remind them that $\begin{bmatrix} a_1, b_1 & a_2, b_2 \\ a_3, b_3 & a_4, b_4 \end{bmatrix}$

is the same as: $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ $\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

and: $\begin{bmatrix} a_1 \; {}^{b_1} & a_2 \; {}^{b_2} \\ a_3 \; {}^{b_3} & a_4 \; {}^{b_4} \end{bmatrix}$

Tutorial N°2

Exercise

Compute all the Nash equilibria of the following game:

player 2

		L	M	R
player 1	T	(7,2)	(2,7)	(3,6)
	B	(2,7)	(7,2)	(4,5)

Solution

Let us, first, see how many possible supports we have:

$$(2^3-1)\cdot(2^2-1) = (8-1)\cdot(4-1) = 7\cdot 3 = 21$$

$\uparrow$ all subsets of $\{L,M,R\}$ minus the $\emptyset$ for player 2.

$\downarrow$ $\{L\}, \{M\}, \{R\}, \{L,M\}, \{L,R\}, \{M,R\}, \{L,M,R\}$

$\uparrow$ all subsets of $\{T,B\}$ minus the $\emptyset$ for player 1.

$\downarrow$ $\{T\}, \{B\}, \{T,B\}$

We will use the support enumeration method. It works due to the characterisation of N.E. (see lecture notes "Introduction to Game Theory")

- Let us see if any player's pure strategies would give us any equilibria:
  - If player 1 chooses T, then player 2 chooses M, but then player 1 would prefer B.
  - If player 1 chooses B, then player 2 chooses L, but then player 1 would prefer T.

  - If player 2 chooses L, then player 1 chooses T, but then pl. 2 would prefer M
  - If player 2 chooses M, then player 1 chooses B, but then pl. 2 would prefer L

- If player 2 chooses R, then player 1 chooses B, but then player 2 would prefer L.

So, we conclude that no pure strategy of any player which would give an equilibrium, i.e. there is no equilibrium where either player has support of size 1 (one strategy - pure).

Therefore, we only need to consider the combinations of the following supports:

$\{T, B\}$ for player 1

$\{L, M\}, \{L, R\}, \{M, R\}, \{L, M, R\}$ for player 2

- Let us try the support $\{T, B\} \times \{L, M\}$. We need:

$$\left.\begin{aligned} u_1(T, p_2) &= u_1(B, p_2) \\ \text{and} \quad u_2(L, p_1) &= u_2(M, p_1) \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} 7p_2(L) + 2p_2(M) &= 2p_2(L) + 7p_2(M) \\ 2p_1(T) + 7p_1(B) &= 7p_1(T) + 2p_1(B) \end{aligned}\right\}$$

$$\Leftrightarrow \left\{\begin{aligned} 5p_2(L) &= 5p_2(M) \\ 5p_1(B) &= 5p_1(T) \end{aligned}\right\}$$

Also, $p_2(L) + p_2(M) = 1$

and $p_1(T) + p_1(B) = 1$

There is a solution which gives: $p_1(T) = p_1(B) = p_2(L) = p_2(M) = 0.5$

and then $u_1(T, p_2) = u_1(B, p_2) = 0.5 \times 7 + 0.5 \cdot 2 = 4.5$

and $u_2(L, p_1) = u_2(M, p_1) = 4.5$

<u>But</u> if pl. 2 deviates to $\{R\}$ then $u_2(R, p_1) = 6 \cdot 0.5 + 5 \cdot 0.5$

$= 3 + 2.5 = 5.5 > 4.5$

So, not an equilibrium.

