From Exact Sequences and Fiber Sequences to Towers and Universal Spectral Sequences

Yang

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Abstract

Short note connecting short exact sequences, (homotopy) fiber/cofiber sequences, and their higher-categorical generalizations: towers/filtrations and spectral sequences. The second part describes a conceptual *universal spectral-sequence machine* in the language of stable ∞-categories and explains how classical spectral sequences (Atiyah–Hirzebruch, Serre/Leray–Serre, Adams, Adams–Novikov) arise as special cases.

1 Exact sequences and fiber/cofiber sequences

In an abelian category a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is the basic building block; it yields a long exact sequence of derived functors (e.g. homology). In homotopy theory the analogue is a *fiber/cofiber sequence*

$$F \longrightarrow E \longrightarrow B$$

with $F \simeq \mathrm{hofib}(E \to B)$ (equivalently $B \simeq \mathrm{cofib}(F \to E)$). In a stable ∞-category these assemble into distinguished triangles and play the role of exact sequences.

Remark 1.1. *Passing from one short exact sequence to a* filtered *object (a finite or infinite sequence of subobjects whose successive quotients are simpler) produces page-by-page approximations captured by spectral sequences. The homotopical analogue of a filtered short exact sequence is therefore a* tower *of fiber/cofiber sequences.*

2 Towers, filtrations and the homotopy spectral sequence

Let $\mathcal{C}$ be a stable ∞-category and consider a (decreasing) filtration

$$\cdots \to F^{p+1}X \to F^pX \to \cdots \to X.$$

Define graded pieces $\mathrm{gr}^pX = \mathrm{cofib}(F^{p+1}X \to F^pX)$. For a homological functor $H_* : \mathcal{C} \to \mathsf{Ab}_{\mathbb{Z}\text{-graded}}$ (e.g. $H_n(-) = \pi_n(-)$ or an extraordinary homology theory) application to the cofiber sequences yields long exact sequences and thus an exact couple. The associated spectral sequence has

$$E^1_{p,q} \cong H_{p+q}(\mathrm{gr}^{-p}X) \quad \Rightarrow \quad H_{p+q}(X),$$

under suitable convergence hypotheses (exhaustivity, connectivity growth, $\varprojlim^1$-vanishing where appropriate).

3 Which object generalizes a single fiber sequence?

A single fiber sequence corresponds to a length-2 filtration; its higher analogue is a *tower of fibers* (a sequence of fibrations or cofibrations). The spectral sequence is the algebraic invariant associated to this tower. Thus:

- short exact sequence $\leftrightarrow$ fiber/cofiber sequence (homotopical);
- chain of short exact sequences / filtration $\leftrightarrow$ tower of fiber/cofiber sequences $\leftrightarrow$ spectral sequence.

4 Towards a "universal" spectral sequence machine

There are many spectral sequences in topology and algebra; a unifying viewpoint is provided by the construction that starts from a *filtered object* in a stable homotopy theory and a homological functor. We describe an abstract *universal spectral sequence* and explain why core classical examples arise from it.

4.1 Abstract construction (stable ∞-categorical)

Fix a stable ∞-category $\mathcal{C}$. Consider the ∞-category of filtered objects

$$\mathrm{Fil}(\mathcal{C}) = \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C}).$$

There is a functor which assigns to each filtered object $F^\bullet X$ its graded pieces $\mathrm{gr}^\bullet X$ (a $\mathbb{Z}$-graded object in $\mathcal{C}$), and composing with a homological functor $H_* : \mathcal{C} \to \mathsf{GrAb}$ yields a graded object of abelian groups. The long exact sequences from cofiber sequences define an exact-couple, and one can functorially pass to the associated spectral sequence (a sequence of pages with differentials).

Abstractly, we obtain a functor

$$\mathrm{SS} : \mathrm{Fil}(\mathcal{C}) \times \mathrm{Hom}(\mathcal{C}, \mathsf{GrAb}) \longrightarrow \{\text{spectral sequences}\},$$

natural in both variables. This may be regarded as the *universal spectral-sequence machine* in the given stable homotopy theory.

4.2 Universal property (informal)

The machine is universal among constructions that assign to a filtered object and a homological functor an exact couple (hence a spectral sequence) compatible with maps of filtered objects and natural transformations of homological functors. More precisely, any functorial spectral-sequence construction factoring through graded pieces and long exact sequences factors (up to canonical equivalence) through SS.

