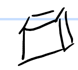
§1.1 - Sets

$\boxed{\text{Def}}$ A <u>set</u> is a collection of objects, called <u>elements</u> or <u>members</u>.

Note: order does not matter!

How to describe sets:

1) if not to big, can list out all elements

<u>Ex</u>) $A = \{1, 2, 3, 4\}$ $\quad$ $C = \{\text{cat}, 1,$  $\}$

$B = \{\text{cat}, \text{dog}, \text{horse}\}$ $\quad$ $D = \{12, \{2, 3\}, 6\}$

We assume no duplicates + delete any repeated elements, so

$$A = \{1, 2, 3, 4\} = \{1, 2, 2, 2, 3, 4\}$$

2) if bigger, can list out the conditions for membership

in larger set S

<u>Ex</u>) $A = \{x \in S \mid x \text{ is an integer multiple of } 5\}$

member name $\quad$ "such that" $\quad$ member condition

This is <u>set-builder notation</u>.

<u>Symbols for common sets</u>

$\mathbb{Z} = \{x \mid x \text{ is an integer}\}$

$\mathbb{Q} = \{x \mid x \text{ is a rational number}\}$

$\mathbb{R} = \{x \mid x \text{ is a real number}\}$

$\mathbb{Z}_{>0} = \{x \mid x \text{ is a non-negative integer}\}$

$\mathbb{Z}_{\geq 0} = \{x \mid x \text{ is a positive integer}\}$

Def If X contains infinitely many elements, we say X is <u>infinite</u>.

Otherwise X is <u>finite</u>.

Ex $\mathbb{Z}$ is infinite

$\{x \in \mathbb{Z}_{>0} \mid x \leq 10^{17}\}$ is finite

Def For a finite set X, the <u>cardinality</u> of X, written $|X|$, is the number of elements in X.

Ex For $A = \{2, 7, 8\}$, $|A| = 3$.

Def For a set X, we write $x \in X$ if x is an element of X. Otherwise we write $x \notin X$

Ex So for A as above, $2 \in A$ and $5 \notin A$.

Def If $X = \{\ \}$, i.e. X has no elements, we say X is <u>empty</u> & call it the empty set, denoted $\emptyset$.

We say sets X and Y are <u>equal</u>, denoted $X = Y$, if they contain the same elements.

Ex $X = \{x \in \mathbb{Z} \mid x \text{ is a multiple of } 2\}$

$Y = \{x \in \mathbb{Z} \mid x \text{ is even}\}$

Then $X = Y$

Q: How to show that sets X, Y are equal?

A: Often we show if $x \in X$, then $x \in Y$ and if $y \in Y$, then $y \in X$!

Ex) $B = \{x \mid x^2 - 7x - 18 = 0\}$ $\quad C = \{9, -2\}$

Q: How to show that sets X, Y are not equal?

A: Find some $x \in X$ where $x \notin Y$ or some $y \in Y$ where $y \notin X$

Ex: $X = \{x \mid x \text{ is an aquatic animal}\}$ $\quad Y = \{y \mid y \text{ is a fish}\}$

$X \neq Y$ because dolphins $\in X$ but dolphins $\notin Y$

Def For sets X, Y if each $x \in X$ is also an element of Y, we say X is a subset of Y, denoted $X \subseteq Y$. Otherwise we write $X \not\subseteq Y$.

Ex) $A = \{x \in \mathbb{R} \mid (x+1)(x-7) = 0\}$

Then $A = \{-1, 7\} \Rightarrow X \subseteq \mathbb{Z}$

Ex) For X, Y as above, $Y \subseteq X$.

Def If $X \subseteq Y$ and $X \neq Y$, we say X is a proper subset of Y, written $X \subset Y$.

Ex $\{\text{types of fish}\} \subseteq \{\text{types of aquatic animals}\}$

Q: How to show, in general, $X \subseteq Y$?
A: Take arbitrary $x \in X$ and prove $x \in Y$.

The set of all subsets of X is the power set, denoted $P(X)$.

Ex $X = \{1,2\}$ $\quad P(X) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

Ex Claim: $P(X) \subseteq P(Y)$ if $X \subseteq Y$

Suppose $S \in P(X)$. Then $S \subseteq X \Rightarrow S \subseteq Y \Rightarrow S \in P(Y)$.

