CRE method of element testing and the Jacobian shape parameters*

John Robinson

Robinson and Associates, Great Bidlake Manor, Bridestowe, Okehampton, Devon EX20 4NT, UK

(Received December 1986)

ABSTRACT

A general method of element testing is presented. The method applies to any shape of element with any number of nodes. The shape parameters for a quadrilateral are shown to be contained within the Jacobian matrix and it is also shown that the determinant of this matrix can be expressed in terms of the shape parameters.

INTRODUCTION

During the period 1968–71 Robinson and Haggenmacher were collaborating on research into the fully-automated force method and stress-based elements at the Lockheed–California Company, USA. During this time, they formulated interchangeability relations[1,2] which enabled stress elements to be used in a displacement method system and displacement elements to be used in a force method system. However, in the latter part of this period, Lockheed started to evaluate NASA's NASTRAN which ultimately replaced the force method system but, through the interchangeability relations, used Lockheed's stress-based elements.

The Lockheed thinking was to use the best part of the displacement method (speed on solution) and the best part of the force method (stress elements). In the evaluation of NASA's NASTRAN, Robinson and Haggenmacher conceived the 'Single Element Test' which has now become a well-recognized and informative test[2–4]. However, in the early days of this test, many academic authorities did not fully appreciate its value since they were displacement-orientated as against stress-orientated. The single element test has identified many deficiencies in elements and continues to do so[5–11]. The single element test enables one to understand fully the load-carrying capabilities of an element, its sensitivity to shape changes, whether or not the element is invariant, and is essential in the development of an element. The single element test helps considerably in the selection of the terms to include in the element's stress/strain field. The single element test also tests for element completeness (constant stress/strain states), rigid body modes and false (spurious) zero energy modes.

Other tests which have been widely used relate to an assemblage of elements, namely, the convergence test and the patch test[7–11,12–14]. In Reference 15 Robinson formalized a set of assessment points and standard tests for element evaluations which were then applied to the elements in a range of finite element systems (MSC/NASTRAN, ANSYS, ASAS, PAFEC and SAP4) with support from the Royal Aircraft Establishment, Farnborough, England. In the formulation and application of these tests it became clear that there was a need for recognized standards in finite element method technology[16] and a 'World FEM Association' was proposed in Reference 17. This led to the formation of NAFEMS (National Agency for Finite Element Methods and Standards[18]), from which various reports on testing procedures have emerged[19–22].

A finite element standards forum is also operating in the USA and tests which have been formulated by this organization can be found in References 23 and 24. A standards organization has also been established in Holland[25].

The original single element test was concerned with the aspect ratio sensitivity but has now been extended to include skew and taper sensitivity. In order to have theoretical solutions available for evaluating skew and taper effects, a new element testing procedure has been developed and referred to as the 'CRE method' (Continuum-Region-Element). This development work was performed under a NAFEMS contract and the full findings are reported in Reference 19. During this contract work, shape parameters were defined and based on a step-by-step drawing procedure which is acceptable to an engineer. It was then suggested that the shape parameters for an element may be contained within its Jacobian matrix. This has now been shown to be the case[26] and it has also been shown that the determinant of the Jacobian matrix can be expressed in terms of the shape parameters. This now gives the program developer a simple means of indicating critical shape parameters for the elements within a model.

When the full testing of an element's shape sensitivity is completed, it will enable a system to check if the combination of an element's shape parameters are critical. The existing knowledge on an element's shape sensitivity will be part of a database of an automated modelling code or expert system.

CRE METHOD

Previous single element tests[2–8] have addressed rectangular elements and rectangles made up of triangles and have investigated aspect ratio sensitivity only. This work introduces a new and general approach to element testing which can be used for any shape of element with any number of nodes. This approach is referred to as the 'Continuum-Region-Element Method' or simply the CRE Method. Although this method is currently being applied to membrane and plate bending elements, it is equally applicable to solid elements.

The CRE Method starts with the idea that we have a rectangular continuum under various loading conditions for which the elasticity (theoretical) solutions are known. The stress and displacement values of a quadrilateral or triangular region within this continuum can be readily calculated from the elasticity solutions for the rectangular continuum. A quadrilateral or triangular region can then be considered as a: (i) single element; (ii) patch of elements, with any number of nodes (see *Figure 1*). In either case, the tests are easy to formulate with elasticity results available for comparison with the finite element results.

In the case of patch tests, the simplest approach is to

* Presented at the International Conference on Reliability of Methods for Engineering Analysis, Swansea, July 1986.

