École Normale Supérieure de Lyon

Internship Report

r-hued coloring of planar graphs

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Contents

Abstract

During the internship, we mainly studied the notion of r-hued coloring. Let $r, k \geq 1$ be two integers. An *r-hued k-coloring* of the vertices of a graph $G = (V, E)$ is a proper k-coloring of the vertices, such that, for all vertex $v \in V$, the number of colors in its neighborhood is at least $\min\{d_G(v), r\}$, where $d_G(v)$ is the degree of v. We proved the existence of an r-hued $(r+1)$-coloring of planar graphs with girth at least 8 for $r \geq 9$.

1 Introduction

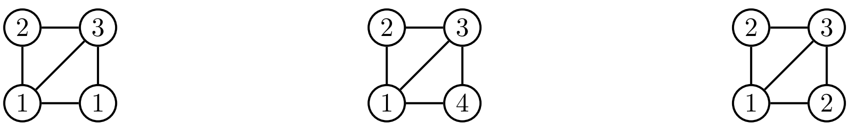
Graphs have been a popular model to solve different kinds of problems in all sort of area. As a result, they are prime objects of study in discrete mathematics. We define a *graph* G as a pair of finite sets $(V(G), E(G))$ where $E(G) \subseteq \{\{u, v\} | u \neq v, (u, v) \in V(G) \times V(G)\}$. We call the elements of $V(G)$ the *vertices* of G, and the elements of $E(G)$, the *edges* of G. A *subgraph* of a graph G is a graph $H = (V(H), E(H))$ where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

A graph is often used as mathematical structure to describe pairwise relationship between objects. This relationship is represented by the edges between two vertices. For convenience, we denote uv the edge $\{u, v\} \in E(G)$. For all edge $e = uv \in E$, we say that u and v are *adjacent* vertices, that u is a *neighbor* of v and vice versa, that e is an *incident edge* of u and v, and that u and v are *incident to* e. The total number of incident edges of v, is the *degree* of v and is denoted $d_G(v)$. We denote the *maximum degree* of a graph G, by $\Delta(G) = \max_{v \in V(G)} d_G(v)$. When there is no ambiguity, we will use Δ (resp. $d(v)$) instead of $\Delta(G)$ (resp. $d_G(v)$).

To understand a graph, we have to understand its structure. An important characteristic of a graph is its connectivity. A *path* P of length n is a sequence of vertices $v_1 v_2 \ldots v_{n+1}$ where $v_i v_{i+1} \in E(G)$ for all $1 \leq i \leq n$. We say that P is a path between v_1 and v_{n+1}, that v_1 and v_{n+1} are incident to P, and that they are at *distance* n from each other. We also call v_1, v_{n+1} the *endvertices* of the path, and $v_2, v_3, \ldots, v_n$ the *internal vertices*. Graph G is *connected* if, for any pair of vertices u and v, there exists a path between u and v. A *component* of a graph G is a maximal connected subgraph. A non-connected graph can be partitioned into multiple components. A *cycle* is a path $v_1 v_2 \ldots v_{n+1}$ where $v_1, v_2, \ldots, v_n$ are distinct vertices and $v_{n+1} = v_1$. A cycle of length n is denoted C_n. The *girth* of a graph is the length of a shortest cycle. A graph is *acyclic* if it does not contain any cycle. A *tree* is a connected acyclic graph. A *forest* is a graph where all of its components are trees. The girth of a forest is infinite by convention.

2 Graph coloring

Graph coloring is one of the most famous problems in graph theory. It is the central object of many current studies. Here, we are interested in vertex coloring problems where we assign colors to vertices under certain constraints. A *k-coloring* of the vertices of a graph G is a map $\phi : V(G) \to \{1, 2, \ldots, k\}$. A k-coloring ϕ is a *proper coloring*, if and only if, for all edge $xy \in E, \phi(x) \neq \phi(y)$. In other words, no two adjacent vertices have the same color. The *chromatic number* of G, denoted $\chi(G)$, is the smallest k so that G has a proper k-coloring. An example of colorings is given in Figure 1.



(i) A non-proper 3-coloring. (ii) A non-optimal proper 4-coloring. (iii) An optimal proper 3-coloring.

Figure 1: A graph G with $\chi(G) = 3$.

A generalization of k-coloring is k-list-coloring. A graph G is *L-list colorable* if for a given list assignment $L = \{L(v) : v \in V(G)\}$ there is a proper coloring ϕ of the vertices such that $\forall v \in V(G), \phi(v) \in L(v)$. If G is L-list colorable for every list assignment with $|L(v)| \geq k$ for all $v \in V(G)$, then G is said to be *k-choosable* or *k-list-colorable*. The *list chromatic number* of a graph G, denoted $\mathrm{ch}(G)$, is the smallest integer k such that G is k-choosable. List coloring can be very different from normal coloring as there exists graphs with a small chromatic number and an arbitrarily large list chromatic number. An example of a proper list-coloring is given in Figure 2.

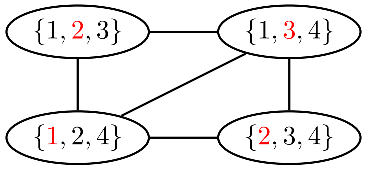
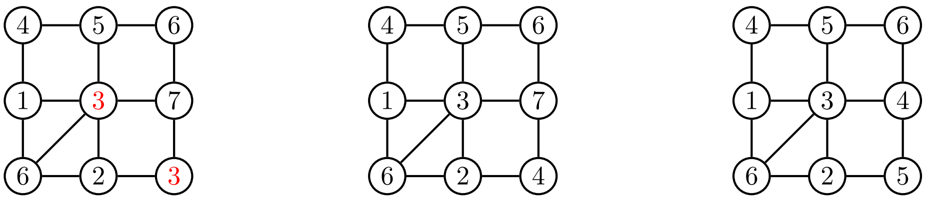


Figure 2: A proper 3-list coloring. The red integer is the chosen color in each list.

3 2-distance coloring

Until now, we have been interested in the "proper" constraint that does not allow two adjacent vertices to have the same color. In 1969, Kramer and Kramer introduced the notion of 2-distance coloring [24, 25]. This notion generalizes the "proper" constraint in the following way: a *2-distance k-coloring* is such that no pair of vertices at distance at most 2 have the same color. Similarly to proper k-list-coloring, we can also define *2-distance k-list-coloring*. The *2-distance chromatic number* of G, denoted $\chi^2(G)$, is the smallest integer k so that G has a 2-distance k-coloring. An example of 2-distance colorings is given in Figure 3.



(i) A proper 7-coloring that is not 2-distance.

(ii) A non-optimal 2-distance 7-coloring.

(iii) An optimal 2-distance 6-coloring.

Figure 3: A graph G with $\chi^2(G) = 6$ and $\chi(G) = 3$.

Observation 1. *For any graph G with maximum degree Δ, $\Delta + 1 \leq \chi^2(G) \leq \Delta^2 + 1$.*

The lower bound is trivial since, in a 2-distance coloring, every neighbor of a vertex v with degree Δ, and v itself must have a different color. As for the upper bound, a greedy algorithm shows that $\chi^2(G) \leq \Delta(G)^2 + 1$. Moreover, this bound is tight for some graphs, for example, Moore graphs of type $(\Delta, 2)$, which are graphs where all vertices have degree Δ, are at distance at most two from each other, and the total number of vertices is $\Delta^2 + 1$. See Figure 4.

One interesting class of graphs to study is planar graphs. A graph is *planar* if we can draw its vertices with points on the plane, and edges with curves intersecting only at its endpoints. When a graph is planar, we sometimes identify the graph with its planar drawing. When G is a planar graph, Wegner conjectured that $\chi^2(G)$ becomes linear in $\Delta(G)$:

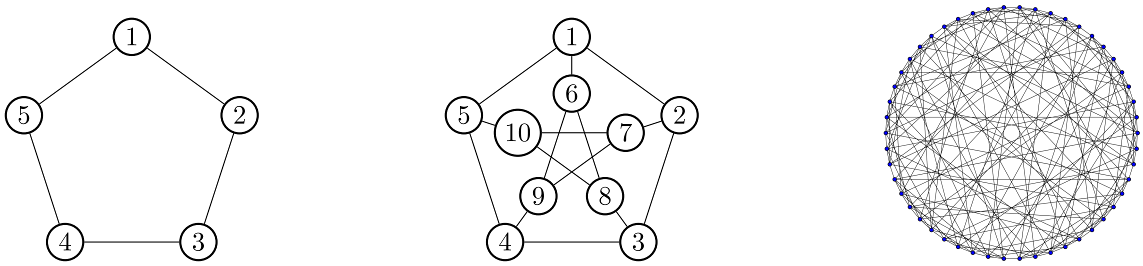
Conjecture 2 (Wegner[33])**.** *Let G be a planar graph with maximum degree Δ. Then,*

$$\chi^2(G) \leq \begin{cases} 7, & \text{if } \Delta \leq 3, \\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7, \\ \lfloor \frac{3\Delta}{2} \rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

Recently, the case $\Delta \leq 3$ was proved by Thomassen[31], and by Hartke, Jahanbekam and Thomas[18] independently. For $\Delta \geq 8$, Havet *et al.* [19] proved that the bound is $\frac{3}{2}\Delta(1 + o(1))$, where $o(1)$ is as $\Delta \to \infty$. This bound is actually tight for the graph in Figure 5i. The coefficient before Δ becomes 1 when the graph becomes "sparser". Here, a "sparse" graph means that it has a "low" number of edges. One way to measure the sparsity of a graph is through its maximum average degree. We define the *average degree* ad of a graph G by $\text{ad}(G) = \frac{2|E|}{|V|}$. The *maximum average degree* $\text{mad}(G)$ is the maximum, over all subgraphs H of G, of $\text{ad}(H)$. Another way to measure the sparsity is through the girth. Intuitively, the higher the girth of a graph is, the sparser it gets. These two measures can actually be linked directly in the case of a planar graphs.

Proposition 3. *For every planar graph G, $(\text{mad}(G) - 2)(g(G) - 2) < 4$.*

Proof. Let H be a subgraph of G such that $\text{mad}(G) = \text{ad}(H) = \frac{2|E(H)|}{|V(H)|}$. Euler's formula states that: $|E(H)| -$

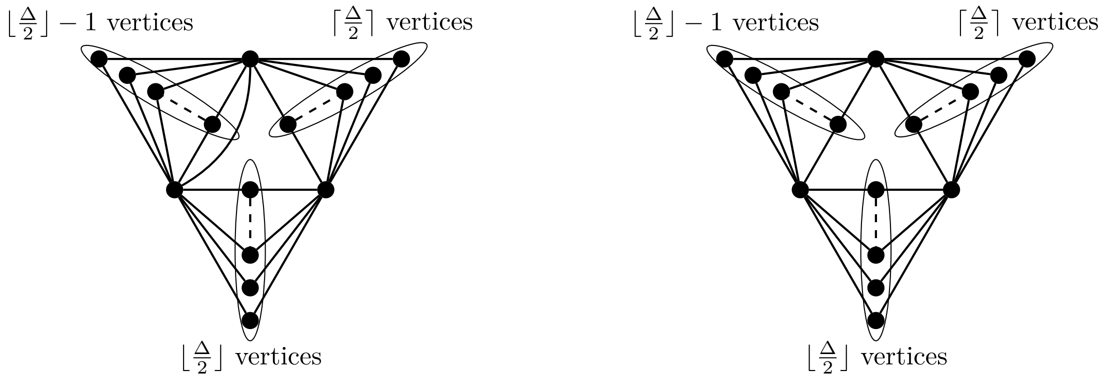


(i) The Moore graph of type (2,2): the odd cycle C_5

(ii) The Moore graph of type (3,2): the Petersen graph.

