

Conditional Investment-Cash Flow Sensitivities and Financing Constraints

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July 20, 2006

Abstract

Kaplan and Zingales (QJE, 1997) study the unconditional sensitivity of investment to cash flow in a static demand for capital framework. We study the sensitivity of investment to cash flow conditional on measures of q in an adjustment costs framework, which is more closely related to the empirical literature on investment and financing constraints. We present a benchmark model in which this conditional investment-cash flow sensitivity increases monotonically with the cost premium for external finance, for firms in a financially constrained regime. Using simulated data, we show that this pattern is found in the standard linear specification that relates investment rates to measures of both cash flow and average q .

JEL Classification: D92, E22, G31.

Key words: Investment, cash flow, financing constraints.

Acknowledgement: We thank Andrew Abel and seminar participants at the NBER Summer Institute and at the Institute for Fiscal Studies for helpful comments. We thank the ESRC for financial support under project RES-000-22-1052. Bond thanks the ESRC Centre for Public Policy at IFS for additional support. Söderbom thanks The Leverhulme Trust for additional support.

1 Introduction

Kaplan and Zingales (1997) study the sensitivity of investment to the availability of internal finance for firms that face different cost premia for external finance, in a one-period model with no costs of adjusting the capital stock. In this framework investment is chosen so that the marginal revenue product of capital is equated with the user cost of capital, i.e. the relevant first order condition is the traditional ‘neoclassical’ marginal productivity condition that describes the demand for capital in a static framework (cf. Jorgenson, 1963). Kaplan and Zingales (1997) show that investment may be more sensitive to the availability of internal funds for firms that face a lower cost premium for external funds, if the marginal revenue product of capital is sufficiently convex.

Although interesting, this analysis is far removed from the framework of most empirical studies of investment and financing constraints, in the tradition of Fazzari, Hubbard and Petersen (1988). These studies typically regress a measure of investment on a measure of q as well as a measure of cash flow, i.e. they estimate the sensitivity of investment to cash flow *conditional* on q or, in some cases, a wider set of control variables. These empirical specifications recognise that, even in the absence of financing constraints, investment is likely to be subject to adjustment costs that prevent the capital stock adjusting continuously to maintain equality between the marginal revenue product and the user cost of capital. The relevant first order condition in a model with strictly convex adjustment costs is that which equates the marginal cost of an additional unit of investment with the shadow value of an additional unit of installed capital (see, for example, Abel, 1980). Notice that the curvature of the marginal revenue product of capital plays no direct role in this condition. Interestingly, the special case of this model that delivers the linear relationship between investment and q , which dominates the

empirical literature, requires marginal adjustment costs that are linear in investment.

We study the sensitivity of investment to the availability of internal finance in a simple model with quadratic adjustment costs. We distinguish between two types of cost premia for external funds: a cost premium that is increasing in the level of external finance used; and a cost premium that is fixed, independent of the level of external finance used. In the former case there are two financial regimes: an unconstrained regime in which investment is financed internally and the shadow value of capital, or marginal q , remains a sufficient statistic for current investment; and a constrained regime in which external funds are the marginal source of finance, and investment displays excess sensitivity to windfall fluctuations in the availability of internal funds. In this constrained regime, there is a straightforward monotonic relationship between the conditional investment-cash flow sensitivity and the severity of the capital market ‘imperfection’, as measured by the slope of the cost schedule for external funds. That is, if we consider two otherwise identical firms with the same adjustment cost function, supply of internal funds, *and* marginal q , the sensitivity of investment to a windfall increase in cash flow will be greater for the firm that faces a more steeply sloping cost of external funds schedule.

In the model with a fixed cost premium for external finance, there are three financial regimes: an unconstrained regime in which investment is financed internally; a constrained regime in which available internal funds are exhausted but the firm chooses to use no external funds; and an external finance regime in which external funds are the marginal source of finance. In this case, if a firm is in the constrained regime, investment increases dollar-for-dollar with small windfall increases in cash flow, regardless of the size of the fixed cost premium for internal

funds. If a firm is in the external finance regime, investment is insensitive to small windfall increases in cash flow, but may be increased by larger cash flow shocks that shift the firm into a different regime. In this model we get a weaker result that, if we consider two otherwise identical firms with the same adjustment cost function, supply of internal funds and marginal q , the sensitivity of investment to a windfall increase in cash flow will be no lower for the firm that faces a higher cost premium for external finance, and will be strictly greater in response to some cash flow shocks.

These results indicate that, at a given level of the shadow value of capital or *marginal q*, otherwise identical firms will display (weakly) greater sensitivity of investment to cash flow if they face a greater cost premium for the use of external finance. We also study the relationship between marginal q and *average q* in these models, to assess the extent to which empirical studies may succeed in controlling for variation in marginal q by including a standard measure of average q , in the presence of financing constraints. Hayashi (1982) showed that, if there is no cost premium for external finance, average q is equal to marginal q if the firm's net revenue function is homogeneous of degree one. More generally we show that with a cost premium for external finance, average q continues to equal marginal q provided the cost premium is also homogeneous of degree one. In this case, our analysis therefore indicates that otherwise identical firms will display (weakly) greater sensitivity of investment to cash flow, at a given level of average q , if they face a greater cost premium for external funds.

We illustrate these results using simulated optimal investment data for a panel of firms with quadratic adjustment costs and a linear homogeneous net revenue function. As expected, the simple linear regression of investment rates on average q and a measure of cash flow indicates no excess sensitivity to cash flow when firms

face no cost premium for external finance. When firms face a linear homogeneous cost premium for external finance, we find a significant positive coefficient on the measure of cash flow in the same specification. More interestingly, this coefficient on cash flow is shown to increase monotonically with the level of the cost premium for external funds. We thus provide a benchmark model in which there is a monotonic relationship between the sensitivity of investment to cash flow, conditional on average q , and the severity of the capital market imperfection. We also note that the structural first order condition for investment can be estimated directly in the presence of this form of capital market imperfection.

The remainder of the paper is organised as follows. Section 2 reviews the sensitivity of investment to windfall fluctuations in cash flow in a static demand for capital framework, and illustrates the result highlighted by Kaplan and Zingales (1997). Section 3 outlines our basic model with convex adjustment costs and discusses the sensitivity of investment to cash flow conditional on marginal q in two special cases. Section 4 considers the relationship between marginal q and average q in these two models. Section 5 presents our results using simulated investment data. Section 6 concludes.

2 A static model

In a setting with no adjustment costs for capital, the first order condition describing the evolution of the optimal capital stock equates the marginal revenue product of capital (MPK) to the user cost of capital (u). The user cost of capital represents the minimum rate of return required for the investment to be value increasing, and reflects the cost of finance. If the firm faces a higher cost for using external funds than for using internal funds, this will be reflected in a higher

required rate of return on investment financed from external sources.¹ This situation is depicted for a case with an increasing marginal cost of external finance in Figure 1.

Here the cost of capital for investment financed internally is denoted u_{INT} . If the firm wants to finance investment spending beyond the level denoted by C , the firm must use increasingly expensive external sources. This increasing cost premium for external funds is reflected in the upward sloping segment of the cost of capital schedule u beyond the investment level C . There are two financing regimes in this framework. If the firm has a marginal revenue product schedule MPK_1 its desired investment spending is low relative to the availability of low cost internal funds.² It finances its preferred level of investment spending I_1 internally, and this level of investment is insensitive to windfall fluctuations in cash flow. More precisely, an increase in the availability of internal funds that leaves the marginal revenue product of capital schedule unchanged, but increases the level of investment that can be financed internally to $C' > C$, has no effect on the optimal level of investment spending for firms in this unconstrained regime.

In contrast, if the firm has the marginal revenue product schedule MPK_2 , its desired investment spending exceeds its supply of low cost internal funds. In this case it finances additional investment beyond C by using more expensive external sources, but the increasing cost of external finance influences its optimal level of investment spending. This firm chooses the level of investment I_2 where the first order condition equating the marginal revenue product and user cost of capital is satisfied. An otherwise identical firm with the same marginal revenue product MPK_2 and the same cost premium schedule for external finance, but with a much

¹See Hubbard (1998), for example, for a discussion of why this ‘pecking order’ assumption may be relevant.

²The figure is drawn for a given inherited level of the capital stock, so there is a one-to-one association between current investment and the current level of the capital stock.

greater supply of low cost internal funds, would instead choose the higher level of investment I_3 . This indicates that the level of investment spending is sensitive to windfall fluctuations in the availability of internal funds, for firms in this financially constrained regime. This sensitivity is illustrated in Figure 2. With more internal funds available, the firm is required to use less external finance, faces a lower required rate of return for all investment levels above C , and optimally chooses a higher level of investment I'_2 . Note that in this model with an increasing marginal cost of external finance, the increase in investment spending ($I'_2 - I_2$) is typically smaller than the windfall increase in cash flow ($C' - C$).

It would appear that the investment spending of firms in the constrained regime will display greater sensitivity to fluctuations in cash flow for firms that face a greater cost premium for external funds, or what might be termed a more severe financing constraint. This is indeed possible, as illustrated in Figure 3. A firm with the marginal product schedule MPK and cost of external funds schedule u_L increases its investment spending from I_L to I'_L in response to a windfall increase in cash flow from C to C' . An otherwise identical firm with the more steeply sloping cost of external funds schedule u_H increases its investment spending by the larger amount ($I'_H - I_H$) as a result of the same cash flow shock. Thus we might expect to find evidence of greater investment-cash flow sensitivity among samples of firms that face a higher cost premium for the use of external sources of finance.

Kaplan and Zingales (1997) have noted that this conclusion depends heavily on the presumed linearity of the marginal revenue product schedule in Figure 3. The opposite result is possible if the firm's marginal revenue product of capital is sufficiently convex. This case is illustrated in Figure 4, where the firm facing the higher cost of external funds schedule u_H increases investment by less in response

to the cash flow shock than the firm which faces the lower cost premium reflected in u_L .

Kaplan and Zingales (1997) correctly conclude that there is not necessarily a monotonic relationship between the sensitivity of investment to windfall fluctuations in the availability of internal finance and the slope of the cost of external finance schedule in a static demand for capital model of this type. They provide a formal analysis of a one-period investment problem with no adjustment costs. In the next section we consider whether a similar result holds in a dynamic investment problem with strictly convex costs of adjustment, which is the basis for the investment-q relation adopted by much of the empirical research in this area, including that presented by Kaplan and Zingales (1997) themselves.

3 A dynamic model with adjustment costs

We study a standard investment problem where the firm chooses investment to maximise the value of its equity V_t given by

$$V_t = E_t \left\{ \sum_{s=0}^{\infty} \beta^s (D_{t+s} - N_{t+s}) \right\} \quad (1)$$

where D_t denotes dividends paid in period t , N_t denotes the value of new equity issued in period t , $\beta < 1$ is the one-period discount factor assumed constant for simplicity, and $E_t[\cdot]$ denotes an expected value given information available at time t .

Dividends and new equity are linked to the firm's net revenue Π_t each period by the sources and uses of funds identity

$$D_t - N_t = \Pi_t - \Phi_t \quad (2)$$

where $\Phi_t = \Phi(N_t, K_t)$ represents additional costs imposed by issuing new equity and K_t is the stock of capital in period t . We follow Kaplan and Zingales (1997) in

not considering debt finance explicity, so that issuing new equity is the only source of external finance considered. Formally we treat $\Phi(N_t, K_t)$ as a transaction fee that must be paid to third parties when new shares are issued. Less formally we can also think of these costs reflecting differential tax treatments, agency costs, or losses imposed on existing shareholders when the firm issues new shares in markets characterised by asymmetric information.³ We assume $\Phi(0, K_t) = 0$, $\Phi_{Nt} = \frac{\partial \Phi_t}{\partial N_t} \geq 0$ and $\Phi_{Kt} = \frac{\partial \Phi_t}{\partial K_t} \leq 0$.

Following the q literature, we assume $\Pi_t = \Pi(K_t, I_t)$ where $K_{t+1} = (1 - \delta)K_t + I_t$, I_t is gross investment in period t (which may be positive or negative), and δ is the rate of depreciation. Notice that investment in period t does not contribute to productive capital until period $t + 1$, so that K_t depends only on past investment decisions. With no cost premium for external finance (i.e. $\Phi_t \equiv 0$), this implies that investment in period t does not respond to serially uncorrelated productivity shocks, although investment does respond to serially correlated productivity shocks that convey information about the (revenue) productivity of capital in period $t + 1$. The dependence of net revenue on investment reflects the presence of adjustment costs, which are assumed to be strictly convex in I_t .

