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An Empirical Research on Identifiability and Q-matrix Design for DINA model

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POLYTECHNIQUE MONTRÉAL
LE GÉNIE EN PREMIÈRE CLASSE

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Motivation

- What is the optimal design of Q-matrix? i.e How to use the least questions to get efficient diagnosis on student profiles?
- Should a test involve only one skill or multiple skills to better diagnose student profiles?
- By which principles should a teacher choose among different options?

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Notations

- Response matrix R

$$\begin{array}{c c} & \begin{array}{ccccccccc} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 & i_9 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

- Q-matrix Q

$$\begin{array}{c c} & \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{array}{c} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \\ i_9 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

- Profile matrix A

$$\begin{array}{c c} & \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Motif

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- Problem: Different parameter setting can lead to same likelihood, thus making true parameter estimation problematic
- Identifiability was researched in
  - multiple diagnosis model comparison
  - Bayesian Knowledge Tracing
  - DINA/DINO model

Definition

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- Definition **Definition (1)** [Casella and Berger, 2002] A parameter θ for a family of distribution $f(x|\theta : \theta \in \Theta$ is *identifiable* if distinct values of θ correspond to distinct pdfs or pmfs. That is, if $\theta \neq \theta'$, then $f(x|\theta)$ is not the same function of x as $f(x|\theta')$.
- Completeness **Definition (2)** [Chen et al., 2015] The matrix Q is *complete* meaning that $\{e_i : i = 1, ..., K\} \subset R_Q$, where R_Q is the set of row vectors of Q and e_i is a row vector such that the i-th element is one and the rest are zero (i.e. a binary unit vector).
- Proposition **Proposition** [Chen et al., 2015] Under the DINA and DINO models, with Q, s and g being known, the population proportional parameter p is *identifiable* if and only if Q is *complete*.

Example

An example of Q-matrix that is not complete

$$\begin{array}{cc} & \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \begin{array}{c} q_1 \\ q_2 \\ q_3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

This Q-matrix does not contain $e_2 : [0, 1, 1]$ or $e_3 : [0, 0, 1]$, and is therefore not complete, even though its items (rows) cover all skills (columns). Using this Q-matrix under DINA model setting entails that the model parameters are not identifiable according to the proposition above, and would in turn compromise student profile diagnosis. In fact, students who only master skill 2 and students who only master skill 3 are indistinguishable under this Q-matrix.

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Basic Settings

- We want to compare different configuration of questions, which corresponds to different Q-matrix Q, each of which is associated with a loss, indicating how bad the configuration is to misspecify student profile
- In our experiment under DINA model, slip s and guess g are known, with each given Q-matrix Q, we first need to compute the estimate of all student profiles P, then compare with true student profiles to obtain the loss
- We use 3-skill case(8 profile categories) for illustration

Model

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- Prior: $\alpha_0 = (1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8)$
- Likelihood:

$$\begin{aligned} L(p, Q, s, g|X) &= P(X|p, Q, s, g) \\ &= \prod_{i=1}^{I} \sum_{\alpha} p_\alpha P(X_i|\alpha, Q, s, g) \\ &= \prod_{i=1}^{I} \sum_{\alpha} p_\alpha \prod_{j=1}^{J} P_j(\alpha)^{X_{ij}} [1 - P_j(\alpha)]^{1-X_{ij}} \end{aligned} \tag{1}$$

- Posterior for student i: $\hat{\alpha}_i = (\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{p}_5, \hat{p}_6, \hat{p}_7, \hat{p}_8)$
- Loss: $loss(Q) = \sum_{i \in \text{students}} \|\hat{\alpha}_i - \alpha_{\text{true}}\|^2$, α_{true} is one-hot encoded

Experiment 1

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Comparison of three strategies

- Strategy 1: Using the identifiability condition (definition (1)) by only repeatedly using the vectors $\{e_i : i = 1, ..., K\}$ (binary unit vectors, or one-hot encodings). Q-matrix used in this strategy is denoted as Q-matrix 1.
- Strategy 2: Using the vectors $\{e_i : i = 1, ..., K\}$ plus an all-one vector $(1, 1, 1)$ (in 3-skill case) or $(1, 1, 1, 1)$ (in 4-skill case). This is inspired by orthogonal array design, which is a commonly seen design of experiments [Montgomery, 2017]. Q-matrix used in this strategy is denoted as Q-matrix 2.
- Strategy 3: Repeatedly using all q-vectors. Q-matrix used in this strategy is denoted as Q-matrix 3.

