13.1 - Curves In Space & Tangents

Particles are defined by parametric functions of t
$x = f(t)$, $y = g(t)$, $z = h(t)$, $t \in I$
where I is the time interval the particle moves in
The vector $r(t) = \vec{OP} = f(t)i + g(t)j + h(t)k$ from the origin is the position vector of the particle.
f, g, h are **component functions** of the position vector
$r(t)$ is a vector-valued function because it returns a vector
The domain, I is the largest set of points t for which all three equations are defined, or
$-\infty < t < \infty$

Limits of Vector-Valued functions

Let $r(t) = f(t)i + g(t)j + h(t)k$ be a vector function with domain D, and let L be a vector. We say that r has **limit** L as t approaches t_0 and write

$$\lim_{t \to t_0} r(t) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that, for all $t \in D$,

$$|r(t) - L| < \epsilon \text{ whenever } 0 < |t - t_0| < \delta$$

A vector function $r(t)$ is **continuous at a point** $t = t_0$ is and only if each component function is continuous there
A vector function is **continuous** if it is continuous at every point in its domain

Derivative of Vector Valued Functions

The vector function $r(t) = f(t)i + g(t)j + h(t)k$ has a **derivative** (is differentiable) at t if f, g, and h have derivatives at t. The derivative is the vector function

$$r'(t) = \frac{dr}{dt} = \lim_{\Delta t \to 0} \frac{r(t + \Delta) - r(t)}{\Delta t} = \frac{df}{dt}i + \frac{dg}{dt}j + \frac{dh}{dt}k$$

A vector function r is differentiable if it is differentiable at every point of its domain.

The curve traced by $r(t)$ is **smooth** if $\frac{dr}{dt}$ is continuous and never 0 (f, g, h have continuous first derivatives and are never simultaneously 0)
A curve is **piecewise smooth** if it is both continuous and made up of smooth curves

Velocity, Speed, Acceleration, Direction

Velocity

$$v = \frac{dr}{dt}$$

Speed

$$\text{Speed } = |v|$$

Acceleration

$$a = \frac{dv}{dt} = \frac{d^2r}{dt^2}$$

Direction

$$\text{Direction} = \frac{v}{|v|}$$

Differentiation Rules

See [Differentiation Cheat Sheet]

Vectors of Constant Length

$$\begin{aligned} r(t) \cdot r(t) = |r(t)|^2 = c^2 \\ \frac{d}{dt}[r(t) \cdot r(t)] = 0 \\ r'(t) \cdot r(t) + r(t) \cdot r'(t) = 0 \\ 2r'(t) \cdot r(t) = 0 \end{aligned}$$

Therefore, The $r(t)$ and $r'(t)$ are orthogonal
If **r** is a differentiable vector function of t and the length of $r(t)$ is constant, then

$$r \cdot \frac{dr}{dt} = 0$$

The converse is also true