(iii) The Moore graph of type (7,2): the Hoffman-Singleton graph.

Figure 4: Examples of Moore graphs for which $\chi^2 = \Delta^2 + 1$.



(i) A graph with girth 3 and $\chi^2 = \lfloor \frac{3\Delta}{2} \rfloor + 1$

(ii) A graph with girth 4 and $\chi^2 = \lfloor \frac{3\Delta}{2} \rfloor - 1$.

Figure 5: Graphs with $\chi^2 \approx \frac{3}{2}\Delta$

$|V(H)| + 2 = |F(H)|$. Since

$$\begin{aligned} \sum_{f \in F(H)} g(H) &\leq \sum_{f \in F(H)} d(f) \\ |F(H)| g(H) &\leq 2|E(H)| \\ |F(H)| &\leq \frac{2|E(H)|}{g(H)} \end{aligned}$$

We have

$$\begin{aligned} |E(H)| - |V(H)| &< \frac{2|E(H)|}{g(H)} \\ \frac{2g(H)}{|V(H)|}|E(H)| - 2g(H) &< \frac{4|E(H)|}{|V(H)|} \\ \mathrm{mad}(G)g(H) - 2g(H) &< 2\,\mathrm{mad}(G) \\ \mathrm{mad}(G)g(H) - 2g(H) - 2\,\mathrm{mad}(G) + 4 &< 4 \\ (\mathrm{mad}(G) - 2)(g(H) - 2) &< 4 \end{aligned}$$

Since $g(H) \geq g(G)$, $(\mathrm{mad}(G) - 2)(g(G) - 2) < 4$. □

As a consequence, any theorem with an upper bound on $\mathrm{mad}(G)$ can be translated to a theorem with a lower bound on $g(G)$ under the condition that G is planar.

In the case of sparse planar graphs, extensive researches have been done and many results have taken the following form: *every planar graph G of girth $g \geq g_0$, and $\Delta(G) \geq \Delta_0$, satisfies $\chi^2(G) \leq \Delta + c(g_0, \Delta_0)$, where $c(g_0, \Delta_0)$ is a constant depending only on g_0 and Δ_0.*
Table 1 shows all known such results on the 2-distance chromatic number of planar graphs with fixed girth, up to our own knowledge.

$\chi^2(G)$ / g_0	$\Delta+1$	$\Delta+2$	$\Delta+3$	$\Delta+4$	$\Delta+5$	$\Delta+6$	$\Delta+7$	$\Delta+8$
3				$\Delta = 3$[31, 18]				
4								
5		$\Delta \geq 10^7$[1][2]	$\Delta \geq 339$[16]	$\Delta \geq 312$[15]	$\Delta \geq 15$[9][1]	$\Delta \geq 12$[8][2]	$\Delta \neq 7, 8$[15]	all Δ[32]
6		$\Delta \geq 17$[4][5]	$\Delta \geq 9$[8][2]		all Δ[11]			
7	$\Delta \geq 16$[20][2]			$\Delta = 4$[13][3]				
8	$\Delta \geq 10$[20][2] $\Delta \geq 9$[4]		$\Delta = 5$[7][3]					
9	$\Delta \geq 8$[3][5]	$\Delta = 5$[7][3]	$\Delta = 3$[14][2]					
10	$\Delta \geq 6$[20][2]							
11		$\Delta = 4$[13][3]						
12	$\Delta = 5$[20][2]	$\Delta = 3$[6][2]						
13								
14	$\Delta \geq 4$[2][5]							
...								
22	$\Delta = 3$[20][2]							

Table 1: The latest results with a coefficient 1 before Δ in the upper bound of χ^2.

For example, the result from line "7" and column "$\Delta + 1$" from Table 1 reads as follows : "*every planar graph G of girth at least 7, and of Δ at least 16, satisfies $\chi^2(G) \leq \Delta + 1$*". The crossed out cases in the first column correspond to the fact that, for $g_0 \leq 6$, there are planar graphs G with $\chi^2(G) = \Delta + 2$ for arbitrarily large Δ[5, 17]. The lack of results for $g \geq 4$ is due to the fact that the graph in Figure 5ii has girth 4, and $\chi^2 = \lfloor \frac{3\Delta}{2} \rfloor - 1$ for all Δ. Finally, many of these results are corollaries of theorems on 2-distance list-colorings or 2-distance colorings of graphs with bounded maximum average degree instead of planar graphs.

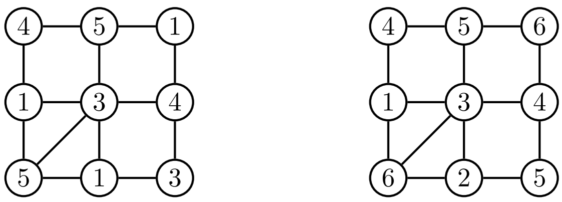
4 r-hued coloring

The "2-distance" condition in 2-distance colorings requires that vertices at distance at most two have different colors. In other words, all neighbors of the same vertex must have different colors. This condition was generalized recently and the notion of r-hued coloring was introduced [28]. Let $r, k \geq 1$ be two integers. An *r-hued k-coloring* of the vertices of G is a proper k-coloring of the vertices, such that all vertices are r-hued. A vertex is r-hued if the number of colors in its neighborhood $N_G(v) = \{x | xv \in E\}$ is at least $\min\{d_G(v), r\}$. The *r-hued chromatic number* of G, denoted $\chi_r(G)$, is the smallest integer k so that G has an r-hued k-coloring.

It is indeed a generalization of 2-distance colorings which correspond to the case $r \geq \Delta$, as all vertices in the same neighborhood will have different colors. More generally, its link to proper coloring and 2-distance coloring resides in the following equation:

$$\chi(G) = \chi_1(G) \leq \chi_2(G) \leq \cdots \leq \chi_\Delta(G) = \chi_{\Delta+1}(G) = \cdots = \chi^2(G) \tag{1}$$

Examples of r-hued colorings are given in Figure 6.



(i) A 2-hued 5-coloring that is not 2-distance.

(ii) A 5-hued 6-coloring which is also 2-distance.

Figure 6: A graph G with $\Delta = 5$.

Observation 4. *For any graph G and any integer r, $r + 1 \leq \chi_r(G) \leq r^2 + 1$.*

[1] Corollaries of r-hued list-colorings of planar graphs (see Section 4).
[2] Corollaries of 2-distance list-colorings of planar graphs.
[3] Corollaries of 2-distance list-colorings of graphs with a bounded maximum average degree.
[4] This is a corollary of our result (see Corollary 7).
[5] Corollaries of 2-distance colorings of graphs with a bounded maximum average degree.

The lower bound is trivial since, in an r-hued coloring, the neighborhood of a vertex v with $d(v) \geq r$ must contain r colors, different from v's. is a proper coloring. The upper bound can be obtained through a greedy algorithm. This upper bound is reached for certain graphs (see Figure 4) when $r \geq \Delta$, but we do not know if it is tight for general graphs when $r < \Delta$. Similar to the 2-distance chromatic number, the r-hued chromatic number is linear in r when it comes to planar graphs. In 2014, Song *et al.* proposed a generalization of Conjecture 2:

Conjecture 5 (Song *et al.*[29])**.** *Let G be a planar graph. Then,*

$$\chi_r(G) \leq \begin{cases} r+3, & \text{if } 1 \leq r \leq 2, \\ r+5, & \text{if } 3 \leq r \leq 7, \\ \lfloor \frac{3r}{2} \rfloor + 1, & \text{if } r \geq 8. \end{cases}$$

Note that Conjecture 5 implies Conjecture 2 except for the case $r = 3$. Moreover, the only extremal known examples reaching the upper bounds of Conjecture 5 are the same as for Conjecture 2 (see Figure 5i). It is less clear what would be the expected upper bound when $r < \Delta$. In 2018, Song and Lai [30] proved that, if $r \geq 8$, then every planar graph verifies $\chi_r(G) \leq 2r + 16$. Similar to 2-distance coloring, the coefficient before r in this upper bound becomes 1 for graphs with a higher girth. Table 2 shows all known results of the following form: *let r and r_0 be integers such that $r \geq r_0$, all planar graph G, of girth $g(G) \geq g_0$, satisfies $\chi_r(G) \leq r + c(g_0, r_0)$, where $c(g_0, r_0)$ is a constant depending only on g_0 and r_0.*

g_0 \ $\chi_r(G)$	$r+1$	$r+2$	$r+3$	$r+4$	$r+5$	$r+6$	$r+7$	$\dots$	$r+10$
3		$r=2$[21][6]	$r=2$[21][7]	$r=2$[23][8]			$r=3$[27]		
4									
5					$r \geq 15$[9][7]				all r[9]
6					$r \geq 3$[26]				
7		$r=2$[23][8]		$r=3$[22][8]					
8	$r \geq 9$[9]								
9	$r \geq 8$[10][8]		$r=3$[22][8]						
10	$r \geq 6$[10][8]								
11									
12	$r \geq 5$[10][8]								
13									
14		$r=3$[12]							

Table 2: The latest results with a coefficient 1 before r in the upper bound of χ_r.

The result from the "9" line and "$r+1$" column reads "for $r \geq 8$, all planar graph G, of girth at least 9, satisfies $\chi_r(G) \leq r+1$". Since an r-hued coloring is a 2-distance coloring when $r \geq \Delta$, some results for 2-distance colorings comes from r-hued colorings. Similarly to 2-distance colorings, many of these results also come from r-hued list-coloring, or r-hued colorings of graphs with a bounded maximum average degree.

5 Our results

We were particularly interested in the case $\chi_r \leq r+1$. Since $r+1$ is a trivial lower bound for χ_r, we studied the class of graphs verifying $\chi_r = r+1$. As a result, we have proven the following theorem:

Theorem 6. *If G is a planar graph with $g(G) \geq 8$, then $\chi_r(G) = r+1$ for $r \geq 9$.*

That is a new result. For $r = \Delta$, we also get the following corollary:

Corollary 7. *If G is a planar graph with $g(G) \geq 8$ and $\Delta(G) \geq 9$, then $\chi^2(G) = \Delta(G)+1$.*

That is an improvement on known results for 2-distance coloring. Our results are shown in red in Tables 1 and 2.

[6] For G connected and different from C_5.
[7] Corollaries of results on r-hued list-colorings of planar graphs.
[8] Corollaries of results on r-hued list-colorings of graphs with a bounded maximum average degree.
[9] This is our result (see Theorem 6).

6 Notations and drawing conventions

Since we will only consider planar graphs, we will denote $F(G)$ the set of faces of a graph G. We denote $d_G(f)$ the size of face $f \in F(G)$. For $v \in V(G)$, the *2-distance neighborhood* of v, denoted $N^*_G(v)$, is the set of 2-distance neighbors of v, which are vertices at distance at most two from v, not including v. We also denote $d^*_G(v) = |N^*_G(v)|$. From now on, we will sometimes drop the argument when it is clear from context.

Some more notations:

- A *d-vertex* (d^+-vertex, d^--vertex) is a vertex of degree d (at least d, at most d). A $(d \leftrightarrow e)$-vertex is a vertex with degree between d and e included.
- A *d-face* (d^+-face, d^--face) is a face of size d (at least d, at most d).
- A *k-path* (k^+-path, k^--path) is a path of length $k+1$ (at least $k+1$, at most $k+1$) where the k internal vertices are 2-vertices.
- A *$(k_1, k_2, \ldots, k_d)$-vertex* is a d-vertex incident to d different paths, where the i^{th} path is a k_i-path for all $1 \leq i \leq d$.

