r-hued coloring of planar graphs

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A graph

A graph

A *graph* G is $(V(G), E(G))$:

- where $E(G) \subseteq \{\{u, v\} | u \neq v, (u, v) \in V(G) \times V(G)\}$,
- $V(G)$ is the set of *vertices*,
- and $E(G)$ is the set of *edges*.

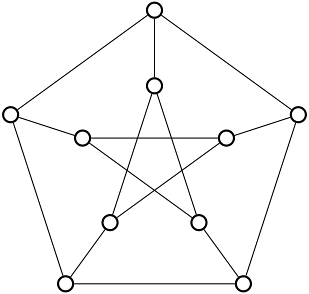

Figure 1: The Petersen graph

Proper k-coloring

A proper k-coloring

A *k-coloring* is a map $\phi : V(G) \to \{1, 2, \ldots, k\}$. A k-coloring ϕ is a *proper coloring*, if and only if, for all edge $xy \in E, \phi(x) \neq \phi(y)$.

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The *chromatic number* of G, denoted $\chi(G)$, is the smallest integer k so that G has a proper k-coloring.

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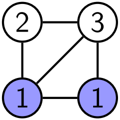
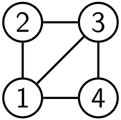
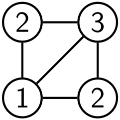


(i) A non-proper 3-coloring.



(ii) A non-optimal proper 4-coloring.



(iii) An optimal proper 3-coloring.

Figure 2: A graph G with $\chi(G) = 3$.

2-distance coloring

A 2-distance k-coloring (Kramer and Kramer, 1969)

A *2-distance k-coloring* is a k-coloring such that no pair of vertices at distance at most 2 has the same color.

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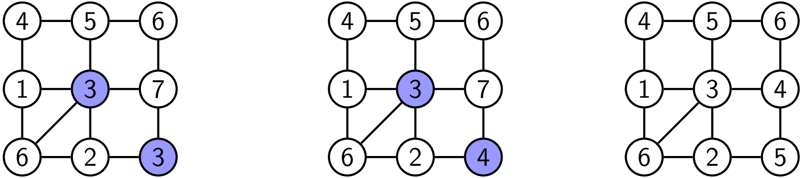
The 2-distance chromatic number

The *2-distance chromatic number* of G, denoted $\chi^2(G)$, is the smallest integer k so that G has a 2-distance k-coloring.



(i) A proper 7-coloring that is not 2-distance.

(ii) A non-optimal 2-distance 7-coloring.

(iii) An optimal 2-distance 6-coloring.

Figure 3: A graph G with $\chi^2(G) = 6$.

2-distance coloring

Observation

For any graph G with maximum degree Δ, $\Delta + 1 \leq \chi^2(G) \leq \Delta^2 + 1$.

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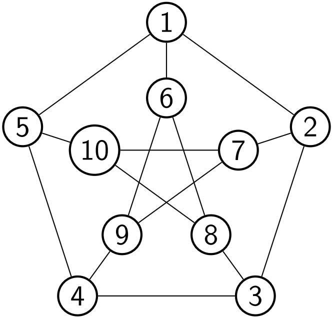

Figure 4: $\chi^2(G) = \Delta^2 + 1$

2-distance coloring

Planar graphs

A graph is *planar* if we can draw its edges without intersections.

2-distance coloring

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Wegner's conjecture, 1977

Let G be a planar graph. Then,

$$\chi^2(G) \leq \begin{cases} 7, & \text{if } \Delta \leq 3, \\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7, \\ \lfloor \frac{3\Delta}{2} \rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

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Thomassen, 2018 and Thomas *et al.*, 2018

Planar graphs with $\Delta \leq 3$ are 2-distance 7-colorable.

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Thomassen, 2018 and Thomas *et al.*, 2018

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Havet *et al.*, 2017

Planar graphs are 2-distance $(\frac{3}{2}\Delta(1 + o(1)))$-colorable when $\Delta \to \infty$.

