COMP323 – Introduction to Computational Game Theory

Bimatrix Games

Giorgos Christodoulou

Department of Computer Science
University of Liverpool

Outline

1. Background: Basic concepts in matrix algebra
2. Strategies and payoffs
3. Equilibria
4. Approximate equilibria

1. Background: Basic concepts in matrix algebra
   - Vectors
   - Matrices
   - Matrix algebra
2. Strategies and payoffs
3. Equilibria
4. Approximate equilibria

Vectors

- A k-dimensional vector $\mathbf{v}$ is an ordered collection of k real numbers $v_1, v_2, \ldots, v_k$ and is written as

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} .$$

- The numbers v_j, for $j = 1, 2, \ldots, k$, are called the components of vector $\mathbf{v}$.

Example: $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \end{bmatrix}$ is a four-dimensional vector. Its first component is 1, its second component is -2, its third component is 0, and its fourth component is 5.

Vectors

Scalar multiplication and vector addition

- Scalar multiplication of a k-dimensional vector $\mathbf{y}$ and a scalar c is defined to be a new k-dimensional vector $\mathbf{z}$, written $\mathbf{z} = c\mathbf{y}$ or $\mathbf{z} = \mathbf{y}c$, whose components are given by $z_j = cy_j$.
- Vector addition of two k-dimensional vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as a new k-dimensional vector $\mathbf{z}$, denoted $\mathbf{z} = \mathbf{x} + \mathbf{y}$, with components given by $z_j = x_j + y_j$.

Note: $\mathbf{y}$ and $\mathbf{x}$ must have the same dimensions for vector addition.

Vectors

Scalar multiplication and vector addition

Examples:

$$4\begin{bmatrix}1\\-2\\0\\5\end{bmatrix} = \begin{bmatrix}4\\-8\\0\\20\end{bmatrix}$$

$$\begin{bmatrix}2\\-1\\6\\0\end{bmatrix} + \begin{bmatrix}-2\\-1\\5\\4\end{bmatrix} = \begin{bmatrix}0\\-2\\11\\4\end{bmatrix}$$

$$\begin{bmatrix}4\\3\\0\end{bmatrix} + \begin{bmatrix}-1\\5\end{bmatrix} \quad \textbf{is not defined} \; .$$

Matrices

- A matrix is defined to be a rectangular array of numbers

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

  whose dimension is m by n (denoted $m \times n$).
- A is called square if $m = n$.
- The numbers a_{ij} are the elements of A.
- Two matrices A and B are said to be equal, written $A = B$, if they have the same dimension and their corresponding elements are equal, i.e., $a_{ij} = b_{ij}$ for all i and j.

Vectors as special cases of matrices

Sometimes it is convenient to think of vectors as merely being special cases of matrices:

- A $k \times 1$ matrix is called a column vector.
- An $1 \times k$ matrix is called a row vector.
- The coefficients in row i of the matrix A determine a row vector

$$A^i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} .$$

- The coefficients in column j of the matrix A determine a column vector

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} .$$

Matrices

Scalar multiplication and addition

- Scalar multiplication of a matrix A and a real number c is defined to be a new matrix B, written $B = cA$ or $B = Ac$, whose elements b_{ij} are given by $b_{ij} = ca_{ij}$.
  Example:
  $$3\begin{bmatrix} 0 & 1 & -2 \\ 4 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 3 & -6 \\ 12 & -3 & 9 \end{bmatrix}.$$
- Addition of two matrices A and B, both with dimension $m \times n$, is defined as a new matrix C, written $C = A + B$, whose elements c_{ij} are given by $c_{ij} = a_{ij} + b_{ij}$.
  Example:
  $$\begin{bmatrix} 7 & -1 & 12 \\ 0 & 6 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -8 \\ 4 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 4 \\ 4 & 12 & -3 \end{bmatrix}.$$
- If A and B do not have the same dimension, then $A + B$ is undefined.

Matrices

Product of matrices

The product of an $m \times p$ matrix A and a $p \times n$ matrix B is defined to be a new $m \times n$ matrix C, written $C = AB$, whose elements c_{ij} are given by

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} \quad .$$

Example:

$$\begin{bmatrix} 2 & 6 & -3 \\ 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 6 \cdot 0 - 3 \cdot 3 & 2 \cdot 2 - 6 \cdot 3 - 3 \cdot 1 \\ 1 \cdot 1 + 4 \cdot 0 + 0 \cdot 3 & 1 \cdot 2 - 4 \cdot 3 + 0 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -17 \\ 1 & -10 \end{bmatrix}$$

Matrices

Product of matrices

- If the number of columns of A does not equal the number of rows of B, then AB is undefined.
- If $\mathbf{x}$ is an m-dimensional row vector and $\mathbf{y}$ is an m-dimensional column vector, then the special case

$$\mathbf{x}\mathbf{y} = \sum_{i=1}^{m} x_i y_i$$

  is referred to as the inner product of $\mathbf{x}$ and $\mathbf{y}$.
- In these terms, the elements c_{ij} of matrix $C = AB$ are found by taking the inner product of the ith row of A with the jth column of B.

Matrices

Transpose of a matrix

The transpose of an $m \times n$ matrix A, denoted A^T, is the $n \times m$ matrix formed by interchanging the rows and columns of A.

Example 1:

$$\begin{bmatrix} 2 & 6 & -3 \\ 1 & 4 & 0 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 \\ 6 & 4 \\ -3 & 0 \end{bmatrix}$$

Example 2: The transpose of a column vector is a row vector (and vice versa):

$$\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}^T = \begin{bmatrix} 1 & -3 & 5 \end{bmatrix}$$

Matrices

Properties

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- A square $(n \times n)$ matrix A is symmetric if $A = A^T$, or, equivalently, if $a_{ij} = a_{ji}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, n$. Examples:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -3 & 5 \\ -3 & 0 & 7 \\ 5 & 7 & 4 \end{bmatrix}$$

Matrix algebra: examples

$$\text{Let} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & 2 & -2 \end{bmatrix}$$

$$\begin{aligned}
\text{Then} \quad \mathbf{x}^T A &= \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ 1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & 2 & 0 \end{bmatrix} \\
A\mathbf{y} &= \begin{bmatrix} 4 & 0 & 1 \\ 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -7 \end{bmatrix} \\
\text{and} \quad \mathbf{x}^T A\mathbf{y} &= (\mathbf{x}^T A)\mathbf{y} = \begin{bmatrix} 9 & 2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = -9 \\
\text{or} \quad \mathbf{x}^T A\mathbf{y} &= \mathbf{x}^T(A\mathbf{y}) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -7 \end{bmatrix} = -9
\end{aligned}$$

Matrix algebra: a general example

- Let $\mathbf{x}$ be an m-dimensional vector and $\mathbf{y}$ be an n-dimensional vector.
- Let A be an $m \times n$ matrix.
- Then $A\mathbf{y}$ is an m-dimensional vector and $A^T\mathbf{x}$ is an n-dimensional vector.
- We denote the ith component of $A\mathbf{y}$ by $(A\mathbf{y})_i$ (similarly for $A^T\mathbf{x}$).

Then we have:

$$
\begin{aligned}
(A\mathbf{y})_i &= \sum_{j=1}^{n} a_{ij} y_j \\
(A^T\mathbf{x})_j &= \sum_{i=1}^{m} a_{ij} x_i \\
\mathbf{x}^T A\mathbf{y} &= \sum_{i=1}^{m} x_i (A\mathbf{y})_i = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j
\end{aligned}
$$

Matrix algebra: a general example

- Note that $\mathbf{x}^T A\mathbf{y}$ is a scalar, so

$$\mathbf{x}^T A\mathbf{y} = (\mathbf{x}^T A\mathbf{y})^T = (A\mathbf{y})^T(\mathbf{x}^T)^T = \mathbf{y}^T A^T \mathbf{x} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}x_i y_j$$

and

$$\mathbf{x}^T A\mathbf{y} = (A^T\mathbf{x})^T\mathbf{y} = (\mathbf{y}^T A^T \mathbf{x})^T = \mathbf{y}^T A^T \mathbf{x} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}x_i y_j \quad .$$

Also,

$$\mathbf{y}^T A^T \mathbf{x} = \sum_{j=1}^{n} y_j (A^T\mathbf{x})_j = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}x_i y_j \quad .$$

1 Background: Basic concepts in matrix algebra

2 Strategies and payoffs
- What is a bimatrix game?
- Pure and mixed strategies
- Expected payoffs
- Symmetric bimatrix games

3 Equilibria

4 Approximate equilibria

Bimatrix games

Recall that a finite, noncooperative strategic game $\Gamma = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$ consists of

1. a finite set of players N,
2. a nonempty finite set of pure strategies S_i for each player $i \in N$ and
3. a payoff function $u_i : \times_{i\in N} S_i \to \mathbb{R}$ for each player $i \in N$, mapping every combination of strategies (one for each player) to a real number.