As a drawing convention for the rest of the figures, black vertices will have a fixed degree, which is represented, and white vertices may have a higher degree than what is drawn.

7 The discharging method

Before proving the result from Theorem 6, we will take a look at the *discharging method*, a classical method used in proving the existence of a coloring. To illustrate this method, we will prove a much weaker result, which is the following:

Theorem 8. *If G is a planar graph with $g(G) \geq 12$ and $\Delta(G) \geq 6$, then $\chi^2(G) = \Delta(G) + 1$.*

In what follows, we will say "coloring" instead of "2-distance $(\Delta + 1)$-coloring" for convenience.

To prove Theorem 8, we will do it according to the following steps:

Step 1: Suppose that there exists a counter-example G and suppose G has the smallest number of vertices.

Step 2: Find *reducible configurations*. A reducible configuration is a graph property P which is forbidden in our counter-example G. In other words, if G satisfies property P (that is, it contains such a configuration), then G can be colored, which is a contradiction. As a result, these reducible configurations will translate into structural properties of G.

Step 3: Assign *charges* to vertices and faces so that the sum of all charges is negative. A charge assignment is a map $\mu : V \cup F \to \mathbb{R}$.

Step 4: Redistribute the charges between the elements of $V \cup F$ according to some defined rules such that the total sum of charges does not change. Using the structural properties of G from **Step 2**, show that this total sum of charges is now non-negative, which is a contradiction.

Proof of Theorem 8. Let us apply the previous proof sketch.

Step 1: Suppose there exists G, a minimal counter-example to Theorem 8. In other words, G is the graph with the minimum number of vertices such that $g \geq 12$, $\Delta \geq 6$, and $\chi^2(G) \geq \Delta + 2$. Without loss of generality, we can assume that G is connected.

Step 2:

Lemma 9. *G has no 1-vertex.*

Proof. If G has a 1-vertex v, then we remove that vertex from G. Since G is minimal, there exists a coloring $\phi_{G'}$ of $G' = G - \{v\}$. Now, we will prove that there exists a coloring ϕ_G of G. For all vertex $v' \neq v$, we choose $\phi_G(v') = \phi_{G'}(v')$. Now for $\phi_G(v)$ we choose a color different from $\phi_G(N^*_G(v))$, which is possible since $d^*_G(v) \leq \Delta$. We obtain a coloring of G, that is a contradiction. □

Lemma 10. *G has no 4^+-paths.*

Proof. Suppose that G contains a 4^+-path $stuvwx$ (see Figure 7).

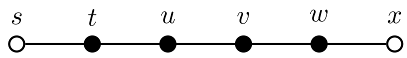


Figure 7: A 4^+-path.

We color $G - \{u, v\}$. Then, it is easy to color u and v since $d^*(u) = d^*(v) = 4 \leq \Delta$.

□

Lemma 11. *Both endvertices of every* 3*-path have degree* Δ*.*

Proof. Suppose there exists a 3-path $stuvw$ where one endvertex, say w, has degree less than Δ (see Figure 8).

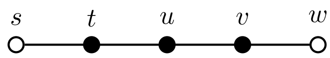


Figure 8: A 3-path with $d(w) \leq \Delta - 1$.

We color $G - \{u, v\}$. Then, we can color v since it sees at most Δ colored vertices in its 2-distance neighborhood. Finally, we can color u as $d^*(u) = 4 \leq \Delta$.

□

Lemma 12. *At least one of the endvertices of a* 2*-path has degree* Δ*, or both of them have degree* $\Delta - 1$*.*

Proof. Suppose by contradiction that there exists a 2-path $uxyw$ with $d(u) \leq \Delta - 1$ and $d(w) \leq \Delta - 2$ (see Figure 9).

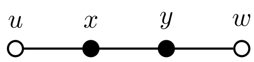


Figure 9: A 2-path with $d(u) \leq \Delta - 1$ and $d(w) \leq \Delta - 2$.

We color $G - \{x, y\}$. Then, we can color x since it sees at most Δ colored vertices in its 2-distance neighborhood. Finally, we can color y as $d^*(y) = d(w) + 2 \leq \Delta$.

□

Step 3: We define the following charge assignment:

$$\mu : v \in V \mapsto 5d(v) - 12 \text{ and } f \in F \mapsto d(f) - 12$$

Now, we need to prove that the total assigned charges is negative. To do so, we use the Euler's formula:

$$\begin{aligned} |V| - |E| + |F| &= 2 \\ -12|V| + 12|E| - 12|F| &= -24 \\ (10|E| - 12|V|) + (2|E| - 12|F|) &= -24 \\ (10 \cdot \frac{1}{2} \sum_{v \in V} d(v) - 12|V|) + (2 \cdot \frac{1}{2} \sum_{f \in F} d(f) - 12|F|) &= -24 \\ \sum_{v \in V} (5d(v) - 12) + \sum_{f \in F} (d(f) - 12) &< 0 \end{aligned}$$

Indeed, we have $\sum_{x \in V \cup F} \mu(x) < 0$.

Step 4: Let us define some special vertex first:

Definition 13. *Consider a vertex $v \in V$ incident to a 2^+-path. We call the 2-vertex, on the 2^+-path, at distance 2 from v, a 2-distance special 2-neighbor.*

The redistribution of the initial charges will be done via *discharging rules*. We will use the following rules:

R0: Every 3-vertex gives charge 1 to its 2-neighbors.

R1: Every 4^+-vertex gives charge 2 to its 2-neighbors.

R2: Every 6^+-vertex gives charge 1 to its 2-distance special 2-neighbors.

Now, let us prove that, after we redistribute the charges according to these rules, we end up with a non-negative amount of charges.

We will denote μ^* the charge assignment after discharging:

Faces: For all $f \in F$, the charges on the faces have not been redistributed by our discharging rules so, $\mu^*(f) = \mu(f) = d(f) - 12 \geq 0$, since $d(f) \geq g \geq 12$.

Vertices: Consider $v \in V$, if:

Case 1: $d(v) \geq 6$

For each incident edge, v has to give charges by **R1** and **R2**. So, v has to give up to charge 3 per edge. As a result,

$$\mu^*(v) \geq \mu(v) - 3d(v) = 5d(v) - 12 - 3d(v) = 2d(v) - 12 \geq 0$$

since $d(v) \geq 6$.

Case 2: $4 \leq d(v) \leq 5$

For each incident edge, v has to give charges by **R1**. So, v has to give up to charge 2 per edge. As a result,

$$\mu^*(v) \geq \mu(v) - 2d(v) = 5d(v) - 12 - 2d(v) = 3d(v) - 12 \geq 0$$

since $d(v) \geq 4$.

Case 3: $d(v) = 3$

For each incident edge, v has to give charges by **R0**. So, v has to give up to charge 1 per edge. As a result,

$$\mu^*(v) \geq \mu(v) - d(v) = 5d(v) - 12 - d(v) = 4d(v) - 12 = 0$$

since $d(v) = 3$.

Case 4: $d(v) = 2$

- v cannot be in a 4^+-path due to Lemma 10.
- If v is in a 3-path, then it either has two 2-neighbors or only one 2-neighbor.

  The endvertices of a 3-path are Δ-vertices due to Lemma 11. So, if v has two 2-neighbors, then it must be a 2-distance special 2-neighbor to both of the endvertices. Since $\Delta \geq 6$, by **R2**, we have

  $$\mu^*(v) \geq \mu(v) + 2 \cdot 1 = 5 \cdot 2 - 12 + 2 = 0.$$

  If v has only one 2-neighbor, then its other neighbor must be a Δ-vertex. By **R1**, we have

  $$\mu^*(v) \geq \mu(v) + 2 = 5 \cdot 2 - 12 + 2 = 0.$$

- If v is in a 2-path, then it has one 2-neighbor and a 3^+-neighbor.

  If v has a 3-neighbor, then v gets charge 1 from **R0**. Due to Lemma 12, the other endvertex of the 2-path must be a Δ-vertex. Then, v gets charge 1 by **R2** since it is a 2-distance special 2-neighbor. So, we have

  $$\mu^*(v) \geq \mu(v) + 1 + 1 = 5 \cdot 2 - 12 + 2 = 0.$$

  If v has a 4^+-neighbor, then v gets charge 2 from **R1**. So, we have

  $$\mu^*(v) \geq \mu(v) + 2 = 5 \cdot 2 - 12 + 2 = 0.$$

- If v is in a 1-path, then it has two 3^+-neighbors.

  Vertex v gets at least charge 1 from both of its neighbors by **R0** and **R1**. So, we have
  $$\mu^*(v) \geq \mu(v) + 2 \cdot 1 = 5 \cdot 2 - 12 + 2 = 0.$$

The graph G has no 1-vertex due to Lemma 9. Thus, we can conclude our proof with a contradiction, since all faces and vertices have a non-negative amount of charges, despite the total of charges being negative in **Step 3**. As a result, there exists no graph G, counter-example to our theorem.

□

8 Proof of Theorem 6

Let us now consider the proof of our main result, namely, if G is a planar graph with $g(G) \geq 8$, then $\chi_r(G) = r+1$ for $r \geq 9$.

Let G be a counterexample to Theorem 6 with the fewest number of edges. The purpose of the proof is to prove that G cannot exist.

Without loss of generality, we can assume that G is connected. Note that G has no vertex of degree 1. Otherwise, we can simply remove the unique edge incident to such vertex v and color the resulting graph with an r-hued coloring ϕ, which is possible due to the minimality of $|E(G)|$. Then, we add the edge back and check the degree of v's unique neighbor x in G. If $d(x) \leq r$, we can choose a color for v different from x's and all of its neighbors' to maintain the r-hued property of the coloring. If $d(x) > r$, then x is already r-hued, so it suffices to choose a color for v different from $\phi(x)$.

This proof will use the discharging method. The Euler formula $|V| - |E| + |F| = 2$ can be rewritten as

$$\sum_{v \in V(G)} (3d(v) - 8) + \sum_{f \in F(G)} (d(f) - 8) = -16. \tag{2}$$

We assign the charge $\mu(v) = 3d(v) - 8$ to a vertex v and $\mu(f) = d(f) - 8$ to a face f. To prove the non-existence of G, we will first describe some structural properties of G, then, based on these, we redistribute the charges preserving their sum and obtaining a non-negative total charge which will contradict Equation (2).

8.1 Structural properties of G

Lemma 14. *Let w be a vertex of V that is adjacent to k vertices u_i ($k \leq d(w)$), each satisfying $d^*(u_i) \leq r+i-1$ for $1 \leq i \leq k$. Then we have $d^*(w) \geq r+k+1$.*

Proof. Suppose by contradiction that w is adjacent to u_i with $d^*_G(u_i) \leq r+i-1$ for $1 \leq i \leq k$, but $d^*_G(w) \leq r+k$. See Figure 10. We remove the edges wu_i for $1 \leq i \leq k$. By minimality of G, let ϕ_H be a r-hued coloring of $H = (V, E \setminus \{wu_1, \ldots, wu_k\})$.

