Homework for Chapter 2 (Theoretical Questions)

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2023 年 11 月 1 日

Problem I.

In this case, $f_0 := f(x_0) = 1$, $f_1 := f(x_1) = \frac{1}{2}$. Using Newton's formula, $p_1(f)$ can be expressed as

$$p_1(f; x) = f_0 + \frac{f_1 - f_0}{x_1 - x_0}(x - x_0) = 1 - \frac{1}{2}(x - 1) = -\frac{1}{2}x + \frac{3}{2}.$$

By $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$, we have

$$\frac{1}{x} - \left(-\frac{1}{2}x + \frac{3}{2}\right) = \frac{1}{\xi^3(x)}(x-1)(x-2) \implies \xi(x) = \sqrt[3]{2x}.$$

Since $\xi(x)$ monotonically increases within $x \in [1,2]$, we have $\min \xi(x) = \sqrt[3]{2}$, $\max \xi(x) = \sqrt[3]{4}$, and $\max f''(\xi(x)) = \max \frac{1}{x} = 1$.

Problem II.

Let $\ell_k(x) = \prod_{i=0, i\neq k}^{n} \frac{(x - x_i)^2}{(x_k - x_i)^2}$. Clearly, $\ell_k(x) \in \mathbb{P}^+_{2n}$. And for every $i \neq k$, we have $\ell_k(x_i) = 0$, $\ell_k(x_k) = 1$. Let

$$P(x) = \sum_{k=0}^{n} f_k \ell_k(x),$$

and we can check that $p(x_i) = f_i,\ i = 1, 2, \ldots, n$.

Problem III.

For $n = 0$, clearly $f[t] = \mathrm{e}^t$. Assume the statement holds for $n - 1$, then by induction hypothesis,

$$\begin{aligned} f[t, t+1, \ldots, t+n] &= \frac{f[t+1, t+2, \ldots, t+n] - f[t, t+1, \ldots, t+n-1]}{n} \\ &= \frac{\frac{(\mathrm{e}-1)^{n-1}}{(n-1)!}\mathrm{e}^{t+1} - \frac{(\mathrm{e}-1^{n-1})}{(n-1)!}\mathrm{e}^{t}}{n} = \frac{(\mathrm{e}-1)^n}{n!}\mathrm{e}^t. \end{aligned}$$

Substitute $t = 0$, we have $f[0, 1, \ldots, n] = \frac{(\mathrm{e}-1)^n}{n!} = \frac{f^{(n)}(\xi)}{n!}$, which implies $\xi = n \ln(\mathrm{e}-1) \approx 0.541n > \frac{n}{2}$. So ξ is located to the right of the midpoint.

Problem IV.

The table of divided differences is as follows:

$$\begin{array}{c|cccc}
0 & 5 & & & \\
1 & 3 & -2 & & \\
3 & 5 & 1 & 1 & \\
4 & 12 & 7 & 2 & \frac{1}{4}
\end{array}$$

By Newton's formula, we have

$$p_3(f;x) = 5 - 2x + x(x-1) + \frac{1}{4}x(x-1)(x-3).$$

We can use argmin $p_3(f;x)$ to estimate $x_{\min}$. By solving $0 = p_3'(f;x) = -2 + 2x - 1 + \frac{1}{4}[(x-1)(x-3) + x(x-1) + x(x-3)] = \frac{3}{4}(x^2 - 3)$, we have $x = \sqrt{3}$ (since x is limited within $(1,3)$). So $x \min \approx \sqrt{3}$.

Problem V.

The following table shows the result:

$$\begin{array}{c|cccccc}
0 & 0 & & & & & \\
1 & 1 & 1 & & & & \\
1 & 1 & 7 & 6 & & & \\
1 & 1 & 7 & 21 & 15 & & \\
2 & 128 & 127 & 120 & 99 & 42 & \\
2 & 128 & 448 & 321 & 201 & 102 & 30
\end{array}$$

So $f[0,1,1,1,2,2] = 30$. Since $f^{(5)}(x) = 2520x^2$, we have $2520\xi^2 = 30$ for some ξ. That is, $\xi \approx 0.1091$.

Problem VI.

The table of divided differences is as follows:

$$\begin{array}{c|ccccc}
0 & 0 & & & & \\
1 & 2 & 1 & & & \\
1 & 2 & -1 & 2 & & \\
3 & 0 & -1 & 0 & \frac{2}{3} & \\
3 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & -\frac{5}{36}
\end{array}$$

which give the Hermite polynomial

$$p(x) = 1 + x - 2x(x-1) + \frac{2}{3}x(x-1)^2 - \frac{5}{36}x(x-1)^2(x-3).$$

We can estimate that $f(2) \approx p(2) = \frac{11}{18}$.

By theorem 2.37 and substitue $x = 2$, we have

$$|f(x) - p(x)| = \left|\frac{f^{(5)}(\xi)}{5!}x(x-1)^2(x-3)^2\right| = \left|\frac{f^{(5)}(\xi)}{60}\right| \leqslant \frac{M}{60}.$$

Problem VII.

The two equations hold for $k = 0$. Assume that they holds for $k - 1$, we have

$$\Delta^k f(x) = \Delta(\Delta^{k-1} f(x)) = \Delta((k-1)!h^{k-1} f[x_0, x_1, \dots, x_{k-1}]) = k!h^k f[x_0, x_1, \dots, x_k],$$
$$\nabla^k f(x) = \nabla(\nabla^{k-1} f(x)) = \nabla((k-1)!h^{k-1} f[x_0, x_{-1}, \dots, x_{-(k-1)}]) = k!h^k f[x_0, x_{-1}, \dots, x_{-k}].$$

Problem VIII.

By definition and the continuity of divided differences, we have

$$\begin{aligned}\frac{\partial}{\partial x_0} f[x_0, x_1, \dots, x_n] &= \lim_{h \to 0} \frac{f[x_0 + h, x_1, \dots, x_n] - f[x_0, x_1, \dots, x_n]}{h} \\ &= \lim_{h \to 0} f[x_0, x_0 + h, x_1, \dots, x_n] \\ &= f[x_0, x_0, x_1, \dots, x_n].\end{aligned}$$

Problem IX.

Write $x = \frac{b-a}{2}x' + \frac{a+b}{2}$, such that $x' \in [-1, 1]$. Then

$$\min \max_{x \in [a,b]} |a_0 x^n + \cdots + a_n| = \min \max_{x' \in [-1,1]} |a_0' x'^n + \cdots + a_n'| = \frac{1}{2^{n-1}}|a_0|$$

by Corollary 2.47.

Problem X.

$$Write \|P_n(z)\|_\infty = \frac{|f_n(z)|_\infty}{|T_n(x)|_\infty} = \frac{1}{|T_n(x)|}.$$

Suppose that $\|P\|_\infty < \|P_n\|_\infty = \frac{1}{|T_n(x)|}$.

Let $r(n) = P(n) - P_n(n)$,

$r(x) = P(x) - P_n(x) = 1 - 1 = 0.$

Thus, $r(n)$ has at least $n + 1$ zero points, leading to contradiction.

Hence for all $P \in P_n^x$, $\|P_n\|_\infty \leq \|P\|_\infty$.