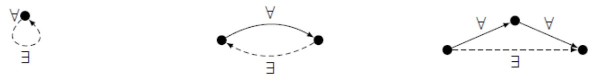
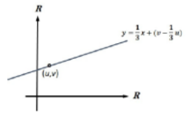
Equivalence Relations with Partial Orders

Definitions/ Proposition	What the result is saying	Formal Definition	Intuition/ Note
Definition 6.1.1	Definition of Partitions	Call $\mathcal{L}$ a partition of a set A if 1) $\mathcal{L}$ is a set of nonempty subsets of A 2) Every element of A is in some element of $\mathcal{L}$ 3) If two elements of $\mathcal{L}$ have a nonempty intersection, then they are equal	Use this to check if a function is well defined
Remark 6.1.2	Symbolic Representation of Above Conditions	1) $\forall S \in \mathcal{L}\ (\emptyset \neq S \subseteq A)$ 2) $\forall x \in A\ \exists S \in \mathcal{L} : (x \in S)$ 3) $\forall S_1, S_2 \in \mathcal{L}\ (S_1 \cap S_2 \neq \emptyset \Rightarrow S_1 = S_2)$	Another way to say this is that $\mathcal{L}$ is a set of mutually disjoint nonempty subsets of A whose union is A
Definition 6.2.1	Equivalence Relations Conditions	Let A be a set and R be a relation on A Reflexivity: 1) R is reflexive if every element of A is R-related to itself $\forall x \in A\ (xRx)$ 2) R is symmetric if x is R-related to y implies y is R-related to x, for all $x, y \in A$ $\forall x, y \in A\ (xRy \Rightarrow yRx)$ 3) R is transitive if x is R-related to y and y is R-related to z implies x is R-related to z, for all $x, y, z \in A$ $\forall x, y, z \in A : (xRy \wedge yRz \Rightarrow xRz)$	 Figure 6.1: Reflexivity, symmetry, and transitivity
Definition 6.2.7	Divide Relation	Let $n \in \mathbb{Z}$, $d \in \mathbb{Z} \setminus \{0\}$ d is said to divide n if $n = dk\ for\ some\ k \in \mathbb{Z}$	Note that $d \neq 0$
Tutorial 5 Question 5	Modulo Relation	Let $k \in \mathbb{Z}^+$. Define the relation $\equiv_k$ on $\mathbb{Z}$ such that $\forall m, n \in \mathbb{Z}$ $m \equiv_k n \Leftrightarrow k\ divides\ m - n$ $\equiv_k$ is also known as the modulo relation The difference between m and n can be divided by k, they have the same ck + b where c is a multiple of k and b is an addition factor that will be minus off	
Tutorial 5 Question 5	Congruent Modulo	The equivalence classes of $\equiv_k$ are congruent modulo There will be k equivalent classes (k different b from above that we can have) and it goes from $[0], [1], \cdots, [k-1]$	
Exercise 6.2.10	Composition of Relations for Transitive	R is transitive $\Leftrightarrow R \circ R \subseteq R$	
Definition 6.2.11	Definition of Equivalence Relation	An equivalence relation is a relation that is: 1) Reflexive 2) Symmetric 3) Transitive	
Tutorial 5 Question 2	Composition of Inverse Relations	Let A and B be sets and R a relation from A to B $R^{-1} \circ R$ is Symmetric	
Proposition 6.2.14	Same Component Relation	Let $\mathcal{L}$ be a partition of a set A. Denote $\sim_{\mathcal{L}}$ the same-component relation with respect to $\mathcal{L}$. $\sim_{\mathcal{L}}$ is an equivalence relation on A $\forall x, y \in A : x \sim_{\mathcal{L}} y \Leftrightarrow x, y \in S\ for\ some\ S \in \mathcal{L}$	
Definition 6.3.1	Definition of Equivalence Classes	Let ~ be an equivalence relation on a set A For each $x \in A$, the equivalence class of x with respect to ~, denoted $[x]_\sim$ is defined to be the set of all elements of A that are ~-related to x $[x]_\sim = \{y \in A : x \sim y\}$	
Lemma 6.3.4	Properties of Equivalence Classes	Let ~ be an equivalence relation on a set A 1) $x \in [x]$ for all $x \in A$ 2) Any equivalence class is nonempty	
Tutorial 5 Question 7	Determine Equivalence Classes of an equivalence relation	We take an arbitrary value as the representative value: for instance (u, v) We put it into the equation and we try to make it in terms of the general terms	