Bimatrix games are a special case of 2-player games:

- $|N| = 2$
- the payoff functions can be described by two real $m \times n$ matrices A and B, where $m = |S_1|$ and $n = |S_2|$.

Bimatrix games

An example

Consider the rock-scissors-paper game:

- Two children simultaneously choose one of three options: rock, paper, or scissors.
- Rock beats scissors, scissors beats paper, and paper beats rock.
- When both play the same, the game is drawn.

We will formulate this game as a bimatrix game.

- We denote the rock, scissors, paper options by R, S, P, respectively.
- The payoff for a win is $+1$, for losing -1, and for a draw 0.

Bimatrix games

An example

The game can be fully described by the following payoff table:

	R	S	P
R	0, 0	1, -1	-1, 1
S	-1, 1	0, 0	1, -1
P	1, -1	-1, 1	0, 0

- The rows represent the choices of the first player.
- The columns represent the choices of the second player.
- In each entry, the first number represents the payoff of the first player and the second number represents the payoff of the second player.
- E.g., when the first player chooses R and the second player chooses P, then the former gets a payoff of -1 and the latter gets a payoff of 1.

Bimatrix games

An example

The game is called a bimatrix game because the payoff table is actually the combination of two matrices:

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

- Each row of each matrix corresponds to a pure strategy (a choice) of the first player.
- Each column of each matrix corresponds to a pure strategy of the second player.
- Each element a_{ij} of matrix A is the payoff to player 1 if she plays her ith strategy and the opponent plays her jth strategy.
- Each element b_{ij} of matrix B is the payoff to player 2 if she plays her jth strategy and the opponent plays her ith strategy.

Bimatrix games

Definition

A bimatrix game is denoted by a pair of matrices, i.e., $\Gamma = (A, B)$, in which:

- The m rows of A and B represent the pure strategies of the first player (the row player).
- The n columns A and B represent the pure strategies of the second player (the column player).
- Then, when the row player chooses strategy i and the column player chooses strategy j, the former gets payoff a_{ij} while the latter gets payoff b_{ij}.

(Mixed) strategies

Recall that a mixed strategy is a probability distribution over the available pure strategies of a player. Given a bimatrix game (A, B) with $m \times n$ payoff matrices A and B:

- A mixed strategy (or simply strategy) for the row player is an m-dimensional vector $\mathbf{x}$ with nonnegative components that sum to 1:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \quad , \quad \sum_{i=1}^{m} x_i = 1 \ , \quad x_i \geq 0 \ \forall i = 1, \cdots, m \ .$$

- A mixed strategy for the column player is such a vector $\mathbf{y}$:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad , \quad \sum_{j=1}^{n} y_j = 1 \ , \quad y_j \geq 0 \ \forall j = 1, \cdots, n \ .$$

Pure strategies

- A pure strategy for the row player can be seen as a special case of a mixed strategy that assigns probability 1 to a single row.
- A pure strategy for the column player can be seen as a special case of a mixed strategy that assigns probability 1 to a single column.
- Hence the pure strategy profile (i, j) can be denoted by the pair of vectors $(\mathbf{x}, \mathbf{y})$ for which

$$x_i = y_j = 1\,, \quad x_t = 0 \ \forall t \neq i\,, \quad y_k = 0 \ \forall k \neq j \ .$$

Support of a strategy

- The support of a mixed strategy is the set of pure strategies that are assigned positive probability.
- Hence, the support of strategy $\mathbf{x}$ of the row player in $m \times n$ bimatrix game $\Gamma = (A, B)$ is

$$\text{Support}_1(\mathbf{x}) = \{i \in \{1, 2, \ldots, m\} : x_i > 0\} \quad .$$

and the support of strategy $\mathbf{y}$ of the column player is

$$\text{Support}_2(\mathbf{y}) = \{i \in \{1, 2, \ldots, n\} : y_j > 0\} \quad .$$

Strategies in bimatrix games: an example

Consider again the rock-scissors-paper game:

	R	S	P
R	0, 0	1, -1	-1, 1
S	-1, 1	0, 0	1, -1
P	1, -1	-1, 1	0, 0

- Assume that the row player plays rock with probability $1/4$ and paper with probability $3/4$, and the column player simply plays paper.
- The strategies of the row and the column players are, respectively,

$$\mathbf{x} = \begin{bmatrix} 1/4 \\ 0 \\ 3/4 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

- The support of the row player is $\{1, 3\}$ (i.e., rows 1 and 3 corresponding to rock and paper) and the support of the column player is the singleton $\{3\}$.

Expected payoff

When the row player chooses mixed strategy $\mathbf{x}$ and the column player chooses $\mathbf{y}$, then

- the row player gets expected payoff

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j a_{ij} = \mathbf{x}^T A\mathbf{y}$$

  and

- the column player gets expected payoff

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j b_{ij} = \mathbf{x}^T B\mathbf{y} \quad .$$

Expected payoffs: an example

	R	S	P
R	0, 0	1, -1	-1, 1
S	-1, 1	0, 0	1, -1
P	1, -1	-1, 1	0, 0

Assume that the row player plays rock with probability $1/4$ and paper with probability $3/4$, and the column player plays rock with probability $1/6$, scissors with probability $1/3$ and paper with probability $1/2$:

$$\mathbf{x} = \begin{bmatrix} 1/4 \\ 0 \\ 3/4 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/2 \end{bmatrix}.$$

Expected payoffs: an example

	R	S	P
R	0, 0	1, -1	-1, 1
S	-1, 1	0, 0	1, -1
P	1, -1	-1, 1	0, 0

The expected payoff for the row player for the strategy profile $(\mathbf{x}, \mathbf{y})$ is

$$
\begin{aligned}
\mathbf{x}^T A \mathbf{y} &= \begin{bmatrix} 1/4 & 0 & 3/4 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1/6 \\ 1/3 \\ 1/2 \end{bmatrix} \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} x_i y_j \\
&= \frac{1}{4} \cdot \frac{1}{6} \cdot 0 + \frac{1}{4} \cdot \frac{1}{3} \cdot 1 + \frac{1}{4} \cdot \frac{1}{2} \cdot (-1) + \frac{3}{4} \cdot \frac{1}{6} \cdot 1 + \frac{3}{4} \cdot \frac{1}{3} \cdot (-1) + \frac{3}{4} \cdot \frac{1}{2} \cdot 0 \\
&= -\frac{1}{6} \; .
\end{aligned}
$$

Expected payoffs: an example

	R	S	P
R	0, 0	1, -1	-1, 1
S	-1, 1	0, 0	1, -1
P	1, -1	-1, 1	0, 0

The expected payoff for the column player for the strategy profile $(\mathbf{x}, \mathbf{y})$ is

$$\begin{aligned}
\mathbf{x}^T B \mathbf{y} &= \begin{bmatrix} 1/4 & 0 & 3/4 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/6 \\ 1/3 \\ 1/2 \end{bmatrix} \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} x_i y_j \\
&= \frac{1}{4} \cdot \frac{1}{6} \cdot 0 + \frac{1}{4} \cdot \frac{1}{3} \cdot (-1) + \frac{1}{4} \cdot \frac{1}{2} \cdot 1 + \frac{3}{4} \cdot \frac{1}{6} \cdot (-1) + \frac{3}{4} \cdot \frac{1}{3} \cdot 1 + \frac{3}{4} \cdot \frac{1}{2} \cdot 0 \\
&= \frac{1}{6} \ .
\end{aligned}$$