The firm maximises V_t subject to this capital accumulation constraint and to non-negativity constraints on dividends and new equity issues, with shadow values λ_t^D and λ_t^N . The problem can be expressed as

$$V_t(K_t) = \max_{I_t, N_t} \left\{ \begin{array}{l} \Pi(K_t, I_t) - \Phi(N_t, K_t) \\ + \lambda_t^D [\Pi(K_t, I_t) + N_t - \Phi(N_t, K_t)] + \lambda_t^N N_t \\ + \beta E_t [V_{t+1}((1 - \delta)K_t + I_t)] \end{array} \right\} \quad (3)$$

Letting $\mu_t^K = \frac{\partial V_t}{\partial K_t}$ denote the shadow value of inheriting one additional unit of installed capital at time t , the first order condition for optimal investment can be

³See, for example, Myers and Majluf (1984).

written as

$$-\Pi_{It} = \frac{\beta E_t [\mu_{t+1}^K]}{1 + \lambda_t^D} = \frac{\lambda_t^K}{1 + \lambda_t^D} \quad (4)$$

where $-\Pi_{It} = \frac{-\partial \Pi_t}{\partial I_t}$ is strictly increasing in the level of investment I_t . If the non-negativity constraint on dividends is not binding ($\lambda_t^D = 0$), this simply equates the marginal cost of investing in an additional unit of capital in period t with the shadow value of an additional unit of capital in period $t + 1$, discounted back to its value in period t . We refer to μ_t^K as the shadow value of capital and to $\lambda_t^K = \beta E_t [\mu_{t+1}^K]$ as the shadow value of investment at time t ; the difference here reflects the timing convention that investment becomes productive with a lag of one period.

Along the optimal path, the evolution of the shadow value of capital is described by the intertemporal condition

$$\mu_t^K = (1 + \lambda_t^D)\Pi_{Kt} - (1 + \lambda_t^D)\Phi_{Kt} + (1 - \delta)\beta E_t [\mu_{t+1}^K] \quad (5)$$

where $\Pi_{Kt} = \frac{\partial \Pi_t}{\partial K_t}$ and $\Phi_{Kt} = \frac{\partial \Phi_t}{\partial K_t}$.

The first order condition for optimal new share issues implies

$$\lambda_t^D = \frac{\Phi_{Nt} - \lambda_t^N}{1 - \Phi_{Nt}} \quad (6)$$

In the case where new shares are issued ($N_t > 0$) and $\lambda_t^N = 0$, this simplifies to give

$$\lambda_t^D = \frac{\Phi_{Nt}}{1 - \Phi_{Nt}} \quad (7)$$

To study the implications we focus on two special cases. The first assumes a strictly increasing cost premium for external finance, similar to the case considered in the previous section. The second considers a different specification of the capital market imperfection, in which there is a fixed cost premium per unit of new equity issued.

3.1 An increasing cost premium

To simplify, we assume that $\Phi(N_t, K_t) = \left(\frac{\phi}{2}\right) \left(\frac{N_t}{K_t}\right)^2 K_t$ where ϕ is a parameter that specifies the slope of the cost premium for external finance. In this case $\Phi_{Nt} = \phi \left(\frac{N_t}{K_t}\right)$, so that the cost premium increases linearly with the amount of external finance raised relative to the size of the firm. In the case where new shares are issued, this gives $\frac{1}{1+\lambda_t^D} = 1 - \phi \left(\frac{N_t}{K_t}\right)$.

The first order condition for investment is then depicted in Figure 5, adapted from Hayashi (1985), which is drawn for a given level of the shadow value of investment λ_t^K . The adjustment cost function used to obtain the linear relationship between investment rates and q makes marginal adjustment costs linear in the investment rate ($\frac{I_t}{K_t}$), giving the linear marginal cost schedules depicted here. As before, levels of investment spending up to C can be financed using low cost internal funds. More precisely, for $I < C$ the firm issues no new equity ($N = 0$) and pays strictly positive dividends ($D > 0$ and $\lambda_t^D = 0$). For $I > C$, the firm issues new equity ($\lambda_t^N = 0$), pays zero dividends ($D = 0$), and λ_t^D is obtained from the first order condition for new equity issues (7). Here this gives

$$\frac{\lambda_t^K}{1 + \lambda_t^D} = \lambda_t^K \left(1 - \phi \left(\frac{N_t}{K_t}\right)\right) \quad (8)$$

as noted above. The curvature of the $\lambda_t^K \left(1 - \phi \left(\frac{N_t}{K_t}\right)\right)$ schedule in the region where $N_t > 0$ reflects the assumption that, as new shares are issued to finance investment spending above the level that can be funded internally, an increasing proportion of the revenue raised is dissipated by the transaction fee paid to third parties, so that $(\frac{N}{K})$ increases at a faster rate than $(\frac{I}{K}) - (\frac{C}{K})$ in this region.

In this model there are again two financing regimes. For a given level of the shadow value of capital or marginal q ,⁴ a firm with the adjustment cost function

⁴Marginal q is usually expressed as the ratio of the shadow value of an additional unit of

$-\Pi_{I1}$ is in the unconstrained regime and chooses the investment rate $\frac{I_1}{K}$ at which the first order condition (4) is satisfied.⁵ A firm with the adjustment cost function $-\Pi_{I2}$ is in the constrained regime and chooses the investment rate $\frac{I_2}{K}$. This firm would choose a higher level of investment if it was less dependent on expensive external finance; if its supply of internal funds was high enough, it would choose the investment rate $\frac{I_3}{K}$. This sensitivity of investment to windfall changes in cash flow for firms in the constrained regime is illustrated in Figure 6. Here a windfall increase in cash flow is one which leaves expected future profitability and hence the shadow value of an additional unit of investment (λ_t^K) unchanged. Formally, given our timing convention, this can be thought of as a serially uncorrelated shock to (revenue) productivity in period t .

Figure 7 considers this investment-cash flow sensitivity for two otherwise identical firms, with the same adjustment cost function, availability of internal funds *and* shadow value of capital, but subject to different cost schedules for external funds. One firm faces a low cost premium represented by ϕ_L , whilst the other firm faces a much higher cost premium represented by ϕ_H . In the constrained regime, a given windfall increase in the availability of internal finance will clearly have a larger impact on the investment spending of the firm that faces the more steeply increasing cost of external finance schedule, and whose investment *conditional on marginal q* is therefore affected much more by reliance on external sources of funds.

This illustrates the main result of this section. In the model with quadratic adjustment costs and a strictly increasing cost of new equity, there is a simple monotonic relationship between the *conditional* sensitivity of investment to wind-

⁵The distinction between the two financial regimes may alternatively be illustrated for a firm with a given adjustment cost schedule by considering different levels of the shadow value of capital.

fall fluctuations in cash flow and the severity of the financing constraint, as reflected in the slope of the cost schedule for external funds, for otherwise identical firms in the financially constrained regime.

The result is obtained by holding constant the shadow value of capital or marginal q . In general, firms with identical technologies but different cost premia for external finance are unlikely to have the same shadow value of capital. Nevertheless this is the kind of conditional investment-cash flow sensitivity that is estimated in regression specifications that relate investment rates to measures of cash flow *and* marginal q . In section 4 we obtain conditions under which marginal q may be measured by the usual measure of average q , even in the presence of a cost premium for external finance. In section 5 we show using simulated investment data that the monotonic relationship between this conditional investment-cash flow sensitivity and the slope of the cost schedule for external funds is found in a model of this form, using the standard linear econometric specification.

This simple monotonic relationship could of course be overturned by introducing sufficient curvature into the marginal adjustment cost schedule $-\Pi_{It}$. This is perfectly consistent with the investment model considered here, but would be inconsistent with the linear specification found in most of the empirical literature on financing constraints and investment. If this possibility were to be taken seriously, the shape of the adjustment cost function would need to be reflected in the functional form specified in the empirical analysis.

3.2 A fixed cost premium

In this section we consider a different specification of the external finance premium, in which external finance is more costly than internal finance, but is available at a fixed cost premium that does not increase with the amount of external funds

used. Formally this can be thought of as a fixed brokerage fee per unit of new equity issued.

Here we assume that $\Phi(N_t, K_t) = \phi N_t$ where ϕ is again a parameter that reflects the size of the cost premium for external finance. In this case $\Phi_{Nt} = \phi$, and in the case where new shares are issued, this gives $\frac{1}{1+\lambda_t^D} = 1 - \phi$.

In the static framework reviewed in section 2, this specification gives a step function for the cost of capital, and a similar result is found for the model with convex adjustment costs. Although apparently simpler, this formulation gives three distinct financial regimes, which are illustrated in Figure 8. For a given level of the shadow value of investment (λ_t^K), a firm with the adjustment cost function $-\Pi_{I1}$ is again in an unconstrained regime where investment is insensitive to windfall fluctuations in cash flow. This firm chooses the investment rate $\frac{I_1}{K}$; investment spending is financed from internal funds, with no new equity issues ($N_t = 0$) and strictly positive dividend payments ($D_t > 0, \lambda_t^D = 0$). A firm with the adjustment cost function $-\Pi_{I2}$ is in a constrained regime where both dividend payments and new share issues are zero. Here investment spending is constrained to the level of available internal funds (C), and locally investment spending will fluctuate dollar-for-dollar with windfall changes in cash flow for firms in this regime. A firm with the adjustment cost function $-\Pi_{I3}$ is in a third regime where additional investment is financed by issuing new equity. The higher cost of external finance influences the optimal level of investment chosen, as indicated by the first order condition (4). Here λ_t^D is given from the optimality condition for new share issues (7), so that

$$\frac{\lambda_t^K}{1 + \lambda_t^D} = \lambda_t^K(1 - \phi) \quad (9)$$

and this firm chooses the investment rate $\frac{I_3}{K}$. If the same firm had access to a sufficiently higher level of internal funds, it would choose the higher investment

rate $\frac{I_5}{K}$. Locally, however, investment spending is insensitive to small windfall fluctuations in cash flow for firms in this regime, as illustrated in Figure 9. The shock to the availability of internal funds must be large enough to move such firms from the third ‘external finance’ regime to the second ‘constrained’ regime in order for their level of investment spending to be affected, as illustrated in Figure 10.

Depending on which regime a firm is in prior to a windfall increase in cash flow, and on the size of the shock, there are six different paths along which the firm’s investment spending may be affected.⁶ We find that investment displays excess sensitivity to cash flow shocks if the firm is initially in the external finance regime and is moved to either of the other regimes, or if the firm is initially in the constrained regime.

When we consider the impact of windfall cash flow shocks on the investment spending of otherwise identical firms that are subject to different cost premia for external finance, there are still more possible combinations to consider. We find several cases in which the effect on investment is strictly greater for the firm with the higher cost premium; two of these possibilities are illustrated in Figures 11 and 12. For the case in Figure 11, the cash flow shock increases the investment rate for the low cost premium (ϕ_L) firm from $\frac{I_3}{K}$ to $\frac{C'}{K}$, whilst the same cash flow shock increases investment for the high cost premium (ϕ_H) firm from the lower rate $\frac{C}{K}$ also to $\frac{C'}{K}$. For the case in Figure 12, the cash flow shock increases investment for the ϕ_L firm from $\frac{I_3}{K}$ to $\frac{I'_3}{K}$, whilst the same shock increases investment for the ϕ_H firm from $\frac{C}{K}$ to $\frac{I'_3}{K}$. There is also a case here in which each firm’s investment increases dollar-for-dollar with the windfall increase in cash flow, as illustrated

⁶A firm in the ‘external finance’ regime may remain in that regime, or move into either of the ‘constrained’ or ‘unconstrained’ regimes. A firm in the ‘constrained’ regime may remain in that regime, or move into the ‘unconstrained’ regime. A firm in the ‘unconstrained’ regime will necessarily remain in that regime following a windfall increase in the availability of internal funds.

in Figure 13. However if we compare otherwise identical firms with the same adjustment cost schedule, supply of internal funds, and shadow value of capital, we find no case in which the effect on investment is strictly greater for the firm with the lower cost premium.

Thus we find that in the model with quadratic adjustment costs and a fixed cost premium for new equity finance, there is a weakly monotonic relationship between the conditional sensitivity of investment to windfall fluctuations in cash flow and the severity of the financing constraint, as reflected in the size of the cost premium for external funds, for otherwise identical firms. The result is again obtained by holding the shadow value of capital or marginal q constant. The following section considers the relationship between marginal q and average q in these models, and hence the extent to which econometric studies may in fact be able to condition on *marginal q* in the presence of financing constraints.

