*COMP323 – Introduction to Computational Game Theory*

# Introduction to Game Theory

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# Outline

# Teaching and Learning Strategies

Lectures (3h per week)

- 10 weeks in total
- slide material uploaded to VITAL
- normally stream-captured (modulo technical problems)

Instructors

- Giorgos Christodoulou: gchristo@liv.ac.uk
- Themistoklis Melissourgos: t.melissourgos@liverpool.ac.uk

Members of the Economics and Computation Group

# Teaching and Learning Strategies

## Tutorials (1h per week)

- Week 2 to Week 11
- Check in Liverpool Life which session you should attend.
- Exercises
  - each week a set of exercises will be uploaded on VITAL
  - you are expected to work on the exercises before the tutorial and come with questions
  - the solutions of **most** exercises will be provided during the tutorial
  - the solutions of the class tests will be discussed during a session (but we are not providing electronic or printed solutions)

## Demonstrators

- Nikos Protopapas: N.Protopapas@liverpool.ac.uk
- Emmanouil Bakopoulos: E.Bakopoulos@liverpool.ac.uk
- George Skretas: G.Skretas@liverpool.ac.uk

# Assessment

## Assessment

- 80 % - Final Exam
- 20 % - Two Class tests (you will be notified in VITAL roughly two weeks ahead)

# Reading Material

- Slides
- Tutorial Material
- Recommended Textbooks (check VITAL)
  - for each topic we cover (see last slide) we will recommend a list of reading resources for further reading

# Objective of game theory

Game theory...

- ...aims to help us understand situations in which decision-makers interact
- ...is the study of mathematical models of conflict and cooperation between rational intelligent decision-makers.

Some applications of game theory:

- firms competing for business;
- political candidates competing for votes;
- animals fighting over prey;
- bidders competing in an auction;
- the role of threats and punishment in long-term relationships.

# Games and solutions

- A game models a situation where two or more individuals (players) have to take some decisions that will influence one another's welfare. I.e., the payoff of each player depends not only on her own decision, but also on the decisions of (a subset of) the other players.
- A game is a description of strategic interaction that includes the constraints on the actions that the players *can* take and the players' interests, but does not specify the actions that the players *do* take.
- A solution is a systematic description of the outcomes that may emerge in a family of games.

# Game theoretic models

There are four basic groups of game theoretic models:

1. strategic games;
2. extensive games with perfect information;
3. extensive games without perfect information;
4. coalitional games.

# Noncooperative and cooperative games

- In strategic and extensive games (with or without perfect information), the sets of possible actions of *individual* players are primitives (noncooperative games).
- In coalitional games, the sets of possible joint actions of *groups* of players are primitives (cooperative games).

# Strategic games and extensive games

- A strategic game is a model of a situation in which each player chooses her plan of action once and for all, and players' decisions are made simultaneously.
  When choosing a plan of action, each player is not informed of the plan of action chosen by any other players.
- An extensive game is a model which specifies the possible orders of events.
  Each player can consider her plan of action not only at the beginning of the game but also whenever she has to make a decision.

# Games with perfect and imperfect information

- In a game with perfect information the players are fully informed about each others' moves.
- In a game with imperfect information the players may be imperfectly informed about each others' moves.

# Assumptions underlying game theory

In game theory, a player (decision-maker) is assumed to be

1. Rational: she makes decisions consistently in pursuit of her own, well-defined objectives.
2. Intelligent: she knows everything about the game, can make any inferences about the situation, and takes into account this knowledge of other decision-makers' behavior (she *reasons strategically*).

# Strategic games

A strategic game (or a game in normal form) is defined by

- a set of players
- for each player, a set of actions
- for each player, preferences over the set of action profiles

  (an action profile is a combination of actions, one for each player)

# Strategic games

A strategic game (or a game in normal form) is defined by

- a set of players
- for each player, a set of actions
- for each player, preferences over the set of action profiles

  (an action profile is a combination of actions, one for each player)

Time is absent from the model of strategic games:

- each player chooses her action once and for all, and
- the players choose their actions "simultaneously", in the sense that no player is informed of the action chosen by any other player when she chooses her action.

# Formulation

A strategic game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is defined by

1. the set $N$ of players
2. the set $S_i$ of actions for each player $i$
3. the payoff function $u_i : \times_{i \in N} S_i \to \mathbb{R}$ for each player $i$, mapping each action profile into a real number

# Example: The Prisoner's Dilemma

## Setting

Two suspects in a major crime are held in separate cells. There is enough evidence to convict each of them of a minor offense, but not enough evidence to convict either of them of the major crime, unless one of them acts as an informer against the other (finks).

- If they both stay quiet, each will be convicted of a minor offense and spend one year in prison.
- If one and only one of them finks, she will be freed and used as a witness against the other, who will spend four years in prison.
- If they both fink, each will spend three years in prison.

# Example: The Prisoner's Dilemma

## Formulation as a strategic game

The situation can be modeled as a strategic game:

- The players are the two suspects: $N = \{1, 2\}$.
- The actions available to each player are to stay quiet or to fink: $S_1 = S_2 = \{Quiet, Fink\}$.

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In order to define the payoff functions of the players, we have to find an ordering of the action profiles for each player.
There are four action profiles:
$(Quiet, Quiet), (Quiet, Fink), (Fink, Quiet), (Fink, Fink)$.

# Example: The Prisoner's Dilemma

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In order to define the payoff functions of the players, we have to find an ordering of the action profiles for each player.
There are four action profiles:
$(Quiet, Quiet), (Quiet, Fink), (Fink, Quiet), (Fink, Fink)$.

- For player 1, $(Fink, Quiet)$ is better than $(Quiet, Quiet)$, which is better than $(Fink, Fink)$, which is better than $(Quiet, Fink)$.
- For player 2, $(Quiet, Fink)$ is better than $(Quiet, Quiet)$, which is better than $(Fink, Fink)$, which is better than $(Fink, Quiet)$.

# Example: The Prisoner's Dilemma

## Formulation as a strategic game

A simple specification is

- $u_1(Fink, Quiet) = 3,\ u_1(Quiet, Quiet) = 2,\ u_1(Fink, Fink) = 1,$
  $u_1(Quiet, Fink) = 0$
- $u_2(Quiet, Fink) = 3,\ u_2(Quiet, Quiet) = 2,\ u_2(Fink, Fink) = 1,$
  $u_2(Fink, Quiet) = 0$

# Example: The Prisoner's Dilemma

## Formulation as a strategic game

A simple specification is

- $u_1(Fink, Quiet) = 3$, $u_1(Quiet, Quiet) = 2$, $u_1(Fink, Fink) = 1$, $u_1(Quiet, Fink) = 0$
- $u_2(Quiet, Fink) = 3$, $u_2(Quiet, Quiet) = 2$, $u_2(Fink, Fink) = 1$, $u_2(Fink, Quiet) = 0$

We can represent the game compactly in a table:

| | | Suspect 2 | |
|---|---|---|---|
| | | *Quiet* | *Fink* |
| Suspect 1 | *Quiet* | (2,2) | (0,3) |
| | *Fink* | (3,0) | (1,1) |

# Example: The Prisoner's Dilemma

## Formulation as a strategic game

| | | Suspect 2 | |
|---|---|---|---|
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| | *Fink* | (3,0) | (1,1) |

The Prisoner's Dilemma models a situation in which

- there are gains from cooperation (each player prefers that both players choose *Quiet* than they both choose *Fink*),
- but each player has an incentive to "free ride" (choose *Fink*) whatever the other player does.

# Example: Games equivalent to the Prisoner's Dilemma

Consider the following two games:

| | X | Y |
|---|---|---|
| X | 3, 3 | 1, 5 |
| Y | 5, 1 | 0, 0 |

| | X | Y |
|---|---|---|
| X | 2, 1 | 0, 5 |
| Y | 3, -2 | 1, -1 |

Does each of the games differ from the Prisoner's Dilemma only in the names of the players' actions, or does it differ also in one or both of the players' preferences?

- The game on the left differs from the Prisoner's Dilemma in both players' preferences. Player 1 prefers (Y, X) to (X, X) to (X, Y) to (Y, Y), for example, which differs from her preference in the Prisoner's Dilemma, whether we let X = *Fink* or X = *Quiet*.
- The game on the right is equivalent to the Prisoner's Dilemma, by letting X = *Quiet* and Y = *Fink*.

# Example: Matching Pennies

## Setting

Two people choose, simultaneously, whether to show the head or the tail of a coin.

- If they show the same side, person 2 pays person 1 $1.
- If they show different sides, person 1 pays person 2 $1.

The game is strictly competitive:

- In each action profile, each player wins as much as the other player loses.
- The players' interests are diametrically opposed: player 1 wants to take the same action as the other player, whereas player 2 wants to take the opposite action.

# Example: Matching Pennies

## Formulation as a strategic game

A strategic game that models this situation, in which the payoffs are equal to the amounts of money involved:

| | | Person 2 | |
|---|---|---|---|
| | | *Head* | *Tail* |
| Person 1 | *Head* | (1,-1) | (-1,1) |
| | *Tail* | (-1,1) | (1,-1) |

# Example: Bach or Stravinsky?

## Setting

- Two people wish to go out together.
- Two concerts are available: one of music by Bach, and one of music by Stravinsky.
- One person prefers Bach and one person prefers Stravinsky.
- If they go to different concerts, each of them is equally unhappy listening to the music of either composer.

