

Assignment 3 of MATP6960: theory questions

Problem 1

In class, we used the Hoeffding's inequality to show some properties of the sampling approximation method for solving chance-constrained problems. The inequality is summarized below.

Hoeffding's inequality. Let Y_1, \dots, Y_N be i.i.d. variables with $\text{Prob}(Y_i \in [a_i, b_i]) = 1$ for all $i = 1, \dots, N$. Then for any $t > 0$,

$$\text{Prob} \left(\sum_{i=1}^N (Y_i - \mathbb{E}[Y_i]) \geq tN \right) \leq \exp \left(-\frac{2N^2 t^2}{\sum_{i=1}^N (b_i - a_i)^2} \right).$$

Use the Hoeffding's inequality to prove that for any $\alpha \in [0, 1]$ and any positive integer N , if $k < \alpha N$, it holds

$$\sum_{i=0}^k \binom{N}{i} \alpha^i (1 - \alpha)^{N-i} \leq \exp \left(-\frac{2(\alpha N - k)^2}{N} \right),$$

where $\binom{N}{i}$ denotes “ N choose i ”.

Problem 2

A function $f(x)$ is called log-concave, if it is positive and $\log(f(x))$ is a concave function. Prove that if $\theta : \mathbb{R} \rightarrow \mathbb{R}$ is a log-concave probability density function, then the cumulative distribution function

$$F(x) = \int_{-\infty}^x \theta(t) dt$$

is also log-concave, provided that F is well-defined for any $x \in \mathbb{R}$.