0264-4401/87/020113-06$2.00

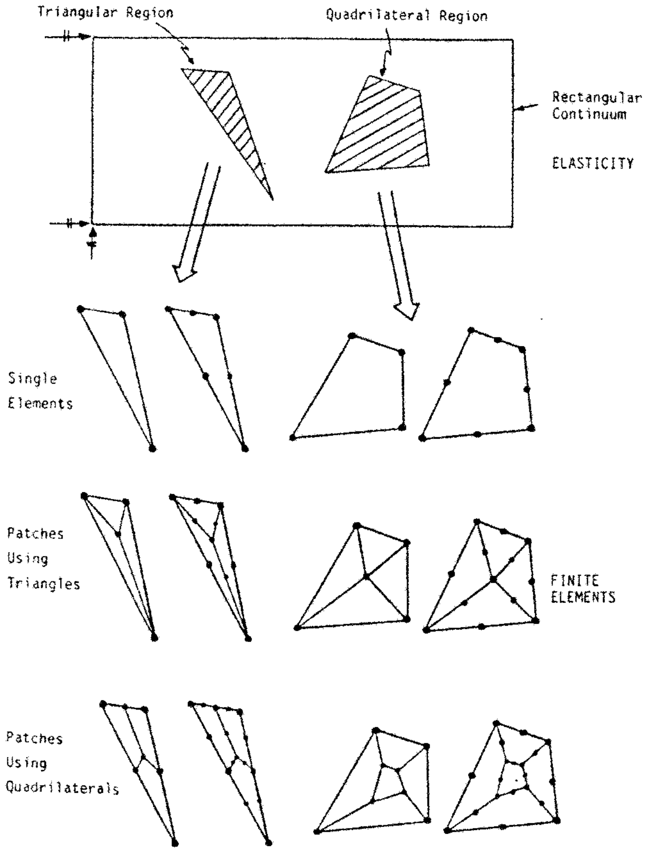


Figure 1 The CRE method

model the complete rectangular continuum as an assemblage of elements.

In the CRE method the applied loading in the tests is in the form of specified displacements which will in general be non-zero, and applied forces. In previous work the specified displacements have been zero.

In the discussion so far, the shape of the regions has been considered as a quadrilaterla or a triangle; clearly the region can be of any shape. The CRE method can therefore be used to establish tests for single elements and patches of any shape for various loading conditions.

A comparison of the CRE method concept for membrane, plate bending and three-dimensional solid elements is shown in *Figure 2*.

SHAPE PARAMETERS FOR A QUADRILATERAL

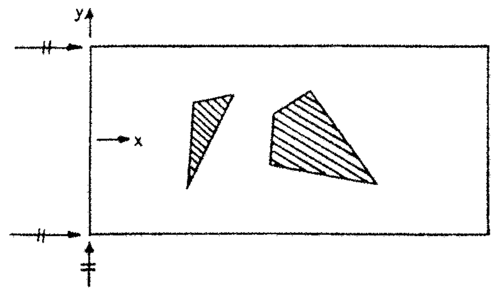
Consider a flat quadrilateral element in a two-dimensional coordinate system (x, y) as shown in *Figure 3*. For finite element modelling we are trying to establish parameters which represent an element shape and then study how these 'shape parameters' affect the results. In this way, guidelines can be formulated for the user or checking procedures can be incorporated within a system to identify 'bad shapes' before running an analysis.

The shape parameters which define a quadrilateral are: (i) aspect ratio; (ii) skew angle; (iii) two tapers. To construct a quadrilateral we can start with a base-line of a given length (four parameters needed). Given an aspect ratio (AR) we can construct a rectangle which can then be skewed using a given skew angle (δ). A quadrilateral can then be constructed using two taper parameters which are taken as the dimensions τ_1 and τ_2. *Figure 4* shows this procedure of constructing a quadrilateral.

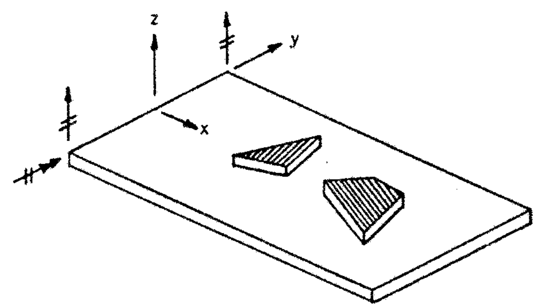
What is now required is the reverse procedure, i.e. for a given quadrilaterla, determine its shape parameters, AR, δ and τ_1 and τ_2.

