6.2 Permutations + Combinations

Def A *permutation* of n distinct elements is an *ordering* of those elements.

Ex) The permutations of $\{A, B, C\}$ are

ABC ACB BAC BCA CAB CBA

$\Rightarrow$ 6 total

How many permutations of $\{A, B, C, D\}$?

Thm There are $n!$ permutations of a set of n distinct elements.

Ex) Thus there are $4!$ permutations of $\{A, B, C, D\}$.

Ex) ① How many permutations of ABCDEF contain DEF in that order?

this amounts to permuting A, B, C, DEF, a 4-element set

$\Rightarrow$ there are $4!$ permutations

② What if they contain DEF in any order?

Then we can first choose where the DEF block lies then secondly permute DEF

$\Rightarrow (4!)(3!)$ ways

Ex) How many ways to seat 6 people around circular table? (seatings up to rotation are =)

place first person anywhere. Then we can just seat the rest clockwise linearly.

$\Rightarrow (6-1)! = 5!$ ways

Def An *r-permutation* of an r-element set is an ordering of an r-element subset. The number of such permutations is denoted $P(n,r)$

Thm $P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$

$$= \frac{n!}{(n-r)!} \quad \text{for } r \leq n$$

Ex How can we select President, VP, Secretary, + treasurer from a committee of 10?

$$P(10,4) = \underset{\text{pres}}{10} \cdot \underset{\text{VP}}{9} \cdot \underset{S}{8} \cdot \underset{T}{7}$$

Def Let $X = \{x_1, \ldots, x_n\}$ have n distinct elt. An *r-combination* of X is an r-elt. subset of X. The number of such subsets is denoted $\binom{n}{r} = C(n,r)$

Thm $C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!\,r!} \qquad r \leq n$

Ex How many ways to choose a committee of 4 from a group of 10 people?

$$\binom{10}{4} = \frac{10!}{6!\,4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!}$$

Ex) How many 8-bit strings have four 1's?
The string is determined by the positions of 1's. Thus we have to choose 4 positions from the 8 total

$$\Rightarrow \binom{8}{4} = \frac{8!}{4!4!} \text{ ways}$$

Ex) Consider $X = \{x \mid x \text{ is in an ordinary deck of cards}\}$

a) How many (unordered) 5 card hands are there?

$$\binom{52}{5}$$

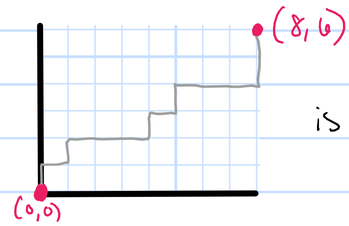
b) How many contain all same suit?

$$4 \cdot \binom{13}{5}$$

c) How many contain 3 of one type + 2 from another (eg. 2 Queen's, 3 7's)

$$13 \cdot \binom{4}{3} \cdot 12 \binom{4}{2}$$

Def) A (shortest) grid walk from $(0,0) \to (m,n)$ where $m,n \in \mathbb{Z}_{\geq 0}$ is path connecting $(0,0)$ to (m,n) using unit steps north and east.

Ex) 

is a grid walk from $(0,0) \to (8,6)$.

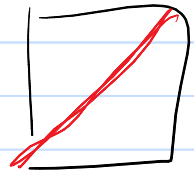
Claim: There are $\binom{n+m}{n}$ grid walks from $(0,0) \to (m,n)$.

Pf

Note that any walk will require m steps east + n steps north. Therefore there will be $n+m$ steps total. Then the total # of grid walks = # ways to choose which of the $n+m$ steps are north steps

$$= \binom{n+m}{n}$$

Now let's focus on the grid walks $(0,0) \to (n,n)$ where the paths never cross the main diagonal!



We say C_n is the number of such walks.

Thm $C_n = \frac{1}{n+1}\binom{2n}{n}$

These are called the Catalan numbers.

6.3 Gen. Permutations + Combinations

Thm Suppose a sequence S of n items has n_1 objects of type 1, n_2 of type 2, ..., n_t of type t. Then the number of orderings is

$$\frac{n!}{n_1!\,n_2!\cdots n_t!}$$

Ex How many strings can we form using

MISSISSIPPI w/ all the letters?

$$\frac{11!}{1!\,4!\,4!\,2!}$$

Thm If X has t elts, the number of k-elt. selections from X, allowing repetitions, is

$$C(k+t-1,\,t-1) = C(k+t-1,\,k)$$

Ex Have piles of R, B, + G cubes w/ ≥ 8 of each.

a) How many ways to select 8 total?

$$\binom{8+3-1}{3-1} = \binom{10}{2}$$

red zone | blue zone | green zone

b) What if we also want ≥ 1 of each color?

$$\binom{5+3-1}{3-1} = \binom{7}{2}$$

Ex a) How many non-neg integer solutions to

$$x_1 + x_2 + x_3 + x_4 = 29$$

$t=4,\ k=29 \Rightarrow \binom{29+4-1}{4-1} = \binom{32}{3}$ solns

b) How many where $x_1 > 0,\ x_2 > 1,\ x_3 > 2,\ x_4 \geq 0$?