Set Operations: On sets X and Y, define

Def the union of X and Y:
$$X \cup Y = \{z \mid z \in X \text{ or } z \in Y\}$$

the intersection of X and Y:
$$X \cap Y = \{z \mid z \in X \text{ and } z \in Y\}$$

If $X \cap Y = \emptyset$, we say X and Y are disjoint.

Ex $X = \{2, 6, 5, 15, 9\}$ $\quad Y = \{3, 8, 6, 15, 18\}$

$X \cup Y = \{2, 3, 5, 6, 8, 9, 15, 18\}$

$X \cap Y = \{6, 15\}$

The <u>difference</u> of X and Y:

$$X - Y = \{ z \mid z \in X \text{ and } z \notin Y \}$$

Often our sets live in a larger set U called the <u>universal set</u>.

The <u>complement</u> of X in U is

$$\overline{X} = \{ x \in U \mid x \notin X \} = U - X$$

<u>Ex</u> For $U = \mathbb{R}$ and $X = \mathbb{Q}$,
$$\overline{X} = \{ x \in \mathbb{R} \mid x \text{ irrational} \}$$

see Theorem 1.1.22 for different ways to express sets.

We call a set of sets a <u>collection</u> of sets.

We write
$$\bigcup_i A_i := \{ x \mid x \in A_i \text{ for some } i \}$$
$$\bigcap_j A_j := \{ x \mid x \in A_j \text{ for each } j \}$$

$\boxed{\text{Def}}$ A collection $S = \{A_1, A_2, \dots\}$ is a <u>partition</u> of a set X if for each $x \in X$, there is some i for which $x \in A_i$ and each pair of sets A_i, A_j are disjoint.

<u>Ex</u> $S = \{ \{1, 4, 2\}, \{5, 3\}, \{6\} \}$ is a partition of $\{1, 2, 3, 4, 5, 6\}$

$\boxed{\text{Def}}$ The <u>Cartesian product</u> $X \times Y$ of sets X and Y, is the set
$$X \times Y = \{ (x, y) \mid x \in X, y \in Y \}$$

§3.1 Functions:

Def A function is an assignment of an element of Y to each element of X.

We write $f: \underset{\text{domain}}{X} \to \underset{\text{codomain}}{Y}$
and $f(x) = y$ if f assigns y to x

Can view f as a subset of $X \times Y$ by instead writing $f = \{(x,y) \mid f \text{ assigns } y \text{ to } x\}$

$\{y \in Y \quad y = f(x) \text{ for some } x \in X\}$ is the range.

Def We define the modulus operator to be the following:
for $x \in \mathbb{Z}_{\geq 0}$ and $y \in \mathbb{Z}_{>0}$ (positive integers), define
$x \bmod y$ to be the remainder from dividing x by y

Ex) $8 \bmod 2 = 0$

$5 \bmod 3 = 2$

$3 \bmod 5 = 3$

Claim: $\{m \in \mathbb{Z} \mid m \bmod n = 0\}$
$= \{m \in \mathbb{Z} \mid m \text{ is an integer multiple of } n\}$

$\boxed{\text{Def}}$ For $x \in \mathbb{R}$,
the floor of x
$\lfloor x \rfloor$ = largest $z \in \mathbb{Z}$ where $z \leq x$ "rounds down"

the ceiling of x
$\lceil x \rceil$ = smallest $z \in \mathbb{Z}$ where $z \geq x$ "rounds up"

$\boxed{\text{Def}}$ $f: X \to Y$ is one-to-one, or injective, if for each $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$

$f: X \to Y$ is onto, or surjective, if for each $y \in Y$, there exists some $x \in X$ where $f(x) = y$.

If f is 1-1 and onto, f is bijective.
If f is bijective, f is invertible.
We write f^{-1} to be the inverse of $f: X \to Y$ where if $f(x) = y$, $f^{-1}(y) = x$, where $f^{-1}: Y \to X$.

For $f: X \to Y$, $g: Y \to Z$,
$g \circ f = g(f)$ is the composition of g w/ f, where for $x \in X$, $y \in Y$, $z \in Z$
if $f(x) = y$, $g(y) = z \Rightarrow$

$$g \circ f(x) = g(f(x)) = g(y) = z$$