Xu_Hong_solutions_HW3

Hong Xu

September 17, 2015

1.

Prove that the total variance of $\boldsymbol{X}$ is the MSE of the mean:

$$E||\boldsymbol{X} - E(\boldsymbol{X})||^2 = \mathrm{tr}(\mathrm{Var}(\boldsymbol{X})).$$

Proof:

We can show that the MSE of the mean: $E||\boldsymbol{X} - E(\boldsymbol{X})||^2 = E(\sum_i (X_i - E(X_i))^2) = \sum_i E((X_i - E(X_i))^2)$

The covariance of $\boldsymbol{X}$: $\mathrm{Var}(\boldsymbol{X}) = E((\boldsymbol{X} - E(\boldsymbol{X}))(\boldsymbol{X} - E(\boldsymbol{X}))^T)$

So the total variance of $\boldsymbol{X}$ equals to $\mathrm{tr}(\mathrm{Var}(\boldsymbol{X})) = \sum_i E((X_i - E(X_i))(X_i - E(X_i))^T) = \sum_i E((X_i - E(X_i))^2)$, which equals to the MSE of the mean.

2.

Suppose $\boldsymbol{Y} \sim N_n(0, I)$ and let A be symmetric. Show that

$$\boldsymbol{Y}^T A \boldsymbol{Y} \stackrel{dist}{=} \sum_i \lambda_i W_i,$$

where $W_i \stackrel{iid}{\sim} \chi_1^2$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.

Proof:

Since A is symmetric, we have the spectral decomposition of $A = PDP^T$,

where P is orthogonal, D is a diagonal matrix $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$

So $\boldsymbol{Y}^T A \boldsymbol{Y} = \boldsymbol{Y}^T PDP^T \boldsymbol{Y} = (P^T \boldsymbol{Y})^T D (P^T \boldsymbol{Y})$

and $P^T \boldsymbol{Y} \sim N_n(P^T 0, P^T IP)$, since P is orthogonal, we have $P^T \boldsymbol{Y} \sim N_n(0, I)$.

With that, we can show $\boldsymbol{Y}^T A \boldsymbol{Y} = \boldsymbol{Y}^T D \boldsymbol{Y} = \sum_i \sum_j \lambda_{ij} Y_i Y_j = \sum_i \lambda_i Y_i^2$.

And $\boldsymbol{Y} \sim N_n(0, I)$,

So, $Y_i \sim N_n(0, 1)$, $W_i = Y_i^2 \sim \chi_1^2$.

So that $\boldsymbol{Y}^T A \boldsymbol{Y} \stackrel{dist}{=} \sum_i \lambda_i W_i$.

3.

a)

Show that this model implies that there is a positive covariance for within-state poverty rates over time, $\mathrm{Cov}[Y_{i1}, Y_{i2}] = \mathrm{Var}[U_i] = \sigma_u^2$.

Proof:

Since $U_i \stackrel{iid}{\sim} \mathcal{N}(0, \sigma_u^2)$ and $\epsilon_{it} \stackrel{ind}{\sim} \mathcal{N}(0, \sigma_{e_t}^2)$, and $Y_{it} = \mu_t + U_i + \epsilon_{it}$,

Then, at time t, $E(Y_{it}) = \mu_t + E(U_i) + E(\epsilon_{it}) = \mu_t + 0 + 0 = \mu_t$.

So, $\text{Cov}\left[Y_{i1}, Y_{i2}\right] = E((Y_{i1} - E(Y_{i1}))(Y_{i2} - E(Y_{i2}))) = E((Y_{i1} - \mu_1)(Y_{i2} - \mu_2)) = E((U_i + \epsilon_{i1})(U_i + \epsilon_{i2})) = E(U_i^2 + \epsilon_{i1}U_i + \epsilon_{i2}U_i + \epsilon_{i1}\epsilon_{i2}) = E(U_i^2) + E(\epsilon_{i1}U_i) + E(\epsilon_{i2}U_i) + E(\epsilon_{i1}\epsilon_{i2})$.

Since U_i and ϵ_{it} are independent, it's easy to show the last three expecations are zero. So, $\text{Cov}\left[Y_{i1}, Y_{i2}\right] = E(U_i^2)$

Also, $\text{Var}(U_i) = E(U_i^2) - E(U_i)^2 = E(U_i^2)$,

So we have $\text{Cov}\left[Y_{i1}, Y_{i2}\right] = E(U_i^2) = \text{Var}(U_i) = \sigma_\mu^2$

b)

Answer:

Since $Y_{it} = \mu_t + U_i + \epsilon_{it}$,

$Y_{it} \sim N(\mu_t, \sigma_\mu^2 + \sigma_{e_t}^2)$.

Obviously Y_{i1} and Y_{i2} follow normaal distribution, so that $\mathbf{Y}_i = (Y_{i1}, Y_{i2})^T$ would be an $\mathcal{MVN}$ distribution.

Its mean vector μ would be $E(\boldsymbol{Y}_i) = (E(Y_{i1}), E(Y_{i2}))^T = (\mu_1, \mu_2)^T$, given the proof we showed in (a).

