13.2 - Integrals of Vector Functions

Integrals of Vector Functions

$R(t)$ is an antiderivative of a vector function $r(t)$ on an interval I if $\frac{dR}{dt} = r$ at each point of I
Every $R(t)$ of $r(t)$ has the form $R + C$ for some constant vector C.
C is a vector instead of a scalar.

$$\text{The indefinite integral (the set of all antiderivatives) is denoted by:} \\ \boxed{\int r(t)\,dt}$$

To integrate a vector function, integrate all of its components.

Definite Integrals of Vector Functions

$$\text{If the components of } r(t) = f(t)i + g(t)j + h(t)k \text{ are integrable over} \\ [a, b]\text{, then so is } r\text{, and the definite integral of } r \text{ from } a \text{ to } b \text{ is:} \\ \int_a^b r(t)\,dt = \left(\int_a^b f(t)\,dt\right)i + \left(\int_a^b g(t)\,dt\right)j + \left(\int_a^b h(t)\,dt\right)k$$

Ideal Projectile Motion

$$\text{Assume } v_0 \text{ is the initial velocity vector and the initial position is the origin.} \\ \text{If } v_0 \text{ makes an angle } \alpha \text{ with the horizontal, then} \\ v_0 = (|v_0|\cos\alpha)i + (|v_0|\sin\alpha)j$$

By Newton's 2^{nd} law, $F = ma$. When the only force is the gravitational force, $-mgj$ (j is the y-axis), then the following is true:

$$m\frac{d^2r}{dt^2} = -mgj \text{ and } \frac{d^2r}{dt^2} = -gj$$

To find r as a function of t solve the following initial value problem:

Differential equation: $$\frac{d^2r}{dt^2} = -gj$$

Initial conditions: $$r = r_0 \text{ and } \frac{dr}{dv} = v_0 \text{ when } t = 0$$

First Integral: $$\frac{dr}{dt} = -(gt)j + v_0$$

Second Integration: $$r = -\frac{1}{2}gt^2 j + v_0 t + r_0$$

Subsititute Initial Velocity Values: $$r = -\frac{1}{2}gt^2 j + (|v_0|\cos\alpha)ti + (|v_0|\sin\alpha)tj + 0$$

Final Equation: $$r = (|v_0|\cos\alpha)ti + \left((|v_0|\sin\alpha)t - \frac{1}{2}gt^2\right)j$$

The components of this are:

$$x = (|v_0|\cos\alpha)t$$

and

$$y = (|v_0|\sin\alpha)t - \frac{1}{2}gt^2$$

Height, Flight Time, and Range

Maximum Height: $$y_{max} = \frac{(v_0 \sin\alpha)^2}{2g}$$

Flight Time: $$t = \frac{2v_0 \sin\alpha}{2g}$$

Range: $$R = \frac{v_0^2}{g}\sin 2\alpha$$

Projectile Motion from a Point

Assuming start point of (x_0, y_0) instead of the origin:

$$r = (x_0 + (|v_0|\cos\alpha)t)i + \left(y_0 + (|v_0|\sin\alpha)t - \frac{1}{2}gt^2\right)j$$