Assignment 1 of MATP6960: theory questions

Problem 1

Recall that the convex conjugate of a closed convex function g is defined as

$$g^*(\mathbf{x}) = \sup_{\mathbf{z}} \left\{ \mathbf{x}^\top \mathbf{z} - g(\mathbf{z}) \right\}.$$

Suppose g is μ -strongly convex on \mathbb{R}^n with $\mu > 0$. Prove that g^* is $\frac{1}{\mu}$ -smooth, i.e., its gradient is $\frac{1}{\mu}$ -Lipschitz continuous. You can use the fact that $\nabla g^*(\mathbf{x}) = \mathbf{z}(\mathbf{x})$, where $\mathbf{z}(\mathbf{x}) = \arg\max_{\mathbf{x}} \left\{ \mathbf{x}^{\top} \mathbf{z} - g(\mathbf{z}) \right\}$.

Problem 2

Recall that a function $f: \mathbb{R}^n \to \mathbb{R}$ is ρ -weakly convex with $\rho \geq 0$, if $f(\cdot) + \frac{\rho}{2} \| \cdot \|^2$ is convex, where $\| \cdot \|$ denotes the Euclidean norm. Define $[x]_+ = \max\{x, 0\}$ for any $x \in \mathbb{R}$.

- 1. Is the function $f(x_1, x_2) = x_2[x_1]_+$ weakly convex? If yes, find the smallest number $\rho \geq 0$ such that $f(x_1, x_2) + \frac{\rho}{2}(x_1^2 + x_2^2)$ is a convex function, and prove it. If not, explain why.
- 2. Is the function $f(x_1, x_2, x_3) = x_3[x_2[x_1]_+]_+$ weakly convex? How about if we restrict the variable $\mathbf{x} = (x_1, x_2, x_3)$ in a unit ball? Justify your answer.