Homework for Chapter 3

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Problem 1.

Solution. The polynomial $p(x)$ must meet the conditions:

$$p(0) = 0,\ p(1) = 1,\ p'(1) = -3,\ p''(1) = 6.$$

Applying Hermite interpolation yields:

$$p(x) = 7x^3 - 18x^2 + 12x.$$

The spline $s(x)$ does not qualify as a natural cubic spline because $s''(0) = -36 \neq 0$. □

Problem 2.

Solution. (a) Let $p_i = s|_{[x_i, x_{i+1}]} \in \mathbb{P}_2$, introducing $3(n-1)$ unknowns in $p_1, ..., p_{n-1}$. The following equations

$$p_i(x_i) = f_i,\ p_i(x_{i+1}) = f_{i+1},\ i = 1, ..., n-1$$

generate $2(n-1)$ conditions. Additionally,

$$p_i'(x_{i+1}) = p_{i+1}'(x_{i+1}),\ i = 1, ..., n-2$$

provide $n-2$ more conditions. Thus, there are $3(n-1)$ unknowns and $3(n-1)-1$ equations.

An extra condition is therefore required.

(b) Assuming $p_i(x) = a_i x^2 + b_i x + c_i$, we derive from the conditions:

$$\begin{cases} x_i^2 a_i + x_i b_i + c_i = f_i \\ x_{i+1}^2 a_i + x_{i+1} b_i + c_i = f_{i+1} \\ 2x_i a_i + b_i = m_i \end{cases}$$

Solving for a_i, b_i, c_i gives:

$$\begin{aligned} a_i &= \frac{f_{i+1} - f_i}{(x_{i+1} - x_i)^2} - \frac{m_i}{x_{i+1} - x_i}, \\ b_i &= \frac{m_i(x_{i+1} + x_i)}{x_{i+1} - x_i} - \frac{2x_i(f_{i+1} - f_i)}{(x_{i+1} - x_i)^2}, \\ c_i &= f_i + \frac{x_i^2(f_{i+1} - f_i)}{(x_{i+1} - x_i)^2} - \frac{m_i x_i x_{i+1}}{x_{i+1} - x_i}. \end{aligned}$$

Thus p_i is uniquely defined.

(c) Determine p_1 using f_1, f_2, m_1. Set $m_2 = p_1'(x_2)$.

Find p_2 using f_2, f_3, m_2. Set $m_3 = p_2'(x_3)$.

⋮

Finally, find p_{n-1} using f_{n-1}, f_n, m_{n-1}.

□

Problem 3.

Solution. Consider $s_2(x) = \alpha x^3 + \beta x^2 + \gamma x + \theta$. The following must hold:

$$s_2(0) = s_1(0) = 1 + c,\ s_2'(0) = s_1'(0) = 3c,\ s_2''(0) = s_1''(0) = 6c,\ s_2(1) = s(1) = -1,\ s_2''(1) = 0.$$

These yield:

$$\begin{cases} \theta = 1 + c, \\ \gamma = 3c, \\ 2\beta = 6c, \\ \alpha + \beta + \gamma + \theta = -1, \\ 6\alpha + 2\beta = 0 \end{cases}.$$

Solving this system gives $c = -\frac{1}{3}$. □

Problem 4.

Solution. (a) The natural cubic spline interpolating f at knots $-1, 0, 1$ is:

$$s(x) = \begin{cases} -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1 & \text{for } x \in [-1, 0], \\ \frac{1}{2}x^3 - \frac{3}{2}x^2 + 1 & \text{for } x \in [0, 1]. \end{cases}$$

(b) The bending energy of s is calculated as:

$$\int_{-1}^{1} [s''(x)]^2\, dx = \int_{-1}^{0} (-3x - 3)^2\, dx + \int_{0}^{1} (3x - 3)^2\, dx = 6.$$

The quadratic polynomial interpolating f at $-1, 0, 1$ is:

$$p(x) = -x^2 + 1.$$

Its bending energy is:

$$\int_{-1}^{1} [p''(x)]^2\, dx = \int_{-1}^{1} 4\, dx = 8 > 6.$$

The bending energy of f is:

$$\int_{-1}^{1} [f''(x)]^2\, dx = \int_{-1}^{1} \left[-\frac{\pi^2}{4} \cos\left(\frac{\pi}{2}x\right)\right]^2 \approx \frac{\pi^4}{16} \approx 6.0881 > 6.$$

□

Problem 5.

Solution.

