$\boxed{7.1}$ Recurrence relations

Ex Consider S defined by $\overset{1}{5}, \overset{2}{8}, \overset{3}{11}, \overset{4}{14}, \overset{5}{17}, \ldots$

$\Rightarrow S_n = 5 + (n-1)\cdot 3 \quad n \geq 1$

so we see

$$\begin{aligned}
S_1 &= 5 \\
S_2 &= 5+3 = S_1 + 3 \\
S_3 &= 5+3+3 = S_2 + 3 \\
S_4 &= 5+3+3+3 = S_3 + 3 \\
S_5 &= 5+3+3+3+3 = S_4 + 3 \\
&\vdots
\end{aligned}$$

$$\Rightarrow \underbrace{S_n = S_{n-1} + 3}_{\text{recurrence relation}} \text{ for } n \geq 2 \text{ where } \underbrace{S_1 = 5.}_{\text{initial conditions}}$$

Def A recurrence relation for the sequence $a_0, a_1, \ldots$ is an equation that relates a_n in terms of (some of) $a_0, a_1, \ldots, a_{n-1}$.
The initial conditions for the sequence are provided values for finitely many terms of a sequence.

Ex Fibonacci numbers f_i for $i \geq 1$

$1, 1, 2, 3, 5, 8, 13, 21, \ldots$

e see $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$
where $f_1 = f_2 = 1$.

Ex) Invest \$1000 at 12% interest, compounded annually. If A_n is the amount after n years, find a rec. rel. + initial cond. to define $\{A_n\}_{n=0}^{\infty}$.

A_{n-1} = amount at the end of $n-1$ years.

Then $A_n = A_{n-1} + (\text{interest})$
$= A_{n-1} + 0.12 \cdot A_{n-1} = 1.12 A_{n-1}$ $\quad n \geq 1$

We know $A_0 = 1000$.
Now we'll use these to deduce a closed formula for A_n.

We'll find a pattern.
For $n=3$,

$$\begin{aligned} A_3 &= 1.12 \cdot A_2 \\ &= 1.12(1.12 A_1) = 1.12^2 A_1 \\ &= 1.12^2(1.12 A_0) = 1.12^3(1000) \end{aligned}$$

In general, we see

$$\begin{aligned} A_n &= 1.12 \cdot A_{n-1} \\ &= 1.12(1.12 A_{n-2}) = 1.12^2 A_{n-2} \\ &\vdots \\ &= 1.12^n \cdot 1000 \end{aligned}$$

$\Rightarrow A_n = 1.12^n \cdot 1000$ for $n \geq 0$.

Ex) $S_n = \left|\{\text{n-bit strings } S \mid S \text{ does not contain the pattern } 111\}\right|$

Find a recurrence rel. + initial conditions that define S.

These strings in S_n are in one of the following cases:

a) begin with 0
b) begin with 10
c) begin with 11

clearly these are disjoint cases

then S_n will the the sum of strings in each case by the Addition Princ.

For a) if the string begins w/ 0, then the substring containing the last $n-1$ bits also doesn't contain 111
$\Rightarrow$ there are S_{n-1} strings in this case

For b) The substring containing the last $n-2$ bits also doesn't contain 111
$\Rightarrow$ there are S_{n-2} strings in this case

For c) we know the 3rd bit will not be 1. The remaining last $n-3$ bits won't contain 111
$\Rightarrow$ there are S_{n-3} strings in this case

$\Rightarrow S_n = S_{n-1} + S_{n-2} + S_{n-3}$ for $n \geq 4$

for initial conditions:

$S_1 = |\{1, 0\}| = 2 \qquad S_2 = |\{10, 11, 01, 00\}| = 4$

$S_3 = |\{000, 001, 010, 011, 100, 101, 110\}| = 7$

Ex| Tower of Hanoi

Have n discs decreasing in diameter stacked on one of 3 pegs.

Let c_n = # moves it takes to move all discs to another peg where a smaller disc can never be placed atop a larger disc

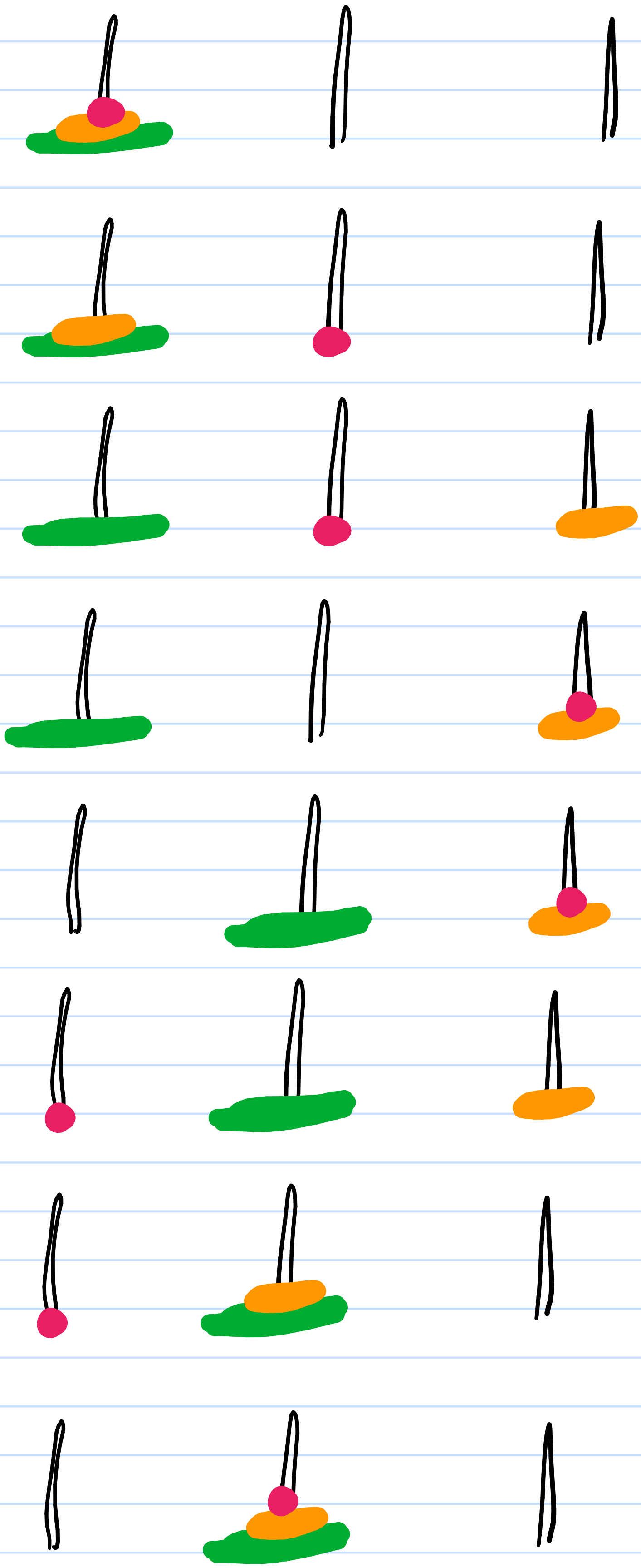
(see next page for $n=3$ example)

We find the recurrence

$$c_n = 2c_{n-1} + 1 \quad n > 1 \text{ where } c_1 = 1$$

We can prove this to be the best possible bound using induction.

Ex) $n=3$



move top $n-1$ discs to another (C_{n-1} steps)

move bottom disc (1 step)

move top $n-1$ discs on top of bottom disc (C_{n-1} steps)