		$[(u,v)]_{\mathcal{L}} = \{(x,y) \in \mathbb{R}^2 : (x,y)\ \mathcal{L}\ (u,v)\}$ $= \{(x,y) \in \mathbb{R}^2 : x - u = 3(y - v)\}$ $= \left\{(x,y) \in \mathbb{R}^2 : y = \frac{1}{3}x + \left(v - \frac{1}{3}u\right)\right\}.$ So $[(u,v)]_{\mathcal{L}}$ is the set of all points on the straight line passing through (u,v) with gradient $1/3$.  Else, if we can categorise them into specific groups that we can easily identify then we do not need to write it in the arbitrary manner	
Tutorial 5 Question 8	For equivalence relations, if they have a relation, their equivalence classes are the same	Let R be an equivalence relation on set X. For any $b, c \in X$ $b\ R\ c \Leftrightarrow [b]_R = [c]_R$	
Lemma 6.3.5	If the intersection is non empty, they are the same equivalence class	Let ~ be an equivalence relation on set A For all $x, y \in A$, if $[x] \cap [y] \neq \emptyset \Rightarrow [x] = [y]$	
Definition 6.3.7	Definition of Quotient	Let A be a set and ~ be an equivalence relation on A. Denote $A/\sim$ as the set of all equivalence classes with respect to ~, i.e. $A/\sim\ = \{[x]_\sim : x \in A\}$	
Theorem 6.3.10	Quotients are Partitions	Let ~ be an equivalence relation on a set A Then $A/\sim$ is a partition of A	
Tutorial 5 Question 7	Determining the Partition Induced by an equivalence relation	We will need to think of a way such that we can create the set so that all the different equivalence classes can be captured. Each equivalence class is a straight line, and it cuts the y-axis at some point, say $(0,c)$; we can use this point to represent the equivalence class. Therefore, the partition of $\mathbb{R}^2$ induced by $\mathcal{L}$ is $\Pi_{\mathcal{L}} = \{[(0,c)]_{\mathcal{L}} : c \in \mathbb{R}\}$. The fact that $\Pi_{\mathcal{L}}$ satisfies the definition of a partition translates to: (a) each line in $\Pi_{\mathcal{L}}$ is a nonempty set of points, (b) the union of all the lines in $\Pi_{\mathcal{L}}$ is $\mathbb{R}^2$, and (c) any two distinct lines in $\Pi_{\mathcal{L}}$ have empty intersection.	
Definition 6.4.1	Definition of Partial Orders/ Total Orders	Let A be a set and R be a relation on A <u>Anti-Symmetric:</u> 1) R is antisymmetric if (If we have a forward relation, we cannot have the backwards relation.) $\forall x, y \in A\ (xRy \wedge yRx \Rightarrow x = y)$ <u>Partial Orders:</u> 2) R is a (non-strict) partial order if R is reflexive, antisymmetric and transitive <u>Comparability:</u> 3) Suppose R is a partial order. Let $x, y \in A$. Then x, y are comparable (under R) if $x\ R\ y\ or\ y\ R\ x$ <u>Total Order:</u> 4) R is a (non-strict) total order or a (non-strict) linear order if R is partial order and every pair of elements is comparable $\forall x, y \in A\ (x\ R\ y \vee y\ R\ x)$	A total order is always a partial order
Definition 6.4.7	Smallest Element of a set A	Let R be a (non-strict) partial order on a set A. A smallest element of A (with respect to the partial order R) is an element $m \in A$ such that $m\ R\ x$ for all $x \in A$	
Theorem 6.4.9	Well-Ordering Principle	3 Conditions to fulfil: 1) $b \in \mathbb{Z}$ 2) $S \subseteq \mathbb{Z}_{\geq b}$ 3) $S \neq \emptyset$ Then S has a smallest element	
Tutorial 5 Question 9	Symmetric and Antisymmetric relations	a) A relation that is symmetric can also be antisymmetric The relation where it is only reflexive and has loops back to itself is both symmetric and antisymmetric b) A relation that is not symmetric does not need to be antisymmetric $R = \{(m,n) \in \mathbb{Z}^2 : m\ divides\ n\}\ OR$ $R = \{(1,2), (2,1), (1,3)\}$	

Functions

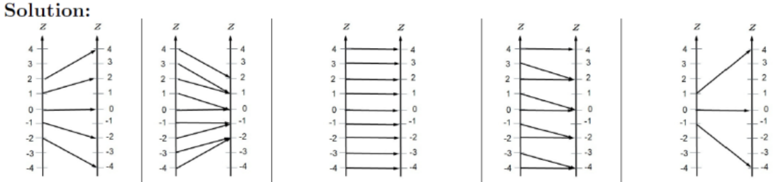
Definitions/ Proposition	What the result is saying	Formal Definition	Intuition/ Note
Definition 7.1.1	Definition of Functions	Let A, B be sets. A function or a map from A to B is a relation f from A to B such that any element of A is f-related to a unique element B (F1) Every element of A is f-related to at least one element of B: $\forall x \in A\, \exists y \in B\ (x, y) \in f$ (F2) Every element of A is f-related to at most one element of B: $\forall x \in A\, \forall y_1, y_2 \in B\ \big((x, y_1) \in f \land (x, y_2) \in f \Rightarrow y_1 = y_2\big)$	Use this to check if a function is well defined
Remark 7.1.2	Negation of above	(~F1) $\exists x \in A\, \forall y \in B\ (x, y) \notin f$ (~F2) $\exists x \in A\, \exists y_1, y_2 \in B\ \big((x, y_1) \in f \land (x, y_2) \in f \land y_1 \neq y_2\big)$	
Definition 7.2.1	Definition of Images/ Range	1) If $x \in A$, then $f(x)$ denotes the unique element $y \in B$ such that $(x, y) \in f$. $f(x)$ is the image of x under f 2) Range of f, denoted by $range(f)$: a. $range(f) = \{f(x): x \in A\}$ b. Set of all images that are produced under the function	
Remark 7.2.2		$(x, y) \in f \Leftrightarrow y = f(x)$	Common remark that is used when we define the images
Definition 7.2.4	Definition of Boolean Functions	A Boolean function is: A function $\{T, F\}^n \to \{T, F\}$ where $n \in \mathbb{Z}^+$	
Proposition 7.2.6	Way to show that 2 functions are equal	Let $f, g: A \to B$. Then $f = g$ if and only if $f(x) = g(x)$ for all $x \in A$	
Proposition 7.3.1	Composition of Functions	Let $f: A \to B$ and $g: B \to C$ Then $g \circ f$ is a function $A \to C$. Moreover, for every $x \in A$: $(g \circ f)(x) = g\big(f(x)\big)$	Note that the codomain of f must be the same as the domain of g
Definition 7.3.3	Definition of Identity Functions	Let A be a set. Then the identity function on A, denoted by id_A: Function $A \to A$ which satisfies for all $x \in A$: $id_A(x) = x$	
Definition 7.4.1	Definition of Surjectivity/ Injectivity/ Bijectivity	Let $f: A \to B$ 1) f is surjective or onto if: $(F^{-1}\ 1)$ $\forall y \in B\, \exists x \in A : y = f(x)$ A surjection is a surjective function For every element in the codomain, there is an element mapping to it 2) f is injective or $one - to - one$ if: $(F^{-1}\ 2)$ $\forall x_1, x_2 \in A\ \big(f(x_1) = f(x_2) \Rightarrow x_1, x_2\big)$ An injection is an injective function For every element in the domain, there is only one element mapping to it 3) f is bijective if it is both surjective and injective. A bijection is a bijective function	 Figure 7.1: Surjectivity (left) and injectivity (right)
Proposition 7.4.3	Inverse of a bijective function is also bijective	If f is a bijection $A \to B$ then f^{-1} is a bijection $B \to A$	
Remark 7.4.5	Negation of Surjective Condition	A function $f: A \to B$ is not surjective if and only if: $\exists y \in B\, \forall x : y \neq f(x)$ We have that there is a y in B such that we cannot find an x to put into the function such that y is an image of x through f	
Remark 7.4.8	Negation of Injective Condition	A function $f: A \to B$ is not injective if and only if $\exists x_1, x_2 \in A : \big(f(x_1) = f(x_2) \land x_1 \neq x_2\big)$ There are 2 elements in the domain such that both of them have the same image under f	
Tutorial 7 Question 1	Requirement for Composite Function to be injective	Let $f: X \to Y$ and $g: Y \to Z$ Suppose $g \circ f$ is injective: 1) f needs to be injective 2) g need not be injective	
Tutorial 7 Question 1	Requirement for Composite Function to be Surjective	Let $f: X \to Y$ and $g: Y \to Z$ Suppose $g \circ f$ is surjective: 1) g needs to be surjective	