- See that

$$B_i^1(x) = \begin{cases} \frac{x - t_{i-1}}{t_i - t_{i-1}} & x \in (t_{i-1}, t_i], \\ \frac{t_{i+1} - x}{t_{i+1} - t_i} & x \in (t_i, t_{i+1}], \\ 0 & \text{otherwise.} \end{cases} \qquad B_{i+1}^1(x) = \begin{cases} \frac{x - t_i}{t_{i+1} - t_i} & x \in (t_i, t_{i+1}], \\ \frac{t_{i+2} - x}{t_{i+2} - t_{i+1}} & x \in (t_{i+1}, t_{i+2}], \\ 0 & \text{otherwise.} \end{cases}$$

And by the recursive definition we have

$$B_i^2(x) = \frac{x - t_{i-1}}{t_{i+1} - t_{i-1}} B_i^1(x) + \frac{t_{i+2} - x}{t_{i+2} - t_i} B_{i+1}^1(x)$$

For $x \in (t_{i-1}, t_i]$,

$$B_i^2(x) = \frac{x - t_{i-1}}{t_{i+1} - t_{i-1}} \cdot \frac{x - t_{i-1}}{t_i - t_{i-1}} + \frac{t_{i+2} - x}{t_{i+2} - t_i} \cdot 0 = \frac{(x - t_{i-1})^2}{(t_{i+1} - t_{i-1})(t_i - t_{i-1})}.$$

For $x \in (t_i, t_{i+1}]$,

$$B_i^2(x) = \frac{(x - t_{i-1})(t_{i+1} - x)}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} + \frac{(t_{i+2} - x)(x - t_i)}{(t_{i+2} - t_i)(t_{i+1} - t_i)}.$$

For $x \in (t_{i+1}, t_{i+2}]$,

$$B_i^2(x) = \frac{x - t_{i-1}}{t_{i+1} - t_{i-1}} \cdot 0 + \frac{t_{i+2} - x}{t_{i+2} - t_i} \cdot \frac{t_{i+2} - x}{t_{i+2} - t_{i+1}} = \frac{(t_{i+2} - x)^2}{(t_{i+2} - t_i)(t_{i+2} - t_{i+1})}.$$

Hence we derived

$$B_i^2(x) = \begin{cases} \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} & x \in (t_{i-1}, t_i], \\ \frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(t_{i+2}-x)(x-t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)} & x \in (t_i, t_{i+1}], \\ \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} & x \in (t_{i+1}, t_{i+2}], \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

- We have

$$\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(x) = \begin{cases} p_1(x) = \frac{2(x-t_{i-1})}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} & x \in (t_{i-1}, t_i], \\ p_2(x) = \frac{t_{i+1}+t_{i-1}-2x}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{t_{i+2}+t_i-2x}{(t_{i+2}-t_i)(t_{i+1}-t_i)} & x \in (t_i, t_{i+1}], \\ p_3(x) = \frac{2(x-t_{i+2})}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} & x \in (t_{i+1}, t_{i+2}], \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

  We have

$$\begin{aligned} p_1(t_i) &= \frac{2(t_i - t_{i-1})}{(t_{i+1} - t_{i-1})(t_i - t_{i-1})} = \frac{2}{t_{i+1} - t_{i-1}} \\ p_2(t_i) &= \frac{t_{i+1} + t_{i-1} - 2t_i}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} + \frac{t_{i+2} + t_i - 2t_i}{(t_{i+2} - t_i)(t_{i+1} - t_i)} \\ &= \frac{t_{i-1} - t_i}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} + \frac{1}{t_{i+1} - t_{i-1}} + \frac{1}{t_{i+1} - t_i} \\ &= \frac{t_{i-1} - t_i + t_{i+1} - t_{i-1}}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} + \frac{1}{t_{i+1} - t_{i-1}} \\ &= \frac{2}{t_{i+1} - t_{i-1}} = p_1(t_i) \end{aligned}$$

  Hence $\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(x)$ is continuous at t_i. Similarly,

$$\begin{aligned} p_3(t_{i+1}) &= \frac{2(t_{i+1} - t_{i+2})}{(t_{i+2} - t_i)(t_{i+2} - t_{i+1})} = -\frac{2}{t_{i+2} - t_i} \\ p_2(t_{i+1}) &= \frac{t_{i+1} + t_{i-1} - 2t_{i+1}}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} + \frac{t_{i+2} + t_i - 2t_{i+1}}{(t_{i+2} - t_i)(t_{i+1} - t_i)} = -\frac{2}{t_{i+2} - t_i} = p_3(t_{i+1}) \end{aligned}$$

  Hence $\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(x)$ is continuous at t_{i+1}.

