ET4386 Estimation and Detection – Signal Detection Project

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1. Detector selection and Implementation

Neyman-Pearson (NP) detector is selected and implemented in our project. As the observations are decided between two hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ and $As[n]$ and $w[n]$ are known, it is a binary hypothesis test with no unknown parameters. There are many detection approaches for a simple binary hypothesis test, such as NP, minimum probability of error, and Bayes risk [1]. For the NP detector data model prerequisites, probability density functions (PDFs) $p(\boldsymbol{x};\, \mathcal{H}_1)$ and $p(\boldsymbol{x};\, \mathcal{H}_0)$ should be known. The Neyman-Pearson criterion is used when the probability of detection (P_d) is maximized subject to the probability of false alarm with a given value [2]. Compared to Bayesian risk, it is used to minimize the average cost. For minimum probability of error, it is to minimize the probability of error.

For the statistics given in this assignment, the signal is known and deterministic and the noise is white Gaussian noise (WGN). The NP detector for a known signal in WGN is the replica-correlator. The derivations are shown below:

Binary detection problem:

$$\mathcal{H}_0\!: x[n] = w[n];$$

$$\mathcal{H}_1\!: x[n] = w[n] + As[n].$$

Where $n = 0,1,\dots,N-1$, $w[n]$ is WGN with the variance σ^2 and $As[n]$ is deterministic and known.

$\mathcal{H}_1$ is decided if the likelihood ratio exceeds a threshold:

$$L(\boldsymbol{x}) = \frac{p(\boldsymbol{x};\, \mathcal{H}_1)}{p(\boldsymbol{x};\, \mathcal{H}_0)} = \frac{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-As[n])^2\right]}{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}}\exp\left(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]\right)} > \lambda.$$

Taking the logarithm:

$$\ln[L(\boldsymbol{x})] = \frac{A}{\sigma^2}\sum_{n=0}^{N-1}x[n]s[n] - \frac{A^2}{2\sigma^2}\sum_{n=0}^{N-1}s^2[n] > ln\lambda.$$

Therefore, the detector we finally selected is:

$$T(\boldsymbol{x}) = \sum_{n=0}^{N-1}x[n]s[n] > \lambda'.$$

Where $\lambda' = \frac{\sigma^2}{A}ln\lambda + \frac{A}{2}\sum_{n=0}^{N-1}s^2[n]$.

The analysis of numerical results for the NP detector implementation in MATLAB (figure 1) is compared with theoretical results and is explained in the next part.

2. Detection Performance

To simplify the description, it is possible to define the total energy of $s[n]$: $\mathcal{E} = \sum_{n=0}^{N-1} s[n]$.

As $x[n]$ follows Gaussian distribution for each n, $T(\boldsymbol{x}) = \sum_{n=0}^{N-1} x[n]s[n]$ also follows Gaussian distribution. The expectation and variance of $T(\boldsymbol{x})$ under hypnosis $\mathcal{H}_0$ and $\mathcal{H}_1$ is shown below:

$$E[T;\mathcal{H}_0] = E\left(\sum_{n=0}^{N-1} w[n]s[n]\right) = 0,$$

$$E[T;\mathcal{H}_1] = E\left(\sum_{n=0}^{N-1} (w[n] + As[n])s[n]\right) = E\left(\sum_{n=0}^{N-1} As^2[n]\right) = A\mathcal{E},$$

$$var[T;\mathcal{H}_0] = var\left(\sum_{n=0}^{N-1} w[n]s[n]\right) = \sum_{n=0}^{N-1} (s^2[n] \cdot var(w[n])) = \sigma^2\mathcal{E},$$

$$var[T;\mathcal{H}_1] = var\left(\sum_{n=0}^{N-1} (w[n] + As[n])s[n]\right) = var\left(\sum_{n=0}^{N-1} As^2[n] + \sum_{n=0}^{N-1} w[n]s[n]\right) = \sigma^2\mathcal{E}.$$

Therefore, $T(\boldsymbol{x}) \sim \begin{cases} \mathcal{H}_0: \mathcal{N}(0, \sigma^2\mathcal{E}) \\ \mathcal{H}_1: \mathcal{N}(A\mathcal{E}, \sigma^2\mathcal{E}) \end{cases}$.