Expected payoffs: an example

	R	S	P
R	0, 0	1, -1	-1, 1
S	-1, 1	0, 0	1, -1
P	1, -1	-1, 1	0, 0

- The expected payoff for the row player if she chooses row 2 (scissors) and the column player plays **y** is

$$(A^T\mathbf{y})_2 = \sum_{k=1}^{3} a_{2k}y_k = (-1)\cdot\frac{1}{6} + 0\cdot\frac{1}{3} + 1\cdot\frac{1}{2} = \frac{1}{3} \ .$$

- The expected payoff for the column player if she chooses column 1 (rock) and the row player plays **x** is

$$(B^T\mathbf{x})_1 = \sum_{t=1}^{3} b_{t1}x_t = 0\cdot\frac{3}{4} + 1\cdot 0 + (-1)\cdot\frac{1}{4} = -\frac{1}{4} \ .$$

Symmetric bimatrix games

A 2-player strategic game is symmetric if

1. the players' sets of pure strategies are the same and
2. the players' payoff functions u_1 and u_2 are such that

$$u_1(s_1, s_2) = u_2(s_2, s_1) \ .$$

That is, a symmetric game does not change when the players change roles. Using the notation of bimatrix games, an $m \times n$ bimatrix game $\Gamma = (A, B)$ is symmetric if

1. $m = n$ and
2. $a_{ij} = b_{ji}$ for all $i, j \in \{1, \ldots, n\}$, or equivalently $B = A^T$.

Symmetric bimatrix games

Examples

Observe that the rock-scissors-paper game is symmetric:

	R	S	P
R	0, 0	1, -1	-1, 1
S	-1, 1	0, 0	1, -1
P	1, -1	-1, 1	0, 0

- For example, if the row player plays scissors and the column player plays rock, then the row player gets -1 and the column player gets 1.
- If the players change roles, so that the row player plays rock and the column player plays scissors, then the payoffs change respectively, so that now the row player gets 1 and the column player gets -1.

Symmetric bimatrix games

Counterexamples

The following games are not symmetric:

	L	M	R
L	0, 1	1, -1	-1, 1
M	-1, 1	0, 0	1, -1
R	1, -1	-1, 1	0, 0

	L	M	R
L	0, 0	1, -1	-1, 1
M	1, 0	0, 0	1, -1
R	1, 0	-1, 1	0, 0

	L	M
L	0, 0	1, -1
M	-1, 1	0, 0
R	1, -1	-1, 1

	L	M
L	0, 0	1, 2
M	1, 2	0, 0

1. Background: Basic concepts in matrix algebra
2. Strategies and payoffs
3. Equilibria
   - Nash equilibria
   - Computing Nash equilibria
   - Existence of Nash equilibrium
4. Approximate equilibria

Nash equilibrium

A Nash equilibrium for a game Γ is a combination of (pure or mixed) strategies, one for each player, such that no player could increase her payoff by unilaterally changing her strategy.
Formally:

Definition

A pair of strategies $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is a *Nash equilibrium* for the bimatrix game $\Gamma = (A, B)$ if

(i) For every (mixed) strategy $\mathbf{x}$ of the row player, $\mathbf{x}^T A\tilde{\mathbf{y}} \leq \tilde{\mathbf{x}}^T A\tilde{\mathbf{y}}$ and

(ii) For every (mixed) strategy $\mathbf{y}$ of the column player, $\tilde{\mathbf{x}}^T B\mathbf{y} \leq \tilde{\mathbf{x}}^T B\tilde{\mathbf{y}}$.

Best responses

A best response for a player is a strategy that maximizes her payoff, given the strategy chosen by the other player.
Formally, given a strategy profile $(\mathbf{x}, \mathbf{y})$ for the $m \times n$ bimatrix game $\Gamma = (A, B)$:

- Strategy $\tilde{\mathbf{x}}$ is a best response for the row player if

$$\tilde{\mathbf{x}}^T A\mathbf{y} \geq \mathbf{x'}^T A\mathbf{y} \quad \forall \mathbf{x}' \ .$$

- Strategy $\tilde{\mathbf{y}}$ is a best response for the column player if

$$\mathbf{x}^T B\tilde{\mathbf{y}} \geq \mathbf{x}^T B\mathbf{y}' \quad \forall \mathbf{y}' \ .$$

Therefore:

Definition

The strategy profile $(\mathbf{x}, \mathbf{y})$ is a *Nash equilibrium* for the bimatrix game $\Gamma = (A, B)$ if $\mathbf{x}$ is a best response of the row player to $\mathbf{y}$ and $\mathbf{y}$ is a best response of the column player to $\mathbf{x}$.

Best responses

A useful characterization

Best responses are characterized by the following combinatorial condition:

Theorem (Nash, 1951)

Let $\mathbf{x}$ and $\mathbf{y}$ be mixed strategies of the row and the column player, respectively. Then $\mathbf{x}$ is a best response to $\mathbf{y}$ if and only if all strategies in the support of $\mathbf{x}$ are (pure) best responses to $\mathbf{y}$.

Proof:

- Let $(A\mathbf{y})_i$ be the ith component of $A\mathbf{y}$, which is the expected payoff to the row player when playing row i.
- Let $u = \max_k (A\mathbf{y})_k$. Then

$$\mathbf{x}^T A\mathbf{y} = \sum_i x_i (A\mathbf{y})_i = u - \sum_i x_i (u - (A\mathbf{y})_i) \ .$$

Best responses

A useful characterization

Proof (continued):

- So $\mathbf{x}^T A\mathbf{y} = u - \sum_i x_i(u - (A\mathbf{y})_i)$.
- The sum $\sum_i x_i(u - (A\mathbf{y})_i)$ is nonnegative, hence $\mathbf{x}^T A\mathbf{y} \leq u$.
- The expected payoff $\mathbf{x}^T A\mathbf{y}$ achieves the maximum u if and only if that sum is zero.
- That is, if $x_i > 0$ implies $(A\mathbf{y})_i = u = \max_k (A\mathbf{y})_k$, as claimed.

Clearly, the same holds for the column player:

Theorem

$\mathbf{y}$ is a best response to $\mathbf{x}$ if and only if all strategies in the support of $\mathbf{y}$ are (pure) best responses to $\mathbf{x}$.

Best responses

Regret

Given a strategy profile $(\mathbf{x}, \mathbf{y})$ of the bimatrix game $\Gamma = (A, B)$

- row player's regret is $\max_i (A\mathbf{y})_i - \mathbf{x}^T A\mathbf{y}$;
- column player's regret is $\max_j (B^T\mathbf{x})_j - \mathbf{x}^T B\mathbf{y}$.

So

- $\mathbf{x}$ is a best response to $\mathbf{y}$ if row player's regret is 0;
- $\mathbf{y}$ is a best response to $\mathbf{x}$ if column player's regret is 0.
- $(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium if each player's regret is 0.

Nash equilibria

Useful characterizations

Based on the characterization of best responses described previously:

Definition

The strategy profile $(\mathbf{x}, \mathbf{y})$ is a *Nash equilibrium* for the $m \times n$ bimatrix game $\Gamma = (A, B)$ if

$$\begin{aligned} \mathbf{x}^T A\mathbf{y} &= \max_{i=1,\dots,m} (A\mathbf{y})_i \quad \text{and} \\ \mathbf{x}^T B\mathbf{y} &= \max_{j=1,\dots,n} (B^T\mathbf{x})_j \end{aligned}$$

Nash equilibria

Useful characterizations

And equivalently:

Definition

The strategy profile $(\mathbf{x}, \mathbf{y})$ is a *Nash equilibrium* for the $m \times n$ bimatrix game $\Gamma = (A, B)$ if

$$x_i > 0 \implies (A\mathbf{y})_i = \max_{t=1,\ldots,m} (A\mathbf{y})_t \quad \forall i = 1, \ldots, m \quad \text{and}$$

$$y_j > 0 \implies (B^T\mathbf{x})_j = \max_{k=1,\ldots,n} (B^T\mathbf{y})_k \quad \forall j = 1, \ldots, n \ .$$

Computing Nash equilibria

Pure Nash equilibria

- Given an $m \times n$ bimatrix game, checking whether a pure Nash equilibrium exists or not can be done efficiently.
- Given the column chosen by the column player, the row player should have no incentive to deviate, i.e., she should choose a row that maximizes her payoff.
- Similarly, given the row chosen by the row player, the row player should choose a row that maximizes her payoff.