4 Marginal q and average q

Hayashi (1982) showed that for a firm with a linear homogeneous revenue function $\Pi(K_t, I_t) = \Pi_{Kt}K_t + \Pi_{It}I_t$, the first order condition (4) and the intertemporal condition (5) can be combined in the absence of financing constraints ($\Phi(N_t, K_t) \equiv 0$) to obtain

$$\mu_t^K = \frac{V_t}{K_t} \quad (10)$$

where V_t is the maximised value of the firm. This implies that the unobserved shadow value of an additional unit of capital can be measured using the average value of capital for a firm that has inherited K_t units of capital from the past. This allows a measure of marginal q to be constructed using average q , the ratio of the maximised value of the firm to the replacement cost of its inherited capital stock.⁷

⁷We discuss the details for our timing convention in the following section.

At least in the absence of share price bubbles,⁸ the numerator of this average q ratio can be measured using the firm's stock market valuation. In the absence of financing constraints, econometric specifications can in principle condition on marginal q in the benchmark case of a linear homogeneous revenue function and strictly convex costs of adjustment.

Combining these optimality conditions in the same way in our model with costly external finance yields the equality

$$\mu_t^K K_t = E_t \left\{ \sum_{s=0}^{\infty} \beta^s (1 + \lambda_{t+s}^D) (\Pi_{t+s} - \Phi_{K,t+s} K_{t+s}) \right\} \quad (11)$$

With no cost premium for external finance ($\Phi(N_t, K_t) \equiv 0$), the shadow value of internal funds (λ_{t+s}^D) and $\Phi_{K,t+s}$ are both identically zero. The sources and uses of funds identity (2) then shows that net revenue Π_{t+s} equals the net cash distribution to stockholders ($D_{t+s} - N_{t+s}$), so that the right hand side of (11) simplifies to the value of the firm V_t as in (1). More generally, we need to consider the relationship between $(1 + \lambda_{t+s}^D) (\Pi_{t+s} - \Phi_{K,t+s} K_{t+s})$ and this net distribution to stockholders.

To obtain the equality between marginal q and average q in this case, we require that the cost premium for external finance is also homogeneous of degree one, so that $\Phi(N_t, K_t) = \Phi_{Nt} N_t + \Phi_{Kt} K_t$. With this assumption, (11) becomes

$$\mu_t^K K_t = E_t \left\{ \sum_{s=0}^{\infty} \beta^s (1 + \lambda_{t+s}^D) (\Pi_{t+s} - \Phi_{t+s} + \Phi_{N,t+s} N_{t+s}) \right\} \quad (12)$$

so that we can focus on the relationship between $(1 + \lambda_{t+s}^D) (\Pi_{t+s} - \Phi_{t+s} + \Phi_{N,t+s} N_{t+s})$ and $(D_{t+s} - N_{t+s})$.

First consider the unconstrained financing regime, in which $D_t > 0$ and $N_t = 0$. This implies that $\lambda_t^D = 0$ and $\Phi_t = 0$, so we have $(1 + \lambda_t^D) (\Pi_t - \Phi_t + \Phi_{Nt} N_t) =$

⁸See Bond and Cummins (2001) and Bond and Söderbom (2006) for further discussion.

Π_t . The sources and uses of funds identity (2) simplifies to $\Pi_t = D_t - N_t$, so we obtain $(1 + \lambda_t^D)(\Pi_t - \Phi_t + \Phi_{Nt}N_t) = D_t - N_t$ for firms in this regime.

Now consider firms that use external finance, so that $D_t = 0$ and $N_t > 0$. This implies that $\lambda_t^N = 0$ and λ_t^D is given from (7), which implies

$$(1 + \lambda_t^D) = \frac{1}{1 - \Phi_{Nt}} \quad (13)$$

Here the sources and uses of funds identity simplifies to $-N_t = \Pi_t - \Phi_t$, which implies

$$(\Pi_t - \Phi_t + \Phi_{Nt}N_t) = -N_t + \Phi_{Nt}N_t = -N_t(1 - \Phi_{Nt}) \quad (14)$$

Combining (13) and (14) gives $(1 + \lambda_t^D)(\Pi_t - \Phi_t + \Phi_{Nt}N_t) = -N_t = D_t - N_t$ for firms in this regime also.

Finally in the case where both $D_t = 0$ and $N_t = 0$, we have $\Phi_t = 0$ so that the sources and uses of funds identity implies $D_t - N_t = \Pi_t = 0$. Again in this case we obtain $(1 + \lambda_t^D)(\Pi_t - \Phi_t + \Phi_{Nt}N_t) = D_t - N_t$.

Since this equality holds regardless of which financial regime the firm happens to be in at any time, we have

$$\begin{aligned} \mu_t^K K_t &= E_t \left\{ \sum_{s=0}^{\infty} \beta^s (1 + \lambda_{t+s}^D) (\Pi_{t+s} - \Phi_{t+s} + \Phi_{N,t+s} N_{t+s}) \right\} \\ &= E_t \left\{ \sum_{s=0}^{\infty} \beta^s (D_{t+s} - N_{t+s}) \right\} = V_t \end{aligned} \quad (15)$$

Consequently the equality between marginal q and average q expressed in (10) continues to hold in this model with financing constraints. In addition to linear homogeneity of the net revenue function $\Pi(K_t, I_t)$, we also require linear homogeneity of the external finance premium $\Phi(N_t, K_t)$.

It can easily be seen that this linear homogeneity condition holds in both of the cases analysed in sections 3.1 and 3.2. The result of this section therefore allows us to state that in the benchmark case of linear homogeneity and quadratic

adjustment costs, we find a monotonic relationship between the size of the cost premium for external finance and the sensitivity of investment to windflow fluctuations in the availability of internal funds, conditional on observable average q . More precisely, if we compare otherwise identical firms with the same adjustment cost function, supply of internal funds and average q , the effect of a windfall increase in cash flow on investment is no lower for the firm that faces the higher cost premium for external finance. At least up to a linear approximation, this is the kind of conditional investment-cash flow sensitivity that is estimated in the empirical literature that relates investment rates to measures of cash flow and average q .

In the next section we use simulated optimal investment data for a parameterised specification of this model to investigate whether this linear approximation is sufficiently adequate for our monotonicity result to describe the behaviour of the estimated coefficient on cash flow in the kind of econometric specification that was used by Fazzari, Hubbard and Petersen (1988) and by many subsequent empirical papers.

5 Results for simulated investment data

5.1 Specification

To generate simulated investment data for this class of models, we require functional forms for the net revenue function and the external cost premium.

Our net revenue function has the form

$$\Pi(K_t, I_t) = A_t K_t - G(K_t, I_t) - I_t \quad (16)$$

where A_t is a stochastic productivity parameter and $G(K_t, I_t)$ denotes costs of adjustment. The relative price of output and capital goods is assumed to be constant, with both prices implicitly normalised to unity.

We assume a stochastic process for $a_t = \ln A_t$ with two components

$$a_t = a_0 + a_t^P + a_t^T \quad (17)$$

with

$$\begin{aligned} a_t^P &= \rho a_{t-1}^P + v_t \\ v_t &\sim \text{iid } N(0, \sigma_v^2) \end{aligned} \quad (18)$$

and

$$a_t^T \sim \text{iid } N(0, \sigma_T^2) \quad (19)$$

The log of productivity thus follows a first order Markov process with both persistent and transitory components. The transitory component does not influence the investment decision if the firm faces no cost premium for external finance, but does affect the availability of internal funds to finance investment spending. We choose parameters $a_0 = -1.6725$, $\rho = 0.8$, $\sigma_v^2 = 0.0225$, and $\sigma_T^2 = 0.0375$, giving serial correlation in a_t of around 0.5.

We assume a standard functional form for adjustment costs

$$G(K_t, I_t) = \frac{b}{2} \left(\frac{I_t}{K_t} - \delta - e_t \right)^2 K_t \quad (20)$$

which is strictly convex in I_t and homogeneous of degree one in (K_t, I_t) . The rate of depreciation is set to $\delta = 0.15$ and e_t is a mean zero adjustment cost shock, distributed as $e_t \sim \text{iid } N(0, \sigma_e^2)$ with $\sigma_e^2 = 0.0016$. Adjustment costs are minimised by setting net investment to zero on average. Since there is also no trend in the productivity process, this generates optimal choices for investment, capital and output with no systematic trends.

With no cost premium for external funds, this gives a convenient linear functional form for the first order condition for investment (4)

$$\frac{I_t}{K_t} = \left(\delta - \frac{1}{b} \right) + \frac{1}{b} (\beta E_t[\mu_{t+1}^K]) + e_t \quad (21)$$

where, as noted earlier, $\beta E_t[\mu_{t+1}^K]$ is marginal q given our timing assumption that current investment becomes productive in period $t + 1$. Using (10), this can be written as

$$\frac{I_t}{K_t} = \left(\delta - \frac{1}{b} \right) + \frac{1}{b} \left(\beta E_t \left[\frac{V_{t+1}}{K_{t+1}} \right] \right) + e_t \quad (22)$$

which simplifies to

$$\frac{I_t}{K_t} = \left(\delta - \frac{1}{b} \right) + \frac{1}{b} \left(\frac{\beta E_t[V_{t+1}]}{K_{t+1}} \right) + e_t \quad (23)$$

since $K_{t+1} = (1 - \delta)K_t + I_t$ is known in period t . We exploit the recursive structure of the value function (1) to obtain

$$\beta E_t[V_{t+1}] = V_t - \Pi_t \quad (24)$$

so that the specification we estimate on the simulated data is

$$\frac{I_t}{K_t} = \left(\delta - \frac{1}{b} \right) + \frac{1}{b} Q_t + e_t \quad (25)$$

where average q is measured as⁹

$$Q_t = \frac{V_t - \Pi_t}{K_{t+1}} \quad (26)$$

The adjustment cost parameter b is set to 5, giving a coefficient on average q of 0.2 in the absence of capital market imperfections. The discount factor β used to generate the simulated investment data is set to 0.95.

Given that the net revenue function (16) is homogeneous of degree one in (K_t, I_t) , the firm's value maximisation problem would have no unique solution in the absence of strictly convex adjustment costs. This requires a different numerical solution method to those that have been used in related papers, which have

⁹Similar results were obtained when using the realised value V_{t+1} to substitute for the expected value $E_t[V_{t+1}]$ in (23), and estimating using instrumental variables dated t and earlier, which are orthogonal to the rational expectations forecast error. In this case the parameter we estimate on $E_t[V_{t+1}]/K_{t+1}$ is β/b .

simulated optimal investment data for models with net revenue functions that are not homogeneous.¹⁰ We choose parameters for the productivity process such that, on average, the firm would not want to expand or to contract in the absence of adjustment costs. The numerical optimisation procedure we use to generate the simulated investment data is described in Appendix A. One of the contributions of this section is thus to provide the first analysis of simulated investment data for a model with quadratic adjustment costs and a linear homogeneous net revenue function, for which structural investment equations like (25) are correctly specified in the absence of capital market imperfections, and which has been a popular specification in the empirical literature.

To extend this analysis to include a cost premium for external funds, we use the increasing cost schedule

$$\Phi(N_t, K_t) = \left(\frac{\phi}{2}\right) \left(\frac{N_t}{K_t}\right)^2 K_t \quad (27)$$

that was considered in section 3.1. Setting $\phi = 0$ gives the benchmark case in which external funds are a perfect substitute for internal funds, and the investment equation (25) is correctly specified. Setting $\phi > 0$ gives cases in which external finance is more costly than internal finance, and the investment spending of firms that are using external finance (i.e. those with $N_t > 0$) is financially constrained in the sense described in section 3.1. We choose values of ϕ and parameters of the productivity process to ensure that a non-negligible proportion of the observations in our simulated datasets are in the constrained regime with $N_t > 0$, and also to ensure that $\Phi_{Nt} = \phi \left(\frac{N_t}{K_t}\right) < 1$, so that firms can always finance additional investment spending by issuing more new shares.

