École Normale Supérieure de Lyon

Internship Report

Proof of the gonality conjecture for linear ear decompositions and start of a generalization to nested ear decompositions

Author:
Xuan Hoang La

Supervisor:
Thi Ha Duong Phan

A report submitted in the fulfillment of the requirements for the first year of the master degree in computer science internship in the Institute of Mathematics of Hanoi

ENS DE LYON

Contents

1 Introduction

In 2007, Baker and Norine introduced the notion of rank of divisors on graphs similarly to the one on Riemann surfaces[3]. Baker also introduced in the same year the Brill-Noether theory for graphs and the special case of the gonality conjecture[2]. Since then, this conjecture has been proven for many different classes of graphs, for instance, metric graphs[2] or graphs of low genus[1]. Ear decompositions have been used to characterize many important classes of graphs. The one that we will be interested in is a subclass of series-parallel graphs which admits a nested ear decomposition. We will introduce all of these notions in the context of a chip firing game[4] and prove the gonality conjecture on linear ear decompositions. This proof will serve as a basis to proving this conjecture on nested ear decompositions, which is still a work in progress.

2 The Chip Firing Game

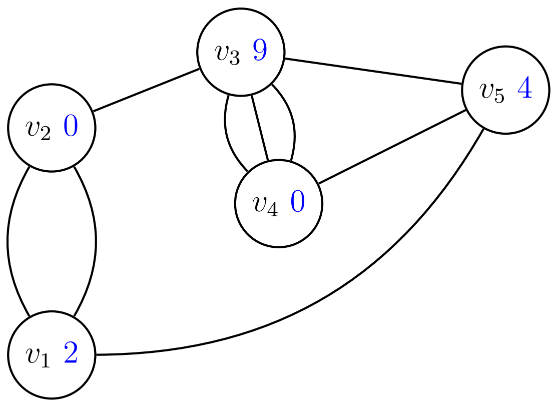
In this section, we will define the chip firing game. This definition is based on a work by R. Cori and Y. Le Borgne [4].

2.1 Configurations and topplings

Let $G = (V, E)$ be a connected multi-graph (see Definition 5.1) with no loops. Let n be the number of vertices and m, the number of edges. $X = \{v_1, v_2, ..., v_n\}$ is the set of vertex and E is a symmetric matrix where $e_{i,j}$ is the number of edges between v_i and v_j. $\varepsilon^{(i)}$ will denote the vector where 1 is assigned to the vertex v_i and 0 is assigned to the others.

Definition 2.1 (Configuration)**.** A configuration, also called a divisor, is a vector of $\mathbb{Z}^n$. Each configuration may be considered as assigning tokens to the vertices. The degree $deg(D)$ of a configuration D is the sum of the D_i's.

Example 1. The configuration $D = \begin{pmatrix} 2 \\ 0 \\ 9 \\ 0 \\ 4 \end{pmatrix}$:

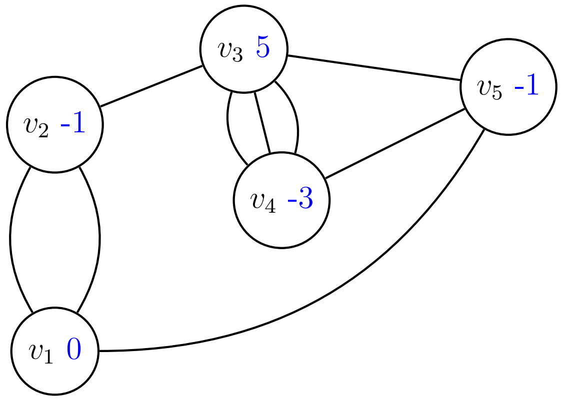


Definition 2.2 (Laplacian configuration)**.** The Laplacian configuration $\Delta^{(i)}$ is given by:

$$\Delta^{(i)} = d_i \varepsilon^{(i)} - \sum_{j=1}^{n} e_{i,j} \varepsilon^{(j)}$$

where $d_i = deg(v_i)$.

Example 2. The Laplacian configuration $\Delta^{(3)} = \begin{pmatrix} 0 \\ -1 \\ 5 \\ -3 \\ -1 \end{pmatrix}$ of the graph from Example 1:

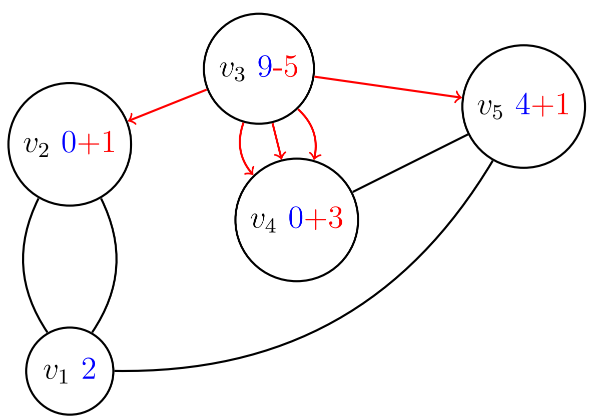


Remarks:

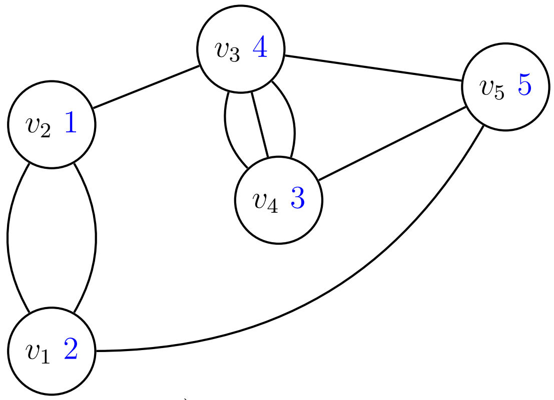
- These configurations corresponds to the rows of the Laplacian matrix (see Example 11).
- The degree of a Laplacian configuration is 0.

Definition 2.3 (Toppling)**.** The transition from a configuration D to the configuration $D - \Delta^{(i)}$ is called a toppling or a firing.

Example 3. The toppling D to $D - \Delta^{(3)}$ where D is the configuration from Example 1:
$D =$



$D - \Delta^{(3)} =$



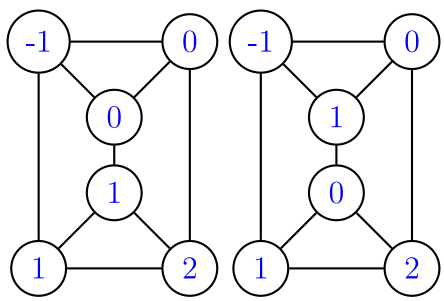
Definition 2.4 (Toppling equivalence)**.** We denote by L_G the subgroup of $\mathbb{Z}^n$ generated by the $\Delta^{(i)}$'s. We say that two configurations D and D' are toppling equivalent, denoted $D \sim_{L_G} D'$, if $D - D' \in L_G$.

Definition 2.5 (Effective configuration)**.** An effective configuration is a configuration with all non negative components, in other words, a configuration D is effective if and only if $D \in \mathbb{N}^n$. We denote $\mathbb{P}$ the set of effective configurations.

Definition 2.6 (L_G.effective configuration)**.** A configuration is L_G.effective if and only if it is equivalent to an effective configuration. We denote $\mathbb{E}$ the set for L_G.effective configurations.

Remark: It is clear that a configuration with negative degree is not L_G.effective since two equivalent configurations have the same degree.

Example 4. Two configurations with positive degree but the one on the left is L_G.effective and the other one is not.



2.2 Configuration rank

Definition 2.7 (Rank of a configuration)**.** The rank $r(D)$ of a configuration D is the integer equal to:

- -1 if D is not L_G.effective.
- If D is L_G.effective, the largest integer r such that for any effective configuration λ of degree D, $D - \lambda$ is L_G.effective.

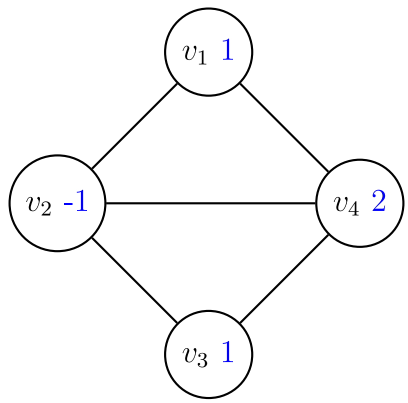
Remarks:

- The rank of a configuration is the maximum number of chips that we can remove (in any way we like) such that the divisor is still "good" (L_G.effective).
- We can also give a compact formula for $r(D)$:

$$r(D) + 1 = min\{deg(\lambda) | \lambda \in \mathbb{P}, D - \lambda \notin \mathbb{E}\}$$

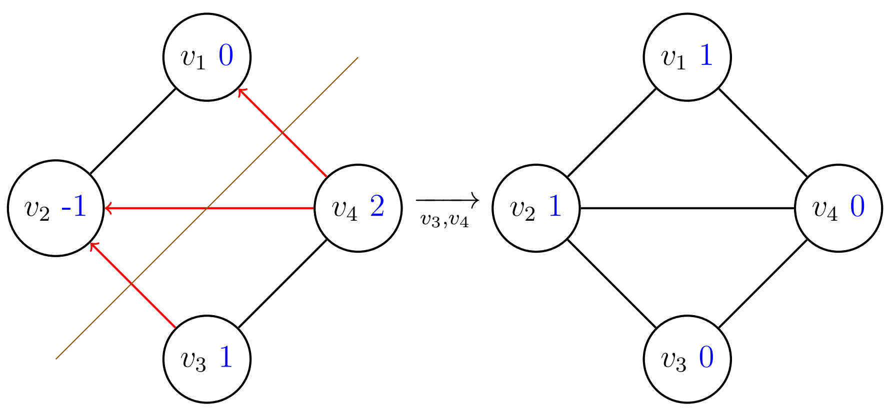
Definition 2.8 (The chip firing game)**.** A chip firing game is a game where, given a graph G, a starting configuration D and a rank r, the goal is to determined if $r(D) \geq r$ by finding for each $\lambda \in \mathbb{P}$ with $deg(\lambda) = 1$, the equivalent effective configuration to $D - \lambda$, which is called a winning position.

Example 5. Let's play the chip firing on the following graph with the following starting configuration D:



The amount of chips that we will be removing is $r = 1$. The goal is then to determine if D has rank at least 1.

- $D - \varepsilon^{(1)}$:



  Note that when we topple vertices consecutively in this case v_3 then v_4, it is the same as toppling the set $\{v_3, v_4\}$, which means that we only care about edges going from a vertex inside of our set to one that is outside since we topple every vertex in our set only once, so the edges between the vertices of the same set do not matter, as the exchange of chips cancel each other out.

- Similarly, $D - \varepsilon^{(2)} = (1, -2, 1, 2) \xrightarrow[v_3, v_4]{} (2, 0, 0, 0)$

- $D - \varepsilon^{(3)} = (1, -1, 0, 2) \xrightarrow[v_1, v_4]{} (0, 1, 1, 0)$
- $D - \varepsilon^{(4)} = (1, -1, 1, 1) \xrightarrow[v_1, v_3, v_4]{} (0, 2, 0, 0)$

So, $r(D) \geq 1$.

3 Proof of the gonality conjecture for linear ear decompositions

From now on, we will only talk about simple graphs which will be denoted by $G = (V, E)$, where $V = \{v_1, v_2, ..., v_n\}$ is the set of vertices and $E \subset V \times V$ is the set of edges.

3.1 The gonality conjecture

The gonality conjecture was first defined on graphs by Baker and Norine in 2007 [2].
Definition 3.1 (Gonality of a graph)**.** We define the gonality $gon(G)$ of a graph as follows:

$$gon(G) = min\{deg(D)|D \in Div(G), r(D) \geq 1\}$$

Gonality conjecture: For all integer $g \geq 0$, the gonality $gon(G)$ of any graph of genus g verifies:

$$gon(G) \leq \lfloor \frac{g+3}{2} \rfloor$$

In the following sections, the notations and the proof are my personal contributions during the internship, with the help and supervision of Dr. PHAN Thi Ha Duong.