Continue

Q-matrix 1
(binary unit vectors)

$$\begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ \dots \\ q_{19} \\ q_{20} \\ q_{21} \end{array} \begin{array}{c} \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \dots & \dots & \dots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Q-matrix 2
(binary unit + all-1s vectors)

$$\begin{array}{c} \\ q_1 \end{array} \begin{array}{c} \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \end{array}$$

Q-matrix 3
(all combinations)

$$\begin{array}{c} \\ q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ \dots \\ q_{15} \end{array} \begin{array}{c} \begin{array}{ccc} k_1 & k_2 & k_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ \dots & \dots & \dots \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Result

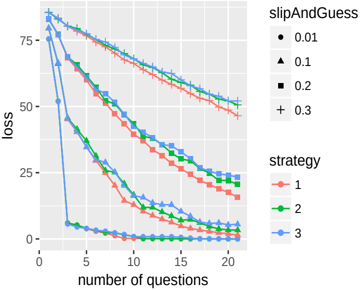

Figure: Three Strategy Comparison on 3-skill case

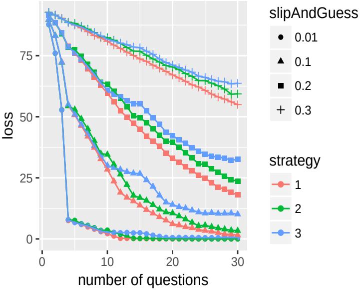

Figure: Three Strategy Comparison on 4-skill case

Experiment 2

Find best configuration

- A brute force approach
- Given number of skills 3, we have number of q-vectors 7, then for a specific number of questions 4, we have $\binom{4+7-1}{7-1} = 210$ possible configurations.

Result

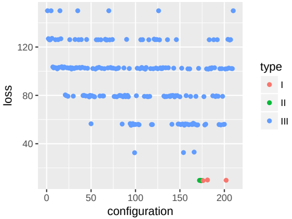

Figure: 3-skill case, slip=guess=0.01, J=4

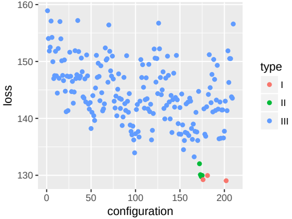

Figure: 3-skill case, slip=guess=0.2, J=4

- type I: Complete and confined, meaning it is only consisted of vectors $\{e_i : i = 1, ..., K\}$.
- type II: Complete but not confined, meaning it not only contains all vectors $\{e_i : i = 1, ..., K\}$, but also contains at least one other q-vector.
- type III: Incomplete Q-matrix.

Result

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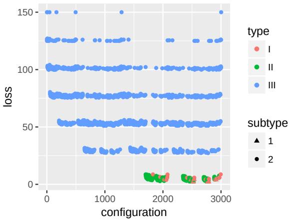

Figure: 3-skill case, slip=guess=0.01, J=8

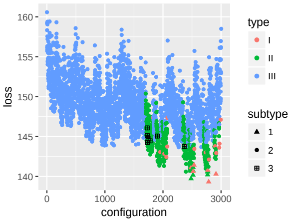

Figure: 3-skill case, slip=guess=0.3, J=8

- subtype 1: Q-matrix contains each component of $\{e_i : i = 1, ..., K\}$ at least twice.
- subtype 2: Other situations (e.g A complete Q-matrix but all the other vectors are just repeated e_1).
- subtype 3: Q-matrix contains all q-vectors.

Result

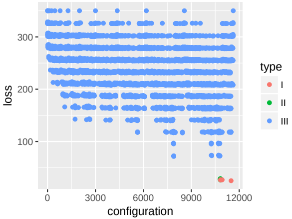

Figure: 4-skill case, slip=guess=0.01, J=5

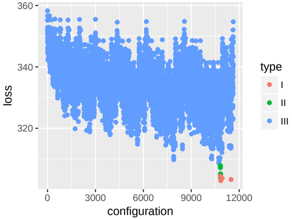

Figure: 4-skill case, slip=guess=0.2, J=5

- type I: Complete and confined, meaning it is only consisted of vectors $\{e_i : i = 1, ..., K\}$.
- type II: Complete but not confined, meaning it not only contains all vectors $\{e_i : i = 1, ..., K\}$, but also contains at least one other q-vector.
- type III: Incomplete Q-matrix.

- **Counter-intuitive:** A good assessment design should contain items with combination of skills
- **Synthetic Dataset:** Impossible to test on real dataset since we do not know the true Q-matrix
- **Restraints on slip and guess:** A different identifiability requirement is involved when slip and guess are unknown

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