Orbifold Tutte Embedding

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Overview

Introduction

Injective Mesh Parameterization A parameterization of a surface mesh is piece-wise linear function defined by $\phi : V \rightarrow \mathbb{R}^2$. Injective means there is no overlap or flip triangle in the parameterization.

Tutte Embedding A Tutte embedding is to place every node into the mean of its neighbors. Convex combination maps (or harmonic maps) generalize to the weighted mean. These methods are only applicable to topological disk (**target domain must be convex**) and torus.

Orbifold Orbifold is a generation of manifold. We can use orbifold structure to apply Tutte Embedding to other type of topology. (Sphere)

Orbifold

An orbifold(for "orbit-manifold") is a generalization of a manifold. An orbifold O is a topological space which has an open cover $\{U_i\}$ closed under finite intersection.

Local Chart Each U_i is associated with a finite group Γ_i which acts on $\tilde{U}_i \subset \mathbb{R}^n$ and a homeomorphism $\phi_i : U_i \cong \tilde{U}_i/\Gamma_i$.

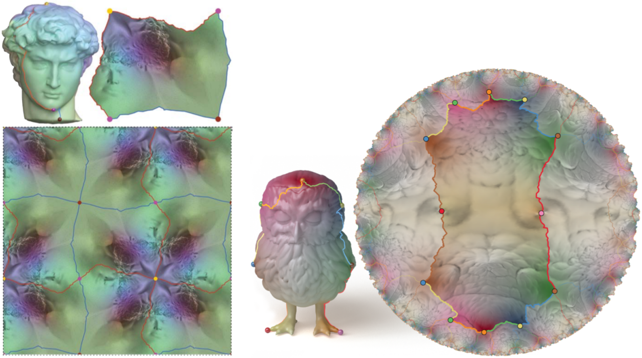
Chart Compatibility Whenever $U_i \subset U_j$, there is a embedding $\tilde{\phi}_{ij} : \tilde{U}_i \hookrightarrow \tilde{U}_j$ and an injective group homomorphism $f_{ij} : \Gamma_i \hookrightarrow \Gamma_j$, where $\forall \gamma \in \Gamma_i, \tilde{\phi}_{ij}(\gamma x) = f_{ij}(\gamma)\tilde{\phi}_{ij}(x)$

Examples

Quotient space by a finite group can be treated as a tiled space by some basic tile.

Euclidean Orbifold An orbifold whose associated groups consist of Euclidean isometries.

Hyperbolic Orbifold An orbifold whose associated groups consist of hypobolic isometries (we mainly discuss Poincare disk model).



Sphere-type Orbifolds

Orbifolds with sphere-type underlying topology. One observation: Only have cone points.

Classification
Define $\delta_i = 2\pi - \Theta_i$.

Euclidean sphere-type orbifold: $\sum_i \delta_i = 2\pi\chi(O) = 4\pi$
$\{\frac{\pi}{2}, \pi, \frac{\pi}{2}\}, \{\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}\}, \{\pi, \frac{2\pi}{3}, \frac{\pi}{3}\}, \{\pi, \pi, \pi, \pi\}$

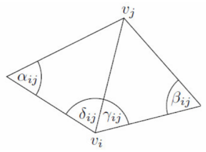
Hyperbolic sphere-type orbifold: $\sum_i \delta_i > 2\pi\chi(O) = 4\pi$
Let $\delta_i = \pi, \#cones > 4$

Convex Combination Maps

Denote $M = (V, H, F)$ an oriented triangular disk-type mesh. $\Phi : V \to \mathbb{R}^2$ is its parameterization. And $W : H \to \mathbb{R}^+$ is a weight function on halfedges. A convex combination map satisfies:

$$\sum_{j \in N_i} w_{ij}(\Phi_i - \Phi_j) = 0, \forall v_i \in Int(M)$$

Tutte Embedding: $w_{ij} = 1$
Harmonic Map: $w_{ij} = \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2}$



Uniquely defined by boundary value, and injective.

Euclidean Orbifold Tutte Embedding

1. Slice: Choose some cone points in $\bar{M}$ and cut the mesh to a disk M.

2. Chart Map: Compute a parameterization of M s.t. it becomes a base domain of some orbifold. It's injective.

3. Orbifold Map: induce a bijective map from $\bar{M}$ to the orbifold.

Euclidean Orbifold Tutte Embedding

$\bar{M}$ is a spherical surface mesh. Assume $\{\bar{v}_i\}_{i=1}^{I}$ is the set of cone points we choose.