Aspect ratio

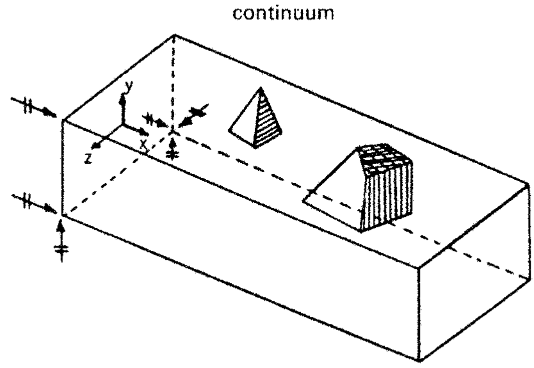
To obtain the aspect ratio of a quadrilateral shape, bisect each boundary and draw a line through the midpoints on opposite boundaries. These lines will intersect at a point '0' as shown in *Figure 5*.



(a) Quadrilateral and triangular regions in a membrane continuum



(b) Quadrilateral and triangular regions in a plate bending continuum



(c) Hexahedron and tetrahedron regions in a three-dimensional solid continuum

Figure 2 CRE method concept for membranes, plates and solids

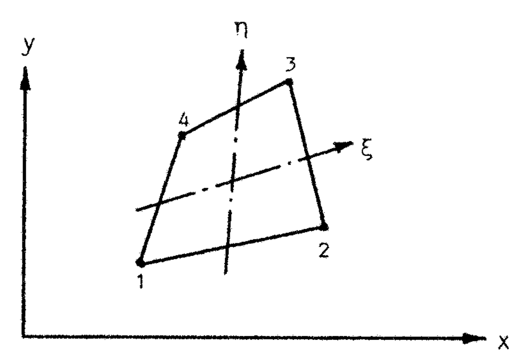


Figure 3 Flat quadrilateral in a two-dimensional system

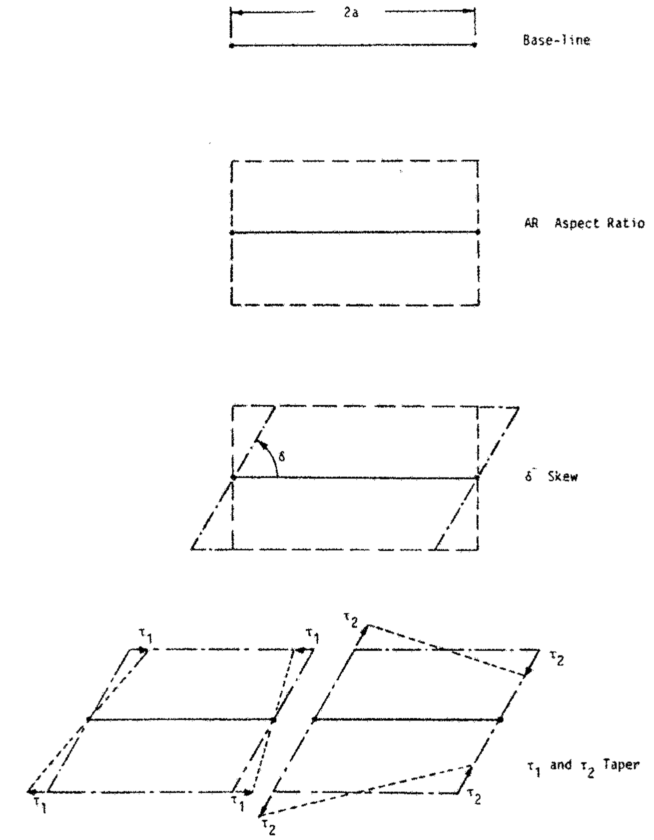


Figure 4 Construction of a quadrilateral

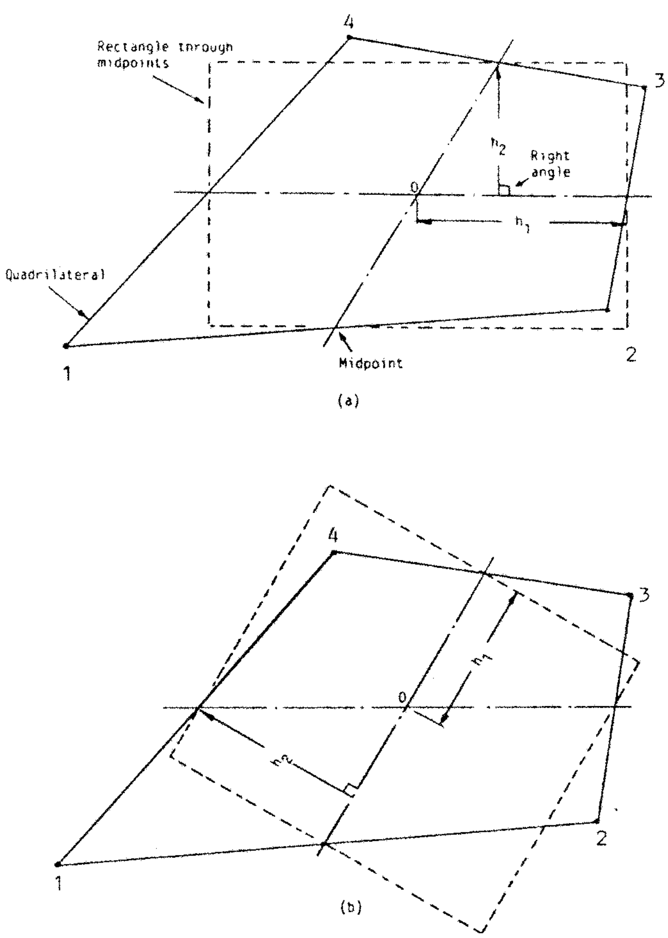


Figure 5 Aspect ratio for quadrilateral

Figures 5a and *5b* show two rectangular constructions which will be used to define the element aspect ratio for a quadrilateral. The aspect ratio (AR) of a rectangle is defined as:

$$AR = \frac{a}{b} \tag{1}$$

where

$$a = h_1 \qquad b = h_2 \qquad h_1 \geqslant h_2 \tag{2}$$

or

$$a = h_2 \qquad b = h_1 \qquad h_1 \quad h_2 \tag{3}$$

For each rectangle, measure the distance h_1 and the perpendicular distance h_2. The largest aspect ratio will be used as the element parameter.

Figure 6 shows a number of elements and their aspect ratio constants.

Skewness

The skewness is measured by the angle between the two lines which pass through the midpoints. *Figure 7* shows the skew angle, δ, and the corresponding parallelogram.

Taper

The distance from a corner node of the parallelogram to its associated corner node of the actual quadrilateral is the same for all nodes. The distance 'c' shown in *Figure 8* is now resolved into two components which are parallel to the lines joining the midpoints (skew axes). These two components are τ_1 and τ_2 which are the previously defined taper parameters (see *Figure 4*).

A simple taper example is shown in *Figure 9*.

