

Remarks on the study of the $X(3872)$ from Effective Field Theory with Pion-Exchange Interaction

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In a recent paper [1], the charmonium state $X(3872)$ is studied in the framework of an effective field theory. In that work it is claimed that (i) the one-pion exchange (OPE) alone provides sufficient binding to produce the X as a shallow bound state at the $D^0\bar{D}^{*0}$ threshold, (ii) short-range dynamics (described by a contact interaction) provides only moderate corrections to the OPE, and (iii) the X -pole disappears as the pion mass is increased slightly and therefore the X should not be seen on the lattice, away from the pion physical mass point, if it were a molecular state. In this paper we demonstrate that the results of Ref. [1] suffer from technical as well as conceptual problems and therefore do not support the conclusions drawn by the authors.

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I. INTRODUCTION

The first evidence for the existence of a narrow ($\Gamma_X < 1.2$ MeV [2]) charmonium-like state $X(3872)$ was reported in 2003 by the Belle Collaboration [3]. The properties of this state are inconsistent with a simple quark–antiquark meson interpretation, and thus it has attracted and is still attracting a lot of attention from both theorists and experimentalists. The $X(3872)$ has the mass [2]

$$M_X = (3871.68 \pm 0.17) \text{ MeV} \quad (1)$$

and therefore it resides within less than 1 MeV from the neutral $D\bar{D}^*$ threshold. The latter fact implies that the admixture of the $D^0\bar{D}^{*0}$ component in the wave function of the X can be substantial. Its quantum numbers are determined to be $J^{PC} = 1^{++}$ [4].

An issue related to the $X(3872)$ as a molecular state heavily discussed in the literature is the nature of the binding forces forming it as a near-threshold state. Pion exchange between charmed mesons was suggested long ago [5, 6] as a mechanism able to bind the isosinglet $D\bar{D}^*$ mesonic system and to form a deuteron-like state near threshold. This model was revisited shortly after the $X(3872)$ discovery [7, 8], while further implications of the nearby pion threshold are discussed in Refs. [9, 10].

Because of the P -wave nature of the $D^*D\pi$ coupling, the OPE potential does not fall off and stays finite at large momenta. As such, it contains short-ranged physics and all loop integrals with the pion exchange are divergent. This observation explains the large regulator dependence observed in Ref. [11]¹, where calculations were performed in the framework of a phenomenological potential model with static D

mesons. However, the D^{*0} mass is very close to the $D^0\pi^0$ threshold and thus the intermediate pion may go on-shell [13]. In consequence, the three-body $D\bar{D}\pi$ unitarity cuts have to be taken into account². The effects of three-body cuts were included in the effective field theory treatments with perturbative pions (the X-EFT) [10] and nonperturbative pions [15] based on Faddeev-type integral equations³. Both approaches were recently extended to investigate the pion mass dependence of the X binding energy, see Ref. [17] for the X-EFT study and Ref. [18] for the nonperturbative calculation. In these works the behaviour of the X binding energy was found to be nontrivial: depending on the interplay of long- and short-range forces the X can either disappear as a bound state or get more bound.

In a recent paper [1], the authors revisit the problem of the binding forces in the $X(3872)$ using an effective field theory approach. They claim that the OPE alone provides sufficient binding to produce the X as a shallow bound state at the $D^0\bar{D}^{*0}$ threshold, while the short-range dynamics (described by a contact interaction) provides only moderate corrections to the OPE. In this paper we demonstrate that the conclusions drawn by the authors of Ref. [1] are incorrect. First, the authors apply dimensional regularization to linearly divergent one-loop integrals and misinterpret their apparent finiteness for $D = 4$, with D being the space-time dimensions, as an ability to disentangle the OPE and the short-range physics in a model-independent way. Furthermore, the authors of Ref. [1] resort to a low-order Padé approximation. We do not only argue that this method leads to inaccurate results but

¹ The role of the short-range contribution of the pion exchange is also discussed in Ref. [12].

² It is shown in Ref. [14] that cut effects are of paramount importance in the $\bar{D}_\alpha D_\beta$ system, if the D_β width is dominated by the S -wave $D_\beta \rightarrow D_\alpha \pi$ decay.

³ In Ref. [16] the role of relativistic corrections in the nonperturbative approach including three-body effects was addressed.

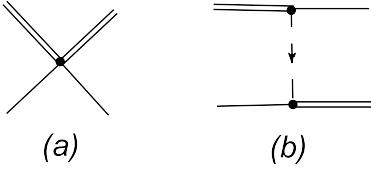


FIG. 1. Tree-level amplitudes. Double, solid, and dashed lines indicate the vector (D^{*0} or \bar{D}^{*0}) mesons, the pseudoscalar (D^0 or \bar{D}^0) mesons, and the pions, respectively. Adapted from Ref. [1].

also demonstrate that it produces a large number of unphysical singularities within the assumed range of applicability of the formalism, so that its predictions, including the emerging S -matrix pole position, cannot have any sensible interpretation. In addition, we show that in Ref. [1] the three-body singularities were treated incorrectly. In short, we demonstrate that the results of Ref. [1] are incorrect and as such are devoid of any physical significance.

The structure of the paper is as follows. In Secs. II and III we provide the expressions for the tree-level and one-loop amplitudes in analogy to those discussed in Ref. [1]. However, instead of using dimensional regularization applied in Ref. [1], we stick to a sharp cut-off regularization in order to make the divergences in the one-loop contributions explicit. In Sec. IV we demonstrate that the entire approach used in Ref. [1] is inconsistent and therefore argue that the results obtained are unreliable. Our conclusions are summarized in Sec.V.

For convenience we stick to the definitions, conventions, and notations of Ref. [1]. In particular, the C -even combination of the states $1 \equiv D\bar{D}^*$ and $2 \equiv D\bar{D}^*$ is chosen in the form

$$|X_+\rangle = \frac{1}{\sqrt{2}}(|\bar{D}D^*\rangle + |D\bar{D}^*\rangle), \quad (2)$$

and thus the $D\bar{D}^*$ scattering amplitude under consideration is

$$T_{++} = \langle X_+ | \hat{T} | X_+ \rangle = \frac{1}{2}(T_{11} + T_{12} + T_{21} + T_{22}). \quad (3)$$

II. THE TREE-LEVEL AMPLITUDE

The general Lagrangian describing four-boson contact interactions is taken as [19]

$$\begin{aligned} \mathcal{L}^{(0)} = & C_2 \left[P^{(Q)\dagger} P^{(\bar{Q})} V_\mu^{(\bar{Q})\dagger} V^{(Q)\mu} \right. \\ & \left. + P^{(\bar{Q})\dagger} P^{(Q)} V_\mu^{(Q)\dagger} V^{(\bar{Q})\mu} \right] \\ & - C_1 \left[P^{(Q)\dagger} P^{(Q)} V_\mu^{(\bar{Q})\dagger} V^{(\bar{Q})\mu} \right. \\ & \left. + P^{(\bar{Q})\dagger} P^{(\bar{Q})} V_\mu^{(Q)\dagger} V^{(Q)\mu} \right], \end{aligned} \quad (4)$$

where $P^{(Q)} = (D^0, D^+)$ and $V^{(Q)} = (D^{*0}, D^{*+})$ are the heavy meson fields, while $P^{(\bar{Q})} = (\bar{D}^0, D^-)$ and $V^{(\bar{Q})} =$

(\bar{D}^{*0}, D^{*-}) are the heavy antimeson fields. The two contact terms C_1 and C_2 enter the scattering amplitude T_{++} in the combination $\lambda = C_2 - C_1$.

The $D^*D\pi$ interaction relevant for the OPE in the $X(3872)$ is described by the Lagrangian

$$\begin{aligned} \mathcal{L}^{(1)} = & 2g_\pi(V_{a\mu}^{(Q)\dagger} P_b^{(Q)} + P_a^{(Q)\dagger} V_{b\mu}^{(Q)}) u_a^\mu \\ & - 2g_\pi(V_{a\mu}^{(\bar{Q})\dagger} P_b^{(\bar{Q})} + P_a^{(\bar{Q})\dagger} V_{b\mu}^{(\bar{Q})}) u_a^\mu, \\ u_\mu = & i(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger), \quad u = \exp\left(\frac{i\phi}{\sqrt{2}f_\pi}\right), \end{aligned} \quad (5)$$

where ϕ is pions matrix, f_π is the pion decay constant, $f_\pi = 92.2$ MeV [2]. The coupling constant g_π is conventionally defined as

$$g_\pi = g\sqrt{M_D M_{D^*}}, \quad (6)$$

where the dimensionless constant g is determined from the strong decay $D^{*+} \rightarrow D^+ \pi^0$. Following Ref. [1] we use $g = 0.3$.

As follows from Lagrangian (5), the D^* decays into $D\pi$ in a P wave, so that the corresponding vertex contains the pion momentum. Then the components of the tree-level amplitude (see Fig. 1) are [1, 20]

$$\begin{aligned} T_{++,SS}^{(0)} = & -\lambda - \frac{2g_\pi^2}{3f_\pi^2} \left[2 - \frac{\mu_\pi^2}{2p^2} \ln\left(1 + \frac{4p^2}{\mu_\pi^2}\right) \right], \\ T_{++,SD}^{(0)} = T_{++,DS}^{(0)} = & \frac{2g_\pi^2}{3\sqrt{2}f_\pi^2} \left[1 - \frac{3\mu_\pi^4}{2p^2} \right. \\ & \left. + \left(\frac{\mu_\pi^2}{2p^2} + \frac{3\mu_\pi^4}{8p^4} \right) \ln\left(1 + \frac{4p^2}{\mu_\pi^2}\right) \right], \\ T_{++,DD}^{(0)} = & \frac{g_\pi^2}{3f_\pi^2} \left[1 - \frac{3\mu_\pi^2}{p^2} - \frac{9\mu_\pi^4}{4p^4} \right. \\ & \left. + \left(\frac{\mu_\pi^2}{2p^2} + \frac{15\mu_\pi^4}{8p^4} + \frac{9\mu_\pi^6}{16p^6} \right) \ln\left(1 + \frac{4p^2}{\mu_\pi^2}\right) \right], \end{aligned} \quad (7)$$

where p is the three-momentum of the D -meson in the centre-of-mass frame of the $D\bar{D}^*$ system,

$$\mu_\pi^2 = m_\pi^2 - \Delta^2, \quad \Delta = M_{D^*} - M_D, \quad (8)$$

and the terms $\propto g_\pi^2$ result from the angular integration of the OPE interaction. To derive Eq. (7) the terms $\simeq \frac{m_\pi}{M_D} p^2$ and $\simeq m_\pi E$ (E is the $D\bar{D}^*$ energy relative to the two-body threshold) in the three-body propagator were dropped. As we will argue below in some more detail this significantly changes the singularity structure of the amplitude. Thus, following the authors of Ref. [1], we consider on-shell D and D^* mesons in their centre-of-mass frame with the incoming D - and D^* -meson momenta

$$p_1^\mu = (E_{\mathbf{p}}, \mathbf{p}), \quad p_2^\mu = (E_{\mathbf{p}}^*, -\mathbf{p}), \quad (9)$$

and their outgoing momenta

$$p_3^\mu = (E_{\mathbf{p}'}, -\mathbf{p}'), \quad p_4^\mu = (E_{\mathbf{p}'}^*, \mathbf{p}'), \quad (10)$$

with $|\mathbf{p}'| = |\mathbf{p}|$. The corresponding D - and D^* -meson energies are

$$E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + M_D^2}, \quad E_{\mathbf{p}}^* = \sqrt{\mathbf{p}^2 + M_{D^*}^2}. \quad (11)$$

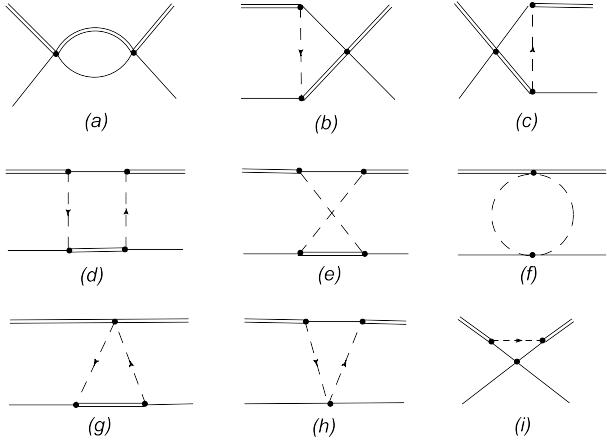


FIG. 2. One loop diagrams. Adapted from Ref. [1].

III. THE ONE-LOOP AMPLITUDE

The diagrams contributing to the one-loop $D\bar{D}^*$ scattering amplitude are depicted in Fig. 2. The amplitude for the diagram (a) corresponds to a contact term (pionless) theory and reads

$$iT_{++}^{(1),a} = \lambda^2 \varepsilon_\mu(p_1) \varepsilon_\nu^*(p_3) \int \frac{d^4 l}{(2\pi)^4} G(p_1 - l) G_{*\mu\nu}(p_2 + l), \quad (12)$$

where, as was explained before, p_1 and p_2 are the incoming D - and D^* -meson momenta, respectively, $\varepsilon_\mu(p_1)$ and $\varepsilon_\nu^*(p_3)$ are the initial and final D^* polarization vectors and

$$G(p) = \frac{1}{p^2 - M_D^2 + i\varepsilon}, \quad G_{*\mu\nu}(p) = \frac{-g_{\mu\nu} + p_\mu p_\nu / M_{D^*}^2}{p^2 - M_{D^*}^2 + i\varepsilon} \quad (13)$$

are the D - and D^* -meson propagators, respectively. In the nonrelativistic limit (to the order $O(p)$) only the spatial Kronecker structure is retained in the D^* propagator, so that the amplitude $T_{++}^{(1),a}$ reads

$$T_{++}^{(1),a} = (\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}^*) T_{++,SS}^{(1),a}, \quad T_{++,SS}^{(1),a} = \lambda^2 I, \quad (14)$$

where the loop function I is given by the integral

$$I = -i\mu^{4-D} \int \frac{d^D l}{(2\pi)^D} \frac{1}{(p_1 - l)^2 - M_D^2 + i\varepsilon} \times \frac{1}{(p_2 + l)^2 - M_{D^*}^2 + i\varepsilon}, \quad (15)$$

which is logarithmically divergent in $D = 4$. Here μ stands for a renormalization scale in dimensional regularization. A straightforward way to deal with this logarithmic divergence is to employ a convenient regularization procedure (sharp cut-off, dimensional regularization, and so on) and finally to absorb the divergent piece into the redefinition of the interaction strength λ (see, for example, Ref. [19] for a contact interaction theory and Ref. [15] for the OPE included).

Here, following Ref. [1], we first perform the integration over the energy l_0 by closing the contour in the upper half of

the l_0 complex plane. The two relevant poles are

$$l_0^{(1)} = E_{\mathbf{p}} - E_{\mathbf{p}-l} + i\varepsilon, \quad l_0^{(2)} = -E_{\mathbf{p}}^* - E_{\mathbf{p}+l}^* + i\varepsilon. \quad (16)$$

However, anticipating the nonrelativistic expansion we retain only the contribution of the pole $l_0^{(1)}$. This yields

$$I = \int \frac{d^{D-1} l}{(2\pi)^{D-1}} \frac{\mu^{4-D}}{2E_{\mathbf{p}-l}[E_{\mathbf{p}-l}^* - (E_{\mathbf{p}} + E_{\mathbf{p}}^* - E_{\mathbf{p}-l})^2]}. \quad (17)$$

Now, treating the D and D^* mesons nonrelativistically, we expand their energies keeping only the leading contribution. Then

$$I = \frac{\mu^{4-D}}{2(M_D + M_{D^*})} \int \frac{d^{D-1} l}{(2\pi)^{D-1}} \frac{1}{(l - \mathbf{p})^2 - \mathbf{p}^2 - i\varepsilon}. \quad (18)$$

Because of the substitution $2E_{\mathbf{p}-l} \approx 2M_D$ made in the denominator, the remaining integral is now linearly divergent in $D = 4$. Finally, using dimensional regularization one arrives at the expression

$$I = \frac{\mu^{4-D} \Gamma(\frac{3-D}{2})}{2(M_D + M_{D^*})(4\pi)^{\frac{D-1}{2}} (-\mathbf{p}^2 - i\varepsilon)^{\frac{3-D}{2}}} \quad (19)$$

which reveals the well-known feature of dimensional regularization to hide power-like divergences, in particular, the linear one. Indeed, formally setting $D = 4$ in Eq. (19) one arrives at the finite result

$$I_{\text{naive}} = \frac{i|\mathbf{p}|}{8\pi(M_D + M_{D^*})} \quad (20)$$

used in Ref. [1] — see Eq. (12) of Ref. [1].

It has to be noticed however that the finite expression (20) is a result of an implicit subtraction hidden by the dimensional regularization scheme. To make the argument more transparent we use the sharp cut-off regularization scheme for the integral (18) to arrive at

$$I = \frac{1}{8\pi(M_D + M_{D^*})} \left(\frac{2}{\pi} \Lambda + i|\mathbf{p}| \right), \quad (21)$$

where Λ is the cut-off parameter, so that

$$T_{++,SS}^{(1),a} = \frac{\lambda^2}{8\pi(M_D + M_{D^*})} \left(\frac{2}{\pi} \Lambda + i|\mathbf{p}| \right). \quad (22)$$

In fact, the power-like divergence manifests itself as a pole at $D = 3$ in Eq. (19). If we subtract this divergence as well, following the power divergence subtraction (PDS) scheme [21], we arrive at the same expression as in Eq. (21) with $2\Lambda/\pi$ replaced by the scale in the PDS scheme.

The diagrams (b)-(i) in Fig. 2 contribute to the one-loop amplitude in the pionfull theory. The authors of Ref. [1] restrict themselves to the order $O(p)$, so that only the contributions from the diagrams (b)-(d) are retained. For the amplitude in Fig. 2(b) one has

$$iT_{++}^{(1),b} = \frac{4\lambda g_\pi^2}{f_\pi^2} \varepsilon_\mu(p_1) \varepsilon_\nu^*(p_3) \int \frac{d^D l}{(2\pi)^D} l_\mu l_\lambda G(p_1 - l) G_{*\nu\lambda}(p_2 + l) D_\pi(l), \quad D_\pi(p) = \frac{1}{p^2 - m_\pi^2 + i\varepsilon}. \quad (23)$$

As before, the integration over the energy l_0 is performed explicitly, and the leading nonrelativistic contribution reads [20]

$$T_{++}^{(1),b} = \frac{2\lambda g_\pi^2 \mu^{4-D}}{f_\pi^2(M_D + M_{D^*})} \int \frac{d^{D-1} l}{(2\pi)^{D-1}} \frac{(\varepsilon \cdot l)(\varepsilon^* \cdot l)}{[(l - p)^2 - p^2 - i\varepsilon][l^2 + \mu_\pi^2 - i\varepsilon]}. \quad (24)$$

Note that here the same approximations were made as in the derivation of the tree-level amplitude (see the discussion after Eq. (7)). In order to avoid implicit subtractions we, again, evaluate the linearly divergent integral in Eq. (24) in $D = 4$ and using the sharp cut-off prescription. The result reads

$$T_{++,SS}^{(1),b} = \frac{\lambda g_\pi^2}{6\pi f_\pi^2(M_D + M_{D^*})} \left(\frac{2}{\pi} \Lambda - \frac{1}{2} \mu_\pi^2 \Gamma_0(|\mathbf{p}|) + i|\mathbf{p}| \right), \quad \Gamma_0(|\mathbf{p}|) = \frac{1}{|\mathbf{p}|} \left[\arctan \frac{2|\mathbf{p}|}{\mu_\pi} + \frac{i}{2} \ln \left(1 + \frac{4\mathbf{p}^2}{\mu_\pi^2} \right) \right]. \quad (25)$$

The diagram depicted in Fig. 2(c) gives the same contribution. Similarly to the amplitude (22), using the dimensional regularization scheme in the amplitude (24) hides the divergence, so that the result reported in Ref. [1] corresponds to the divergent term $\propto \Lambda$ in parenthesis implicitly subtracted — see Eq. (13) of Ref. [1].

Finally, the amplitude for the box diagram depicted in Fig. 2(d) reads

$$T_{++}^{(1),d} = \frac{16g_\pi^4}{f_\pi^4} \mu^{4-D} \varepsilon_\mu(p_1) \varepsilon_\nu^*(p_3) \int \frac{d^D l}{(2\pi)^D} l_\mu l_\lambda (l+q)_\nu (l+q)_\sigma G(p_1 - l) G_{*\nu\lambda}(p_2 + l) D_\pi(l+q) D_\pi(l), \quad q = p_3 - p_1 \quad (26)$$

or, after performing the integration over l_0 and retaining only the leading contribution [20],

$$T_{++}^{(1),d} = \frac{8g_\pi^4 \mu^{4-D} \varepsilon_i \varepsilon_k^*}{f_\pi^4(M_D + M_{D^*})} \int \frac{d^{D-1} l}{(2\pi)^{D-1}} \frac{l_i (l+q)_k l \cdot (l+q)}{[(l - p)^2 - p^2 - i\varepsilon][(l+q)^2 + \mu_\pi^2 - i\varepsilon][l^2 + \mu_\pi^2 - i\varepsilon]}. \quad (27)$$

In $D = 4$ with the sharp cut-off regularization this gives

$$T_{++,SS}^{(1),d} = \frac{4g_\pi^4 \Lambda}{3\pi^2 f_\pi^4(M_D + M_{D^*})} + \left(T_{++,SS}^{(1),d} \right)_{\text{fin}}, \quad (28)$$

where the finite part $\left(T_{++,SS}^{(1),d} \right)_{\text{fin}}$ is quoted in Eq. (35) below. As before, only this finite part survives if the dimensional regularization scheme is naively applied to the linearly divergent integral in Eq. (27), as it is done in Ref. [1].

Combining the results (22), (25), and (28), one can find for the one-loop amplitude $T_{++,SS}^{(1)}$ to the order $O(p)$:

$$T_{++,SS}^{(1)} = \frac{\Lambda}{\pi^2(M_D + M_{D^*})} \left(\frac{1}{4} \lambda^2 + \lambda \frac{2g_\pi^2}{3f_\pi^2} + \frac{4g_\pi^4}{3f_\pi^4} \right) + \left(T_{++,SS}^{(1)} \right)_{\text{fin}}, \quad (29)$$

where $\left(T_{++,SS}^{(1)} \right)_{\text{fin}}$ is the finite amplitude used in Ref. [1] instead of the full one-loop amplitude (29). Notice that the divergent piece $\propto \Lambda$ contains contributions both from the contact interaction as well as from the OPE. In order to renormalize the one-loop amplitude (29) the constant contact counterterm has to be added to $T_{++,SS}^{(1)}$, the divergent part of which,

$$\delta T_{++,SS}^{(1)} = -\frac{\Lambda}{\pi^2(M_D + M_{D^*})} \left(\frac{1}{4} \lambda^2 + \lambda \frac{2g_\pi^2}{3f_\pi^2} + \frac{4g_\pi^4}{3f_\pi^4} \right), \quad (30)$$

does not vanish even in the limit $\lambda = 0$,

$$\delta T_{++,SS}^{(1)}(\lambda = 0) = -\frac{4g_\pi^4 \Lambda}{3\pi^2 f_\pi^4(M_D + M_{D^*})}. \quad (31)$$

Thus, contrary to the claim of Ref. [1], setting $\lambda = 0$ does not imply that only the OPE interaction is left, since the contact

operator (31) is added to the OPE. As was explained before, this contact operator is hidden (added implicitly) in Ref. [1] by using the dimensional regularization scheme for linearly divergent integrals. Alternatively to the sharp cut-off scheme used above, one can resort to the PDS scheme [21] and subtract the power divergence in $D = 3$. This would reveal the divergence and make the subtraction explicit. Furthermore, had the authors of Ref. [1] proceeded beyond the one-loop approximation, divergences would have become explicit, too.

Therefore, the conclusion one is led to is that the OPE potential in the $D\bar{D}^*$ system is well defined in the sense of an effective field theory only in connection with a contact operator. Thus the conclusion drawn in Ref. [1] that “the pion exchange interaction is the main reason for the system to be bound” has to be considered as model- and scheme-dependent.

IV. PADÉ APPROXIMATION AND THE WOULD-BE X POLE

In addition to the conceptual problem outlined in the previous section, the work of Ref. [1] also suffers from a severe technical problem as we explain in this section. Based on a particular method of unitarization, Ref. [1] reports the existence of a dynamically generated S -matrix pole which is interpreted as the $X(3872)$. In particular, for $\lambda = 0$, the pole resides at

$$p_0 = -15.46 + i24.62 \text{ MeV}. \quad (32)$$

In this section we provide very strong evidence that this pole is an artifact most probably caused by using Padé approximants of a too low order as well as an incorrect treatment of three-body effects. In particular, to formulate our argument we focus on a regime where the pion is sufficiently heavy that the D^* is stable which implies $\mu_\pi^2 > 0$ (see Eq. (8)). In this regime no three-body cuts need to be considered when solving the equations. In addition, the S -matrix needs to be consistent with the general theorems on two-body scattering. We show that in this regime the S -matrix of Ref. [1] contains many singularities most of those being unphysical and very close to the threshold. This suggests that the formalism used in Ref. [1] should not be used in the regime $\mu_\pi^2 < 0$ either. This statement is further supported by the observation that the three-body effects are treated inconsistently in Ref. [1].

Using Padé approximation is a well known technique to solve integral equations in the context of few-body problems (see, for example, the very pedagogical presentation in chapter 2.7.3 of the textbook Ref. [22]). In general it is argued that one can approximate the physical amplitude $f(E, \xi)$ — represented via an infinite series with an increasing number of insertions of the scattering potential scaled by the strength parameter ξ (where $\xi = 1$ refers to the physical situation) — by the rational function $f_{[N, M]}(E, \xi) = P_N(\xi)/Q_M(\xi)$, where $P_N(\xi)$ and $Q_M(\xi)$ are polynomials in ξ of order N and M , respectively, with energy-dependent coefficients. In particular, for the case of NN scattering in the spin-singlet S -wave

channel at 12 MeV above threshold (see Tab. 2.1 in Ref. [22]) the Padé series fully converges only after the inclusion of 10 terms (which implies $N=5, M=4$). Using only 5 iterations as input ($N=2, M=2$) the series is still off by 50%. The absolute value of the real part of the amplitude is too large by a factor 50 at $N = 1$ and $M = 0$. In the three-body case, the situation is similar (see Tab. 3.2 in Ref. [22] for instance). Naturally, the order of the Padé approximation needed to gain an acceptable accuracy is increased dramatically in case when near-threshold singularities are present in the amplitude. It should be stressed at this point that the formalism used in Ref. [1] pretends to be able to describe a near-threshold bound-state pole and nevertheless it refers to $N = 0$ and $M = 1$. In addition to the low accuracy that one should expect in this case we will demonstrate below that for these small values of N and M the amplitude contains several unphysical singularities within the assumed range of applicability of the formalism.

For the physical pion mass, the parameter μ_π is purely imaginary

$$\mu_\pi = -i\sqrt{\Delta^2 - m_\pi^2} = -i44.36 \text{ MeV}; \quad (33)$$

the $D\bar{D}^*$ -threshold as well as the mass of the $X(3872)$ are located above the $D\bar{D}\pi$ threshold. A pole search in this kinematic regime is technically quite demanding. We therefore study the pole structure of the equations for larger pion masses with $\mu_\pi^2 > 0$. This simplifies the analysis significantly and still allows us to show that the equations used in Ref. [1] are ill-behaved. For real values of μ_π , the D^* does not have phase space to decay into $D\pi$, so that three-body ($D\bar{D}\pi$) intermediate states can not go on-shell. For simplicity, in this section we use $\lambda = 0$.

The equation underlying the results of Ref. [1] are the [0,1] Padé approximation (Eq. (15) of Ref. [1]) given by

$$T_{++}^{\text{phy}} = T_{++}^{(0)} \cdot [T_{++}^{(0)} - T_{++}^{(1)}]^{-1} \cdot T_{++}^{(0)}, \quad (34)$$

where the tree-level amplitude $T_{++,SS}^{(0)}$ is quoted in Eq. (7) while, for $\lambda = 0$, the one-loop amplitude $T^{(1)}$ is given entirely by the box diagram (d) in Fig. 2 and reads [20]

$$\begin{aligned} T_{++,SS}^{(1)} = & \frac{g_\pi^4}{12\pi f_\pi^4(M_D + M_{D^*})} \left\{ -4\mu_\pi - \frac{\mu_\pi^3}{\mathbf{p}^2} - \left(4|\mathbf{p}| + \frac{6\mu_\pi^2}{|\mathbf{p}|} \right) \arctan \frac{2|\mathbf{p}|}{\mu_\pi} + \left(\frac{\mu_\pi^5}{2\mathbf{p}^4} + \frac{3\mu_\pi^7}{8\mathbf{p}^6} \right) \ln \left(1 + \frac{\mathbf{p}^2}{\mu_\pi^2} \right) \right. \\ & + \left(4|\mathbf{p}| + \frac{4\mu_\pi^2}{|\mathbf{p}|} - \frac{\mu_\pi^4}{2|\mathbf{p}|^3} - \frac{3\mu_\pi^6}{4|\mathbf{p}|^5} \right) \arctan \left(\frac{\mu_\pi |\mathbf{p}|}{\mu_\pi^2 + 2\mathbf{p}^2} \right) \\ & + \left(\frac{\mu_\pi^2}{|\mathbf{p}|^3} + \frac{\mu_\pi^6}{2|\mathbf{p}|^5} + \frac{3\mu_\pi^8}{16|\mathbf{p}|^7} \right) \left[\text{ImLi}_2 \left(\frac{2\mathbf{p}^2 - i|\mathbf{p}|\mu_\pi}{\mu_\pi^2 + 4\mathbf{p}^2} \right) + \text{ImLi}_2 \left(\frac{-2\mathbf{p}^2 + i|\mathbf{p}|\mu_\pi}{\mu_\pi^2 + 4\mathbf{p}^2} \right) \right] \\ & \left. + i|\mathbf{p}| \left[3 - \frac{\mu_\pi^2}{\mathbf{p}^2} + \frac{3\mu_\pi^4}{4\mathbf{p}^4} - \left(\frac{\mu_\pi^2}{\mathbf{p}^2} + \frac{\mu_\pi^4}{4\mathbf{p}^4} + \frac{3\mu_\pi^6}{8\mathbf{p}^6} \right) \ln \left(1 + \frac{4\mathbf{p}^2}{\mu_\pi^2} \right) + \left(\frac{\mu_\pi^4}{4\mathbf{p}^4} + \frac{\mu_\pi^6}{8\mathbf{p}^6} + \frac{3\mu_\pi^8}{64\mathbf{p}^8} \right) \ln^2 \left(1 + \frac{4\mathbf{p}^2}{\mu_\pi^2} \right) \right] \right\}, \end{aligned} \quad (35)$$

where Li_2 is the dilogarithm function⁴. The other components, $T_{++,SD}^{(1)} = T_{++,DS}^{(1)}$ and $T_{++,DD}^{(1)}$, take a complicated form, similar to Eq. (35) [20], and we do not quote them here.

⁴ Notice that in Eq. (35) the divergent piece is subtracted, as was explained

in Sect. III above.

For real values of μ_π one easily verifies that the tree-level and the one-loop amplitudes satisfy the perturbative two-body unitarity condition (Eq. (14) of Ref. [1])

$$\text{Im}T_{++}^{(1)} = T_{++}^{(0)} \frac{|\mathbf{p}|}{8\pi\sqrt{s}} T_{++}^{(0)*}, \quad (36)$$

which, for $T_{++,SS}^{(1)}$, takes the form

$$\text{Im}T_{++,SS}^{(1)} = \frac{|\mathbf{p}|}{8\pi\sqrt{s}} \left(\left| T_{++,SS}^{(0)} \right|^2 + \left| T_{++,SD}^{(0)} \right|^2 \right). \quad (37)$$

With this input T_{++}^{phy} defined in Eq. (34) is consistent with two-body unitarity.

The authors of Ref. [1] claim that they do not find any pole of the S -matrix for $\mu_\pi^2 > 0$. In particular, they claim that “When m_π is larger than Δ (142 MeV), there is no bound state or resonance pole.” Meanwhile, by an explicit calculation one can demonstrate that the equation

$$\det \left(T_{++}^{(0)} - T_{++}^{(1)} \right) = 0 \quad (38)$$

for real μ_π does possess multiple solutions similar to the solution (32). For example, for $\mu_\pi = 44.36$ MeV, the following near-threshold solutions exist:

$$\begin{aligned} &\pm 11.80 + i22.47 \text{ MeV}, \quad \pm 23.87 - i20.44 \text{ MeV}, \\ &\pm 29.94 + i13.24 \text{ MeV}, \quad \pm 10.74 - i0.03 \text{ MeV}, \\ &\pm 10.66 + i0.02 \text{ MeV}. \end{aligned} \quad (39)$$

Each solution above corresponds to a pole of the physical amplitude (34). In particular, the pole

$$p_1 = -11.80 + i22.47 \text{ MeV} \quad (40)$$

looks very similar to the pole (32) reported in Ref. [1] and, naively, leads to a similar interpretation as a bound state. It is easy to see, however, that such an interpretation is misleading. Indeed, if the three-body threshold is not open, the bound state pole in the complex momentum plane must reside on the positive half of the imaginary axis (this corresponds to a pole below threshold on the real axis on the first Riemann sheet of the complex energy plane) in accordance with general principles of Quantum Mechanics. The fact that p_1 from Eq. (40) possesses a real part (and quite a large one!) can only be ascribed to a shortcoming of the low-order Padé approximation used in the calculations to produce spurious poles which, as a matter of principle, cannot be interpreted as observable objects. The latter statement can be given additional strong support based on the following argument: by changing the sign of the OPE, and thus making it repulsive, one would expect all physical poles to go away from the near-threshold region. Meanwhile, solutions similar to those from Eq. (39) continue to exist.

In addition to the issues already mentioned, the equations of Ref. [1] suffer from an incorrect treatment of three-body effects. First of all, in three-body systems there is typically a subtle cancellation between the imaginary parts that come from the single particle (D^*) self-energies embedded in the three-body system and those that come from the three-particle

$(D\bar{D}\pi)$ propagators [23]. In Ref. [1] the cuts related to the D^* self-energy are omitted altogether. Thus, the imaginary parts from the three-body effects are calculated in an inconsistent manner and are therefore incorrect.

In addition, since the recoil terms are dropped in the three-body propagators, the $D\bar{D}\pi$ three-body cut is effectively converted into a two-body cut — notice the singularity at $\mathbf{p}^2 = -\mu_\pi^2/4$ in the tree-level amplitude (7) as well as in the one-loop amplitude (35) (for a different reaction this is discussed in some detail in Ref. [24]). This pronounces the $D\bar{D}\pi$ singularity in the very near threshold regime way too strongly, since a two-body cut scales as \sqrt{E} while a three-body singularity scales as E^2 . This again shows that the treatment of the three-body dynamics is not correct in Ref. [1].

V. CONCLUSIONS

In this paper we demonstrated that the results of Ref. [1] suffer from technical as well as conceptual problems and therefore do not support the conclusions drawn by the authors. We pointed out several flaws of that paper.

First, we demonstrate that the separation of the short-ranged physics and the OPE which the authors of Ref. [1] dwell on at some length is not possible as a matter of principle. Thus the conclusions made in Ref. [1] that the OPE alone provides sufficient binding to produce the X as a shallow bound state at the $D^0\bar{D}^{*0}$ threshold and that the short-range dynamics (described by a contact interaction) provides only moderate corrections to the OPE are incorrect. As proven above, the OPE potential in the $D\bar{D}^*$ system is well defined in the sense of an effective field theory only in connection with a contact operator.

Second, based on results found for the two- and three-nucleon system, we argued that the low-order Padé approximation employed in Ref. [1] to construct the physical amplitude cannot be expected to be reliable.

Third, the equations were shown to produce a significant number of unphysical singularities very close to the threshold at least in the kinematic regime where the D^* is stable. We argue that the same should also happen in the physical regime where the D^* is unstable and that there is no reason to expect that the pole reported in Ref. [1] is physical.

Fourth, we argued that in Ref. [1] the three-body effects are treated incorrectly and inconsistently.

Therefore in this paper we have demonstrated that the results of Ref. [1] are based on an inconsistent and incomplete set of equations that produces a large number of unphysical singularities in the claimed range of applicability of the formalism. Instead of the formalism of Ref. [1] it appears more appropriate to solve the full scattering equation including the one-pion exchange either perturbatively [10] or non-perturbatively [15]. Not only do these formalisms not suffer from the mentioned inconsistencies, they also allow for a systematic, controlled study of the light-quark mass dependence of the $X(3872)$. While in Ref. [1] the $X(3872)$ is claimed to disappear to the second sheet unavoidably as soon as the light-quark masses are increased slightly, in both Ref. [17]

as well as Ref. [18] for perturbative and non-perturbative pions, respectively, it is demonstrated that field theoretic consistency demands the inclusion of a quark-mass-dependent contact term in the theory and it is the sign of that contact term that decides on the fate for the $X(3872)$ pole as the pion mass is increased. As a consequence, both Ref. [17] as well as Ref. [18] are consistent with the so far only existing lattice calculation for the $X(3872)$ [25], while Ref. [1] is not. It should be stressed that these findings of Refs. [17, 18] for the $D\bar{D}^*$ system are in full analogy to those for the nucleon–nucleon system [26–28].

To summarize, we have demonstrated that none of the claims of Ref. [1] listed in the introduction holds. Especially, under the assumption that the $X(3872)$ is a $D\bar{D}^*$ molecular state, the pion mass dependence of its pole position is expected to depend strongly on the pion mass dependence of the $D\bar{D}^*$ interaction at short range. Furthermore, a more deeply bound X -state for increased pion masses as found in Ref. [25] does not contradict a molecular nature of the X . One

might hope to eventually reveal important information on the structure of the potential responsible for the binding of the $X(3872)$, once its pion mass dependence is mapped out using lattice QCD from calculations along the lines of Ref. [28].

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