Assignment 1 of MATP6960: theory questions

Problem 1

Recall that the convex conjugate of a closed convex function g is defined as

$$g^*(\mathbf{x}) = \sup_{\mathbf{z}} \left\{ \mathbf{x}^\top \mathbf{z} - g(\mathbf{z}) \right\}.$$

Suppose g is μ-strongly convex on $\mathbb{R}^n$ with $\mu > 0$. Prove that g^* is $\frac{1}{\mu}$-smooth, i.e., its gradient is $\frac{1}{\mu}$-Lipschitz continuous. You can use the fact that $\nabla g^*(\mathbf{x}) = \mathbf{z}(\mathbf{x})$, where $\mathbf{z}(\mathbf{x}) = \arg\max_{\mathbf{z}} \left\{ \mathbf{x}^\top \mathbf{z} - g(\mathbf{z}) \right\}$.

Problem 2

Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is ρ-weakly convex with $\rho \geq 0$, if $f(\cdot) + \frac{\rho}{2}\|\cdot\|^2$ is convex, where $\|\cdot\|$ denotes the Euclidean norm. Define $[x]_+ = \max\{x, 0\}$ for any $x \in \mathbb{R}$.

1. Is the function $f(x_1, x_2) = x_2[x_1]_+$ weakly convex? If yes, find the smallest number $\rho \geq 0$ such that $f(x_1, x_2) + \frac{\rho}{2}(x_1^2 + x_2^2)$ is a convex function, and prove it. If not, explain why.

2. Is the function $f(x_1, x_2, x_3) = x_3 \left[x_2[x_1]_+\right]_+$ weakly convex? How about if we restrict the variable $\mathbf{x} = (x_1, x_2, x_3)$ in a unit ball? Justify your answer.