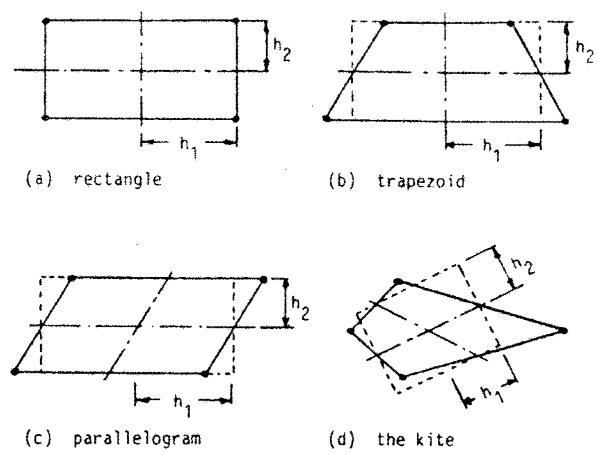


Figure 6 Some aspect ratio constants

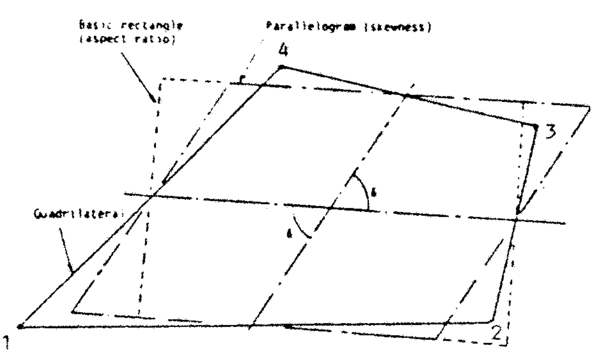


Figure 7 Skew angle for quadrilateral

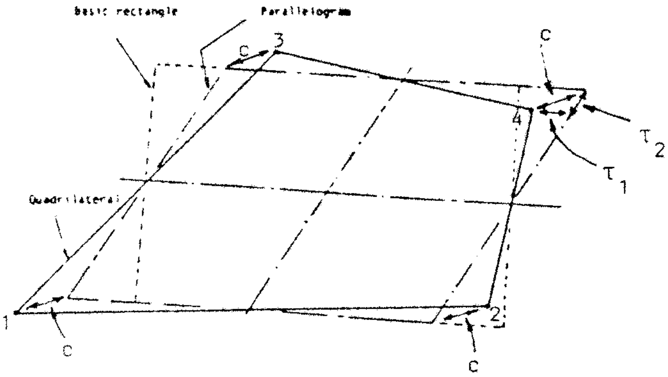


Figure 8 Tapers for quadrilateral

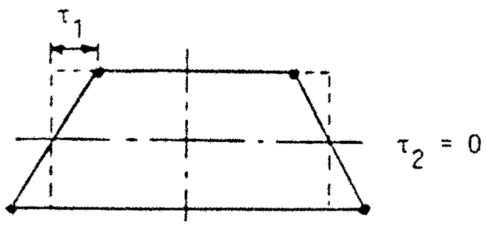


Figure 9 Taper parameters for a trapezoid

SHAPE PARAMETERS AND THE JACOBIAN MATRIX

In the previous section the four shape parameters were based on a step-by-step drawing procedure for constructing a flat quadrilateral. This section takes a new look at the definition of the shape parameters and shows that they can be expressed in terms of coefficients in simple polynomials which express the shape of a quadrilateral. These polynomial coefficients have a simple physical meaning and are functions of the corner point coordinates of the flat or projected quadrilateral.

It is shown how the shape parameters can be obtained from the Jacobian matrix. In future automated analysis and design, it will be necessary to take each element of the finite element model in turn and calculate its shape parameters. These parameters will then be used to scan an already established database from shape sensitivity tests for the specific element within the FEM system being used to ascertain its acceptance in the model. In other words, previous testing of the element will have shown that certain combinations of the shape parameters should be avoided. This existing knowledge will be part of the database of an automated modelling code or expert system.

A New Look at Shape Parameters

Figure 3 shows a flat (projected) quadrilateral in a general local xy-system. The shape of the quadrilateral can be written in the form of interpolation functions[27] as:

$$x=\sum_{i=1}^{4} P_i(\xi,\eta)x_i \tag{4}$$

and

$$y=\sum_{i=1}^{4} P_i(\xi,\eta)y_i \tag{5}$$

where