4.3 Encoding classical spectral sequences

- **Atiyah–Hirzebruch spectral sequence (AHSS).** Take $\mathcal{C} = \mathrm{Sp}$, $H_* = E_*(-)$ an extraordinary homology theory represented by a spectrum E, and filter a CW-spectrum X by its skeletal filtration. Then $\mathrm{gr}^p X \simeq \bigvee S^p \wedge X^p$ and $E^1_{p,q} = E_{p+q}(\mathrm{gr}^{-p} X)$ recovers the AHSS.
- **Serre / Leray–Serre spectral sequence.** For a fibration $F \to E \to B$ use the skeletal filtration on the base B and pull back to E; applying homology gives the Leray–Serre spectral sequence via the same universal machine.

- **Adams spectral sequence.** Work in Sp and fix a homology theory E. The Adams (or Adams–Novikov) spectral sequence arises by taking an E-Adams resolution or the cobar/bar cosimplicial resolution of the E_*-comodule E_*X, which produces a tower/filtered object in the derived category of comodules (or in Sp after realizing the cosimplicial object). Applying the universal machine yields the Adams E_2-page expressed as Ext groups over the Hopf algebroid (E_*, E_*E).

5 Remarks and variants

- The construction works equally well in model categories where one passes to a homotopy coherent model for $\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C})$ and takes homotopy colimits/limits as needed.
- Some spectral sequences (e.g. Grothendieck spectral sequence for composite derived functors) take a slightly different input (composite functors and derived functors) but can be recovered from the universal viewpoint by expressing derived functors via totalizations of appropriate bicomplexes or cosimplicial objects and then filtering.
- Convergence issues live at the heart of the machine: completeness, boundedness, connectivity growth, and $\lim^1$-vanishing hypotheses determine which spectral sequences converge strongly to the intended target.

6 Freudenthal Suspension Theorem in the Language of Dérivateurs

We now illustrate how the Freudenthal Suspension Theorem can be formulated and proved entirely within the formalism of pointed strong dérivateurs. This clarifies its categorical nature and exemplifies how universal spectral sequences interact with stabilization phenomena.

6.1 Setup

Let $\mathbb{D}$ be a pointed, strong dérivateur associated to a pointed simplicial (or topological) model category. In particular:

- homotopy colimits and limits exist;
- there are suspension and loop functors Σ, Ω defined via homotopy pushouts and pullbacks;
- spherical objects $S^n = \Sigma^n S^0$ define homotopy groups $\pi_n^{\mathbb{D}}(X) = [S^n, X]$ in $\mathrm{Ho}(\mathbb{D}(e))$.

Let $\mathrm{Stab}(\mathbb{D})$ denote the stabilization of $\mathbb{D}$.

6.2 Statement

If $X \in \mathbb{D}(e)$ is r-connected, then the unit map $X \to \Omega^\infty \Sigma^\infty X$ induces isomorphisms on π_k for $k \leq 2r$ and a surjection for $k = 2r + 1$.

6.3 Sketch of Proof

The proof uses the James filtration inside the dérivateur, producing a spectral sequence $E^1_{n,t} = \pi_t(X^{\wedge n}) \Rightarrow \pi_t(\Omega\Sigma X)$, which collapses in low degrees by connectivity estimates.

7 From Freudenthal Suspension to Brown Representability

Freudenthal suspension provides the homotopical mechanism that bridges unstable and stable homotopy theory.

7.1 Stabilization

Iterating suspension yields a sequence $X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to \cdots$ whose homotopy colimit defines $\Omega^\infty\Sigma^\infty X$.

7.2 Brown Representability

Brown representability asserts that every reduced cohomology theory is represented by a spectrum E with $E^n(X) \cong [\Sigma^\infty X, \Sigma^n E]$.

7.3 Operations as Maps of Spectra

Natural cohomology operations correspond to maps of spectra, e.g. $H\mathbb{F}_2 \to \Sigma^k H\mathbb{F}_2$.

Conclusion

Freudenthal suspension justifies stabilization; Brown representability formalizes it.

8 Concluding perspective

The *universal spectral-sequence machine* is the functorial association of a spectral sequence to a filtered object plus a homological functor inside a stable homotopy theory. It packages and explains the common origin of classical spectral sequences: they differ mainly by the choice of filtration and the chosen homological functor; the machine controls the formal pages and differentials and the convergence conditions encode the needed connectivity/completeness hypotheses for computation.

9 References (select)

- J. Lurie, *Higher Algebra*, especially §1.2.2 on filtered objects and spectral sequences.
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- R. Adams, *Stable Homotopy and Generalised Homology.*
- J. P. May, *A concise course in algebraic topology* (James construction, looping/suspension filtrations).