

# Production of the $X(3870)$ at the $\Upsilon(4S)$ by the Coalescence of Charm Mesons

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## Abstract

If the recently-discovered charmonium state  $X(3870)$  is a loosely-bound molecule of the charm mesons  $D^0$  and  $\bar{D}^{*0}$  or  $\bar{D}^0$  and  $D^{*0}$ , it can be produced in  $e^+e^-$  annihilation at the  $\Upsilon(4S)$  resonance by the coalescence of charm mesons produced in the decays of  $B^+$  and  $B^-$  mesons. Remarkably, in the case of 2-body decays of the  $B$  mesons, the leading contribution to the coalescence probability depends only on hadron masses and on the width and branching fractions of the  $B$  meson. As the binding energy  $E_b$  of the molecule goes to zero, the coalescence probability scales as  $E_b^{1/2} \log(E_b)$ . Unfortunately, the coalescence probability is also suppressed by two powers of the ratio of the width to the mass of the  $B$  meson, and is therefore many orders of magnitude too small to be observed in current experiments at the  $B$  factories.

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The recent unexpected discovery of a narrow charmonium resonance near 3.87 GeV presents a challenge to our understanding of heavy quarkonium. The new charmonium state  $X(3870)$  was discovered by the Belle collaboration in electron-positron collisions through the  $B$ -meson decay  $B^\pm \rightarrow K^\pm X$  followed by the decay  $X \rightarrow J/\psi \pi^+ \pi^-$  [1]. Its mass was measured to be  $M_X = 3872.0 \pm 0.6 \pm 0.5$  MeV [1]. It is narrow compared to other charmonium states above the threshold for decay into  $D\bar{D}$ : the upper bound on the width is  $\Gamma_X < 2.3$  MeV. The discovery has been confirmed by the CDF collaboration who observed  $X$  through  $J/\psi \pi^+ \pi^-$  events in proton-antiproton collisions and measured its mass to be  $M_X = 3871.4 \pm 0.7 \pm 0.4$  MeV [2].

The proposed interpretations of the  $X(3870)$  include a D-wave charmonium state with quantum numbers  $J^{PC} = 2^{--}$  or  $2^{-+}$ , an excited P-wave charmonium state with  $J^{PC} = 1^{++}$  or  $1^{+-}$ , a “hybrid charmonium” state in which a gluonic mode has been excited, and a  $D^0 \bar{D}^{*0}$  or  $\bar{D}^0 D^{*0}$  molecule [3–12]. This last possibility is motivated by the fact that the  $X(3870)$  is extremely close to the threshold  $3871.2 \pm 0.7$  MeV for decay into the charmed mesons  $D^0$  and  $\bar{D}^{*0}$ .

If the  $X(3870)$  is an S-wave  $D^0 \bar{D}^{*0} / \bar{D}^0 D^{*0}$  molecule, its binding energy is smaller than any other hadron that can be interpreted as a 2-body bound state of hadrons. For two hadrons whose low-energy interactions are mediated by pion exchange, the natural low-energy scale for the binding energy of a molecule is  $m_\pi^2/(2m)$ , where  $m$  is the reduced mass of the two hadrons. The natural low-energy scale for a  $D^0 \bar{D}^{*0}$  molecule is about 10 MeV. The measurements of the mass of the  $X(3870)$  imply that its binding energy (which is positive by definition) is  $E_b = -0.5 \pm 0.9$  MeV. Thus  $E_b$  is likely to be less than 0.4 MeV, which is much smaller than the natural low-energy scale.

The tiny binding energy of the  $X(3870)$  implies that the  $D^0 \bar{D}^{*0}$  scattering length  $a$  is unnaturally large compared to the natural scale  $1/m_\pi$ . The molecule therefore has properties that depend on  $a$  but are insensitive to other details of the interactions of  $D^0$  and  $\bar{D}^{*0}$ , a phenomenon called “low-energy universality.” The universal prediction for the binding energy is

$$E_b \equiv m_D + m_{D^*} - M_X \simeq \frac{1}{2m_{DD^*}a^2}, \quad (1)$$

where  $m_{DD^*} = m_{D^0}m_{D^{*0}}/(m_{D^0} + m_{D^{*0}})$  is the reduced mass of the two constituents. The universal prediction for the momentum space wavefunction of the  $D^0 \bar{D}^{*0}$  or  $\bar{D}^0 D^{*0}$  is

$$\psi(k) \simeq \frac{(8\pi/a)^{1/2}}{\mathbf{k}^2 + 1/a^2} \quad |\mathbf{k}| \ll m_\pi, \quad (2)$$