$$\begin{aligned} P_1&=\tfrac{1}{4}(1-\xi)(1-\eta)\\ P_2&=\tfrac{1}{4}(1+\xi)(1-\eta)\\ P_3&=\tfrac{1}{4}(1+\xi)(1+\eta)\\ P_4&=\tfrac{1}{4}(1-\xi)(1+\eta)\end{aligned} \tag{6}$$

and ξ and η are non-dimensional coordinates with limits of ± 1. The non-dimensional coordinates of points 1, 2, 3 and 4 are $(-1,-1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$, respectively.

Equations (4) and (5) show that *eight* parameters are needed to define a quadrilateral. In this case, the corner point coordinates:

$$\begin{matrix} x_1 & x_2 & x_3 & x_4\\ y_1 & y_2 & y_3 & y_4\end{matrix}$$

The use of interpolation functions is very elegant but mathematical elegance can hide the practical significance of various quantities. An alternative form of shape representation is to use the simple and basic polynomials[27]:

$$\begin{aligned} x&=e_1+e_2\xi+e_3\eta+e_4\xi\eta\\ y&=f_1+f_2\xi+f_3\eta+f_4\xi\eta\end{aligned} \tag{7}$$

where the e and f coefficients are related to the nodal coordinates x_1 to x_4 and y_1 to y_4 by:

$$\begin{aligned} e_1&=\tfrac{1}{4}(x_1+x_2+x_3+x_4)\\ e_2&=\tfrac{1}{4}(-x_1+x_2+x_3-x_4)\\ e_3&=\tfrac{1}{4}(-x_1-x_2+x_3+x_4)\\ e_4&=\tfrac{1}{4}(x_1-x_2+x_3-x_4)\\ f_1&=\tfrac{1}{4}(y_1+y_2+y_3+y_4)\\ f_2&=\tfrac{1}{4}(-y_1+y_2+y_3-y_4)\\ f_3&=\tfrac{1}{4}(-y_1-y_2+y_3+y_4)\\ f_4&=\tfrac{1}{4}(y_1-y_2+y_3-y_4)\end{aligned} \tag{8}$$

Again, there are eight parameters (e_1 to e_4 and f_1 to f_4). The physical significance of the e and f coefficients is shown in *Figure 10*. From *Figure 10a* it is clear that e_1 and f_1 define an origin (translation of axes); *Figure 10b* shows that e_2 and f_3 define the size of a rectangle (aspect ratio); *Figure 10c* demonstrates that e_3 and f_2 give two rotations (*Skew* and rotation of axes); *Figures 10d* and *10e* show that e_4 and f_4 give *Two Tapers*.