This game is also referred to as the *Battle of Sexes*.

# Example: Bach or Stravinsky?

## Formulation as a strategic game

A strategic game that models this situation:

| | | Person 2 | |
|---|---|---|---|
| | | *Bach* | *Stravinsky* |
| Person 1 | *Bach* | (2,1) | (0,0) |
| | *Stravinsky* | (0,0) | (1,2) |

# Symmetric games

An $n$-person strategic game is symmetric if

- each player has the same set of actions and
- each player's payoff depends only on her action and that of her opponents, not on whether she is player 1, 2, ..., or $n$.

Formally:

**Definition**

A symmetric strategic game is a game $\Gamma = \langle N, (S)_{i \in N}, (u_i)_{i \in N} \rangle$ such that, for all actions $a \in S$ and for all action profiles of $n-1$ players $\mathbf{s} \in S^{n-1}$,

$$u_i(a, \mathbf{s}) = u_j(a, \mathbf{s}) \quad \forall i, j \in N \ .$$

Examples:

- *Prisoner's Dilemma* is a 2-person symmetric game.
- *Matching Pennies* and *Bach or Stravinsky* are not symmetric.

# Solutions of strategic games

What actions will be chosen by the players in a strategic game?

- We wish to assume that each player chooses the best available action.
- However, the best action for any given player depends, in general, on the other players' actions.
- So, when choosing an action, a player must have in mind the actions the other players will choose.

# Solutions of strategic games

What actions will be chosen by the players in a strategic game?

- We wish to assume that each player chooses the best available action.
- However, the best action for any given player depends, in general, on the other players' actions.
- So, when choosing an action, a player must have in mind the actions the other players will choose.

Based on the above, the main solution concept for a strategic game is the Nash equilibrium:

- Each player chooses her best available action, given the actions chosen by all the other players.

# Pure Nash equilibrium

## Definition

A Nash equilibrium is a combination of actions, one for each player, such that no player can increase her payoff by unilaterally changing her action:

> A pure Nash equilibrium is an action profile **s** with the property that no player $i$ can do better by choosing an action different from $s_i$, given that every other player $j$ adheres to $s_j$.

A Nash equilibrium corresponds to a steady state: if every one else adheres to it, no individual wishes to deviate from it.

# Pure Nash equilibrium

## Definition

Formally, let

- $\mathbf{s} = (s_i)_{i \in N}$ be an action profile
- $(s'_i, \mathbf{s}_{-i})$ be the action profile that results from $\mathbf{s}$ when player $i \in N$ switches to her action $s'_i \in S_i$, while the rest of the players preserve their actions.

# Pure Nash equilibrium

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- $\mathbf{s} = (s_i)_{i \in N}$ be an action profile
- $(s'_i, \mathbf{s}_{-i})$ be the action profile that results from $\mathbf{s}$ when player $i \in N$ switches to her action $s'_i \in S_i$, while the rest of the players preserve their actions.

Then:

**Definition**

A pure Nash equilibrium of a strategic game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is an action profile $\mathbf{s} = (s_i)_{i \in N}$ such that, for all players $i \in N$,

$$u_i(\mathbf{s}) \geq u_i(s'_i, \mathbf{s}_{-i}) \quad \text{for all } s'_i \in S_i \ .$$

# Pure Nash equilibrium

## Existence and uniqueness

Note: The definition implies neither that a strategic game necessarily has a pure Nash equilibrium, nor that it has at most one.
Examples in the following show that

- some games have a single pure Nash equilibrium,
- some possess no pure Nash equilibrium, and
- others have many pure Nash equilibria.

# Pure Nash equilibrium

## Example 1: The Prisoner's Dilemma

| | | Suspect 2 | |
|---|---|---|---|
| | | *Quiet* | *Fink* |
| Suspect 1 | *Quiet* | (2,2) | (0,3) |
| | *Fink* | (3,0) | (1,1) |

The action pair (*Fink*, *Fink*) is a pure Nash equilibrium because

1. given that player 2 chooses *Fink*, player 1 is better off choosing *Fink* than *Quiet* (looking at the right column of the table we see that *Fink* yields player 1 a payoff of 1 whereas *Quiet* yields her a payoff of 0), and
2. given that player 1 chooses *Fink*, player 2 is better off choosing *Fink* than *Quiet* (looking at the bottom row of the table we see that *Fink* yields player 2 a payoff of 1 whereas *Quiet* yields her a payoff of 0).

# Pure Nash equilibrium

## Example 1: The Prisoner's Dilemma

| | | Suspect 2 | |
|---|---|---|---|
| | | *Quiet* | *Fink* |
| Suspect 1 | *Quiet* | (2,2) | (0,3) |
| | *Fink* | (3,0) | (1,1) |

No other action profile is a Nash equilibrium:

- (*Quiet*, *Quiet*) is not an equilibrium because when player 2 chooses *Quiet*, player 1's payoff to *Fink* exceeds her payoff to *Quiet*.
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- (*Quiet*, *Fink*) is not an equilibrium, because when player 2 chooses *Fink*, player 1's payoff to *Fink* exceeds her payoff to *Quiet*.

# Pure Nash equilibrium

## Example 1: The Prisoner's Dilemma

| | | Suspect 2 | |
|---|---|---|---|
| | | *Quiet* | *Fink* |
| Suspect 1 | *Quiet* | (2,2) | (0,3) |
| | *Fink* | (3,0) | (1,1) |

- In summary, (*Fink*, *Fink*) is the only pure Nash equilibrium of the game.
- Actually, action *Fink* is the best action for each player not only if the other player chooses her equilibrium action (*Fink*), but also if she chooses her other action (*Quiet*).
- In most games however, a player's Nash equilibrium action does not satisfy this condition: the action is optimal if the other players choose their equilibrium actions, but some other action is optimal if the other players choose nonequilibrium actions.

# Pure Nash equilibrium

## Example 2: Matching Pennies

Pure Nash equilibria do not always exist:

| | | Person 2 | |
|---|---|---|---|
| | | *Head* | *Tail* |
| Person 1 | *Head* | (1,-1) | (-1,1) |
| | *Tail* | (-1,1) | (1,-1) |

# Pure Nash equilibrium

## Example 2: Matching Pennies

Pure Nash equilibria do not always exist:

| | | Person 2 | |
|---|---|---|---|
| | | *Head* | *Tail* |
| Person 1 | *Head* | (1,-1) | (-1,1) |
| | *Tail* | (-1,1) | (1,-1) |

There is no pure Nash equilibrium.

# Pure Nash equilibrium

## Example 3: Bach or Stravinsky?

Multiple pure Nash equilibria may exist:

| | | Person 2 | |
|---|---|---|---|
| | | *Bach* | *Stravinsky* |
| Person 1 | *Bach* | (2,1) | (0,0) |
| | *Stravinsky* | (0,0) | (1,2) |

# Pure Nash equilibrium

## Example 3: Bach or Stravinsky?

Multiple pure Nash equilibria may exist:

| | | Person 2 | |
|---|---|---|---|
| | | *Bach* | *Stravinsky* |
| Person 1 | *Bach* | (2,1) | (0,0) |
| | *Stravinsky* | (0,0) | (1,2) |

There are two pure Nash equilibria: (*Bach*, *Bach*) and (*Stravinsky*, *Stravinsky*).

# Pure Nash equilibrium

## Example 4: Guessing two-thirds of the average

- Each of three people announces an integer from 1 to $K$.
- If the integers are different, the person whose integer is closest to $\frac{2}{3}$ of the average of the three integers wins \$1.
- If two or more integers are the same, \$1 is split equally between those whose integer is closest to $\frac{2}{3}$ of the average integer.

# Pure Nash equilibrium

## Example 4: Guessing two-thirds of the average

- Each of three people announces an integer from 1 to $K$.
- If the integers are different, the person whose integer is closest to $\frac{2}{3}$ of the average of the three integers wins \$1.
- If two or more integers are the same, \$1 is split equally between those whose integer is closest to $\frac{2}{3}$ of the average integer.

Question 1: Is there any integer $k$ such that the action profile $(k, k, k)$, in which every person announces the same integer $k$, is a pure Nash equilibrium?

Question 2: Is any other profile a pure Nash equilibrium?

# Pure Nash equilibrium

## Example 4: Guessing two-thirds of the average

Question 1: Is there any integer $k$ such that the action profile $(k, k, k)$, in which every person announces the same integer $k$, is a pure Nash equilibrium?

- If all three players announce the same integer $k \geq 2$ then any one of them can deviate to $k - 1$ and obtain \$1 (since her number is now closer to $\frac{2}{3}$ of the average than the other two) rather than $\$\frac{1}{3}$. Thus no such action profile is a Nash equilibrium.
- If all three players announce 1, then no player can deviate and increase her payoff; thus $(1, 1, 1)$ is a Nash equilibrium.

# Pure Nash equilibrium

## Example 4: Guessing two-thirds of the average

Question 2: Is any other profile a pure Nash equilibrium?

Consider an action profile in which not all three integers are the same; denote the highest by $k^*$.