		2) f need not be surjective	
Proposition 7.4.11	Definition of Inverse Function	Let $f: A \to B$ and $g: B \to A$ $$g = f^{-1} \Leftrightarrow \forall x \in A\ \forall y \in B\ \big(g(y) = x \Leftrightarrow y = f(x)\big)$$	When we can rewrite it in terms of y as the subject and a function of x
Tutorial 7 Question 2	Alternative way to prove inverse functions identity and bijective	Consider a function $f: X \to Y$ and $g: Y \to X$ Such that: 1) $g \circ f = id_X$ 2) $f \circ g = id_Y$ Then: 1) f is bijective 2) $g = f^{-1}$	
Tutorial 7 Question 3	Involution Functions	Let X be a nonempty set and $f: X \to X$ such that $f \circ f = id_X$ 1) f is bijective 2) $f^{-1} = f$	
Proposition 7.4.14	Composition of a function and its inverse	Let f be a bijection $A \to B$. Then 1) $f^{-1} \circ f = id_A$ 2) $f \circ f^{-1} = id_B$	Think of which one we are going from first and we will go back to the original one
Tutorial 6 Question 8	Properties of Identity Function	Let X and Y be sets and let $f: X \to Y$ and $g: Y \to X$ such that $g \circ f = id_X$ If f is surjective then g is injective	

Useful Functions for Examples on Surjectivity and Injectivity:

$f(n) = 2n, \quad g(n) = \left\lfloor \frac{n}{2} \right\rfloor \ \forall n \in \mathbb{Z}$

Solution:



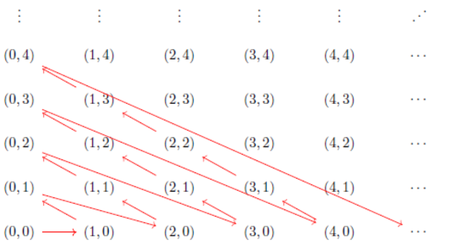
f	g	$g \circ f$	$f \circ g$	$f \circ f$
inj.	not inj.	inj.	not inj.	inj.
$f(n) = f(k)$ $\Rightarrow 2n = 2k$ $\Rightarrow n = k$	$g(0) = 0$ $= g(1)$	$(g \circ f)(n) = (g \circ f)(k)$ $\Rightarrow g(2n) = g(2k)$ $\Rightarrow \lfloor \frac{2n}{2} \rfloor = \lfloor \frac{2k}{2} \rfloor$ $\Rightarrow n = k$	$(f \circ g)(0) = 0$ $= (f \circ g)(1)$	$(f \circ f)(n) = (f \circ f)(k)$ $\Rightarrow f(2n) = f(2k)$ $\Rightarrow 4n = 4k$ $\Rightarrow n = k$
not surj.	surj.	surj.	not surj.	not surj.
$f(n) \neq 1$ for any $n \in \mathbb{Z}$	Given $y \in \mathbb{Z}$, let $x = 2y$; then $g(x) = \lfloor \frac{x}{2} \rfloor$ $= \lfloor y \rfloor$ $= y$	Given $y \in \mathbb{Z}$, let $x = y$; then $(g \circ f)(x) = g(f(x))$ $= g(2x) = \lfloor \frac{2x}{2} \rfloor = x$	$(f \circ g)(n) \neq 1$ for any $n \in \mathbb{Z}$	$(f \circ f)(n) \neq 1$ for any $n \in \mathbb{Z}$

$\text{range}(f) = \{2n : n \in \mathbb{Z}\}, \quad \text{range}(g) = \mathbb{Z}, \quad \text{range}(g \circ f) = \mathbb{Z},$

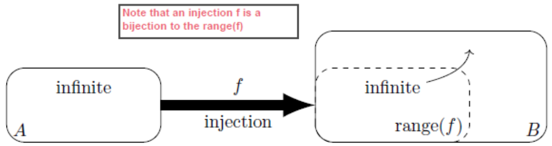
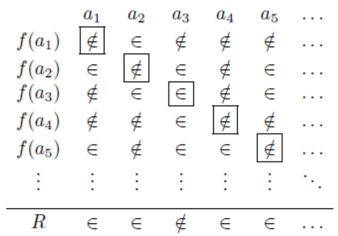
$\text{range}(f \circ g) = \{2n : n \in \mathbb{Z}\}, \quad \text{range}(f \circ f) = \{4n : n \in \mathbb{Z}\}.$