If the local axes of the quadrilateral are now defined as shown in *Figure 11*, then

$$e_1=f_1=f_2=0 \tag{9}$$

and the functions defining the quadrilateral become:

$$\begin{aligned} x&=e_2\xi+e_3\eta+e_4\xi\eta\\ y&=f_3\eta+f_4\xi\eta\end{aligned} \tag{10}$$

The meaning of the coefficients in (10) is now shown in *Figure 12* and clearly identifies the required shape parameters as:

$$\text{aspect ratio}=\frac{e_2}{f_3} \quad \text{or} \quad \frac{f_3}{e_2}\,\text{(largest)} \tag{11}$$

$$\text{skew}=\frac{e_3}{f_3} \tag{12}$$

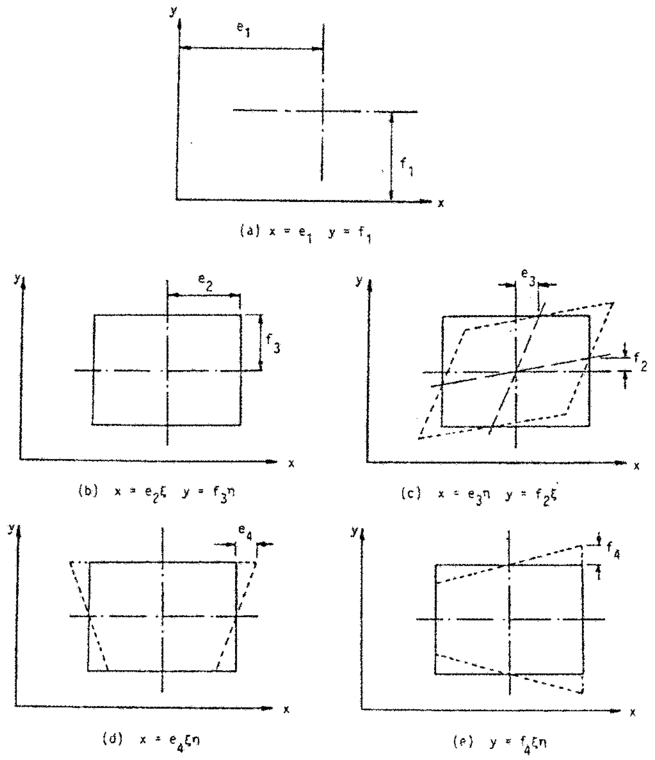


Figure 10 Physical meaning of the *e* and *f* coefficients

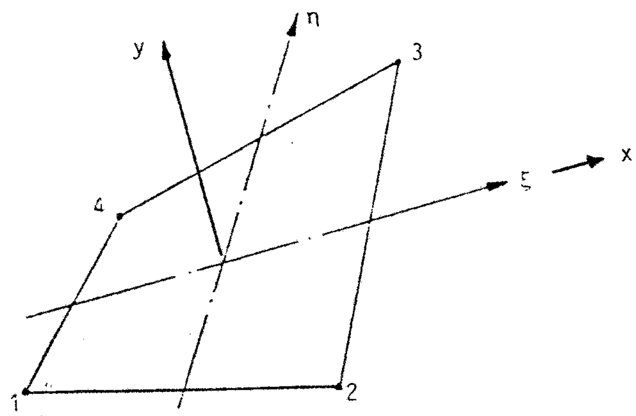


Figure 11 Recommended local axes

$$\text{taper in the } x\text{-direction } (T_x) = \frac{f_4}{f_3} \tag{13}$$

$$\text{taper in the } y\text{-direction } (T_y) = \frac{e_4}{e_2} \tag{14}$$

It should be noted that a quadrilateral has two sets of shape parameters depending on which of the oblique axes is taken as the local x-axis, although the skew parameter is the same in each case. When calculating the shape parameters the absolute values should perhaps be given.

