14.2 - Limits and Continuity in Higher Dimensions

Limits for Functions of Two Variables

Definition:

Suppose that every open circular disk centered at (x_0, y_0) contains a point in the domain of f other than (x_0, y_0) itself.

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|d(x,y) - L| < \epsilon \text{ whenever } 0 < \sqrt{rt(x - x_0)^2 + (y - y_0)^2} < \delta$$

THEOREM 1—Properties of Limits of Functions of Two Variables

The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y) + g(x,y)] = L + M$
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y) - g(x,y)] = L - M$
3. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y) \cdot g(x,y)] = L \cdot M$
5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$, $M \neq 0$
6. *Power Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer, and if n is even, we assume that $L > 0$.
8. *Composition Rule:* If $h(z)$ is continuous at $z = L$, then $\lim_{(x,y)\to(x_0,y_0)} h(f(x,y)) = h(L)$.

Continuity

Definition:

A function $f(x, y)$ is continuous at the point (x_0, y_0) if

1. f is defined at (x_0, y_0)
2. $\lim_{x,y) \to (x_0,y_0)} f(x, y)$ exists, and
3. $\lim_{x,y) \to (x_0,y_0)} f(x, y) = f(x_0, y_0)$
   A function is continuous if it is continuous at every point in its domain.

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composition $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also continuous at (x_0, y_0).

EVT

Extreme Value Theorem holds on higher dimension functions if the domain is closed and bounded.