where the normalization is  $\int d^3k/(2\pi)^3 |\psi(k)|^2 = 1$ . This wavefunction has been exploited by Voloshin to calculate the momentum distributions for the decays  $X \rightarrow D^0 \bar{D}^0 \pi^0$  and  $X \rightarrow D^0 \bar{D}^0 \gamma$  [6]. The universal prediction for the amplitude for elastic scattering of  $\bar{D}^0$  and  $D^{*0}$  in the center-of-momentum frame with total energy  $E$  is

$$\mathcal{A}[\bar{D}^0 D^{*0} \rightarrow \bar{D}^0 D^{*0}] \simeq \frac{8\pi m_D m_{D^*}}{m_{DD^*}(-1/a - i|\mathbf{k}|)} \quad |\mathbf{k}| \ll m_\pi, \quad (3)$$

where  $|\mathbf{k}| = [2m_{DD^*}(E - m_D - m_{D^*})]^{1/2}$ . Other consequences of low-energy universality have been discussed in Ref. [9]. One consequence is that as the scattering length  $a$  increases, the probabilities for components of the wavefunction other than  $D^0 \bar{D}^{*0}$  or  $\bar{D}^0 D^{*0}$  decrease

as  $1/a$ . In the limit  $a \rightarrow \infty$ , it becomes a pure molecular state. If it has charge conjugation  $\mathcal{C} = \pm$ , the state is a superposition  $(|D^0 \bar{D}^{*0}\rangle \pm |\bar{D}^0 D^{*0}\rangle)/\sqrt{2}$ .

One of the challenges for the interpretations of the  $X(3870)$  as a  $D^0 \bar{D}^{*0}/\bar{D}^0 D^{*0}$  molecule is to understand its production rate. The large  $D^0 \bar{D}^{*0}$  scattering length implies that a  $D^0$  and  $\bar{D}^{*0}$  with relative momentum small compared to  $m_\pi$  have very strong interactions. One way to produce  $X$  is therefore to produce  $\bar{D}^0$  and  $D^{*0}$  with small relative momentum which then undergo a final-state interaction that binds them to form  $X$ . An example of such a process is the decay of  $\Upsilon(4S)$  into  $B^+ B^-$ , followed by decays of the  $B^+$  and  $B^-$  into states containing  $\bar{D}^0$  and  $D^{*0}$ , respectively. There is a small probability that the  $\bar{D}^0$  and  $D^{*0}$  will be produced with sufficiently small relative momentum for them to coalesce into  $X$ . In this paper, we calculate the leading contribution to the coalescence probability in the case of 2-body decays of the  $B^+$  and  $B^-$ . We show that the coalescence probability scales as  $E_b^{1/2} \log E_b$  as the binding energy of  $X$  goes to 0. Remarkably, the coefficient of  $E_b^{1/2} \log E_b$  depends only on hadron masses and on the width and branching fractions of the  $B$  meson.

We consider the decay  $\Upsilon(4S) \rightarrow X h h'$ , where  $h$  and  $h'$  are light hadrons. This process can proceed via the decay  $\Upsilon(4S) \rightarrow B^+ B^-$ , followed by the 2-body decays  $B^+ \rightarrow \bar{D}^0 h$  and  $B^- \rightarrow D^{*0} h'$ , followed by the coalescence  $\bar{D}^0 D^{*0} \rightarrow X$ . This process can also proceed through a second pathway consisting of the 2-body decays  $B^+ \rightarrow \bar{D}^{*0} h$  and  $B^- \rightarrow D^0 h'$  followed by the coalescence  $\bar{D}^{*0} D^0 \rightarrow X$ . In principle, these two pathways can interfere. However, we shall see that the momentum configurations of  $X$ ,  $h$  and  $h'$  are completely determined by the masses of the hadrons and they are different for the two pathways. Thus there is no interference between the amplitudes.

The decay process  $\Upsilon(4S) \rightarrow X h h'$  is complicated because there are many relevant energy and momentum scales and they range over many orders of magnitude. The mass  $M_\Upsilon$  of the  $\Upsilon(4S)$  is larger than the binding energy  $E_b$  of  $X$  by more than 4 orders of magnitude and the width  $\Gamma_B$  of the  $B$  meson is smaller by about 10 orders of magnitude. We expect the rate for this decay to go to 0 in the limit  $E_b \rightarrow 0$  (with  $E_b \gg \Gamma_B$ ), because the  $\bar{D}^0$  and  $D^{*0}$  must have relative momentum  $k$  of order  $(m_{D^*} E_b)^{1/2} \approx 1/a$  in order to bind to form  $X$  and such small relative momentum accounts for a decreasing fraction of the total phase space available to the  $\bar{D}^0$  and  $D^{*0}$ . Our calculation shows that there are contributions to the rate that scale as  $E_b^{1/2}$ . They include contributions from relative momentum  $k$  ranging from the scale  $1/a$  to the scale  $m_\pi$ . The contributions from  $k \sim 1/a$  are constrained by low-energy universality, and we expect these to be calculable in terms of the scattering length. The contributions from  $k \sim m_\pi$  necessarily involve the full complications of low-energy QCD. Fortunately we find that there is a logarithmic contribution coming from the range  $1/a \ll k \ll m_\pi$ , which is also governed by low-energy universality. This logarithmic term dominates in the limit  $E_b \rightarrow 0$ . The logarithmic term in the decay rate for  $\Upsilon(4S) \rightarrow X h h'$  is calculated in Appendix A. The branching ratio that measures the coalescence probability