We consider the behaviour of the estimated coefficients on both average q and

¹⁰See, for example, Gomes (2001) and Cooper and Ejarque (2003).

the cash flow variable in the standard ‘excess sensitivity’ test specification

$$\frac{I_t}{K_t} = \left(\delta - \frac{1}{b} \right) + \frac{1}{b} Q_t + \gamma \left(\frac{C_t}{K_t} \right) + e_t \quad (28)$$

Cash flow (C_t) is measured as $A_t K_t - G(K_t, I_t)$, while output (Y_t) is measured as $A_t K_t$. The null hypothesis $\gamma = 0$ corresponds to the case with no cost premium for external funds. More generally, the coefficient γ estimates the sensitivity of investment spending to cash flow conditional on average q . However this simple linear specification imposes the restriction that this conditional investment-cash flow sensitivity is common to all the observations in the sample. When firms face a cost premium for external finance, this linear model is certainly mis-specified; we know that the conditional sensitivity of investment to cash flow should be positive for firms using external finance in period t , but should also be zero for firms that are not using external finance in period t .¹¹

Our analysis of investment equations estimated on these simulated datasets will thus indicate whether our theoretical result on the monotonic relationship between conditional investment-cash flow sensitivity and the cost premium for external funds *for observations in the constrained regime* is useful for understanding the behaviour of the estimates of conditional investment-cash flow sensitivity that are commonly reported in the empirical literature. In section 5.3 we also note how the cost premium parameter ϕ can be estimated directly from a correctly specified structural investment equation for models with linear homogeneity and an increasing cost premium for external funds of the form considered here.

¹¹While some papers such as Bond and Meghir (1994) have attempted to use current financial policy information to classify *observations* to different regimes, this approach has not been common in the empirical literature.

5.2 Results

We generate simulated panel datasets for samples with 2000 firms observed for 16 periods. The generated data has the expected time series properties for a model with a linear homogeneous net revenue function, so that in the absence of adjustment costs firms would have no optimal size.¹² The logs of the firm value and capital stock series are integrated of order one, while the investment rates and average q series are integrated of order zero; indicating that firm value and capital stocks are cointegrated in this framework. The mean of the simulated average q variable is close to one; the mean of the investment rates is close to 0.15, the rate of depreciation; and there are no systematic trends in the capital stocks or other measures of firm size.

Column (i) of Table 1 reports the OLS estimates of model (25) for a sample in which there is no cost premium for external finance. The intercept coefficient is close to the theoretical value of -0.05, and the coefficient on average q is close to the theoretical value of 0.2. While similar results were obtained using 2SLS with a variety of instrument sets, there is no indication that the average q variable defined in (26) is correlated with the *iid* adjustment cost shocks (e_t).

Column (ii) of Table 1 reports the OLS estimates of model (28), with a linear cash flow term included, for the same sample. As expected under the null of perfect capital markets, the baseline average q model is correctly specified, and there is no evidence of ‘excess sensitivity’ of investment to cash flow. For this specification, there is also no correlation between the cash flow variable (C_t/K_t) and the adjustment cost shocks, so that the OLS estimates correctly indicate that there is no sensitivity of investment to cash flow, conditional on average q .

Column (iii) of Table 1 estimates this model for a sample in which all firms face

¹²See, for example, Lucas (1967).

a cost premium for external finance, with the parameter ϕ set to 1.6. Firms issue new equity in 28.4% of the observations in this sample, so we expect to find some evidence of ‘excess sensitivity’ to cash flow. This is reflected in a lower coefficient on average q and a significantly positive coefficient on the cash flow variable in the OLS estimates of model (28).

Column (iv) of Table 1 repeats this experiment for a sample in which all firms face a higher cost premium for external funds, with $\phi = 4$. The fraction of the observations with firms in the constrained regime is slightly lower in this case at 27.7%. Nevertheless any effect of this is dominated by the greater sensitivity of investment to cash flow, conditional on average q , for the firms in the constrained regime in this sample. The estimated coefficient on average q is lower here than in column (iii), while the estimated coefficient on the cash flow variable is considerably higher. This comparison thus suggests that the estimates of conditional investment-cash flow sensitivities obtained from these simple linear specifications follow the monotonic pattern that we obtained theoretically in section 3.1 for firms in the financially constrained regime of this model.

Table 2 confirms that these differences in the estimated coefficients on the cash flow variable, for firms facing different cost premia for external finance, are significantly different from each other. In column (i), half the firms in the sample face no cost premium for external funds, while half the firms face an increasing cost schedule for external funds with $\phi = 4$. The researcher is assumed to know *a priori* which firms are (always) ‘unconstrained’ and which firms are (potentially) ‘constrained’. All the terms in model (28) are interacted with a binary dummy variable *Dum* equal to one for the firms with a positive cost premium for external finance, and zero otherwise. This allows all the coefficients to be different for the two sub-samples, and mimics the kind of ‘sample splitting’ test that is common

in the empirical literature. The results indicate that the estimated coefficient on average q for the sub-sample facing a cost premium is significantly lower than the coefficient on average q for the sub-sample facing no cost premium; while the coefficient on the cash flow variable is significantly higher for the sub-sample whose investment spending is financially constrained in the periods when they use external finance. As expected, the coefficient on cash flow is not significantly different from zero for the sub-sample whose investment spending is never financially constrained.

Column (ii) of Table 2 considers a similar exercise where half the firms in the sample face a low cost premium for external funds ($\phi_L = 1.6$) and the remaining firms face a higher cost premium ($\phi_H = 4$). In this case the investment spending of firms in both sub-samples will be sensitive to the availability of internal funds, conditional on average q , in the periods when they are reliant on external funds. However, as shown in section 3.1, we expect the conditional sensitivity of investment to cash flow to be higher in this regime for the firms which face a higher cost premium for external finance. As expected, and consistent with the results of Table 1, we find that the coefficient on the cash flow variable is significantly different from zero for both sub-samples. More interestingly, these results indicate that the estimated coefficient on the cash flow variable is not just higher but significantly higher for the sub-sample that face the larger cost premium for external funds.

This analysis thus suggests that the monotonic relationship between the slope of the cost premium for external funds and conditional investment-cash flow sensitivity for firms in the financially constrained regime, that we showed theoretically in section 3.1, can be detected by estimates of conditional investment-cash flow sensitivity obtained from the simple linear specifications that have been commonly used in the empirical literature. Of course these linear models are clearly

mis-specified if the conditional investment-cash flow sensitivity is present only for a subset of the observations on firms with a positive cost premium for external funds, in the periods when they are reliant on external finance. In the next section we note how the cost premium parameter (ϕ) can be estimated directly from a correctly specified structural model derived from the first order condition for investment (4), in the case where firms have a linear homogeneous net revenue function and face an increasing cost premium for external funds of the form specified in (27).

5.3 A structural specification with costly external finance

Combining the first order condition for investment (4) with the first order condition for new shares (6), and using the form of the cost premium (27) as in (8), gives the condition

$$-\Pi_{It} = \beta E_t[\mu_{t+1}^K] \left(1 - \phi \left(\frac{N_t}{K_t} \right) \right) \quad (29)$$

Using the forms of the net revenue function (16) and the adjustment cost function (20) then gives

$$\frac{I_t}{K_t} = \left(\delta - \frac{1}{b} \right) + \frac{1}{b} \left[\beta E_t[\mu_{t+1}^K] \left(1 - \phi \left(\frac{N_t}{K_t} \right) \right) \right] + e_t \quad (30)$$

or

$$\frac{I_t}{K_t} = \left(\delta - \frac{1}{b} \right) + \frac{1}{b} (\beta E_t[\mu_{t+1}^K]) - \frac{\phi}{b} \left(\beta E_t[\mu_{t+1}^K] \left(\frac{N_t}{K_t} \right) \right) + e_t \quad (31)$$

which reduces to (21) when the cost premium parameter $\phi = 0$. As expected marginal q , as conventionally defined for the case of perfect capital markets (i.e. $\beta E_t[\mu_{t+1}^K]$ given our timing assumptions), is not a sufficient statistic for investment rates in the model with an increasing cost premium for external funds.

Given linear homogeneity we can again replace marginal q by an observable

measure of average q , giving

$$\frac{I_t}{K_t} = \left(\delta - \frac{1}{b} \right) + \frac{1}{b} Q_t - \frac{\phi}{b} \left(Q_t \left(\frac{N_t}{K_t} \right) \right) + e_t \quad (32)$$

where average q is again given by (26). This model can be estimated given data on investment rates, average q , and the value of new shares issued. The parameters are all structural parameters of the adjustment cost function or the cost premium function, and the error term is again the stochastic shock to the rate of investment at which adjustment costs are minimised.

Several points can be noted about this specification. For firms paying positive dividends and issuing no new shares, this reduces to the standard specification under perfect capital markets given in (25). The additional term in (32) is an interaction between average q and new equity, which is zero when no external finance is used. This interaction term has a negative coefficient, consistent with the result that at a given level of q , firms using high cost external finance will choose lower investment rates than firms with sufficient low cost internal funds to finance all their investment spending (see Figure 5, noting that $(I_2/K) < (I_3/K)$). The cost premium parameter ϕ is identified from the coefficient on this additional interaction term.

The linear cash flow term (C_t/K_t) included in the excess sensitivity test specification (28) is negatively correlated with $Q_t \left(\frac{N_t}{K_t} \right)$, and thereby positively correlated with the omitted variable $-\frac{\phi}{b} \left(Q_t \left(\frac{N_t}{K_t} \right) \right)$ that is relevant when $\phi > 0$. This is consistent with the positive coefficients found on the included linear cash flow terms in columns (iii) and (iv) of Table 1. The correlation between (C_t/K_t) and $Q_t \left(\frac{N_t}{K_t} \right)$ is -0.306 and -0.337 respectively in these simulated datasets.

Finally the use of external finance depends in part on the realisation of the adjustment cost shock (e_t). All else equal, firms experiencing adjustment cost shocks that make them want to undertake additional investment are more likely

to be in the financially constrained regime of the model, with $N_t > 0$ (see Figure 5, comparing $-\Pi_{I1}$ and $-\Pi_{I2}$). This suggests that the interaction term is likely to be correlated with the error term in (32), and this was found to be the case in our simulated data. For example, the correlation between $Q_t \left(\frac{N_t}{K_t} \right)$ and e_t is around 0.56 in the samples used in Table 1. Consequently OLS estimates of the coefficients in (32) are biased and inconsistent. However we can exploit the structure of the model to obtain valid and informative instruments. Given our timing assumptions, the available instruments that are orthogonal to iid adjustment cost shocks include current and lagged average q , lagged values of the interaction term, and current output. Since current output reflects the current shock to the persistent component of the productivity process (i.e. v_t in (18) above), and this productivity innovation also affects both Q_t and N_t , current output is expected to be an informative instrument. The correlation between current output and the interaction term is around -0.15 in the samples used in Table 1.

Table 3 presents 2SLS estimates of model (32) using these instrumental variables. The three columns use the same simulated datasets that were used in Table 1, with values of the cost premium parameter ϕ set to zero, 1.6 and 4 respectively, and common to all firms in the generated samples. The expected values of the coefficient $(-\phi/b)$ are thus zero in column (i), -0.32 in column (ii) and -0.8 in column (iii). The estimated coefficients on the linear average q terms are close to their expected value of 0.2 in all three samples. In column (i), the estimated coefficient on the interaction term is not significantly different from zero, correctly indicating that the firms in this sample do not face a cost premium for external funds. In column (ii), the estimated coefficient on the interaction term is significantly different from zero, and close to its expected value of -0.32. In column (iii), the estimated coefficient on the interaction term is again significantly different

from zero, and not significantly different from its expected value of -0.8.

These instrumental variables estimates of the structural model (32) thus permit reliable inference about the presence of a cost premium for external finance, and provide a reasonable quantitative guide to the size of this cost premium in different samples. Of course this specific model, in which new equity is the only source of external finance, is much too simple to be useful in practice. Nevertheless our results in this section suggest that this is a promising direction for further research, incorporating debt finance and ideally a richer set of financial policies available to firms, such as the accumulation of liquid financial assets. Hennessy, Levy and Whited (2005) suggest some tractable ways of extending this kind of structural model, and present some interesting results for publicly traded US corporations.

6 Conclusions

In contrast to Kaplan and Zingales (1997), we find that in a dynamic investment problem with quadratic costs of adjustment, there is a monotonic relationship between conditional investment-cash flow sensitivity and the severity of the financing constraint, as reflected in the size or slope of the cost premium for external finance. In particular we provide a benchmark specification in which a higher cost premium for one group of firms would be reflected in a greater sensitivity of investment to windfall fluctuations in cash flow, conditional on average q , for observations in the constrained financing regime, than would be expected for an otherwise identical group of firms with a lower cost premium for external finance. Our adjustment costs framework is more closely related to the empirical literature on investment and financing constraints than the static demand for capital framework analysed by Kaplan and Zingales (1997). Results using simulated investment data suggest that, if adjustments costs are quadratic and linear homogeneity is satisfied,

a higher cost premium for external funds would also be reflected in a higher coefficient on a cash flow variable in the typical econometric specification that relates investment rates to measures of both cash flow and average q .