- Suppose only one player names $k^*$; denote the other integers named by $k_1$ and $k_2$, with $k_1 \geq k_2$.
- The $\frac{2}{3}$ of the average of the three integers is $\frac{2}{9}(k^* + k_1 + k_2)$.
- $k^*$ is further from $\frac{2}{3}$ of the average than is $k_1$ (some simple calculations are needed to see this: consider separately the cases where $k_1 \geq \frac{2}{9}(k^* + k_1 + k_2)$ and $k_1 < \frac{2}{9}(k^* + k_1 + k_2)$).
- Hence the player who names $k^*$ does not win, and is better off naming $k_2$, in which case she obtains a share of the prize.
- Thus no such action profile is a Nash equilibrium.

# Pure Nash equilibrium

## Example 4: Guessing two-thirds of the average

Question 2: Is any other profile a pure Nash equilibrium?

Consider an action profile in which not all three integers are the same; denote the highest by $k^*$.

- Now suppose two player name $k^*$, and the third player names $k < k^*$.
- The $\frac{2}{3}$ of the average of the three integers is $\frac{4}{9}k^* + \frac{2}{9}k$. We have $\frac{4}{9}k^* + \frac{2}{9}k < \frac{1}{2}(k^* + k)$, hence $k$ is closer to the $\frac{2}{3}$ of the average than is $k^*$.
- So the player who names $k$ is the sole winner.
- Then, either of the other players can switch to $k$ and obtain a share of the prize.
- Thus no such action profile is a Nash equilibrium.

We conclude that $(1, 1, 1)$ is the only pure Nash equilibrium of this game.

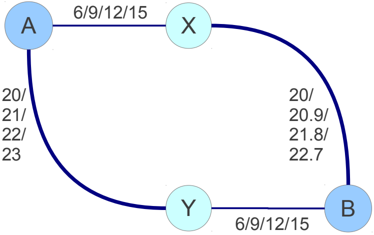
# Pure Nash equilibrium

## Example 5: Choosing a route

- Four people must drive from A to B at the same time. Each of them must choose a route.
- Two routes are available, one via X and one via Y.
- The roads from A to X, and from Y to B, are both short and narrow; in each case, one car takes 6 minutes, and each additional car increases the travel time *per car* by 3 minutes.
- The roads from A to Y, and from X to B, are long and wide; on A to Y one car takes 20 minutes, and each additional car increases the travel time *per car* by 1 minute; on X to B one car takes 20 minutes, and each additional car increases the travel time *per car* by 0.9 minutes.

# Pure Nash equilibrium

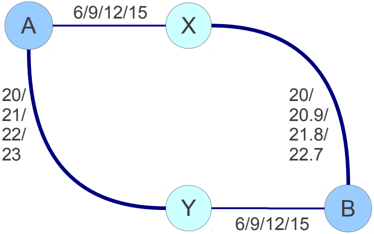
## Example 5: Choosing a route




- Getting from A to B: the numbers beside each road are the travel times per car when 1, 2, 3, or 4 cars take that road.
- For example, if two cars drive from A to X, then each car takes 9 minutes.

# Pure Nash equilibrium

## Example 5: Choosing a route




Formulation as a strategic game:

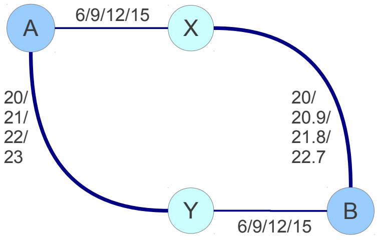
**Players:** The four people.

**Actions:** The set of actions of each person is $\{X, Y\}$ (the route via X and the route via Y).

**Payoffs:** Each player's payoff is the negative of her travel time.

# Pure Nash equilibrium

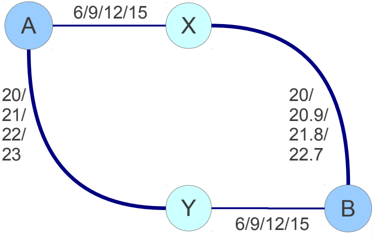
## Example 5: Choosing a route




- Assume two people take each route. For any such action profile, each person's travel time is either 29.9 or 30 minutes (depending on the route she takes).
- If a person taking the route via X switches to the route via Y her travel time becomes $22 + 12 = 34$ minutes; if a person taking the route via Y switches to the route via X her travel time becomes $12 + 21.8 = 33.8$ minutes.

# Pure Nash equilibrium

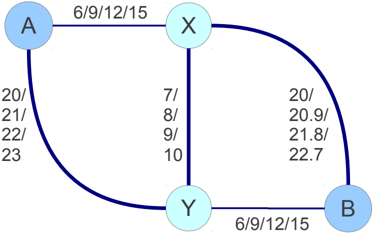
## Example 5: Choosing a route




- For any other allocation of people to routes, at least one person can decrease her travel time by switching routes.
- Thus the set of Nash equilibria is the set of action profiles in which two people take the route via X and two people take the route via Y.

# Pure Nash equilibrium

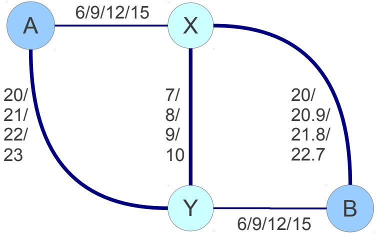
## Example 5: Choosing a route




- Now suppose that a relatively short, wide road is built from Y to X, giving each person four options to travel from A to B.
- Which are the Nash equilibria of this new situation?
- Does each person's travel time improve in the new equilibrium?

# Pure Nash equilibrium

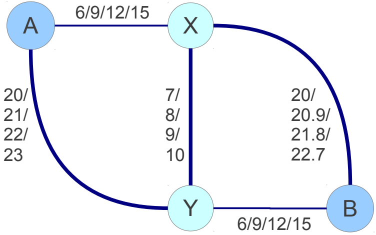
## Example 5: Choosing a route




- There is no equilibrium in which the new road is not used, because the only equilibrium before the new road is built has two people taking each route, resulting in a total travel time for each person of either 29.9 or 30 minutes.
- However, if a person taking A-X-B switches to the new road at X and then takes Y-B her total travel time becomes $9 + 7 + 12 = 28$ minutes.

# Pure Nash equilibrium

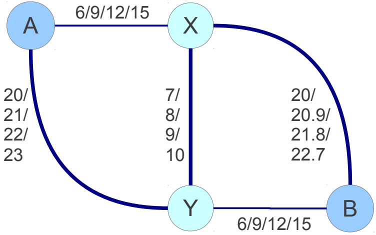
## Example 5: Choosing a route




- In any Nash equilibrium, one person takes A-X-B, two people take A-X-Y-B, and one person takes A-Y-B.
- For this assignment, each person's travel time is 32 minutes.
- No person can change her route and decrease her travel time.

# Pure Nash equilibrium

## Example 5: Choosing a route




- For every other allocation of people to routes at least one person can switch routes and reduce her travel time.
- Thus in the equilibrium with the new road every person's travel time increases, from either 29.9 or 30 minutes to 32 minutes.

# Strategic games in which players may randomize

- Recall that a pure Nash equilibrium does not always exist.
- The notion of Mixed Nash equilibrium or simply Nash equilibrium is a generalization of pure Nash equilibrium that models a *stochastic* steady state of a strategic game: we allow each player to choose a probability distribution over the set of her actions rather than restricting her to choose a single deterministic action.
- Payoff functions are naturally extended to capture expectation.
- The idea behind mixed Nash equilibrium is the same as the idea behind pure Nash equilibrium.
- Every strategic game (with finite player and actions sets) possesses at least one (mixed) Nash equilibrium.

# (Mixed) Strategies

A strategy $p_i$ for player $i \in N$ is a probability distribution over the set of her actions:

$$
\begin{aligned}
p_i &: S_i \to [0,1] \\
\sum_{s_i \in S_i} p_i(s_i) &= 1
\end{aligned}
$$

The set of strategies of player $i$ is denoted by $\Delta(S_i)$.

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The set of strategies of player $i$ is denoted by $\Delta(S_i)$.

A pure strategy is a strategy that poses probability 1 to a specific action $s_i \in S_i$: $(p_i(s_i) = 1)$, and is denoted by $s_i$ for simplicity.

# Expected payoffs

- A strategy profile is a combination of strategies, one for each player: $\mathbf{p} = (p_i)_{i \in N}$
- Given a strategy profile $\mathbf{p} = (p_i)_{i \in N}$, the expected payoff of player $i \in N$ is the expected value of her payoff function, i.e., the sum, over all action profiles, of the payoff of $i$ in the action profile times the probability of the action profile occurring:

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$$u_i(\mathbf{p}) = \sum_{s_1 \in S_1} \cdots \sum_{s_n \in S_n} \prod_{j=1}^{n} p_j(s_j) u_i(s_1, \ldots, s_n)$$

# Expected payoffs

## Example (1/2)

| | | Suspect 2 | |
|---|---|---|---|
| | | $Q$ | $F$ |
| Suspect 1 | $Q$ | (2,2) | (0,3) |
| | $F$ | (3,0) | (1,1) |

If Suspect 1 chooses strategy $p_1(Q) = 3/4$, $p_1(F) = 1/4$ and if Suspect 2 chooses strategy $p_2(Q) = 1/3$, $p_2(F) = 2/3$, then

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If Suspect 1 chooses strategy $p_1(Q) = 3/4$, $p_1(F) = 1/4$ and if Suspect 2 chooses strategy $p_2(Q) = 1/3$, $p_2(F) = 2/3$, then

$$\begin{aligned} u_1(p_1, p_2) &= \frac{3}{4} \cdot \frac{1}{3} \cdot u_1(Q, Q) + \frac{3}{4} \cdot \frac{2}{3} \cdot u_1(Q, F) + \\ &\quad \frac{1}{4} \cdot \frac{1}{3} \cdot u_1(F, Q) + \frac{1}{4} \cdot \frac{2}{3} \cdot u_1(F, F) \end{aligned}$$

