Homework for Chapter 2 (Theoretical Questions)

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Problem I.

It is clear that $\mathcal{C}[a,b]$ is a vector space. We need to show that it is an inner product space.

(1) Real positivity:

$$\forall u \in \mathcal{C}[a,b], \quad \langle u,u\rangle = \int_a^b \rho(t)u(t)\overline{u(t)}\mathrm{d}t = \int_a^b \rho(t)|u(t)|^2\mathrm{d}t \geq 0.$$

(2) Definiteness:

$$\langle u,u\rangle = 0 \Leftrightarrow \rho(t)|u(t)|^2 = 0 \Leftrightarrow |u(t)|^2 = 0 \Leftrightarrow u = 0$$

(3) Linearity in the first slot:

$$\forall u,v,w \in \mathcal{C}[a,b], \quad \langle u+w,v\rangle = \int_a^b \rho(t)(u(t)+w(t))\overline{v(x)}\mathrm{d}t = \int_a^b \rho(t)u(t)\overline{v(t)}\mathrm{d}t + \int_a^b \rho(t)w(t)\overline{w(t)}\mathrm{d}t = \langle u,v\rangle + \langle w,v\rangle$$

$$\forall c \in \mathcal{C}, \quad \forall u,v \in \mathcal{C}[a,b], \quad \langle cu,v\rangle = c\int_a^b \rho(t)u(t)\overline{v(t)}\mathrm{d}t = c\,\langle u,v\rangle$$

(4) conjugate symmetry:

$$\forall u,v \in \mathcal{C}[a,b] \quad \langle u,v\rangle = \int_a^b \rho(t)u(t)\overline{v(t)}\mathrm{d}t = \overline{\int_a^b \overline{\rho(x)u(x)\overline{v(x)}}\mathrm{d}x} = \overline{\int_a^b \rho(x)v(x)\overline{u(x)}\mathrm{d}x} = \overline{\langle v,u\rangle}$$

We also need to verify the requirements of a norm,

(1) real positivity:

$$\|u\|_2 = \left(\int_a^b \rho(t)|u(t)|^2\mathrm{d}t\right)^{1/2} \geq 0, \quad \text{with } \|u\|_2 = 0 \Leftrightarrow u \equiv 0.$$

(2) homogeneity:

$$\forall c \in \mathbb{C}, \quad \|cu\|_2 = \left(\int_a^b \rho(t)|cu(t)|^2\mathrm{d}t\right)^{1/2} = |c|\left(\int_a^b \rho(t)|u(t)|^2\mathrm{d}t\right)^{1/2} = |c|\,\|u\|_2$$

(3) triangle inequality:

$$\forall u,v \in \mathcal{C}[a,b], \quad \|u+v\|_2 = \left(\int_a^b \rho(x)|u(x)+v(x)|^2\mathrm{d}x\right)^{1/2}$$

$$\leq \left(\int_a^b \rho(x)|u(x)|^2\mathrm{d}x\right)^{1/2} + \left(\int_a^b \rho(x)|v(x)|^2\mathrm{d}x\right)^{1/2} = \|u\|_2 + \|v\|_2$$

Problem II.

By definition, $T_n(x) = \cos(n\arccos(x))$,

(a) For any m, n, we have

$$\begin{aligned}
\langle T_m, T_n\rangle &= \int_{-1}^{1} \rho(t)T_n(t)\overline{T_m(t)}dt \\
&= \int_{-1}^{1} \frac{\cos(n\arccos t)\cos(m\arccos t)}{\sqrt{1-t^2}}dt \\
&= \int_0^{\pi} \cos(m\theta)\cos(n\theta)d\theta \\
&= \int_0^{\pi} \frac{\cos(m\theta+n\theta)}{\cos(m\theta-n\theta)}d\theta \\
&= \begin{cases} \cos\dfrac{\pi}{2}, & m=n\neq 0 \\ 0, & m\neq n \\ \pi, & m=n=0 \end{cases}
\end{aligned}$$

Therefore, T_n are orthogonal.

(b) We have $T_0(x) = 1$, $T_1(x) = x$ and $T_2(x) = 2x^2 - 1$. After normalization, we obtain $T_0^*(x) = \dfrac{1}{\sqrt{\pi}}$, $T_1^*(x) = \sqrt{\dfrac{\pi}{2}}x$ and $T_2^*(x) = \sqrt{\dfrac{2}{\pi}}(2x^2-1)$.

Problem III.

(a) With the basis (T_0^*, T_1^*, T_2^*),the Fourier coefficients are $\langle y, T_0^*\rangle = \dfrac{2}{\sqrt{\pi}}$, $\langle y, T_1^*\rangle = 0$, $\langle y, T_2^*\rangle = -\dfrac{2}{3}\sqrt{\dfrac{2}{\pi}}$, the approximate function is $\hat{\phi}(x) = \dfrac{2}{\sqrt{\pi}}T_0^* + 0T_1^* + -\dfrac{2}{3}\sqrt{\dfrac{2}{\pi}}T_2^* = \dfrac{10}{3\pi} - \dfrac{8}{3\pi}x^2$.