$t=4,\ k = 29-1-2-3 = 23$

$\Rightarrow \binom{23+4-1}{4-1} = \binom{26}{3}$ solns

$\boxed{6.7}$ Binomial Coeff.

Ex) $(a+b)^3 = (a+b)(a+b)(a+b)$

$$= aaa + aab + aba + abb + baa + bab + bba + bbb$$

$$= a^3 + 3a^2b + 3ab^2 + b^3$$

In general $(a+b)^n = \underbrace{(a+b)(a+b)\cdots(a+b)}_{n\text{-times}}$

Idea: $a^{n-k}b^k$ appears as a term from choosing b from k factors $(a+b)$ and a from $n-k$ factors.

$\Rightarrow$ can be chosen in $\binom{n}{k}$ ways

$\Rightarrow$ $a^{n-k}b^k$ appears $\binom{n}{k}$ times

$\boxed{\text{Theorem}}$ (Binomial) If $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}_{>0}$, then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Ex) Expand $(3x-2y)^4$ using the binomial thm.

Let $a = 3x$, $b = -2y$

Then $(3x-2y)^4 = (a+b)^4$

$$= \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^2 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}a^0b^4$$

$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$= 3^4x^4 - 2^3\cdot 3^3x^3y + 2^3 3^3 x^2y^2 - 3\cdot 2^5 xy^3 + 2^4y^4$$

Ex) Find the coeff of a^5b^4 in the expansion of $(a+b)^9$.

The corresponding monomial in the theorem is
$$a^{n-k}b^k = a^5b^4 \Rightarrow k=4,\ n=9$$
$$\Rightarrow \text{the coefficient } \binom{n}{k} = \binom{9}{4} = \frac{9!}{4!5!}$$

writing out these coeffs in the following way is Pascal's triangle

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ & & \vdots & & & & & & \vdots & & \end{array}$$

$\boxed{\text{Theorem}}$ For $1 \le k \le n$, $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Ex) Prove $(\star)$ $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ using Binomial thm

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} (1)^{n-k} (1)^k = (1+1)^n = 2^n$$

Ex) Prove $(\star)$ combinatorially.

Pf We can do this by proving LHS & RHS are different ways of counting elts in the same set

RHS: $2^n = |P(X)| = \#$ of subsets of X

LHS $= \sum_{k=0}^{n} \left| \{ S \subseteq X \mid |S| = k \} \right| = \#$ of subsets of X

so we are done.

Ex) Prove (⊗) inductively

Base case: For $n=0$, we know $2^0 = 1 = \binom{0}{0}$, so we are done

Ind assump: Assume $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ for some n.

Then $\sum_{k=0}^{n+1} \binom{n+1}{k} = \binom{n+1}{0} + \left(\sum_{k=1}^{n} \binom{n+1}{k}\right) + \binom{n+1}{n+1}$

$= 1 + \left(\sum_{k=1}^{n} \binom{n}{k-1} + \binom{n}{k}\right) + 1$

$= 2 + \sum_{k=1}^{n} \binom{n}{k-1} + \sum_{k=1}^{n} \binom{n}{k}$

by reindexing $= 2 + \left(\sum_{k=0}^{n} \binom{n}{k}\right) - \binom{n}{n} + \left(\left(\sum_{k=0}^{n} \binom{n}{k}\right) - \binom{n}{0}\right)$

by Ind Assump $= 2 + (2^n - 1) + (2^n - 1) = 2 \cdot 2^n = 2^{n+1}$

so the result is proven

Ex) Prove $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$

Pf

We will use that $\binom{i}{k} = \binom{i+1}{k+1} - \binom{i}{k+1}$

Then $\sum_{i=k}^{n} \binom{i}{k} = \binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k}$

$= 1 + \binom{k+2}{k} - \binom{k+1}{k+1} + \binom{k+3}{k+1} - \binom{k+2}{k+1} + \dots + \binom{n+1}{k+1} - \binom{n}{k+1}$

$= \binom{n+1}{k+1}$ by cancelling

6.8 Pigeonhole Principle

Claim: If we have 7 pigeons in 4 pigeonholes, we know at least 2 pigeons are in the same hole.

Theorem (Pigeonhole Principle) If n pigeons fly into k pigeonholes and $k < n$, some pigeonhole contains at least two pigeons

Ex) If there are 10 students w/ first names Alice, Bernard, + Charles + last names Lee, McDuff, + Ng. Show at least 2 people have same 1st + last names.

pf

There are $3 \cdot 3 = 9 < 10$ possible names. If names are the holes + people are the pigeons, ≥ 2 must have the same names.

Ex) Prove this again by contradiction.