# Expected payoffs

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$$
\begin{aligned}
u_1(p_1, p_2) &= \frac{3}{4} \cdot \frac{1}{3} \cdot u_1(Q, Q) + \frac{3}{4} \cdot \frac{2}{3} \cdot u_1(Q, F) + \\
&\quad \frac{1}{4} \cdot \frac{1}{3} \cdot u_1(F, Q) + \frac{1}{4} \cdot \frac{2}{3} \cdot u_1(F, F) \\
&= \frac{3}{4} \cdot \frac{1}{3} \cdot 2 + \frac{3}{4} \cdot \frac{2}{3} \cdot 0 + \frac{1}{4} \cdot \frac{1}{3} \cdot 3 + \frac{1}{4} \cdot \frac{2}{3} \cdot 1 \\
&= \frac{11}{12}
\end{aligned}
$$

# Expected payoffs

## Example (2/2)

| | | Suspect 2 | |
|---|---|---|---|
| | | $Q$ | $F$ |
| Suspect 1 | $Q$ | (2,2) | (0,3) |
| | $F$ | (3,0) | (1,1) |

If Suspect 1 chooses strategy $p_1(Q) = 3/4$, $p_1(F) = 1/4$ and if Suspect 2 chooses her pure strategy $Q$, then

# Expected payoffs

## Example (2/2)

| | | Suspect 2 | |
|---|---|---|---|
| | | $Q$ | $F$ |
| Suspect 1 | $Q$ | (2,2) | (0,3) |
| | $F$ | (3,0) | (1,1) |

If Suspect 1 chooses strategy $p_1(Q) = 3/4$, $p_1(F) = 1/4$ and if Suspect 2 chooses her pure strategy $Q$, then

$$u_2(p_1, Q) = \frac{3}{4} \cdot u_2(Q, Q) + \frac{1}{4} \cdot u_2(F, Q)$$

# Expected payoffs

## Example (2/2)

| | | Suspect 2 | |
|---|---|---|---|
| | | $Q$ | $F$ |
| Suspect 1 | $Q$ | (2,2) | (0,3) |
| | $F$ | (3,0) | (1,1) |

If Suspect 1 chooses strategy $p_1(Q) = 3/4$, $p_1(F) = 1/4$ and if Suspect 2 chooses her pure strategy $Q$, then

$$
\begin{aligned}
u_2(p_1, Q) &= \frac{3}{4} \cdot u_2(Q, Q) + \frac{1}{4} \cdot u_2(F, Q) \\
&= \frac{3}{4} \cdot 2 + \frac{1}{4} \cdot 0 \\
&= \frac{3}{2}
\end{aligned}
$$

# (Mixed) Nash equilibria

Formally, let

- $\mathbf{p} = (p_i)_{i \in N}$ be an strategy profile (determining a strategy for each player)
- $(p'_i, \mathbf{p}_{-i})$ be the strategy profile that results from $\mathbf{p}$ when player $i \in N$ switches to her strategy $p'_i \in \Delta(S_i)$, while the rest of the players preserve their strategies.

# (Mixed) Nash equilibria

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Then:

**Definition**

A Nash equilibrium is a strategy profile $\mathbf{p}$ such that for each player $i$ and for each strategy $p'_i$ of player $i$, $u_i(\mathbf{p}) \geq u_i(p'_i, \mathbf{p}_{-i})$.

# (Mixed) Nash equilibria

Equivalently:

**Definition**

A Nash equilibrium is a strategy profile $\mathbf{p}$ such that for each player $i$ and for each action $s_i$ of player $i$, $u_i(\mathbf{p}) \geq u_i(s_i, \mathbf{p}_{-i})$.

The second definition follows from the fact that

$$\max_{p'_i \in \Delta(S_i)} u_i(p'_i, \mathbf{p}_{-i}) = \max_{s'_i \in S_i} u_i(s'_i, \mathbf{p}_{-i}) \ .$$

# A useful characterization of Nash equilibria

- A player's expected payoff to a strategy profile is a weighted average of her expected payoffs to her pure strategies, where the weight attached to each pure strategy is the probability assigned to it by the player. Symbolically:

$$u_i(\mathbf{p}) = \sum_{s_i \in S_i} p_i(s_i) u_i(s_i, \mathbf{p}_{-i}) \ .$$

# A useful characterization of Nash equilibria

- Now let $\mathbf{p}$ be an equilibrium. Then, player $i$'s expected payoffs to the pure strategies to which $p_i$ assigns positive probability equal $u_i(\mathbf{p})$, i.e.,the expected payoff of $i$ in the equilibrium $\mathbf{p}$. (If any were smaller, then the weighted average would be smaller!)

We conclude that:

- the expected payoff to each action to which $p_i$ assigns positive probability is $u_i(\mathbf{p})$ and
- the expected payoff to every other action (to which $p_i$ assigns zero probability) is at most $u_i(\mathbf{p})$.

# A useful characterization of Nash equilibria

Conversely, if these conditions are satisfied for every player $i$, then $\mathbf{p}$ is a Nash equilibrium. Recall that

$$u_i(\mathbf{p}) = \sum_{s_i \in S_i} p_i(s_i) u_i(s_i, \mathbf{p}_{-i}) \ ,$$

- the expected payoff to $p_i$ is $u_i(\mathbf{p})$ and
- the expected payoff to any other strategy is at most $u_i(\mathbf{p})$, because it is a weighted average of $u_i(\mathbf{p})$ and numbers that are at most $u_i(\mathbf{p})$.

# A useful characterization of Nash equilibria

## Formulation

The support of strategy $p_i$ of player $i$ is the subset of actions of $i$ where $p_i$ poses strictly positive probability:

$$\text{Support}(p_i) = \{s_i \in S_i \ : \ p_i(s_i) > 0\}$$

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**Theorem**

A strategy profile $\mathbf{p}$ is a Nash equilibrium if and only if, for all players $i$ and for all $s_i \in S_i$,

$$s_i \in \text{Support}(p_i) \implies s_i \in \arg\max_{s \in S_i} u_i(s, \mathbf{p}_{-i}) \ .$$

# A useful characterization of Nash equilibria

## Formulation

**Theorem**

$\mathbf{p} = (p_i)_{i\in N}$ is a Nash equilibrium iff, $\forall i$ and $\forall s_i \in S_i$,

$$s_i \in \text{Support}(p_i) \implies s_i \in \arg\max_{s\in S_i} u_i(s, \mathbf{p}_{-i})$$

Proof.

( $\implies$ ) Assume there exists $i$ and $s_i$ such that $p_i(s_i) > 0$ and $u_i(s_i, \mathbf{p}_{-i}) < \max_{s\in S_i} u_i(s, \mathbf{p}_{-i})$. Then

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$$\begin{aligned} u_i(\mathbf{p}) &= \sum_{s \in S_i} p_i(s) u_i(s, \mathbf{p}_{-i}) \\ &< \sum_{s \in S_i} p_i(s) \max_{s' \in S_i} u_i(s', \mathbf{p}_{-i}) \end{aligned}$$

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**Theorem**

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# A useful characterization of Nash equilibria

## Formulation

**Theorem**

$\mathbf{p} = (p_i)_{i \in N}$ is a Nash equilibrium iff, $\forall i$ and $\forall s_i \in S_i$,

$$s_i \in \mathrm{Support}(p_i) \implies s_i \in \arg\max_{s \in S_i} u_i(s, \mathbf{p}_{-i})$$

Proof.

$(\Longleftarrow)$ Assume $\forall i$ and $\forall s_i$ that

$$p_i(s_i) > 0 \implies u_i(s_i, \mathbf{p}_{-i}) = \max_{s \in S_i} u_i(s, \mathbf{p}_{-i}).$$ Then

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$p_i(s_i) > 0 \implies u_i(s_i, \mathbf{p}_{-i}) = \max_{s \in S_i} u_i(s, \mathbf{p}_{-i})$. Then

$$\begin{aligned} u_i(\mathbf{p}) &= \sum_{s \in S_i} p_i(s) u_i(s, \mathbf{p}_{-i}) \\ &= \sum_{s \in S_i} p_i(s) \max_{s' \in S_i} u_i(s', \mathbf{p}_{-i}) \\ &= \max_{s' \in S_i} u_i(s', \mathbf{p}_{-i}) \quad \text{so } \mathbf{p} \text{ is an equilibrium} \end{aligned}$$

# A useful characterization of Nash equilibria

## Example: choosing numbers

- Players 1 and 2 each choose a positive integer up to $K$.
- If the players choose the same number, then player 2 pays $1 to player 1.
- If the players choose different numbers, no payment is made.

We will show that:

1. the game has a Nash equilibrium in which each player chooses each positive integer up to $K$ with probability $1/K$, and
2. the game has no other Nash equilibria.

# A useful characterization of Nash equilibria

## Example: choosing numbers

- To show that the pair of strategies $((1/K, \ldots, 1/K), (1/K, \ldots, 1/K))$ is a Nash equilibrium, it suffices to verify the conditions of the theorem stated previously.
- Given that each player's strategy specifies a positive probability for every action, it suffices to show that each action of each player yields the same expected payoff.

