贝叶斯统计导论大作业

2020 年 6 月 2 日

1 理论推导

MCMC 需要用到后验分布如下：

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\begin{aligned}
p(x_{j(\gamma^C)}|\gamma) &\int p(x_j|\sigma_{0j}^2)\cdot p(\sigma_{0j}^2|a_0,b_0)d\sigma_{0j}^2 \\
&= \frac{1}{\sqrt{2\pi}}\int\left(\frac{1}{\sigma_{0j}}\right)^n \exp\left(-\frac{x_j^2}{2\sigma_{0j}^2}\right)\cdot IG(\sigma_{0j}^2;a_0,b_0)d\sigma_{0j}^2 \\
p(\boldsymbol{x}_j(\gamma^C)|\gamma) &= (2\pi)^{-n/2}\frac{b_0^{a_0}}{\Gamma(a_0)}\int(\sigma_{0j}^2)^{(-a_0-1-n/2)}\exp\left(-\frac{\boldsymbol{x}_j^T\boldsymbol{x}_j+2b_0}{2\sigma_{0j}^2}\right)d\sigma_{0j}^2 \\
&= (2\pi)^{-n/2}\frac{b_0^{a_0}}{\Gamma(a_0)}\frac{\Gamma(a_0')}{b_0'^{a_0'}}, a_0'=a_0+\frac{n}{2}, b_0'=\frac{\boldsymbol{x}_j^T\boldsymbol{x}_j}{2}+b_0 \\
p\left(\mathbf{x}_{jk(\gamma)}|\mathbf{Z}_{\mathbf{k}(\delta_\mathbf{k})},\mu_{\mathbf{0jk}},\boldsymbol{\delta},\boldsymbol{\gamma}\right) &= \iint p(\boldsymbol{x}_{jk}|\boldsymbol{Z}_{k(\delta_k)},\beta_{kj})\cdot p(\beta_{jk})\cdot p(\sigma_{jk}^2;a_k,b_k)d\beta_{jk}d\sigma_{jk}^2 \\
&= \frac{b_k^{a_k}}{\Gamma(a_k)}\iint(2\pi)^{-n_k/2}(\sigma_{jk}^2)^{-a_k+n_k/2}h^{-\delta_k/2}(2\pi)^{-\delta_k/2}(\sigma_{jk}^2)^{-\delta_k/2} \\
&\cdot\exp\left(-\frac{\beta_{kj}^T\Sigma^{-1}\beta_{kj}+\boldsymbol{m}^T\boldsymbol{m}-2\beta_k^TZ_k^T\boldsymbol{m}+2b_k}{2\sigma_{jk}^2}\right)d\beta_kd\sigma_{jk}^2 \\
&= \frac{b_k^{a_k}}{\Gamma(a_k)}\iint(2\pi)^{-n_k/2}(\sigma_{jk}^2)^{-a_k+n_k/2}h^{-\delta_k/2}(2\pi)^{-\delta_k/2}(\sigma_{jk}^2)^{-\delta_k/2}|\Sigma|^{-1/2}|\Sigma|^{1/2} \\
&\cdot\exp\left(-\frac{(\beta_{kj}-\Sigma\cdot Z_k^T\cdot\boldsymbol{m})^T\Sigma^{-1}(\beta_{kj}-\Sigma\cdot Z_k^T\cdot\boldsymbol{m})}{2\sigma_{jk}^2}\right)d\beta_k \\
&\cdot\exp\left(-\frac{\boldsymbol{m}^T\boldsymbol{m}-\boldsymbol{m}^TZ_k\Sigma Z_k^T\boldsymbol{m}+2b_k}{2\sigma_{jk}^2}\right)d\sigma_{jk}^2 \\
&= \frac{b_k^{a_k}}{\Gamma(a_k)}(2\pi)^{-n_k/2}\int(\sigma_{jk}^2)^{-a_k+n_k/2}h^{-\delta_k/2}|\Sigma|^{1/2} \\
&\cdot\exp\left(-\frac{\boldsymbol{m}^T\boldsymbol{m}-\boldsymbol{m}^TZ_k\Sigma Z_k^T\boldsymbol{m}+2b_k}{2\sigma_{jk}^2}\right)d\sigma_{jk}^2 \\
&= \left(\frac{1}{\sqrt{2\pi}}\right)^{n_k}h^{-\delta_k/2}\frac{\Gamma\left(a_k'\right)}{\Gamma\left(a_k\right)}\frac{b_k^{a_k}}{b_k'^{a_k'}}|\Sigma|^{1/2}
\end{aligned}
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其中

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\begin{aligned}
\Sigma &= (\mathbf{Z}_{k(\delta_k)}^T\mathbf{Z}_{k(\delta_k)}+h^{-1}\mathbf{I}_{|\boldsymbol{\delta}_k|})^{-1} \\
\boldsymbol{m} &= \boldsymbol{x}_{jk}-\mathbf{1}_{n_k}\mu_{0jk} \\
a_k' &= a_k+n_k/2 \\
b_k' &= b_k+\boldsymbol{m}^T(I_{n_k}-Z_k\Sigma Z_k^T)\boldsymbol{m}/2
\end{aligned}
$$