• Let us try the support $\{T,B\} \times \{L,R\}$. We need:

$$\left.\begin{array}{l} w_1 = u_1(T, p_2) = u_1(B, p_2) \\ w_2 = u_2(L, p_1) = u_2(R, p_2) \end{array}\right\} \Leftrightarrow$$

$$\left\{\begin{array}{l} 7p_2(L) + 3p_2(R) = 2p_2(L) + 4p_2(R) \\ 2p_1(T) + 7p_1(B) = 6p_1(T) + 5p_1(B) \end{array}\right\} \Leftrightarrow$$

$$\left\{\begin{array}{l} 5p_2(L) = p_2(R) \\ \cancel{2}p_1(B) = 2\cancel{4}p_1(T) \end{array}\right\}$$

Also, $p_2(L) + p_2(R) = 1 \Rightarrow 6p_2(L) = 1 \Rightarrow p_2(L) = \frac{1}{6}$

and $p_2(R) = \frac{5}{6}$

and $p_1(T) + p_1(B) = 1 \Rightarrow 3p_1(T) = 1 \Rightarrow p_1(T) = \frac{1}{3}$

and $p_1(B) = \frac{2}{3}$

Then, $w_1 = 7 \cdot \frac{1}{6} + 3 \cdot \frac{5}{6} = \frac{22}{6} = \frac{11}{3}$

and $w_2 = 2 \cdot \frac{1}{3} + 7 \cdot \frac{2}{3} = \frac{16}{3}$

and also $u_2(M, p_1) = 7 \cdot \frac{1}{3} + 2 \cdot \frac{2}{3} = \frac{11}{3} < w_2$

So, this <u>is</u> an equilibrium.

- Let us try the support $\{T,B\} \times \{M,R\}$. We need:

$$\begin{cases} w_1 = u_1(T, p_2) = u_1(B, p_2) \\ w_2 = u_2(M, p_1) = u_2(R, p_1) \end{cases} \Leftrightarrow$$

$$\begin{cases} 2p_2(M) + 3p_2(R) = 7p_2(M) + 4p_2(R) \\ 7p_1(T) + 2p_1(B) = 6p_1(T) + 5p_1(B) \end{cases} \Leftrightarrow$$

$$\begin{cases} 5p_2(M) + p_2(R) = 0 \\ p_1(T) = 3p_1(B) \end{cases}$$

no solution!!! since $p_2(M), p_2(R) > 0$.

- Let us try the support $\{T,B\} \times \{L,M,R\}$. We need:

$$\begin{cases} w_1 = u_1(T, p_2) = u_1(B, p_2) \\ w_2 = u_2(L, p_1) = u_2(M, p_1) = u_3(R, p_1) \end{cases} \Leftrightarrow$$

$$\begin{cases} 7p_2(L) + 2p_2(M) + 3p_2(R) = 2p_2(L) + 7p_2(M) + 4p_2(R) \\ 2p_1(T) + 7p_1(B) = 7p_1(T) + 2p_1(B) = 6p_1(T) + 5p_1(B) \end{cases} \Leftrightarrow$$

$$\begin{cases} 5p_2(L) = 5p_2(M) + p_2(R) \\ \cancel{5}p_1(T) = \cancel{5}p_1(B) \\ 2\cancel{4}p_1(T) = \cancel{2}p_1(B) \\ p_1(T) = 3p_1(B) \end{cases}$$

conflicting - no solution!

Homework

Find all the Nash equilibria of the following game (The Going-to-the-Movies Deadlock). (Hint: use the iterated elimination of strictly dominated strategies method: see Ch. 4.4 of "Game Theory: Decisions, Interaction and Evolution – James N. Webb", and Ch 4.2 of "A Course in Game Theory – Martin J. Osborne, Ariel Rubinstein". This method uses the fact that no strictly dominated action is in the support of strategies in a Nash equilibrium. After that, apply the support enumeration method.)

T \ F	I	J	R
I	8, 1	2, −2	2, 4
J	3, −2	9, 5	3, 4
R	−2, −2	−2, 2	4, 7

Answer: 3 Nash equilibria:

(J, J), (R, R), $\left(\left(0, \frac{5}{6}, \frac{1}{6}\right), \left(0, \frac{1}{12}, \frac{11}{12}\right)\right)$

We will first eliminate in a sequential way the strictly dominated actions, in order to simplify the game. No such action can be in the support of strategies in a Nash equilibrium (see given bibliography for this question).

Since we are looking for combinations of strategies (profiles) that are Nash equilibria, we can discard the strictly dominated strategies when applying the support enumeration method. Eliminating strictly dominated actions helps us, since we have to consider significantly less pairs of supports in the support enumeration method.

