Assignment 3 of MATP6960: theory questions

Problem 1

In class, we used the Hoeffding's inequality to show some properties of the sampling approximation method for solving chance-constrained problems. The inequality is summarized below.

Hoeffding's inequality. Let $Y_1, \ldots, Y_N$ be i.i.d. variables with $\text{Prob}(Y_i \in [a_i, b_i]) = 1$ for all $i = 1, \ldots, N$. Then for any $t > 0$,

$$\text{Prob}\left(\sum_{i=1}^{N}(Y_i - \mathbb{E}[Y_i]) \geq tN\right) \leq \exp\left(-\frac{2N^2t^2}{\sum_{i=1}^{N}(b_i - a_i)^2}\right).$$

Use the Hoeffding's inequality to prove that for any $\alpha \in [0, 1]$ and any positive integer N, if $k < \alpha N$, it holds

$$\sum_{i=0}^{k}\left(\begin{array}{c} N \\ i \end{array}\right)\alpha^i(1-\alpha)^{N-i} \leq \exp\left(-\frac{2(\alpha N - k)^2}{N}\right),$$

where $\left(\begin{array}{c} N \\ i \end{array}\right)$ denotes "N choose i".

Problem 2

A function $f(x)$ is called log-concave, if it is positive and $\log\big(f(x)\big)$ is a concave function. Prove that if $\theta : \mathbb{R} \to \mathbb{R}$ is a log-concave probability density function, then the cumulative distribution function

$$F(x) = \int_{-\infty}^{x} \theta(t)dt$$

is also log-concave, provided that F is well-defined for any $x \in \mathbb{R}$. You can assume θ is a continuously differentiable function though the conclusion holds for a general θ.