3.2 Linear ear decompositions

Definition 3.2 (Ear)**.** An ear of a graph is a path $P = (u_1, u_2, ..., u_p)$, where every vertex has degree 2 with the eventual exception of the two endpoints which may coincide:

$$\forall i \notin \{1, p\}, deg(u_i) = 2$$

Definition 3.3 (Ear decomposition)**.** An ear decomposition of a graph is a decomposition of its set of vertices $V = \bigcup_{i=1}^{g} V_i$, where V_1 is a cycle and $\forall 1 < i \leq g, V_i = (v_{i,1}, v_{i,2}, ..., v_{i,p_i})$ are ears such that the endpoints of any ear belong to earlier ears:

$$\forall 1 < i \leq g, v_{i,1}, v_{i,p_i} \in \bigcup_{k=1}^{i-1} V_k$$

and the internal vertices do not

$$\forall 1 < i \leq g, \forall j \notin \{v_{i,1}, v_{i,p_i}\}, v_{i,j} \notin \bigcup_{k=1}^{i-1} V_k$$

A proper ear decomposition is an ear decomposition where, for each ear, the endpoints are distinct:

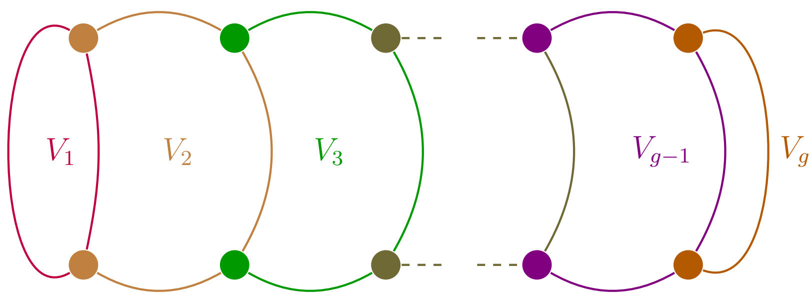
$$\forall 1 < i \leq g, v_{i,1} \neq v_{i,p_i}$$

Remark: Note that if G has an ear decomposition then its genus is equal to its number of ears, the first cycle included.

Definition 3.4. We define Γ as the set of graphs that admit linear ear decomposition, which is a proper ear decomposition such that the endpoints of the i-th ear belong to the $(i-1)$-th ear:

$$\forall 1 < i \leq g, v_{i,1}, v_{i,p_i} \in V_{i-1}$$

Example 6. This is what a graph in Γ looks like:



Theorem 1. *The gonality conjecture is true for graphs in* Γ.

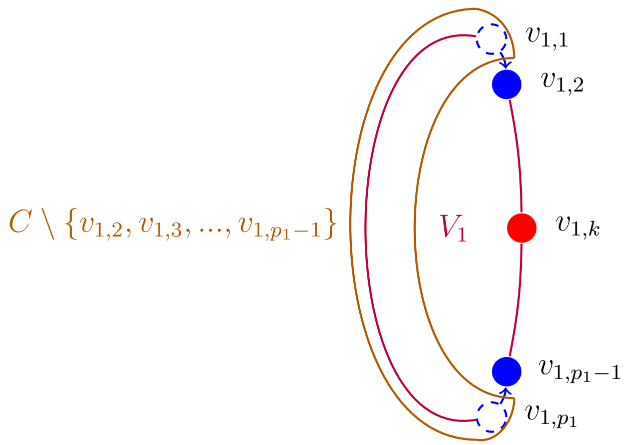
Proof. First, note that to prove the gonality conjecture, it suffices to prove that there exists a divisor D with $deg(D) = \lfloor \frac{g+3}{2} \rfloor$ that has rank at least 1. In terms of a chip firing game, there exists a configuration with $\lfloor \frac{g+3}{2} \rfloor$ chips such that we can always end up in a winning position if any vertex were to lose a chip. We will call such a configuration, a winnable starting configuration.

For convenience, we will consider a divisor (or configuration) as a tuple of elements of V where each element corresponds to the placement of a chip on the corresponding vertex. Let us look at winnable starting configurations for graphs in Γ with small genus ($g \in \{1, 2, 3\}$).

For $g = 1$:
We have a simple cycle and $\lfloor \frac{g+3}{2} \rfloor = \lfloor \frac{1+3}{2} \rfloor = 2$ chips. Placing these two chips on any distinct vertices (one on each) will give us a winnable starting configurations. Let us consider one such configuration and denote the cycle $C = \{v_{1,1}, v_{1,2}, ..., v_{1,p_1}, v_{1,p_1+1}, ..., v_{1,p_1+l}\}$ where $v_{1,p_1+l} = v_{1,1}$ and both $v_{1,1}$ and v_{1,p_1} contain one chip each.

If $v_{1,1}$ or v_{1,p_1} loses a chip, it is clear that no one vertex ends up in debt so we have a winning position. Now, if another vertex loses a chip, say $v_{1,k}$, we will show that we can clear its debt by a sequence of topplings. Suppose that $1 < k < p_1$, recall that toppling a set of vertices only move chips from the vertices incident to the edges going outside of the set. With that in mind, let us see what happens when we topple $C \setminus \{v_{1,2}, v_{1,3}, ..., v_{1,p_1-1}\}$.

We will end up in the configuration $(v_{1,2}, v_{1,p_1-1})$:



As a result, if we topple consecutively everything except the internal nodes of the path between the vertices with the chips, then we will move the chips towards the vertex in debt. More formally, by toppling the sets

$$C \setminus \{v_{1,2}, v_{1,3}, ..., v_{1,p_1-1}\}$$

then

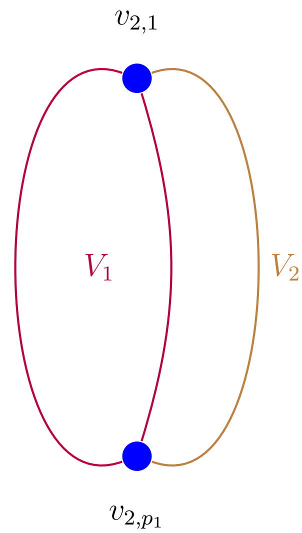
$$C \setminus \{v_{1,3}, v_{1,4}, ..., v_{1,p_1-2}\}$$

until

$$C \setminus \{v_{1,1+k'}, v_{1,3}, ..., v_{1,p_1-k'}\}$$

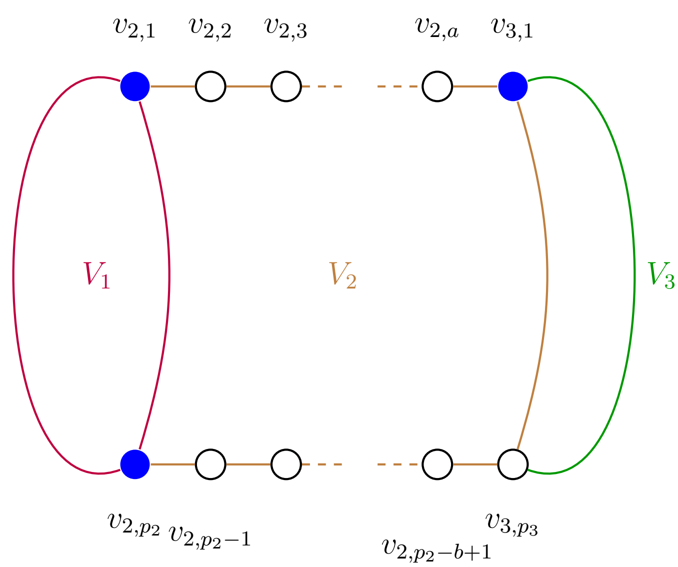
where $k' = min\{i | 1 + i = k - 1 \text{ or } p_1 - i = k + 1\}$, we will cover $v_{1,k}$'s debt and end up in a winning position. The same strategy works when $p_1 < k < p_1 + l$.

For $g = 2$:
In the previous case, we have proven that if a vertex, that lies on a path, where all internal nodes have degree 2, between two vertices with one chip each, loses a chip, then we can cover its debt. Since we have $\lfloor \frac{g+3}{2} \rfloor = \lfloor \frac{2+3}{2} \rfloor = 2$ chips. It suffices to put these chips on $v_{2,1}$ and v_{2,p_2}, we will call this configuration D_2:

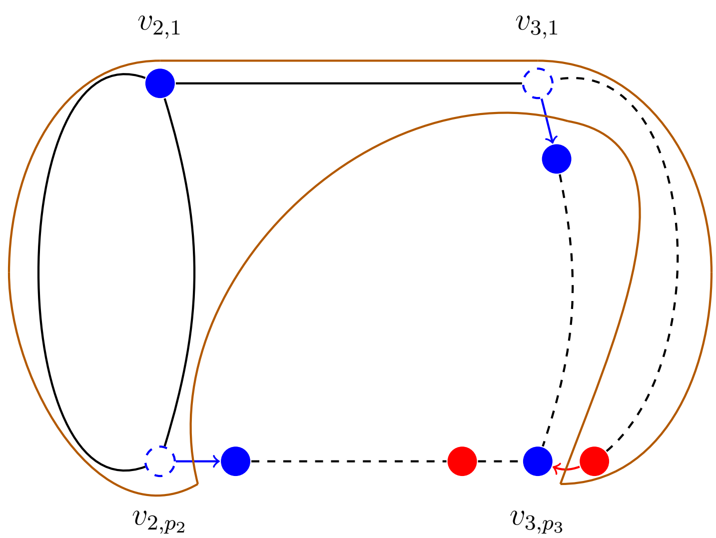


We can notice that if a vertex of V_1 loses a chip, since $v_{2,1}, v_{2,p_2} \in V_1$ and V_1 is a cycle, such a vertex will lie in between $v_{2,1}$ and v_{2,p_2}. And if it is in V_2, since V_2 is an ear and $v_{2,1}$ and v_{2,p_2} are its endpoints, we once again can cover its debt.

For $g = 3$:
By renumbering the vertices of V_3 if necessary, we can suppose that $v_{3,1}$ is closer to $v_{2,1}$ than v_{2,p_2}. We will denote the distance between $v_{2,1}$ and $v_{3,1}$ a and the distance between v_{2,p_2} and v_{3,p_3} b. Note that if $a = 0$, then $v_{2,1} = v_{3,1}$ and if $b = 0$ then $v_{2,p_2} = v_{3,p_3}$. Since we have $\lfloor \frac{g+3}{2} \rfloor = \lfloor \frac{3+3}{2} \rfloor = 3$ chips, we will put two chips on $v_{2,1}$ and v_{2,p_2}, one on each, like in D_2. The last chip, we will put it on $v_{3,1}$ if $a \geq b$, or on v_{3,p_3} otherwise, this configuration will be denoted D_3. Suppose that $a \geq b$, we have the following graph:



Note that $v_{2,a+1} = v_{3,1}$ and $v_{2,p_2-b} = v_{3,p_3}$.
We will prove that D_3 is a winnable starting position. If a vertex in V_1 or between $v_{2,1}$ and $v_{3,1}$ were to lose a chip, we have already seen that we can cover its debt. However, if the one to lose a chip is located somewhere between v_{2,p_2} and $v_{3,1}$, then the same strategy will not succeed in moving the chips toward the vertex in debt as the a path between v_{2,p_2} and $v_{3,1}$ necessarily runs through a vertex with degree 3 and this is what happens when you apply this strategy:



The dashed lines shows the paths where we have not shown that we can cover its vertices debt.
As you can see, if we topple everything except the internal nodes of a path between v_{2,p_2} and $v_{3,1}$ then it may help with the debt found between v_{2,p_2} and v_{3,p_3} but it creates another one between $v_{3,1}$ and v_{3,p_3} and we fall into the same problem.

Nevertheless, we can still prove that D_3 is a winnable starting configuration:
Suppose the vertex losing a chip is between v_{2,p_2} and v_{3,p_3}, say v_{2,p_2-k} for some $1 \leq k \leq b$. By toppling

$$V_1$$

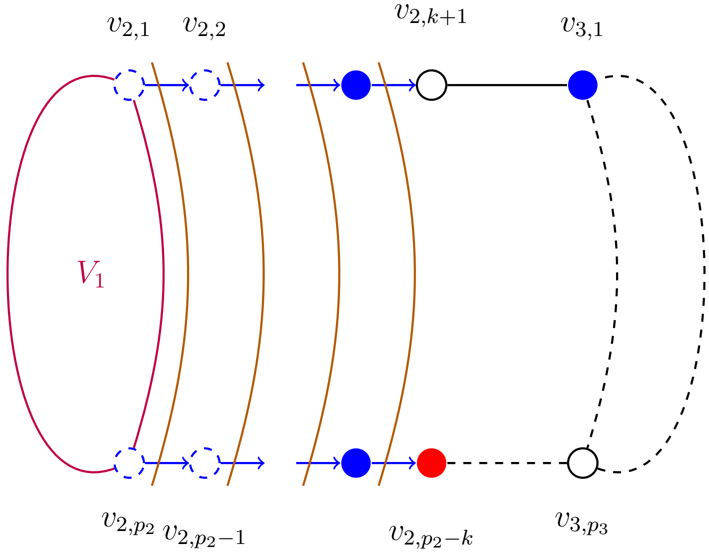
then

$$V_1 \bigcup \{v_{2,2}\} \bigcup \{v_{2,p_2-1}\}$$

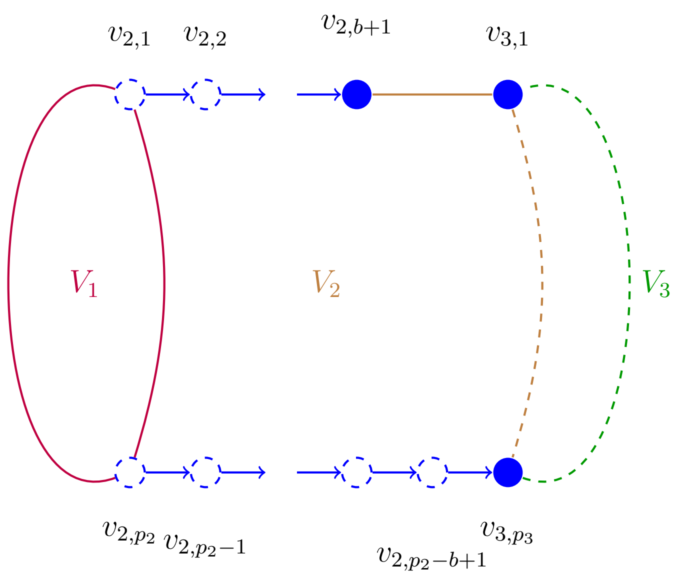
until

$$V_1 \bigcup \{v_{2,2}, v_{2,3}, ..., v_{2,k}\} \bigcup \{v_{2,p_2-1}, v_{2,p_2-2}, ..., v_{2,p_2-k+1}\}$$

then we will cover v_{2,p_2-k} debt:



Note that these topplings are only possible since $a \geq b \geq k$. With this strategy, we can cover every vertex between v_{2,p_2} and v_{3,p_3}. Notice that with this strategy, we can also reach the equivalent configuration $D_3' = (v_{3,1}, v_{3,p_3}, v_{2,b+1})$:

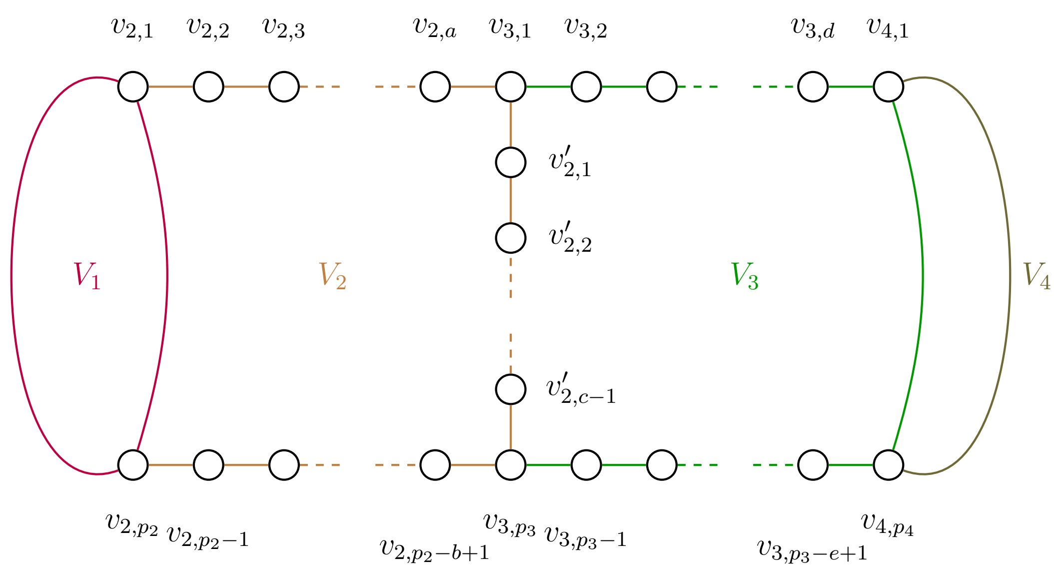


Note that from this configuration, whether a vertex in V_3 or a vertex in V_2, between $v_{3,1}$ and v_{3,p_3} loses a chip, we can cover its debts the same way we did for the case $g = 1$. Now, if $a \leq b$, we can use the same strategy by symmetry.

Let us prove the following induction hypothesis for all $i \geq 4$:
***H*(*i*): "For all $G \in \Gamma$ with genus i, G has two equivalent winnable starting**

configurations D_i and D'_i that verify $v_{i-2,1}, v_{i-2,p_{i-2}} \in D_i$ and $v_{i,1}, v_{i,p_i} \in D'_i$."

Let us prove $H(4)$:
By renumbering the vertices if necessary, we will suppose that the vertices $(v_{i,1})_{i\in\{2,3,4\}}$ are "on the same side". We will denote the distance between $v_{2,1}$ and $v_{3,1}$ a, between v_{2,p_2} and v_{3,p_3} b, between $v_{3,1}$ and v_{3,p_3} c, between $v_{3,1}$ and $v_{4,1}$ d and finally between v_{3,p_3} and v_{4,p_4} e:



Once again, note that $v'_{2,0} = v_{3,1}$, $v'_{2,c} = v_{3,p_3}$, $v_{3,d+1} = v_{4,1}$ and $v_{3,p_3-e} = v_{4,p_4}$. Also, a, b, c, d are nonnegative but c is positive since this ear decomposition is proper ($v_{3,1} \neq v_{3,p_3}$).
We have 3 chips in total. We will start by putting two chips on $v_{2,1}$ and v_{2,p_2}, one on each. The third chip's placement will depend on a, b, c, d and e. Let us split this problem into different cases:

If $a = b$:
First, note that the two chips on $v_{2,1}$ and v_{2,p_2} already cover V_1. By using the same strategy that we used when $g = 3$, we can cover everything between $v_{2,1}$ and $v_{3,1}$ and between v_{2,p_2} and v_{3,p_3}. Formally, the topplings are

$$V_1$$

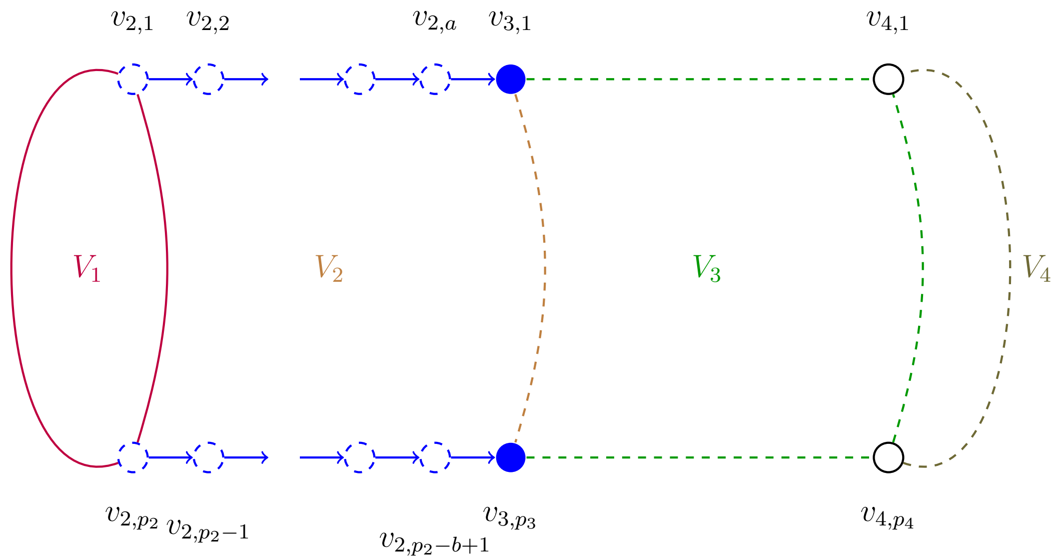
then

$$V_1 \bigcup \{v_{2,2}\} \bigcup \{v_{2,p_2-1}\}$$

until

$$V_1 \bigcup \{v_{2,2}, v_{2,3}, ..., v_{2,a}\} \bigcup \{v_{2,p_2-1}, v_{2,p_2-2}, ..., v_{2,p_2-b+1}\}$$

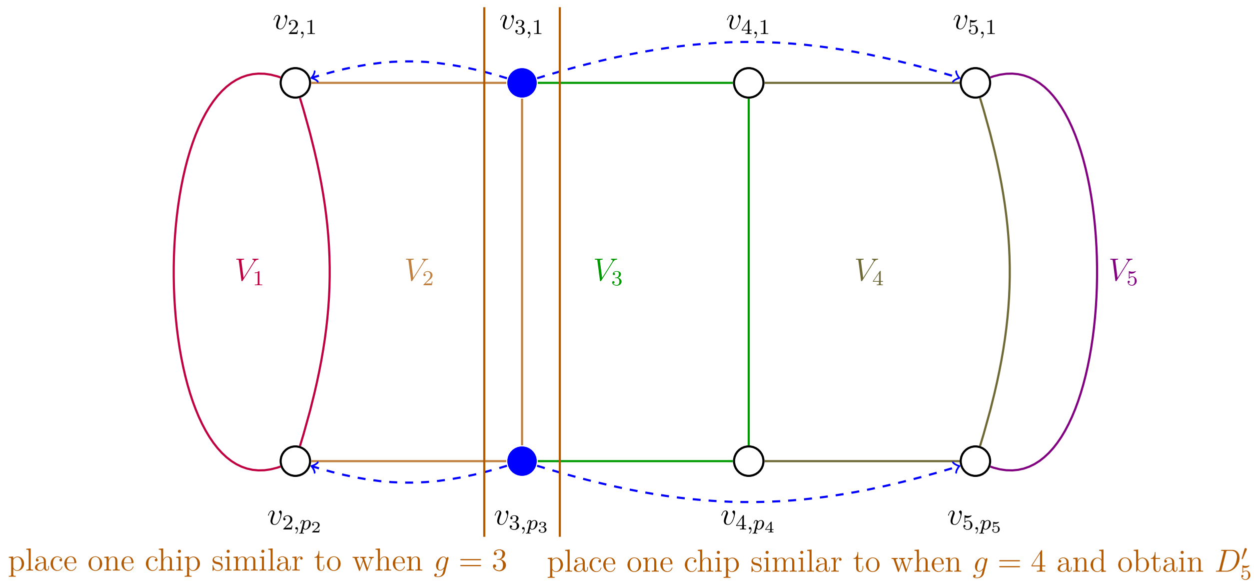
which is possible since $a = b$.



Since the chips are at $v_{3,1}$ and v_{3,p_3}, everything in between is covered. Moreover, you may recognize that we are in a similar situation to the one for $g = 3$. As a result, placing the third chip on $v_{4,1}$ when $d \geq e$ and on v_{4,p_4} otherwise will work in the same way. We will call the following configuration $D_4 = (v_{2,1}, v_{2,p_2}, v_{4,k})$ where $k = 1$ if $d \geq e$ and $k = p_4$ otherwise. Similarly to how we have proven that D_3 is a winnable starting configuration and $D_3 \sim D_3'$, we have here the existence of $D_4' \sim D_4$ such that $v_{4,1}, v_{4,p_4} \in D_4'$ and both are winnable starting configurations, hence $H(4)$.

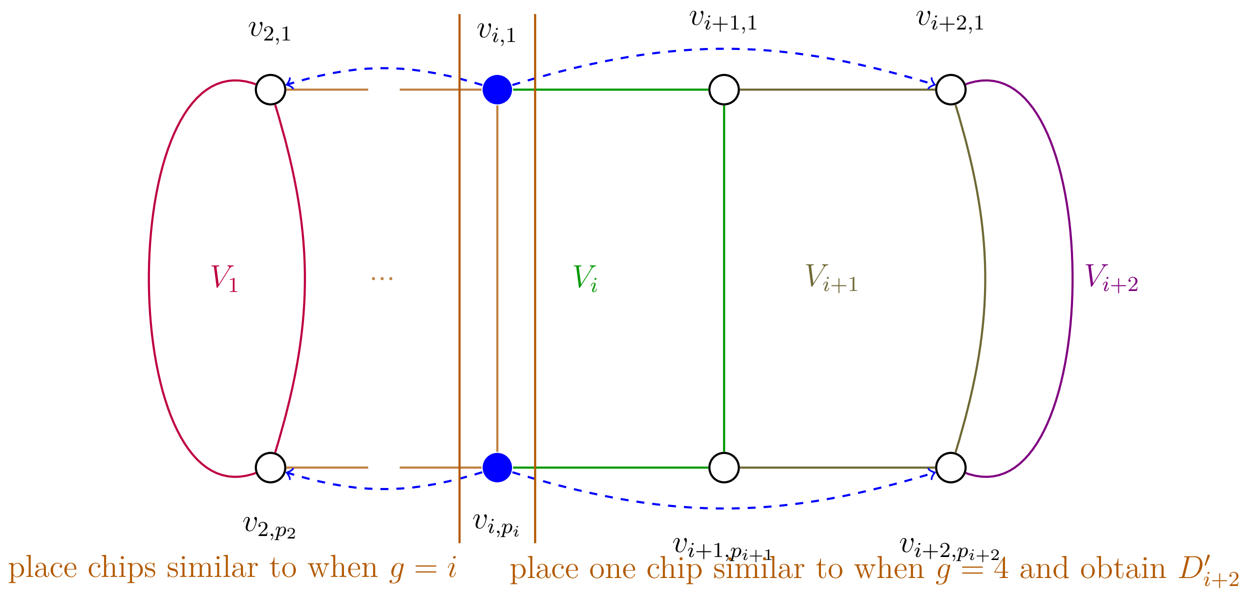
For the other cases, we use the same kind of techniques, you can see the complete proof in the Appendix (5.1).