$|\boldsymbol{\delta}_k|$ 表示 $\delta_k = 1$ 的元素个数。

$$
\begin{aligned}
p(\boldsymbol{\mu}_{0k(\gamma)}|\boldsymbol{\gamma}) = & \iint p(\mu_{0k(\gamma)}|\nu_{k(\gamma)}, h_1\Gamma_{0k(\gamma)}) \cdot p(\mu_{k(\gamma)}|m_{0k(\gamma)}, h_1\Gamma_{0k(\gamma)}) d\nu_{k(\gamma)} \\
& \cdot \text{Inv-Wishart}(\Gamma_{0k(\gamma);d_k,Q}) d\Gamma_{0k(\gamma)} \\
= & \iint (2\pi)^{-p_\gamma/2} |h_1\Gamma_{0k(\gamma)}|^{-1/2} \exp\left(-\frac{1}{2}(\mu_{0k(\gamma)} - \nu_{0k(\gamma)})^T (h_1\Gamma_{0k(\gamma)})^{-1} (\mu_{0k(\gamma)} - \nu_{0k(\gamma)})\right) \\
& \cdot (2h_1\pi)^{-p_\gamma/2} |\Gamma_{0k(\gamma)}|^{-1/2} \exp\left(-\frac{1}{2}(\nu_{0k(\gamma)} - m_{0k(\gamma)})^T (h_1\Gamma_{0k(\gamma)})^{-1} (\nu_{0k(\gamma)} - m_{0k(\gamma)})\right) d\nu_{k(\gamma)} \\
& \cdot \text{Inv-Wishart}(\Gamma_{0k(\gamma);d_k,Q}) d\Gamma_{0k(\gamma)} \\
= & \iint (2\pi)^{-p_\gamma/2} |h_1\Gamma_{0k(\gamma)}/2|^{-1/2} \\
& \cdot \exp\left(-\frac{1}{2}\left(\nu_{0k(\gamma)} - \frac{\mu_{0k(\gamma)} + m_{0k(\gamma)}}{2}\right)^T \left(\frac{h_1}{2}\Gamma_{0k(\gamma)}\right)^{-1} \left(\nu_{0k(\gamma)} - \frac{\mu_{0k(\gamma)} + m_{0k(\gamma)}}{2}\right)\right) d\nu_{k(\gamma)} \\
& \cdot (4h_1\pi)^{-p_\gamma/2} |\Gamma_{0k(\gamma)}|^{-1/2} \exp\left(-\frac{(\mu_{0k(\gamma)} - m_{0k(\gamma)})^T (h_1\Gamma_{0k(\gamma)})^{-1} (\mu_{0k(\gamma)} - m_{0k(\gamma)})}{4}\right) \\
& \cdot \text{Inv-Wishart}(\Gamma_{0k(\gamma);d_k,Q}) d\Gamma_{0k(\gamma)} \\
= & \int (4h_1\pi)^{-p_\gamma/2} |\Gamma_{0k(\gamma)}|^{-(d_k+p_\gamma+2)/2} \exp\left(-\frac{\text{tr}((2h_1 S_+ Q)\Gamma_{0k(\gamma)}^{-1})}{2}\right) \cdot \frac{|Q|^{d_k/2}}{2^{d_k p_\gamma/2}\Gamma_{p_\gamma}(d_k/2)} d\Gamma_{0k(\gamma)} \\
= & (h_1^*\pi)^{-p_\gamma/2} \frac{|Q|^{d_k/2}}{|Q+S|^{(d_k+1)/2}/h_1^*} \frac{\Gamma((d_k+1)/2)}{\Gamma((d_k - p_\gamma + 1)/2)} \\
p(\boldsymbol{\mu}_{0jk(\gamma^c)}|\boldsymbol{\gamma}) = & \int p(\mu_{0jk(\gamma^c)}|\sigma_j^2) \cdot p(\sigma_j^2|\tilde{a}_j, \tilde{b}_j) d\sigma_j^2 \\
= & \frac{1}{\sqrt{2\pi}} \frac{\tilde{b}_j^{\tilde{a}_j}}{\tilde{b}_{jk}^{\prime\tilde{a}_j'}} \frac{\Gamma\left(\tilde{a}_j'\right)}{\Gamma\left(\tilde{a}_j\right)}
\end{aligned}
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其中

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\begin{aligned}
S &= (\mu_{0k(\gamma)} - m_{0k(\gamma)})(\mu_{0k(\gamma)} - m_{0k(\gamma)})^T \\
h_1^* &= 2h_1 \\
\tilde{a}_j' &= \tilde{a}_j + 1/2 \\
\tilde{b}_{jk}' &= \tilde{b}_j + \frac{(\mu_{0jk} - m_{0jk})^2}{2}
\end{aligned}
$$