The procedure is as follows:

- For each row $i = 1, \ldots, m$ and for each column $j = 1, \ldots n$, we check whether $a_{ij} = \max_t a_{tj}$ and $b_{ij} = \max_k b_{ik}$.
- If both conditions hold, then (i, j) is a pure Nash equilibrium.
- We have $m \cdot n$ pure strategy profiles to check.

Computing Nash equilibria

Pure Nash equilibria

Example: Let us find all the pure Nash equilibria (PNE) of the game

	L	M	R
U	5, 3	2, 7	0, 4
D	5, 5	5, -1	-4, -2

1. (U, L) is not a PNE because, given U, player 2 prefers M to L $(7 > 3)$.
2. (U, M) is not a PNE because, given M, player 1 prefers D to U $(5 > 2)$.
3. (U, R) is not a PNE because, given U, player 2 prefers M to R $(7 > 4)$.
4. (D, L) is a PNE because no player has an incentive to deviate $(5 \geq 5$ and $5 > -1, 5 > -2)$.
5. (D, M) is not a PNE because, given D, player 2 prefers L to M $(5 > -1)$.
6. (D, R) is not a PNE because, given U, player 1 prefers L to R $(5 > -2)$.

Computing Nash equilibria

Pure Nash equilibria

Example: Does the rock-scissors-paper game possess a pure Nash equilibrium?

	R	S	P
R	0, 0	1, -1	-1, 1
S	-1, 1	0, 0	1, -1
P	1, -1	-1, 1	0, 0

We can easily see that the answer is no.

Computing Nash equilibria

Mixed Nash equilibria

To find the mixed Nash equilibria of an $m \times n$ bimatrix game (A, B), we use the following characterization we have already proved:

Definition

$(\mathbf{x}, \mathbf{y})$ is a *Nash equilibrium* if

$$x_i > 0 \quad \Longrightarrow \quad (A\mathbf{y})_i = \max_{t=1,\ldots,m} (A\mathbf{y})_t \quad \forall i = 1, \ldots, m \quad \text{and}$$

$$y_j > 0 \quad \Longrightarrow \quad (B^T\mathbf{x})_j = \max_{k=1,\ldots,n} (B^T\mathbf{y})_k \quad \forall j = 1, \ldots, n \ .$$

- This states that, in a Nash equilibrium, each player assigns positive probability only to her pure strategies that maximize her payoff.
- So, the expected payoffs for all pure strategies in the support of a player must be equal and maximal (given the mixed strategy of the other player).

Computing Nash equilibria

Mixed Nash equilibria

Thus the procedure to find all Nash equilibria is as follows:

- For each possible support of player 1 and for each possible support of player 2, check if there is solution to the system of equations of the definition above.
- If such a solution exists and corresponds to probabilities (i.e., all x_k's are non-negative and sum up to 1, and so are all y_k's, then an equilibrium is found.
- We have $(2^m - 1)(2^n - 1)$ possible cases to consider, since there are $2^m - 1$ possible supports for the row player and $2^n - 1$ possible supports for the column player.

Computing Nash equilibria

Mixed Nash equilibria

Example

	L	M	R
U	6, 1	1, 6	2, 5
D	1, 6	6, 1	3, 4

Let us check if there exists a Nash equilibrium with supports $\{U, D\}$ and $\{L, M\}$. So let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & 0 \end{bmatrix}^T$.
$(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium iff all the following conditions hold:

$$\begin{aligned}
(A\mathbf{y})_1 &= (A\mathbf{y})_2 \\
(B^T\mathbf{x})_1 &= (B^T\mathbf{x})_2 \geq (B^T\mathbf{x})_3 \\
x_1 + x_2 &= 1 \\
y_1 + y_2 &= 1 \\
x_1, x_2, y_1, y_2 &\geq 0 \ .
\end{aligned}$$

Computing Nash equilibria

Mixed Nash equilibria

Example (continued)

	L	M	R
U	6, 1	1, 6	2, 5
D	1, 6	6, 1	3, 4

We have, equivalently,

$$\begin{aligned} (A\mathbf{y})_1 &= (A\mathbf{y})_2 \\ 6 \cdot y_1 + 1 \cdot y_2 &= 1 \cdot y_1 + 6 \cdot y_2 \\ y_1 = y_2 &= 1/2 \end{aligned}$$

and

$$\begin{aligned} (B^T\mathbf{x})_1 &= (B^T\mathbf{x})_2 \\ 1 \cdot x_1 + 6 \cdot x_2 &= 6 \cdot x_1 + 1 \cdot x_2 \\ x_1 = x_2 &= 1/2 \ . \end{aligned}$$

Computing Nash equilibria

Mixed Nash equilibria

Example (continued)

	L	M	R
U	6, 1	1, 6	2, 5
D	1, 6	6, 1	3, 4

But then

$$(B^T\mathbf{x})_1 = (B^T\mathbf{x})_2 = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 6 = \frac{7}{2}$$

and

$$(B^T\mathbf{x})_3 = \frac{1}{2}\cdot 5 + \frac{1}{2}\cdot 4 = \frac{9}{2} > \frac{7}{2} = (B^T\mathbf{x})_1$$

so $(\mathbf{x}, \mathbf{y})$ is not a Nash equilibrium.

Computing Nash equilibria

Mixed Nash equilibria

Example (continued)

	L	M	R
U	6, 1	1, 6	2, 5
D	1, 6	6, 1	3, 4

Now let us check supports $\{U, D\}$ and $\{M, R\}$. So let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} 0 & y_2 & y_3 \end{bmatrix}^T$.
$(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium iff all the following conditions hold:

$$\begin{aligned} (A\mathbf{y})_1 &= (A\mathbf{y})_2 \\ (B^T\mathbf{x})_2 &= (B^T\mathbf{x})_3 \geq (B^T\mathbf{x})_1 \\ x_1 + x_2 &= 1 \\ y_2 + y_3 &= 1 \\ x_1, x_2, y_2, y_3 &\geq 0 . \end{aligned}$$

Computing Nash equilibria

Mixed Nash equilibria

Example (continued)

	L	M	R
U	6, 1	1, 6	2, 5
D	1, 6	6, 1	3, 4

We have, equivalently,

$$\begin{aligned}(A\mathbf{y})_1 &= (A\mathbf{y})_2 \\ y_2 + 2y_3 &= 6y_2 + 3y_3 \\ y_2 + 2(1-y_2) &= 6y_2 + 3(1-y_2) \\ y_2 &= -1/4\end{aligned}$$

which is not an acceptable solution (negative probability is impossible), so $(\mathbf{x}, \mathbf{y})$ is not an equilibrium.