Cardinality

Definitions/ Proposition	What the result is saying	Formal Definition	Intuition/ Note
Proposition 8.1.1	(1) The composition maintains the relationships	(2) If f and g are surjective, so is $g \circ f$ (3) If f and g are injective, so is $g \circ f$ (4) If f and g are bijective, so is $g \circ f$, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$	
Theorem 8.1.2 (Pigeonhole Principle)	Upper bound on number of elements in A if there is an injection	Let $A = \{x_1, \dots, x_n\}$ and $B = \{y_1, \dots, y_n\}$ where $n, m \in \mathbb{Z}_{\geq 0}$. Where the x's are different, and y's are different If there is an injection $A \to B$, $n \leq m$	If we have an injection, it means that the set is one-to-one. Where each element in the domain is mapped to specifically one element in the codomain. However, there could be elements that are not mapped in the codomain. Therefore, $n \leq m$
Theorem 8.1.3 (Dual Pigeonhole Principle)	Lower bound on number of elements in A if there is a surjection	Let $A = \{x_1, \dots, x_n\}$ and $B = \{y_1, \dots, y_n\}$ where $n, m \in \mathbb{Z}_{\geq 0}$. Where the x's are different, and y's are different If there is a surjection $A \to B$, $n \geq m$	If we have a surjection, it means that the set is onto. It means that for all the elements in the codomain, we have at least one element being mapped to it. However, we can have more than 1 element mapped to each of them. Therefore, $n \geq m$
Theorem 8.1.4	Number of elements in A if there is a bijection	Let $A = \{x_1, \dots, x_n\}$ and $B = \{y_1, \dots, y_n\}$ where $n, m \in \mathbb{Z}_{\geq 0}$. Where the x's are different, and y's are different $n = m$ if and only if there is a bijection $A \to B$	If we have a bijection, it means that every element in the domain and codomain has exactly one other element pairing. Therefore, if we can have that match, they will have the same number of elements
Definition 8.2.1	If there is a bijection, the 2 sets have the same cardinality	(Cantor) A set A is said to have the same cardinality as a set B if there is a bijection $A \to B$	
Proposition 8.2.3	Cardinality is an equivalence relation	Let A, B, C be sets (1) A has the same cardinality as A (Reflexivity) (2) If A has the same cardinality as B, then B has the same cardinality as A (Symmetry) (3) If A has the same cardinality as B, and B has the same cardinality as C, then A has the same cardinality as C (Transitivity)	
Tutorial 7 Question 5	Proving Equivalence Classes have the same Cardinality	Find a function that is bijective from one equivalence class to another 1) Prove that it is Well-Defined 2) Prove the function is bijective	
Definition 8.2.4	Definition of finite and infinite sets	A set A is finite if it has the same cardinality as $\{1,2 \dots, n\}$ for some $n \in \mathbb{Z}_{\geq 0}$. n is the cardinality or size of set A, denoted by |A| A set is *infinite* if it is not finite	<u>To prove infinite sets:</u> Show that no function that can map $\{1, 2, \dots, n\} \to A$ (No bijection)
Definition 8.3.1	Definition of countability. Note that this just adds in the infinite set of $\mathbb{Z}^+$	A set is countable if it is finite or it has the same cardinality as $\mathbb{Z}^+$ A set is *uncountable* if it is not countable	To prove finite sets if they are countable: If a set is countable, show that there is no bijection to $\mathbb{Z}^+$
Tutorial 7 Question 9	Surjection from $\mathbb{Z}^+$ can be used to prove countability	X is countable if and only if: 1) It is empty 2) There is a surjection $\mathbb{Z}^+ \to X$	Surjection is whereby we have a set that can be mapped onto another. By Dual PHP, since there is a surjection, $\mathbb{Z}^+$ has more elements than X which means that it is countable
Tutorial 8	Injection from X to $\mathbb{Z}^+$ can be used to prove countability	X is countable if and only if: 1) There is an injection $X \to \mathbb{Z}^+$	Injection is whereby each of the elements in the latter set can only have one element coming from it. By PHP, we know that the latter set has more than or equal number of elements than the former. Therefore, $\mathbb{Z}^+$ has more elements than X which means that it is countable
Tutorial 8 Question 2	Union of countable sets is countable	Let $A_0, A_1, A_2 \cdots$ be countable sets $$\cup_{i=1}^{\infty} A_i$$ Is countable	
Proposition 8.3.4	Existence of countable infinite sets under an infinite set	Every infinite set B has a countable infinite subset	
Proposition 8.3.5	Any subset of a countable set is countable	Any subset A of a countable set B is countable	
Definition 8.4.1	Definition of odd and even numbers	An integer is even if it is $2x$ for some $x \in \mathbb{Z}$ An integer is odd if it is $2x + 1$ for some $x \in \mathbb{Z}$	
Fact 8.4.2		Any integer is either even or odd, but not both	
Proposition 8.4.3		$\mathbb{Z}$ is countable	We have a bijection from $\mathbb{Z} \to \mathbb{Z}^+$. Set the positive numbers from $\mathbb{Z}$ to the even numbers in $\mathbb{Z}^+$ and negative numbers from $\mathbb{Z}$ to the odd numbers in $\mathbb{Z}^+$
Theorem 8.4.4	Cartesian Product of set of Positive Integers is countable	(Cantor 1877) $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is countable	We can lay out the elements in a grid and count it in a zig zag manner.

			
Proposition 8.4.5	All possible strings of 0 and 1 is countable	$\{0, 1\}^*$ is countable	
Corollary 8.4.6		The set of all computer programs is computable	This follows from {0, 1}* being countable since all computer programs can be represented by that
Tutorial 8 Question 3	$\mathbb{Q}$ is countable	$\mathbb{Q} = \left\{r \in \mathbb{R}: \exists m \in \mathbb{Z}\ \exists n \in \mathbb{Z}^+\ r = \frac{m}{n}\right\}$	Make use of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$
Tutorial 8 Question 4	Cross Products of Countable and Infinite sets are countable	Let Y and Z be sets that are countable and infinite $Y \times Z$ is countable	Make use of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$