Let us prove $H(5)$:
First, when $g = 5$, we have 4 chips. Let start by placing two of those on $v_{3,1}$ and v_{3,p_3}, one on each. Note that for $g = 3$, there is a way to place a chip to move from D_3 to D_3' and cover everything. We can do the same by placing a chip the same way, somewhere in V_1 and V_2 to cover them both. Similarly, for $g = 4$, we have seen that we also only need one chip to move from D_4 to D_4' and cover everything. We will do the same by placing a chip somewhere in V_3 and V_4 to cover the rest. We can call it D_5 and to obtain D_5', we simply need to do the same kind of topplings as for $g = 4$ when we went from D_4 to D_4'.



Suppose $H(i)$ is true for some $i \geq 4$, let us prove $H(i+2)$:

Let $G \in \Gamma$ such that G has genus $g = i+2$. We have $\lfloor \frac{i+5}{2} \rfloor$ chips. Since $H(i)$ is true, we know that there is a winnable starting configuration D'_i such that $v_{i,1}, v_{i,p_i} \in D'_i$. We will place our chips according to D'_i. This will allow us to cover $V_1, V_2, ..., V_{i-1}$. Moreover, since D'_i only uses at most $\lfloor \frac{i+3}{2} \rfloor$ chips, we have one chip left and we can place it somewhere in V_i or V_{i+1} the same way we did to move from D_4 to D'_4 when $g = 4$, and call it D_{i+2}. By doing so, we will be able to cover the rest as well as obtaining D'_{i+2}.



To conclude, for all $G \in \Gamma$, G has a winnable starting configuration, or in other words, the gonality conjecture is verified for G.

□

4 Generalization of the proof of the gonality conjecture on nested ear decompositions

4.1 Nested ear decompositions

Definition 4.1 (Tree ear decomposition)**.** A tree ear decomposition is a proper ear decomposition where the endpoints of every ear belong to a same earlier ear:

$$\forall 1 < i \leq g, \exists! j < i, v_{i,1}, v_{i,p_i} \in V_j$$

Definition 4.2 (Nested ear decomposition)**.** A nested ear decomposition is a tree ear decomposition where, for every ear, the different pairs of endpoints located on that ear form nested intervals:

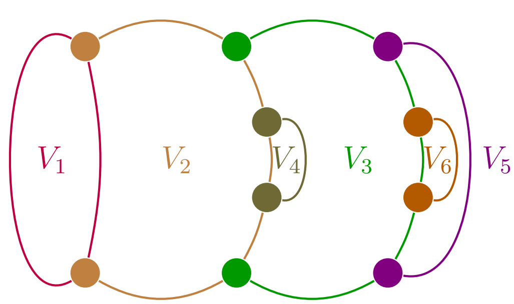
$$\forall 1 \leq i \leq g, \forall (j_1, j_p), (k_1, k_p) \in \{(l_1, l_p) | \exists i \leq l \leq g, v_{l,1} = v_{i,l_1}, v_{l,p_l} = v_{i,l_p}\},$$

$$j_1 \leq k_1 \leq k_p \leq j_p \text{ (1) or } k_1 \leq j_1 \leq j_p \leq k_p \text{ (2)}$$

For this condition to be well defined, we suppose that $V_1 = (v_{1,1}, v_{1,2}, ..., v_{1,p_1})$ is an ear and $v_{1,1}$ and v_{1,p_1} are linked by an edge.

Remark: When having two pairs of endpoints $v_j = (v_{j,1}, v_{j,p_j})$ and $v_k = (v_{k,1}, v_{k,p_k})$ that belong to the same ear, suppose that they correspond respectively to the pairs of integers (j_1, j_p) and (k_1, k_p), we call condition (1) "v_k is nested in v_j" and (2) "v_j is nested in v_k".

Example 7. An example of a nested ear decomposition:



Each colored arc represents an ear, the endpoints are highlighted and $v_{1,1} = v_{2,1}$ and $v_{1,p_1} = v_{2,p_2}$.

Theorem 2. *The gonality conjecture is true for graphs that admit a nested ear decomposition.*

The following sections will provide us with smaller results that will help us to prove this theorem

4.2 Weighted trees

From now on, we will only talk about graphs with a nested ear decomposition so we will define a graph by its decomposition as in, if a graph has multiple different nested ear decompositions then each decomposition is considered a different graph. Note that a nested ear decomposition that consists of only one or two ears is also a linear ear decomposition for which we have already proven that the gonality conjecture is true. As a result, we will only consider the set N of nested ear decompositions of genus at least 3.

Definition 4.3 (Weighted binary trees)**.** We define the set Θ as follows:
$T = (g, V, E, w_V, w_E, n_1, n_2, n_3, n_4)$ is in Θ if it verifies all of the following conditions:

1. $g \geq 3$.
2. (V, E) is a tree such that:
   - $V = \{v_2, v_3, ..., v_g\}$.
   - $\forall v \in V, 1 \leq deg(v) \leq 3$.
3. $w_V : V \to \{1, 2, ..., g-1\} \times \mathbb{N}^*$ is such that:
   - $\forall 1 < i \leq g, w_V(v_i)_0 < i$.
   - For all i, j such that $i < j$ and $(v_i, v_j) \in E, w_V(v_i)_0 = w_V(v_j)_0$ or $w_V(v_j)_0 = i$.
4. $w_E : E \to \mathbb{N}^2$ verifies:
   For all $1 < i \leq g$, let $E_i = \{(v_j, v_k) \in E | w_V(v_j)_0 = w_V(v_k)_0 = i$ or $w_V(v_k)_0 = j = i\}$, then $\sum_{e \in E_i} w_E(e)_0 + w_E(e)_1 < w_V(v_i)_1$. Similarly, we define $E_1 = \{(v_j, v_k) \in E | w_V(v_j)_0 = w_V(v_k)_0 = 1\}$ and we need $n_2 + n_3 + \sum_{e \in E_1} w_E(e)_0 + w_E(e)_1 < n_1$.
5. $n_1 \geq 1$.
6. $n_4 > 1$.

Remarks:

- We work with undirected graphs so $(v_i, v_j) = (v_j, v_i) \in E$.
- When rooted, a graph in Θ is a binary tree with weighted edges and nodes.

Lemma 1. *Let $G \in N$, G has a nested ear decomposition $\bigcup_{i=1}^{g} V_i$ where g is G's genus. The following map is an bijection from N to Θ:*

$$\theta : N \to \Theta$$

$$G = (V, E) \mapsto (g, V', E', w_V, w_E, n_1, n_2, n_3, n_4)$$

where

- *g is G's genus.*
- *$V' = \{v_2, v_3, ..., v_g\}$ where $\forall 1 < i \leq g, v_i = (v_{i,1}, v_{i,p_i})$.*

- $\forall 1 < j < k \leq g, (v_j, v_k) \in E'$ *if*

$$\exists! 1 \leq i \leq g, \exists (j_1, j_p), (k_1, k_p) / v_j = (v_{i,j_1}, v_{i,j_p}), v_k = (v_{i,k_1}, v_{i,k_p}),$$

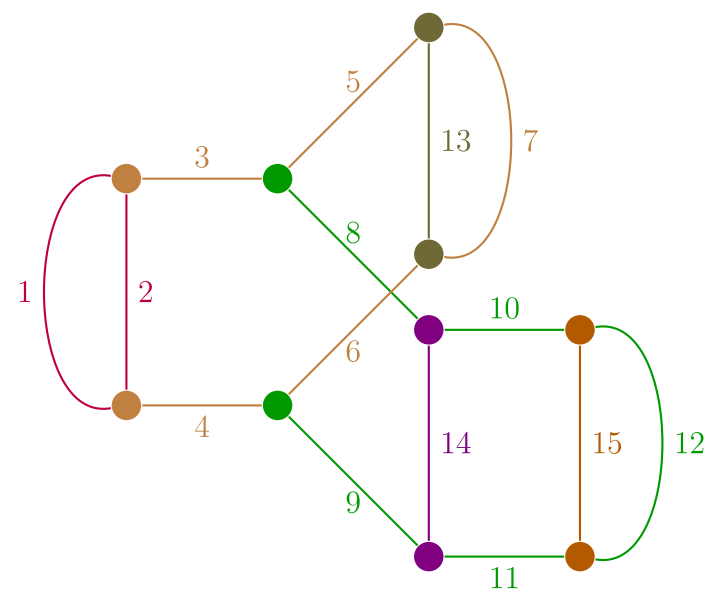
$$(j_1 \leq k_1 \leq k_p \leq j_p \text{ and } \forall m \in \{j_1{+}1, j_1{+}2, ..., k_1{-}1, k_p{+}1, k_p{+}2, ..., j_p{-}1\}, deg(v_{i,m}) = 2)$$

$$\text{or } (k_1 \leq j_1 \leq j_p \leq k_p \text{ and } \forall m \in \{k_1{+}1, k_1{+}2, ..., j_1{-}1, j_p{+}1, j_p{+}2, ..., k_p{-}1\}, deg(v_{i,m}) = 2)$$

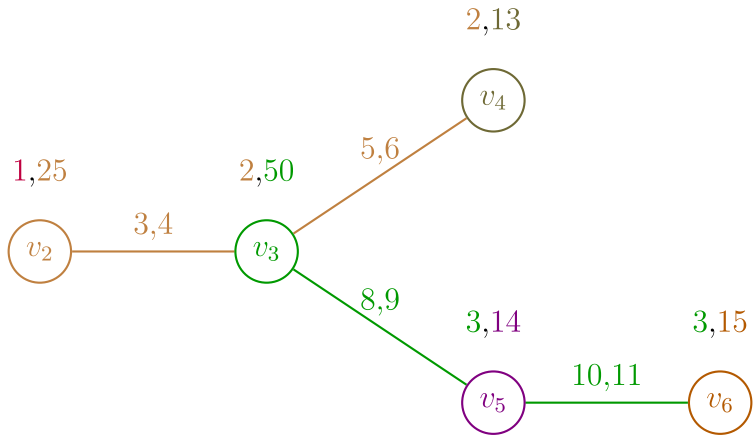
In other words, v_k is nested in v_j or v_j is nested in v_k and there are no endpoints in between. For convenience, we will simply call this condition "v_k is nested in v_j" or "v_j is nested in v_k" and if either one are true, then we say "v_j and v_k are nested".

- $\forall 1 < i \leq g, w_V(v_i) = (j, p_i - 1)$ *where* $j < i$ *is such that* $v_i \in V_j^2$*.*
- $\forall (v_j, v_k) \in E', w_E(v_j, v_k) = (|k_1 - j_1|, |k_p - j_p|)$ *where* (j_1, j_p) *and* (k_1, k_p) *are such that* $\exists! 1 \leq i \leq g, \exists (j_1, j_p), (k_1, k_p) / v_j = (v_{i,j_1}, v_{i,j_p}), v_k = (v_{i,k_1}, v_{i,k_p})$*.*
- $n_1 = p_1 - 1$, $n_2 = min\{i | deg(v_{1,1+i}) > 2\}$, $n_3 = min\{j | deg(v_{1,p_1-j}) > 2\}$ *and* $n_4 = k$ *where* k *is the unique index such that* $v_{1,1+n_2} = v_{k,1}$ *and* $v_{1,p_1-n_3} = v_{k,p_k}$*.*

Example 8. The same graph from Example 7 with the length of the paths added can be drawn as follows:



The corresponding tree in Θ is the following: $g = 6, n_1 = 3, n_2 = 0, n_3 = 0, n_4 = 2$



Remarks:

- Given a pair v_i and $j < i$ such that $v_i \in V_j^2$, there can be at most two different pairs that can be nested in v_i, one on V_j, the other on V_i, as there can be no other endpoints in between. As for pairs in which v_i is nested, there is always such a pair (except if $i = 1$) and the uniqueness comes from the same argument.