for  $\bar{D}^0 D^{*0} \rightarrow X$  or  $\bar{D}^{*0} D^0 \rightarrow X$  is

$$\begin{aligned}
& \frac{\Gamma[\Upsilon(4S) \rightarrow Xhh']}{\Gamma[\Upsilon(4S) \rightarrow \bar{D}^0 D^{*0} hh'] + \Gamma[\Upsilon(4S) \rightarrow \bar{D}^{*0} D^0 hh']} \\
&= \frac{2\pi M_X m_B^8}{m_{DD^*} M_\Upsilon (M_\Upsilon^2 - 4m_B^2)^{1/2}} \left( \frac{2E_b}{m_{DD^*}} \right)^{1/2} \log \left( \frac{m_\pi^2}{2m_{DD^*} E_b} \right) \left( \frac{\Gamma_B}{m_B} \right)^2 \\
&\times \sum \mathcal{B}[B^+ \rightarrow \bar{D}^0 h] \mathcal{B}[B^- \rightarrow D^{*0} h'] \frac{J(M_\Upsilon, m_B, M_X, m_D, m_{D^*}, m_h, m_{h'})}{\lambda^{1/2}(m_B, m_D, m_h) \lambda^{1/2}(m_B, m_{D^*}, m_{h'})} \\
&\times \left( \sum \mathcal{B}[B^+ \rightarrow \bar{D}^0 h] \mathcal{B}[B^- \rightarrow D^{*0} h'] \right)^{-1}, \tag{4}
\end{aligned}$$

where  $J(M_\Upsilon, \dots)$  is the function of hadron masses given in (A21) and the function  $\lambda(m_1, m_2, m_3)$  is given after (A4). The sum in the numerator and the denominator is over two terms, the one shown and a second term obtained by replacing  $\bar{D}^0$  and  $D^{*0}$  by  $\bar{D}^{*0}$  and  $D^0$ . Notice that the expression (4) depends only on hadron masses and on the width  $\Gamma_B$  and branching fractions of the  $B$ -meson. If  $h$  and  $h'$  are each others antiparticles such as  $\pi^+$  and  $\pi^-$ , the branching fractions cancel between the numerator and denominator. If we take the binding energy of  $X$  to be  $E_b = 0.1$  MeV, the branching ratios in (4) for the cases  $hh' = (\pi^+\pi^-, \rho^+\rho^-, K^+K^-, K^{*+}K^{*-})$  are  $(1.1, 1.3, 1.2, 1.3) \times 10^{-24}$ . For any other combination of  $h = (\pi^+, \rho^+, K^+, K^{*+})$  and  $h' = (\pi^-, \rho^-, K^-, K^{*-})$ , the branching ratio in (4) depends on  $B^+$  branching fractions but it is in the range from 1.2 to  $1.4 \times 10^{-24}$ .

We can get a simple expression that can be used to estimate the order of magnitude of the branching ratio by neglecting the light hadron masses  $m_h$  and  $m'_h$ , and making the approximations  $m_{D^*} - m_D \ll M_X$  and  $M_X \ll m_B$ . In this limit, the function  $J(p_Q, p'_Q)$  given by (A21) approaches

$$J(p_Q, p'_Q) \longrightarrow \frac{\pi M_X}{m_B^2 (M_\Upsilon^2 - 4m_B^2)^{1/2}}. \tag{5}$$

The branching ratio in (4) then reduces to

$$\begin{aligned}
& \frac{\Gamma[\Upsilon(4S) \rightarrow Xhh']}{\Gamma[\Upsilon(4S) \rightarrow \bar{D}^0 D^{*0} hh'] + \Gamma[\Upsilon(4S) \rightarrow \bar{D}^{*0} D^0 hh']} \\
&\longrightarrow \frac{8\pi^2 m_B^2 M_X}{M_\Upsilon (M_\Upsilon^2 - 4m_B^2)} \left( \frac{8E_b}{M_X} \right)^{1/2} \log \left( \frac{2m_\pi^2}{M_X E_b} \right) \left( \frac{\Gamma_B}{m_B} \right)^2. \tag{6}
\end{aligned}$$