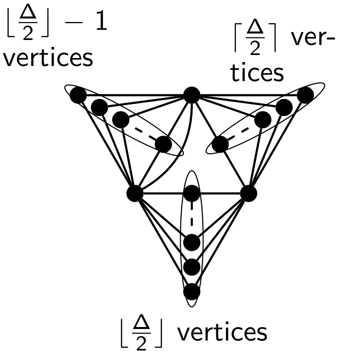

Figure 5: A graph with $\chi^2 = \lfloor \frac{3\Delta}{2} \rfloor + 1$

State of the art

Coefficient 1 before Δ

Every planar graph G of girth $g \geq g_0$, and maximum degree $\Delta \geq \Delta_0$, satisfies $\chi^2(G) \leq \Delta + c(g_0, \Delta_0)$, where $c(g_0, \Delta_0)$ is a constant depending only on g_0 and Δ_0.

g_0 \ $\chi^2(G)$	$\Delta+1$	$\Delta+2$	$\Delta+3$	$\Delta+4$	$\Delta+5$	$\Delta+6$	$\Delta+7$	$\Delta+8$
3				$\Delta = 3$				
4								
5		$\Delta \geq 10^7$	$\Delta \geq 339$	$\Delta \geq 312$	$\Delta \geq 15$	$\Delta \geq 12$	$\Delta \neq 7, 8$	all Δ
6		$\Delta \geq 17$	$\Delta \geq 9$		all Δ			
7	$\Delta \geq 16$			$\Delta = 4$				
8	$\Delta \geq 10$ $\Delta \geq 9$		$\Delta = 5$					
9	$\Delta \geq 8$	$\Delta = 5$	$\Delta = 3$					
10	$\Delta \geq 6$							
11		$\Delta = 4$						
12	$\Delta = 5$	$\Delta = 3$						
13								
14	$\Delta \geq 4$							
22	$\Delta = 3$							

Table 1: Results from almost 20 different papers.

r-hued coloring

An r-hued coloring (Montgomery, 2001)

An *r-hued k-coloring* of G is a proper k-coloring, such that, for all vertex, the number of colors in its neighborhood is at least $\min\{d_G(v), r\}$.

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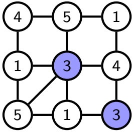
The r-hued chromatic number

The *r-hued chromatic number* of G, denoted $\chi_r(G)$, is the smallest integer k so that G has an r-hued k-coloring.

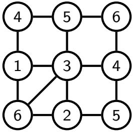
The link between r-hued coloring, proper coloring, and 2-distance coloring:

$$\chi(G) = \chi_1(G) \leq \chi_2(G) \leq \cdots \leq \chi_\Delta(G) = \chi_{\Delta+1}(G) = \cdots = \chi^2(G)$$

r-hued coloring



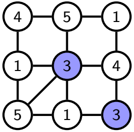
(i) A 2-hued 5-coloring that is not 2-distance.



(ii) A 5-hued 6-coloring which is also 2-distance.

Figure 6

r-hued coloring



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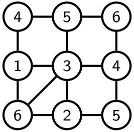

(ii) A 5-hued 6-coloring which is also 2-distance.

Figure 6

Observation

For any graph G and any integer r, $r+1 \leq \chi_r(G) \leq r^2+1$.

Song *et al.*'s conjecture, 2014

Let G be a planar graph. Then,

$$\chi_r(G) \leq \begin{cases} r+3, & \text{if } 1 \leq r \leq 2, \\ r+5, & \text{if } 3 \leq r \leq 7, \\ \lfloor \frac{3r}{2} \rfloor + 1, & \text{if } r \geq 8. \end{cases}$$

State of the art

Coefficient 1 before r

Let r and r_0 be integers such that $r \geq r_0$, all planar graph G, of girth $g(G) \geq g_0$, satisfies $\chi_r(G) \leq r + c(g_0, r_0)$, where $c(g_0, r_0)$ is a constant depending only on g_0 and r_0.

g_0 \ $\chi_r(G)$	$r+1$	$r+2$	$r+3$	$r+4$	$r+5$	$r+6$	$r+7$	$\dots$	$r+10$
3		$r=2^*$	$r=2$				$r=3$		
4									
5					$r \geq 15$				all r
6					$r \geq 3$				
7		$r=2$		$r=3$					
8	$r \geq 9$								
9	$r \geq 8$		$r=3$						
10	$r \geq 6$								
11									
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14		$r=3$							

Table 2: Results from almost 10 different papers

Our results

Theorem

If G is a planar graph with $g(G) \geq 8$, then $\chi_r(G) = r + 1$ for $r \geq 9$.

Corollary

If G is a planar graph with $g(G) \geq 8$ and $\Delta(G) \geq 9$, then $\chi^2(G) = \Delta(G) + 1$.