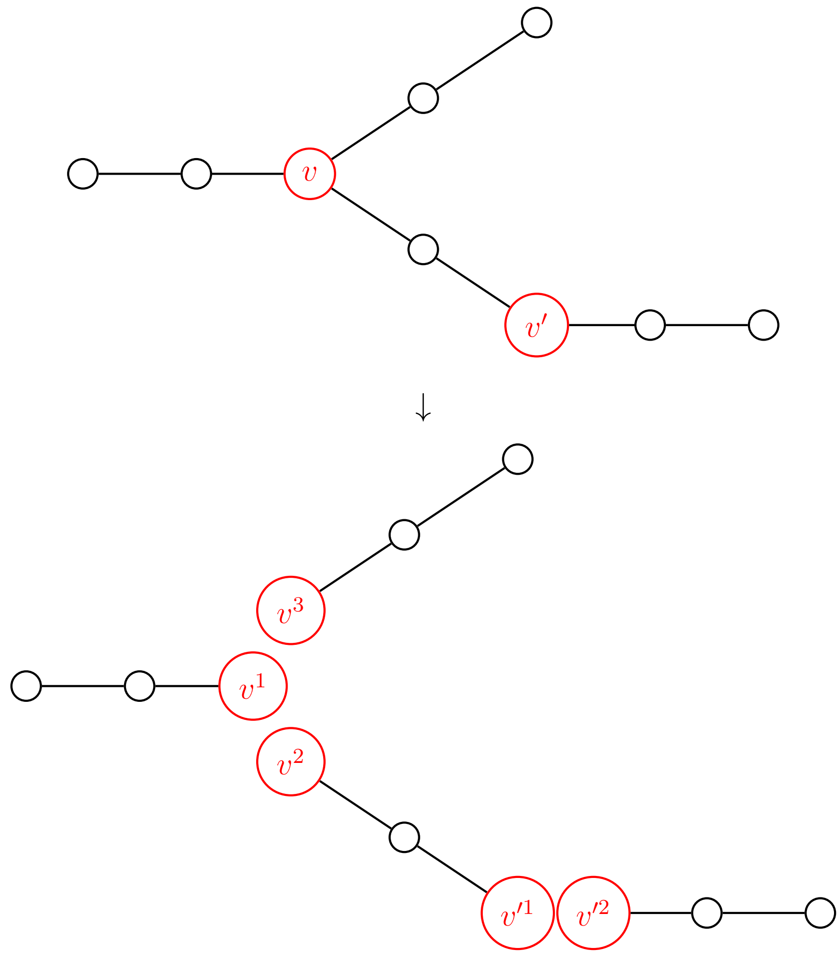
- You may have noticed that we could have defined $V = \{v_1, v_2, ..., v_g\}$ and n_1, n_2, n_3, n_4 would correspond to $n_1 = w_V(v_1)_1$ and $(n_2, n_3) = w_E(v_1, v_{n_4})$. However, v_1 was not made into a vertex in this tree since we want to keep a certain symmetry: every vertex with degree one represents a pair of endpoints with two paths, where all internal vertices have degree 2, between them.

You can see the proof in the Appendix (5.2).

4.3 Breakers

Definition 4.4 (Breaker)**.** Let $T = (*, V, E, *, *, *, *, *, *) \in \Theta$, we define a breaker as a vertex v with $deg(v) \geq 2$ that "breaks" (V, E) into components with an even number of edges. More precisely, let $E_v = \{e_1, e_2, ..., e_{deg(v)}\}$ be the edges incident to v, if we replace v by $v^1, v^2, ..., v^{deg(v)}$ and each edge $e_i = (v, u_i)$ by $e'_i = (v^i, u_i)$, then $(V \backslash \{v\} \cup \{v^1, v^2, ..., v^{deg(v)}\}, E \backslash \{e_1, e_2, ..., e_{deg(v)}\} \cup \{e'_1, e'_2, ..., e'_{deg(v)}\})$ will be a forest where each tree has an even number of edges.

Example 9. The breakers are highlighted in red:



Remark: A tree does not always have breakers, for example, a tree with an odd number of edges.

Lemma 2. *If $T = (*, V, E, *, *, *, *, *, *) \in \Theta$ is such that $|E|$ is even and V has no breaker, then there exists an unique vertex with degree 2.*

You can see the proof in the Appendix (5.3)

Definition 4.5 (Unbreakable even tree)**.** Let $T = (*, V, E, *, *, *, *, *, *) \in \Theta$ such that $|E|$ is even and V has no breaker, due to the previous lemma, we have the existence of an unique $r \in V$ with degree 2. We will call (V, E) an unbreakable even tree and r its root.

Remark: A rooted unbreakable even tree is a full binary tree.

Given any tree in Θ, we can break it into unbreakable even trees where each has one vertex with degree 2. These trees will be called "the components" of our original tree. In the following sections, we will take a closer look at a component and prove the gonality conjecture on it.

4.4 Tree components

Lemma 3. *Let $T = (*, V, E, *, *, *, *, *, *) \in \Theta$ such that (V, E) is an unbreakable even tree, there exists a configuration D that verifies the gonality conjecture on $\theta^{-1}(T)$ such that, for every leaf in (V, E), the corresponding pair of endpoints verifies:*
There is an equivalent configuration to D that has at least one chip on each endpoint of this pair.
Lemma 4. *The previous lemma still hold if we had an unbreakable even tree to which we added an additional vertex.*

The idea to prove these lemma is based on the proof of the gonality conjecture on linear ear decompositions. The complete proof is still a work in progress.

4.5 Proof scheme for the gonality conjecture on nested ear decompositions

Theorem 3. *The gonality conjecture is true for nested ear decompositions.*

Proof. Let G be a graph with a nested ear decomposition. If G has genus 2 or less, we have already seen that G has a linear ear decomposition which means that the gonality conjecture is true for G. If G has genus $g \geq 3$, we can identify G and one of its nested ear decomposition in N.

Let $T = \theta(G)$ and suppose T has an even number of edges. We then break T into components which are unbreakable even tree. We start by placing two chips on a pair of endpoints v corresponding to a leaf of any component C of T. Due to lemma 3, we know that there exists a configuration D verifying the gonality conjecture on C that also has two chips on v. We then place the necessary number of chips according to this winning configuration as to cover any possible debt that can occur in C. For the another component, say C', since T is a tree, there exists an unique path from v to C', and it must run through a breaker, say b'. We know that there is also a winning configuration D' that has two chips on b', we then place our chips according to D' except for the two chips on b'. We can do this for all other components. Suppose that we have enough chips, by placing our chips with this strategy, we are sure to cover C. Then, for any adjacent component, as in components that share a breaker b with C, since a breaker is a leaf in a component, we know that there exists an equivalent configuration that places two chips on b, so, since we have place our chips accordingly to a winning configuration with two chips on b, we know that we can also cover any debt in this adjacent component. And so on, we can so the same inductively for other components thus proving that we can cover any debt located anywhere in G.
Back to the number of chips, let $m_1, m_2, ..., m_k$ be the number of edges in each component. Note that by construction of T, the genus is the total number of edges plus 2 so the number of chips allowed is $\lfloor \frac{m_1+m_2+...+m_k+5}{2} \rfloor = \frac{m_1+m_2+...+m_k}{2} + 2$ since m_i is even for all $1 \leq i \leq k$. Without counting the two starting chips that we placed in C, in the i-th component, we placed $\lfloor \frac{m_i+5}{2} \rfloor - 2 = \frac{m_i}{2}$. Finally, we have placed exactly the allowed number of chips thus concluding our proof for the case where T has an even number of edges.

Now, if T has an odd number of edges. The idea is to remove a vertex in T then apply the even case. Then, we add back the vertex to where it belongs and apply the lemma 4. As for the number of chips, adding one more vertex gives us one more chip compared to the even case and that chip is added to the one component with an odd number of edges. □

References

[1] S. Atanasov and D. Ranganathan. A note on Brill–Noether existence for graphs of low genus. *ArXiv e-prints*, 2016.

[2] M. Baker. Specialization of linear systems from curves to graphs. *ArXiv Mathematics e-prints*, 2007.

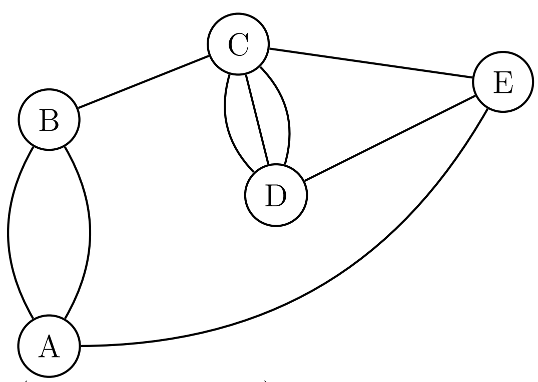
[3] M. Baker and S. Norine. Riemann-Roch and Abel-Jacobi theory. *Advances in Mathematics*, 215(2):766–788, 2007.

[4] R. Cori and Y. Le Borgne. The Riemann-Roch theorem for graphs and the rank in complete graphs. *ArXiv Mathematics e-prints*, 2013.

5 Appendix

Definition 5.1 (Multi-graph)**.** A *multi-graph* is a graph that is permitted to have parallel edges, that is, edges with the same end nodes.

Example 10.



Definition 5.2 (Laplacian matrix)**.** The Laplacian matrix L of a graph is defined by:

$$L_{i,j} = \begin{cases} deg(x_i) & \text{if } i = j \\ -e_{i,j} & \text{otherwise} \end{cases}$$

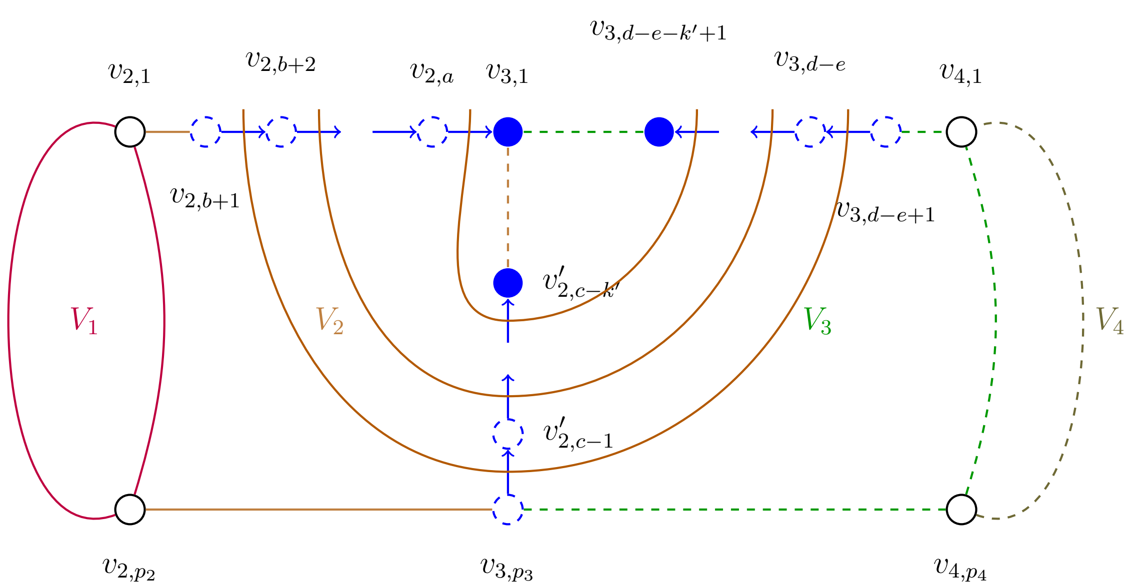
Example 11. The Laplacian matrix of the graph from Example 10:

$$L = \begin{bmatrix} 3 & -2 & 0 & 0 & -1 \\ -2 & 3 & -1 & 0 & 0 \\ 0 & -1 & 5 & -3 & -1 \\ 0 & 0 & -3 & 4 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{bmatrix}$$

5.1 Proof for the case $H(4)$ for linear ear decompositions

If $a > b$:

- **If $d > e$:**
  Let $D_4 = (v_{2,1}, v_{2,p_2}, v_{3,d-e+1})$ and let us prove that it is a winnable starting configuration. First, by toppling
  $$V_1$$
  then
  $$V_1 \bigcup \{v_{2,2}\} \bigcup \{v_{2,p_2-1}\}$$
  until
  $$V_1 \bigcup \{v_{2,2}, v_{2,3}, ..., v_{2,b}\} \bigcup \{v_{2,p_2-1}, v_{2,p_2-2}, ..., v_{2,p_2-b+1}\}$$
  which is possible since $a > b$, we will move the chips respectively from $v_{2,1}$ and v_{2,p_2} to $v_{2,b+1}$ and v_{3,p_3} like what we did for $g = 3$. Let us denote this configuration $D_4'' = (v_{2,b+1}, v_{3,p_3}, v_{3,1+d-e})$. From this configuration, we can topple
  $$V \setminus (\{v_{2,b+2}, v_{2,b+3}, ..., v_{2,a}\} \bigcup \{v'_{2,1}, v'_{2,2}, ..., v'_{2,c-1}\} \bigcup \{v_{3,1}, v_{3,2}, ..., v_{3,d-e}\})$$
  then
  $$V \setminus (\{v_{2,b+3}, ..., v_{2,a}\} \bigcup \{v'_{2,1}, ..., v'_{2,c-2}\} \bigcup \{v_{3,1}, ..., v_{3,d-e-1}\})$$
  until
  $$V \setminus (\{v_{2,b+k'}, ..., v_{2,a}\} \bigcup \{v'_{2,1}, ..., v'_{2,c-k'+1}\} \bigcup \{v_{3,1}, ..., v_{3,d-e-k'+2}\})$$
  where $k = min(a-b, c, d-e+1)$. Suppose that $k = a - b$, we will have the following:



From this point, you can see that everything between $v_{3,1}$ and $v_{3,d-e-k'+1}$, as well as everything between $v_{3,1}$ and $v'_{2,c-k'}$, can be covered. From D''_4, we can also topple

$$V_1 \bigcup V_2 \bigcup \{v_{3,2}, v_{3,3}, ..., v_{3,d-e+1}\}$$

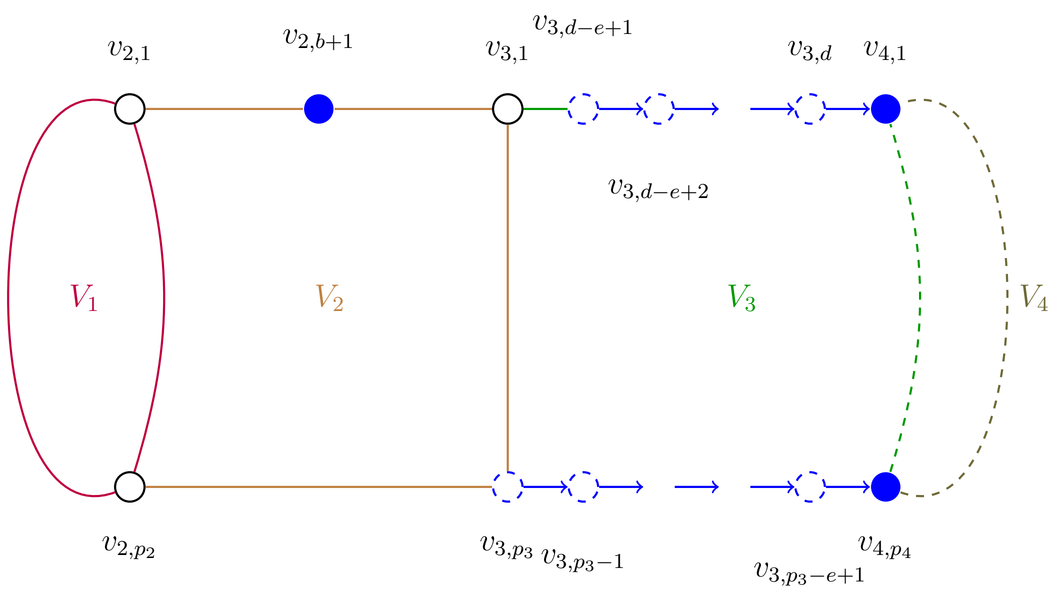
then

$$V_1 \bigcup V_2 \bigcup \{v_{3,2}, v_{3,3}, ..., v_{3,d-e+1}\} \bigcup \{v_{3,d-e+2}\} \bigcup \{v_{3,p_3-1}\}$$

until

$$V_1 \bigcup V_2 \bigcup \{v_{3,2}, v_{3,3}, ..., v_{3,d-e+1}\} \bigcup \{v_{3,d-e+2}, ..., v_{3,d}\} \bigcup \{v_{3,p_3-1}, ..., v_{3,p_3-e+1}\}$$

to move the chips respectively from $v_{3,d-e+1}$ and v_{3,p_3} to $v_{4,1}$ and v_{4,p_4} and we will call this configuration D'_4:



Notice that by doing so, we will cover everything between $v_{3,d-e+1}$ and $v_{4,1}$ as well as everything between v_{3,p_3} and v_{4,p_4}. Finally, once the chips are at $v_{4,1}$ and v_{4,p_4}, everything between these two vertices can also be covered, whether it is on V_3 or on V_4. As a result, we have $H(4)$.

- **If $d \leq e$:**
  - ⋄ **If $a - b \geq e - d$:**
    - ∗ **If $c \geq e - d$:**
      Let $D_4 = (v_{2,1}, v_{2,p_2}, v'_{2,e-d})$ which is possible since $c \geq e - d$. Once again, we begin by toppling

$$V_1$$

then

$$V_1 \bigcup \{v_{2,2}\} \bigcup \{v_{2,p_2-1}\}$$

until

$$V_1 \bigcup \{v_{2,2}, v_{2,3}, ..., v_{2,b}\} \bigcup \{v_{2,p_2-1}, v_{2,p_2-2}, ..., v_{2,p_2-b+1}\}$$

which is possible since $a > b$. By doing so, we have moved the chips respectively from $v_{2,1}$ and v_{2,p_2} to $v_{2,b+1}$ and v_{3,p_3} and covered everything on the way. Moreover, we can also cover everything between $v'_{2,e-d}$ and v_{3,p_3} from this configuration.
From there, we will topple

$$V_1 \bigcup \{v_{2,p_2-1}, v_{2,p_2-2}, ..., v_{2,p_2-b+1}\} \bigcup \{v_{2,2}, ..., v_{2,b}\}$$

$$\bigcup \{v_{3,p_3}\} \bigcup \{v_{2,b+1}\} \bigcup \{v'_{2,e-d}, v'_{2,e-d+1}, ..., v'_{2,c-1}\}$$

then

$$V_1 \bigcup \{v_{2,p_2-1}, v_{2,p_2-2}, ..., v_{2,p_2-b+1}\} \bigcup \{v_{2,2}, ..., v_{2,b}\}$$

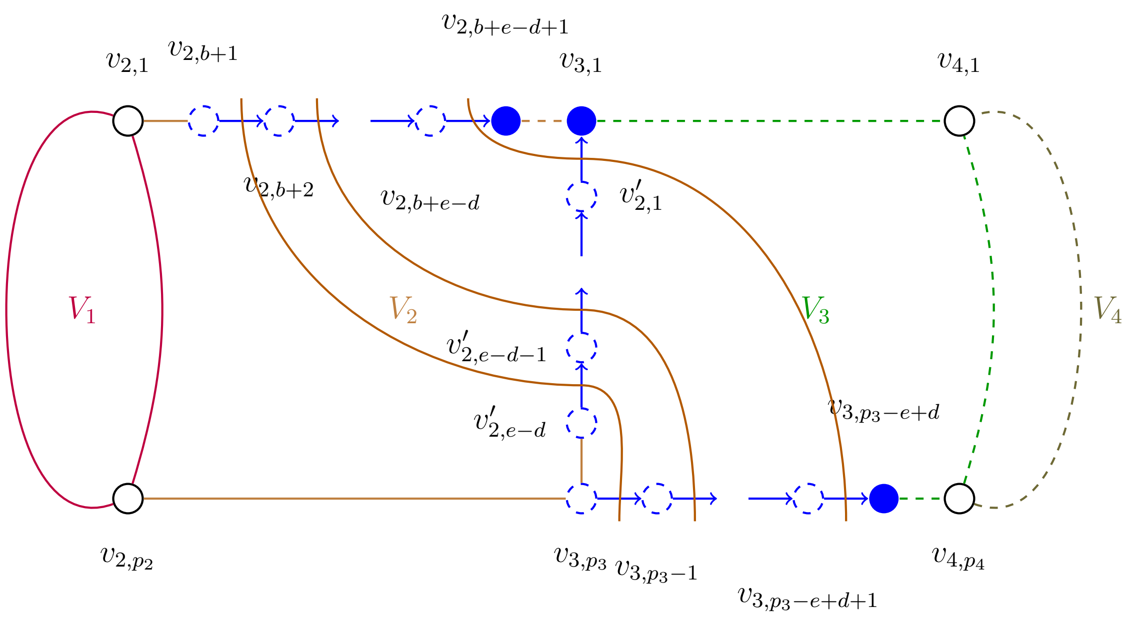
$$\bigcup \{v_{3,p_3}, v_{3,p_3-1}\} \bigcup \{v_{2,b+1}, v_{2,b+2}\} \bigcup \{v'_{2,e-d-1}, v'_{2,e-d}, ..., v'_{2,c-1}\}$$

until

$$V_1 \bigcup \{v_{2,p_2-1}, v_{2,p_2-2}, ..., v_{2,p_2-b+1}\} \bigcup \{v_{2,2}, ..., v_{2,b}\}$$

$$\bigcup \{v_{3,p_3}, v_{3,p_3-1}, ..., v_{3,p_3-e+d+1}\} \bigcup \{v_{2,b+1}, v_{2,b+2}, ..., v_{2,b+e-d}\} \bigcup \{v'_{2,1}, v'_{2,2}, ..., v'_{2,c-1}\}$$

which is possible since $a \geq b + e - d$.



We will call this configuration D''_4. You may notice that we covered almost everything on V_2 (from D''_4, we can also cover the rest of V_2 that are in between $v_{2,b+e-d+1}$ and $v_{3,1}$) and covered partially the path from v_{3,p_3} to v_{4,p_4}.

Now, similarly to what we did to go from D''_4 to D'_4 when $d > e$, we can do the same here by toppling

$$V_1 \bigcup V_2 \bigcup \{v_{3,p_3-1}, v_{3,p_3-2}, ..., v_{3,p_3-e+d}\}$$

then

$$V_1 \bigcup V_2 \bigcup \{v_{3,p_3-1}, v_{3,p_3-2}, ..., v_{3,p_3-e+d-1}\} \bigcup \{v_{3,2}\}$$

until

$$V_1 \bigcup V_2 \bigcup \{v_{3,p_3-1}, v_{3,p_3-2}, ..., v_{3,p_3-e+1}\} \bigcup \{v_{3,2}, v_{3,3}, ..., v_{3,d}\}$$

These topplings will cover everything between $v_{3,1}$ and $v_{4,1}$ as well as everything between v_{3,p_3} and v_{4,p_4} and they will move the chips respectively from $v_{3,1}$ and v_{3,p_3-e+d} to $v_{4,1}$ and v_{4,p_4} resulting in the configuration $(v_{2,b+e-d+1}, v_{4,1}, v_{4,p_4})$ that we will call D'_4. Finally, from D'_4, we can cover the rest of V_3 as well as V_4.

* **If $\boldsymbol{c < e - d}$:**
Let $D_4 = (v_{2,1}, v_{2,p_2}, v_{3,p_3-e+d+c})$ which is possible since $e > c + d$. As usual, we will first move the two chips respectively from $v_{2,1}$ and v_{2,p_2} to $v_{2,b+1}$ and v_{3,p_3} and cover everything in between. Once we are there, you may notice that we can cover everything between v_{3,p_3} and $v_{3,p_3-e+d+c+1}$. Now, we will topple

$$V_1 \bigcup \{v_{2,2}, v_{2,3}..., v_{2,b+1}\} \bigcup \{v_{3,p_3}, v_{3,p_3-1}..., v_{3,p_3-e+d+c}\}$$

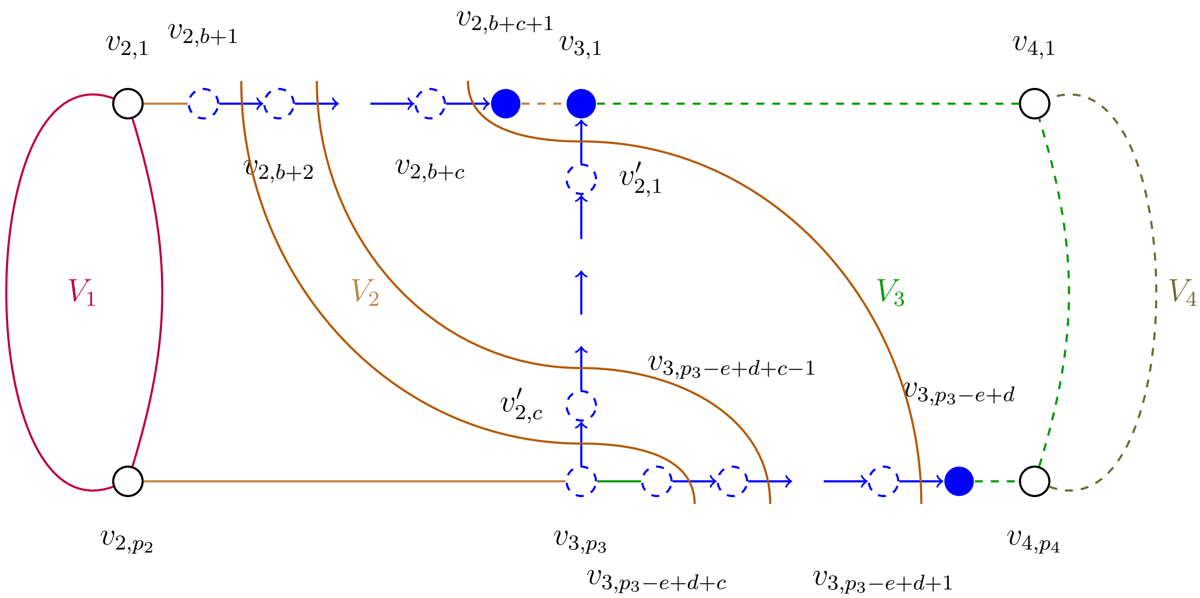
then

$$V_1 \bigcup \{v_{2,2}, v_{2,3}, ..., v_{2,b+2}\} \bigcup \{v_{3,p_3}, v_{3,p_3-1}, ..., v_{3,p_3-e+d+c-1}\} \bigcup \{v'_{2,c-1}\}$$

until

$$V_1 \bigcup \{v_{2,2}, v_{2,3}, ..., v_{2,b+c}\} \bigcup \{v_{3,p_3}, v_{3,p_3-1}, ..., v_{3,p_3-e+d+1}\} \bigcup \{v'_{2,c-1}, v'_{2,c-2}, ..., v'_{2,1}\}$$

which is possible since $a > b + c$ and $e \geq d$.