If we take the binding energy to be 0.1 MeV, this estimate for the branching ratio is  $6.3 \times 10^{-25}$ , which is within a factor of 2 of the more accurate results calculated using (4). This estimate applies equally well if  $h$  or  $h'$  is replaced by a multiparticle state of light hadrons or a lepton pair whose invariant mass is small compared to  $m_B$ . We conclude that the inclusive branching fraction for  $\Upsilon(4S) \rightarrow X(3870)$  via this coalescence mechanism is about 24 orders of magnitude smaller than the product of the inclusive branching fractions for  $B^+ \rightarrow \bar{D}^0$  and  $B^+ \rightarrow \bar{D}^{*0}$ .

We have calculated the leading contribution to the probability for charm mesons produced by the decay of  $\Upsilon(4S)$  to coalesce into  $X(3870)$ . Remarkably, this coalescence probability can be expressed completely in terms of hadron masses and the width and branching fractions of the  $B$  meson. Unfortunately there is a suppression factor of  $(\Gamma_B/m_B)^2$  that makes the

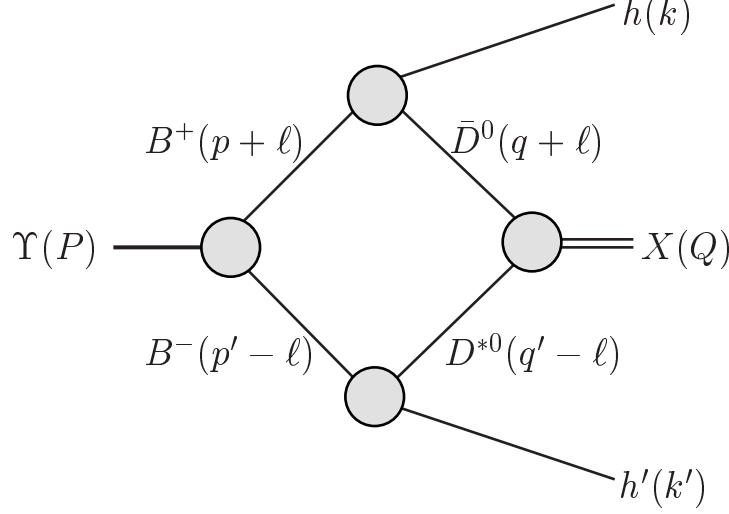


FIG. 1: Feynman diagram for the amplitude of  $\Upsilon(4S) \rightarrow Xhh'$  via the first pathway.

rate for  $\Upsilon(4S) \rightarrow Xhh'$  many orders of magnitude too small to be observed at the current  $B$  factories.

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## APPENDIX A: CALCULATION OF THE RATE FOR $\Upsilon(4S) \rightarrow Xhh'$

In this appendix, we calculate the rate for the decay  $\Upsilon(4S) \rightarrow X(3870) + h + h'$ , where  $h$  and  $h'$  are light hadrons. This decay proceeds through two pathways: the decay  $\Upsilon(4S) \rightarrow B^+B^-$  followed by the 2-body decays  $B^+ \rightarrow \bar{D}^0h$  and  $B^- \rightarrow D^{*0}h'$  followed by the coalescence process  $\bar{D}^0D^{*0} \rightarrow X$ , and the pathway obtained by replacing  $\bar{D}^0$  and  $D^{*0}$  by  $\bar{D}^{*0}$  and  $D^0$ . As we shall see, the two pathways do not interfere. The amplitude for the first pathway can be represented by the 1-loop Feynman diagram with meson lines shown in Fig. 1. We denote the  $\Upsilon(4S)$  simply by  $\Upsilon$ . The momenta of  $\Upsilon$ ,  $X$ ,  $h$  and  $h'$  are  $P$ ,  $Q$ ,  $k$  and  $k'$ , respectively. The momenta of the virtual  $B^+$ ,  $B^-$ ,  $\bar{D}^0$  and  $D^{*0}$  mesons are  $p + \ell$ ,  $p' - \ell$ ,  $q + \ell$  and  $q' - \ell$ , respectively, where  $\ell$  is the loop momentum. Momentum conservation requires  $P = p + p'$ ,  $p = k + q$ ,  $p' = k' + q'$ , and  $q + q' = Q$ . We can choose  $q$  and  $q'$  to be

$$\begin{aligned} q^\mu &= \frac{m_D}{M_X} Q^\mu, \\ q'^\mu &= \frac{m_{D^*}}{M_X} Q^\mu, \end{aligned} \tag{A1}$$

so that they are on the mass shells of the  $\bar{D}^0$  and  $D^{*0}$ :  $q^2 = m_D^2$  and  $q'^2 = m_{D^*}^2$ . Since the binding energy of  $X$  is so tiny, the momenta (A1) are consistent with the constraint  $q + q' = Q$ .