The false alarm probability can be written as:

$$P_{fa} = Pr(T > \lambda'; \mathcal{H}_0) = Q\left(\frac{\lambda'}{\sqrt{\sigma^2\mathcal{E}}}\right).$$

Expressing threshold λ' in terms of P_{fa} gives:

$$\lambda' = \sqrt{\sigma^2\mathcal{E}}Q^{-1}(P_{fa}).$$

The detection probability is:

$$P_d = Pr(T > \lambda'; \mathcal{H}_1) = Q\left(\frac{\lambda' - A\mathcal{E}}{\sqrt{\sigma^2\mathcal{E}}}\right) = Q\left(Q^{-1}(P_{fa}) - A\sqrt{\frac{\mathcal{E}}{\sigma^2}}\right).$$

To indicate the relationship between the detection probability and the energy of the signal, the figure below is drawn by MATLAB. The colored lines in the Figure 1 are the theoretical SNR-P_d lines that illustrate the formula above with 5 different values of P_{fa}. At the same time, after finding the values of the threshold by the values of P_{fa}, the detector outputs of 10000 groups of data with length 10 for 30 different values of A is compared with the threshold. The detection result is compared with the data “mask” to evaluate the probability of detection. P_d is calculated by the ratio of the number of the output ‘1’s in correct positions and the total number of ‘1’s in the data “mask”. Thus, the P_d is also calculated in a numerical way and the SNR-P_d lines are shown in Figure 1 with black dashed line.

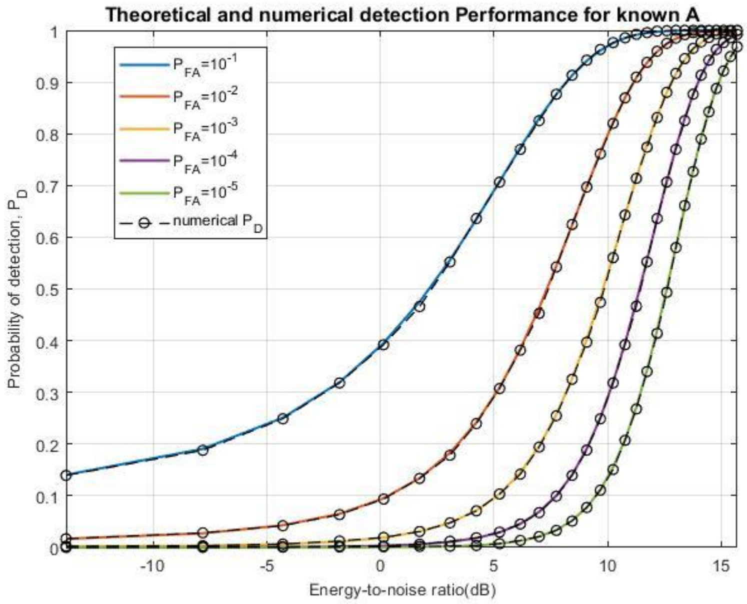

Figure 1 Theoretical and numerical detection performance for known A versus SNR

In Figure 1, it is obvious that the black dashed lines almost coincide with colored lines which proves the correctness of the calculation above in section 1 and 2. The difference among the colored lines also indicates that with the growing of the signal energy and the decrease of the false alarm probability, the detection probability will improve.

3. Unknown signal analysis

Generalized likelihood ratio test (GRLT) approach is used in our project for signal detection with an unknown parameter A. The derivations are shown below:

Binary detection problem:

$$\mathcal{H}_0: x[n] = w[n]$$

$$\mathcal{H}_1: x[n] = w[n] + As[n]$$

Where $n = 0,1,\dots,N-1$, $w[n]$ is WGN with the variance σ^2, $s[n]$ is deterministic and known, and A is unknown.

Firstly, maximum likelihood estimation (MLE) of A is calculated:

$$p(\boldsymbol{x}; \hat{A}) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left(x[n] - \hat{A}s[n]\right)^2\right]$$

$$\frac{\partial \ln p(\boldsymbol{x}; \hat{A})}{\partial \hat{A}} = \frac{\partial}{\partial \hat{A}}\left(\ln\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} - \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left(x[n] - \hat{A}s[n]\right)^2\right) = \frac{1}{\sigma^2}\sum_{n=0}^{N-1} x[n]s[n] - \frac{\hat{A}}{\sigma^2}\sum_{n=0}^{N-1} s[n]^2 = 0$$

$$MLE\ of\ A: \hat{A} = \frac{\sum_{n=0}^{N-1} x[n]s[n]}{\sum_{n=0}^{N-1} s[n]^2}$$

Then, the GLRT becomes:

$$L_G(\boldsymbol{x}) = \frac{p(\boldsymbol{x};\ \hat{A}, \mathcal{H}_1)}{p(\boldsymbol{x};\ \mathcal{H}_0)} = \frac{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left(x[n] - \hat{A}s[n]\right)^2\right]}{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left(-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} x^2[n]\right)} > \gamma$$