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Appendix A: The dynamic programming model

This appendix describes the numerical optimisation procedure used to generate the simulated investment data analysed in Section 5. The value of the firm is given by the Bellman equation

$$V(K_t, a_{t-1}^P; v_t, a_t^T, e_t) = \max_{I_t, K_{t+1}, N_t} \left\{ \begin{array}{l} \Pi(K_t, a_{t-1}^P, I_t, N_t; v_t, a_t^T, e_t) \\ + \beta E_t [V(K_{t+1}, a_t^P; v_{t+1}, a_{t+1}^T, e_{t+1})] \end{array} \right\}$$

subject to the capital evolution constraint

$$K_{t+1} = I_t + (1 - \delta) K_t; \quad (33)$$

and the non-negativity constraints for dividends and new share issues

$$\Pi(K_t, a_{t-1}^P, I_t, N_t; v_t, a_t^T, e_t) + N_t \geq 0; \quad N_t \geq 0 \quad (34)$$

where $\Pi(K_t, a_{t-1}^P, I_t, N_t)$ is the net revenue function

$$\Pi(K_t, a_{t-1}^P, I_t, N_t; v_t, a_t^T, e_t) = A(a_{t-1}^P; v_t, a_t^T) K_t - G(K_t, I_t; e_t) - I_t - \Phi(N_t, K_t)$$

and $A(a_{t-1}^P; v_t, a_t^T)$, $G(K_t, I_t; e_t)$ and $\Phi(N_t, K_t)$ are parameterized as described in Section 5. Hence, at time t , K_t and a_{t-1}^P are state variables and I_t , K_{t+1} and N_t are control variables. It is convenient to plug in the capital evolution constraint (33) into the Bellman equation, in order to reduce the number of control variables from three to two

$$V(K_t, a_{t-1}^P; v_t, a_t^T, e_t) = \max_{K_{t+1}, N_t} \left\{ \begin{array}{l} \Pi(K_t, a_{t-1}^P, I_t(K_{t+1}), N_t; v_t, a_t^T, e_t) \\ + \beta E_t [V(K_{t+1}, a_t^P; v_{t+1}, a_{t+1}^T, e_{t+1})] \end{array} \right\} \quad (35)$$

with $I_t(K_{t+1}) = K_{t+1} - (1 - \delta) K_t$, subject to the non-negativity constraints (34).

As far as we know, it is not possible to solve for (K_{t+1}, N_t) analytically as a function of the state variables, the productivity innovations and the adjustment cost shock in this model. Therefore we use numerical methods to simulate optimal investment data.

Solving the Bellman equation using value function iteration

We solve the firm's optimisation problem (35), subject to (34), using value function iteration. Conveniently, because the value function is homogeneous of

degree one in capital, it is sufficient to evaluate the value function at one arbitrary level of K_t , say \bar{K} ¹³

$$\begin{aligned} V(\bar{K}, a_{t-1}^P; v_t, a_t^T, e_t) &= \max_{K_{t+1}, N_t} \left\{ \begin{array}{l} \Pi(\bar{K}, a_{t-1}^P, I_t(K_{t+1}), N_t; v_t, a_t^T, e_t) \\ + \beta E_t [V(K_{t+1}, a_t^P; v_{t+1}, a_{t+1}^T, e_{t+1})] \end{array} \right\} \\ V(\bar{K}, a_{t-1}^P; v_t, a_t^T, e_t) &= \max_{K_{t+1}, N_t} \left\{ \begin{array}{l} \Pi(\bar{K}, a_{t-1}^P, I_t(K_{t+1}), N_t; v_t, a_t^T, e_t) \\ + \beta \frac{K_{t+1}}{\bar{K}} E_t [V(\bar{K}, a_t^P; v_{t+1}, a_{t+1}^T, e_{t+1})] \end{array} \right\} \quad (36) \end{aligned}$$

Once a solution conditional on $K_t = \bar{K}$ has been obtained, the value conditional on general K_t can be obtained by linear extrapolation

$$V(K_t, a_{t-1}^P; v_t, a_t^T, e_{t+1}) = V(\bar{K}, a_{t-1}^P; v_t, a_t^T, e_{t+1}) \frac{K_t}{\bar{K}}$$

Naturally, linear homogeneity is crucial for this to be an appropriate approach.

The principles of our value function iteration algorithm are as follows.

1. Start with a guess for the true value function $V(\bar{K}, a_{t-1}^P; v_t, a_t^T, e_t)$. Call this guess V^1 . Use it on the right-hand side of the Bellman equation (36), and compute $E_t[V(\bar{K}, a_t^P; v_{t+1}, a_{t+1}^T, e_{t+1})]$ by integrating over the innovations $v_{t+1}, a_{t+1}^T, z_{t+1}$ (more on this step below). Solve for K_{t+1} and N_t , imposing the non-negativity constraints (34) (more on this step below too).
2. Update the guess for the true value function using the solution (K_{t+1}, N_t) obtained in the previous step. Call this updated guess V^2 . Check if $V^2 = V^1$. If true, we have converged to the true function and so iteration can stop; if not go to step 3.
3. For $j = 3, 4, \dots$, use V^{j-1} on the right-hand side of the Bellman equation and calculate the optimal choice rule (K_{t+1}, N_t) , subject to (34). Update the guess of the value function, V^j . Check if $V^j = V^{j-1}$. If true, there is convergence and so iteration stops; if not, set $j = j + 1$ and repeat step 3.

¹³An alternative, equivalent approach would be to divide through by K_t and rewrite the Bellman equation (35) as

$v(A_{t-1}^P/K_t; v_t, a_t^T, e_t) = \max_{(I_t/K_t), (N_t/K_t)} \pi(A_{t-1}^P/K_t, I_t/K_t, N_t/K_t; v_t, a_t^T, e_t) + (I_t/K_t + 1 - \delta) \beta E_t [v(A_t^P/K_{t+1}; v_{t+1}, a_{t+1}^T, e_{t+1})],$
where $v \equiv V/K$ and $\pi \equiv \Pi/K$ (see Bloom, 2005).

To implement this method we need to deal with three main difficulties: first, while discretisation of the state space is a necessity in numerical dynamic programming, we need to allow for the fact that we are dealing with continuous variables (e.g. capital); second, we need a way of calculating the expected value of the firm in the next period; third, we need a way of imposing the non-negativity constraints (34). We now discuss these issues in turn.

The approximation of the value function

We deal with the first problem mentioned above by discretising the state space, setting the chosen values equal to the optimal nodes of a Chebyshev polynomial. We then interpolate between nodes using the Chebyshev iterative formula. Conditional on the innovations v_t, a_t^T, e_t we write the Chebyshev polynomial approximation of the value function as

$$V(\bar{K}, a_{t-1}^P; v_t, a_t^T, e_t) \simeq \sum_{j=0}^n \varphi_j^{v, a^T, e} T_j \left(\frac{2a_{t-1}^P - a^{hi} - a^{lo}}{a^{hi} - a^{lo}} \right)$$

where $\varphi_j^{v, a^T, e}$, $j = 0, 1, \dots, n$ are the Chebyshev coefficients, a^{hi} and a^{lo} are upper and lower limits on a_{t-1}^P , and¹⁴

$$\begin{aligned} T_0(a^P) &= 1 \\ T_1(a^P) &= \frac{2a^P - a^{hi} - a^{lo}}{a^{hi} - a^{lo}} \\ T_{i+1}(a^P) &= 2 \left(\frac{2a^P - a^{hi} - a^{lo}}{a^{hi} - a^{lo}} \right) T_i(a^P) - T_{i-1}(a^P), \quad i = 2, \dots, n-1 \end{aligned}$$

Conditional on v_t, a_t^T, e_t the value function V is evaluated at m values of a_{t-1}^P : a^1, a^2, \dots, a^m , where

$$\begin{aligned} a^k &= (\theta^k + 1) \left(\frac{a^{hi} - a^{lo}}{2} \right) + a^{lo}, \quad k = 1, \dots, m \\ \theta^k &= -\cos \left(\frac{2k-1}{2m}\pi \right), \quad k = 1, \dots, m \end{aligned}$$

¹⁴The truncation of the distribution of the persistent component of productivity is introduced for computational reasons. We set $a^{lo} = -4\sigma_a$ and $a^{hi} = 4\sigma_a$, which given that a is normally distributed implies that the likelihood of truncation is approximately 0.000006.

In our applications, we set $m = 5$. Increasing m improves the accuracy of the approximation but also increases the computational time. Increasing m to 9 had very marginal effects on our results, probably because our value functions are smooth and monotonic.

The Chebyshev polynomial approximation of the value function has been used in previous work by Fafchamps and Pender (1997). For more details on this approach, see Chapter 6 in Judd (1998).

The expected value in the next period

To compute the expected value of the firm in the next period we use numerical integration, specifically a Gauss-Hermite quadrature. This involves evaluating the value function at a finite number of values for the random variables $v_{t+1}, a_{t+1}^T, e_{t+1}$, and summing the results using a set of weights. The weights and the positions of the nodes are determined by the Gauss-Hermite quadrature. Conditional on a_t^P , $E_t[V(\bar{K}, a_t^P; v_{t+1}, a_{t+1}^T, e_{t+1})]$ is given by

$$\begin{aligned} E_t[V(\bar{K}, a_t^P; v_{t+1}, a_{t+1}^T, e_{t+1})] &= (2\pi\sigma_v^2)^{-1/2} (2\pi\sigma_{a^T}^2)^{-1/2} (2\pi\sigma_e^2)^{-1/2} \\ &\quad \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(\bar{K}, a_t^P; v_{t+1}, a_{t+1}^T, e_{t+1}) \\ &\quad \times e^{-v^2/(2\sigma_v^2)} e^{-(a^T)^2/(2\sigma_{a^T}^2)} e^{-e^2/(2\sigma_e^2)} dv da^T de \end{aligned}$$

since v, a^T, e are normally distributed and independent of each other. An analytical solution cannot be obtained, so we use numerical integration based on Gauss-Hermite quadrature

$$\begin{aligned} E[V(\bar{K}, a_t^P; v_{t+1}, a_{t+1}^T, e_{t+1})] &\simeq \pi^{-3/2} \sum_{q1=1}^Q \omega_{q1} \sum_{q2=1}^Q \omega_{q2} \sum_{q3=1}^Q \omega_{q3} V(\bar{K}, a_t^P; \sqrt{2}\sigma_v x_{q1}, \sqrt{2}\sigma_{a^T} x_{q2}, \sqrt{2}\sigma_e x_{q3}) \end{aligned}$$

where Q is the number of quadrature nodes, and ω_q and x_q are fixed weights and nodes, respectively (see Judd, 1998, pp.261-262). We set $Q = 3$ throughout.

Imposing the non-negativity constraints

To impose the non-negativity constraints on dividends and new issues

$$\Pi(K_t, a_{t-1}^P, I_t, N_t; v_t, a_t^T, e_t) + N_t \geq 0; \quad N_t \geq 0 \quad (37)$$

we use a two stage procedure as follows.

1. First, we solve for optimal investment without imposing the constraint (37) and setting $N_t = 0$. At the resulting level of investment, we check if (37) holds (i.e. we check if $\Pi(K_t, a_{t-1}^P, I_t, N_t; v_t, a_t^T, e_t) \geq 0$). If $\Pi_t \geq 0$, we conclude that the non-negativity constraint on dividends does not bind, and take the resulting K_{t+1} , and $N_t = 0$, to be optimal. However, if $\Pi_t < 0$, the solution is not permissible. In this case we proceed to stage 2, which is as follows.
2. Grid search on N_t .
 - (a) Set $j = 1$. Start at $N_t = 0$ and obtain solutions (if real solutions exist) for investment that satisfy $\Pi_t = 0$ (conditional on N_t , solutions for which $\Pi_t = 0$ are always superior to solutions for which $\Pi_t > 0$). Because of quadratic adjustment costs, there may exist two real solutions. If so, pick the level of investment that is associated with the highest firm value. Store this firm value, denoted V^1 .
 - (b) Set $j = j + 1$. Increase N_t by a small amount, and obtain solutions (if real solutions exist) for investment that satisfy $\Pi_t + N_t = 0$. Store the relevant firm value, denoted V^j . Compare to the firm value obtained in the previous grid search step, i.e. V^{j-1} . If $V^j < V^{j-1}$, then go to (c). If $V^j \geq V^{j-1}$, repeat step (b).
 - (c) Stop grid search. The solution (K_{t+1}, N_t) obtained in the previous grid search step (i.e. step $j - 1$) is taken to be optimal.