The decay rate can be written as

$$\begin{aligned} \Gamma[\Upsilon \rightarrow Xhh'] &= \frac{1}{2M_\Upsilon} \int \frac{d^3k}{(2\pi)^3 2k_0} \frac{d^3k'}{(2\pi)^3 2k'_0} \frac{d^3Q}{(2\pi)^3 2Q_0} \\ &\quad \times (2\pi)^4 \delta^{(4)}(P - k - k' - Q) |\mathcal{A}[\Upsilon \rightarrow Xhh']|^2. \end{aligned} \tag{A2}$$

The amplitude for the decay through the first pathway is

$$\begin{aligned} \mathcal{A}_1[\Upsilon \rightarrow X h h'] &= \int \frac{d^4 \ell}{(2\pi)^4} \mathcal{A}[\Upsilon \rightarrow B^+ B^-] \mathcal{A}[B^+ \rightarrow \bar{D}^0 h] \mathcal{A}[B^- \rightarrow D^{*0} h'] \mathcal{A}[\bar{D}^0 D^{*0} \rightarrow X] \\ &\times \frac{i}{(p + \ell)^2 - m_B^2 + i m_B \Gamma_B} \frac{i}{(p' - \ell)^2 - m_B^2 + i m_B \Gamma_B} \\ &\times \frac{i}{(q + \ell)^2 - m_D^2 + i\epsilon} \frac{i}{(q' - \ell)^2 - m_{D^*}^2 + i\epsilon}. \end{aligned} \quad (\text{A3})$$

The rate depends crucially on the width  $\Gamma_B$  of the  $B$  meson, so the effect of the width must be included in the propagators of the  $B^+$  and  $B^-$ .

Since the loop integral is dominated by very small momenta, we can neglect any momentum dependence of the amplitudes  $\mathcal{A}$  for  $\Upsilon \rightarrow B^+ B^-$ ,  $B^+ \rightarrow \bar{D}^0 h$  and  $B^- \rightarrow D^{*0} h'$ . They can be approximated by the amplitudes for the on-shell decays. For example, the amplitude for  $B^+ \rightarrow \bar{D}^0 h$  is determined by the branching fraction for that decay:

$$\mathcal{B}[B^+ \rightarrow \bar{D}^0 h] = \frac{1}{16\pi} |\mathcal{A}[B^+ \rightarrow \bar{D}^0 h]|^2 \frac{\lambda^{1/2}(m_B, m_D, m_h)}{m_B^3 \Gamma_B}, \quad (\text{A4})$$

where  $\lambda(x, y, z) = x^4 + y^4 + z^4 - 2(x^2 y^2 + y^2 z^2 + z^2 x^2)$ . The amplitude for the coalescence process  $\bar{D}^0 D^{*0} \rightarrow X$  can be deduced from the amplitude (3) for the scattering process  $\bar{D}^0 D^{*0} \rightarrow \bar{D}^0 D^{*0}$ . This amplitude has a pole in the total energy  $E$  at the mass  $M_X = m_D + m_{D^*} - E_b$ , where  $E_b$  is the binding energy given by (1). Its behavior near the pole is

$$\mathcal{A}[\bar{D}^0 D^{*0} \rightarrow \bar{D}^0 D^{*0}] \longrightarrow -\frac{8\pi m_D m_{D^*}}{m_{D^*}^2 a} \frac{1}{E - (m_D + m_{D^*} - E_b)}. \quad (\text{A5})$$

The residue is proportional to the square of the amplitude for  $\bar{D}^0 D^{*0} \rightarrow X$ :

$$\mathcal{A}[\bar{D}^0 D^{*0} \rightarrow X] = \left( \frac{16\pi M_X m_D m_{D^*}}{m_{D^*}^2 a} \right)^{1/2}. \quad (\text{A6})$$

Our strategy is to manipulate the decay rate into a form that includes as a factor the differential decay rate for  $\Upsilon \rightarrow B^+ B^-$ . The first step is to integrate over the component  $\ell_0$  of the loop momentum. The dominant contribution to the integral over  $\ell_0$  in (A3) comes from the particle poles in the propagators for the  $\bar{D}^0$  and  $D^{*0}$  mesons:

$$\int \frac{d\ell_0}{2\pi} \frac{1}{(q + \ell)^2 - m_D^2 + i\epsilon} \frac{1}{(q' - \ell)^2 - m_{D^*}^2 + i\epsilon} = \frac{i}{4E_D E_{D^*} (E_D + E_{D^*} - Q_0)}. \quad (\text{A7})$$

In the rest frame of  $X$ , the energies are  $E_D = (m_D^2 + \ell^2)^{1/2}$ ,  $E_{D^*} = (m_{D^*}^2 + \ell^2)^{1/2}$  and  $Q_0 = M_X$ . Expanding the denominator to lowest order in  $\ell$  and  $1/a$ , (A7) becomes

$$\frac{1}{4E_D E_{D^*} (E_D + E_{D^*} - Q_0)} \simeq \frac{m_{D D^*}}{2m_D m_{D^*}} \frac{1}{\ell^2 + 1/a^2}, \quad (\text{A8})$$

which is proportional to the momentum-space wavefunction  $\psi(\ell)$  in (2).

