Freudenthal Suspension Theorem as a Special Case of Stabilization in ∞-Categories

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Abstract

We present a modern perspective on the classical Freudenthal Suspension Theorem by interpreting it as a special case of the stabilization of pointed ∞-categories. This approach highlights the connection between classical homotopy theory and stable homotopy theory, and provides a natural framework for understanding suspension as a functorial process in higher category theory.

1 Introduction

Let X be a connected pointed space. The classical *Freudenthal Suspension Theorem* states that the suspension map

$$\pi_n(X) \longrightarrow \pi_{n+1}(\Sigma X)$$

is an isomorphism for $n < 2\,\mathrm{conn}(X)$ and a surjection for $n = 2\,\mathrm{conn}(X)$. Here, $\mathrm{conn}(X)$ denotes the connectivity of X.

We now reinterpret this result in the modern context of ∞-categories.

2 Stabilization of Pointed ∞-Categories

Let $\mathcal{C}$ be a pointed ∞-category. Its *stabilization* is the universal stable ∞-category

$$\mathrm{Stab}(\mathcal{C})$$

equipped with an exact functor

$$\Sigma^\infty : \mathcal{C} \longrightarrow \mathrm{Stab}(\mathcal{C}),$$

called the *suspension spectrum functor*. The right adjoint

$$\Omega^\infty : \mathrm{Stab}(\mathcal{C}) \longrightarrow \mathcal{C}$$

is called the *infinite loop space functor*. Together they form an adjunction

$$\Sigma^\infty : \mathcal{C} \rightleftarrows \mathrm{Stab}(\mathcal{C}) : \Omega^\infty.$$

Definition 2.1 (Unit of the Adjunction)**.** For $X \in \mathcal{C}$, the *unit map* of the adjunction is

$$\eta_X : X \longrightarrow \Omega^\infty \Sigma^\infty X.$$

3 Freudenthal as Connectivity of the Unit

Let X be r-connected. The classical Freudenthal theorem can be seen as the statement that the unit η_X is *highly connected*:

Theorem 3.1 (Freudenthal via Stabilization)**.** *For X an r-connected pointed space, the unit map*

$$\eta_X : X \longrightarrow \Omega^\infty \Sigma^\infty X$$

induces isomorphisms on homotopy groups

$$\pi_n(X) \xrightarrow{\cong} \pi_n(\Omega^\infty \Sigma^\infty X) \quad \textit{for } n \leq 2r$$

and a surjection for $n = 2r + 1$.

Sketch of Proof. Consider the colimit description:

$$\pi_k(\Omega^\infty \Sigma^\infty X) = \varinjlim_n \pi_{k+n}(\Sigma^n X),$$

where the maps in the colimit are the classical suspension maps. By the classical Freudenthal theorem, these maps are isomorphisms in the stable range $k + n \leq 2(r + n)$, i.e. $k \leq 2r + n$. For fixed k, the colimit stabilizes at finite n, giving the desired isomorphisms and surjections. Hence the connectivity of the unit recovers the Freudenthal suspension theorem. □

4 Commutative Diagram

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "\Sigma"] & \Omega^\infty \Sigma^\infty X \\
& \Omega \Sigma X \arrow[u, dashed]
\end{tikzcd}$$

This diagram shows the classical suspension map factoring through the infinite loop space construction.

5 Conclusion

From the perspective of ∞-categories, Freudenthal's theorem is naturally explained as a connectivity statement about the unit of the stabilization adjunction. This viewpoint generalizes to other contexts in stable homotopy theory and provides a bridge between classical homotopy theory and modern categorical methods.

Formalization Perspective. The above relationship has been formalized in the setting of Homotopy Type Theory (HoTT) and Lean. In type-theoretic language, the Blakers–Massey theorem establishes the connectivity of a pushout square, and Freudenthal appears as the special case corresponding to the suspension pushout. This provides a bridge between classical homotopy theory and modern formal proof assistants such as Lean, Coq, or Rocq.