Diagonalization

Definitions/ Proposition	What the result is saying	Formal Definition	Intuition/ Note
Example 9.1.1		$\mathbb{Z}^+$ is infinite. There is no function $\mathbb{Z}^+ \to \{1, 2, \ldots, n\}$, where $n \in \mathbb{Z}_{\geq 0}$, can be injective	
Lemma 9.1.2	Properties if they have the same cardinality	Let A and B be sets of the same cardinality (1) A is *finite* if and only if B is *finite* (2) A is *countable* if and only if B is *countable*	
Tutorial 8 Question 5	Interactions between countable and uncountable sets	(a) If a set X has an uncountable subset, then X is also uncountable (b) If A is uncountable and B is countable, then $A \backslash B$ is uncountable	(a) Make use of the contrapositive of Proposition 8.3.5 (b) Make use of $A = (A \backslash B) \cup (A \cap B)$ and $(A \cap B) \subseteq B$
Proposition 9.1.3	Properties of subset of finite sets	Any subset A of a finite set B is finite	Can be used to prove that if the subset of a set is infinite then the set is infinite. Because if the set is finite, then any subset should be finite as well (Corollary 9.1.4)
Tutorial 8 Question 6	If X is finite, then X has countably many subsets		
Corollary 9.1.4	Injecting an infinite set into another set makes the latter infinite as well	A set B is infinite if there is an injection *f* from some infinite set A to B	Make use of the bijection from range(f). An injective function is also a bijection to its range 
Theorem 9.2.1	Cardinality of Power Sets	(Cantor 1891) No set A has the same cardinality as $\mathcal{P}(A)$	Useful for finding uncountable sets Make use of the diagonalization argument  Figure 9.2: Illustration of Cantor's diagonal argument
Corollary 9.2.2	Power set of countable infinite set is uncountable	Let A be a countable infinite set. Then $\mathcal{P}(A)$ is uncountable	Making use of Theorem 9.2.1 and proving that $\mathcal{P}(A)$ is infinite and does not have the same cardinality as $\mathbb{Z}^+$
Corollary 9.2.3	Power set of all possible strings of 0, 1 is uncountable	$\mathcal{P}(\{0,1\}^*)$ is uncountable	
Corollary 9.2.4		The set $\mathcal{S}$ of all functions $\{0,1\}^* \to \{0,1\}$ has the same cardinality as $\mathcal{P}(\{0,1\}^*)$ Consequently, this $\mathcal{S}$ is uncountable	$\mathcal{S}$ is the set of all functions that maps each of the strings to 0, 1. We can view the mapping as whether we include it in the subset of {0,1}*. Therefore, it will cover all the possible subsets
Corollary 9.3.1	Injection from an uncountable set makes the latter set uncountable as well	A set B is uncountable if there is an injection f from some uncountable set A to B	Similar argument to Corollary 9.1.4 & use Lemma 9.1.2(2)
Corollary 9.3.3	There exists a function that maps from {0, 1}* to {0,1} that no program can compute	There is a function $\{0,1\}^* \to \{0,1\}$ that cannot be computed by any program	Note that every program can only compute 1 function and therefore since we have uncountably many functions and countably many programs, it must mean that there are some functions that cannot be computed
Theorem 9.3.4	Proof for uncomputable functions	(Turing 1936). There is no program that computes the function $h: \{0,1\}^* \to \{0,1\}$ satisfying $h(\sigma) = \begin{cases} 1, & \text{if } \sigma \text{ is a program that does not stop on the empty input} \\ 0, & \text{otherwise} \end{cases}$ for all $\sigma \in \{0,1\}^*$	
Tutorial 8 Question 8	Finite subset of an infinite set	If B is a finite subset of an infinite set C There are uncountably many countable sets X such that $B \subseteq X \subseteq C$	Take a countable infinite subset D of C, then add B to the subsets of D

Combinations

Definitions/ Proposition	What the result is saying	Formal Definition	Intuition/ Note
Addition Rule		If A and B are finite sets that are disjoint (i.e. $A \cap B = \emptyset$) , then $A \cup B$ is finite and $\lvert A \cup B\rvert = \lvert A\rvert + \lvert B\rvert$	When 2 sets are disjoint and finite, they will not have any overlapping elements, therefore, the total number of elements in the union is the sum of the individual number of elements
Lemma 3.9	(i) Union of finite sets is finite (ii) Number of elements in the difference if a set is a subset of the other	Let Y be a finite set (i) If X is finite, then $X \cup Y$ is finite (ii) If $X \subseteq Y$, then $\lvert Y - X\rvert = \lvert Y\rvert - \lvert X\rvert$	
Useful Note 1		Given 2 sets X and Y $$Y = (Y \backslash X) \cup (Y \cap X)$$ $Y \backslash X$ and $Y \cap X$ are disjoint	Useful when we want to use arguments for infinite or finite sets
Useful Note 2		Given 2 sets X and Y $$X \cup Y = (X \backslash Y) \cup Y$$ $(X \backslash Y)$ *and* Y are disjoint	Useful when we want to use arguments for infinite or finite sets
Theorem 3.10	Union of any number of finite sets is finite	Suppose $A_1, A_2, \ldots$ are finite sets. Then $A_1 \cup A_2 \cup \ldots \cup A_n$ is finite for any $n \geq 2$	
Theorem 3.11	Inclusion Exclusion Rule	Let A, B, C be finite sets $$\lvert A \cup B\rvert = \lvert A\rvert + \lvert B\rvert - \lvert A \cap B\rvert$$ $$\lvert A \cup B \cup C\rvert = \lvert A\rvert + \lvert B\rvert + \lvert C\rvert - \lvert A \cap B\rvert - \lvert B \cap C\rvert - \lvert A \cap C\rvert + \lvert A \cap B \cap C\rvert$$	Make use of disjoint and addition rule
Multiplication Rule	Number of possible combinations for x,y pairs	Consider the 2-tuple (x, y) Suppose that there are m possible values for x and, For each choice of x, there are n possible choices for y There are mn possible choices for (x, y) $$\lvert A_1 \times A_2\rvert = \lvert A_1\rvert\lvert A_2\rvert$$	
Theorem 3.12	Extension of Multiplication Rule to n sets	Suppose $A_1, A_2, \ldots$ are finite sets Then $\lvert A_1 \times A_2 \times \ldots \times A_n\rvert = \lvert A_1\rvert\lvert A_2\rvert \cdots \lvert A_n\rvert$ for any $n \geq 2$	
Theorem 3.13		Let $x \in \mathbb{R}, x \neq 1$. Then $1 + x + x^2 + \cdots + x^n = \frac{x^{n+1}-1}{x-1}$ for any $n \in \mathbb{Z}^+$ If x = 1, $1 + x + x^2 + \cdots + x^n = n + 1$	Make use of sum of n terms for Geometric Series
Corollary 3.14	Number of strings with length smaller than ℓ	Suppose Γ is an alphabet and $\lvert\Gamma\rvert = s, s > 1$. If $\ell \in \mathbb{Z}^+$, then there are $\frac{s^{\ell}-1}{s-1}$ over Γ with length smaller than ℓ	
Definition	Definition for Factorial	For $n \in \mathbb{N}$, n factorial is $$n! = \begin{cases} 1, & if\ n = 0 \\ n(n-1)!, & if\ n \geq 1 \end{cases}$$	Recursive definition
Definition	Definition of Permutation	Let $n \in \mathbb{Z}^+$ and $S = \{x_1, \ldots, x_n\}$ A bijection $f: \{1, \ldots, n\} \to \{1, \ldots, n\}$ is called a permutation and the string $x_{f(1)} \cdots x_{f(n)}$ is also called a permutation of S. A bijection $\emptyset: \emptyset \to \emptyset$ is also a permutation, the empty string is a permutation of $\emptyset$	When we can give each of the elements an index to rearrange their position, each of the ways is 1 permutations
Theorem 3.15	Total number of permutations	Let S be a set with n elements, $n \in \mathbb{N}$ Then there are $n!$ permutations of S If there are duplicate values with $k_1, k_2, \ldots, k_n$ of the duplicates: There are $\frac{n!}{k_1 \times k_2 \times \cdots \times k_n}$ total permutations	This is the total number of orderings in which we can order n elements. We can think of it as having n choices for first position n-1 choices for second and so on till the last
Definition	Definition for combinations	Let $r, n \in \mathbb{N}$ and let S be a set of n elements A subset of r elements is called an $r - combination$ of S Number of ways to choose r elements from n elements $\binom{n}{r}$ denotes the number of r-combinations of S $$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$ Number of Subsets of a Power Set: $\lvert\mathcal{P}(S)\rvert = 2^{\lvert S\rvert}$	Intuition for number of subsets of a Powerset: Order all elements as 0 and 1. If we want to include in the subset, indicate 1, 0 otherwise. We have 2 choices for each element and we have n elements

