14.4 - The Chain Rule

If $w = f(x)$ is a differentiable function of x, and $x = g(t)$ is a differentiable function of t, then w is a differentiable function of t, and $\frac{dw}{dt}$ can be calculated by the formula

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$$

Functions of Two Variable

If $w = f(x, y)$ is differentiable and if $x - x(t), y = y(t)$ are differentiable functions of t, then the composition $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$$

or

\frac{dw}{dt} =\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}

Functions Defined on Surfaces

Consider the temperature function $w = f(x, y, z)$ at points (x, y, z) on Earth's surface x, y, and z can be represented of functions of r and s that give the points' longitudes and latitudes.
If $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$ the temperature function is

$$w = f(g(r, s), h(r, s), k(r, s))$$

If all functions are differentiable, then w has partial derivatives with respect to r and s , given by:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Implicit Differentiation

Suppose that $F(x, y)$ differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then, at any point where $F_y \neq 0$:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Extension to three variables:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_y} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Implicit Function Theorem

The above equations are valid if the partial derivatives F_x, F_y, and F_z are continuous throughout an open region R containing the point (x_0, y_0, z_0), and if for a constant c, $F(x_0, y_0, z_0) = c$ and $F_z(x_0, y_0, z_0) \neq 0$.