Table 1. Excess Sensitivity Tests

	(i) $\phi = 0$	(ii) $\phi = 0$	(iii) $\phi = 1.6$	(iv) $\phi = 4$
$const$	-0.0481 (.0014)	-0.0483 (.0015)	-0.0495 (.0014)	-0.0516 (.0013)
Q_t	0.1983 (.0012)	0.1986 (.0020)	0.1896 (.0019)	0.1839 (.0018)
$\frac{C_t}{K_t}$		0.0003 (.0055)	0.0412 (.0052)	0.0755 (.0049)
R^2	0.43	0.43	0.47	0.50

Sample size: $N = 2000$ $T = 16$ Observations = 32,000

Ordinary least squares estimates

Table 2. Split Sample Tests

	(i)	(ii)
	$\phi_L = 0$ $\phi_H = 4$	$\phi_L = 1.6$ $\phi_H = 4$
<i>const</i>	-0.0476 (.0020)	-0.0521 (.0019)
Q_t	0.1987 (.0026)	0.1944 (.0026)
$\frac{C_t}{K_t}$	-0.0034 (.0073)	0.0303 (.0071)
Dum	-0.0001 (.0028)	0.0044 (.0026)
$Dum * Q_t$	-0.0190 (.0038)	-0.0148 (.0037)
$Dum * \left(\frac{C_t}{K_t}\right)$	0.0824 (.0104)	0.0487 (.0101)
R^2	0.46	0.48

Sample size: $N = 2000$ $T = 16$ Observations = 32,000

50% of firms have $\phi = \phi_L$ ($Dum = 0$); 50% of firms have $\phi = \phi_H$ ($Dum = 1$)

Ordinary least squares estimates

Table 3. Structural Model Estimates

	(i) $\phi = 0$	(ii) $\phi = 1.6$	(iii) $\phi = 4$
$const$	-0.0470 (.0017)	-0.0504 (.0019)	-0.0521 (.0022)
Q_t	0.1985 (.0014)	0.2003 (.0014)	0.2017 (.0015)
$Q_t * \left(\frac{N_t}{K_t} \right)$	-0.1173 (.0812)	-0.3300 (.1115)	-0.6657 (.1384)
p	0.18	0.79	0.52

Sample size: $N = 2000$ $T = 16$ Observations = 32,000

Two stage least squares estimates

Instrumental variables: $Q_t, Q_{t-1}, Q_{t-1} * \left(\frac{N_{t-1}}{K_{t-1}} \right), Y_t$

p is the p-value of the Sargan test of over-identifying restrictions

Figure 1 Static Demand for Capital

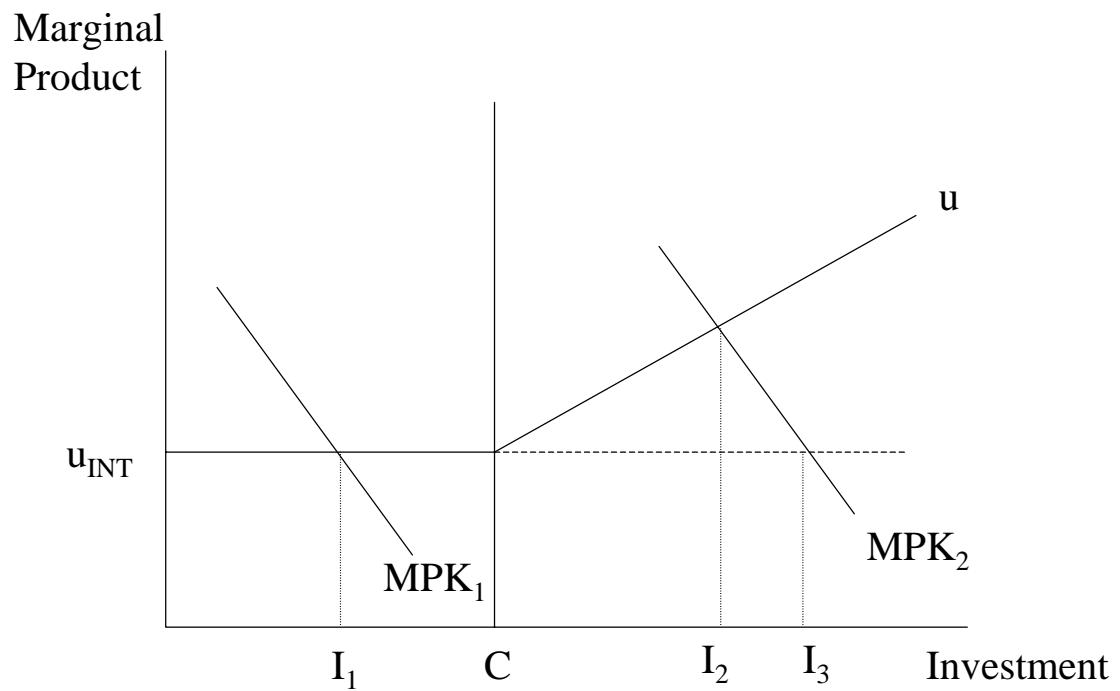


Figure 1:

Figure 2 A Cash Flow Shock

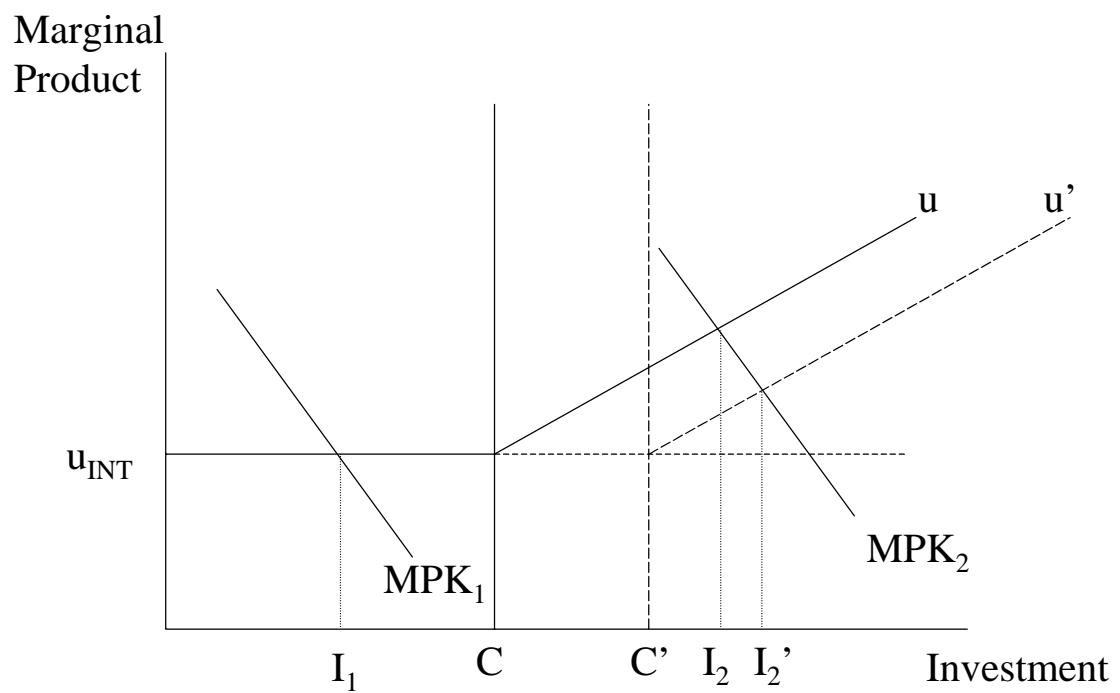


Figure 2:

Figure 3 Cost Premia $u_H > u_L$

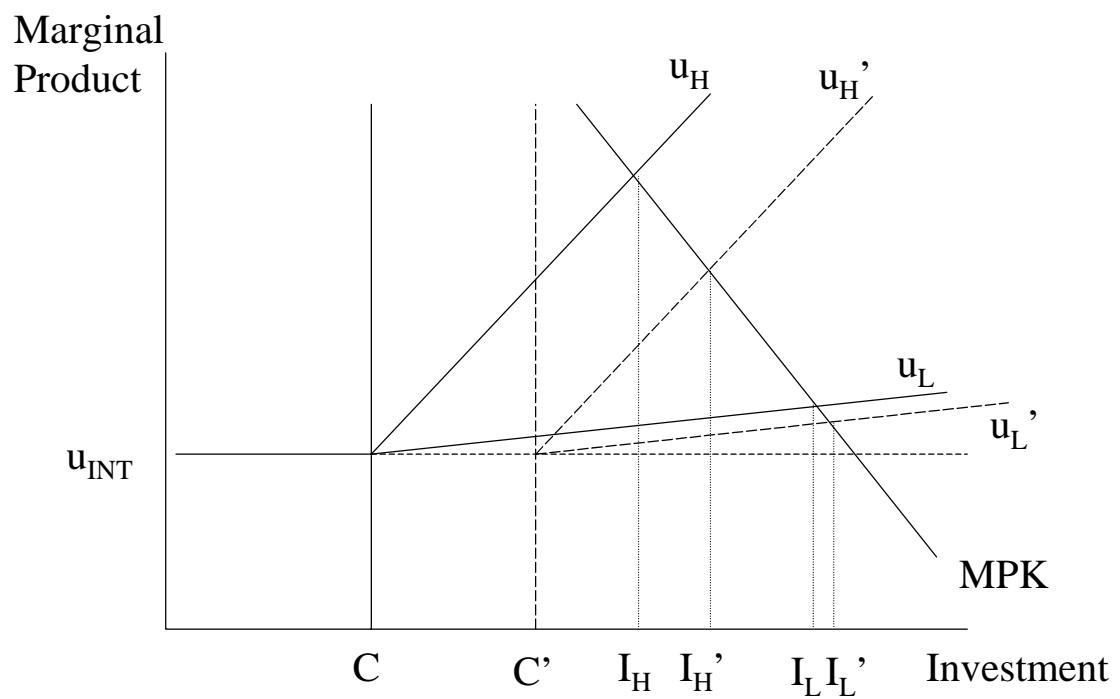


Figure 3:

Figure 4 The Kaplan-Zingales Case

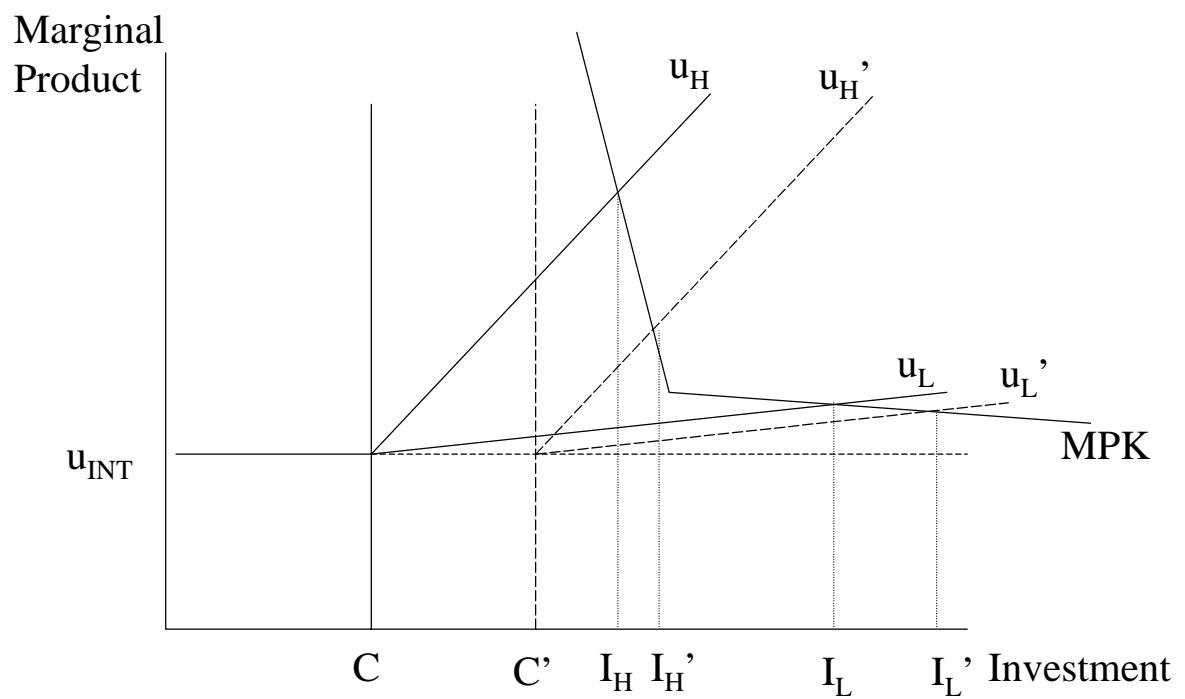


Figure 4:

