then there is $\xi \in [x_n, \alpha]$ such that

$$0 = f(x_n) + (\alpha - x_n) f'(\xi).$$

That is, $f(x_n) = f'(\xi) e_n$. Then

$$e_{n+1} = x_{n+1} - \alpha = x_n - \frac{f(x_n)}{f'(x_n)} - \alpha = e_n - \frac{f(x_n)}{f'(x_n)} = e_n \left(1 - \frac{f'(\xi_n)}{f'(x_0)}\right).$$

By comparing with $e_{n+1} = C e_n^s$, we have

$$s = 1, \qquad C = 1 - \frac{f'(\xi)}{f'(x_0)}.$$

Problem V.

The interation clearly converges when $x_0 = 0$. And, if $|x_0| < \frac{\pi}{2}$, then $|x_n| = |\arctan x_{n-1}| < \frac{\pi}{2}$, so it is a bounded sequence.

For $x_0 \in \left(0, \frac{\pi}{2}\right)$, we have $x_{n+1} - x_n = \arctan x_n - x_n < 0$, so the sequence decreases. By the monotone convergence theorem, the sequence converges. For $x_0 \in \left(-\frac{\pi}{2}, 0\right)$, we have similar results since $x < \arctan x$ when $x < 0$.

Problem VI.

Write $x_1 = \frac{1}{p}$, $x_{n+1} = \frac{1}{x_n + p}$. It suffices to show $x := \lim_{n \to \infty} x_n$ exists.

By induction, it is easy to show that $x_n \in (0, 1)$, $n \in \mathbb{Z}_+$.

Let $f(x) = \frac{1}{x + p}$, and then $f'(x) = -\frac{1}{(x+p)^2}$. It follows that

$$\lambda := \max_{x \in [0,1]} |f'(x)| < \frac{1}{p^2} < 1.$$

By theorem 1.39, f has a unique fixed point α on $[0, 1]$, and the sequence $\{x_n\}$ converges to α. From $\alpha = \frac{1}{\alpha + p}$, we see that $x = \alpha = \frac{-p + \sqrt{p^2 + 4}}{2}$.

Problem VII.

Let α be the root in $[a_0, b_0]$. We have similar estimates as in Problem I and Problem II:

$$\text{Relative error } = \frac{|\alpha - l_n|}{|\alpha|} \leqslant \frac{l_n}{2^n|\alpha|} = \frac{b_0 - a_0}{2^n|\alpha|}.$$

Assume the relative error can be controlled within ε, we have

$$n \geqslant \frac{\log(b_0 - a_0) - \log\varepsilon - \log|\alpha|}{\log 2} - 1.$$

However, $n \to \infty$ as $\varepsilon \to 0^+$, leading to contradiction. Thus the relative error can't be appropriately measured.