# A useful characterization of Nash equilibria

## Example: choosing numbers

- To show that the pair of strategies $((1/K, \ldots, 1/K), (1/K, \ldots, 1/K))$ is a Nash equilibrium, it suffices to verify the conditions of the theorem stated previously.
- Given that each player's strategy specifies a positive probability for every action, it suffices to show that each action of each player yields the same expected payoff.
- Player 1's expected payoff to each pure strategy is $1/K$, because with probability $1/K$ player 2 chooses the same number, and with probability $1 - 1/K$ player 2 chooses a different number.
- Similarly, player 2's expected payoff to each pure strategy is $-1/K$, because with probability $1/K$ player 1 chooses the same number, and with probability $1 - 1/K$ player 2 chooses a different number.
- Thus the pair of strategies is Nash equilibrium.

# A useful characterization of Nash equilibria

## Example: choosing numbers

- Let $(\mathbf{p}, \mathbf{q})$ be a Nash equilibrium, where $\mathbf{p}$ and $\mathbf{q}$ are vectors, the $j$th components of which are the probabilities assigned to the integer $j$ by each player.
- Given that player 2 uses strategy $\mathbf{q}$, player 1's expected payoff if she chooses the number $k$ is $q_k$ . Hence if $p_k > 0$ then we need $q_k \geq q_j$ for all $j$, so that, in particular, $q_k > 0$ ($q_j$ cannot be zero for all $j$!).
- But player 2's expected payoff if she chooses the number $k$ is $-p_k$ , so given $q_k > 0$ we need $p_k \leq p_j$ for all $j$, and, in particular, $p_k \leq 1/K$ ($p_j$ cannot exceed $1/K$ for all $j$!).
- We conclude that any probability $p_k$ that is positive must be at most $1/K$. The only possibility is that $p_k = 1/K$ for all $k$. A similar argument implies that $q_k = 1/K$ for all $k$.

# Best responses

Consider a strategic game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$.

- Fix some player $i \in N$.
- Fix a (partial) strategy profile $\mathbf{p}_{-i} \in \times_{j \neq i} \Delta(S_j)$ of the other players.
- A best response of player $i$ to $\mathbf{p}_{-i}$ is a strategy of $i$ that maximizes her payoff, given the strategies $\mathbf{p}_{-i}$ of the other players.

Formally:

**Definition (Best-response function)**

The best-response function $BR_i : \times_{j \neq i} \Delta(S_j) \to 2^{S_i}$ of player $i$ maps a strategy profile of all players except $i$ to a subset of actions of player $i$, so that

$$BR_i(\mathbf{p}_{-i}) = \{s_i \in S_i \ : \ s_i \in \arg\max_{s \in S_i} \{u_i(s, \mathbf{p}_{-i})\}\} \ .$$

# Best responses

It is straightforward to see that

1. Any probability distribution on the best-response actions of player $i$, i.e., any $p_i \in \Delta(BR_i(\mathbf{p}_{-i}))$, maximizes player $i$'s payoff:

$$u_i(p_i, \mathbf{p}_{-i}) \geq u_i(p', \mathbf{p}_{-i})) \quad \forall p' \in \Delta(S_i) \ .$$

2. The strategy profile $\mathbf{p} = (p_i)_{i \in N}$ is a Nash equilibrium of $\Gamma$ if and only if, for all players $i$, if $p_i(s_i) > 0$ for some $s_i \in S_i$, then $s_i \in BR_i(\mathbf{p}_{-i})$.

So, in a Nash equilibrium, each player's strategy is a best response to the other players' strategies.

# Existence and computation of Nash equilibria

**Theorem (Nash, 1951)**

Every finite game (i.e., a game with a finite number of players and with finite action sets) has at least one Nash equilibrium.

# Existence and computation of Nash equilibria

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**Theorem (Chen and Deng, 2006)**

The problem of computing a Nash equilibrium is PPAD-complete, even for games involving only two players.

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More details next week!

# Existence and computation of Nash equilibria

**Theorem (Nash, 1951)**

Every finite game (i.e., a game with a finite number of players and with finite action sets) has at least one Nash equilibrium.

Note:

- This result is of no help in finding equilibria.
- The finiteness of the number of actions of each player is only sufficient for the existence of an equilibrium, not necessary: many games in which the players have infinitely many actions possess Nash equilibria.
- A player's strategy in a Nash equilibrium may assign probability 1 to a single action; if every player does so, then the equilibrium corresponds to a pure Nash equilibrium.

# Symmetric Nash equilibrium

**Definition**

A strategy profile $\mathbf{p}$ in a symmetric strategic game (in which each player has the same set of actions) is a symmetric Nash equilibrium if it is a (pure or mixed) Nash equilibrium and $p_i$ is the same for every player $i$.

**Theorem**

*Every symmetric strategic game in which each player's set of actions is finite has a symmetric Nash equilibrium.*

# Illustration: Bargaining

- Two players bargain over the division of a pie of size 10.
- The players simultaneously make demands; the possible demands are the non-negative *even* integers up to 10.
- If the demands sum to 10, then each player receives her demand.
- If the demands sum to less than 10, then each player receives her demand plus half of the pie that remains after both demands have been satisfied.
- If the demands sum to more than 10, then neither player receives any payoff.

We will find all the symmetric Nash equilibria in which each player assigns positive probability to at most two demands.

# Illustration: Bargaining

## Symmetric equilibria of support size 1 (pure)

The game:

| | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| 0 | 5, 5 | 4, 6 | 3, 7 | 2, 8 | 1, 9 | 0, 10 |
| 2 | 6, 4 | 5, 5 | 4, 6 | 3, 7 | 2, 8 | 0, 0 |
| 4 | 7, 3 | 6, 4 | 5, 5 | 4, 6 | 0, 0 | 0, 0 |
| 6 | 8, 2 | 7, 3 | 6, 4 | 0, 0 | 0, 0 | 0, 0 |
| 8 | 9, 1 | 8, 2 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| 10 | 10, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |

By inspection it has a single symmetric pure strategy Nash equilibrium, (10, 10).

# Illustration: Bargaining

## Symmetric equilibria of support size 2

Now consider situations in which the common mixed strategy assigns positive probability to two actions.

- Suppose that player 2 assigns positive probability only to 0 and 2.
- Then player 1's payoff to her action 4 exceeds her payoff to either 0 or 2. Thus there is no symmetric equilibrium in which the actions assigned positive probability are 0 and 2.
- By a similar argument we can rule out equilibria in which the actions assigned positive probability are any pair except 2 and 8, or 4 and 6.

# Illustration: Bargaining

## Symmetric equilibria of support size 2

- If the actions to which player 2 assigns positive probability are 2 and 8 then player 1's expected payoffs to 2 and 8 are the same if the probability player 2 assigns to 2 is $\frac{2}{5}$ (and the probability she assigns to 8 is $\frac{3}{5}$).
- Given these probabilities, player 1's expected payoff to her actions 2 and 8 is $\frac{16}{5}$, and her expected payoff to every other action is less than $\frac{16}{5}$.
- Thus the pair of mixed strategies in which every player assigns probability $\frac{2}{5}$ to 2 and $\frac{3}{5}$ to 8 is a symmetric mixed strategy Nash equilibrium.
- Similarly, the game has a symmetric mixed strategy equilibrium in which each player assigns probability $\frac{4}{5}$ to the demand of 4 and probability $\frac{1}{5}$ to the demand of 6.

# Illustration: Reporting a crime

- A crime is observed by a group of $n$ people.
- Each person would like the police to be informed, but prefers that someone else make the phone call.
- Suppose each person attaches the value $v$ to the police being informed and bears the cost $c$ if she makes the call, where $v > c > 0$.

# Illustration: Reporting a crime

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Formulation as a strategic game:

Players: the $n$ people.

Actions: Each player's set of actions is $\{Call, Don't\ call\}$.

Payoffs: Each player's payoff function assigns

- 0 to the profile in which no one calls;
- $v - c$ to any profile in which she calls;
- $v$ to any profile in which at least one person calls, but she does not.

# Illustration: Reporting a crime

## Pure Nash equilibria

- The game has $n$ pure Nash equilibria, in which exactly one person calls:
  - If the person who calls switches to not calling, her payoff falls from $v - c > 0$ to 0.
  - If any other person switches to calling, her payoff falls from $v$ to $v - c$.

# Illustration: Reporting a crime

## Pure Nash equilibria

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  - If the person who calls switches to not calling, her payoff falls from $v - c > 0$ to 0.
  - If any other person switches to calling, her payoff falls from $v$ to $v - c$.
- The game has no other pure Nash equilibrium:
  - If no one calls, then any person can switch to calling and raise her payoff from 0 to $v - c$.
  - If two or more persons call, then any of them can switch to not calling and raise her payoff from $v - c$ to $v$.

