CS180 Homework 3

Due: March 7, 11:59pm

1. (20 pt) Let $G = (V, E)$ be an undirected graph and each edge $e \in E$ is associated with a positive weight ℓ_e. For simplicity we assume weights are distinct. For each of the following statement answer True or False. If the answer is false, give a counter example. Otherwise briefly justify why it is true.

   (a) Let P be a shortest path between two nodes s, t. Now, suppose we replace each edge weight ℓ_e with $(\ell_e)^2$, then P is still a shortest path between s and t.

   (b) Let T be a minimum spanning tree for the graph with the original weight. Suppose we replace each edge weight ℓ_e with $(\ell_e)^2$, then T is still a minimum spanning tree.

2. (20 pt) Give an example where Dijkstra's algorithm fails when there are edges of negative weight even if there are no negative cycle. (A negative cycle means a cycle $v_1, \ldots, v_r$ where $v_1 = v_r$ and the total weights $\sum_{i=1}^{r-1} \ell_{v_i, v_{i+1}}$ is negative).

3. (30 pt) Given an undirected weighted graph G with n nodes and m edges, and we have used Prim's algorithm to construct a minimum spanning tree T. Suppose the weight of one of the tree edge ($(u, v) \in T$) is changed from w to w', design an algorithm to verify whether T is still a minimum spanning tree. Your algorithm should run in $O(m)$ time, and explain why your algorithm is correct. You can assume all the weights are distinct.

   (Hint: When an edge is removed, nodes of T will break into two groups. Which edge should we choose in the cut of these two groups?)

4. (30 pt) Suppose you're consulting for a bank that's concerned about fraud detection, and they come to you with the following problem. They have a collection of n bank cards that they've confiscated, suspecting them of being used in fraud. Each bank card is a small plastic object, containing a magnetic stripe with some encrypted data, and it corresponds to a unique account in the bank. Each account can have many bank cards corresponding to it, and we'll say that two bank cards are equivalent if they correspond to the same account.

   It's very difficult to read the account number off a bank card directly, but the bank has a high-tech "equivalence tester" that takes two bank cards and, after performing some computations, determines whether they are equivalent.

   Their question is the following: among the collection of n cards, is there a set of more than $n/2$ of them that are all equivalent to one another? Assume that the only feasible operations you can do with the cards are to pick two of them and plug them in to the equivalence tester. Show how to decide the answer to their question with only $O(n \log n)$ invocations of the equivalence tester.

   (Hint: divide and conquer)

★ Homework assignments are due on the exact time indicated. Please submit your homework using the Gradescope system. Email attachments or other electronic delivery methods are not acceptable. To learn how to use Gradescope, you can:

- 1. Watch the one-minute video with complete instructions from here: https://www.youtube.com/watch?v=-wemznvGPfg
- 2. Follow the instructions to generate a PDF scan of the assignments: http://gradescope-static-assets.s3-us-west-2.amazonaws.com/help/submitting_hw_guide.pdf
- 3. **Make sure you start each problem on a new page.**

- ★ We recommend to use LaTeX, LyX or other word processing software for submitting the homework. This is not a requirement but it helps us to grade the homework and give feedback. For grading, we will take into account both the correctness and the clarity. Your answer are supposed to be in a simple and understandable manner. Sloppy answers are expected to receiver fewer points.
- ★ Unless specified, you should justify the correctness and time complexity of your algorithm.
- ★ **Your answer to each problem should be written in 2 (A4 or letter sized) pages.** If you are typing for the homework, a font size of 12 points or larger must be used. If you are writing, the font size should not be too small to affect the readability. You will get 5 points penalty for each additional page used. For all these problems, the correct solution can be easily written in 1 page.