Taking the logarithm of both sides:

$$\ln\left(L_G(\boldsymbol{x})\right) = \frac{\hat{A}}{\sigma^2}\sum_{n=0}^{N-1} x[n]s[n] - \frac{\hat{A}^2}{2\sigma^2}\sum_{n=0}^{N-1} s^2[n] > \ln\gamma$$

Taking into the MLE $\hat{A} = \frac{\sum_{n=0}^{N-1} x[n]s[n]}{\sum_{n=0}^{N-1} s[n]^2}$:

$$\frac{1}{2\sigma^2}\frac{(\sum_{n=0}^{N-1} x[n]s[n])^2}{\sum_{n=0}^{N-1} s^2[n]} > \ln\gamma$$

$\mathcal{H}_1$ then can be decided if:

$$T'(\boldsymbol{x}) = \left|\sum_{n=0}^{N-1} x[n]s[n]\right| \geq \gamma'$$

It is obvious that $T'(\boldsymbol{x}) = T(\boldsymbol{x})$, the calculation of the false alarm probability P'_{fa} and detection probability P'_d can be decided according to P_{fa} and P_d.

$$P'_{fa} = \Pr\{T' \geq \gamma'; \mathcal{H}_0\} = 2Q\left(\frac{\gamma'}{\sqrt{\sigma^2\mathcal{E}}}\right)$$

$$\gamma' = \sqrt{\sigma^2\mathcal{E}}Q^{-1}\left(\frac{P'_{fa}}{2}\right)$$

$$P'_d = Pr(T' > \gamma'; \mathcal{H}_1) = Q\left(\frac{\gamma' - A\mathcal{E}}{\sqrt{\sigma^2\mathcal{E}}}\right) + 1 - Q\left(\frac{-\gamma' - A\mathcal{E}}{\sqrt{\sigma^2\mathcal{E}}}\right) = Q\left(\frac{\gamma' - A\mathcal{E}}{\sqrt{\sigma^2\mathcal{E}}}\right) + Q\left(\frac{\gamma' + A\mathcal{E}}{\sqrt{\sigma^2\mathcal{E}}}\right)$$

$$P'_d = Q\left(Q^{-1}\left(\frac{P'_{fa}}{2}\right) - A\sqrt{\frac{\mathcal{E}}{\sigma^2}}\right) + Q\left(Q^{-1}\left(\frac{P'_{fa}}{2}\right) + A\sqrt{\frac{\mathcal{E}}{\sigma^2}}\right)$$

Similar as section 2, the experimental detection probabilities are calculated and being illustrated together with the theoretical P'_d in Figure 2.

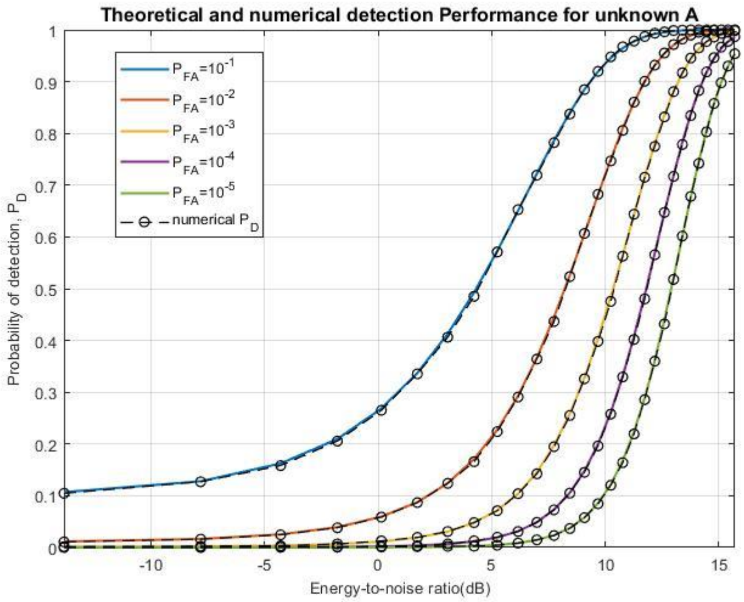

Figure 2 Theoretical and numerical detection performance for unknown A versus SNR

Again, the black dashed lines almost coincide with colored lines that could proves the correctness of the calculation. The SNR-P_d line follows basically the same trend as in the case of known A which also implied the efficiency of this detector. However, the P_d' corresponding to each P_{fa} and SNR is slightly lower than P_d with known A.