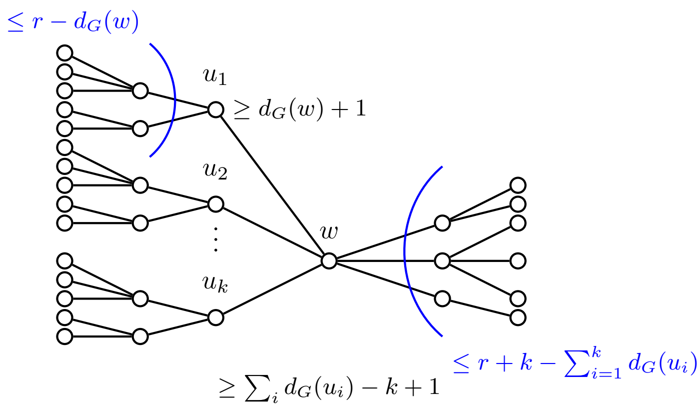


Figure 10: The configuration of Lemma 14.

We uncolor the vertex w and the vertices u_i for $1 \leq i \leq k$. We extend then ϕ to G as follows :

1. We define $\phi_G(v) = \phi_H(v)$ for all $v \in V \setminus \{w, u_1, \ldots, u_k\}$.

2. We define $\phi_G(w)$ to be a color different from all of those of the vertices of $F_w = \bigcup_{i=1}^{k} N_G(u_i) \backslash \{w\} \bigcup N_H^*(w)$. We have $|F_w| = \sum_{i=1}^{k}(d_G(u_i) - 1) + d_H^*(w) = \sum_{i=1}^{k}(d_G(u_i) - 1) + d_G^*(w) - \sum_{i=1}^{k} d_G(u_i) = d_G^*(w) - k$. By hypothesis, we have $d_G^*(w) \leq r + k$ and thus $|F_w| \leq r$. Thus, we have $r + 1$ colors and at most r are forbidden, so it remains at least one color for w.

3. We then define $\phi_G(u_k)$ to be a color different from those that appear on $F_{u_k} = N_H^*(u_k) \cup N_H(w) \cup \{w\}$. Since $d_G^*(u_i) \leq r+i-1$, we have $d_H^*(u_i) \leq r+i-1-d_G(w)$. Therefore, we have $|F_{u_k}| = d_H^*(u_k)+d_H(w)+1 \leq (r + k - 1 - d_G(w)) + d_H(w) + 1 = (r + k - 1 - d_G(w)) + (d_G(w) - k) + 1 = r$. So it remains at least one color for u_k.

4. One by one (from $k - 1$ to 1), we define $\phi_G(u_i)$ to be a color different from those that appear on $F_{u_i} = N_H^*(u_i) \cup N_H(w) \cup \{w, u_{i+1}, u_{i+2}, \ldots, u_k\}$. Using similar argument as the previous subcase, $|F_{u_i}| \leq r$ and thus it remains at least one color for each u_i.

Observe that we 2-distance colored the vertices w, $u_1, \ldots, u_k$. Hence the obtained coloring ϕ_G is r-hued. □

Lemma 15. *Graph G has no 4^+-paths.*

Proof. Suppose G contains a 4-path $stuvwx$ (see Figure 11). Then $d^*(u) = d^*(v) = 4 < r$ which contradicts Lemma 14. □

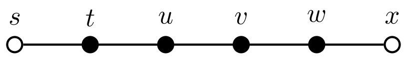


Figure 11: A 4-path.

Lemma 16. *Both endvertices of a 3-path have degree r.*

Proof. Suppose that G contains a 3-path $stuvw$ (see Figure 12). Since $d^*(u) = 4 \leq r$, we have $d^*(v) \geq r + 2$ due to Lemma 14. Moreover, $d^*(v) = d(w) + 2$, so $d(w) \geq r$. Suppose now that $d(w) > r$. Let ϕ be an r-hued coloring of $G' = G - \{u, v\}$ (by minimality of G). Whatever color we choose for v, vertex w is r-hued since $|\phi(N_{G'}(w))| \geq \min(d_G(w) - 1, r) \geq r = \min(d_G(w), r)$. It suffices to choose $\phi(v)$ different from $\phi(w)$ (to have a proper coloring) and from $\phi(t)$ (to make sure that u is r-hued). Finally, we 2-distance color u (the obtained coloring is proper, and the vertices t and v are also r-hued). □

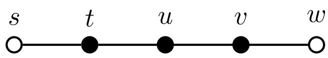


Figure 12: A 3-path.

Lemma 17. *At least one of the endvertices of a 2-path has degree r or both of them have degree $r - 1$.*

Proof. Consider a 2-path $uxyw$ (see Figure 13). Suppose by contradiction that $d(w) \neq r$ and $d(u) \notin \{r - 1, r\}$.

If $d(u) \leq r - 2$, then $d^*(x) = d(u) + 2 \leq r$. So, by Lemma 14, $d^*(y) = d(w) + 2 \geq r + 2$ meaning that $d(w) > r$. By minimality of G, we color $G - \{x, y\}$. Observe that w is already r-hued. We 2-distance color x (u and y become r-hued), and we color y with a color different from that of u, x, and w (x becomes r-hued).

If $d(u) \geq r + 1$, then we color $G - \{x, y\}$. Observe that u is r-hued. Either $d(w) \geq r + 1$ (in that case w is already r-hued) and we color y with a color different from that of w and u, or $d(w) \leq r - 1$ and we 2-distance color y. Finally we color x with a color different from the colors of u, y, and w. □

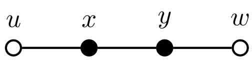


Figure 13: A 2-path.

Lemma 18. *Graph G has no cycles consisting of 3-paths.*

Proof. Suppose that G contains a cycle consisting of k 3-paths (see Figure 14). We remove all vertices $v_{4i+1}, v_{4i+2}, v_{4i+3}$ for $0 \leq i \leq k-1$. Consider a coloring of the resulting graph. We color greedily $v_1, v_3, v_5, \ldots, v_{4k-1}$. This is possible since each of them has at least two choices of color (as $d(v_0) = d(v_4) = \cdots = d(v_{4(k-1)}) = r$ due to Lemma 16) and by 2-choosability of even cycles (see Proposition 28 in Appendix). This procedure ensures that every vertex with even index is r-hued. Finally, it is easy to color greedily $v_2, v_6, \ldots, v_{4k-2}$ since they each have at most four forbidden colors (ensuring that every vertex with odd index is r-hued). $\square$

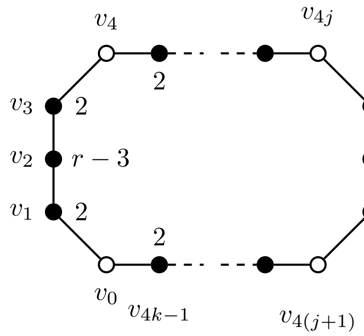


Figure 14: A cycle consisting of consecutive 3-paths.

Lemma 19. *Let $v \in V$ such that $3 \leq d(v) \leq \lfloor \frac{r+1}{2} \rfloor$. Then v cannot be a $(2, 1^+, 1^+, \ldots, 1^+)$-vertex.*

Proof. Suppose that G contains a vertex v with $3 \leq d(v) \leq \lfloor \frac{r+1}{2} \rfloor$ that is a $(2, 1^+, 1^+, \ldots, 1^+)$-vertex. Let w be a neighbor of v that belongs to a 2-path. See Figure 15. We have $d^*(w) = d(v) + 2$ and $d^*(v) = 2d(v)$. Moreover, as $d(v) \leq \lfloor \frac{r+1}{2} \rfloor$, it follows that $d^*(w) \leq r$ since $r > 3$. Thus, $d^*(v) \geq r+2$ by Lemma 14. Since $d(v)$ is an integer and $2d(v) \geq r+2$, $d(v) \geq \lceil \frac{r+2}{2} \rceil$ which contradicts $d(v) \leq \lfloor \frac{r+1}{2} \rfloor$. $\square$

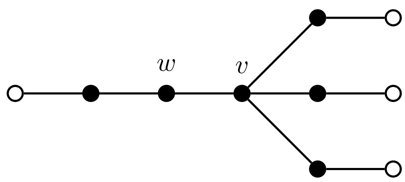


Figure 15: A $(2, 1^+, \ldots, 1^+)$-vertex v with $3 \leq d(v) \leq \lfloor \frac{r+1}{2} \rfloor$.

Lemma 20. *Graph G does not contain the configurations depicted by Figure 16.*

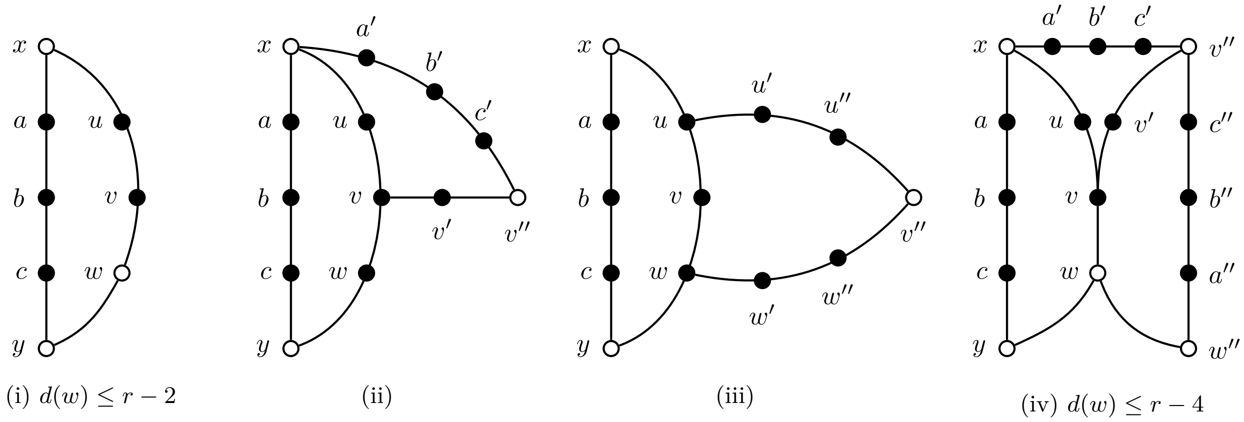


Figure 16: Configurations of Lemma 20 (black vertices have fixed degree, which is represented on the figure).

Proof. Recall that the endvertex of a 3-path always have degree r by Lemma 16. Also, at least one endvertex of a 2-path has degree r unless they both have degree $r-1$ by Lemma 17. Thus, x, y, and v'' always have degree r in what follows ($r \geq 9$).

(a) Consider the configuration depicted on Figure 16i where $d(w) \leq r-2$.

By minimality of G, let ϕ be an r-hued coloring of $G' = G - \{a, b, u, v\}$. Let us start coloring a and u. Both vertices have $r-2+1 = r-1$ restrictions coming from x. Additionally, a (resp. u) has one restriction from c (resp. w). As $\phi(c) \neq \phi(w)$ (since $d(y) = r$), one can color a and u with two distinct colors. Finally, b and v can always be 2-distance colored since b only has four restrictions on its number of colors, and v always has at least one choice of color as $d(w) \leq r-2$. The obtained coloring is r-hued. That contradiction completes the proof.

(b) Consider the configuration depicted on Figure 16ii.