Definition	Definition for r-permutations	Let $r, n \in \mathbb{Z}^+, r \leq n$ and let $f: \{1, \cdots, n\} \rightarrow \{1, \cdots, n\}$ be a permutation If $S = \{x_1, \cdots, x_n\}$ then $x_{f(1)} \cdots x_{f(r)}$ is an r-permutation for S. Number of ways to choose r elements that are ordered nP_r is the number of r-permutations of S $$nP_r = \binom{n}{r} r! = \frac{n!}{(n-r)!}$$	Think of it that we have n choices for the first, n-1 choices for the second and n – (r-1) for the rth index and we multiply them together Note that this is not really useful when we have repetitions. If there are then we need to count individually
Tutorial 9 Question 1		For any $n, r \in \mathbb{Z}^+$ and $1 \leq r \leq n$ $$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$$	
Tutorial 9 Question 2		For $n \in \mathbb{Z}^+$ $$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$ $$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^r \binom{n}{r} + \cdots + (-1)^n \binom{n}{n} = 0$$	Use Binomial Theorem
Tutorial 9 Question 3	Binomial Theorem	For any $x, y \in \mathbb{R}$ and $n \in \mathbb{Z}^+$ $$(x+y)^n = \binom{n}{0} x^n y^0 + \cdots + \binom{n}{r} x^{n-r} y^r + \cdots + \binom{n}{n} x^0 y^n$$	
Tutorial 9 Question 4		For any $n, r \in \mathbb{N}$ $$\binom{0}{r} + \binom{1}{r} + \binom{2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$$	
Tutorial 9 Question 5	Vandermonde's Identity	$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0}$$	
Tutorial 9 Question 9	Counting for Types of Relations	Suppose A and B are nonempty finite sets, $\|A\| = n$ and $\|B\| = k$ <u>Number of Relations from $A \rightarrow B$:</u> $$2^{\|A\|\|B\|} = 2^{nk}$$ <u>Number of Functions from A to B:</u> $$\|B\|^{\|A\|} = k^n$$ <u>Number of Boolean Functions for m variables:</u> $A = \{T, F\}^m$ and $B = \{T, F\}$ $$\|A\| = 2^m, \|B\| = 2 \Rightarrow \|B\|^{\|A\|} = 2^{2^m}\ Boolean\ Functions$$ <u>Number of Injective Functions:</u> Assuming that k > n $$k(k-1)\cdots(k-n+1) = kP_n$$ <u>Number of Surjective Functions:</u> $$T(n,k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$ <u>Number of Bijective Functions:</u> Assuming that k = n $$Total = \begin{cases} n!, & if\ n = k \\ 0, & if\ n \neq k \end{cases}$$ <u>Number of Functions that are not injective and not surjective:</u> $$\begin{aligned} &\#(\sim injective \wedge \sim surjective) \\ &= \#functions - (\#injective + \#surjective - \#bijective) \end{aligned}$$	
Tutorial 10 Question 1	Counting Types of Binary Relations	Suppose A is a set with n elements. $\|A\| = n$ Each node has 4 choices $(\rightarrow, \leftarrow, \rightleftarrows, no\ edge)$ <u>Number of Binary Relations on A:</u> $$2^{n^2}$$	

		<u>Number of Reflexive Relations:</u> $1^n 4^{\binom{n}{2}}$ <u>Number of Symmetric Relations:</u> $2^{n+\binom{n}{2}} = 2^{\left(\frac{n(n+1)}{2}\right)}$ <u>Number of Antisymmetric Relations:</u> $2^n 3^{\binom{n}{2}}$ <u>Number of Antisymmetric and Symmetric:</u> There are no $x \rightarrow y$ edges, only possible that they have loops or not $2^n 1^{\frac{n}{2}}$ <u>Not Reflexive and not Symmetric:</u> (number of relations – number of reflexive or symmetric relations) = (number of relations – ((Number of reflexive relations) + (Number of symmetric relations) – (Number of reflexive and symmetric relations)) $= 2^{n^2} - 4^{\binom{n}{2}} + 2^{\frac{n(n+1)}{2}} - 2^{\binom{n}{2}}$	

Graphs

Definitions/ Proposition	What the result is saying	Formal Definition	Intuition/ Note
Tutorial 10	Number of edges for complete graph	Total Number of Edges between pairs of nodes $= \binom{n}{2} = \frac{n(n-1)}{2}$ Total number of loops $= n = \binom{n}{1}$ Total number of edges with loops $= n\binom{n}{2}$	
Definition	Definition of Finite graphs	Consider an undirected graph $G = (V, E)$ V – Set of all the vertices E – Set of edges where each of the connecting edges is represented by {a, b} where the edge connects vertices a and b G is **trivial** if and only if $\lvert V\rvert = 1$ G is **finite** if V is finite G is **infinite** if V is infinite	
Definition	Definition of Complete Graph	A complete graph for n nodes, denoted K_n is: 1) Undirected graph 2) With an edge between every pair of nodes 3) And a loop at each node	
Tutorial 10 Question 6	Definition of Complement of Graphs	Let $G = (V, E)$ be a loopless undirected graph The complement of G is the loopless graph $\bar{G} = (V, F)$ where $\{u, v\} \in F$ if and only if $\{u, v\} \notin E$	
Tutorial 10 Question 6	Relationship between Complement of Graphs and Connected Graphs	For any G, G and $\bar{G}$ cannot both be unconnected	
Definition	Definition of Subgraphs	Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be undirected graphs H is a **subgraph** of G (or G contains H) if and only if: 1) The set of vertices is a subset of the original graph: $V_H \subseteq V_G$; and 2) The set of edges is a subset of the original graph:$E_H \subset E_G$ H is a **proper subgraph** of G if and only if 1) H is subgraph of G; and 2) $H \neq G$	
Tutorial 10	Number of Subgraphs of complete graphs	<u>Number of Subgraphs K_n have for $n \geq 3$:</u> $$\lvert A\rvert = \sum_{i=1}^{n} \lvert A_i\rvert = 2n + \sum_{k=2}^{n} \binom{n}{i} 2^{\binom{i}{1}+\binom{i}{2}}.$$ <u>Number of Proper subgraphs of K_n:</u> Note that there is only one graph which is not a proper subgraph $$(2n-1) + \sum_{k=2}^{n} \binom{n}{i} 2^{\binom{i}{1}+\binom{i}{2}}$$	
Definition	Definition of Paths	Let $G = (V, E)$ be an undirected graph and $p \geq 2$ A subgraph of the form $\left(\{x_1, \cdots, x_p\}, \left\{\{x_1, x_2\}, \{x_2, x_3\}, \cdots, \{x_{\{p-1\}}, x_{\{p\}}\}\right\}\right)$ is called a **path** between x_1 and x_p in G; A **path** is when there are edges connecting from 1 node to another This path has **length** $p - 1$ Note: 1) No repetition of nodes along a path 2) Repetition makes the path ambiguous 3) Paths have no loops 4) $p \geq 2 \Rightarrow$ Paths must have at least one edge	
Definition	Definition of Isomorphic Graphs	Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be finite loopless undirected graphs $G \simeq H$ $(Isopmorhic)$ if and only if:	