If not for the loop momenta, the product of the  $B^+$  propagator in (A3) and its complex conjugate could be expressed as a Breit-Wigner resonance factor. If we take into account the loop momenta, that product can be approximated as

$$\begin{aligned} & \frac{i}{(p+\ell)^2 - m_B^2 + im_B\Gamma_B} \left( \frac{i}{(p+\ell')^2 - m_B^2 + im_B\Gamma_B} \right)^* \\ & \simeq \frac{1}{p \cdot (\ell - \ell') + (\ell^2 - \ell'^2)/2 + im_B\Gamma_B} \frac{im_B\Gamma_B}{(p^2 - m_B^2)^2 + (m_B\Gamma_B)^2} \\ & \simeq \frac{1}{k \cdot (\ell - \ell') + im_B\Gamma_B} i\pi\delta(p^2 - m_B^2). \end{aligned} \quad (\text{A9})$$

In the second line, we expressed the product of propagators in terms of a difference between propagators and took the limit  $\ell \rightarrow 0$  and  $\ell' \rightarrow 0$  in that difference to get a Breit-Wigner resonance factor. In the third line, we took the limit  $\Gamma_B \rightarrow 0$  in the resonance factor to get a delta function. We also used the relations  $\ell^2 = -2q \cdot \ell$  and  $\ell'^2 = -2q \cdot \ell'$ , which follow from the fact that  $q$ ,  $q + \ell$  and  $q + \ell'$  are all on the mass-shell of the  $\bar{D}^0$  meson. The inner product  $k \cdot (\ell - \ell')$  in the denominator of (A9) can be expanded in powers of the momenta  $\ell$  and  $\ell'$ . The terms  $q \cdot \ell$  and  $q \cdot \ell'$  are already second order and could be neglected. However, for the purpose of evaluating the integral over  $\ell$ , it is more convenient to use the fact the  $q \cdot \ell$  and  $q \cdot \ell'$  are second order to replace  $k^\mu$  by a vector  $k_Q^\mu$  whose  $\mu = 0$  component vanishes in the rest frame of  $X$ :

$$k_Q^\mu \equiv k^\mu - \frac{k \cdot Q}{Q^2} Q^\mu. \quad (\text{A10})$$

The expression for  $q$  in (A10) implies  $k_Q^\mu = p_Q^\mu$ . Thus our approximation for the inner product in the denominator of (A9) can be written

$$k \cdot (\ell - \ell') \simeq p_Q \cdot (\ell - \ell'). \quad (\text{A11})$$

We can integrate in the momentum  $p$  of the  $B^+$  using the identity

$$1 = \int \frac{dp^2}{2\pi} \int \frac{d^3p}{(2\pi)^3 2p_0} (2\pi)^4 \delta^{(4)}(p - k - q). \quad (\text{A12})$$

The integral over  $p^2$  can be evaluated using the delta function in (A9). After similar manipulations involving the  $B^-$  momentum, our decay rate through the first pathway can be written

$$\begin{aligned} \Gamma_1[\Upsilon \rightarrow X h h'] &= \int d\Gamma[\Upsilon \rightarrow B^+ B^-] |\mathcal{A}[B^+ \rightarrow \bar{D}^0 h]|^2 |\mathcal{A}[B^- \rightarrow D^{*0} h']|^2 \\ &\times \int (2\pi)^4 \delta^{(4)}(p - q - k) \frac{d^3k}{(2\pi)^3 2k_0} (2\pi)^4 \delta^{(4)}(p' - q' - k') \frac{d^3k'}{(2\pi)^3 2k'_0} \\ &\times \frac{\pi M_X}{m_D m_{D^* a}} \int I \frac{d^3Q}{(2\pi)^3 2Q_0}, \end{aligned} \quad (\text{A13})$$

where  $d\Gamma[\Upsilon \rightarrow B^+ B^-]$  is the differential decay rate for  $\Upsilon$  into  $B^+$  and  $B^-$  with momenta  $p$  and  $p'$ . In the rest frame of  $X$ , the factor  $I$  in (A13) is given by the integral

$$I = - \int \frac{d^3\ell}{(2\pi)^3} \frac{d^3\ell'}{(2\pi)^3} \frac{1}{-\mathbf{p}_Q \cdot (\ell - \ell') + i\epsilon} \frac{1}{\mathbf{p}'_Q \cdot (\ell - \ell') + i\epsilon} \frac{1}{\ell^2 + 1/a^2} \frac{1}{\ell'^2 + 1/a^2}. \quad (\text{A14})$$

We have replaced the terms  $im_B\Gamma_B$  in the propagators by  $i\epsilon$ , because the integral has a well-behaved limit as  $\Gamma_B \rightarrow 0$ .