Step 1: Check for strictly dominated actions and eliminate them.

(a) Action "I" of player E is strictly dominated by her action "R". Therefore "I" can be eliminated.

The simplified game is now:

T \ E	J	R
I	2,-2	2,4
J	9,5	3,4
R	-2,2	4,7

Observe that at this point no action of player E is strictly dominated.

(b) Action "I" of player T is strictly dominated by her action "J". Therefore "I" can be eliminated.

The simplified game is now:

T \ F	J	R
J	9,5	3,4
R	-2,2	4,7

Observe that at this point no action of any player is strictly dominated. So, we can move on to applying the support enumeration method in order to find all Nash equilibria.

* Note that we could <u>not</u> have done first (b) and then (a) !

Step 2: Apply the support enumeration method on the simplified game.

All Nash equilibria of the original game are the same as those of the simplified 2×2 game.

Possible supports of player T: {J}, {R}, {J, R}.

Possible supports of player F: {J}, {R}, {J, R}.

We will examine all possible pairs of supports to find which ones admit a Nash equilibrium.

player T player F

(a) $(\{J\},\{J\})$: This combination in fact defines a pure strategy profile.
One can check and see that this $\boxed{\text{is a PNE}}$

(b) $(\{J\},\{R\})$: Not a PNE. (player F would like to deviate to "J")

(c) $(\{R\},\{J\})$: Not a PNE (player T would like to deviate to "J")

(d) $(\{R\},\{R\})$: Indeed the profile (R,R) $\boxed{\text{is a PNE.}}$

(e) $(\{J\},\{J,R\})$:

player F : It should hold: $U_F(J,J) = U_F(J,R) \iff 5=4$ ✗
contradiction.

So, no NE

(f) $(\{R\},\{J,R\})$:

player F : It should hold: $U_F(R,J) = U_F(R,R) \iff 2 = 7$ ✗
contradiction

So, no NE.

(g) $(\{J,R\},\{J\})$:

player T : It should hold: $U_T(J,J) = U_T(R,J) \iff 9 = -2$ ✗
contradiction

So, no NE.

(h) $(\{J,R\},\{R\})$:

player (T) : It should hold: $u_T(J,R) = u_T(R,R) \Longleftrightarrow 3=4$ ✗ contradiction

So, no NE.

(i) $(\{J,R\},\{J,R\})$:

T \ F	J	R
J	9, 5	3, 4
R	-2, 2	4, 7

player (T): $u_T(J,q) = u_T(R,q)$

$\Longleftrightarrow 9\cdot q(J) + 3\cdot q(R) = -2\cdot q(J) + 4\cdot q(R)$

$\Longleftrightarrow 11\cdot q(J) = q(R)$ (1)

probability condition: $q(J)+q(R)=1$ (2)

$(1),(2) \Rightarrow 11\cdot q(J) = 1 - q(J) \Rightarrow \boxed{q(J) = \frac{1}{12}} \xRightarrow{(2)} \boxed{q(R) = \frac{11}{12}}$ ✓

player (F) : $u_F(p,J) = u_F(p,R)$

$\Longleftrightarrow 5\cdot p(J) + 2\cdot p(R) = 4\cdot p(J) + 7\cdot p(R)$

$\Longleftrightarrow p(J) = 5\cdot p(R)$ (3)

probability condition: $p(J)+p(R)=1$ (4)

$(3),(4) \Rightarrow p(J) = 5\cdot(1-p(J)) \Rightarrow p(J) = 5 - 5\cdot p(J)$

$\Rightarrow 6\cdot p(J) = 5 \Rightarrow \boxed{p(J) = \frac{5}{6}} \xRightarrow{(4)} \boxed{p(R) = \frac{1}{6}}$ ✓

So, this is a NE.

Therefore, the Nash equilibria of the original 3×3 game are the ones of the simplified 2×2 game, namely:

(J,J), (R,R), $\left(\left(0,\frac{5}{6},\frac{1}{6}\right),\left(0,\frac{1}{12},\frac{11}{12}\right)\right)$