References

[1] S. Kay and S. Kay, in *Fundamentals of Statistical Signal Processing: Detection theory*, Upper Saddle River, NJ: Prentice Hall PTR, 2008, pp. 416–418.

[2] S. Kay and S. Kay, in *Fundamentals of Statistical Signal Processing: Detection theory*, Upper Saddle River, NJ: Prentice Hall PTR, 2008, pp. 94.

Appendix - MATLAB codes

```
clear;
clc;
close all;
load('Data.mat');
s_square = sum (s.*s);
for i=1:1:30
    SNR(i) = 10*log10( (A(i)^2) * s_square / sigmaw2);
end
Pfa = [0.1, 0.01, 0.001, 0.0001, 0.00001];

%Calculate theoretical Pd-SNR curve under 5 Pfa values
for i=1:1:30
    for j=1:1:5
        Pd_1(i,j) = qfunc( qfuncinv( Pfa(j) ) -  A(i)*sqrt( s_square / sigmaw2));
        Pd_3(i,j) = qfunc( qfuncinv( Pfa(j)/2 )-  A(i)*sqrt( s_square / 
sigmaw2))+qfunc( qfuncinv( Pfa(j)/2 )+ A(i) * sqrt(s_square / sigmaw2));
    end
end


%Calculate numerical Pd-SNR curve under 5 Pfa values
for i=1:1:5
    lambda_1(i) = qfuncinv(Pfa(i))*sqrt(sigmaw2*s_square);
    lambda_2(i) = qfuncinv(Pfa(i)/2)*sqrt(sigmaw2*s_square);
end
sum_mask = sum(mask);
sum_T2=zeros(5,30,10000);
sum_T4=zeros(5,30,10000);
for i=1:1:30
    for j=1:1:10000
        T=0;
        for l=1:1:10
            T=T+s(l)*x(j,l,i);
        end
        for k=1:1:5
            if T>lambda_1(k)
                sum_T2(k,i,j)=1;
            end
            if abs(T)>lambda_2(k)
                sum_T4(k,i,j)=1;
            end
        end
    end
```

```
    for m=1:1:5
        A_sum2=0;
        A_sum4=0;
        for n=1:1:10000
            A_sum2=A_sum2+sum_T2(m,i,n)*mask(n);
            A_sum4=A_sum4+sum_T4(m,i,n)*mask(n);
        end
        Pd_2(i,m)=A_sum2/sum_mask;
        Pd_4(i,m)=A_sum4/sum_mask;
    end
end

figure(1);
plot(SNR,Pd_1(:,1),SNR,Pd_1(:,2),SNR,Pd_1(:,3),SNR,Pd_1(:,4),SNR,Pd_1(:,5),'Linewid
th',1.5);
hold on;
plot(SNR,Pd_2(:,1),'k--o',SNR,Pd_2(:,2),'k--o',SNR,Pd_2(:,3),'k--
o',SNR,Pd_2(:,4),'k--o',SNR,Pd_2(:,5),'k--o','Linewidth',0.9);
title('Theoretical and numerical detection Performance for known A','FontSize',12)
xlabel('Energy-to-noise ratio(dB)','FontSize',10);
ylabel('Probability of detection, P_D','FontSize',10);
legend('P_{FA}=10^{-1}','P_{FA}=10^{-2}','P_{FA}=10^{-3}','P_{FA}=10^{-
4}','P_{FA}=10^{-5}' ,'numerical P_D');
axis([min(SNR) max(SNR) 0 1]);
grid on;


figure(2);
plot(SNR,Pd_3(:,1),SNR,Pd_3(:,2),SNR,Pd_3(:,3),SNR,Pd_3(:,4),SNR,Pd_3(:,5),'Linewid
th',1.5);
hold on;
plot(SNR,Pd_4(:,1),'k--o',SNR,Pd_4(:,2),'k--o',SNR,Pd_4(:,3),'k--
o',SNR,Pd_4(:,4),'k--o',SNR,Pd_4(:,5),'k--o','Linewidth',0.9);
title('Theoretical and numerical detection Performance for unknown
A','FontSize',12)
xlabel('Energy-to-noise ratio(dB)','FontSize',10);
ylabel('Probability of detection, P_D','FontSize',10);
legend('P_{FA}=10^{-1}','P_{FA}=10^{-2}','P_{FA}=10^{-3}','P_{FA}=10^{-
4}','P_{FA}=10^{-5}' ,'numerical P_D');
axis([min(SNR) max(SNR) 0 1]);
grid on;
```