Note that we obtain a very similar configuration to D''_4 in the previous case when $c \geq e - d$. The end of the proof for this case is the same and we obtain $D'_4 = (v_{2,b+c+1}, v_{4,1}, v_{4,p_4})$.

- ◇ **If $a - b < e - d$:** Let $f = min(a-b, c)$ and $D_4 = (v_{2,1}, v_{2,p_2}, v_{3,p_3-e+d+f})$ which is possible since $e > d+f$. We begin by moving the chips respectively from $v_{2,1}$ and v_{2,p_2} to $v_{2,b+1}$ and v_{3,p_3} while covering everything in between. Once there, everything between v_{3,p_3} and $v_{3,p_3-e+d+f}$ can also be covered. Now, we topple

$$V_1 \bigcup \{v_{2,p_2-1}, v_{2,p_2-1}, ..., v_{2,p_2-b+1}\}$$

$$\bigcup \{v_{2,2}, v_{2,3}, ..., v_{2,b+1}\} \bigcup \{v_{3,p_3}, v_{3,p_3-1}, ..., v_{3,p_3-e+d+f}\}$$

then

$$V_1 \bigcup \{v_{2,p_2-1}, v_{2,p_2-1}, ..., v_{2,p_2-b+1}\}$$

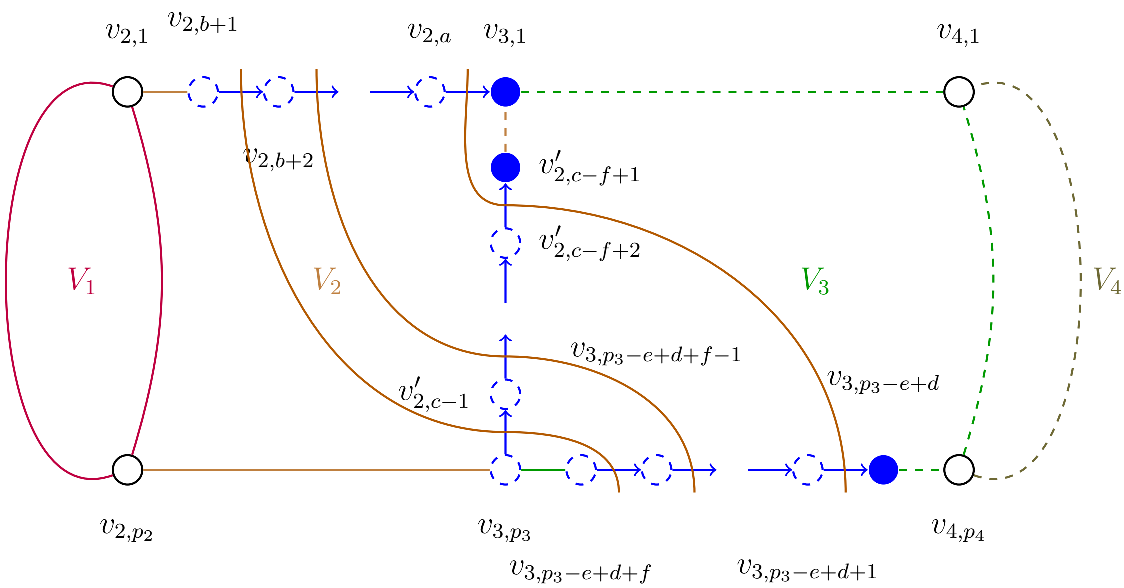
$$\bigcup \{v_{2,2}, v_{2,3}, ..., v_{2,b+2}\} \bigcup \{v_{3,p_3}, v_{3,p_3-1}, ..., v_{3,p_3-e+d+f-1}\} \bigcup \{v'_{2,c-1}\}$$

until

$$V_1 \bigcup \{v_{2,p_2-1}, v_{2,p_2-1}, ..., v_{2,p_2-b+1}\}$$

$$\bigcup \{v_{2,2}, v_{2,3}, ..., v_{2,b+f}\} \bigcup \{v_{3,p_3}, v_{3,p_3-1}, ..., v_{3,p_3-e+d+1}\} \bigcup \{v'_{2,c-1}, v'_{2,c-2}, ..., v'_{2,c-f+1}\}$$

which is possible since $f = min(a-b, c)$. If $f = c$, we have exactly the same case as in $a - b \geq e - d > c$. Now, if $f = a - b$, we get:



Note that everything between $v_{3,1}$ and $v'_{2,c-f+1}$ can be covered. Then, we can finish by moving the chips respectively from $v_{3,1}$ and v_{3,p_3-e+d} to $v_{4,1}$ and v_{4,p_4} like usual and obtain $D'_4 = (v'_{2,c-f+1}, v_{4,1}, v_{4,p_4})$ which can cover the rest.

If $a < b$:
The reasoning is exactly the same by symmetry.

5.2 Proof for the bijection between nested ear decompositions and weighted trees

Proof. First, let us prove by induction on g that given $G \in N$ with genus g, $\theta(G)$ is well defined.

For $g = 3$:

1. $g \geq 3$.

2. $V' = \{v_2, v_3\}$ and $E' = (v_2, v_3)$ since $v_2 \in V_1^2$ and if $v_3 \in V_1^2$, then v_2 and v_3 are nested. Otherwise, $v_3 \in V_2^2$ in which case v_2 and v_3 are also nested so (V', E') is a tree. Moreover, $deg(v_2) = deg(v_3) = 1$.

3. $w_V(v_2)_0 < 2$ and $w_V(v_3)_0 < 3$ by definition. If $v_3 \in V_1^2$, then $w_V(v_2)_0 = w_V(v_3)_0 = 1$. Otherwise, $v_3 \in V_2^2$ so $w_V(v_3)_0 = 2$.

4. If $v_3 \in V_2^2$, then E_1 is empty so we need $n_2 + n_3 < n_1$ which is the case by definition of n_2 and n_3, which exist since $v_2 \in V_1^2$ so there is at least a vertex in V_1 with degree greater than 2, and the strict inequality comes from the fact that the decomposition is proper. Also, $E_2 = \{(v_2, v_3)\}$ and $w_E(v_2, v_3)_0 + w_E(v_2, v_3)_1 < w_V(v_2)_1$ since v_3 is nested

in v_2 so the sum of the distances between their endpoints is strictly smaller than the length of the ear V_2. We can prove the same things if $v_3 \in V_1^2$.

5. $n_1 = p_1 - 1 \geq 1$.

6. $n_4 = 2$ or $3 > 1$.

Suppose that for all $H \in N$ with genus $g \geq 3$, $\theta(H)$ is well defined, let us consider $G \in N$ with genus $g+1$:

1. $g+1 > g \geq 3$.

2. $G = \bigcup_{i=1}^{g+1} V_i$ so we can define $H = \bigcup_{i=1}^{g} V_i$. We denote

$$\theta(G) = (V(G), E(G), w_{V(G)}, w_{E(G)}, n_1(G), n_2(G), n_3(G))$$

and similarly for $\theta(H)$.
Since G has genus $g+1, V(G) = \{v_2, v_3, ..., v_{g+1}\}$. Now consider v_{g+1}. There exists an unique $1 \leq i \leq g$ such that $v_{g+1} \in V_i^2$. Since the different pairs of endpoints in V_i form nested intervals, there exists an unique $1 \leq j \leq g$ such that v_{g+1} is nested in v_j:

   - If $j = 1$, then there exists an unique pair $v_k \in V_1^2$ $(1 < k \leq g)$ nested in v_{g+1}. Thus, $|E(G)| = |E(H) \cup \{(v_k, v_{g+1})\}| = |E(H)| + 1 = |V(H)|$ since $(V(H), E(H))$ is a tree as we can apply the induction hypothesis to H. Moreover, $|V(H)| = |V(G)| - 1$ and $(V(G), E(G))$ stays connected since $(V(H), E(H))$ is connected. Thus, $(V(G), E(G))$ is a tree.

   - If $j > 1$ and there is no pair nested in v_{g+1}, then $|E(G)| = |E(H) \cup \{(v_j, v_g+1)\}| = |V(H)| - 1 + 1 = |V(H)| = |V(G)| - 1$ with the same argument as above. So, $(V(G), E(G))$ is a tree.

   - If $j > 1$ and there is a pair v_k nested in v_{g+1}, then that pair is unique since there is no pair on V_{g+1} so the only way it exists is that it is located on the same ear V_i. Moreover, on the same ear, there can only be one pair that is nested in v_{g+1}. As a result, $|E(G)| = |E(H) \cup \{(v_j, v_{g+1}), (v_{g+1}, v_k)\} \setminus \{(v_j, v_k)\}|$ since, without v_{g+1}, v_k would be nested in v_j. Once again, we can apply the induction hypothesis to H and conclude: $|E(G)| = |E(H)| + 2 - 1 = |V(H)| = |V(G)| - 1$ and $(V(G), E(G))$ is still connected so it is a tree.

   As for the degree of a vertex, since the graph is connected, it is at least 1 and we have already noted that, given any pair, there is at most 1 pair in which it is nested and 2 pairs that are nested in it which means that the corresponding degree is at most 3.

3. $\forall 1 < i \leq g+1, w_V(v_i)_0 < i$ by definition of the decomposition.
By applying the induction hypothesis on H, we have: for all i, j such that $i < j$ and $(v_i, v_j) \in E(H), w_V(v_i)_0 = w_V(v_j)_0$ or $w_V(v_j)_0 = i$. Previously, we have two deduced three possible cases for $E(G)$. Let $1 \leq i \leq g$ be such that $v_{g+1} \in V_i^2$ and $1 \leq j \leq g$ be such that v_{g+1} is nested in v_j:

   - If $j = 1$, then $i = 1$ and there exists an unique pair $v_k \in V_1^2$ $(1 < k \leq g)$ nested in v_{g+1} and $E(G) = E(H) \cup \{(v_k, v_{g+1})\}$ where $w_{V(G)}(v_{g+1})_0 = w_{V(G)}(v_k)_0 =$

$w_{V(H)}(v_k)_0 = 1$.

- If $j > 1$ and there is no pair nested in v_{g+1}, then $E(G) = E(H) \cup \{(v_j, v_g + 1)\}$ where $w_{V(G)}(v_{g+1})_0 = j$ if $i = j$ or $w_{V(G)}(v_{g+1})_0 = w_{V(G)}(v_j)_0 = i$ if $i \neq j$.
- If $j > 1$ and there is an unique pair $v_k \in V_i^2$ nested in v_{g+1}, then $E(G) = E(H) \cup \{(v_j, v_{g+1}), (v_{g+1}, v_k)\} \setminus \{(v_j, v_k)\}$ where $w_{V(G)}(v_{g+1})_0 = w_{V(G)}(v_k)_0 = j$ if $i = j$ and $w_{V(G)}(v_{g+1})_0 = w_{V(G)}(v_k)_0 = w_V(v_j)_0 = i$ if $i \neq j$.