In order to evaluate  $I$ , we first combine both the denominators  $\ell^2 + m^2$  and  $\ell'^2 + m^2$  where  $m = 1/a$  and the denominators  $-\mathbf{p}_Q \cdot (\ell - \ell') + i\epsilon$  and  $\mathbf{p}'_Q \cdot (\ell - \ell') + i\epsilon$  into squared denominators using Feynman parameters  $x$  and  $z$ . We then combine the squared denominators using an integral over  $y$ :

$$\begin{aligned} & \frac{1}{-\mathbf{p}_Q \cdot (\ell - \ell') + i\epsilon} \frac{1}{\mathbf{p}'_Q \cdot (\ell - \ell') + i\epsilon} \frac{1}{\ell^2 + m^2} \frac{1}{\ell'^2 + m^2} \\ &= \int_0^1 dx \frac{1}{[x\ell^2 + (1-x)\ell'^2 + m^2]^2} \int_0^1 dz \frac{1}{[\mathbf{C}(z) \cdot (\ell - \ell') + i\epsilon]^2} \\ &= 6 \int_0^1 dx \int_0^1 dz \int_0^\infty dy \frac{y}{[x\ell^2 + (1-x)\ell'^2 + m^2 + y\mathbf{C}(z) \cdot (\ell - \ell') + i\epsilon]^4}, \end{aligned} \quad (\text{A15})$$

where  $\mathbf{C}(z) = -z\mathbf{p}_Q + (1-z)\mathbf{p}'_Q$ . The integrals over the momenta  $\ell$  and  $\ell'$  can be simplified by first shifting them and then rescaling them by factors of  $x$  and  $1-x$ :

$$\begin{aligned} & \int \frac{d^3\ell}{(2\pi)^3} \frac{d^3\ell'}{(2\pi)^3} \frac{1}{[x\ell^2 + (1-x)\ell'^2 + m^2 + y\mathbf{C} \cdot (\ell - \ell') + i\epsilon]^4} \\ &= \int \frac{d^3\ell}{(2\pi)^3} \frac{d^3\ell'}{(2\pi)^3} \frac{1}{[x\ell^2 + (1-x)\ell'^2 + m^2 - y^2\mathbf{C}^2/(4x(1-x)) + i\epsilon]} \\ &= x^{-3/2}(1-x)^{-3/2} \int \frac{d^3\ell}{(2\pi)^3} \frac{d^3\ell'}{(2\pi)^3} \frac{1}{[\ell^2 + \ell'^2 + m^2 - y^2\mathbf{C}^2/(4x(1-x)) + i\epsilon]^4}. \end{aligned} \quad (\text{A16})$$

The integrals over  $y$  and then  $x$  can now be evaluated analytically. The resulting expression for  $I$  is

$$I = 4\pi \int_0^1 dz \frac{1}{\mathbf{C}(z)^2} \int \frac{d^3\ell}{(2\pi)^3} \frac{d^3\ell'}{(2\pi)^3} \frac{1}{[\ell^2 + \ell'^2 + 1/a^2]^3}. \quad (\text{A17})$$

The integral over  $\ell$  and  $\ell'$  has a logarithmic ultraviolet divergence. It can be regularized by subtracting the integral with  $m = 1/a$  replaced by an ultraviolet cutoff  $\Lambda \gg m$ :

$$\int \frac{d^3\ell}{(2\pi)^3} \frac{d^3\ell'}{(2\pi)^3} \frac{1}{[\ell^2 + \ell'^2 + m^2]^3} - (m \rightarrow \Lambda) = \frac{1}{64\pi^3} \log \frac{\Lambda}{m}. \quad (\text{A18})$$

The appropriate choice for the cutoff is  $\Lambda = m_\pi$ , since the region of validity of the expression (2) for the wavefunction of  $X$  is  $|\mathbf{k}| \ll m_\pi$ . The integral over  $z$  in (A17) can be expressed in a manifestly Lorentz-invariant form. Our final expression for the integral  $I$  in (A14) is

$$I = \frac{1}{16\pi^2} \log(am_\pi) J(p_Q, p'_Q), \quad (\text{A19})$$

where  $J(p_Q, p'_Q)$  is a Lorentz-invariant function of  $p_Q$  and  $p'_Q$  defined by the integral

$$J(p_Q, p'_Q) = \int_0^1 dz \frac{-1}{[zp_Q - (1-z)p'_Q]^2}. \quad (\text{A20})$$