- We konw $\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(x)$ is continuous, and is a linear function at each interval $(t_{i-1}, t_i], (t_i, t_{i+1}]$ and $(t_{i+1}, t_{i+2}]$. And we have that

$$\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(t_{i-1}) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}x} B_i^2(t_i) = \frac{2}{t_{i+1} - t_{i-1}} > 0.$$

  So by the property of linear function,

$$\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(x) > 0, \quad x \in (t_{i-1}, t_i]$$

  Morever,

$$\frac{\mathrm{d}}{\mathrm{d}x} B_i^2(t_{i+1}) = -\frac{2}{t_{i+2} - t_i} < 0$$

Hence by the property of linear function, there is unique $x^* \in (t_i, t_{i+1})$ such that $\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x^*) = 0$. It follows the below eqution.

$$\frac{t_{i+1} + t_{i-1} - 2x^*}{t_{i+1} - t_{i-1}} + \frac{t_{i+2} + t_i - 2x^*}{t_{i+2} - t_i} = 0$$

Solve it and we got

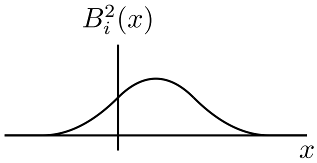
$$x^* = \frac{t_{i+2}t_{i+1} - t_i t_{i-1}}{(t_{i+2} + t_{i+1}) - (t_i + t_{i-1})}.$$

- By (c) we know that:

$$\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) > 0, \quad x \in (t_{i-1}, x^*)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) < 0, \quad x \in (x^*, t_{i+2})$$

  Also $B_i^2(t_{i-1}) = B_i^2(t_{i+2}) = 0$. And $B(x^*) < 1$ could be verified by a trivial computation. Hence $B_i^2(x) \in [0, 1)$.

- Clearly the image of $B_i^2(x)$ with different i could be obtained by translation. So we just draw with $i = 0$.



□

Problem 6.

Solution. For $x \in (t_{i-1}, t_i]$, by Lagrange's formula we have:

$$\begin{aligned}
[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 =& \frac{(t_i - x)^2}{(t_i - t_{i-1})(t_i - t_{i+1})(t_i - t_{i+2})} + \frac{(t_{i+1} - x)^2}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)(t_{i+1} - t_{i+2})} \\
&+ \frac{(t_{i+2} - x)^2}{(t_{i+2} - t_{i-1})(t_{i+2} - t_i)(t_{i+2} - t_{i+1})} \\
=& \frac{(x - t_{i-1})^2}{(t_{i+2} - t_{i-1})(t_{i+1} - t_{i-1})(t_i - t_{i-1})} = \frac{B_i^2(x)}{t_{i+2} - t_{i-1}}
\end{aligned}$$

For $x \in (t_i, t_{i+1}]$, by Lagrange's formula we have:

$$\begin{aligned}
[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 =& \frac{(t_{i+1} - x)^2}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)(t_{i+1} - t_{i+2})} + \frac{(t_{i+2} - x)^2}{(t_{i+2} - t_{i-1})(t_{i+2} - t_i)(t_{i+2} - t_{i+1})} \\
=& \frac{B_i^2(x)}{t_{i+2} - t_{i-1}}
\end{aligned}$$

For $x \in (t_{i+1}, t_{i+2}]$, by Lagrange's formula we have:

$$\begin{aligned}
[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 =& \frac{(t_{i+2} - x)^2}{(t_{i+2} - t_{i-1})(t_{i+2} - t_i)(t_{i+2} - t_{i+1})} \\
=& \frac{B_i^2(x)}{t_{i+2} - t_{i-1}}
\end{aligned}$$

Hence we verified

$$(t_{i+2} - t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 = B_i^2(x)$$

in the support of $B_i^2(x)$. And clearly, the equation is also right when $B_i^2(x)$ vanishes. □

Problem 7.

Solution. By the Theorem on derivates of B-splines, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}B_i^{n+1}(x) = \frac{(n+1)B_i^n(x)}{t_{i+n}-t_{i-1}} - \frac{(n+1)B_{i+1}^n(x)}{t_{i+n+1}-t_i}, \qquad n = 1, 2, ...$$

Integral to both side, we have:

$$\int_{t_{i-1}}^{t_{i+n+1}} \frac{\mathrm{d}}{\mathrm{d}x}B_i^{n+1}(x)\mathrm{d}x = \int_{t_{i-1}}^{t_{i+n+1}} \frac{(n+1)B_i^n(x)}{t_{i+n}-t_{i-1}} - \frac{(n+1)B_{i+1}^n(x)}{t_{i+n+1}-t_i}\mathrm{d}x, \qquad n = 1, 2, ...$$