By minimality of G, let ϕ be an r-hued coloring of $G' = G - \{a, b, c, u, v, w, a', b', c', v'\}$. Observe first that, since $d^*(b) < r+1, d^*(v) < r+1, d^*(b') < r+1$, vertices b, v, b' can be 2-distance colored at the end. Vertices a, u, a' have the same $r-2$ restrictions coming from x ; they must be colored with the last three available colors, say $\alpha_1, \alpha_2, \alpha_3$. Similarly c and w (resp. c' and v') have the same $r-1$ restrictions coming from y (resp. v'') ; they must be colored with the last two available colors, say β_1 and β_2 (resp. γ_1 and γ_2). Now, if β_1 does not occur in $\{\alpha_1, \alpha_2, \alpha_3\}$, then one can sequentially color c with β_1, then w, v', u, c', a', and a. So by symmetry, we have $\{\beta_1, \beta_2\} \subset \{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\gamma_1, \gamma_2\} \subset \{\alpha_1, \alpha_2, \alpha_3\}$. If follows that $\{\beta_1, \beta_2\}$ and $\{\gamma_1, \gamma_2\}$ have at least one common element, say $\beta_1 = \gamma_1$. Hence we color the vertices as follows : c with β_1, w with β_2, v' with $\gamma_1 = \beta_1$, c' with γ_2 (which may be equal to β_2), a' with β_1, a with β_2, and u with the color of $\{\alpha_1, \alpha_2, \alpha_3\} \setminus \{\beta_1, \beta_2\}$. That leads to an r-hued coloring of G, a contradiction.

(c) Consider the configuration depicted on Figure 16iii.

By minimality of G, let ϕ be an r-hued coloring of $G' = G - \{a, b, c\}$. Since $d^*(b) < r+1$, $d^*(v) < r+1$, $d^*(u') < r+1$, $d^*(w') < r+1$, b can be 2-distance colored and the vertices v, u', w' can be 2-distance recolored at the end if necessary. Vertex a (resp. c) has r restrictions coming from x and u (resp. y and w). If they can be colored differently, then we obtain an r-hued coloring of G. So, they must have the same available color left, say α. Without loss of generality, say $\phi(u) = \beta$ and $\phi(w) = \gamma$. Since ϕ is r-hued, α, β, γ are all distinct. Moreover at least one of u'' and w'' has a color distinct from α ; by symmetry say $\phi(u'') \neq \alpha$. We now recolor u with α, we color a with β, c with α, we 2-distance color b and as well u', v, w' if necessary. That leads to an r-hued coloring of G, a contradiction.

(d) Consider the configuration depicted on Figure 16iv where $d(w) \leq r-4$.

By minimality of G, let ϕ be an r-hued coloring of $G' = G - \{a', b', c'\}$. Recall that $d(w) \leq r-4$; so $d^*(v) < r+1$. The same holds for $d^*(b)$ and $d^*(b')$, so vertices v, b, b' can be 2-distance recolored at the end. Vertex a' (resp. c') has r restrictions coming from x, a, u (resp. v'', v', c''). If a' and c' can be colored differently, then we can obtain an r-hued coloring of G. So, they must have the same available color left, say α. Let β be the color of u and γ the one of a. Since ϕ is r-hued, α, β, γ are all distinct. If $\phi(c) \neq \alpha$, then we recolor a with α, a' with γ, and c' with α. It follows that $\phi(c) = \alpha$. Now observe that, as $d(y) = d(v") = r$, we have $\phi(w) \neq \alpha$ and $\phi(v') \neq \alpha$ (as α is the available color for c'). So we recolor u with α ; we color a' with β and c' with α. It remains to 2-distance recolor v if necessary and to 2-distance color b'. That leads to an r-hued coloring of G, a contradiction.

□

Lemma 21. *Given a $(2,1,0)$-vertex v having a 7-neighbor, the endvertex of the 1-path (distinct from v) is a 8^+-vertex.*

Proof. Suppose G contains a $(2,1,0)$-vertex v having three neighbors a, b, c such that a belongs to a 2-path, b belong to a 1-path vbd, and such that c has degree 7 and d has degree at most 7. See Figure 17. Let ϕ be an r-hued coloring of $G' = G - \{a, b, v\}$. Let us sequentially 2-distance color v, b, and a. The obtained coloring is r-hued, a contradiction. □

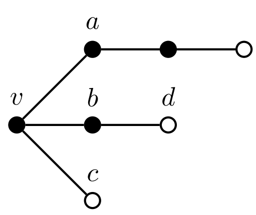


Figure 17: A $(2,1,0)$-vertex having a 7-neighbor.

8.2 Discharging rules

In this section, we define the discharging procedure that contradicts the structural properties of G (see Lemmas 14 to 21) showing that G does not exist.

Definition 22 (Small, medium, and large 2-vertex)**.** *A 2-vertex v is said to be*

- large *if it is adjacent to two 3^+-vertices,*
- medium *if it is adjacent to exactly one 2-vertex,*
- small *if it is adjacent to two 2-vertices.*

Definition 23 (Bridge vertex)**.** *A large 2-vertex is called a* bridge *if it has a 3-neighbor and a 8^+-neighbor.*

Definition 24 (Sponsor)**.** *Consider the set of 3-paths in G. By Lemma 18, the graph induced by the edges of the 3-paths is a forest $\mathcal{F}$. For each tree of $\mathcal{F}$, we choose an arbitrary root. Each small 2-vertex v is assigned a* unique *sponsor* *which is the r-vertex corresponding to the grandson of v. See Figure 18.*

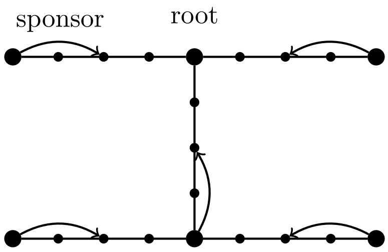


Figure 18: The sponsor assignment in a tree consisting of 3-paths.

Definition 25 (Special and non-special vertices)**.** *A $(3 \leftrightarrow 5)$-vertex is said to be* special *if it has at least two r-neighbors and* non-special *otherwise.*

We first assign to each vertex v the charge $\mu(v) = 3d(v) - 8$ and to each face f the charge $\mu(f) = d(f) - 8$. By Equation (2), the total sum of the charges is negative. We then apply the following discharging rules (R1 to R9):

Vertices to vertices:

R0 (see Figure 19):

- (i) Every 3^+-vertex gives 1 to its large 2-neighbors, and 2 to its medium 2-neighbors.
- (ii) Every sponsor gives 1 to its small 2-neighbors.
- (iii) Every 8^+-vertex gives 1 to its adjacent bridges.

R1 (see Figure 20):

- (i) Every 8^+-vertex gives 2 to its 3-neighbors.
- (ii) Every $(5 \leftrightarrow 7)$-vertex v gives 1 to its 3-neighbors.
- (iii) Every bridge gives 1 to its 3-neighbor.

R2 (see Figure 21):

- (i) Every 8^+-vertex gives 2 to its 4-neighbors.
- (ii) Every $(6 \leftrightarrow 7)$-vertex gives 1 to its 4-neighbors.

R3 (see Figure 22): Every 8^+-vertex gives 2 to its 5-neighbors.

R4 (see Figure 23): Every special vertex gives 1 to its r-neighbors.

Vertices to faces:

R5 (see Figure 24): Each 8-face $f = v_1v_2 \dots v_8$ with $d(v_1) = d(v_7) = r$, $3 \le d(v_4) \le 5$ and $d(v_2) = d(v_3) = d(v_5) = d(v_6) = 2$, receives charge $\frac{1}{2}$ from v_1 and v_7.

R6 (see Figure 27): Let $f = xabcywvu$ be an 8-face where $xabcy$ is a 3-path.

(i) If $xuvw$ is a 2-path with $d(w) \geq r-1$, then y gives $\frac{1}{2}$ to f.

(ii) If xuv is a 1-path with $d(v) \geq 4$, then x gives $\frac{1}{2}$ to f.

(iii) If xuv is a 1-path with $d(v) = 3$ and $d(w) \leq 5$, then v gives $\frac{1}{2}$ to f.

(iv) If xuv is a 1-path with $d(v) = 3$ and $d(w) \geq 6$, y gives $\frac{1}{2}$ to f.

(v) If $d(u) \geq 6$ and $d(w) \geq 3$, then x gives $\frac{1}{2}$ to f.

(vi) If $4 \leq d(u) \leq 5$ and $d(w) \geq 3$, then u gives $\frac{1}{2}$ to f.

(vii) If $d(u) = 3$ and $d(v) \geq 3$, then u gives $\frac{1}{2}$ to f.

(viii) If u is a $(1,1,0)$-vertex, or a $(1,0,0)$-vertex, with $d(v) = 2$, and $d(w) \geq 3$, then u gives $\frac{1}{2}$ to f.

Faces to faces:

R7 (see Figure 25): Let $f = xabcywvvu$ be an 8-face where $xabcy$ is a 3-path, and u and w are $(2,1,0)$-vertices (with the 1-path in common). Let u', u'', and u''' (reps. w', w'', and w''') be, respectively, the 1-distance, 2-distance and 3-distance neighbor of u (resp. w) along its incident 2-path. We also suppose that $u''' \neq w'''$. Let f' be the 9^+-face incident to $u'''u''u'uvww'w''w'''$. Face f' gives $\frac{1}{2}$ to f.

Faces to vertices:

R8 (see Figure 26): Each face f gives $\frac{1}{2}$ to each of its incident small 2-vertices[10].

R9 (see Figure 24): Each 8^+-face f incident to a path $v_1 v_2 \ldots v_7$ as described in **R5** gives 1 to v_4.

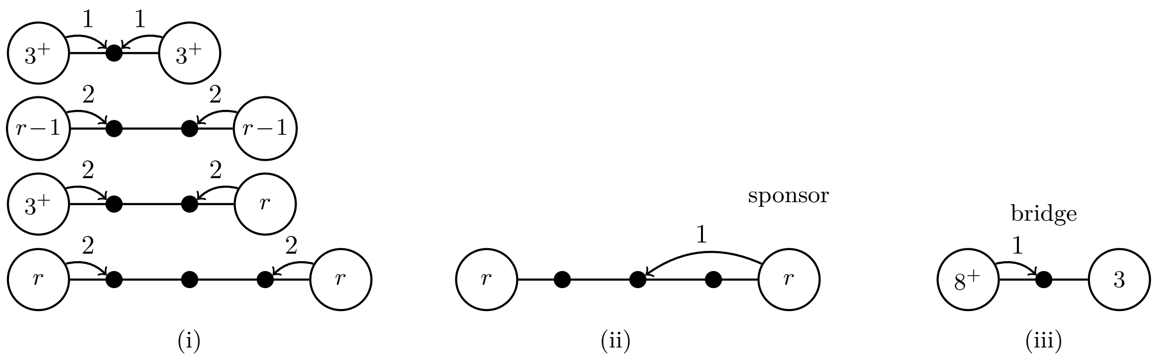


Figure 19: **R0**

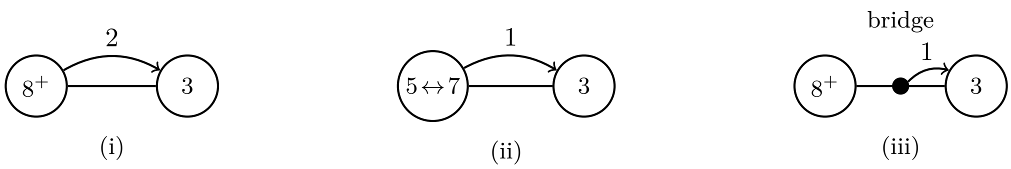


Figure 20: **R1**

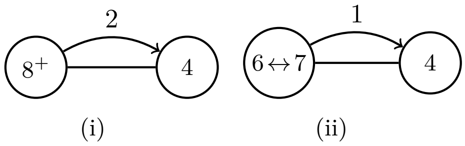


Figure 21: **R2**

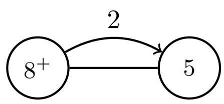


Figure 22: **R3**

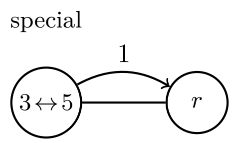


Figure 23: **R4**

[10] f gives $\frac{1}{2}$ twice to a small 2-vertex if that vertex is only incident to f.

