Assignment 4 of MATP6610/4820

(Due on March-19-2019 in class)

Problem 1

Consider the problem

$$\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}}\, f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{A}\mathbf{x} - \mathbf{b}^\top \mathbf{x}, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix, and $\mathbf{b} \in \mathbb{R}^n$.

1. Let $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Starting from $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, do by hand two cycles of the coordinate descent method for (1).

2. Given $\bar{\mathbf{x}}$ and i, let

   $$\hat{x}_i = \arg\min_{x_i} f(\bar{\mathbf{x}}_{<i}, x_i, \bar{\mathbf{x}}_{>i}).$$

   Derive an explicit formula of $\hat{x}_i$, which should depend on $\mathbf{A}, \mathbf{b}$ and $\bar{\mathbf{x}}$.

3. **(Only for 6610 students:)** Let $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ be the iterate sequence given by the coordinate descent method for (1). Show that for any initial point $\mathbf{x}^{(0)}$, $f(\mathbf{x}^{(k)})$ converges to the optimal objective value.

   Hint: first show that

   $$f(\mathbf{x}_{<i}^{k+1}, x_i^{k+1}, \mathbf{x}_{>i}^{k}) = f(\mathbf{x}_{<i}^{k+1}, x_i^{k}, \mathbf{x}_{>i}^{k}) - \frac{a_{ii}}{2}(x_i^k - x_i^{k+1})^2;$$

   secondly, show that

   $$f(\mathbf{x}^{k+1}) \le f(\mathbf{x}^k) - \frac{c}{2}\|\mathbf{x}^k - \mathbf{x}^{k+1}\|^2,\ \forall k \ge 0,$$

   where $c = \min_i a_{ii} > 0$, and a_{ii} is the i-th diagonal element of $\mathbf{A}$. Thirdly, noting $f(\mathbf{x}^k) \ge \min_{\mathbf{x}} f(\mathbf{x})$, show $\sum_{k=0}^{\infty} \|\mathbf{x}^k - \mathbf{x}^{k+1}\|^2 < \infty$ and thus $\mathbf{x}^k - \mathbf{x}^{k+1} \to \mathbf{0}$ as $k \to \infty$. Fourth, let $\bar{\mathbf{x}}$ be any limit point of $\{\mathbf{x}^k\}_{k\ge 0}$, and show $\mathbf{A}\bar{\mathbf{x}} = \mathbf{b}$ and thus $\bar{\mathbf{x}}$ is an optimal solution. Finally, use the monotonicity of $\{f(\mathbf{x}^k)\}_{k\ge0}$ to argue $f(\mathbf{x}^k) \to f(\bar{\mathbf{x}})$ as $k \to \infty$.

Problem 2

Consider the Lasso problem:

$$\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}}\, f(\mathbf{x}) = \frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{b}\|^2 + \lambda\|\mathbf{x}\|_1, \tag{2}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$ are given.

Use the instructor's provided file `CD_Lasso.m` to write a Matlab function `CD_Lasso` with input $\mathbf{A}$, $\mathbf{b}$, initial vector $\mathbf{x}0$, parameter λ, and maximum iteration number `maxit`. Also test your function by running the provided test file `test_CD_Lasso.m` and compare to the instructor's function. Print your code and the results you get.