# Illustration: Reporting a crime

## Symmetric (mixed) Nash equilibrium

- The game is symmetric, so it must have a symmetric Nash equilibrium.
- The game has no symmetric pure Nash equilibrium, so it must have a symmetric mixed Nash equilibrium.
- In any such equilibrium, each person's expected payoff to calling is equal to her expected payoff to not calling.
- Denote $p$ the probability with which each person calls ($0 < p < 1$) in a symmetric Nash equilibrium, and let $\mathbf{p} = (p)_{i \in N}$.
  Equilibrium condition: For each person $i$,

$$u_i(Call, \mathbf{p}_{-i}) \quad = \quad u_i(Don't\ call, \mathbf{p}_{-i})$$

# Illustration: Reporting a crime

## Symmetric (mixed) Nash equilibrium

Now:

$$u_i(Call, \mathbf{p}_{-i}) = v - c$$

and

$$\begin{aligned} u_i(Don't\ call, \mathbf{p}_{-i}) &= 0 \cdot \Pr\{\text{no one else calls}\} \\ &\quad + v \cdot \Pr\{\text{at least one else calls}\} \\ &= v \cdot (1 - \Pr\{\text{no one else calls}\}) \end{aligned}$$

and the equilibrium condition gives

$$v - c = v \cdot (1 - \Pr\{\text{no one else calls}\})$$

# Illustration: Reporting a crime

## Symmetric (mixed) Nash equilibrium

$$
\begin{aligned}
v - c &= v \cdot (1 - \Pr\{\text{no one else calls}\}) \\
\frac{c}{v} &= \Pr\{\text{no one else calls}\} \\
\frac{c}{v} &= (1-p)^{n-1} \\
p &= 1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}} \in (0,1) \;.
\end{aligned}
$$

# Illustration: Reporting a crime

## Symmetric (mixed) Nash equilibrium

We conclude that the game has a unique mixed strategy Nash equilibrium, in which each person calls with probability

$$p = 1 - \left(\frac{c}{v}\right)^{\frac{1}{n-1}} \quad .$$

Remarks:

- As $n$ increases, the probability $p$ that any given person calls decreases.
- As $n$ increases, the probability that at least one person calls also decreases.
- The larger the group, the less likely the police are informed of the crime!

# Computing all Nash equilibria

- In a Nash equilibrium, if two different actions of player $i$ both have positive probability, then they must both give her the same expected payoff, which must be maximum.
- Although there are infinitely many mixed strategy profiles, there are only finitely many subsets of $\times_{i \in N} S_i$ that can be supports of equilibria.
- So, we can search for equilibria by sequentially considering various guesses as to what the support may be and looking for equilibria with each guessed support.

# Computing all Nash equilibria

Let $\times_{i \in N} D_i$ be our current guess of the support. If there is an equilibrium $\mathbf{p} = (p_i)_{i \in N}$ with support $\times_{i \in N} D_i$, then there must exist numbers $(\omega_i)_{i \in N}$ such that:

$$
\begin{aligned}
u_i(s_i, \mathbf{p}_{-i}) &= \omega_i \quad \forall i \in N, \forall s_i \in D_i \\
p_i(e_i) &= 0 \quad \forall i \in N, \forall e_i \in S_i \setminus D_i \\
\sum_{s_i \in S_i} p_i(s_i) &= 1 \quad \forall i \in N
\end{aligned}
$$

# Computing all Nash equilibria

Let $\times_{i \in N} D_i$ be our current guess of the support. If there is an equilibrium $\mathbf{p} = (p_i)_{i \in N}$ with support $\times_{i \in N} D_i$, then there must exist numbers $(\omega_i)_{i \in N}$ such that:

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p_i(e_i) &= 0 \quad \forall i \in N, \forall e_i \in S_i \setminus D_i \\
\sum_{s_i \in S_i} p_i(s_i) &= 1 \quad \forall i \in N
\end{aligned}
$$

$\sum_{i \in N}(|S_i| + 1)$ equations in the same number of unknowns

# Computing all Nash equilibria

- Given the guessed support $\times_{i \in N} D_i$, we can find all solutions to the system.
- These solutions do not necessarily give equilibria:
  1. No solutions may exist.
  2. A solution may fail to be a strategy profile, if some $p_i(s_i)$ is negative. So we must require

$$p_i(s_i) \geq 0 \quad \forall i \in N, \forall s_i \in D_i \;.$$

  3. A solution may fail to be an equilibrium if some player $i$ has some other action outside $D_i$ that would give her better payoff, so we must require

$$\omega_i \geq u_i(e_i, \mathbf{p}_{-i}) \quad \forall i \in N, \forall e_i \in S_i \setminus D_i \;.$$

# Computing all Nash equilibria

- Given the guessed support $\times_{i \in N} D_i$, we can find all solutions to the system.
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  3. A solution may fail to be an equilibrium if some player $i$ has some other action outside $D_i$ that would give her better payoff, so we must require

     $$\omega_i \geq u_i(e_i, \mathbf{p}_{-i}) \quad \forall i \in N, \forall e_i \in S_i \setminus D_i \; .$$

- If we find a solution $(\mathbf{p}, \omega)$ that satisfies the above conditions, then $\mathbf{p}$ is a Nash equilibrium.

# Computing all Nash equilibria

- Given the guessed support $\times_{i\in N} D_i$, we can find all solutions to the system.
- These solutions do not necessarily give equilibria:
  1. No solutions may exist.
  2. A solution may fail to be a strategy profile, if some $p_i(s_i)$ is negative. So we must require
  $$p_i(s_i) \geq 0 \quad \forall i \in N\,, \forall s_i \in D_i \ \ .$$
  3. A solution may fail to be an equilibrium if some player $i$ has some other action outside $D_i$ that would give her better payoff, so we must require
  $$\omega_i \geq u_i(e_i, \mathbf{p}_{-i}) \quad \forall i \in N\,, \forall e_i \in S_i \setminus D_i \ \ .$$
- If we find a solution $(\mathbf{p}, \omega)$ that satisfies the above conditions, then $\mathbf{p}$ is a Nash equilibrium.
- Nash's theorem: there is at least one support for which all conditions will be satisfied!

# Computing all Nash equilibria

## An example (1/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

# Computing all Nash equilibria

## An example (1/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

There are $(2^3 - 1) \cdot (2^2 - 1) = 21$ possible supports:

- $\{T\}, \{B\}, \{T, B\}$ for the row player
- $\{L\}, \{M\}, \{R\}, \{L, M\}, \{L, R\}, \{M, R\}, \{L, M, R\}$ for the column player

# Computing all Nash equilibria

## An example (2/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

We begin by considering pure strategies (supports of size 1):

- If the row player chooses T, then the column player would choose M, but then the row player would prefer B.

# Computing all Nash equilibria

## An example (2/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

We begin by considering pure strategies (supports of size 1):

- If the row player chooses T, then the column player would choose M, but then the row player would prefer B.
- If the row player chooses B, then the column player would choose L, but then the row player would prefer T.

# Computing all Nash equilibria

## An example (2/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

We begin by considering pure strategies (supports of size 1):

- If the row player chooses T, then the column player would choose M, but then the row player would prefer B.
- If the row player chooses B, then the column player would choose L, but then the row player would prefer T.
- . . . Similarly for the column player.

# Computing all Nash equilibria

## An example (2/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

We begin by considering pure strategies (supports of size 1):

- If the row player chooses T, then the column player would choose M, but then the row player would prefer B.
- If the row player chooses B, then the column player would choose L, but then the row player would prefer T.
- . . . Similarly for the column player.

Therefore, there is no equilibrium where either player has support of size 1.
So, it suffices to consider the supports

1. $\{T, B\}$ for the row player
2. $\{L, M\}, \{L, R\}, \{M, R\}, \{L, M, R\}$ for the column player

# Computing all Nash equilibria

## An example (3/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us first try the support $\{T, B\} \times \{L, M, R\}$. We need:

$$\begin{aligned} \omega_1 &= u_1(T, p_2) = u_1(B, p_2) \\ \omega_1 &= 7p_2(L) + 2p_2(M) + 3p_2(R) = 2p_2(L) + 7p_2(M) + 4p_2(R) \end{aligned}$$

# Computing all Nash equilibria

## An example (3/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us first try the support $\{T, B\} \times \{L, M, R\}$. We need:

$$\begin{aligned} \omega_1 &= u_1(T, p_2) = u_1(B, p_2) \\ \omega_1 &= 7p_2(L) + 2p_2(M) + 3p_2(R) = 2p_2(L) + 7p_2(M) + 4p_2(R) \end{aligned}$$

and

$$\begin{aligned} \omega_2 &= u_2(L, p_1) = u_2(M, p_1) = u_2(R, p_1) \\ \omega_2 &= 2p_1(T) + 7p_1(B) = 7p_1(T) + 2p_1(B) = 6p_1(T) + 5p_1(B) \end{aligned}$$

# Computing all Nash equilibria

## An example (3/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us first try the support $\{T, B\} \times \{L, M, R\}$. We need:

$$\begin{aligned} \omega_1 &= u_1(T, p_2) = u_1(B, p_2) \\ \omega_1 &= 7p_2(L) + 2p_2(M) + 3p_2(R) = 2p_2(L) + 7p_2(M) + 4p_2(R) \end{aligned}$$

and

$$\begin{aligned} \omega_2 &= u_2(L, p_1) = u_2(M, p_1) = u_2(R, p_1) \\ \omega_2 &= 2p_1(T) + 7p_1(B) = 7p_1(T) + 2p_1(B) = 6p_1(T) + 5p_1(B) \end{aligned}$$

and $p_1(T) + p_1(B) = 1 = p_2(L) + p_2(M) + p_2(R)$

# Computing all Nash equilibria

## An example (3/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us first try the support $\{T, B\} \times \{L, M, R\}$. We need:

$$\begin{aligned}\omega_1 &= u_1(T, p_2) = u_1(B, p_2)\\ \omega_1 &= 7p_2(L) + 2p_2(M) + 3p_2(R) = 2p_2(L) + 7p_2(M) + 4p_2(R)\end{aligned}$$

and

$$\begin{aligned}\omega_2 &= u_2(L, p_1) = u_2(M, p_1) = u_2(R, p_1)\\ \omega_2 &= 2p_1(T) + 7p_1(B) = 7p_1(T) + 2p_1(B) = 6p_1(T) + 5p_1(B)\end{aligned}$$

and $p_1(T) + p_1(B) = 1 = p_2(L) + p_2(M) + p_2(R)$. No solution!