Figure 5 Q Model - for given λ^K

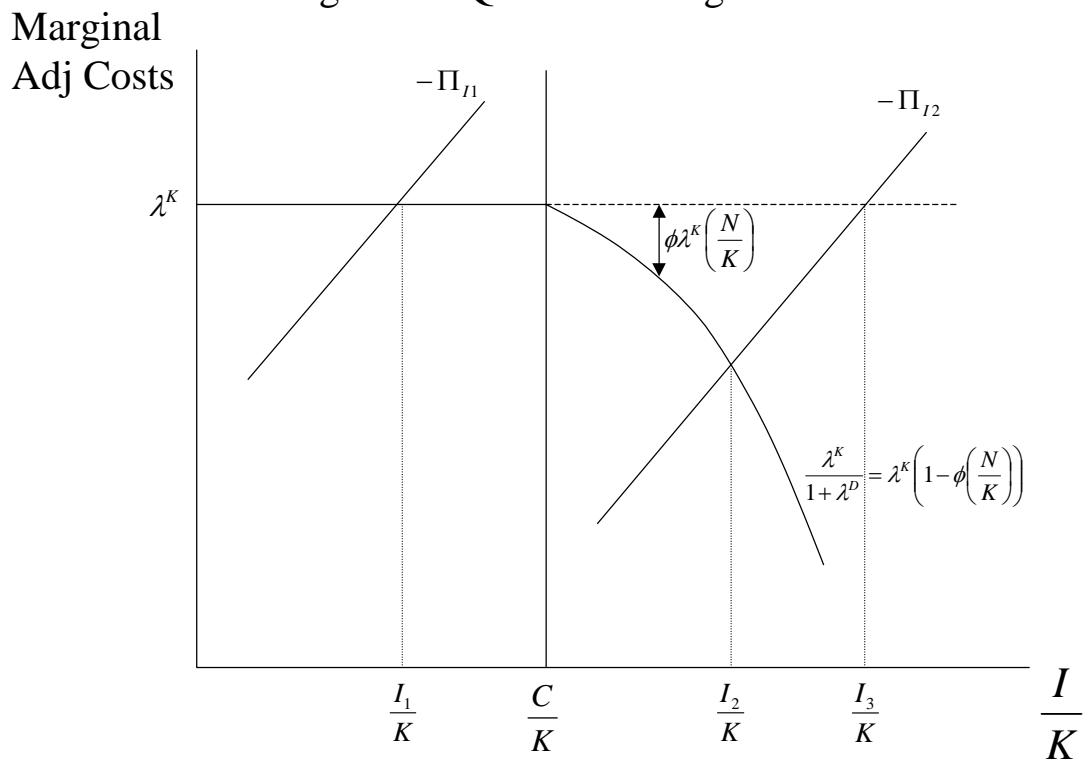


Figure 5:

Figure 6 A Cash Flow Shock

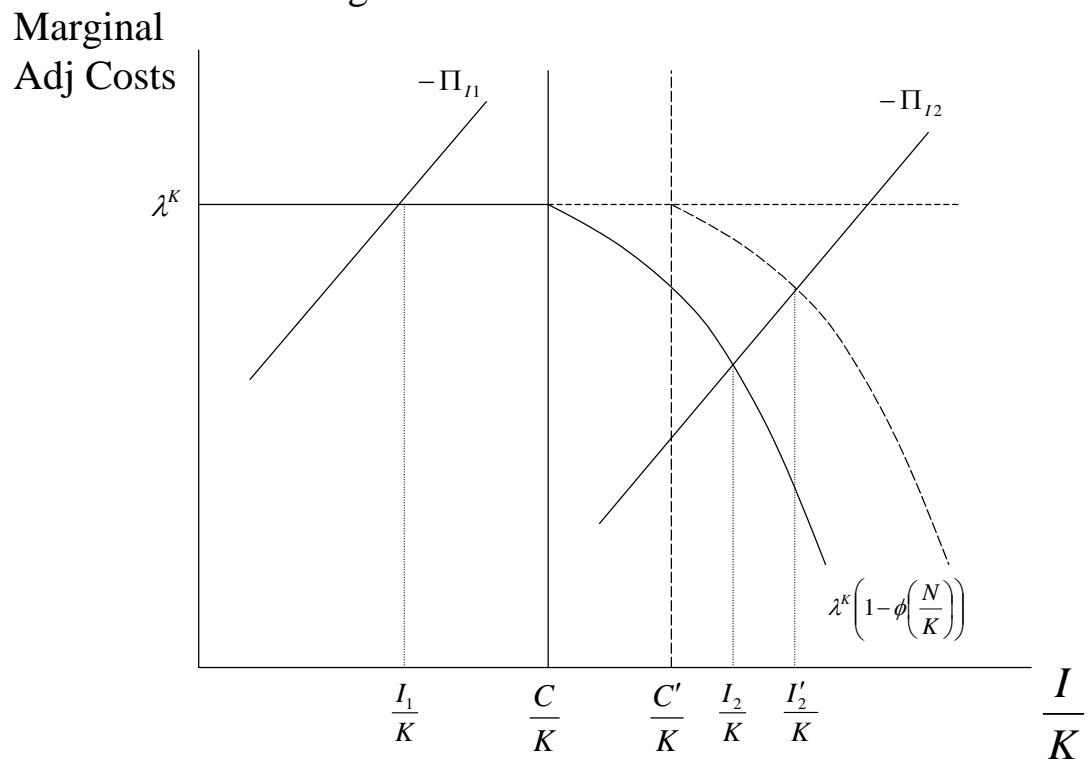


Figure 6:

Figure 7 Cost Premia $\phi_H > \phi_L$

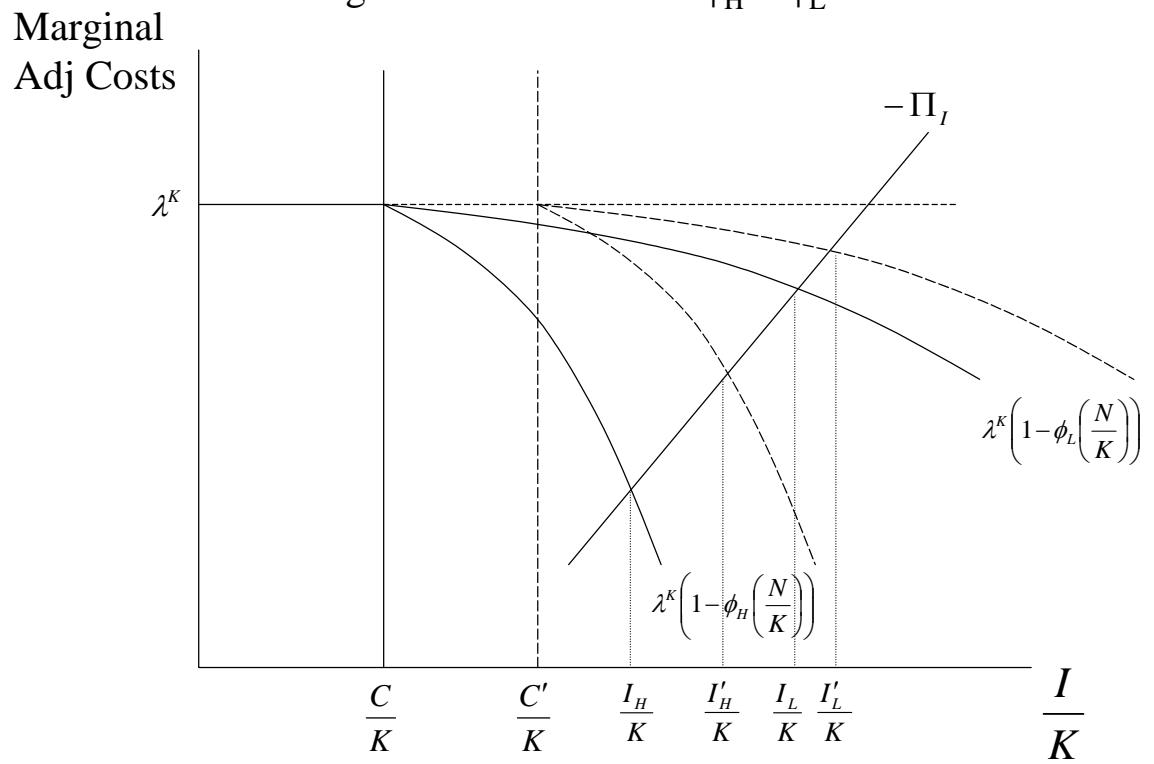


Figure 7:

Figure 8 Fixed Cost Premium

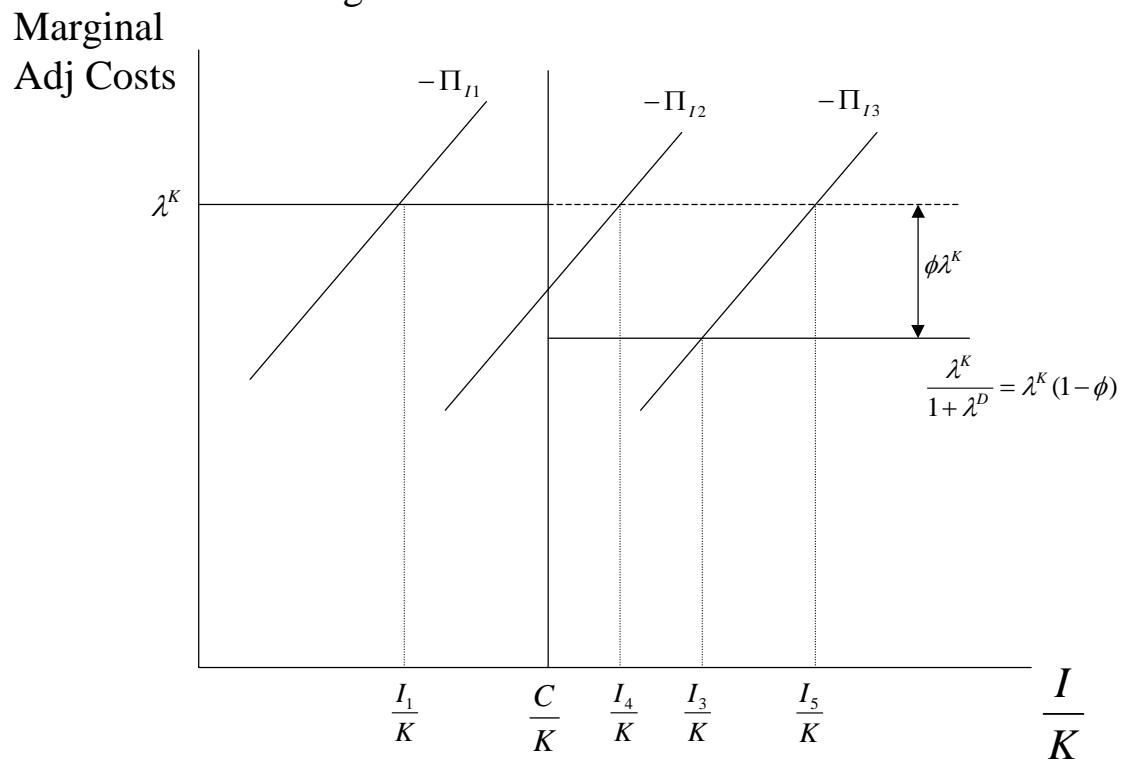


Figure 8:

Figure 9 A Small Cash Flow Shock

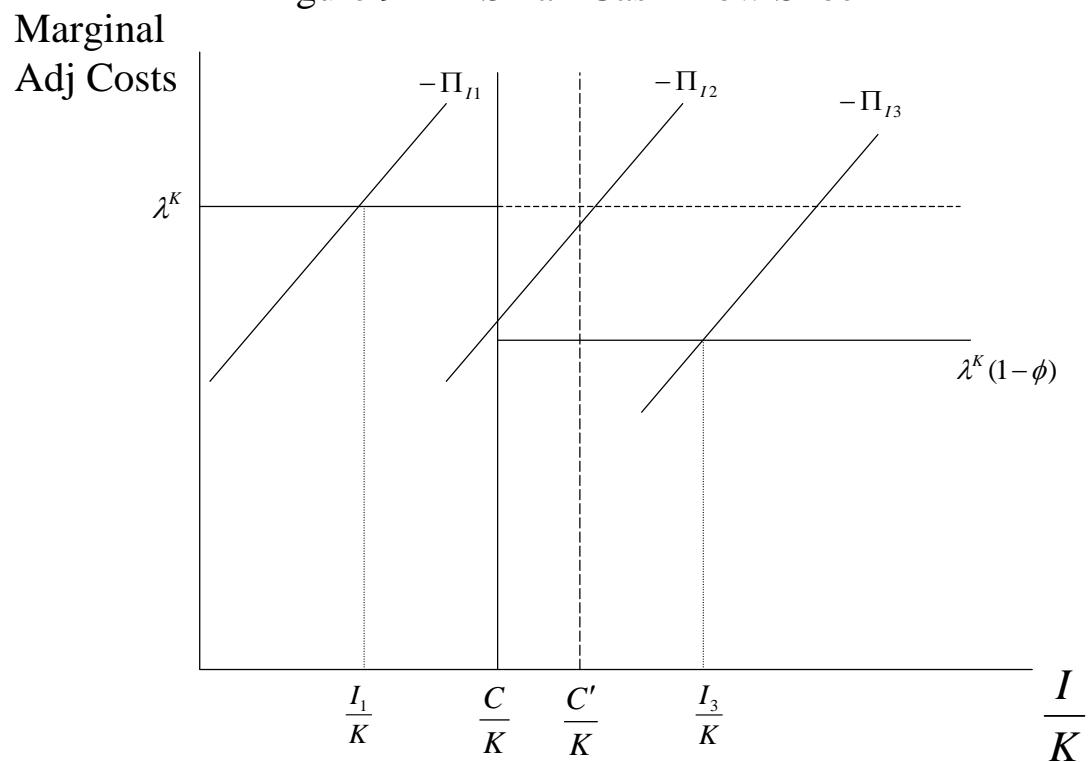


Figure 9:

Figure 10 A Larger Cash Flow Shock

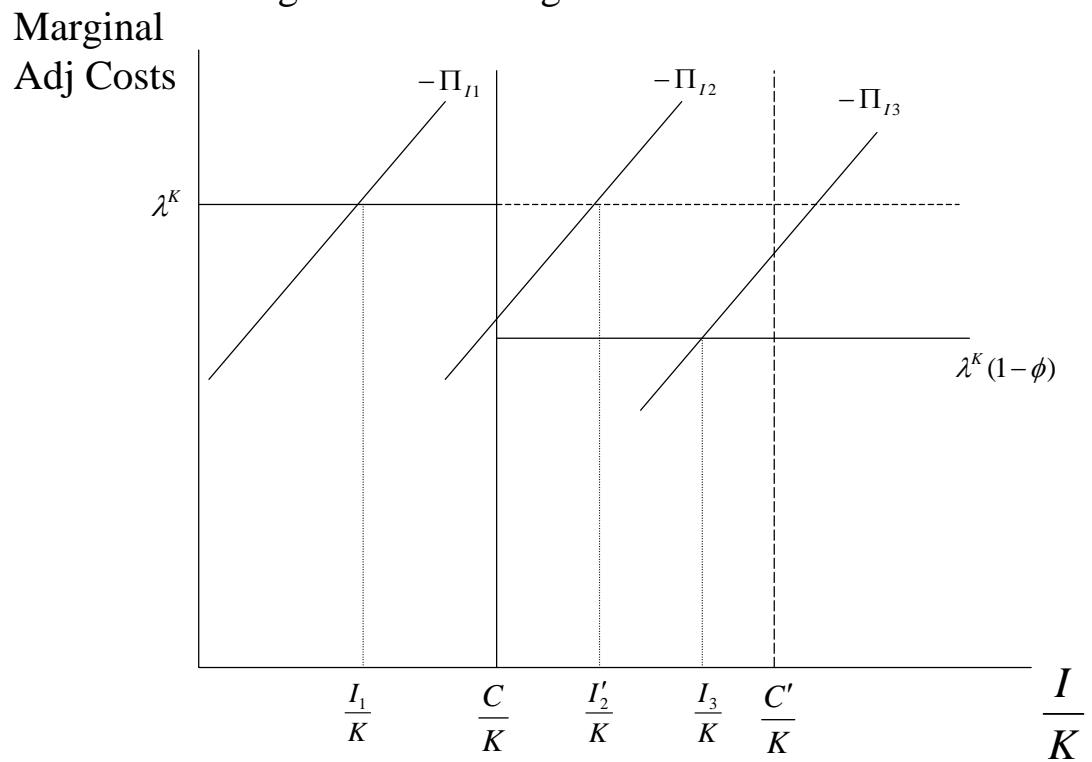


Figure 10:

Figure 11 Cost Premia $\phi_H > \phi_L$

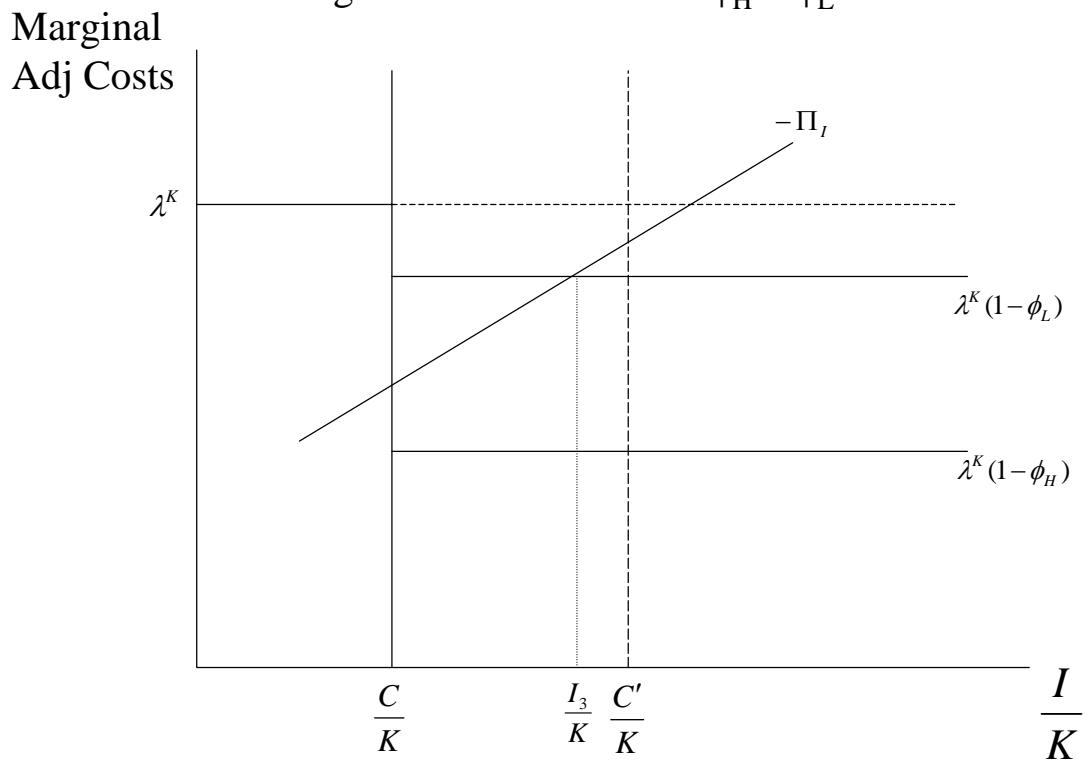


Figure 11:

Figure 12 Cost Premia $\phi_H > \phi_L$

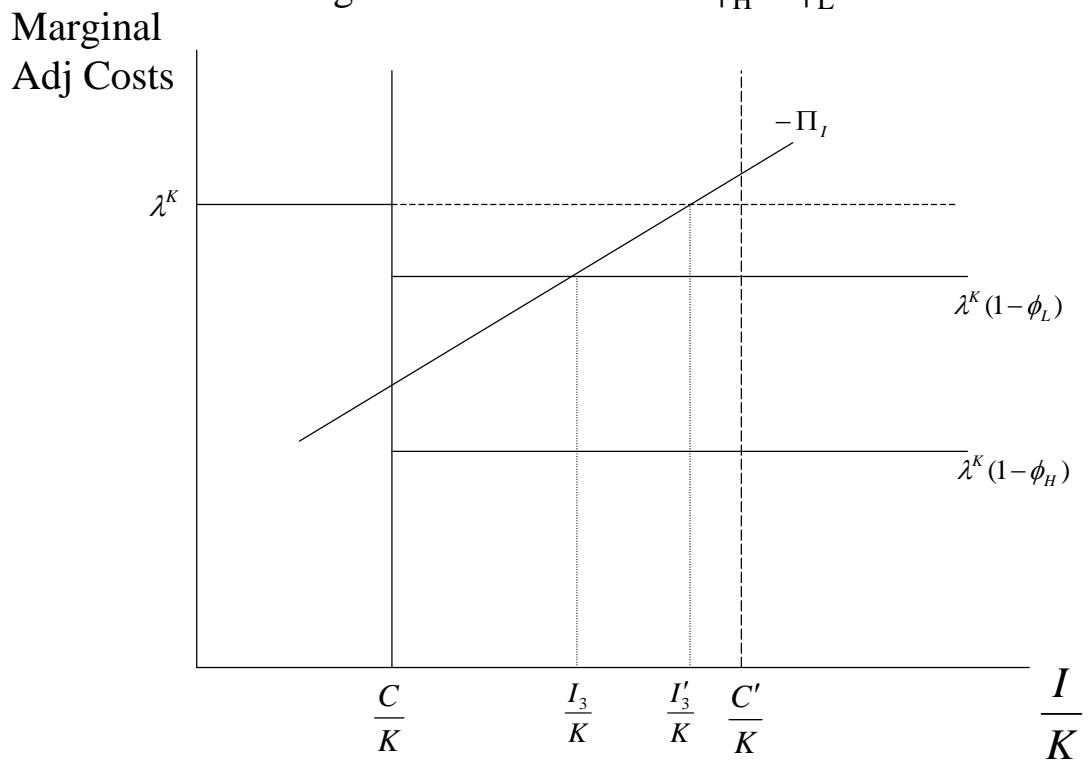


Figure 12:

Figure 13 Cost Premia $\phi_H > \phi_L$

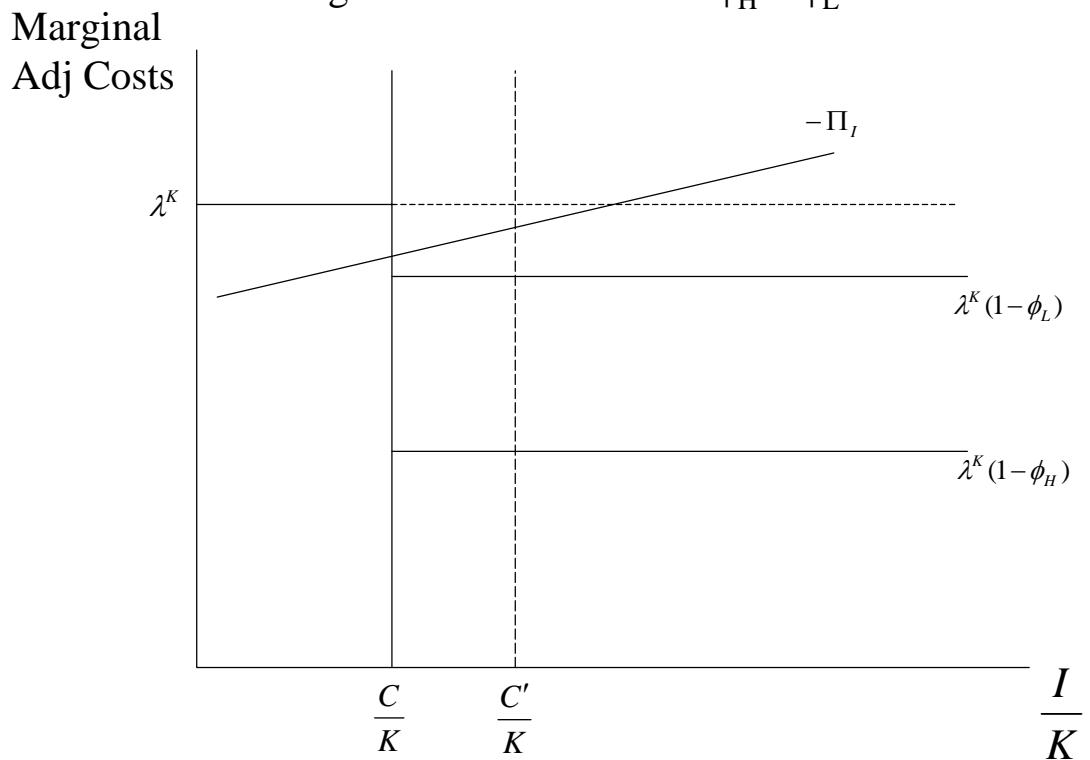


Figure 13: