Decay $X(3872) \to \pi^0\pi^+\pi^-$ and S-wave $D^0\bar{D}^0 \to \pi^+\pi^-$ scattering length

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The isospin-breaking decay $X(3872) \to (D^*\bar{D} + \bar{D}^*D) \to \pi^0 D\bar{D} \to \pi^0\pi^+\pi^-$ is discussed. In its amplitude there is a triangle logarithmic singularity, due to which the dominant contribution to $BR(X(3872) \to \pi^0\pi^+\pi^-)$ comes from the production of the $\pi^+\pi^-$ system in a narrow interval of the invariant mass $m_{\pi^+\pi^-}$ near the value of $2m_{D^0} \approx 3.73$ GeV. The analysis shows that $BR(X(3872) \to \pi^0\pi^+\pi^-)$ can be expected at the level of 10^{-3}–10^{-4}. This estimate includes, in particular, the assumption that the S-wave inelastic scattering length $|\alpha''_{D^0\bar{D}^0\to\pi^+\pi^-}| \approx 1/(2m_{D^{*+}}) \approx 0.25$ GeV^{-1}.

I. INTRODUCTION

The state $X(3872)$ [or $\chi_{c1}(3872)$ [1]] was first observed in 2003 by the Belle Collaboration in the process $B \to K(X(3872) \to \pi^+\pi^- J/\psi)$ [2]. Then it was observed in many other experiments in other processes and decay channels [1]. The $X(3872)$ is a narrow resonance in non-$(D^{*0}\bar{D}^0 + \bar{D}^{*0}D^0)$ decay channels, $\Gamma_X < 1.2$ MeV [3], and its mass coincides practically with the $D^{*0}\bar{D}^0$ threshold [1]. It has the quantum numbers $I^G(J^{PC}) = 0^+(1^{++})$ [1, 4, 5]. In addition to decay into $\pi^+\pi^- J/\psi$ [2, 6, 7], the $X(3872)$ also decays into $\omega J/\psi$ [8–11], $D^{*0}\bar{D}^0 + c.c.$ [12, 13], $\gamma J/\psi$ [14–16], $\gamma\psi(2S)$ [14–16], and $\pi^0\chi_{c1}(1P)$ [17, 18]. The nature of $X(3872)$ remains the subject of much discussion; see, for example, Refs. [14–33]. Of course, new experiments will allow making a more definite choice between different interpretations.

The search for $X(3872)$ in decay channels that do not contain charmed particles or charmonium states [i.e., in channels other than $D^{*0}\bar{D}^0 + c.c.$, $D^0\bar{D}^0\pi^0$, $\pi^+\pi^- J/\psi$, $\omega J/\psi$, $\gamma J/\psi$, $\gamma\psi(2S)$, $\pi^+\pi^-\eta_c(1S)$, $\pi^+\pi^-\chi_{c1}(1P)$, and $\pi^0\chi_{c1}(1P)$] is of great interest [1, 25–35]. For example, the $c\bar{c} = \chi_{c1}(2P)$ scenario predicts a significant number of various two gluon decays $X(3872) \to$ (gluon + gluon) $\to$ light hadrons [26–30]. The situation here is qualitatively the same as for the decays $\chi_{c1}(1P) \to$ (gluon + gluon) $\to$ light hadrons. In this way, only one channel has been explored so far [1]. Namely, the LHCb Collaboration undertook a search for the decay $X(3872) \to p\bar{p}$, which resulted in the following restriction [34]:

$$\frac{BR(B^+ \to X(3872)K^+) \times (BR(X(3872) \to p\bar{p})}{BR(B^+ \to J/\psi K^+) \times (BR(J/\psi \to p\bar{p})} < 0.25 \times 10^{-2}. \quad (1)$$

Hence, in view of $BR(B^+ \to J/\psi K^+) \times (BR(J/\psi \to p\bar{p}) \approx 2.14 \times 10^{-6}$ [1] and $0.9 \times 10^{-4} < BR(B^+ \to X(3872)K^+) < 2.7 \times 10^{-4}$ [1, 36], it follows that

$$BR(X(3872) \to p\bar{p}) < 0.6 \times 10^{-4}. \quad (2)$$

Taking into account a sizable contribution of the $D^{*0}\bar{D}^0 + \bar{D}^{*0}D^0$ channel (and also the channels containing the charmonium states) to the $X(3872)$ decay rate, one can conclude that the above relation is in satisfactory agreement (at least not in contradiction) with what is observed in the decays of the $\chi_{c1}(1P)$ meson: $BR(\chi_{c1}(1P) \to p\bar{p}) = (7.60 \pm 0.34) \times 10^{-5}$ [1]. Note that the $\chi_{c1}(1P)$ has only one decay into $\gamma J/\psi$ containing $c\bar{c}$ quarks in the final state. It is also proposed to investigate the $X(3872)$ coupling to the $p\bar{p}$ channel in the reaction $p\bar{p} \to X(3872) \to \pi^+\pi^- J/\psi$ with the PANDA detector [35].

We propose to obtain an experimental limit on the probability of the decay $X(3872) \to \pi^0\pi^+\pi^-$ and, if lucky, to register this decay. According to our estimate, the branching ratio of the decay $X(3872) \to \pi^0\pi^+\pi^-$ can be expected at the level of 10^{-3}–10^{-4} due to the transition mechanism $X(3872) \to (D^*\bar{D} + \bar{D}^*D) \to \pi^0 D\bar{D} \to \pi^0\pi^+\pi^-$. In this case, the main contribution to $BR(X(3872) \to \pi^0\pi^+\pi^-)$ comes from the production of $\pi^+\pi^-$ pairs in a narrow interval of the invariant mass $m_{\pi^+\pi^-}$ near the value of $2m_{D^0} \approx 3.73$ GeV.

As for the nature of X(3872), our calculations implicitly imply for this state the conventional $c\bar{c}$ nature, i.e., that it is a compact charmonium state similar to the states $\chi_{c1}(1P)$, $\psi(2S)$, $\psi(3770)$, and so on, and to describe its decays one can use the effective phenomenological Lagrangian approach [25–30].

II. ESTIMATE OF $BR(X(3872) \to \pi^0\pi^+\pi^-)$

The decay $X(3872) \to (D^{*0}\bar{D}^0 + \bar{D}^{*0}D^0) \to \pi^0 D^0\bar{D}^0$ (see Fig. 1) is one of the main decay channels of the $X(3872)$ resonance [1]. Because of the final state interaction among D^0 and $\bar{D}^0$ mesons, i.e., due to the S-wave transition $D^0\bar{D}^0 \to \pi^+\pi^-$, the isospin breaking decay $X(3872) \to (D^{*0}\bar{D}^0 + \bar{D}^{*0}D^0) \to \pi^0 D^0\bar{D}^0 \to \pi^0\pi^+\pi^-$ is induced (see Fig. 2).

The amplitudes of such triangle diagrams, as in Fig. 2, may contain logarithmic singularities that can produce some enhancement in the mass spectra. The conditions for the appearance of such singularities in the physical region of the reaction were repeatedly deduced in various forms and discussed in the literature; see, for example, Refs. [37–45] and also the very recent work [46]. For the considered mechanism of the $X \to \pi^0\pi^+\pi^-$ decay, these conditions are reduced to the following relations.

If the virtual invariant mass squared of the $X(3872)$ resonance s_1 falls in the range

$$2(m^2_{D^{*0}} + m^2_{D^0}) - m^2_{\pi^0} = (3.87193\ \text{GeV})^2 > s_1$$

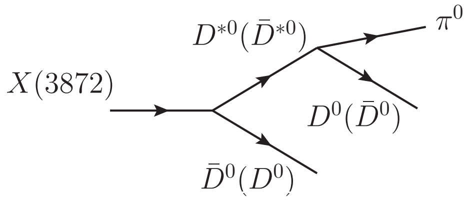

Figure 1: The diagram of the decay $X(3872) \to D^0\bar{D}^0\pi^0$. The four-momenta of $X(3872)$, D^0, $\bar{D}^0$, and π^0 are, respectively, p_1, p_D, $p_{\bar{D}}$, and p_π; the four-momenta of the intermediate D^{*0} and $\bar{D}^{*0}$ are k_1 and k_2, respectively.

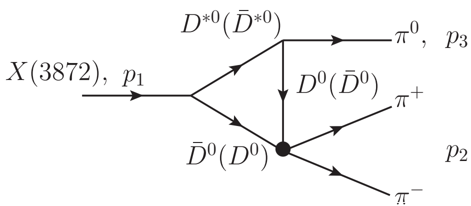

Figure 2: The diagram of the decay $X(3872) \to (D^{*0}\bar{D}^0 + \bar{D}^{*0}D^0) \to \pi^0 D^0\bar{D}^0 \to \pi^0\pi^+\pi^-$. In the $X(3872)$ mass region, all intermediate particles in the triangle loop can be near or directly on the mass shell. As a consequence, a logarithmic singularity in the imaginary part of the amplitude emerges in the hypothetical case of the stable D^{*0} meson when the conditions (3) and (4) are fulfilled. The four-momenta of corresponding particles are denoted as p_1, p_2, and p_3; $p_1^2 = s_1$ is the squared invariant mass of the $X(3872)$ resonance or of the final $\pi^0\pi^+\pi^-$ system; $p_2^2 = s_2 = m^2_{\pi^+\pi^-}$ is the squared invariant mass of the final $\pi^+\pi^-$ system; and $p_3^2 = m^2_{\pi^0}$.

$$> (m_{D^{*0}} + m_{D^0})^2 = (3.87168\ \text{GeV})^2, \qquad (3)$$

then, in the range of the invariant mass squared of the $\pi^+\pi^-$ system $s_2 = m^2_{\pi^+\pi^-}$

$$\frac{m_{D^0}}{m_{D^{*0}}}(m^2_{D^{*0}} + m^2_{D^0} - m^2_{\pi^0}) + 2m^2_{D^0} = (3.7299\ \text{GeV})^2$$
$$> s_2 > 4m^2_{D^0} = (3.72966\ \text{GeV})^2, \qquad (4)$$

the imaginary part of the amplitude of the diagram in Fig. 2 contains the triangle logarithmic singularity [37–46]. Below, we see that this singularity leads to the resonancelike enhancement in the $\pi^+\pi^-$ mass spectrum at $\sqrt{s_2} = m_{\pi^+\pi^-} \approx 2m_{D^0} \approx 3.73$ GeV, i.e., near the $D^0\bar{D}^0$ threshold.

The decay $X(3872) \to \pi^0\pi^+\pi^-$ can also be produced via the charged intermediate states, $X(3872) \to (D^{*+}D^- + D^{*-}D^+) \to \pi^0 D^+D^- \to \pi^0\pi^+\pi^-$ (see Fig. 3). From the isotopic symmetry for the coupling constants (C invariance of the amplitudes is implied), it follows that the contributions of the diagrams in Figs. 2 and 3 exactly compensate each other and the isospin breaking decay $X(3872) \to \pi^0\pi^+\pi^-$ is absent, if $m_{D^{*+}} = m_{D^{*0}}$ and $m_{D^+} = m_{D^0}$. However, the $D^{*0}\bar{D}^0$ and

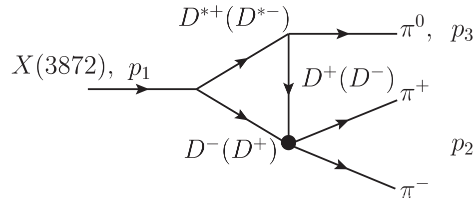

Figure 3: The diagram of the decay $X(3872) \to \pi^0\pi^+\pi^-$ corresponding to the charged intermediate state contributions, $X(3872) \to (D^{*+}D^- + D^{*-}D^+) \to \pi^0 D^+D^- \to \pi^0\pi^+\pi^-$.

$D^{*+}D^-$ thresholds in the variable $\sqrt{s_1}$ differ by 8.23 MeV ($m_{D^{*0}} + m_{\bar{D}^0} = 3.87168$ GeV, $m_{D^{*+}} + m_{D^-} = 3.87991$ GeV) and the $D^0\bar{D}^0$ and D^+D^- thresholds in the variable $\sqrt{s_2}$ differ by 9.644 MeV ($2m_{D^0} = 3.72966$ GeV, $2m_{D^\pm} = 3.73930$ GeV). Therefore, in the region of the variables $\sqrt{s_1}$ and $\sqrt{s_2}$ that is significant for the decay $X(3872) \to \pi^0\pi^+\pi^-$ (i.e., for $\sqrt{s_1} \approx m_X \approx m_{D^{*0}} + m_{\bar{D}^0}$, where m_X is the nominal mass of the $X(3872)$ equal to 3.87169 GeV [1], and $\sqrt{s_2} \approx 2m_{\bar{D}^0} \approx 3.73$ GeV), the contributions from the neutral (see Fig. 2) and charged (see Fig. 3) intermediate states weakly compensate each other and the contribution of the diagram in Fig. 2 dominates.

We write the differential probability for the decay of the virtual state $X(3872)$ to $\pi^0\pi^+\pi^-$ in the form

$$\frac{d^2BR(X \to \pi^0\pi^+\pi^-; s_1, s_2)}{d\sqrt{s_1}d\sqrt{s_2}}$$
$$= \frac{2\sqrt{s_1}}{\pi}\frac{\sqrt{s_1}}{|D_X(s_1)|^2}\frac{d\Gamma(X \to \pi^0\pi^+\pi^-; s_1, s_2)}{d\sqrt{s_2}}, \qquad (5)$$

where $D_X(s_1)$ is the inverse propagator of the $X(3872)$ resonance [25, 27, 28] that takes into account the couplings of $X(3872)$ with the $D^*\bar{D} + \bar{D}^*D$ decay channels as well as with all non-$(D^*\bar{D} + \bar{D}^*D)$ decay channels; and $d\Gamma(X \to \pi^0\pi^+\pi^-; s_1, s_2)/d\sqrt{s_2}$ is the $X \to \pi^0\pi^+\pi^-$ differential decay width in the variable $\sqrt{s_2} = m_{\pi^+\pi^-}$ caused by the sum of the diagrams in Figs. 2 and 3.

The $X(3872)$ resonance propagator constructed in Refs. [25, 27, 28] has good analytical and unitary properties. The inverse propagator $D_X(s_1)$ has the form [25, 27, 28]

$$D_X(s_1) = m_X^2 - s_1$$
$$+ \sum_{ab}[\text{Re}\Pi_X^{ab}(m_X^2) - \Pi_X^{ab}(s_1)] - im_X\Gamma_{non}, \qquad (6)$$

where $\Gamma_{non} = \Sigma_i\Gamma_i$ is the total width of the $X(3872)$ decay to all non-$(D^*\bar{D} + \bar{D}^*D)$ channels which in the narrow region of the $X(3872)$ peak ($\Gamma_X < 1.2$ MeV [1, 3]) is approximated by a constant; $ab = D^{*0}\bar{D}^0$, $\bar{D}^{*0}D^0$, $D^{*+}D^-$, $D^{*-}D^+$. At $s_1 > (m_a + m_b)^2$

$$\Pi_X^{ab}(s_1) = \frac{g_A^2}{16\pi}\left[\frac{m_{ab}^{(+)}m_{ab}^{(-)}}{\pi s_1}\ln\frac{m_b}{m_a} + \rho_{ab}(s_1)\right.$$

$$\times\left(i-\frac{1}{\pi}\ln\frac{\sqrt{s_1-m_{ab}^{(-)\,2}}+\sqrt{s_1-m_{ab}^{(+)\,2}}}{\sqrt{s_1-m_{ab}^{(-)\,2}}-\sqrt{s_1-m_{ab}^{(+)\,2}}}\right)\Bigg], \quad (7)$$

where $\rho_{ab}(s_1) = \sqrt{s_1-m_{ab}^{(+)\,2}}\sqrt{s_1-m_{ab}^{(-)\,2}}/s$, $m_{ab}^{(\pm)} = m_a \pm m_b$, $m_a > m_b$,

$$\mathrm{Im}\,\Pi_X^{ab}(s_1) = \sqrt{s_1}\Gamma_{X\to ab}(s_1) = \frac{g_A^2}{16\pi}\rho_{ab}(s_1), \quad (8)$$

and g_A is the coupling constant of X with the $D^{*0}\bar{D}^0$ channel. At $m_{ab}^{(-)\,2} < s_1 < m_{ab}^{(+)\,2}$

$$\Pi_X^{ab}(s_1) = \frac{g_A^2}{16\pi}\left[\frac{m_{ab}^{(+)}m_{ab}^{(-)}}{\pi s_1}\ln\frac{m_b}{m_a}\right.$$
$$\left.-\rho_{ab}(s_1)\left(1-\frac{2}{\pi}\arctan\frac{\sqrt{m_{ab}^{(+)\,2}-s_1}}{\sqrt{s_1-m_{ab}^{(-)\,2}}}\right)\right], \quad (9)$$

where $\rho_{ab}(s_1)=\sqrt{m_{ab}^{(+)\,2}-s_1}\sqrt{s_1-m_{ab}^{(-)\,2}}/s_1$. If $s_1 \le m_{ab}^{(-)\,2}$, then $\rho_{ab}(s_1)=\sqrt{m_{ab}^{(+)\,2}-s_1}\sqrt{m_{ab}^{(-)\,2}-s_1}/s_1$, and

$$\Pi_X^{ab}(s_1) = \frac{g_A^2}{16\pi}\left[\frac{m_{ab}^{(+)}m_{ab}^{(-)}}{\pi s_1}\ln\frac{m_b}{m_a}\right.$$
$$\left.+\rho_{ab}(s_1)\frac{1}{\pi}\ln\frac{\sqrt{m_{ab}^{(+)\,2}-s_1}+\sqrt{m_{ab}^{(-)\,2}-s_1}}{\sqrt{m_{ab}^{(+)\,2}-s_1}-\sqrt{m_{ab}^{(-)\,2}-s_1}}\right]. \quad (10)$$

The sum of the probabilities of the $X(3872)$ decay to all modes satisfies the unitarity [25, 27, 28]

$$BR(X\to(D^{*0}\bar{D}^0+c.c.))$$
$$+BR(X\to(D^{*+}D^-+c.c))+\Sigma_i BR(X\to i)=1. \quad (11)$$

The coupling of the $X(3872)$ with the $D^{*0}\bar{D}^0$ system was introduced in Refs. [25–28] by means of the Lagrangian

$$L_{XD^{*0}\bar{D}^0}(x) = g_A X^\mu(D_\mu^{*0}\bar{D}^0+\bar{D}_\mu^{*0}D^0) \quad (12)$$

and the range of possible values of the coupling constant $g_A^2/(16\pi)$ was determined from the analysis of the experimental data [3, 6, 8, 9, 13, 15].

To describe the amplitudes of the $D^*\to D\pi^0$ decays, we use the expression

$$V_{D^*D\pi^0} = g_{D^*D\pi^0}(\epsilon_{D^*}, p_{\pi^0}-p_D)\,, \quad (13)$$

where ϵ_{D^*} is the polarization four-vector of the D^* meson, p_{π^0} and p_D are the four-momenta of π^0 and D, respectively; $g_{D^{*+}D^+\pi^0} = -g_{D^{*0}D^0\pi^0}$.

The effective vertex of the $X(3872) \to (D^*\bar{D}+\bar{D}^*D)\to\pi^0 D\bar{D}\to\pi^0\pi^+\pi^-$ transition corresponding to the sum of the diagrams in Figs. 2 and 3, in which the $\pi^+\pi^-$ system is produced in the S wave, can be written as

$$V_{X\pi^0\pi^+\pi^-} = G_{X\pi^0\pi^+\pi^-}(s_1,s_2)(\epsilon_X, p_3-p_2)$$
$$= 2\frac{\bar{g}}{16\pi}[F_0(s_1,s_2)-F_+(s_1,s_2)]\,, \quad (14)$$

where the invariant amplitude $G_{X\pi^0\pi^+\pi^-}(s_1,s_2)$ is used below [see, Eq. (19)] to compactly write the expression for the energy dependent differential width of the $X\to\pi^0\pi^+\pi^-$ decay; ϵ_X is the polarization four-vector of the $X(3872)$, the amplitudes $F_0(s_1,s_2)$ and $F_+(s_1,s_2)$ describe the contributions from the neutral and charged intermediate $D^*\bar{D}$ states, respectively, and

$$\bar{g}{=}g_A\, g_{D^{*0}D^0\pi^0}\, g_{D^0\bar{D}^0\pi^+\pi^-}. \quad (15)$$

We assume the S-wave amplitudes of the processes $D^0\bar{D}^0\to\pi^+\pi^-$ and $D^+\bar{D}^-\to\pi^+\pi^-$ (entering in the amplitudes of the diagrams in Figs. 2 and 3) to be equal and approximate them in the region of the $D\bar{D}$ thresholds by an s_2-independent constant $g_{D^0\bar{D}^0\pi^+\pi^-}$.

Taking into account Eqs. (12)–(14), the amplitude $F_0(s_1,s_2)$ can be written in the form

$$F_0(s_1,s_2) = \frac{i}{\pi^3}\,\epsilon_{X\mu}\int\frac{\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{m_{D^{*0}}^2}\right)(2p_{3\nu}-k_\nu)}{(k^2-m_{D^{*0}}^2+i\varepsilon)}$$
$$\times\frac{d^4k}{((p_1-k)^2-m_{D^0}^2+i\varepsilon)((k-p_3)^2-m_{\bar{D}^0}^2+i\varepsilon)}. \quad (16)$$

The four-vector under the integral sign we transform as follows

$$\left(-g_{\mu\nu}-\frac{k_\mu k_\nu}{m_{D^*}^2}\right)(2p_{3\nu}-k_\nu) = -2p_{3\mu}+k_\mu\left(m_{D^{*0}}^2\right.$$
$$\left.-m_{D^0}^2+m_{\pi^0}^2\right)/m_{D^{*0}}^2 - k_\mu((k-p_3)^2-m_{\bar{D}^0}^2)/m_{D^{*0}}^2. \quad (17)$$

This shows that after reducing the numerator and denominator in Eq. (16) by the factor $((k-p_3)^2-m_{\bar{D}^0}^2)$, the divergent part of the integral is proportional to $p_{1\mu}$ [i.e., the four-moment of the $X(3872)$ resonance] and does not contribute to $F_0(s_1,s_2)$ because $(\epsilon_X,p_1)=0$. For the numerical calculation of the amplitudes $F_0(s_1,s_2)$ in Eq. (16), we use the method developed in Refs. [47, 48]. Note that the part of the contribution from the second term in (17), $k_\mu(m_{D^{*0}}^2-m_{D^0}^2+m_{\pi^0}^2)/m_{D^{*0}}$, which after integration turns out to be proportional to $p_{3\mu}$, gives a negligible contribution to $F_0(s_1,s_2)$ in the $\sqrt{s_1}$ and $\sqrt{s_2}$ region under consideration. Thus we put

$$F_0(s_1,s_2) = -2(\epsilon_X,p_3)\frac{i}{\pi^3}\int\frac{d^4k}{(k^2-m_{D^{*0}}^2+i\varepsilon)}$$
$$\times\frac{1}{((p_1-k)^2-m_{D^0}^2+i\varepsilon)((k-p_3)^2-m_{\bar{D}^0}^2+i\varepsilon)}. \quad (18)$$

The amplitude $F_+(s_1,s_2)$ is obtained from Eq. (18) by replacing the masses of neutral D^* and D mesons by the masses of their charged partners.

Using Eq. (14) we express the differential width $d\Gamma(X\to\pi^0\pi^+\pi^-;s_1,s_2)/d\sqrt{s_2}$ in terms of the invariant amplitude $G_{X\pi^0\pi^+\pi^-}(s_1,s_2)$.

$$\frac{d\Gamma(X\to\pi^0\pi^+\pi^-;s_1,s_2)}{d\sqrt{s_2}} = \frac{2}{3}\,\frac{|G_{X\pi^0\pi^+\pi^-}(s_1,s_2)|^2}{4\pi}\,\frac{p^3(s_1,s_2)}{s_1}\,\frac{\rho(s_2)}{16\pi}\,\frac{2\sqrt{s_2}}{\pi}\,, \quad (19)$$

where

$$p(s_1,s_2)=\frac{\sqrt{s_1^2-2s_1(s_2+m^2_{\pi^0})+(s_2-m^2_{\pi^0})^2}}{2\sqrt{s_1}}, \quad (20)$$

$$\rho(s_2)=\sqrt{1-4m^2_{\pi^+}/s_2}. \quad (21)$$

The width of the decay $X\to\pi^0\pi^+\pi^-$ as a function of s_1 has the form

$$\Gamma(X\to\pi^0\pi^+\pi^-;s_1) = \int_{2m_{\pi^+}}^{\sqrt{s_1}-m^0_\pi}\frac{d\Gamma(X\to\pi^0\pi^+\pi^-;s_1,s_2)}{d\sqrt{s_2}}d\sqrt{s_2}\,, \quad (22)$$

and the probability of this decay is given by the expression

$$BR(X\to\pi^0\pi^+\pi^-) = \int_{3m_\pi}^{\infty}\frac{2\sqrt{s_1}}{\pi}\,\frac{\sqrt{s_1}\Gamma(X\to\pi^0\pi^+\pi^-;s_1)}{|D_X(s_1)|^2}d\sqrt{s_1}\,. \quad (23)$$

Equations (22) and (23) indicate the kinematically allowable limits of integration. In fact, the main contributions in Eqs. (22) and (23) are concentrated in much smaller intervals.

We now estimate the coupling constants $g_{D^{*0}D^0\pi^0}$ and $g_{D^0\bar D^0\pi^+\pi^-}$.

For the total decay width of the D^{*0} meson, only its upper limit is known so far: $\Gamma_{D^{*0}}<2.1$ MeV [1]. On the other hand, the total decay width of the D^{*+} meson and the branching ratio of the $D^{*+}\to(D\pi)^+$ decay are well known [1]: $\Gamma_{D^{*+}}\approx 83.6$ keV, $BR(D^{*+}\to(D\pi)^+)\approx 98.4\%$. Assuming the isotopic symmetry for the coupling constants $g_{D^*D\pi}$, we have

$$\frac{m^2_{D^{*0}}\Gamma_{D^{*0}\to D^0\pi^0}}{p^3_{D^0\pi^0}}=\frac{m^2_{D^{*+}}\Gamma_{D^{*+}\to(D\pi)^+}}{2p^3_{D^0\pi^+}+p^3_{D^+\pi^0}}\,, \quad (24)$$

where $p_{D\pi}$ denotes the momentum of the final D or π meson in the D^* rest frame. From here we find the decay width $\Gamma_{D^{*0}\to D^0\pi^0}\approx 36\,\text{keV}$ and the coupling constant $g^2_{D^{*0}D^0\pi^0}/(4\pi)=3m^2_{D^{*0}}\Gamma_{D^{*0}\to D^0\pi^0}/(2p^3_{D^0\pi^0})\approx 2.8$. Using also the value of $BR(D^{*0}\to D^0\pi^0)\approx 64.7\%$ [1], we get an estimate for the total decay width of the D^{*0} meson: $\Gamma_{D^{*0}}\approx 55.6$ keV. Here we note in passing the following. As the examples [49–54] show, the instability of the vector mesons in the intermediate states (i.e., the finiteness of their total widths) is important to take into account when estimating the contributions of logarithmic triangle singularities. In this case, $\Gamma_{D^{*0}}$ is small. Nevertheless, its accounting in the D^{*0} propagator (by replacing $m^2_{D^{*0}}\to m^2_{D^{*0}}-im_{D^{*0}}\Gamma_{D^{*0}}$) noticeably smoothes the logarithmic singularity in the amplitude of the diagram in Fig. 2 and the computed width $\Gamma(X(3872)\to\pi^0\pi^+\pi^-;m_X)$ is reduced by approximately 30% as compared to that for $\Gamma_{D^{*0}}=0$. In a similar way, we take into account the width $\Gamma_{D^{*\pm}}$ in the $D^{*\pm}$ propagator.

The constant $g_{D^0\bar D^0\pi^+\pi^-}$ is associated with the annihilation cross section $\sigma_{D^0\bar D^0\to\pi^+\pi^-}$ at the $D^0\bar D^0$ threshold and with the corresponding inelastic scattering length $\alpha''_{D^0\bar D^0\to\pi^+\pi^-}$ by the relations:

$$\frac{k\,\sigma_{D^0\bar D^0\to\pi^+\pi^-}}{4\pi}=|\alpha''_{D^0\bar D^0\to\pi^+\pi^-}|=q\left|\frac{g_{D^0\bar D^0\pi^+\pi^-}}{8\pi\sqrt{s_2}}\right|^2, \quad (25)$$

where k and q are momenta of the D^0 and π^+ mesons, respectively, in the center-of-mass frame of the reaction $D^0\bar D^0\to\pi^+\pi^-$. In the $D^0\bar D^0$ threshold domain of interest to us, $q/s_2\approx 1/(4m_{D^0})$. At present, the values in Eq. (25), which characterizes the S-wave $D^0\bar D^0\to\pi^+\pi^-$ annihilation at rest, are completely unknown. If we naively put the inelastic scattering length $|\alpha''_{D^0\bar D^0\to\pi^+\pi^-}|\approx 1/(2m_{D^{*+}})\approx 1/(4\,\text{GeV})$ (which is in dimensionless units $m_{\pi^+}|\alpha''_{D^0\bar D^0\to\pi^+\pi^-}|\approx 0.0347$), then $|g_{D^0\bar D^0\pi^+\pi^-}/(8\pi)|^2$ is approximately equal to ≈ 1.8. We use this value in further evaluations. It is clear that our rough estimate is related to considerations about the $D^0\bar D^0$ annihilation radius. An experiment will show whether this value is reasonable or not. For comparison, we note that the tree $D^0\bar D^0\to\pi^+\pi^-$ annihilation amplitude caused by the charged D^* exchange leads to $|\alpha''_{D^0\bar D^0\to\pi^+\pi^-}|$, which is about 15 times greater than our estimate, due to the large coupling constant $g^2_{D^{*+}D^0\pi^+}/(4\pi)\approx 5.6$ (see note [55]).

Figure 4 shows an example of the $\pi^+\pi^-$ mass spectrum in the decay $X(3872)\to\pi^0\pi^+\pi^-$, i.e., $d\Gamma(X\to\pi^0\pi^+\pi^-;s_1,s_2)/d\sqrt{s_2}$ as a function of $\sqrt{s_2}$, calculated with use of Eq. (19) at $\sqrt{s_1}=m_X=3.87169$ GeV and the coupling constant of $X(3872)$ with the $D^{*0}\bar D^0$ channel $g^2_A/(16\pi)=0.25$ GeV^2 (other possible values for $g^2_A/(16\pi)$ are discussed below). The integration $d\Gamma(X\to\pi^0\pi^+\pi^-;m^2_X,s_2)/d\sqrt{s_2}$ over $\sqrt{s_2}$ in the region of 35 MeV wide, i.e., from $m_X-m_{\pi^0}-0.035$ GeV $=3.70171$ GeV to $m_X-m_{\pi^0}=3.73671$ GeV, results in $\Gamma(X\to\pi^0\pi^+\pi^-;m^2_X)\approx 3$ keV. However, as can be seen from Fig. 5, this is in fact the maximal value of the $X(3872)\to\pi^0\pi^+\pi^-$ decay width in the $X(3872)$ resonance region. The width $\Gamma(X\to\pi^0\pi^+\pi^-;s_1)$ is a sharply changing function of $\sqrt{s_1}$. Two peaks in $\Gamma(X\to\pi^0\pi^+\pi^-;s_1)$ located near the $D^{*0}\bar D^0$ and $D^{*+}D^-$ thresholds (see Fig. 5) are manifestations of the logarithmic singularities in the amplitudes of the diagrams in Fig. 2 (the left peak) and in Fig. 3 (the right peak) [56]. The most important contribution to $BR(X\to\pi^0\pi^+\pi^-)$ [see Eq. (23)] comes from the left peak. The right peak in

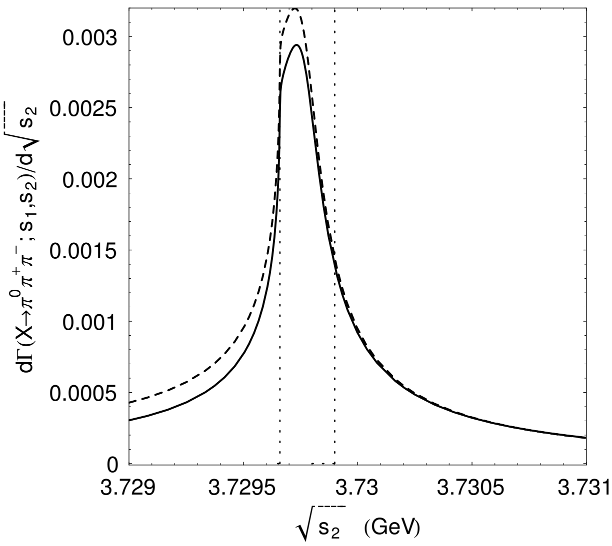

Figure 4: An example of the $\pi^+\pi^-$ mass spectrum $d\Gamma(X \to \pi^0\pi^+\pi^-; s_1, s_2)/d\sqrt{s_2}$ constructed with the use of Eq. (19) at $\sqrt{s_1} = m_X = 3.87169$ GeV and $g_A^2/(16\pi) = 0.25$ GeV^2. The solid curve corresponds to the sum of the diagrams in Figs. 2 and 3. The dashed curve shows the contribution from the diagram in Fig. 2 only. The $\sqrt{s_2}$ values between which [according to Eq. (4)] the amplitude of the $X(3872) \to (D^{*0}\bar{D}^0 + \bar{D}^{*0}D^0) \to \pi^0 D^0\bar{D}^0 \to \pi^0\pi^+\pi^-$ decay contains the logarithmic singularity, in the hypothetical case of the stable D^{*0} meson, are shown by the dotted vertical lines. In so doing, the singularity itself is located at $\sqrt{s_2} = 3.72982$ GeV (see note [57]).

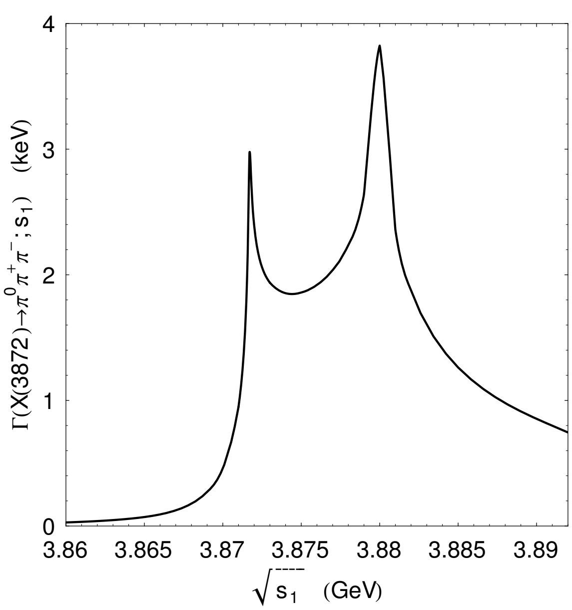

Figure 5: The width $\Gamma(X \to \pi^0\pi^+\pi^-; s_1)$ as a function of $\sqrt{s_1}$. The constructed example corresponds to $g_A^2/(16\pi) = 0.25$ GeV^2.

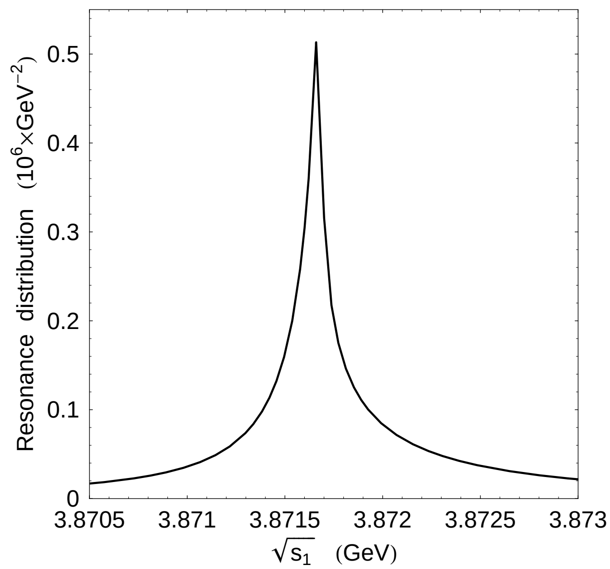

Figure 6: The resonance distribution $2s_1/(\pi|D_X(s_1)|^2)$ at $g_A^2/(16\pi) = 0.25$ GeV^2 and $\Gamma_{non} = 1$ MeV.

$\Gamma(X \to \pi^0\pi^+\pi^-; s_1)$ practically does not work as it is located far on the right tail of the $X(3872)$ resonance and its contribution to $BR(X \to \pi^0\pi^+\pi^-)$ is strongly suppressed by the $X(3872)$ propagator module squared.

We now present numerical estimates for $BR(X \to \pi^0\pi^+\pi^-)$ using as a guide the values of g_A obtained in Refs. [25, 27, 28]. Figure 6 shows an example of the resonance distribution $2s_1/(\pi|D_X(s_1)|^2)$ calculated at $m_X = 3.87169$ GeV [1], $g_A^2/(16\pi) = 0.25$ GeV^2, and $\Gamma_{non} = 1$ MeV. Weighting with this distribution the energy dependent width $\Gamma(X \to \pi^0\pi^+\pi^-; s_1)$ shown in Fig. 5, we find, according to Eq. (23), that for the above values of the parameters $BR(X \to \pi^0\pi^+\pi^-) \approx 5 \times 10^{-4}$. Estimates for $BR(X \to \pi^0\pi^+\pi^-)$ for different values of $g_A^2/(16\pi)$ and Γ_{non}, which we vary in a fairly wide but reasonable range, are given in Table I at $m_X = 3.87169$ GeV [1].

Table I: $BR(X((3872) \to \pi^0\pi^+\pi^-)$ in units of 10^{-4} for five values of $g_A^2/(16\pi)$ and three values of Γ_{non}; $m_X = 3.87169$ GeV.

$g_A^2/(16\pi)$ (in GeV^2)	$= 0.1$	$= 0.2$	$= 0.25$	$= 0.5$	$= 1.0$
$\Gamma_{non} = 0.5$ MeV	7.42	8.42	8.35	7.10	5.19
$\Gamma_{non} = 1$ MeV	3.93	4.99	5.14	4.88	3.84
$\Gamma_{non} = 2$ MeV	1.93	2.70	2.89	3.07	2.67

It is not yet clear whether the mass of the $X(3872)$ state lies slightly above or slightly below the $D^{*0}\bar{D}^0$ threshold. The ± 0.17 MeV uncertainty that the Particle Data Group [1] indicates allows for both possibilities. Tables II and III show the estimates for $BR(X \to \pi^0\pi^+\pi^-)$ at the same values of $g_A^2/(16\pi)$ and Γ_{non} as in Table I

Table II: The same as Table I but for $m_X = 3.87169 + 0.00017$ GeV.

$g_A^2/(16\pi)$ (in GeV^2)	$= 0.1$	$= 0.2$	$= 0.25$	$= 0.5$	$= 1.0$
$\Gamma_{non} = 0.5$ MeV	6.45	6.97	6.82	5.63	3.94
$\Gamma_{non} = 1$ MeV	3.76	4.60	4.68	4.30	3.27
$\Gamma_{non} = 2$ MeV	1.93	2.64	2.80	2.89	2.45

Table III: The same as Table I but for $m_X = 3.87169 - 0.00017$ GeV.

$g_A^2/(16\pi)$ (in GeV^2)	$= 0.1$	$= 0.2$	$= 0.25$	$= 0.5$	$= 1.0$
$\Gamma_{non} = 0.5$ MeV	8.04	11.2	12.2	14.7	16.3
$\Gamma_{non} = 1$ MeV	3.91	5.57	6.08	7.37	8.20
$\Gamma_{non} = 2$ MeV	1.86	2.73	3.01	3.70	4.12

but for $m_X = 3.87169 \pm 0.00017$ GeV.

III. CONCLUSION

The above analysis shows that $BR(X(3872) \to \pi^0\pi^+\pi^-)$ can be expected at the level of 10^{-3}–10^{-4}. The dominant contribution to $BR(X(3872) \to \pi^0\pi^+\pi^-)$ comes from the production of the $\pi^+\pi^-$ system in a narrow (no more than 20 MeV wide) interval of the invariant mass $m_{\pi^+\pi^-}$ near the value of $2m_{D^0} \approx 3.73$ GeV. The $\pi^+\pi^-$ events with such an invariant mass can serve as a signature of the decay $X(3872) \to (D^{*0}\bar{D}^0 + \bar{D}^{*0}D^0) \to \pi^0 D^0\bar{D}^0 \to \pi^0\pi^+\pi^-$.

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[56] The contributions of the diagrams in Figs. 2 and 3 to the width $\Gamma(X \to \pi^0\pi^+\pi^-; s_1)$ begin to cancel each other at s_1 just below the $D^{*0}\bar{D}^0$ threshold and just above the $D^{*+}D^-$ threshold. This practically does not occur between the thresholds due to significantly different phases of the neutral and charged intermediate states. The phase of the $X \to \pi^0\pi^+\pi^-$ transition amplitude changes between the $D^{*0}\bar{D}^0$ and $D^{*+}D^-$ thresholds by about 90°, just as is the case between the K^+K^- and $K^0\bar{K}^0$ thresholds in the $a^0_0(980) - f_0(980)$ mixing amplitude [53] or in the amplitude of the decay $\eta(1405) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ [54].

[57] The logarithmic singularities in the amplitude are at $x_1^2 + x_2^2 + x_3^2 + 2x_1x_2x_3 - 1 = 0$, where $x_1 = (s_1 - m^2_{D^{*0}} - m^2_{D^0})/(2m_{D^{*0}}m_{D^0})$, $x_2 = (s_2 - 2m^2_{D^0})/(2m^2_{D^0})$, $x_3 = (m^2_{\pi^0} - m^2_{D^{*0}} - m^2_{D^0})/(2m^2_{D^{*0}}m^2_{D^0})$ (see, for example, Refs. [38–40, 45, 49–52, 54]). Only one solution of this equation falls into the region defined by Eqs. (3) and (4).