# Computing all Nash equilibria

## An example (4/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us now try the support $\{T, B\} \times \{M, R\}$. We need:

$$\omega_1 = 2p_2(M) + 3p_2(R) = 7p_2(M) + 4p_2(R)$$

# Computing all Nash equilibria

## An example (4/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us now try the support $\{T, B\} \times \{M, R\}$. We need:

$$\omega_1 = 2p_2(M) + 3p_2(R) = 7p_2(M) + 4p_2(R)$$

and

$$\omega_2 = 7p_1(T) + 2p_1(B) = 6p_1(T) + 5p_1(B)$$

# Computing all Nash equilibria

## An example (4/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us now try the support $\{T, B\} \times \{M, R\}$. We need:

$$\omega_1 = 2p_2(M) + 3p_2(R) = 7p_2(M) + 4p_2(R)$$

and

$$\omega_2 = 7p_1(T) + 2p_1(B) = 6p_1(T) + 5p_1(B)$$

and $p_1(T) + p_1(B) = 1 = p_2(M) + p_2(R)$

# Computing all Nash equilibria

## An example (4/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us now try the support $\{T, B\} \times \{M, R\}$. We need:

$$\omega_1 = 2p_2(M) + 3p_2(R) = 7p_2(M) + 4p_2(R)$$

and

$$\omega_2 = 7p_1(T) + 2p_1(B) = 6p_1(T) + 5p_1(B)$$

and $p_1(T) + p_1(B) = 1 = p_2(M) + p_2(R)$
Solution: $p_1(T) = 3/4, p_1(B) = 1/4, p_2(M) = -1/4, p_2(R) = 5/4$

# Computing all Nash equilibria

## An example (4/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us now try the support $\{T, B\} \times \{M, R\}$. We need:

$$\omega_1 = 2p_2(M) + 3p_2(R) = 7p_2(M) + 4p_2(R)$$

and

$$\omega_2 = 7p_1(T) + 2p_1(B) = 6p_1(T) + 5p_1(B)$$

and $p_1(T) + p_1(B) = 1 = p_2(M) + p_2(R)$

Solution: $p_1(T) = 3/4, p_1(B) = 1/4, p_2(M) = -1/4, p_2(R) = 5/4$

Negative solution, so this is not an equilibrium!

# Computing all Nash equilibria

## An example (5/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us now try the support $\{T, B\} \times \{L, M\}$. We need:

$$\omega_1 = 7p_2(L) + 2p_2(M) = 2p_2(L) + 7p_2(M)$$

# Computing all Nash equilibria

## An example (5/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us now try the support $\{T, B\} \times \{L, M\}$. We need:

$$\omega_1 = 7p_2(L) + 2p_2(M) = 2p_2(L) + 7p_2(M)$$

and

$$\omega_2 = 2p_1(T) + 7p_1(B) = 7p_1(T) + 2p_1(B)$$

# Computing all Nash equilibria

## An example (5/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us now try the support $\{T, B\} \times \{L, M\}$. We need:

$$\omega_1 = 7p_2(L) + 2p_2(M) = 2p_2(L) + 7p_2(M)$$

and

$$\omega_2 = 2p_1(T) + 7p_1(B) = 7p_1(T) + 2p_1(B)$$

and $p_1(T) + p_1(B) = 1 = p_2(L) + p_2(M)$

# Computing all Nash equilibria

## An example (5/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us now try the support $\{T, B\} \times \{L, M\}$. We need:

$$\omega_1 = 7p_2(L) + 2p_2(M) = 2p_2(L) + 7p_2(M)$$

and

$$\omega_2 = 2p_1(T) + 7p_1(B) = 7p_1(T) + 2p_1(B)$$

and $p_1(T) + p_1(B) = 1 = p_2(L) + p_2(M)$
Solution: $p_1(T) = p_1(B) = p_2(L) = p_2(M) = 0.5$, $\omega_1 = \omega_2 = 4.5$.

# Computing all Nash equilibria

## An example (5/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us now try the support $\{T, B\} \times \{L, M\}$. We need:

$$\omega_1 = 7p_2(L) + 2p_2(M) = 2p_2(L) + 7p_2(M)$$

and

$$\omega_2 = 2p_1(T) + 7p_1(B) = 7p_1(T) + 2p_1(B)$$

and $p_1(T) + p_1(B) = 1 = p_2(L) + p_2(M)$
Solution: $p_1(T) = p_1(B) = p_2(L) = p_2(M) = 0.5$, $\omega_1 = \omega_2 = 4.5$.
But $u_2(R, p_1) = 6 \cdot 0.5 + 5 \cdot 0.5 = 5.5 > 4.5 = \omega_2$, so this is not an equilibrium!

# Computing all Nash equilibria

## An example (6/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us now try the support $\{T, B\} \times \{L, R\}$. We need:

$$\omega_1 = 7p_2(L) + 3p_2(R) = 2p_2(L) + 4p_2(R)$$

# Computing all Nash equilibria

## An example (6/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us now try the support $\{T, B\} \times \{L, R\}$. We need:

$$\omega_1 = 7p_2(L) + 3p_2(R) = 2p_2(L) + 4p_2(R)$$

and

$$\omega_2 = 2p_1(T) + 7p_1(B) = 6p_1(T) + 5p_1(B)$$

# Computing all Nash equilibria

## An example (6/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us now try the support $\{T, B\} \times \{L, R\}$. We need:

$$\omega_1 = 7p_2(L) + 3p_2(R) = 2p_2(L) + 4p_2(R)$$

and

$$\omega_2 = 2p_1(T) + 7p_1(B) = 6p_1(T) + 5p_1(B)$$

and $p_1(T) + p_1(B) = 1 = p_2(L) + p_2(R)$

# Computing all Nash equilibria

## An example (6/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us now try the support $\{T, B\} \times \{L, R\}$. We need:

$$\omega_1 = 7p_2(L) + 3p_2(R) = 2p_2(L) + 4p_2(R)$$

and

$$\omega_2 = 2p_1(T) + 7p_1(B) = 6p_1(T) + 5p_1(B)$$

and $p_1(T) + p_1(B) = 1 = p_2(L) + p_2(R)$

Solution: $p_1(T) = 1/3, p_1(B) = 2/3, p_2(L) = 1/6, p_2(R) = 5/6,$
$\omega_1 = 11/3,\ \omega_2 = 16/3.$

# Computing all Nash equilibria

## An example (6/6)

| | L | M | R |
|---|---|---|---|
| T | 7, 2 | 2, 7 | 3, 6 |
| B | 2, 7 | 7, 2 | 4, 5 |

Let us now try the support $\{T, B\} \times \{L, R\}$. We need:

$$\omega_1 = 7p_2(L) + 3p_2(R) = 2p_2(L) + 4p_2(R)$$

and

$$\omega_2 = 2p_1(T) + 7p_1(B) = 6p_1(T) + 5p_1(B)$$

and $p_1(T) + p_1(B) = 1 = p_2(L) + p_2(R)$
Solution: $p_1(T) = 1/3, p_1(B) = 2/3, p_2(L) = 1/6, p_2(R) = 5/6,$
$\omega_1 = 11/3,\ \omega_2 = 16/3.$
Also, $u_2(M, p_1) = 7 \cdot 1/3 + 2 \cdot 2/3 = 11/3 \leq \omega_2 = 16/3$, so this is an equilibrium!

# Strict domination

## Definition

In a strategic game, one (pure or mixed) strategy of a player strictly dominates an action (pure strategy) of that player if it is superior, no matter what the other players do:

### Definition

In a strategic game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, player $i$'s strategy $p_i$ strictly dominates her action $s_i \in S_i$ if, for every action profile $\mathbf{s}_{-i} \in \times_{j \in N \setminus \{i\}} S_j$ of the other players,

$$u_i(p_i, \mathbf{s}_{-i}) > u_i(s_i, \mathbf{s}_{-i}) \ .$$

We say that the action $s_i$ is strictly dominated.

### Property

A strictly dominated action is not used with positive probability in any Nash equilibrium.

# Strict domination

## Examples

Recall Prisoner's Dilemma:

| | | Suspect 2 | |
|---|---|---|---|
| | | *Quiet* | *Fink* |
| Suspect 1 | *Quiet* | (2,2) | (0,3) |
| | *Fink* | (3,0) | (1,1) |

The action (pure strategy) *Fink* strictly dominates the action *Quiet*: regardless of her opponent's action, a player prefers the outcome when she chooses *Quiet*.