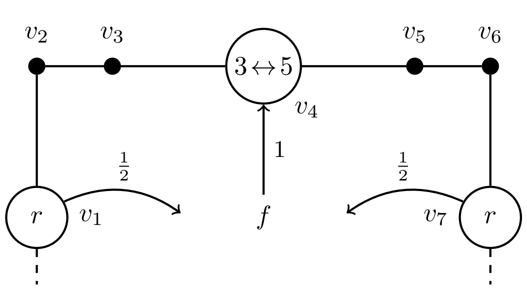


Figure 24: **R5** and **R9**

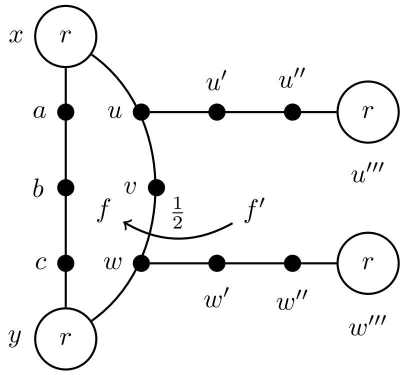


Figure 25: **R7**

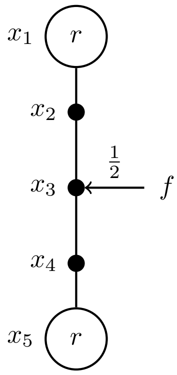


Figure 26: **R8**

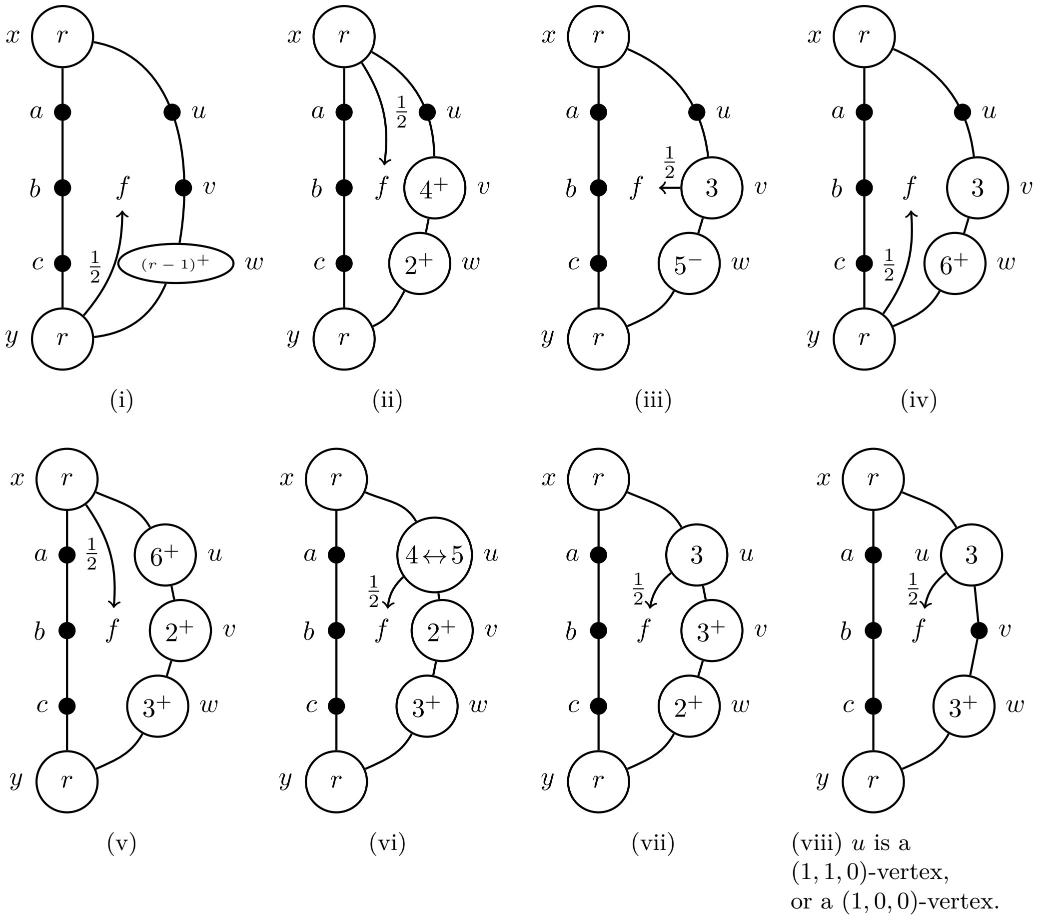


Figure 27: **R6**

8.3 Verifying that charges on vertices and faces are non-negative

Let μ^* be the assigned charges after the discharging procedure. In what follows, we prove that: $\forall x \in V(G) \cup F(G), \mu^*(x) \geq 0$.

8.3.1 Faces

Let f be a face of G. Recall that $\mu(f) = d(f) - 8$. We consider two cases according to the length of f:

Case 1: $d(f) \geq 9$

Note that f gives charge only by **R7**, **R8** and **R9**. By **R9**, face f may give 1 at most $\frac{d(f)}{6}$ times. Similarly by **R8**, face f may give $\frac{1}{2}$ at most $\frac{d(f)}{4}$ times; and by **R7**, face f may give $\frac{1}{2}$ at most $\frac{d(f)}{8}$

times. Observe that except the r-vertices (u'', w'', x_1, x_5, v_1, v_7), all other vertices in the configurations depicted by these rules, are pairwise distinct. Therefore, assuming that **R9** (resp. **R8**, **R7**) is applied i (resp. j, k) times, we must have $d(f) \geq 6i + 4j + 8k$.

Observe that: $\mu^*(f) \geq d(f) - 8 - i - \frac{j}{2} - \frac{k}{2} \geq 6i + 4j + 8k - 8 - i - \frac{j}{2} - \frac{k}{2} \geq 5i + \frac{7}{2}j + \frac{15}{2}k - 8 \geq 0$ when $i \geq 2$ or $k \geq 2$ or $j \geq 3$ or ($j \geq 1$ and $i = 1$) or ($j \geq 1$ and $k = 1$). Now observe that for the remaining cases: $\mu^*(f) \geq d(f) - 8 - i - \frac{j}{2} - \frac{k}{2} \geq 1 - i - \frac{j}{2} - \frac{k}{2} \geq 0$ when $(i, j, k) = (1, 0, 0)$ or $(i, j, k) = (0, 0, 1)$ or $(i, j, k) = (0, 2^-, 0)$. It follows that $\mu^*(f) \geq 0$.

Case 2: $d(f) = 8$

Suppose f is not incident to a 3-path. It follows that f is involved only in **R5** and **R9** ; hence $\mu^*(f) \geq d(f) - 8 + 2 \cdot \frac{1}{2} - 1 = 8 - 8 + 1 - 1 = 0$.

Suppose that f is incident to a 3-path. By Lemma 18, f has only one such path on its boundaries. Face f gives once $\frac{1}{2}$ by **R8** (and **R9** cannot be applied). We show now that f receives $\frac{1}{2}$ by **R6** and **R7**. Let $f = xabcywvu$ where $xabcy$ is a 3-path.

- If f is also incident to a 2-path, say w.l.o.g. $xuvw$, then f gets $\frac{1}{2}$ by **R6**(i) (see Figure 27(i)). Note that the case where $d(w) \leq r - 2$ does not occur by Lemma 20(i).

$$\mu^*(f) \geq d(f) - 8 - \frac{1}{2} + \frac{1}{2} = 8 - 8 - \frac{1}{2} + \frac{1}{2} = 0.$$

- If f is incident to a 1-path of the form xuv, then f gets $\frac{1}{2}$ by **R6**(ii), (iii), or (iv) (see Figure 27(ii), (iii), (iv))).

$$\mu^*(f) \geq 0 - \frac{1}{2} + \frac{1}{2} = 0.$$

- If f is incident to a 1-path of the form uvw and $d(u) > 3$, then f gets $\frac{1}{2}$ from **R6**(v) or (vi) (see Figure 27(v), (vi)). If $d(u) = 3$, then u is either a $(1, 1, 0)$-vertex, or a $(1, 0, 0)$-vertex, or a $(2, 1, 0)$-vertex. By symmetry, the same reasoning holds for w. If one of them is a $(1, 1, 0)$-vertex, or a $(1, 0, 0)$-vertex, then f gets $\frac{1}{2}$ by **R6**(viii) (see Figure 27(viii)). If both of them are $(2, 1, 0)$-vertices, then we are in Configuration **R7** (see Figure 25) with $u''' \neq w'''$ by Lemma 20(iii). In that case, f also receives $\frac{1}{2}$. So, we have in all cases:

$$\mu^*(f) \geq 0 - \frac{1}{2} + \frac{1}{2} = 0.$$

- In the remaining case, f receives $\frac{1}{2}$ by **R6**(v), (vi) or (vii) (see Figure 27(v), (vi), (vii)).

$$\mu^*(f) \geq 0 - \frac{1}{2} + \frac{1}{2} = 0.$$

8.3.2 Vertices

Case 1: $d(v) \geq 8$
Suppose first that $d(v) \neq r$. Observe that v is involved in **R0**(i) and (iii), **R1**(i), **R2**(i), **R3** and v gives at most 2 to each adjacent vertex by **R0**(i), **R1**(i), **R2**(i), **R3** or a combination of **R0**(i) and (iii) (in the case of a bridge). Hence,

$$\mu^*(v) \geq 3d(v) - 8 - 2d(v) = d(v) - 8 \geq 0.$$

Suppose now that $d(v) = r$. Additionally, v also gives charges to faces by **R5** and **R6** and to sponsored small 2-vertices by **R0**(ii). Using the same idea as before, we show that v gives at most 2 along each incident edge.

When **R5** is applied to v, w.l.o.g. $v_1 = v$ in Figure 24, one sends $\frac{1}{2}$ to f via the edge v_1v_8. The edge v_1v_8 belongs to two faces, hence v_1v_8 may be involved twice by **R5**. If v_8 has degree at least 6, no additional charge transits via v_1v_8. If v_8 is a $(3 \leftrightarrow 5)$-vertex, then v_1 gives 2 to v_8 by **R1**(i), **R2**(i), and **R3**, but it receives 1 by **R4** since v_8 would be special as v_1, v_7 are r-vertices. If v_8 has degree 2, then only 1 may transit by **R0**(i). In all cases, at most 2 transits from v_1 along v_1v_8.

Consider now that **R6** is applied to v. As previously, we show that the charge $\frac{1}{2}$ is given to f via a particular edge on which at most 2 transits. Rule **R6** is applied to v in the cases **R6**(i), **R6**(ii), **R6**(iv), and **R6**(v). Observe that no charge is given to 6^+-vertices. Hence charge $\frac{1}{2}$ transits (at most twice) along edge yw in **R6**(i) and **R6**(iv), along edge xu in **R6**(v). In case **R6**(ii), charge $\frac{1}{2}$ transits (at most twice) along edge xu and $x = v$ gives 1 to u by **R0(i)**. Again at most 2 transits along each incident edge.

Finally, vertex v can sponsor (at most) one small 2-vertex by the definition of the sponsor relation and **R0**(ii). It follows that:

$$\begin{aligned}\mu^*(v) &\geq 3d(v) - 8 - 2d(v) - 1 \\ &\geq d(v) - 9 = r - 9 \geq 0\end{aligned}$$

Case 2: $d(v) = 7$
Observe that v may send 1 by **R1**(ii), **R2**(ii), and **R0**(i) in the case of the 1-path, and may send 2 by **R0**(i) in the case of the 2-path. As $\mu(v) = 13$, $\mu^*(v) \geq 0$ except in the case where v is incident to seven 2-paths, but in that case $d^*(v) = 14$, contradicting Lemma 14 (that implies $d^*(v) \geq 17$).