Computing Nash equilibria

Mixed Nash equilibria

Example (continued)

	L	M	R
U	6, 1	1, 6	2, 5
D	1, 6	6, 1	3, 4

Now let us check supports $\{U, D\}$ and $\{L, R\}$. So let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & 0 & y_3 \end{bmatrix}^T$.
$(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium iff all the following conditions hold:

$$\begin{aligned}
(A\mathbf{y})_1 &= (A\mathbf{y})_2 \\
(B^T\mathbf{x})_1 &= (B^T\mathbf{x})_3 \geq (B^T\mathbf{x})_2 \\
x_1 + x_2 &= 1 \\
y_1 + y_3 &= 1 \\
x_1, x_2, y_1, y_3 &\geq 0 \ .
\end{aligned}$$

Computing Nash equilibria

Mixed Nash equilibria

Example (continued)

	L	M	R
U	6, 1	1, 6	2, 5
D	1, 6	6, 1	3, 4

We have, equivalently,

$$\begin{aligned}
(A\mathbf{y})_1 &= (A\mathbf{y})_3 \\
6 \cdot y_1 + 2 \cdot y_3 &= 1 \cdot y_1 + 3 \cdot y_3 \\
6y_1 + 2(1 - y_1) &= y_1 + 3(1 - y_1) \\
y_1 &= 1/6 \\
y_3 &= 5/6 \; .
\end{aligned}$$

Computing Nash equilibria

Mixed Nash equilibria

Example (continued)

	L	M	R
U	6, 1	1, 6	2, 5
D	1, 6	6, 1	3, 4

Also, for the column player:

$$
\begin{aligned}
(B^T\mathbf{x})_1 &= (B^T\mathbf{x})_3 \\
1 \cdot x_1 + 6 \cdot x_2 &= 5 \cdot x_1 + 4 \cdot x_2 \\
x_1 + 6(1 - x_1) &= 5x_1 + 4(1 - x_1) \\
x_1 &= 1/3 \\
x_2 &= 2/3 \ .
\end{aligned}
$$

Computing Nash equilibria

Mixed Nash equilibria

Example (continued)

	L	M	R
U	6, 1	1, 6	2, 5
D	1, 6	6, 1	3, 4

Then

$$(B^T\mathbf{x})_1 = (B^T\mathbf{x})_3 = 1 \cdot \frac{1}{3} + 6 \cdot \frac{2}{3} = \frac{13}{3}$$

and

$$(B^T\mathbf{x})_2 = 6 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = \frac{7}{3} < \frac{13}{3} = (B^T\mathbf{x})_1$$

so in this case the solution $(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium.

Computing Nash equilibria

Mixed Nash equilibria

The rock-scissors-paper game

	R	S	P
R	0, 0	1, -1	-1, 1
S	-1, 1	0, 0	1, -1
P	1, -1	-1, 1	0, 0

Let us consider full supports, i.e., $\{R, S, P\}$ for both players.
So let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T$.
$(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium iff all the following conditions hold:

$$\begin{aligned} (A\mathbf{y})_1 &= (A\mathbf{y})_2 = (A\mathbf{y})_3 \\ (B^T\mathbf{x})_1 &= (B^T\mathbf{x})_2 = (B^T\mathbf{x})_3 \\ x_1 + x_2 + x_3 &= 1 \\ y_1 + y_2 + y_3 &= 1 \\ x_1, x_2, x_3, y_1, y_2, y_3 &\geq 0 \ . \end{aligned}$$

Computing Nash equilibria

Mixed Nash equilibria

The rock-scissors-paper game

	R	S	P
R	0, 0	1, -1	-1, 1
S	-1, 1	0, 0	1, -1
P	1, -1	-1, 1	0, 0

For the row player we get the system of equations:

$$\begin{aligned} 0 \cdot y_1 + 1 \cdot y_2 + (-1) \cdot y_3 &= (-1) \cdot y_1 + 0 \cdot y_2 + (-1) \cdot y_3 \\ (-1) \cdot y_1 + 0 \cdot y_2 + (-1) \cdot y_3 &= 1 \cdot y_1 + (-1) \cdot y_2 + 0 \cdot y_3 \\ y_1 + y_2 + y_3 &= 1 \ , \end{aligned}$$

whose solution is $y_1 = y_2 = y_3 = 1/3$.
Similarly, we can show that $x_1 = x_2 = x_3 = 1/3$.
Note: It can be shown that this is the unique equilibrium of the game.

Existence of Nash equilibrium

Nash's Theorem

Every game with finite number of players and finite number of pure strategies for each player has at least one Nash equilibrium (involving pure or mixed strategies).

A general proof of Nash's theorem relies on the use of a fixed point theorem (e.g., Brouwer's or Kakutani's). Roughly:

- For some compact set $\mathbf{S}$ and a map $f : \mathbf{S} \rightarrow \mathbf{S}$ that satisfies various conditions, the map has a fixed point $p \in \mathbf{S}$, i.e., such that $f(p) = p$.
- The proof of Nash's theorem follows by showing that the best response map satisfies the necessary conditions for it to have a fixed point.

Existence of Nash equilibrium

2×2 bimatrix games

We will provide a self-contained proof of Nash's theorem for 2×2 bimatrix games. Consider a 2×2 bimatrix game with arbitrary payoffs:

	L	R
U	a, b	c, d
D	e, f	g, h

First we consider pure Nash equlibria:

1. If $a \geq e$ and $b \geq d$ then (U, L) is a Nash equilibrium.
2. If $e \geq a$ and $f \geq h$ then (D, L) is a Nash equilibrium.
3. If $c \geq g$ and $d \geq b$ then (U, R) is a Nash equilibrium.
4. If $g \geq c$ and $h \geq f$ then (D, R) is a Nash equilibrium.

Existence of Nash equilibrium

2×2 bimatrix games

	L	R
U	a, b	c, d
D	e, f	g, h

There is no pure Nash equilibrium if either

1. $a < e$ and $f < h$ and $g < c$ and $d < b$, or
2. $a > e$ and $f > h$ and $g > c$ and $d > b$.

In these cases we look for a mixed Nash equilibrium.

- Let $\mathbf{x} = \begin{bmatrix} p \\ 1-p \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} q \\ 1-q \end{bmatrix}$.
- $(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium if and only if

$$(A\mathbf{y})_1 = (A\mathbf{y})_2 \quad \text{and} \quad (B^T\mathbf{x})_1 = (B^T\mathbf{x})_2 \ .$$

Existence of Nash equilibrium

2×2 bimatrix games

	L	R
U	a, b	c, d
D	e, f	g, h

We have:

$$\begin{aligned} A\mathbf{y} &= \begin{bmatrix} a & c \\ e & g \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix} = \begin{bmatrix} aq + c(1-q) \\ eq + g(1-q) \end{bmatrix} \\ B^T\mathbf{x} &= \begin{bmatrix} b & f \\ d & h \end{bmatrix} \begin{bmatrix} p \\ 1-p \end{bmatrix} = \begin{bmatrix} bp + f(1-p) \\ dp + h(1-p) \end{bmatrix} \quad , \end{aligned}$$

and $(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium if and only if

$$aq + c(1-q) = eq + g(1-q) \quad \text{and} \quad bp + f(1-p) = dp + h(1-p) \ .$$

Existence of Nash equilibrium

2×2 bimatrix games

Equivalently:

$$q = \frac{c-g}{c-g+e-a}$$

and

$$p = \frac{h-f}{h-f+b-d}.$$

Recall the two cases where there is no pure Nash equilibrium:

1. $a < e$ and $f < h$ and $g < c$ and $d < b$, or
2. $a > e$ and $f > h$ and $g > c$ and $d > b$.

In both cases, $0 < p, q < 1$ as required for a mixed Nash equilibrium.

Existence of symmetric Nash equilibrium

We now will prove that every symmetric 2×2 bimatrix game has at least one symmetric Nash equilibrium, i.e., an equilibrium of the form $(\mathbf{x}, \mathbf{x})$.
Consider a 2×2 symmetric bimatrix game with arbitrary payoffs:

	S	T
S	a, a	b, c
T	c, b	d, d

First we consider pure Nash equilibria:

1. If $a \geq c$ then (S, S) is a symmetric Nash equilibrium.
2. If $d \geq b$ then (T, T) is a symmetric Nash equilibrium.
3. If $a < c$ and $d < b$ then there is no symmetric pure Nash equilibrium, so we will look for a mixed strategy Nash equilibrium.

Existence of symmetric Nash equilibrium

	S	T
S	a, a	b, c
T	c, b	d, d

- Let $\mathbf{x} = \begin{bmatrix} p \\ 1-p \end{bmatrix}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
- $(\mathbf{x}, \mathbf{x})$ is a symmetric Nash equilibrium if and only if

$$(A\mathbf{x})_1 = (A\mathbf{x})_2 \quad .$$

We have:

$$A\mathbf{x} \quad = \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p \\ 1-p \end{bmatrix} = \begin{bmatrix} ap + b(1-p) \\ cp + d(1-p) \end{bmatrix} \quad ,$$

Existence of symmetric Nash equilibrium

Hence $(\mathbf{x}, \mathbf{x})$ is a symmetric Nash equilibrium if and only if

$$\begin{aligned} ap + b(1-p) &= cp + d(1-p) \\ p &= \frac{b-d}{c-a+b-d} \end{aligned}$$

- Recall that, if there is no pure symmetric Nash equilibrium, then $a < c$ and $d < b$:
- So $0 < p < 1$ as required for a mixed Nash equilibrium.

1 Background: Basic concepts in matrix algebra

2 Strategies and payoffs

3 Equilibria

4 Approximate equilibria
- Definitions
- 3/4-approximate Nash equilibrium
- 1/2-approximate Nash equilibrium

The emergence of Nash equilibrium approximations

- (Chen and Deng; 2006) Computing a Nash equilibrium is PPAD-complete, even for bimatrix games.
- Hence, we seek for ϵ-approximate Nash equilibria, in which no player can improve her payoff by more than ϵ by deviating.
- (Chen, Deng and Teng; 2006) Computing a $\frac{1}{n^{\Theta(1)}}$-approximate Nash equilibrium is PPAD-complete.
- (Lipton, Markakis and Mehta; 2004) It is conjectured that it is unlikely that finding an ϵ-approximate Nash equilibrium is PPAD-complete when ϵ is an absolute constant.

Approximate equilibria

Recall: Given a bimatrix game $\Gamma = (A, B)$ and a strategy profile $(\mathbf{x}, \mathbf{y})$,

- Row player's regret is $\max_i (A\mathbf{y})_i - \mathbf{x}^T A\mathbf{y}$.
- Column player's regret is $\max_j (B^T \mathbf{x}_j) - \mathbf{x}^T B\mathbf{y}$.

Then,

$(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium if and only if both players have regret 0.

In an approximate Nash equilibrium, the above condition is relaxed:

$(\mathbf{x}, \mathbf{y})$ is an ϵ-approximate Nash equilibrium if and only if both players have regret at most ϵ.

Approximate equilibria

Definition

Equivalently:

Definition

$(\mathbf{x}, \mathbf{y})$ is an ϵ-approximate Nash equilibrium of the $m \times n$ bimatrix game $\Gamma = (A, B)$ if and only if

$$\begin{aligned} \mathbf{x}^T A\mathbf{y} &\geq (A\mathbf{y})_i - \epsilon \quad \forall i = 1, \ldots, m \quad \text{and} \\ \mathbf{x}^T B\mathbf{y} &\geq (B^T\mathbf{x})_j - \epsilon \quad \forall j = 1, \ldots, n \ . \end{aligned}$$

- Note: This is an additive approximation.
- We consider bimatrix games with positively normalized matrices: each element (payoff) is in the range $[0, 1]$.

Positively normalized games

We will show that every pair of equilibrium strategies of a bimatrix game does not change upon multiplying all the entries of a payoff matrix by a constant, and upon adding the same constant to each entry.

- Consider the $n \times m$ bimatrix game $\Gamma = (A, B)$ and let c, d be two arbitrary positive real constants.
- Suppose that $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is a Nash equilibrium for Γ
- Let $\mathbf{x}$ and $\mathbf{y}$ be any strategy of the row and column player respectively.
- Now consider the game $\Gamma' = (cA, dB)$. Then it holds that

$$\mathbf{x}^T(cA)\tilde{\mathbf{y}} = c\mathbf{x}^T A\tilde{\mathbf{y}} \leq c\tilde{\mathbf{x}}^T A\tilde{\mathbf{y}} = \tilde{\mathbf{x}}^T(cA)\tilde{\mathbf{y}}$$

  and, similarly,

$$\tilde{\mathbf{x}}^T(dB)\mathbf{y} \leq \tilde{\mathbf{x}}^T(dB)\tilde{\mathbf{y}} \ .$$

Positively normalized games

- Now suppose that $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is an ϵ-approximate Nash equilibrium for Γ.
- Then

$$\mathbf{x}^T(cA)\hat{\mathbf{y}} \leq \hat{\mathbf{x}}^T(cA)\hat{\mathbf{y}} + c\epsilon$$

  and

$$\hat{\mathbf{x}}^T(dB)\mathbf{y} \leq \hat{\mathbf{x}}^T(dB)\hat{\mathbf{y}} + d\epsilon \ .$$

- Hence Γ and Γ' have precisely the same set of Nash equilibria; furthermore, any ϵ-Nash equilibrium for Γ is a $\ell\epsilon$-Nash equilibrium for Γ' (where $\ell = \max\{c, d\}$) and vice versa.

Positively normalized games

- Now let C be an $n \times m$ matrix such that, for all columns j, $c_{ij} = c_j$ for all i.
- Similarly, let D be an $n \times m$ matrix such that, for all rows i, $d_{ij} = d_i$ for all j.
- Note that, for every pair of strategies $\mathbf{x}, \mathbf{y}$,

$$\mathbf{x}^T C\mathbf{y} = \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} x_i y_j = \sum_{j=1}^{n} y_j \sum_{i=1}^{m} c_j x_i = \sum_{j=1}^{n} c_j y_j$$

and

$$\mathbf{x}^T D\mathbf{y} = \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_i y_j = \sum_{i=1}^{m} x_i \sum_{j=1}^{n} d_i y_j = \sum_{i=1}^{m} d_i x_i \quad .$$

- Consider now the game $\Gamma'' = (C + A, D + B)$.

Positively normalized games

- Then, for all $\mathbf{x}$,

$$\mathbf{x}^T(C+A)\tilde{\mathbf{y}} = \mathbf{x}^T C\tilde{\mathbf{y}} + \mathbf{x}^T A\tilde{\mathbf{y}} \leq \sum_{j=1}^{n} c_j \tilde{y}_j + \tilde{\mathbf{x}}^T A\tilde{\mathbf{y}} = \tilde{\mathbf{x}}^T(C+A)\tilde{\mathbf{y}}$$

  and similarly, for all $\mathbf{y}$,

$$\tilde{\mathbf{x}}^T(D+B)\mathbf{y} \leq \tilde{\mathbf{x}}^T(D+B)\tilde{\mathbf{y}} \ .$$

- Also, for all $\mathbf{x}$ it holds that

$$\mathbf{x}^T(C+A)\hat{\mathbf{y}} = \mathbf{x}^T C\hat{\mathbf{y}} + \mathbf{x}^T A\hat{\mathbf{y}} \leq \sum_{j=1}^{n} c_j \hat{y}_j + \hat{\mathbf{x}}^T A\hat{\mathbf{y}} + \epsilon = \hat{\mathbf{x}}^T(C+A)\hat{\mathbf{y}} + \epsilon$$

  and similarly, for all $\mathbf{y}$,

$$\hat{\mathbf{x}}^T(D+B)\mathbf{y} \leq \hat{\mathbf{x}}^T(D+B)\hat{\mathbf{y}} + \epsilon \ .$$

- Thus Γ and Γ'' are equivalent as regards their sets of Nash equilibria, as well as their sets of ϵ-Nash equilibria.

How to find a 3/4-approximate Nash equilibrium

Kontogiannis, Panagopoulou, & Spirakis, 2006

Basic idea: Given an $m \times n$ bimatrix game $\Gamma = (A, B)$:

1. Take the maximum element a_{i_1,j_1} of the row player's payoff matrix A.
2. Take the maximum element b_{i_2,j_2} of the column player's payoff matrix B.
3. The row player plays rows i_1 and i_2 with probability $1/2$ each, and the column player plays columns j_1 and j_2 with probability $1/2$ each.
4. Then the resulting strategy profile $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$, for which

$$\begin{aligned} \hat{x}_{i_1} = \hat{x}_{i_2} &= \frac{1}{2} \\ \hat{x}_t &= 0 \quad \forall t \neq i_1, i_2 \\ \hat{y}_{j_1} = \hat{y}_{j_2} &= \frac{1}{2} \\ \hat{y}_t &= 0 \quad \forall t \neq j_1, j_2 \ , \end{aligned}$$

is a 3/4-approximate Nash equilibrium for Γ.

How to find a 3/4-approximate Nash equilibrium

Kontogiannis, Panagopoulou, & Spirakis, 2006

Illustration:

$1, 1/2$	$0, 1$	$0, 0$
$1, 0$	$0, 1/2$	$1, 1$
$0, 1$	$1, 0$	$0, 1$

- Consider the bimatrix game above.

How to find a 3/4-approximate Nash equilibrium

Kontogiannis, Panagopoulou, & Spirakis, 2006

Illustration:

$1, 1/2$	$0, 1$	$0, 0$
$1, 0$	$0, 1/2$	$1, 1$
$0, 1$	$1, 0$	$0, 1$

- Consider the bimatrix game above.
- Find an entry that maximizes the payoff of the row player.

How to find a 3/4-approximate Nash equilibrium

Kontogiannis, Panagopoulou, & Spirakis, 2006

Illustration:

$1, 1/2$	$0, 1$	$0, 0$
$1, 0$	$0, 1/2$	$1, 1$
$0, 1$	$1, 0$	$0, 1$

- Consider the bimatrix game above.
- Find an entry that maximizes the payoff of the row player.
- Find an entry that maximizes the payoff of the column player.

How to find a 3/4-approximate Nash equilibrium

Kontogiannis, Panagopoulou, & Spirakis, 2006

Illustration:

$1, 1/2$	$0, 1$	$0, 0$
$1, 0$	$0, 1/2$	$1, 1$
$0, 1$	$1, 0$	$0, 1$

- Consider the bimatrix game above.
- Find an entry that maximizes the payoff of the row player.
- Find an entry that maximizes the payoff of the column player.
- The row player chooses the highlighted rows with probability $1/2$ each.
- The column player chooses the highlighted columns with probability $1/2$ each.

How to find a 3/4-approximate Nash equilibrium

Kontogiannis, Panagopoulou, & Spirakis, 2006

Illustration (continued):

1, 1/2	0, 1	0, 0
1, 0	0, 1/2	1, 1
0, 1	1, 0	0, 1

- Using bimatrix games notation:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1/2 & 1 & 0 \\ 0 & 1/2 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

$$x = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}, \quad y = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}.$$

How to find a 3/4-approximate Nash equilibrium

Kontogiannis, Panagopoulou, & Spirakis, 2006

Illustration (continued): We have:

$$\begin{aligned}
A\mathbf{y} &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} \\
B^T\mathbf{x} &= \begin{bmatrix} 1/2 & 0 & 1 \\ 1 & 1/2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1/2 \\ 1/2 \end{bmatrix} \\
\mathbf{x}^T A\mathbf{y} &= \begin{bmatrix} 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{4} \\
\mathbf{x}^T B\mathbf{y} &= (B^T\mathbf{x})^T\mathbf{y} = \begin{bmatrix} 3/4 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = \frac{5}{8} .
\end{aligned}$$

How to find a 3/4-approximate Nash equilibrium

Kontogiannis, Panagopoulou, & Spirakis, 2006

Illustration (continued): Therefore

$$\max_i (A\mathbf{y})_i - \mathbf{x}^T A\mathbf{y} \;=\; 1 - \frac{1}{4} = \frac{3}{4}$$

and

$$\max_j (B^T\mathbf{x})_j - \mathbf{x}^T B\mathbf{y} \;=\; \frac{3}{4} - \frac{5}{8} = \frac{1}{8} \;.$$

So $(\mathbf{x}, \mathbf{y})$ is a 3/4-approximate Nash equilibrium.

How to find a 3/4-approximate Nash equilibrium

Kontogiannis, Panagopoulou, & Spirakis, 2006

Formally:

Lemma

Consider an $m \times n$ bimatrix game $\Gamma = (A, B)$ and let

$$\begin{aligned} a_{i_1,j_1} &= \max_{i,j} a_{i,j} \\ b_{i_2,j_2} &= \max_{i,j} b_{i,j} \quad . \end{aligned}$$

Then the pair of strategies $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ where

$$\hat{x}_{i_1} = \hat{x}_{i_2} = \hat{y}_{j_1} = \hat{y}_{j_2} = \frac{1}{2}$$

is a $\frac{3}{4}$-Nash equilibrium for Γ.

How to find a 3/4-approximate Nash equilibrium

Kontogiannis, Panagopoulou, & Spirakis, 2006

Proof: First observe that

$$
\begin{aligned}
\hat{\mathbf{x}}^T A \hat{\mathbf{y}} &= \sum_{i=1}^{m} \sum_{j=1}^{n} \hat{x}_i \hat{y}_j a_{ij} \\
&= \hat{x}_{i_1} \hat{y}_{j_1} a_{i_1,j_1} + \hat{x}_{i_1} \hat{y}_{j_2} a_{i_1,j_2} + \hat{x}_{i_2} \hat{y}_{j_1} a_{i_2,j_1} + \hat{x}_{j_1} \hat{y}_{j_1} a_{i_2,j_2} \\
&= \frac{1}{4} \left(a_{i_1,j_1} + a_{i_1,j_2} + a_{i_2,j_1} + a_{i_2,j_2} \right) \geq \frac{1}{4} a_{i_1,j_1} \quad , \\
\hat{\mathbf{x}}^T B \hat{\mathbf{y}} &= \sum_{i=1}^{m} \sum_{j=1}^{n} \hat{x}_i \hat{y}_j b_{ij} \\
&= \hat{x}_{i_1} \hat{y}_{j_1} b_{i_1,j_1} + \hat{x}_{i_1} \hat{y}_{j_2} b_{i_1,j_2} + \hat{x}_{i_2} \hat{y}_{j_1} b_{i_2,j_1} + \hat{x}_{j_1} \hat{y}_{j_1} b_{i_2,j_2} \\
&= \frac{1}{4} \left(b_{i_1,j_1} + b_{i_1,j_2} + b_{i_2,j_1} + b_{i_2,j_2} \right) \geq \frac{1}{4} b_{i_2,j_2} \quad .
\end{aligned}
$$

How to find a 3/4-approximate Nash equilibrium

Kontogiannis, Panagopoulou, & Spirakis, 2006

Proof (continued): Now observe that, for any (mixed) strategies $\mathbf{x}$ and $\mathbf{y}$ of the row and column player respectively,

$$\mathbf{x}^T A\hat{\mathbf{y}} \leq a_{i_1,j_1} \quad \text{and} \quad \hat{\mathbf{x}}^T B\mathbf{y} \leq b_{i_2,j_2}$$

and recall that $a_{ij}, b_{ij} \in [0,1]$ for all i,j. Hence

$$\mathbf{x}^T A\hat{\mathbf{y}} \leq a_{i_1,j_1} = \frac{1}{4}a_{i_1,j_1} + \frac{3}{4}a_{i_1,j_1} \leq \hat{\mathbf{x}}^T A\hat{\mathbf{y}} + \frac{3}{4}$$

and

$$\hat{\mathbf{x}}^T B\mathbf{y} \leq b_{i_2,j_2} = \frac{1}{4}b_{i_2,j_2} + \frac{3}{4}b_{i_2,j_2} \leq \hat{\mathbf{x}}^T B\hat{\mathbf{y}} + \frac{3}{4} \ .$$

Thus $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is a $\frac{3}{4}$-Nash equilibrium for Γ.

How to find an 1/2-approximate Nash equilibrium

Daskalakis, Mehta, & Papadimitriou, 2006

Basic idea: Given an $m \times n$ bimatrix game $\Gamma = (A, B)$:

1. Choose an arbitrary pure strategy for the row player (say row i).
2. Take a best-response pure strategy to i for the column player (say column j).
3. Take a best-response pure strategy to j for the row player (say row k).
4. The row player plays rows i and k with probability 1/2 each, and the column player plays column j with probability 1.
5. Then the resulting strategy profile $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$, for which

$$\begin{aligned} \hat{x}_i = \hat{x}_k &= \frac{1}{2} \\ \hat{x}_t &= 0 \quad \forall t \neq i, k \\ \hat{y}_j &= 1 \\ \hat{y}_t &= 0 \quad \forall t \neq j \ , \end{aligned}$$

is an 1/2-approximate Nash equilibrium for Γ.

How to find an 1/2-approximate Nash equilibrium

Daskalakis, Mehta, & Papadimitriou, 2006

Illustration:

$1/2, 1/2$	$0, 1$	$1, 0$
$1, 0$	$1/2, 1/2$	$0, 1$
$0, 1$	$1, 0$	$1/2, 1/2$

- Consider the bimatrix game above.

How to find an 1/2-approximate Nash equilibrium

Daskalakis, Mehta, & Papadimitriou, 2006

Illustration:

$1/2, 1/2$	$0, 1$	$1, 0$
$1, 0$	$1/2, 1/2$	$0, 1$
$0, 1$	$1, 0$	$1/2, 1/2$

- Consider the bimatrix game above.
- Choose an arbitrary row.

How to find an 1/2-approximate Nash equilibrium

Daskalakis, Mehta, & Papadimitriou, 2006

Illustration:

$1/2, 1/2$	$0, 1$	$1, 0$
$1, 0$	$1/2, 1/2$	$0, 1$
$0, 1$	$1, 0$	$1/2, 1/2$

- Consider the bimatrix game above.
- Choose an arbitrary row.
- Take a best response for the column player.

How to find an 1/2-approximate Nash equilibrium

Daskalakis, Mehta, & Papadimitriou, 2006

Illustration:

$1/2, 1/2$	$0, 1$	$1, 0$
$1, 0$	$1/2, 1/2$	$0, 1$
$0, 1$	$1, 0$	$1/2, 1/2$

- Consider the bimatrix game above.
- Choose an arbitrary row.
- Take a best response for the column player.
- Take a best response for the row player.

How to find an 1/2-approximate Nash equilibrium

Daskalakis, Mehta, & Papadimitriou, 2006

Illustration:

$1/2, 1/2$	$0, 1$	$1, 0$
$1, 0$	$1/2, 1/2$	$0, 1$
$0, 1$	$1, 0$	$1/2, 1/2$

- Consider the bimatrix game above.
- Choose an arbitrary row.
- Take a best response for the column player.
- Take a best response for the row player.
- The row player chooses the highlighted rows with probability $1/2$ each.
- The column player chooses the highlighted column with probability 1.

How to find an 1/2-approximate Nash equilibrium

Daskalakis, Mehta, & Papadimitriou, 2006

Illustration (continued):

$1/2, 1/2$	$0, 1$	$1, 0$
$1, 0$	$1/2, 1/2$	$0, 1$
$0, 1$	$1, 0$	$1/2, 1/2$

- Using bimatrix games notation:

$$A = \begin{bmatrix} 1/2 & 0 & 1 \\ 1 & 1/2 & 0 \\ 0 & 1 & 1/2 \end{bmatrix}, \quad B = \begin{bmatrix} 1/2 & 0 & 1 \\ 1 & 1/2 & 0 \\ 0 & 1 & 1/2 \end{bmatrix},$$

$$x = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}, \quad y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

How to find an 1/2-approximate Nash equilibrium

Daskalakis, Mehta, & Papadimitriou, 2006

Illustration (continued): We have:

$$
\begin{aligned}
A\mathbf{y} &= \begin{bmatrix} 1/2 & 0 & 1 \\ 1 & 1/2 & 0 \\ 0 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix} \\
B^T\mathbf{x} &= \begin{bmatrix} 1/2 & 0 & 1 \\ 1 & 1/2 & 0 \\ 0 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1/2 \\ 1/2 \end{bmatrix} \\
\mathbf{x}^T A\mathbf{y} &= \begin{bmatrix} 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix} = 0 \\
\mathbf{x}^T B\mathbf{y} &= (B^T\mathbf{x})^T\mathbf{y} = \begin{bmatrix} 3/4 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}
\end{aligned}
$$

How to find an 1/2-approximate Nash equilibrium

Daskalakis, Mehta, & Papadimitriou, 2006

Illustration (continued): Therefore

$$\max_i (A\mathbf{y})_i - \mathbf{x}^T A\mathbf{y} \quad = \quad \frac{1}{2} - 0 = \frac{1}{2}$$

and

$$\max_j (B^T\mathbf{x})_j - \mathbf{x}^T B\mathbf{y} \quad = \quad \frac{3}{4} - \frac{1}{2} = \frac{1}{4} \ .$$

So $(\mathbf{x}, \mathbf{y})$ is an 1/2-approximate Nash equilibrium.

How to find an 1/2-approximate Nash equilibrium

Daskalakis, Mehta, & Papadimitriou, 2006

Formal proof:

- Recall: i is an arbitrary row, j is a best-response column to j, and k is a best-response row to j, and $\hat{x}_i = \hat{x}_k = 1/2$ and $\hat{y}_j = 1$.
- The row player's payoff under $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is

$$\hat{\mathbf{x}}^T A \hat{\mathbf{y}} \quad = \quad \sum_{t=1}^{m} \sum_{r=1}^{n} \hat{x}_t \hat{y}_r a_{rt} = \frac{1}{2} a_{ij} + \frac{1}{2} a_{kj} \quad .$$

- By construction, one of her best responses to $\hat{y}$ is to play the pure strategy on row k, which gives a payoff of a_{kj}.
- Hence her regret (incentive to defect) is equal to the difference:

$$a_{kj} - \left(\frac{1}{2} a_{ij} + \frac{1}{2} a_{kj} \right) \quad = \quad \frac{1}{2} a_{kj} - \frac{1}{2} a_{ij} \leq \frac{1}{2} a_{kj} \leq \frac{1}{2} \quad .$$

How to find an 1/2-approximate Nash equilibrium

Daskalakis, Mehta, & Papadimitriou, 2006

Proof (continued):

- The column player's payoff under $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is

$$\hat{\mathbf{x}}^T B \hat{\mathbf{y}} \;=\; \sum_{t=1}^{m} \sum_{r=1}^{n} \hat{x}_t \hat{y}_r b_{rt} = \frac{1}{2} b_{ij} + \frac{1}{2} b_{kj} \;.$$

- Let j' be a best-response pure strategy (column) to $\hat{\mathbf{x}}$, giving her a payoff of $\frac{1}{2} b_{ij'} + \frac{1}{2} b_{kj'}$.
- Hence the regret of the column player is equal to the difference:

$$\begin{aligned}\left(\frac{1}{2} b_{ij'} + \frac{1}{2} b_{kj'}\right) - \left(\frac{1}{2} b_{ij} + \frac{1}{2} b_{kj}\right) \;&=\; \frac{1}{2}\left(b_{ij'} - b_{ij}\right) + \frac{1}{2}\left(b_{kj'} - b_{kj}\right) \\ &\leq\; 0 + \frac{1}{2}\left(b_{kj'} - b_{kj}\right) \leq \frac{1}{2} \;.\end{aligned}$$

(The first inequality follows from the fact that column j was a best response to row i, by the first step of the construction.)

Some other results on approximate Nash equilibria

- (Chen, Deng and Teng, 2006) Computing a $\frac{1}{n^{\Theta(1)}}$-Nash equilibrium is PPAD-complete.
- (Lipton, Markakis and Mehta, 2004) For any constant $\epsilon > 0$, there exists an ϵ-Nash equilibrium that can be computed in quasi-polynomial ($n^{O(\ln n)}$) time.
- It is conjectured that it is unlikely that finding an ϵ-Nash equilibrium is PPAD-complete when ϵ is an absolute constant.
- The best known polynomial-time constant approximation achieves $\epsilon = 0.3393$ (Tsaknakis and Spirakis, 2008).

Further reading

- J. N. Webb: Game Theory: Desicions, Interaction and Evolution. Springer, 2007.
- M. J. Osborne: An Introduction to Game Theory. Oxford University Press, 2004.
- R. B. Myerson: Game Theory: Analysis of Conflict. Harvard University Press, 1991.
- S. C. Kontogiannis, P. N. Panagopoulou, P. G. Spirakis: Polynomial algorithms for approximating Nash equilibria of bimatrix games. WINE 2006, pp. 286–296.
- C. Daskalakis, A. Mehta, C. H. Papadimitriou: A note on approximate Nash equilibria. WINE 2006, pp. 297–306.
- R. J. Lipton, E. Markakis, A. Mehta: Playing large games using simple strategies. EC 2003, pp. 36–41.
- H. Tsaknakis, P. G. Spirakis: An optimization approach for approximate Nash equilibria. WINE 2007, pp: 42–56.