# Strict domination

## Examples

The following matrix gives the payoffs of player 1 (row player) in a strategic game.

| | $L$ | $R$ |
|---|---|---|
| $T$ | 1 | 1 |
| $M$ | 4 | 0 |
| $B$ | 0 | 3 |

We will find all strategies of player 1 that strictly dominate $T$:

- Denote the probability that player 1 assigns to $T$ by $p$ and the probability she assigns to $M$ by $r$ (so that the probability she assigns to $B$ is $1 - p - r$).
- A mixed strategy of player 1 strictly dominates $T$ if and only if $1 \cdot p + 4 \cdot r + 0 \cdot (1 - p - r) > 1$ and $1 \cdot p + 0 \cdot r + 3 \cdot (1 - p - r) > 1$.
- Equivalently, if and only if $1 - 4r < p < 1 - \frac{3}{2}r$.

# Weak domination

## Definition

### Definition

In a strategic game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, player $i$'s strategy $p_i$ weakly dominates her action $s_i \in S_i$ if, for every action profile $\mathbf{s}_{-i} \in \times_{j \in N \setminus \{i\}} S_j$ of the other players,

$$u_i(p_i, \mathbf{s}_{-i}) \geq u_i(s_i, \mathbf{s}_{-i}) \ ,$$

and, for some action profile $\mathbf{s}_{-i} \in \times_{j \in N \setminus \{i\}} S_j$ of the other players,

$$u_i(p_i, \mathbf{s}_{-i}) > u_i(s_i, \mathbf{s}_{-i}) \ .$$

We say that the action $s_i$ is weakly dominated.

# Weak domination

## Properties

Weakly (as well as strictly) dominated actions do not necessarily exist.

Note that, unlike strictly dominated actions,

A weakly dominated action may be used with positive probability in a Nash equilibrium.

However:

**Proposition**

Every finite strategic game has a Nash equilibrium in which no player's strategy is weakly dominated.

# Dominant actions

## Definition

A dominant strategy occurs when one pure strategy (action) is better than any other strategy for one player, no matter how that player's opponents may play. Formally:

**Definition**

A pure strategy $s_i \in S_i$ is dominant for player $i$ in the strategic game $\Gamma = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ if

$$u_i(s_i, \mathbf{s}_{-i}) > u_i(s_i', \mathbf{s}_{-i})$$

for all $s_i' \neq s_i \in S_i$ and for all $\mathbf{s}_{-i} \in \times_{j \neq i} S_j$.

In other words, a dominant action is an action that strictly dominates all other actions of a player.

# Dominant actions

## Properties

Note:

- Dominant strategies do not necessarily exist.
- If all players have a dominant action, then their combination is the unique pure Nash equilibrium.
- If a dominant strategy exists for one player in a game, then that player will play that (pure) strategy in each of the game's Nash equilibria.

# Nash equilibrium refinements

## Strict Nash equilibrium

- The definition of a pure Nash equilibrium requires only that the outcome of a deviation be *no better* (rather than *worse*) for the deviant than the equilibrium outcome.
- An equilibrium is strict if each player's equilibrium action is *better* than all her other actions, given the other players' actions.

**Definition**

A **strict Nash equilibrium** is a pure strategy profile $\mathbf{s} = (s_i)_{i \in N}$ such that for each player $i$,

$$u_i(\mathbf{s}) > u_i(s'_i, \mathbf{s}_{-i}) \quad \forall s'_i \in S_i, \;\; s'_i \neq s_i \;\; .$$

# Nash equilibrium refinements

## Strong Nash equilibrium

### Definition

A **strong Nash equilibrium** is a Nash equilibrium such that there is no nonempty set of players who could all gain by deviating together to some other combination of strategies that is jointly feasible for them, when the other players who are not in this set are expected to stay with their equilibrium strategies.

# Oligopoly

- How does the outcome of competition among the firms in an industry depend on the characteristics of the demand for the firms' output, the firms' cost functions, and the number of firms?
- Will the benefits of technological improvements be passed on to consumers?
- Will a reduction in the number of firms generate a less desirable outcome?

$\Longrightarrow$ We need a model of the interaction between firms competing for the business of consumers: models of **oligopoly** (competition between a small number of sellers).

1. Cournot's model of oligopoly
2. Bertrand's model of oligopoly

# Cournot's model of oligopoly

## General model

- A single good is produced by $n$ firms.
- The cost to firm $i$ of producing $q_i$ units of the good is $C_i(q_i)$, where $C_i$ is an increasing function (more output is more costly to produce).
- All the output is sold at a single price, determined by the demand for the good and the firms' total output.
- Specifically, if the firms' total output is $Q$ then the market price is $P(Q)$; $P$ is called the "inverse demand function".
- Assume that $P$ is a decreasing function when it is positive: if the firms' total output increases, then the price decreases (unless it is already zero).
- If the output of each firm $i$ is $q_i$, then the price is $P(q_1 + \cdots + q_n)$, so that firm $i$'s revenue is $q_i P(q_1 + \cdots + q_n)$.
- Thus firm $i$'s profit, equal to its revenue minus its cost, is

$$\pi_i(q_1, \ldots, q_n) = q_i P(q_1 + \cdots + q_n) - C_i(q_i) \ .$$

# Cournot's oligopoly game

Cournot suggested that the industry be modeled as the following strategic game:

- Players: The firms.
- Actions: Each firm's set of actions is the set of its possible outputs (nonnegative numbers).
- Payoffs: Each firm's payoff is represented by its profit $\pi_i$.

# Cournot's oligopoly game

## Example

Suppose there are **two** firms (the industry is a duopoly), each firm's cost function is $C_i(q_i) = cq_i$ for all $q_i$, and the inverse demand function is linear where it is positive, given by

$$P(Q) = \begin{cases} a - Q & \text{if } Q \leq a \\ 0 & \text{if } Q > a \end{cases}$$

where $a > 0$ and $c \geq 0$ are constants, and $c < a$.
Firm 1's profit is

$$\begin{aligned} \pi_1(q_1, q_2) &= q_1(P(q_1 + q_2) - c) \\ &= \begin{cases} q_1(a - c - q_1 - q_2) & \text{if } q_1 + q_2 \leq a \\ -cq_1 & \text{if } q_1 + q_2 > a \end{cases} \end{aligned}$$

# Cournot's oligopoly game

## Example

- To find the Nash equilibria in this example, we should find the firms' best response functions.
- To find firm 11s best response to any given output $q_2$ of firm 2, we need to study firm 1's profit as a function of its output $q_1$ for given values of $q_2$.
- By setting the derivative of firm 1's profit with respect to $q_1$ equal to zero and solving for $q_1$, we can find firm 1's best response to any given input $q_2$:

$$b_1(q_2) = \begin{cases} \frac{1}{2}(a - c - q_2) & \text{if } q_2 \leq a - c \\ 0 & \text{if } q_2 > a - c \end{cases} .$$

- Because firm 2's cost function is the same as firm 1's, its best response function $b_2$ is also the same: for any number $q$, we have $b_2(q) = b_1(q)$.

# Cournot's oligopoly game

## Example

- A Nash equilibrium is a pair $(q_1^*, q_2^*)$ of outputs for which $q_1^*$ is a best response to $q_2^*$, and $q_2^*$ is a best response to $q_1^*$:

$$q_1^* = b_1(q_2^*)\,, \quad q_2^* = b_2(q_1^*) \ .$$

- (Unique) solution:

$$q_1^* = q_2^* = \frac{1}{3}(a - c) \ .$$

- The total output in this equilibrium is $2/3(a - c)$.
- The price at which output is sold is $P(2/3(a - c)) = 1/3(a + 2c)$.
- As $a$ increases (meaning that consumers are willing to pay more for the good), the equilibrium price and the output of each firm increases.
- As $c$ (the unit cost of production) increases, the output of each firm falls and the price rises.

# Bertrand's model of oligopoly

- In Cournot's game, each firm chooses an output; the price is determined by the demand for the good in relation to the total output produced.
- In Bertrand's model of oligopoly, each firm chooses a price, and produces enough output to meet the demand it faces, given the prices chosen by all the firms.

Setting:

- A single good is produced by $n$ firms; each firm can produce $q_i$ units of the good at a cost of $C_i(q_i)$.
- It is convenient to specify demand by giving a demand function $D$, rather than an inverse demand function as we did for Cournot's game.
- The interpretation of $D$ is that if the good is available at the price $p$ then the total amount demanded is $D(p)$.

# Bertrand's model of oligopoly

- If the firms set different prices then all consumers purchase the good from the firm with the lowest price, which produces enough output to meet this demand.
- If more than one firm sets the lowest price, all the firms doing so share the demand at that price equally.
- A firm whose price is not the lowest price receives no demand and produces no output.
- Note: a firm does not choose its output strategically; it simply produces enough to satisfy all the demand it faces, given the prices, even if its price is below its unit cost, in which case it makes a loss.

# Bertrand's oligopoly game

Bertrand's oligopoly game is the following strategic game:

- Players: The firms.
- Actions: Firm $i$'s set of actions is the set of possible prices (nonnegative numbers $p_i$).
- Payoffs: If firm $i$ is one of $m$ firms setting the lowest price, its profit is

$$\frac{p_i D(p_i)}{m} - C_i\left(\frac{D(p_i)}{m}\right) \quad .$$

  If some firm's price is lower than $p_i$, firm $i$'s profit is zero.

# Further reading

- Martin J. Osborne: An Introduction to Game Theory. Oxford University Press, 2004.
- Martin J. Osborne and Ariel Rubinstein: A Course in Game Theory. The MIT Press, 1994.
- Roger B. Myerson: Game Theory: Analysis of Conflict. Harvard University Press, 1991.