(b) Since

$$G(1,x,x^2) = \begin{bmatrix} \langle 1,1\rangle & \langle 1,x\rangle & \langle 1,x^2\rangle \\ \langle 1,1\rangle & \langle 1,x\rangle & \langle 1,x^2\rangle \\ \langle 1,1\rangle & \langle 1,x\rangle & \langle 1,x^2\rangle \end{bmatrix} = \begin{bmatrix} \pi & 0 & \frac{\pi}{2} \\ 0 & \frac{\pi}{2} & 0 \\ \frac{\pi}{2} & 0 & \frac{3\pi}{2} \end{bmatrix},$$

$$c = (\langle y,1\rangle, \langle y,x\rangle, \langle y,x^2\rangle)^{\mathrm{T}} = (2,0,3)^{\mathrm{T}},$$

By solving the equation $G^{\mathrm{T}}a = c$, we obtain $a = (\dfrac{10}{3\pi}, 0, -\dfrac{8}{3\pi})^{\mathrm{T}}$. Hence the approximate function is $\hat{\phi}(x) = \dfrac{10}{3\pi} - \dfrac{8}{3\pi}x^2$.

Problem IV.

(a) Using the monomials $(1, x, x^2)$, with inner product$\langle u, v\rangle = \sum_i^{12} u(t_i)v(t_i)$, we have

$$u_1 = v_1 = 1, \|v_1\| = \sqrt{12}, u_1^* = \frac{1}{2\sqrt{3},}$$

$$v_2 = u_2 - \langle u_2, u_1^*\rangle\, u_1^* = x - \frac{13}{2}, u_1^* = \frac{1}{\sqrt{143}}(x - \frac{13}{2}),$$

$$v_3 = u_3 - \langle u_3, u_1^*\rangle\, u_1^* - \langle u_3, u_2^*\rangle\, u_2^* = x^2 - 13x + \frac{91}{3}, u_3^* = \sqrt{\frac{3}{4004}}(x^2 - 13x + \frac{91}{3}).$$

(b) The best approximate function is

$$\begin{aligned}\hat{\varphi}(x) &= \langle y, u_1^*\rangle\, u_1^* + \langle y, u_2^*\rangle\, u_2^* + \langle y, u_3^*\rangle\, u_3^* \\ &= \frac{831}{\sqrt{3}}u_1^* + \frac{589}{\sqrt{143}}u_2^* + \frac{12068\sqrt{3}}{\sqrt{4004}}u_3^* \\ &\approx 9.042x^2 - 113.4266x + 386.0013\end{aligned}$$

(c) The orthonormal polynomials can be reused but the normal equation cannot be reused. Due to we need to recalculated G and solving equation but the previous method just renew index of basis, therefore orthonormal polynomials has advantage over normal equations.

Problem V.

Proof of Thm 5.60. (PDI-1) For any $x \in \mathbb{F}^n$, we have

$$AA^+x = A\left(\sum_{j=1}^{r} \frac{1}{\sigma_j}\langle x, v_j\rangle\, u_j\right) = A\frac{\sigma_j}{\sigma_j}x = Ax.$$

Hence $AA^+A = A$ since x is arbitrary.

(PDI-2) For $j = 1, 2, \ldots, r$, $A^+AA^+v_j = A^+A\frac{1}{\sigma_j}u_j = A^+v_j$; for $j = r+1, r+2, \ldots, n$, $A^+AA^+v_j = 0 = A^+v_j$.

(PDI-3) From above conclusions, for $v \in \mathbb{F}^m$,

$$AA^+v = \operatorname{diag}(1, \ldots, 1, 0, \ldots, 0)\begin{pmatrix} v_1 \\ \vdots \\ v_r \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_m \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Hence $(AA^+)^* = AA^+$. Similarly we have $(A^+A)^* = A^+A$ $\square$

Proof of Lemma 5.61. Let A be a matrix with linearly independent columns. Then the product A^*A is invertible. To show this, consider any vector x in the kernel of A^*A, such that $A^*Ax = 0$. Pre-multiplying by x^* gives $x^*A^*Ax = 0$, which implies $(Ax)^*(Ax) = 0$. Therefore, $Ax = 0$. Since A has linearly independent columns, this implies $x = 0$, and the kernel of A^*A only contains the zero vector, making A^*A full rank and invertible.

Now, for any matrix A with linearly independent columns, the pseudoinverse A^+ is given by $A^+ = (A^*A)^{-1}A^*$. This is a left inverse of A because:

$$\begin{aligned} A^+A &= (A^*A)^{-1}A^*A \\ &= I. \end{aligned}$$

Similarly, if A has linearly independent rows, then AA^* is invertible, and the pseudoinverse A^+ is given by $A^+ = A^*(AA^*)^{-1}$, which is a right inverse of A because:

$$\begin{aligned} AA^+ &= AA^*(AA^*)^{-1} \\ &= I. \end{aligned}$$

□