Jacobian Matrix and the Shape Parameters

The Jacobian matrix[27] for a flat (projected) quadrilateral is given by:

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{15}$$

This can be expressed in terms of the e and f coefficients through (7). Hence

$$[J] = \begin{bmatrix} (e_2 + e_4\eta) & (f_2 + f_4\eta) \\ (e_3 + e_4\xi) & (f_3 + f_4\xi) \end{bmatrix} \tag{16}$$

If the Jacobian is first of all evaluated at $\xi = \eta = 0$ to give $[J_0]$, that is,

$$[J_0] = \begin{bmatrix} e_2 & f_2 \\ e_3 & f_3 \end{bmatrix} \tag{17}$$

this gives the values of the coefficients:

$$\begin{aligned} e_2 &= J_0(1, 1) \\ e_3 &= J_0(2, 1) \\ f_2 &= J_0(1, 2) = 0 \\ f_3 &= J_0(2, 2) \end{aligned} \tag{18}$$

For the suggested choice of local axes $f_2 = 0$, see (9).

If the Jacobian is now evaluated at $\xi = \eta = 1$ this gives

$$[J_1] = \begin{bmatrix} (e_2 + e_4) & (f_2 + f_4) \\ (e_3 + e_4) & (f_3 + f_4) \end{bmatrix} \tag{19}$$

Therefore,

$$\begin{aligned} e_4 &= J_1(1, 1) - J_0(1, 1) \\ f_4 &= J_1(1, 2) \end{aligned} \tag{20}$$

This shows how the shape parameters can be evaluated from the Jacobian matrix.

Determinant of the Jacobian and the Shape Parameters

The determinant of the Jacobian matrix of (16) can be written as:

$$\det[J] = c_1 + c_2\xi + c_3\eta \tag{21}$$

where, using the recommended local axes,

$$\begin{aligned} c_1 &= e_2 f_3 \\ c_2 &= e_2 f_4 \\ c_3 &= e_4 f_3 - e_3 f_4 \end{aligned} \tag{22}$$

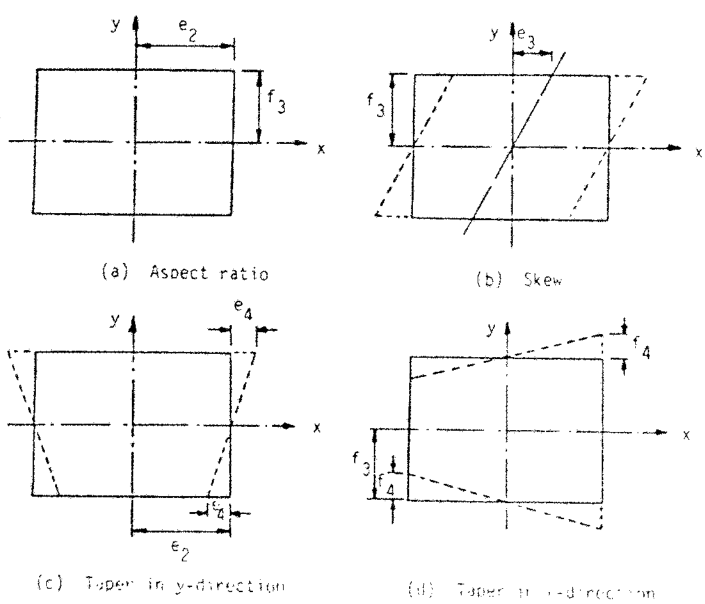


Figure 12 Shape parameters

The determinant has the dimensions of area and can now be expressed in terms of the shape parameters using the definitions of (1)–(14). Hence,

$$\det[J] = f_3^2(AR)\left(1 + T_x\xi + \left(T_y - \frac{\text{Skew}}{AR}T_x\right)\eta\right) \quad (23)$$

where AR is aspect ratio, T_x is the taper in the x-direction, T_y is the taper in the y-direction and f_3 is a half side length of the basic rectangle.

The coefficients in the determinant are therefore a function of the shape parameters. The c_1 coefficient is concerned only with the basic rectangle and will always be positive. The c_2 coefficient contains the taper in the x-direction which introduces one distortion. However, the c_3 coefficient introduces the skew and the two tapers.

FINAL COMMENT

More comprehensive developments of the CRE method and the application of it for both quadrilateral and triangular elements can be found in Robinson and Associates contractors' report to NAFEMS[19]. Reference 26 gives a more detailed and extended presentation of the work on the Jacobian shape parameters and covers element warpage and convexity.