For the left side, we have:

$$\mathrm{LHS} = B_i^{n+1}(t_{i+n+1}) - B_i^{n+1}(t_{i-1}) = 0 - 0 = 0.$$

For the right side, we have:

$$\mathrm{RHS} = (n+1)\left(\int_{t_{i-1}}^{t_{i+n}} \frac{B_i^n(x)}{t_{i+n}-t_{i-1}}\mathrm{d}x - \int_{t_i}^{t_{i+n+1}} \frac{B_{i+1}^n(x)}{t_{i+n+1}-t_i}\mathrm{d}x\right)$$

Then we got

$$\int_{t_{i-1}}^{t_{i+n}} \frac{B_i^n(x)}{t_{i+n}-t_{i-1}}\mathrm{d}x = \int_{t_i}^{t_{i+n+1}} \frac{B_{i+1}^n(x)}{t_{i+n+1}-t_i}\mathrm{d}x$$

Hence the scaled integral of $B_i^n(x)$ over its support is independent of i. □

Problem 8.

Solution. [(a)]

By the definition,

$$\tau_2(x_i, x_{i+1}, x_{i+2}) = x_i^2 + x_{i+1}^2 + x_{i+2}^2 + x_i x_{i+1} + x_i x_{i+2} + x_{i+1}x_{i+2}.$$

Make a table of divided difference as following.

$$\begin{array}{c|lll}
x_i & x_i^4 & & \\
x_{i+1} & x_{i+1}^4 & (x_{i+1}^2 + x_i^2)(x_{i+1}+x_i) & \\
x_{i+2} & x_{i+2}^4 & (x_{i+2}^2 + x_{i+1}^2)(x_{i+2}+x_{i+1}) & \frac{(x_{i+2}^2+x_{i+1}^2)(x_{i+2}+x_{i+1})-(x_{i+1}^2+x_i^2)(x_{i+1}+x_i)}{x_{i+2}-x_i}
\end{array}$$

Then the result follows from

$$\begin{aligned}
&\frac{(x_{i+2}^2 + x_{i+1}^2)(x_{i+2}+x_{i+1}) - (x_{i+1}^2 + x_i^2)(x_{i+1}+x_i)}{x_{i+2}-x_i} \\
=&\frac{(x_{i+2}^3 - x_i^3) + x_{i+1}(x_{i+2}^2 - x_i^2) + x_{i+1}^2(x_{i+2}-x_i)}{x_{i+2}-x_i} \\
=&(x_{i+2}^2 + x_{i+2}x_i + x_i^2) + x_{i+1}(x_{i+2}+x_i) + x_{i+1}^2 \\
=&\tau_2(x_i, x_{i+1} + x_{i+2}).
\end{aligned}$$

2. By the lemma on recursive relations of complete symmetric polynomials, we have

$$\begin{aligned}
&(x_{i+n+1} - x_i)\tau_{m-n-1}(x_i, ..., x_{i+n+1}) \\
=&\tau_{m-n}(x_i, ..., x_{i+n+1}) - \tau_{m-n}(x_i, ..., x_{i+n}) - x_i\tau_{m-n-1}(x_i, ..., x_{i+n+1}) \\
=&\tau_{m-n}(x_{i+1}, ..., x_{i+n+1}) + x_i\tau_{m-n-1}(x_i, ..., x_{i+n+1}) - \tau_{m-n}(x_i, ..., x_{i+n}) - x_i\tau_{m-n-1}(x_i, ..., x_{i+n+1}) \\
=&\tau_{m-n}(x_{i+1}, ..., x_{i+n+1}) - \tau_{m-n}(x_i, ..., x_{i+n}).
\end{aligned}$$

Now we prove the theorem by induction. For $n = 0$, clearly

$$\tau_m(x_i) = [x_i]x^m = x_i^m.$$

Now we suppose the theorem is true for some $0 \leq n < m$. Then for $n + 1$, we have

$$\begin{aligned}
\tau_{m-n-1}(x_i, ..., x_{i+n+1}) &= \frac{\tau_{m-n}(x_{i+1}, ..., x_{i+n+1}) - \tau_{m-n}(x_i, ..., x_{i+n})}{x_{i+n+1} - x_i} \\
&= \frac{[x_{i+1}, ..., x_{i+n+1}]x^m - [x_i, ..., x_{i+n}]}{x_{i+n+1} - x_i} \\
&= [x_i, ..., x_{i+n+1}]x^m
\end{aligned}$$

Then the theorem is proved by induction.

□