Pf Suppose no 2 people have same 1st + last names.

But there are 9 possible names

$\Rightarrow$ must be ≤ 9 people

but there are 10 people $\rightarrow\leftarrow$

Theorem (Pigeonhole Principle, reformulated)

If $f: X \to Y$ where X, Y finite, and $|X| > |Y|$, then $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$ where $x_1 \neq x_2$.

Ex) Show that if we have 151 math courses numbered between 1 + 300 (inclusive), at least 2 are numbered consecutively.

Pf

Call these numbers

$c_1, c_2, \ldots, c_{151}$.

Then $\{c_1+1, c_2+1, \ldots, c_{150}+1\} \subseteq [1, 300]$

$$|\{c_1, c_2, \ldots, c_{151}\}| + |\{c_1+1, c_2+1, \ldots, c_{150}+1\}| = 301$$

By PP, at least 2 of these must coincide.

Since $c_1, c_2, \ldots, c_{151}$ are distinct, there must be some c_k+1 such that $c_k+1 = c_j$ for some $j \in [1, 151]$.

$\Rightarrow c_{j-1}, c_j$ are consecutive.

Ex) The library has a list of 89 sitcom DVD's, marked available (A) or unavailable (U). Show that there are ≥ 2 A items on the list exactly 9 items apart.

Ex)

let a_i be the position of i^{th} A item.

$\{a_1, a_2, \ldots, a_{50}\}$ are the positions of A items

Consider $\{a_1+9, a_2+9, \ldots + a_{50}+9\}$.

Then in the above sets, there are 100 positions, with values in $[1, 89+9] = [1, 98]$.

Thus, ≥ 2 must coincide, so

Some $a_i = a_k + 9 \Rightarrow a_i - a_k = 9$ for some i, k.

Thm (Pigeonhole Principle - Part 3)
Let $f: X \to Y$ for X, Y finite sets. Suppose $|X| = n$ and $|Y| = m$. Let $k = \lceil n/m \rceil$. Then there are at least k distinct values $a_1, \ldots, a_k \in X$ such that

$$f(a_1) = f(a_2) = \cdots = f(a_k).$$

Ex I have 8 pairs of socks, each a different color, loose in my laundry basket. How many socks must I grab to ensure I have chosen at least one matching pair?

$8+1$ (worst case, the first 8 were all different, so the 9^{th} will ensure a pair)

What if I also have matching gloves for each pair of socks also loose in the basket. How many pieces of clothing must I grab before I know I'll have a full matching set?

$8+8+8+1$ (worst case the first $3 \cdot 8$ were all different, so $3 \cdot 8 + 1$ will ensure a match)

Ex Show that in our 61 class of 210 students (assuming all are in year 1, 2, 3, or 4), there exist at least 96 in year 1, 65 in year 2, 40 in year 3, 9 in year 4.

If not there are less than or equal to

$$(96-1) + (65-1) + (40-1) + (9-1) = 206 < 210$$

students. Therefore the result follows by PP.

Ex There are 34 times at which UCLA can schedule a class. If there are 846 different classes, what is the minimum number of rooms they need?

By PP, there will be a time with at least $\left\lceil \frac{846}{34} \right\rceil = 25$ classes, so they need at least 25 rooms.

Challenge

Ex) Suppose UCLA pairs roomates based on compatibility. Show that among 6 students, there are either 3 mutually compatible, or 3 mutually incompatible.
(C) (I)

pf

Call them $S_1, S_2, S_3, S_4, S_5, S_6$.

pairs below

(S_1, S_2) (S_1, S_3) (S_1, S_4) (S_1, S_5) (S_1, S_6)

are either C or I.

Here, in PP-P3 $X = \{(S_1, S_2), (S_1, S_3) \ldots, (S_1, S_6)\}$

and $Y = \{C, I\}$

Thus $k = \lceil 5/2 \rceil = 3$

$\Rightarrow$ there are at least 3 pairs in X with same output

$\Rightarrow$ 3 C's or 3 I's

Case 1: 3 C's

Suppose they are (S_1, S_i) (S_1, S_j) (S_1, S_k)

If any (S_i, S_j) (S_j, S_k) (S_i, S_k) is C, (say (S_i, S_j)) then

$\{(S_i, S_j), (S_1, S_i), (S_1, S_j)\}$ is mutually compat.

Otherwise, $\{(S_i, S_j), (S_j, S_k), (S_i, S_k)\}$ is mutually incompat.

Case 2: 3 I's

Suppose they are (S_1, S_i) (S_1, S_j) (S_1, S_k)

If any (S_i, S_j) (S_j, S_k) (S_i, S_k) is I, (say (S_i, S_j)) then

$\{(S_i, S_j), (S_1, S_i), (S_1, S_j)\}$ is mutually incompat.

Otherwise, $\{(S_i, S_j), (S_j, S_k), (S_i, S_k)\}$ is mutually compat.