		1) There is a permutation $\pi: V_G \to V_H$ such that $\{u,v\} \in E_G \leftrightarrow \{\pi(u), \pi(v)\} \in E_H$ 2) There is a bijection between their vertex sets 3) When the bijection maps adjacent vertices to adjacent vertices <u>To show that 2 graphs are not isomorphic:</u> 1) Graphs have different no. of vertices/edges 2) Vertices of graphs have different orders 3) Graphs have different connectivity properties	
19/20 Sem 2 Question 3 (X)	Number of Isomorphic Graphs	Choose a specific shape first and then try to choose the various edges	
Definition	Definition of Connected Graphs	An undirected graph is connected if and only if: 1) It is trivial (1 Vertex); OR 2) There is a path between any two distinct nodes	
Tutorial 10 Question 5	Connection for the number of edges and connected graphs	If a loopless undirected graph has 1) n vertices, where $n \geq 2$ 2) And more than $\binom{n-1}{2}$ edges Then it is connected	
Definition	Binary Relation for transitive closure	Let R be a binary relation on a nonempty set A. For $n \in \mathbb{Z}^+$ $$R_n = \begin{cases} R, & if\ n = 1 \\ R \circ R_{n-1}, & if\ n \geq 2 \end{cases}$$	
Theorem 4.1	Paths under transitive closure	Let $G = (V, E)$ be an undirected graph with $\lvert V\rvert \geq 2$, and $R = \{(b,c) \in V \times V \mid b \neq c\ and\ \{b,c\} \in E\}$ Consider two different nodes x and y in G, and $n \in \mathbb{Z}^+$ (i) If there a path length n between x and y, then $(x,y) \in R_n$ (ii) If $(x,y) \in R_n$, then there is a path of length at most n between x and y	(i) If there is a path of length n, then it must mean that if we compose the graph n times, we will have an edge between x and y since we are taking away the first few edges each time (ii) If it is in the transitive closure of n, it means that there is a path of length n but there could be a shorter path within it
Definition	Definition of Transitive Closure	Let R be a binary relation on a set A. The transitive closure of R is $R_+ = \cup_{n=1}^{\infty} R_n$	If they under the transitive closure, they are in the same equivalence class
Tutorial 10 Question 7	Transitivity of a Graph and the Transitive Closure	Let R be a binary relation on a set R is transitive if and only if $R_+ \subseteq R$	
Corollary 4.2	Definition for connected graphs under transitive closure	Let $G = (V, E)$ be an undirected graph with $\lvert V\rvert \geq 2$, and $R = \{(b,c) \in V \times V \mid b \neq c\ and\ \{b,c\} \in E\}$ Then G is connected if and only if 1) $(x,y) \in R_+$ for any $x, y \in V$ such that $x \neq y$.	There should be an edge between any 2 nodes in the transitive closure of the graph for the graph to be connected
Definition	Definition of Cycle	Let $n \in \mathbb{Z}$, $n \geq 3$. A **cycle** is an undirected graph of the form: $(\{x_1, x_2, \dots, x_n\}, \{\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{n-1}, x_n\}, \{x_n, x_1\}\})$ There should be an edge coming back to the first node and it will complete a cycle	
Definition	Definition of Cyclic/ Acyclic Graphs	An undirected graph is **cyclic** if: 1) It contains a loop 2) Or contains a cycle Otherwise, the graph is **acyclic**	
Definition	Definition of Cycles for Directed Graphs	For $n \geq 2$, a directed graph $(\{v_1, v_2, \cdots, v_n\}, \{(v_1, v_2), \cdots, (v_{n-1}, v_n), (v_n, v_1)\}$ is called a cycle	
Definition	Definition of Cyclic/ Acyclic for Directed Graphs	A directed graph $G = (V, D)$ is cyclic if it: 1) Contains a loop 2) Or Contains a cycle as a subgraph Otherwise, the graph is acyclic	
Tutorial 11 Question 9	Relationship between acyclic directed graphs and its edges	Suppose a directed graph $G = (V, D)$ is acyclic Then D is antisymmetric	Note that the converse is False (i.e. If D is antisymmetric, it does not imply that the directed graph is acyclic)
Tutorial 11 Question 9	Relationship between Partial Order and Cycles of directed graphs	Suppose a directed graph $G = (V, D)$ If D is a partial order, then G does not contain any cycles	We can use this to arrange the edges in the graph for a partial order, so they all point in one direction
Tutorial 11 Question 9	Existence of graph with n nodes and $\frac{1}{2}n(n-1) = \binom{n}{2}$ edges	For any $n \geq 2$,, there is a directed acyclic graph with: 1) n nodes; 2) And $\frac{1}{2}n(n-1)$ edges	
Tutorial 11 Question 9	Connection between Cyclic and number of edges for directed graphs	For any directed graph with: 1) n nodes 2) More than $\frac{1}{2}n(n-1)$ edges	Consider 2 cases where G has a loop and has no loops. For loop case, it is already cyclic, for the no loop case, make use of that fact that $\frac{1}{2}n(n-1) = \binom{n}{2}$ which is the different choices for