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If G is a planar graph with $g(G) \geq 8$, then $\chi_r(G) = r + 1$ for $r \geq 9$.

Corollary

If G is a planar graph with $g(G) \geq 8$ and $\Delta(G) \geq 9$, then $\chi^2(G) = \Delta(G) + 1$.

Theorem (An example)

If G is a planar graph with $g(G) \geq 24$ and $\Delta(G) \geq 3$, then $\chi^2(G) = \Delta(G) + 1$.

The discharging method (on planar graphs):

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1: Suppose that there exists a counter-example G and suppose that G has the smallest number of vertices.

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2: Study the structural properties of G.

3: Assign charges to vertices and faces so that the sum of all charges is negative thanks to the Euler's formula ($|V| - |E| + |F| = 2$).

4: Redistribute the charges without changing the total sum, and show that we obtain a non-negative final amount, thanks to the structural properties, which is a contradiction.

The discharging method: Step 1

Theorem (An example)

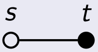
If G is a planar graph with $g(G) \geq 24$ and $\Delta(G) \geq 3$, then $\chi^2(G) = \Delta(G) + 1$.

Step 1: Suppose that there exists G, the graph with the minimum number of vertices such that $g \geq 24$, $\Delta \geq 3$, and $\chi^2(G) \geq \Delta + 2$.

The discharging method: Step 2

Step 2: Structural properties of G.

Lemma

- *G is connected.*
- *G has no* 

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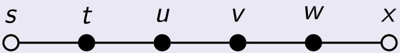
- *G has no* 

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The discharging method: Step 3

Step 3: Assign charges to vertices and faces so that the total sum is negative thanks to the Euler's formula ($|V| - |E| + |F| = 2$).

Charge assignment

We define the following charge assignment μ:

$$v \mapsto 11d(v) - 24 \text{ and } f \mapsto d(f) - 24$$

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Proof.

$$\begin{aligned}
|V| - |E| + |F| &= 2 \\
-24|V| + 24|E| - 24|F| &= -48 \\
(22|E| - 24|V|) + (2|E| - 24|F|) &= -48 \\
(22 \cdot \frac{1}{2} \sum_{v \in V} d(v) - 24|V|) + (2 \cdot \frac{1}{2} \sum_{f \in F} d(f) - 24|F|) &= -48 \\
\sum_{v \in V} (11d(v) - 24) + \sum_{f \in F} (d(f) - 24) &< 0
\end{aligned}$$

The discharging method: Step 4

Step 4: Redistribute the charges to obtain a non-negative sum.

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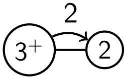
Discharging rules:

R0:



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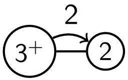
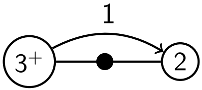
R0: 

R1: 

Step 4: Redistribute the charges to obtain a non-negative sum.

Faces: $d(f) - 24 \geq 0$ since $g \geq 24$.

$$\sum_{v \in V}(11d(v) - 24) + \sum_{f \in F}(d(f) - 24) < 0$$

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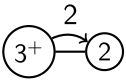

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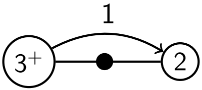

The discharging method: Step 4

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$$\sum_{v\in V}(11d(v)-24)$$
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Faces: $d(f)-24\geq 0$ since $g\geq 24$.

Vertices: If

- $d(v)\geq 3$
  $\mu^*(v)=\mu(v)-3d(v)$
  $=8d(v)-24\geq 0.$

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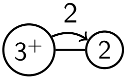

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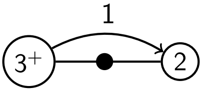


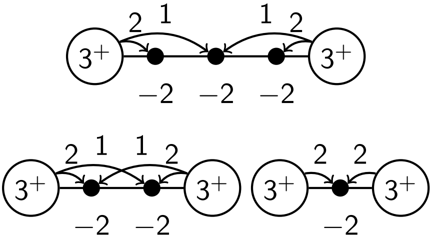
Step 4: Redistribute the charges to obtain a non-negative sum.

$$\sum_{v\in V}(11d(v)-24)$$
$$+\sum_{f\in F}(d(f)-24)<0$$

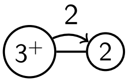
Faces: $d(f)-24\geq 0$ since $g\geq 24$.