The integral can be evaluated analytically:

$$J(p_Q, p'_Q) = \frac{1}{D(p_Q, p'_Q)} \left( \arctan \frac{-P_Q \cdot p_Q}{D(p_Q, p'_Q)} + \arctan \frac{-P_Q \cdot p'_Q}{D(p_Q, p'_Q)} \right), \quad (\text{A21})$$

where  $p_Q$ ,  $p'_Q$  and  $P_Q$  are all defined by (A10) and  $D(p_Q, p'_Q) = [p_Q^2 p'^2_Q - (p_Q \cdot p'_Q)^2]^{1/2}$ . In the rest frame of  $X$  where  $p_Q$  and  $p'_Q$  are spacelike,  $D(p_Q, p'_Q) = |\mathbf{p}_Q \times \mathbf{p}'_Q|$ .

It remains only to evaluate the integrals over  $k$ ,  $k'$  and  $Q$  in (A13). The integral over  $k$  can be evaluated by using the identity

$$\int \frac{d^3 k}{(2\pi)^3 2k_0} (2\pi)^4 \delta^{(4)}(p - q - k) = 2\pi \delta((p - q)^2 - m_h^2), \quad (\text{A22})$$

and similarly for  $k'$ . The two remaining delta functions can then be used to evaluate the integral over  $Q$ :

$$\int \frac{d^3 Q}{(2\pi)^3 2Q_0} 2\pi \delta((p - q)^2 - m_h^2) 2\pi \delta((p' - q')^2 - m_h'^2) = \frac{M_X^2}{4m_D m_{D^*} M_Y (M_Y^2 - 4m_B^2)^{1/2}}. \quad (\text{A23})$$

Our final expression for the decay rate through the first pathway is

$$\begin{aligned} \Gamma_1[\Upsilon \rightarrow X h h'] &= \Gamma[\Upsilon \rightarrow B^+ B^-] |\mathcal{A}[B^+ \rightarrow \bar{D}^0 h]|^2 |\mathcal{A}[B^- \rightarrow D^{*0} h']|^2 \\ &\times \frac{M_X J(p_Q, p'_Q)}{64\pi m_{DD^*} M_Y (M_Y^2 - 4m_B^2)^{1/2}} \frac{\log(am_\pi)}{am_{DD^*}}. \end{aligned} \quad (\text{A24})$$

The function  $J(p_Q, p'_Q)$  given explicitly in (A21) is a Lorentz-invariant scalar function of the momenta  $p$ ,  $p'$  and  $Q$ . The Lorentz invariants are  $p^2 = p'^2 = m_B^2$ ,  $Q^2 = M_X^2$  and

$$\begin{aligned} p \cdot Q &= \frac{M_X}{2m_D} (m_B^2 + m_D^2 - m_h^2), \\ p' \cdot Q &= \frac{M_X}{2m_{D^*}} (m_B^2 + m_{D^*}^2 - m_{h'}^2), \\ p \cdot p' &= \frac{1}{2} (M_Y^2 - 2m_B^2). \end{aligned}$$

The decay rate through the second pathway is obtained by replacing  $\bar{D}^0$  and  $D^{*0}$  by  $\bar{D}^{*0}$  and  $D^0$ . The inner products between the momenta  $k$ ,  $k'$  and  $Q$  of the final-state particles are completely determined by the hadron masses. Since these inner products are different for the two pathways, they produce distinct momentum configurations and they therefore cannot interfere.

The rate for the decay  $\Upsilon \rightarrow B^+ B^-$  followed by the decays  $B^+ \rightarrow \bar{D}^0 h$  and  $B^- \rightarrow D^{*0} h'$  is

$$\Gamma_1[\Upsilon \rightarrow \bar{D}^0 D^{*0} h h'] = \Gamma[\Upsilon \rightarrow B^+ B^-] \mathcal{B}[B^+ \rightarrow \bar{D}^0 h] \mathcal{B}[B^- \rightarrow D^{*0} h']. \quad (\text{A25})$$

The expression for the branching fraction  $\mathcal{B}[B^+ \rightarrow \bar{D}^0 h]$  is given in (A4). The amplitude  $\mathcal{A}[B^+ \rightarrow \bar{D}^0 h]$  in (A24) can be eliminated in favor of  $\mathcal{B}[B^+ \rightarrow \bar{D}^0 h]$ .

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