Case 3: $d(v) = 6$
Vertex v may give 1 (resp. 2, 1, 1) by **R0**(i) in the case of the 1-path (resp. **R0**(i) in the case of the 2-path, **R1**(ii), **R2**(ii)). As $\mu(v) = 10$, $\mu^*(v) \geq 0$ except in the case where v gives 2 to each of five of its neighbors and gives at least 1 to its last neighbor, but in that case $d^*(v) \leq 14$, contradicting Lemma 14 (that implies $d^*(v) \geq 15$).

Case 4: $d(v) = 5$
Vertex v may give 1 (resp. 2, 1, 1, $\frac{1}{2}$) by **R0**(i) in the case of the 1-path (resp. **R0**(i) in the case of the 2-path, **R1**(ii), **R4** when it is a special vertex, and **R6**(vi)) and may receive 2 (resp. 1) by **R3**(i) (resp. **R9**). Recall $\mu(v) = 7$.

Suppose that **R6**(vi) is applied to v (v plays the role of u in Figure 27(vi)). Let us use the notations of Figure 27(vi). Hence u gives $\frac{1}{2}$ to f (let say via the edge ux). It may give 1 to x by **R4** (if u is special), and receives 2 from x by **R3**. Moreover **R6**(vi) may be applied to the two faces incident to ux. When we sum the charges transiting along ux, u may give at most $2 \cdot \frac{1}{2} - 2 + 1 = 0$. Hence in the following we consider that, if **R6**(vi) is applied to u, no charge is transferred along ux.

By Lemma 19(i), v is not a $(2, 1^+, 1^+, 1^+, 1^+)$-vertex. Hence v is incident to at most four 2-paths. If v is incident to four 2-paths, then v receives 1 from three incident faces by **R9** and may give at most 2, 2, 2, 2, 1 along incident edges ; so $\mu^*(v) \geq 7 + 3 - 4 \cdot 2 - 1 = 1$. If v is incident to exactly three 2-paths, then v receives at least 1 by **R9** and may give at most 2, 2, 2, 1, 1 along incident edges ; so $\mu^*(v) \geq 7 + 1 - 3 \cdot 2 - 2 \cdot 1 = 0$. If v is incident to at most two 2-paths, then $\mu^*(v) \geq 7 - 2 \cdot 2 - 3 \cdot 1 = 0$.

Case 5: $d(v) = 4$
Vertex v may give 1 (resp. 2, 1, $\frac{1}{2}$) by **R0**(i) in the case of the 1-path (resp. **R0**(i) in the case of the 2-path, **R4**, **R6**(vi)) and may receive 2 (resp. 1, 1) by **R2**(i) (resp. **R2**(ii), **R9**). Recall $\mu(v) = 4$.

As for degree 5 vertices, if **R6**(vi) is applied to v, then no charge is transferred along the edge linking v and the r-vertex.

By Lemma 19(i), v is not a $(2, 1^+, 1^+, 1^+)$-vertex. Hence v is incident to at most three 2-paths. If v is incident to three 2-paths, then v is not special, v receives 1 from two incident faces by **R9** and gives 2, 2, 2, 0 along incident edges ; so $\mu^*(v) = 4 + 2 \cdot 1 - 3 \cdot 2 = 0$. If v is incident to at most one 2-path, then $\mu^*(v) \geq 4 - 2 - 3 \cdot 1 = 0$. Suppose now that v is incident to two 2-paths. If v is not incident to a 1-path, then we are done as $\mu^*(v) = 4 - 2 \cdot 2 = 0$ if v is not special and $\mu^*(v) = 4 - 2 \cdot 2 - 2 \cdot 1 + 2 \cdot 2 = 2$ otherwise. So consider that v is incident to exactly one 1-path by Lemma 19(i) and so is not special. The 3^+-neighbor of v has degree at least 6 (otherwise it contradicts Lemma 14, $d^*(v) \leq 11$ while we must have $d^*(v) \geq 12$), then it gives at least 1 to v by **R2** and so $\mu^*(v) \geq 4 + 1 - 2 \cdot 2 - 1 = 0$.

Case 6: $d(v) = 3$
Vertex v may give 1 (resp. 2, $\frac{1}{2}$, 1) by **R0**(i) in the case of the 1-path (resp. **R0**(i) in the case of the 2-path, **R6**, **R4**) and may receive 2 (resp. 1, 1, 1) by **R1**(i) (resp. **R1**(ii), **R1**(iii), **R9**). Recall $\mu(v) = 1$. By Lemma 19(i), v is not a $(2, 1^+, 1^+)$-vertex. Let us examine all possible configurations for v.

- Suppose that v is a $(2, 2, 0)$-vertex. Let v_1, v_2, and u be the two 2-neighbors and 3^+-neighbor of v respectively. Since v is not special, **R4** does not apply. Vertex v does not fall into any configuration of **R6**, so **R6** does not apply. Vertex v gives 2 to each of its 2-neighbor by **R0**(ii). By Lemma 17, the other endvertices of the two 2-paths are r-vertices ; so v falls into the configuration in **R9** and receives 1 from an incident face. Moreover, v_1 and v_2 satisfy $d^*(v_i) = 5 \leq r$ ($i = 1, 2$). By Lemma 14,

$d^*(v) \geq 12$ and $d^*(v) = d(u) + 4$, so $d(u) \geq 8$. By **R1**(i), v receives 2 from u. In total, we have

$$\mu^*(v) \geq 1 - 2 \cdot 2 + 1 + 2 = 0.$$

– Suppose that v is a $(2, 1, 0)$-vertex. Let v_1, v_2, and u be the two 2-neighbors (where v_1 belongs to the 2-path and v_2 belongs to the 1-path) and 3^+-neighbor of v respectively. As previously, v is not special. Vertex v_1 has $d^*(v_1) = 5 \leq r$. By Lemma 14, $d^*(v) \geq 11$, and $d^*(v) = d(u) + 4$, so $d(u) \geq 7$. It follows that **R6** does not apply (in particular **R6**(iii)).

  If $d(u) \geq 8$, then v receives 2 from u by **R1**(i). Hence, by **R0**(i) and **R1**(i), we have:

$$\mu^*(v) \geq 1 - 2 - 1 + 2 = 0.$$

  If $d(u) = 7$, then v receives 1 from u by **R1**(ii). Moreover, the neighbor of v_2 (different from v) has degree at least 8 by Lemma 21. Hence v receives 1 from v_2 by **R1**(iii). It follows that:

$$\mu^*(v) \geq 1 - 2 - 1 + 1 + 1 = 0.$$

– Suppose that v is a $(2, 0, 0)$-vertex. Let x_1, x_2 be the 0-path neighbors of v and v_1 be the 2-path neighbor of v.

  Suppose first that v is not concerned by **R6**(vii) (i.e. v only gives charge to vertices). Vertex v_1 satisfies $d^*(v_1) = 5 \leq r$. By Lemma 14, $d^*(v) \geq r + 2$. Since $d^*(v) = d(x_1) + d(x_2) + 2$, we have $d(x_1) + d(x_2) \geq r \geq 9$. W.l.o.g. x_1 has degree at least 5. Note that, if v is non-special, then **R4** does not apply and u receives at least 1 from x_1 by **R1**(i) or **R1**(ii); if v is special, then $d(x_1) = d(x_2) = r$, v gives 1 to x_1 and x_2 by **R4** and receives 2 from x_1 and x_2 by **R1**(i). In both case, we can consider that v receives at least 1 from x_1. So

$$\mu^*(v) \geq 1 - 2 + 1 = 0.$$

  Suppose now that **R6**(vii) is applied to v. Observe that **R6**(vii) is applied once. If v is non-special, then v receives 2 from its r-neighbor by **R1**(i); if it is special, by the same arguments as in the previous paragraph, we can consider that v receives 1 from both x_1 and x_2 (by **R1**(i) and **R4**). So

$$\mu^*(v) \geq 1 - 2 - \frac{1}{2} + 2 > 0.$$

– Suppose that v is a $(1, 1, 1)$-vertex. Note that only **R0**(i), **R1**(iii), and **R6**(iii) may concern v. Vertex v gives 1 to each 2-neighbor by **R0**(i) and $\frac{1}{2}$ to at most one incident face by **R6**(iii) and Lemma 20(ii). Let vxw be a 1-path incident to v. We have $d^*(v) = 6 \leq r$. It follows that $d^*(x) \geq 11$ by Lemma 14 and as $d^*(x) = d(w) + 3$, we have $d(w) \geq 8$, meaning that **R1**(iii) applies. Thus,

$$\mu^*(v) \geq 1 - 3 \cdot 1 - \frac{1}{2} + 3 \cdot 1 > 0.$$

– Suppose that v is a $(1, 1, 0)$-vertex. Let vv_1w_1 and vv_2w_2 be the two 1-path incident to v and let u be the 3^+-neighbor of v. Note that v is not special, and it may be concerned by **R0**(i), **R1**, **R6**(iii), and **R6**(viii).

  Suppose first that v is not concerned by **R6** (i.e. v only gives charge to vertices). By **R0**(i), v gives to 1 each of its 2-neighbors.

  If $d(u) \geq 5$, then we have by **R1**(i) and **R1**(ii):

$$\mu^*(v) \geq 1 - 2 \cdot 1 + 1 = 0.$$

  If $d(u) \leq 4$, then $d^*(v) = 8 \leq r$. By Lemma 14, $d^*(v_1) \geq 11$. As $d^*(v_1) = d(w_1) + 3$, we have $d(w_1) \geq 8$ meaning that v receives 1 from v_1 by **R1**(iii) (and from v_2 by symmetry). Hence,

$$\mu^*(v) \geq 1 - 2 \cdot 1 + 2 \cdot 1 > 0.$$

  Suppose that **R6**(iii) or **R6**(viii) is applied to v.

Assume we are in configuration **R6**(viii). Vertex v gives 1 to each of its 2-neighbor and $\frac{1}{2}$ to at most three incident faces (by a combination of **R6**(iii) and **R6**(viii)), and receives 2 from u by **R1**(i). If it gives charge to three faces, then w_1 and w_2 are also endvertices of a 3-path, meaning that they are of degree $r \geq 8$. By **R1**(iii), v receives 1 from each bridge v_1 and v_2. Thus,

$$\mu^*(v) \geq 1 - 2 \cdot 1 - 3 \cdot \frac{1}{2} + 2 + 2 \cdot 1 > 0.$$

Now, if v only gives charge to at most two faces, then we have:

$$\mu^*(v) \geq 1 - 2 \cdot 1 - 2 \cdot \frac{1}{2} + 2 = 0.$$

Assume we are in configuration **R6**(iii) (only, otherwise we are in the previous case). Let us reuse the notations of Figure 27. Observe that either w has degree 2 and u and w are two bridges (since x and y are r-vertices), or w is a $(3 \leftrightarrow 5)$-vertex and the endvertices of the 1-paths incident to v (different from v) are 8^+-vertices by Lemma 14 implying that the 2-neighbors of v are bridges. Hence if **R6**(iii) is applied at most twice, we have by **R0**(i) and **R1**(iii):

$$\mu^*(v) \geq 1 - 2 \cdot 1 - 2 \cdot \frac{1}{2} + 2 \cdot 1 = 0.$$

Now, if **R6**(iii) is applied three times, then we obtain the configuration depicted by Figure 16iv which is forbidden by Lemma 20.