		The graph must be cyclic	the pairings of {x, y} and with PHP conclude that it must be cyclic to have more than $\frac{1}{2}n(n-1)$ edges
Theorem 4.3	Cyclic graphs if they are connected undirected with no loops	Let G be a connected undirected graph with no loops G is cyclic if and only if: 1) There are 2 distinct nodes with more than one path between them	
Definition	Definition of Connected Components	Let G be an undirected graph and H a connected subgraph of G H is a connected component of G if: 1) H is a connected subgraph of G 2) G does not contain another subgraph H' such that H is a proper subgraph of H' a. Vertices of H is not a subset of H'; **AND** b. Edges of H is not a subset of H'	
Tutorial 10	Connection between Disconnected graphs and connected components	If a graph is disconnected, it can be written as a union of its connected components $G = G_1 \cup \cdots \cup G_k$	
Tutorial 10 Question 8	Connection between Equivalence Relation, Connected Component and Complete Graph	Let R be an equivalence relation on a nonempty A, and let G be the undirected graph representing R Every connected component of G is a complete graph	
Tutorial 10 Question 9	Connection between Partitions and Connected Components	Consider an undirected graph G, whose connected components $H_1, \cdots, H_k$, where $k \geq 2$ Suppose $G = (V, E)$ and $H_1 = (V_1, E_1), \cdots, H_k = (V_k, E_k)$ $(V_1, \cdots, V_k)$ is a partition of V	Note that $\{E_1, E_2, \cdots, E_k\}$ is not a partition of E because E_i could be empty
Theorem 4.4	Relationship between Connected Components and Paths	Let x and y be distinct nodes in an undirected graph G Then there is a path in G between x and y if and only if: 1) x and y are in the same connected component of G	
Corollary 4.5	Relationship between Connected Components and Equivalence Classes	Let A be a non empty set and R an equivalence relation on A Let G be the undirected graph representing R, and suppose $x\ and\ y$ are different nodes in G Then x and y are in the same equivalence class under R if and only if: 1) x and y are in the same connected component in G	The partition of A induced by the equivalence relation R consists of connected components in the graph representing R The proof works for infinite graphs as well
Definition	Definition of Tree	A tree is a: 1) Connected 2) Acyclic 3) Undirected Graph	
Tutorial 11 Question 2	Number of Trees with n nodes	<u>General Formula for trees with n nodes:</u> For n = 1, 1 For $n \geq 2, n^{n-2}$	Note that labelling non-isomorphic trees gives us trees
Definition	Definition of Forest	A forest is: 1) Acyclic Graph Note that each of the connected components are trees themselves	
Theorem 4.6 (Tree Theorem)	Properties of Trees	Let $G = (V, E)$ be a finite connected undirected graph Then the following are equivalent: 1) G is a Tree 2) Removing any edge disconnected G 3) $\|E\| = \|V\| - 1$	Note that the graph needs to be connected or else the statements will not be equivalent to each other. Even if one of them holds, the others may not hold
Tutorial 11 Question 4	Lower bound for number of edges for Connected Graphs	Let $G = (V, E)$ be an undirected graph If G is connected, then $\|E\| \geq \|V\| - 1$	Make use of the fact that it is connected and Theorem 4.7 that states that it has a spanning tree to use the properties of a tree Note the converse is not true
Tutorial 11 Question 5	Upper bound for number of edges for acyclic graphs	Let $G = (V, E)$ be an undirected graph If G is acyclic, then $\|E\| \leq \|V\| - 1$	Try to make use of tree property. The only thing missing is the connected property, we can write a disconnected graph as a union of its connected components. The connected components are trees themselves. Note the converse is not true
Tutorial 11 Question 6	Trees if we only have loopless undirected graph	Loopless undirected graph is a tree if and only if there is exactly one path between every path of nodes	
Definition	Definition of Spanning Trees	Let G be an undirected graph A spanning tree is: 1) A subgraph of G; and	

		2) Is a tree; and 3) Contains all nodes of G	
Theorem 4.7	Existence of a spanning tree in any connected undirected graph	Every finite connected undirected graph has a spanning tree	
Tutorial 11 Question 7	Number of spanning tress for a complete graph	Cayley's Formula $= n^{n-2}$	
Definition	Definition of Rooted Trees	A rooted tree is: 1) A tree; and 2) Has a distinguished node called the root	
Definition	Definitions of Terms under a rooted tree	Let r be the root of a rooted tree T <u>Levels:</u> The level of r is 0 The level of any node $x \neq r$ is the number of edges in the (unique) path from r to x <u>Height:</u> Let level(x) denote the level of x The **height** of T is the **maximum** level of any node in T <u>Ancestor:</u> Consider any x If there is any node that can be reached before going to x, it is an ancestor of x Any node y, $y \neq x$, on the path from r to x (including y = r) is called an ancestor of x <u>Descendant:</u> Consider any x If y is an ancestor of x, then x is a descendant of y <u>Parent/ Child:</u> If x is a descendant of p and the $level(x) = level(p) + 1$ (If they are just 1 level apart) p is called the parent of x x is called the child of p <u>Internal Node:</u> A node that has a child is an internal node <u>Leaf:</u> A node with no children is called a leaf	
Tutorial 11 Question 8	Connection between number of leaves and the number of parents for a tree with exactly b children	Consider a rooted tree T in which every parent has exactly b children If T has L leaves and p parents: $L = (b-1)p + 1$	For each of the parents, there will be 1 less leaf because there needs to be 1 of them to become a parent. We add 1 back because the final parent at the bottom will no longer have any parent under it
Definition	Definition of Binary Tree	A binary tree is: 1) A rooted tree 2) Where every node has at most 2 children	
Theorem 4.8	Relationship between number of leaves and height of trees	For any binary tree with m leaves and height h, $m \leq 2^h$	The upper limit will be that we have 2 children for all parents. It will be like a chain reaction for the number of children from the level 1 we have 2 parents and they each have 2 children and it goes down to level h and we will have 2^h leaves (children)
Theorem 4.9	Relationship when we have binary trees and all parents have exactly 2 children	Consider any binary tree T Where every parent has exactly 2 children If T has m leaves, then it has m-1 parents	