

Freudenthal Suspension Theorem as a Special Case of Stabilization in ∞ -Categories

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Abstract

We present a modern perspective on the classical Freudenthal Suspension Theorem by interpreting it as a special case of the stabilization of pointed ∞ -categories. This approach highlights the connection between classical homotopy theory and stable homotopy theory, and provides a natural framework for understanding suspension as a functorial process in higher category theory.

1 Introduction

Let X be a connected pointed space. The classical *Freudenthal Suspension Theorem* states that the suspension map

$$\pi_n(X) \longrightarrow \pi_{n+1}(\Sigma X)$$

is an isomorphism for $n < 2 \operatorname{conn}(X)$ and a surjection for $n = 2 \operatorname{conn}(X)$. Here, $\operatorname{conn}(X)$ denotes the connectivity of X .

We now reinterpret this result in the modern context of ∞ -categories.

2 Stabilization of Pointed ∞ -Categories

Let \mathcal{C} be a pointed ∞ -category. Its *stabilization* is the universal stable ∞ -category

$$\operatorname{Stab}(\mathcal{C})$$

equipped with an exact functor

$$\Sigma^\infty : \mathcal{C} \longrightarrow \operatorname{Stab}(\mathcal{C}),$$

called the *suspension spectrum functor*. The right adjoint

$$\Omega^\infty : \text{Stab}(\mathcal{C}) \longrightarrow \mathcal{C}$$

is called the *infinite loop space functor*. Together they form an adjunction

$$\Sigma^\infty : \mathcal{C} \rightleftarrows \text{Stab}(\mathcal{C}) : \Omega^\infty.$$

Definition 2.1 (Unit of the Adjunction). For $X \in \mathcal{C}$, the *unit map* of the adjunction is

$$\eta_X : X \longrightarrow \Omega^\infty \Sigma^\infty X.$$

3 Freudenthal as Connectivity of the Unit

Let X be r -connected. The classical Freudenthal theorem can be seen as the statement that the unit η_X is *highly connected*:

Theorem 3.1 (Freudenthal via Stabilization). *For X an r -connected pointed space, the unit map*

$$\eta_X : X \longrightarrow \Omega^\infty \Sigma^\infty X$$

induces isomorphisms on homotopy groups

$$\pi_n(X) \xrightarrow{\cong} \pi_n(\Omega^\infty \Sigma^\infty X) \quad \text{for } n \leq 2r$$

and a surjection for $n = 2r + 1$.

Sketch of Proof. Consider the colimit description:

$$\pi_k(\Omega^\infty \Sigma^\infty X) = \varinjlim_n \pi_{k+n}(\Sigma^n X),$$

where the maps in the colimit are the classical suspension maps. By the classical Freudenthal theorem, these maps are isomorphisms in the stable range $k + n \leq 2(r + n)$, i.e. $k \leq 2r + n$. For fixed k , the colimit stabilizes at finite n , giving the desired isomorphisms and surjections. Hence the connectivity of the unit recovers the Freudenthal suspension theorem. \square

4 Commutative Diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \Omega^\infty \Sigma^\infty X \\ & \searrow \Sigma & \uparrow \text{---} \\ & & \Omega \Sigma X \end{array}$$

This diagram shows the classical suspension map factoring through the infinite loop space construction.

5 Conclusion

From the perspective of ∞ -categories, Freudenthal's theorem is naturally explained as a connectivity statement about the unit of the stabilization adjunction. This viewpoint generalizes to other contexts in stable homotopy theory and provides a bridge between classical homotopy theory and modern categorical methods.