- Suppose that v is a $(1,0,0)$-vertex. Let u, v_1, and v_2 be its 2-neighbor and the two 3^+-neighbors of v, respectively. First note that each time **R4** applies, **R1**(i) also applies, leading v to receive 1. Vertex v gives also 1 to u by **R0**(i). Hence if **R6** does not apply, then $\mu^*(v) \geq 0$. Suppose now that **R6**(iii), (vii) or (viii) is applied to v.

  If **R6**(vii) or **R6**(viii) is applied to v, then (at least) one of the 3^+-neighbors of v is an r-vertex. If v is special, then **R4** and **R1**(i) are applied twice and so v gains 2. Otherwise, **R1**(i) is applied once and not **R4**, and again v gains 2. It follows

$$\mu^*(v) \geq 1 - 1 - 3 \cdot \frac{1}{2} + 2 > 0.$$

  Suppose now only **R6**(iii) is applied to v. Observe that **R6**(iii) may be applied at most twice. Vertex v receives 1 from the bridge by **R1**(iii). Hence,

$$\mu^*(v) \geq 1 - 1 - 2 \cdot \frac{1}{2} + 1 = 0.$$

- Suppose that v is a $(0,0,0)$-vertex. Observe that each time **R4** is applied, so is **R1**(i), and v gains 1. Vertex v may give charge to faces only by **R6**(vii) and in that case it receives at least 1 from its r-neighbor. It follows that:

$$\mu^*(v) \geq 1 - 3 \cdot \frac{1}{2} + 1 > 0.$$

Case 7: $d(v) = 2$

We have $\mu(v) = -2$. Vertex v receives 2 by **R0**(i) unless v is a small 2-vertex. When v is small, it receives 1 from its sponsor by **R0**(ii) and twice $\frac{1}{2}$ from incident faces by **R8**. Now if v is a bridge, then it also gives 1 to a 3-vertex by **R1**(iii), but it also receives 1 from **R0**(iii). In all case, $\mu^*(v) = 0$.

To sum up, we have proven that we started out with a negative total number of charge, and after the discharging procedure that preserves this sum, we end up with a non-negative one, a contradiction.

9 Future work

In this Master thesis, we studied the class of planar graphs satisfying $\chi_r = r + 1$ (the minimum possible number of colors). There are many axis of research left to explore in this domain. For instance, we have also worked on the following theorem:

Theorem 26. *If G is a planar graph with $g(G) \geq 7$, then $\chi_r(G) = r + 1$ for $r \geq 15$.*

The lower bound on r may even be improved to $r \geq 14$. The ideas of the proofs resemble that of Theorem 6, but they are much more complex and are still being reviewed. Note that, if these results are valid, they will also imply better results for 2-distance colorings, since when $r \geq \Delta$, the corresponding r-hued coloring is also a 2-distance coloring. For graphs with girth $g \leq 6$, we know that there is no $(r+1)$-coloring when $r \geq \Delta$[5, 17]. However, the case $r < \Delta$ needs further inspection:

Problem 27. *For a fixed integer $r \geq 2$, does there exist a planar graph G with girth $g \leq 6$ and maximum degree $\Delta > r$ for which $\chi_r(G) \geq r + 2$?*

The other end of the spectrum is to find a large enough girth for which G has an r-hued $(r+1)$-coloring with $r \geq 2$.

A direct improvement of our results is to do the same with list-coloring. The main work to be done is to color our reducible configurations with the additional constraints from list-coloring.

Our effort was mainly put into improving lower bounds on r, while still maintaining $\chi_r = r+1$. However, there could be a stopping point, as in, if r is small enough, then $\chi_r \geq r + 2$. Searching for such graphs will provide an indication on the tightness and optimality of our bound.

We studied mainly planar graphs and its planarity intervenes mainly when we discharge faces of the graphs. Adding the constraint of discharging only on vertices (without using the faces) would be a natural next step. If possible, this would yield the same type of result, but for a more general class of graphs – one with a bounded maximum average degree.

Another line of research is to look at classes of graphs with $\chi_r = r + c_0$ for $c_0 > 1$ a small integer.

In all of these coloring problems, we often use a minimal (in terms of edges, vertices or the sum of both) graph that is a counter-example to our theorem. These are called *critical* graphs with respect to χ_r. Focusing our study on the structure of these graphs is also an interesting direction to take, as it may provide general properties that will be useful in proving bounds on χ_r.

Finally, due to the similarities between the notions of 2-distance and r-hued coloring, we believe that the recent results of $\chi_r \leq 2r + 16$ for $r \geq 8$ by Song and Lai [30], could potentially be improved using what we already know through the findings of Havet *et al.* ($\chi^2 \leq \frac{3}{2}(1 + o(1))\Delta$ for $\Delta \to \infty$[19]).

References

[1] M. Bonamy, D. Cranston, and L. Postle. Planar graphs of girth at least five are square (Δ+2)-choosable. *Journal of Combinatorial Theory, Series B*, 134:218–238, 2019.

[2] M. Bonamy, B. Lévêque, and A. Pinlou. 2-distance coloring of sparse graphs. *Discrete Mathematics*, 38:155–160, 2011.

[3] M. Bonamy, B. Lévêque, and A. Pinlou. 2-distance coloring of sparse graphs. *Journal of Graph Theory*, 77(3), 2014.

[4] M. Bonamy, B. Lévêque, and A. Pinlou. Graphs with maximum degree $\Delta \geq 17$ and maximum average degree less than 3 are list 2-distance (Δ + 2) -colorable. *Discrete Mathematics*, 317:19–32, 2014.

[5] O.V. Borodin, A.N. Glebov, A.O. Ivanova, T.K. Neutroeva, and V.A. Tashkinov. Sufficient conditions for the 2-distance $\Delta + 1$-colorability of plane graphs. *Sibirskie Elektronnye Matematicheskie Izvestiya*, 1:129–141, 2004.

[6] O.V. Borodin and A.O. Ivanova. List 2-facial 5-colorability of plane graphs with girth at least 12. *Discrete Mathematics*, 312:306–314, 2012.

[7] Y. Bu, X. Lv, and X. Yan. The list 2-distance coloring of a graph with $\Delta(G) = 5$. *Discrete Mathematics, Algorithms and Applications*, 7(2), 2015.

[8] Y. Bu and C. Shang. List 2-distance coloring of planar graphs without short cycles. *Discrete Mathematics, Algorithms and Applications*, 8(1), 2016.

[9] Y. Bu and J. Zhu. *Channel Assignment with r-Dynamic Coloring: 12th International Conference, AAIM 2018, Dallas, TX, USA, December 3–4, 2018, Proceedings*, pages 36–48. 2018.

[10] Y. Bu and J. Zhu. List r-dynamic coloring of graphs with small maximum average degree. *Discrete Applied Mathematics*, 258:254–263, 2019.

[11] Y. Bu and X. Zhu. An optimal square coloring of planar graphs. *Journal of Combinatorial Optimization*, 24:580, 2012.

[12] J. Cheng, H.-J. Lai, K. J. Lorenzen, R. Luo, J. C. Thompson, and C.-Q. Zhang. r-hued coloring of sparse graphs. *Discrete Applied Mathematics*, 237:75–81, 2018.

[13] D. Cranston, R. Erman, and R. Skrekovski. Choosability of the square of a planar graph with maximum degree four. *Australian Journal of Combinatorics*, 59(1):86–97, 2014.

[14] D. Cranston and S. Kim. List-coloring the square of a subcubic graph. *Journal of Graph Theory*, 1:65–87, 2008.

[15] W. Dong and W. Lin. On 2-distance coloring of plane graphs with girth 5. *Discrete Applied Mathematics*, 217:495–505, 2017.

[16] W. Dong and B. Xu. 2-distance coloring of planar graphs with girth 5. *Journal of Combinatorial Optimization*, 34:1302, 2017.

[17] Z. Dvořák, D. Kràl, P. Nejedlỳ, and R. Škrekovski. Coloring squares of planar graphs with girth six. *European Journal of Combinatorics*, 29(4):838–849, 2008.

[18] S.G. Hartke, S. Jahanbekam, and B. Thomas. The chromatic number of the square of subcubic planar graphs. arXiv:1604.06504, 2018.

[19] F. Havet, J. van den Heuvel, C. McDiarmid, and B. Reed. List colouring squares of planar graphs. arXiv:0807.3233, 2017.

[20] A.O. Ivanova. List 2-distance (Δ+1)-coloring of planar graphs with girth at least 7. *Journal of Applied and Industrial Mathematics*, 5(2):221–230, 2011.

[21] S.-J. Kim, S.J. Lee, and W.-J. Park. Dynamic coloring and list dynamic coloring of planar graphs. *Discrete Applied Mathematics*, 161:2207–2212, 2013.

[22] S.-J. Kim and B. Park. List 3-dynamic coloring of graphs with small maximum average degree. *Discrete Mathematics*, 341:1406–1418, 2018.

[23] S.-J. Kim and W.-J. Park. List dynamic coloring of sparse graphs. *Combinatorial Optimization and Applications*, 6831:156–162, 2011.

[24] F. Kramer and H. Kramer. Ein Färbungsproblem der Knotenpunkte eines Graphen bezüglich der Distanz p. *Revue Roumaine de Mathématiques Pures et Appliquées*, 14(2):1031–1038, 1969.

[25] F. Kramer and H. Kramer. Un problème de coloration des sommets d'un graphe. *Comptes Rendus Mathématique Académie des Sciences, Paris.*, 268:46–48, 1969.

[26] H.-J. Lai, H. Song, and J.-L. Wu. On r-hued coloring of planar graphs with girth at least 6. *Discrete Applied Mathematics*, 198:251–263, 2016.

[27] S. Loeb, T. Mahoney, B. Reiniger, and J. Wise. Dynamic coloring parameters for graphs with given genus. *Discrete Applied Mathematics*, 235:129–141, 2018.

[28] B. Montgomery. Ph.d. dissertation. *West Virginia University*, 2001.

[29] H. Song, S.-H. Fan, Y. Chen, L. Sun, and H.-J. Lai. On r-hued coloring of K_4-minor free graphs. *Discrete Mathematics*, 315-316:47–52, 2014.

[30] H. Song and H.-J. Lai. Upper bound of r-hued colorings of planar graphs. *Discrete Applied Mathematics*, 243:262–369, 2018.

[31] C. Thomassen. The square of a planar cubic graph is 7-colorable. *Journal of Combinatorial Theory, Series B*, 128:192–218, 2018.

[32] W.Dong and W. Lin. An improved bound on 2-distance coloring plane graphs with girth 5. *Journal of Combinatorial Optimization*, 32(2):645–655, 2016.

[33] G. Wegner. Graphs with given diameter and a coloring problem. *Technical report, University of Dormund*, 1977.

A List-coloring

The following theorem comes from list-coloring and will be useful for the proof of Lemma 18.
Proposition 28 (2-choosability of even cycles)**.** *An even cycle is* 2*-choosable.*

Proof of Proposition 28. Consider a cycle $v_1v_2\dots v_{2l}$. If all of the v_i's have the same list of size 2, it suffices to alternate the chosen color for each vertex, which is possible since it is an even cycle. If at least one v_i has a different list from v_{i-1}, we start by coloring v_i with a color not in v_{i-1}'s list. Then we continue by coloring $v_{i+1}, v_{i+2}, \dots$ (each has at least one avaible color left) until we reach v_{i-1}. The only color restriction on v_{i-1} comes from v_{i-2}, so we can finish coloring v_{i-1}. □