Its Σ would be:

$$\mathbf{\Sigma} = \begin{bmatrix} \text{Var}(Y_{i1}) & \text{Cov}(Y_{i1}, Y_{i2}) \\ \text{Cov}(Y_{i2}, Y_{i1}) & \text{Var}(Y_{i2}) \end{bmatrix}$$

Given the proof in a) $\text{Cov}\left[Y_{i1}, Y_{i2}\right] = E(U_i^2) = \text{Var}(U_i) = \sigma_\mu^2$,

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_\mu^2 + \sigma_{e_1}^2 & \sigma_\mu^2 \\ \sigma_\mu^2 & \sigma_\mu^2 + \sigma_{e_2}^2 \end{bmatrix}$$

c)

```r
# skip the first row
saipe98 <- read.csv("CPS98T.TXT", skip = 1, sep="")
saipe05 <- read.csv("CPS05T.TXT", skip = 1, sep="")
# for children aged 5-17, CPS poverty is at column cps98.1 or cps05.1
cps98 <- saipe98[, "cps98.1"]
cps05 <- saipe05[, "cps05.1"]
# construct a data frame
cps <- cbind(cps98, cps05)
```

d)

```r
## find the mean poverty rate
colMeans(cps)
```

```
##    cps98    cps05
## 16.97407 15.69192
```

e)

```r
## find the covariance matrix for Y, i.e. cps
cov(cps)
```

```
##          cps98    cps05
## cps98 39.92843 27.59969
## cps05 27.59969 33.05264
```

f)

In b) we showed:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_\mu^2 + \sigma_{e_1}^2 & \sigma_\mu^2 \\ \sigma_\mu^2 & \sigma_\mu^2 + \sigma_{e_2}^2 \end{bmatrix}$$

and

$$\mu = (\mu_1, \mu_2)^T$$

So,

$\mu_1 = 16.97407$

$\mu_2 = 15.69192$

$\sigma_\mu^2 = 27.59969$

$\sigma_{e_1}^2 = 39.92843 - \sigma_\mu^2 = 39.92843 - 27.59969 = 12.32874$

$\sigma_{e_2}^2 = 39.92843 - \sigma_\mu^2 = 33.05264 - 27.59969 = 5.45295$

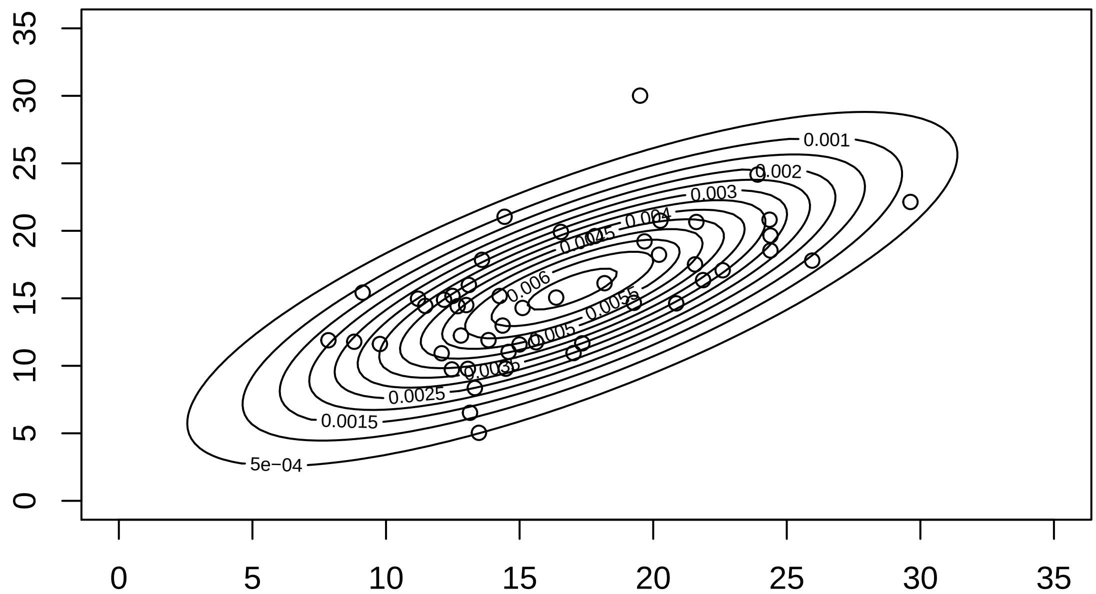
g)

```r
require(mvtnorm)
```

```
## Loading required package: mvtnorm
```

```r
x.points <- seq(0, 35, length.out = 100)
y.points <- x.points
z <- matrix(0, 100, 100)
mu <- colMeans(cps)
sigma <- cov(cps)

for (i in 1:100) {
  for (j in 1:100) {
    z[i, j] <- dmvnorm(c(x.points[i], y.points[j]), mean = mu, sigma = sigma)
  }
}
contour(x.points, y.points, z)
# add points
points(cps[,1], cps[,2])
```




```
## The multivariate normal looks like a good fit to the data
## given the density and the rotation the data points appear.
```