Assignment 7 of MATP6610/4820

(Due on April-12-2019 in class)

Problem 1

Consider the Least Absolute Deviation (LAD) problem:

$$\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}}\, \|\mathbf{A}\mathbf{x}-\mathbf{b}\|_1 \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$ are given. In class, we showed that the LAD problem is equivalent to

$$\underset{\mathbf{x}\in\mathbb{R}^n,\mathbf{z}\in\mathbb{R}^m}{\text{minimize}}\, \|\mathbf{z}\|_1, \text{subject to } \mathbf{A}\mathbf{x}-\mathbf{z}-\mathbf{b}=\mathbf{0}. \tag{2}$$

The augmented Lagrangian function of (2) is

$$\mathcal{L}_\rho(\mathbf{x},\mathbf{z},\mathbf{y}) = \|\mathbf{z}\|_1 + \mathbf{y}^\top(\mathbf{A}\mathbf{x}-\mathbf{z}-\mathbf{b}) + \frac{\rho}{2}\|\mathbf{A}\mathbf{x}-\mathbf{z}-\mathbf{b}\|^2,$$

where $\rho > 0$ is the penalty parameter, and $\mathbf{y} \in \mathbb{R}^m$ is the multiplier vector. The iterative update schemes of ADMM on solving (2) is

$$\mathbf{x}^{k+1} = \arg\min_{\mathbf{x}\in\mathbb{R}^n} \mathcal{L}_\rho(\mathbf{x},\mathbf{z}^k,\mathbf{y}^k), \tag{3a}$$

$$\mathbf{z}^{k+1} = \arg\min_{\mathbf{z}\in\mathbb{R}^m} \mathcal{L}_\rho(\mathbf{x}^{k+1},\mathbf{z},\mathbf{y}^k), \tag{3b}$$

$$\mathbf{y}^{k+1} = \mathbf{y}^k + \rho(\mathbf{A}\mathbf{x}^{k+1}-\mathbf{z}^{k+1}-\mathbf{b}) \tag{3c}$$

1. Assume $\mathbf{A}$ has column full rank, i.e., $\mathbf{A}^\top\mathbf{A}$ is nonsingular. Give the closed-form solution of $\mathbf{x}^{k+1}$ in the update (3a). The solution should involve the inverse of $\mathbf{A}^\top\mathbf{A}$.

2. Recall that we have derived the proximal mapping of $\lambda\|\cdot\|_1$ for any $\lambda > 0$ in class. Let's denote it as $\mathbf{prox}_{\lambda\|\cdot\|_1}$. Use the proximal mapping to derive a closed form solution of $\mathbf{z}^{k+1}$ in the update (3b). In the solution, λ should be $1/\rho$.

3. Suppose $(\mathbf{x}^{k+1},\mathbf{z}^{k+1},\mathbf{y}^{k+1})$ is the output. At this point, give the violation of primal and dual feasibility.

4. Use the instructor's provided file `ADMM_LAD.m` to write a Matlab function `ADMM_LAD` with input $\mathbf{A}$, $\mathbf{b}$, initial vector $\mathbf{x}0$, stopping tolerance `tol`, maximum number of iterations `maxit`, and penalty parameter $\rho > 0$. Also test your function by running the provided test file `test_ADMM_LAD.m` and compare to the instructor's function. Print your code and the results you get.