We can only "see" the orbifold by a **chart**: Find a path $\bar{v}_1 \to \bar{v}_2 \to \dots \to \bar{v}_I$. Cut the mesh to disk topology M along this slice to get a chart.

Each vertex $\bar{v}_i$ on the slice will be split into two vertices v_i and $v_{i'}$, they are **equivalent** under a rotation $R_{ii'}$ of some cone point.

Our goal: get $\bar{\Phi} : \bar{M} \to O$ from $\Phi : M \to \mathbb{R}^2$

Sphere-type Euclidean Orbifolds

$\left\{\frac{\pi}{2}, \pi, \frac{\pi}{2}\right\}$ (i) $\quad$ $\left\{\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}\right\}$ (ii) $\quad$ $\left\{\pi, \frac{2\pi}{3}, \frac{2\pi}{6}\right\}$ (iii) $\quad$ $\{\pi, \pi, \pi, \pi\}$ (iv)

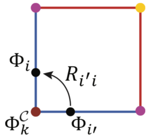
Linear System for Euclidean Orbifold

The singularities becomes $\{v_1, v_2, ...v_J\}$. Assign them prescribed coordinates $\{\Phi_i^C\}_{i=1}^J$

For inner vertex v_i:

$$\sum_{j \in N_i} (\Phi_i - \Phi_j) = 0$$

For each pair of boundary vertex v_i and $v_{i'}$:



$$\sum_{j \in N_i} w_{ij}(\Phi_i - \Phi_j) + \sum_{j \in N_{i'}} w_{i'j} R_{i'i}(\Phi_i' - \Phi_j) = 0$$

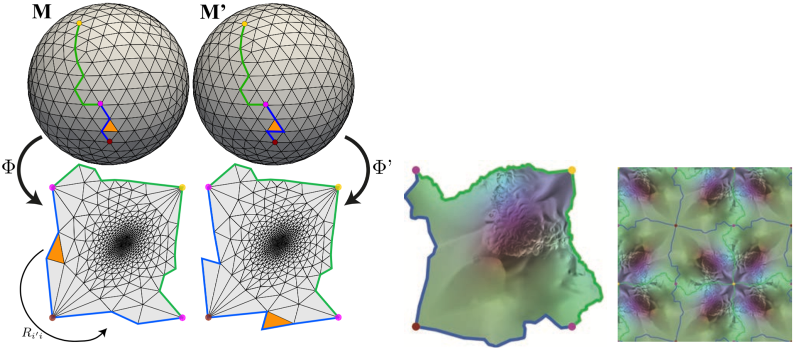
$$R_{i'i}(\Phi_{i'} - \Phi_k^C) - (\Phi_i - \Phi_k^C) = 0$$

Orbifold Map

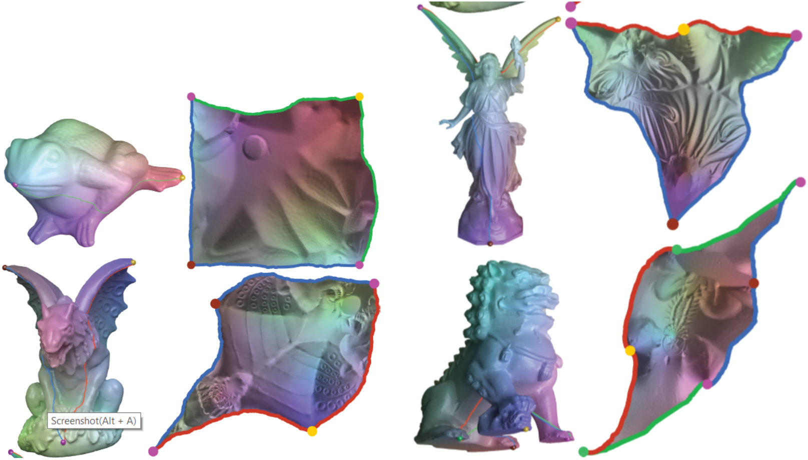
Construct $\bar{\Phi} : \bar{M}$ as follows:

$$\bar{\Phi}(\bar{v}_i) = [\Phi(v_i)]$$

This map is invariant of the choice of slice:



Results



Hyperbolic Convex Combination Map

Non-linear!
Assume $w_{ij} = w_{ji} > 0$.
Minimize

$$E(\Phi) = \frac{1}{2} \sum_{(i,j) \in E} = w_{ij} d(\Phi_i, \Phi_j)^2$$

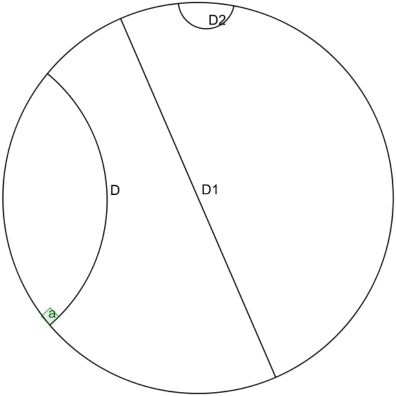
$d(\Phi_i, \Phi_j)$ - geodesic distance

Poincare Disk Model

Hyperbolic metric:

$$ds^2 = \frac{4|dz|^2}{(1 - |z|^2)^2}$$

Geodesic curves:



Geodesic distance:

$$d(z, w) = \text{arccos}(1 + 2\frac{|z - w|^2}{(1 - |z|^2)(1 - |w|^2)})$$

Isometries: Mobius transformations ($D^2 \to D^2$)

$$m(z) = e^{i\theta}\frac{z - p_0}{1 - \bar{p}_0 z}$$

Determined by two points.

Sphere-type Hyperbolic Orbifolds

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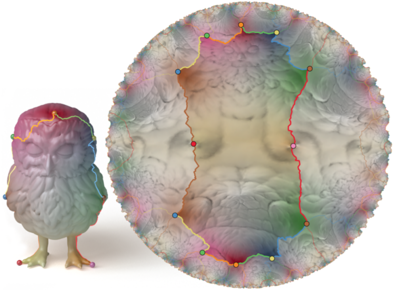

Hyperbolic Orbifold Tutte Embedding

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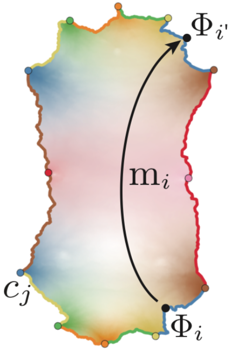
Optimization



$$
\begin{aligned}
& \min E(\Phi) \\
s.t.\quad & \Phi_j = c_j, \quad j \in cones \\
& \Phi_{i'} = m_i(\Phi_i), \quad (i, i')\ equivalent.
\end{aligned}
$$

Reformulate Gradient

Insert $\Phi_{i'} = m_i(\Phi_i)$ into the energy expression.

$$\nabla_{\Phi_i} E = \sum_{j \in N_i} \nabla_{\Phi_i} w_{ij} d(\Phi_i, \Phi_j)^2 + \sum_{j \in N_{i'}} \nabla_{\Phi_i} w_{i'j} d(\Phi_i, m_i^{-1}(\Phi_j))^2$$

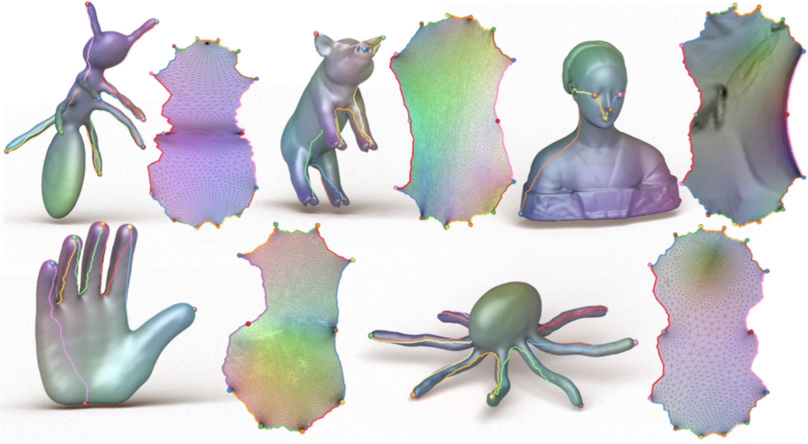
Multiple gradients pointwise by $(1 - |\Phi|^2)^2/4$ - hyperbolic length on the disk.
Solve by first-order optimization algorithms (e.g. LBFGS)

Properties

Injective chart map. $\Phi : M \to D^2$ is injective.
Bijective orbifold map. $\bar{\Phi}(\bar{v}_i) = [\Phi(v_i)]$ is bijective from $\bar{M}$ to O.
This map is invariant of the choice of slice, too.

Results

$k = 7$ with $\Theta_i = \pi$:



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