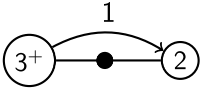
Vertices: If

- $d(v)\geq 3$
  $\mu^*(v)=\mu(v)-3d(v)$
  $=8d(v)-24\geq 0.$
- $d(v)=2$



Discharging rules:

R0: 

R1: 

A comparison

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- 8 lemmas,

Lemma 20. *Graph G does not contain the configurations depicted by Figure 16.*

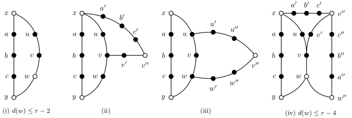

Figure 16: Configurations of Lemma 20 (black vertices have fixed degree, which is represented on the figure).

A comparison

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Lemma 20. *Graph G does not contain the configurations depicted by Figure 16.*

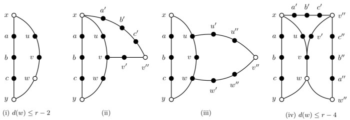


Figure 16: Configurations of Lemma 20 (black vertices have fixed degree, which is represented on the figure).

- 22 rules,

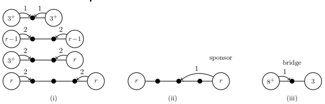


Figure 19: **R0**

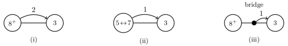


Figure 20: **R1**

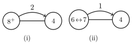


Figure 21: **R2**

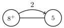


Figure 22: **R3**

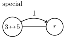


Figure 23: **R4**

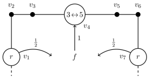


Figure 24: **R5** and **R9**

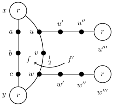


Figure 25: **R7**

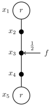


Figure 26: **R8**

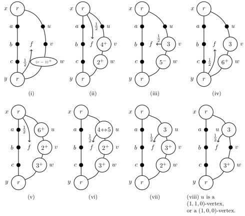


Figure 27: **R6**

A comparison

For our proof:

- 8 lemmas,

Lemma 20. *Graph G does not contain the configurations depicted by Figure 16.*

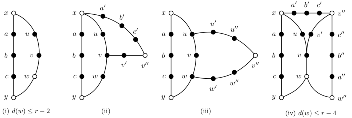
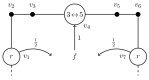
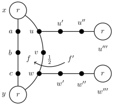
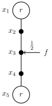


Figure 16: Configurations of Lemma 20 (black vertices have fixed degree, which is represented on the figure).







- 22 rules,

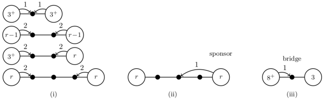


Figure 19: **R0**

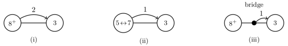


Figure 20: **R1**

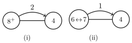


Figure 21: **R2**

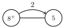


Figure 22: **R3**

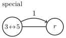


Figure 23: **R4**

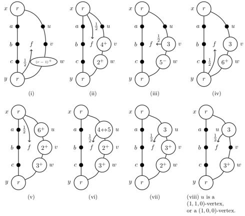


Figure 27: **R6**

- 11 pages of proof.

Future work

g_0 \ $\chi_r(G)$	$r+1$	$r+2$	$r+3$	$r+4$	$r+5$	$r+6$	$r+7$	$\dots$	$r+10$
3	?	$r=2^*$	$r=2$	$r=2$			$r=3$		
4	?								
5	?				$r\geq 15$				all r
6	?				$r\geq 3$				
7	$r\geq 14$	$r=2$		$r=3$					
8	$r\geq 9(8)$								
9	$r\geq 8$		$r=3$						
10	$r\geq 6$								
11									
12	$r\geq 5$								
13									
14		$r=3$							
?	$r\geq 2$								

Future work

$\chi_r(G)$ \ g_0	$r+1$	$r+2$	$r+3$	$r+4$	$r+5$	$r+6$	$r+7$	$\dots$	$r+10$
3	?	$r=2^*$	$r=2$	$r=2$			$r=3$		
4	?								
5	?				$r\geq 15$				all r
6	?				$r\geq 3$				
7	$r\geq 14$	$r=2$		$r=3$					
8	$r\geq 9(8)$								
9	$r\geq 8$		$r=3$						
10	$r\geq 6$								
11									
12	$r\geq 5$								
13									
14		$r=3$							
?	$r\geq 2$								

Question:

How does χ_r behave when $r \ll \Delta$ compare to χ^2 $(r \geq \Delta)$?