4. Let us define $E_i(G)$ for all $1 \leq i \leq g$ like in Definition 4.3. With the same notation as before:

   - If $j = 1$, then $i = 1$ and we have $v_k \in V_1^2$ such that $E(G) = E(H) \cup \{(v_k, v_{g+1})\}$ or more precisely, $E_1(G) = E_1(H) \cup \{(v_k, v_{g+1})\}$. So,

$$n_1(G) + n_2(G) + \sum_{e \in E_1(G)} w_{E(G)}(e)_0 + w_{E(G)}(e)_1$$

$$= n_1(G) + w_{E(G)}((v_k, v_{g+1}))_0 + n_2(G) + w_{E(G)}((v_k, v_{g+1}))_1 + \sum_{e \in E_1(H)} w_{E(H)}(e)_0 + w_{E(H)}(e)_1$$

$$= n_1(H) + n_2(H) + \sum_{e \in E_1(H)} w_{E(H)}(e)_0 + w_{E(H)}(e)_1 < n_1(H) = n_1(G).$$

   - If $j > 1$ and there is no pair nested in v_{g+1}, then $E(G) = E(H) \cup \{(v_j, v_g + 1)\}$ and $E_j(G) = E_j(H) \cup \{(v_j, v_{g+1})\} = \{(v_j, v_{g+1})\}$ if $i = j$. So,

$$w_{E(G)}((v_j, v_{g+1}))_0 + w_{E(G)}((v_j, v_{g+1}))_1 = a + b$$

where $v_{j,1+a} = v_{g+1,1}$ and $v_{j,p_j - b} = v_{g+1,p_{g+1}}$ and $a + b < p_j - 1 = w_{V(H)}(v_j)_1 = w_{V(G)}(v_j)_1$ since $1 + a < p_j - b$ as v_{g+1} is nested in v_j and $v_{g+1,1} \neq v_{g+1,p_{g+1}}$.
If $i \neq j$, then $E_i(G) = E_i(H) \cup \{(v_j, v_{g+1})\}$. So,

$$\sum_{e \in E_i(G)} w_{E(G)}(e)_0 + w_{E(G)}(e)_1$$

$$= w_{E(G)}((v_j, v_{g+1}))_0 + w_{E(G)}((v_j, v_{g+1}))_1 + \sum_{e \in E_i(H)} w_{E(H)}(e)_0 + w_{E(H)}(e)_1$$

$$= a + b + \sum_{e \in E_i(H)} w_{E(H)}(e)_0 + w_{E(H)}(e)_1$$

where $v_{i,1+a+\sum_{e \in E_i(H)} w_{E(H)}(e)_0} = v_{g+1,1}$ and $v_{i,p_i - b - \sum_{e \in E_i(H)} w_{E(H)}(e)_1} = v_{g+1,p_{g+1}}$. Note that $v_{i,1+\sum_{e \in E_i(H)} w_{E(H)}(e)_0} = v_{j,1}$ and $v_{i,p_i - \sum_{e \in E_i(H)} w_{E(H)}(e)_1} = v_{j,p_j}$ since, without v_{g+1}, there is nothing nested in v_j that is on V_i. So,

$$a + b < p_i - \sum_{e \in E_i(H)} w_{E(H)}(e)_0 + w_{E(H)}(e)_1 - 1$$

since v_{g+1} is nested in v_j and $v_{g+1,1} \neq v_{g+1,p_{g+1}}$. In other words, we have

$$\sum_{e \in E_i(G)} w_{E(G)}(e)_0 + w_{E(G)}(e)_1 < p_i - 1 = w_{V(G)}(v_i)_1.$$

- If $j > 1$ and there is an unique pair $v_k \in V_i^2$ nested in v_{g+1}, then $E(G) = E(H) \cup \{(v_j, v_{g+1}), (v_{g+1}, v_k)\} \setminus \{(v_j, v_k)\}$ and $E_i(G) = E_i(H) \cup \{(v_j, v_{g+1}), (v_{g+1}, v_k)\} \setminus \{(v_j, v_k)\}$. So,

$$\sum_{e \in E_i(G)} w_{E(G)}(e)_0 + w_{E(G)}(e)_1 = \sum_{e \in E_i(H)} w_{E(H)}(e)_0 + w_{E(H)}(e)_1$$

$$-w_{E(H)}(v_j, v_k)_0 + w_{E(G)}(v_j, v_{g+1})_0 + w_{E(G)}(v_{g+1}, v_k)_0$$

$$-w_{E(H)}(v_j, v_k)_1 + w_{E(G)}(v_j, v_{g+1})_1 + w_{E(G)}(v_{g+1}, v_k)_1$$

$$= \sum_{e \in E_i(H)} w_{E(H)}(e)_0 + w_{E(H)}(e)_1 < w_{V(H)}(v_i)_1 = w_{V(G)}(v_i)_1$$

5. $n_1(G) = n_1(H) \geq 1$.
6. Once again, with the same notations:
   - If $j = 1$ then $n_4(G) = g + 1 > 1$.
   - If $j > 1$ then $n_4(G) = n_4(H) > 1$.

Now, we will prove the bijectivity of the map:

Let $T = (g, V', E', w_V, w_E, n_1, n_2, n_3, n_4) \in \Theta$. We can build an unique $G = (V, E)$ such that $\theta(G) = T$:

- $V = \bigcup_{i=1}^{g} V_i$ where $g \geq 3$ and $\forall 1 < i \leq g, |V_i| = |\{v_{i,1}, v_{i,2}, ..., v_{i,w_V(v_i)_1+1}\}| \geq 2$ since $w_V(v_i)_1 \geq 1$ and $|V_1| = |\{v_{1,1}, v_{1,2}, ..., v_{1,n_1+1}\}| \geq 2$ since $n_1 \geq 1$.
- $\forall 1 < i \leq g, w_V(v_i)_0 < i$ and $v_i \in V^2_{w_V(v_i)_0}$.
- For all $1 < i \leq g$, let us consider $(e_1 = (e_{1,0}, e_{1,1}), e_2, ..., e_{|E_i|})$ where E_i is defined in Definition 4.3, $\forall 1 \leq j \leq |E_i|, e_j \in E_i$ and $e_{1,0} = v_i$ and $\forall 1 \leq j < |E_1|, e_{j,1} = e_{j+1,0}$. We have $\forall 1 \leq j \leq |E_i|, v_{e_{j,1},1} = v_{i,1+\sum_{k=1}^{j} w_E(e_k)_0}$ and $v_{e_{j,1},p_{e_{j,1}}} = v_{i,p_{e_{1,1}}-\sum_{k=1}^{j} w_E(e_k)_1}$. This ensures that the pairs of endpoints are nested two by two and it is possible thanks to the fact that $\sum_{e\in E_i} w_E(e)_0 + w_E(e)_1 < w_V(v_i)$ (the strict inequality is necessary for the endpoints, of the most nested in ear on V_i, to be different).
- Similarly, for the ears located on V_1, we consider $(e_1, e_2, ..., e_{|E_1|})$ where $\forall 1 \leq j \leq |E_1|, e_j \in E_1$ and $e_{1,0} = v_{n_4}$ and $\forall 1 \leq j < |E_1|, e_{j,1} = e_{j+1,0}$. We have $v_{e_{1,0},1} = v_{1,1+n_2}$ and $\forall 1 \leq j \leq |E_1|, v_{e_{j,1},1} = v_{i,1+n_2+\sum_{k=1}^{j} w_E(e_k)_0}$ and $v_{e_{1,0},p_{e_{1,0}}} = v_{1,p_1-n_3}$ and $v_{e_{j,1},p_{e_{j,1}}} = v_{i,p_{e_{1,1}}-n_3-\sum_{k=1}^{j} w_E(e_k)_1}$. Similarly, n_1, n_2, n_3, n_4 are well conditioned so that this is possible.
- Finally, $\forall 1 \leq i \leq g, \forall 1 \leq j \leq w_V(v_i)_1, (v_{i,j}, v_{i,j+1}) \in E$ and $(v_{1,1}, v_{1,p_1}) \in E$.

□

Remark: The proof of the bijectivity of θ gives us the way to rebuild a nested ear decomposition from any given tree $T \in \Theta$. We will call it $\theta^{-1} : \Theta \to N$.

5.3 Proof of the existence of unbreakable even trees

Proof. We call R the set of vertices with degree 2. We consider the partition $U_1 \cup U_2 \cup ... \cup U_{|R|+1} = V \setminus R$ where, for all $1 \leq i \leq |R|+1$, the vertices in U_i are connected. We also consider the partition $E_1 \cup E_2 \cup ... \cup E_{|R|+1} = E$ where each E_i is the set of all edges incident to vertices in U_i. We call R_i the subset of R of vertices that are neighbors to vertices in U_i. Now, we consider the trees $T_i = (U_i \cup R_i, E_i)$ for $1 \leq i \leq |R|+1$. Note that $\tau = (\{T_1, T_2, ..., T_{|R|+1}\}, \{(T_i, T_j) | R_i \cap R_j \neq \varnothing\})$ is a tree since (V, E) is a tree.

The purpose of the proof is to prove that τ has only two nodes of degree 1, which means that $|R| = 1$. We will do it in two steps:

1. First, we will prove that every vertex in τ has an odd degree.
2. Then, we will prove that there is no vertex with degree at least 3.

1. We suppose that τ is rooted at a vertex of degree 1. In other words, every vertex has 1 parent and the rest of its neighbors are its children except for the root who has, in this case, one child and no parent. We call d τ's depth (the root is at depth 0) and we will prove
   $H(k)$:"Every vertex at depth k has an odd number of neighbors and the sum of edges in the trees of its descendants is even" for every $1 \leq k \leq d$.

   - Every T at depth d has degree 1 and no descendant so $H(d)$ is true.
   - Suppose that $H(k)$ is true for some $k \leq d$, let us consider a vertex T_i at depth $k-1$. Suppose by contradiction that T_i has an odd number of children, say C, which means that, with its parent, it has an even number of neighbors. Note that for every $1 \leq a < b \leq |R|+1$, $|R_a \cap R_b| = 0$ or 1, otherwise, we would find a cycle in (V, E). So, let $\{v\} = R_i \cap R_j$ where T_j is T_i's parent. Each of the trees contains only vertices with degree 1 or 3, so they have an odd number of edges (this result can be proven easily by induction). We call the number of edges in T_i's children $m_1, m_2, ..., m_C$. Each one of its children verifies $H(k)$ so their respective descendants trees's number of edges sum up to even numbers that we will call $S_1, S_2, ..., S_C$. If we count the total number of edges in T_i and its descendants, we have:

     $$|E_i| + m_1 + m_2 + ... + m_C + S_1 + S_2 + ... + S_C = 0 \pmod 2$$

     $S_1, S_2, ..., S_C$ are even so $S_1 + S_2 + ... + S_C$ is even. $m_1, m_2, ..., m_C$ are odd so $m_1 + m_2 + ... + m_C$ is odd since C is odd. $|E_i|$ is odd. Finally, the sum of two odd numbers and an even one is even. As a result, v is a breaker as it breaks (V, E) into two trees and we have just proved that one has an even number of edges and so is the other since $|E|$ is even. This contradiction means that C is even. In other words, T_i has an odd number of neighbors. Moreover, the sum of edges in its descendants trees is:

     $$m_1 + m_2 + ... + m_C + S_1 + S_2 + ... + S_C = 0 \pmod 2$$

   Since the root also has degree 1, this means that every vertex in τ has an odd degree.

2. Suppose that there exists a vertex $T_i = (U_i \cup R_i, E_i)$ with degree at least 3. Note that every vertex in R_i corresponds to a leaf in T_i. We suppose that T_i is rooted at r where $\{r\} = R_i \cap R_j$ with T_j as T_i's parent in τ. We will prove the existence of a breaker in T_i. To do so, we will prove that there exists a pair of leaves in R_i such that their lowest common ancestor in T_i is not an ancestor to any other leaf in R_i. Suppose by contradiction that for any pair of leaves in R_i, there exists another leaf in R_i which is also a descendant to the lowest common ancestor of the other two. Take two leaves $v, v' \in R_i$ and call their common ancestor a, we know that there is $v'' \in R_i$ whose ancestor is also a. Recall that vertices in T_i has degree 1 or 3 so when rooted, r's child is the root to a full binary tree. a cannot be r since r has only one child so r's child should also be ancestor to both v and v', thus contradicting the fact that a is the lowest common ancestor. So, a has two children, say b and c. a is the lowest common ancestor to v and v' so b is an ancestor to, say v, but not v' and c is an ancestor to v' and not v. Since a is also an ancestor to v'', v'' must be a descendant of b or c. By symmetry, we can suppose that is b. As a consequence, v and v'' lowest common ancestor is either b or a descendant of b which means that it has depth strictly greater than a's. If we continue with v and v'', we have the existence of another leaf in R_i with the same ancestor. With the same reasoning, we can prove that the lowest common ancestor of two of these three leaves has a strictly greater depth than the previous considered ancestor. And so on, inductively, we would have built a sequence of vertices of depth strictly increasing to infinity which is impossible since T_i is finite. Thus, there exists a pair of leaves $v, v' \in R_i$ whose lowest common ancestor in T_i is ancestor to no other in R_i.
   We will call this common ancestor a and we will prove that it is a breaker. When we break (V, E) at a, we obtain three different trees that we will call $A_v, A_{v'}, A_r$ where A_v is the one containing v, $A_{v'}$ is the one containing v' and A_r is the third one which contains r. Let $(V(A_v), E(A_v)) = A_v$. v is the root to another tree that we will call T_k. We will call S the sum of all edges in T_k's descendants trees. The number of edges in A_v is:

$$|E(A_v)| = |E_k| + S + |E(A_v) \cap E_i| = 0 \pmod 2$$

   $|E(A_v) \cap E_i|$ is odd since every vertex incident to the edges in $E(A_v) \cap E_i$ still has degree 1 or 3. $|E_k|$ is also odd by the same argument and S is even, which we have already proven thanks to the induction in 1. As a result, $|E(A_v)|$ is even. The same can be done to prove that $A_{v'}$ contains an even number of edges and since the total number of edges $|E|$ is even, the same goes for A_r. Finally, we have proven that a is a breaker but since (V, E) has no breaker, τ do not have any vertex with degree at least 3.

τ is a tree with only vertices with degree 1 so τ is simply two vertices linked by an edge. In other words, $|R| = 1$. $\square$