REFERENCES

1 Robinson, J. and Haggenmacher, G. W. Basis for element interchangeability in finite element programs. *Third Conf. Matrix Methods in Struct. Mech., Wright-Patterson Air Force Base* (1971)

2 Robinson, J. *Integrated Theory of Finite Element Methods*, John Wiley, Chichester (1973)

3 Robinson, J. and Haggenmacher, G. W. Warped semi-monocoque quadrilateral membrane element for the finite element methods, *Report No. 23676*, Lockheed–California Company (1970)

4 Robinson, J. and Haggenmacher, G. W. Evaluation of calac semi-monocoque membrane element, *Report No. 24055*, Lockheed–California Company (1970)

5 Robinson, J. A single element test, *Computer Methods in Applied Mechanics and Engineering*, North-Holland, Amsterdam, Vol. 7, pp. 191–200 (1976)

6 Robinson, J. Single element test exposes commercial elements, *Proc. NASTRAN Users' Conf., Munich* (May 1976)

7 Robinson, J. and Haggenmacher, G. W. LORA—An accurate four-node stress plate bending element, *Int. J. Num. Meth. Eng.*, **14**, 296–306 (1979)

8 Robinson, J. and Blackham, S. An evaluation of lower order membranes as contained in the MSC/NASTRAN, ASAS and PAFEC Systems, *Report to RAE, Farnborough* (MOD Contract No. A93b/494) (September 1979)

9 Robinson, J. and Blackham, S. An evaluation of plate bending elements: MSC/NASTRAN, ASAS, PAFEC, ANSYS, SAP4, *Robinson and Associates Report* (1981)

10 Robinson, J. Single element test for aspect ratio sensitivity of solids, *Finite Element News*, Issue No. 1 (February) (1986)

12 Haggenmacher, G. W. A case for stress field elements, *Fourth World Cong. Finite Element Methods, Interlaken*, Robinson and Associates (1984)

13 Robinson, J. Application of Haggenmacher's membrane skew tests, *Eur. MSC/NASTRAN Users' Conf., Munich* (May 1984)

14 Robinson, J. An evaluation of skew sensitivity of thirty-three plate bending elements in nineteen FEM systems,
(i) series of *Finite Element News* articles by FEM users (1983–1986)
(i) Finite Element Standards Forum (Book 2), *26th Structures, Structural Dynamics, and Materials Conf., Orlando* (April 1985)
(iii) *Proc. Int. Semin. CAD React. Struct.* (SMiRT-8 Post Conference), Brussels (August 1985)
(iv) *Nucl. Eng. Design*, **90**, (1) (1985)

15 Robinson, J. Element evaluation—a set of assessment points and standard tests, *Second World Cong. Finite Element Methods, Bournemouth*, Robinson and Associates (1978)

16 Robinson, J. A need for recognized standards in FEM technology, *Proc. Eighth IKOSS Int. Finite Element Congr., Baden-Baden* (November 1979)

17 Robinson, J. A World FEM Association, *Third World Cong. Finite Element Methods, Beverly Hills*, Robinson and Associates (1981)

18 Mair, W. M. National Agency for Finite Element Methods and Standards, *Finite Element News*, Issue No. 3 (June), pp. 16–17 (1983)

19 Robinson, J. Basic and shape sensitivity tests for membrane and plate bending finite elements, *NAFEMS Contractors' Rep.* (January 1985)

20 Burrows, D. J. Benchmark tests in thermal stress analysis using finite element methods, *NAFEMS Contractors' Rep.* (July 1985)

21 Kamoulakos, A., Davies, G. A. O. and Hitchings, D. Benchmark tests for various finite element assemblies, *NAFEMS Contractors' Rep.* (August 1985)

22 Morris, A. J. Shell finite element evaluation tests, *NAFEM'S Contractors' Rep.* (August 1985)

23 MacNeal, R. H. and Harder, R. L. A proposed standard set of problems to test finite element accuracy, MacNeal–Schwendler Corp. (1984)

24 Finite Element Standards Forum, Books 1 and 2, *26th Structures, Structural Dynamics, and Materials Conf., Orlando* (1985)

25 Braam, H. Finite element standards group for Holland, *Finite Element News* Issue No. 1 (February), p. 36 (1985)

26 Robinson, J. A new look at element shape parameters, *Robinson and Associates Report* (April 1985); presented to the NAFEMS Steering Committee; *Finite Elements in Analysis and Design*, North-Holland, Amsterdam, in press

27 Robinson, J. *Understanding Finite Element Stress Analysis*, Robinson and Associates (1981)