

The $c\bar{c}$ interaction above threshold and the radiative decay $X(3872) \rightarrow J/\psi\gamma$

A.M. Badalian* and Yu.A. Simonov†
Institute of Theoretical and Experimental Physics, Moscow, Russia

B.L.G. Bakker‡

Department of Physics and Astronomy, Vrije Universiteit, Amsterdam, The Netherlands
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Radiative decays of $X(3872)$ are studied in single-channel approximation (SCA) and in the coupled-channel (CC) approach, where the decay channels $D\bar{D}^*$ are described with the string breaking mechanism. In SCA the transition rate $\tilde{\Gamma}_2 = \Gamma(2^3P_1 \rightarrow \psi\gamma) = 71.8$ keV and large $\tilde{\Gamma}_1 = \Gamma(2^3P_1 \rightarrow J/\psi\gamma) = 85.4$ keV are obtained, giving for their ratio the value $R_{\psi\gamma} = \frac{\tilde{\Gamma}_2}{\tilde{\Gamma}_1} = 0.84$. In the CC approach three factors are shown to be equally important. First, the admixture of the 1^3P_1 component in the normalized wave function of $X(3872)$ due to the CC effects. Its weight $c_X(E_R) = 0.200 \pm 0.015$ is calculated. Secondly, the use of the multipole function $g(r)$ instead of r in the overlap integrals, determining the partial widths. Thirdly, the choice of the gluon-exchange interaction for $X(3872)$, as well as for other states above threshold. If for $X(3872)$ the gluon-exchange potential is taken the same as for low-lying charmonium states, then in the CC approach $\Gamma_1 = \Gamma(X(3872) \rightarrow J/\psi\gamma) \sim 3$ keV is very small, giving the large ratio $R_{\psi\gamma} = \frac{\mathcal{B}(X(3872) \rightarrow \psi(2S)\gamma)}{\mathcal{B}(X(3872) \rightarrow J/\psi\gamma)} \gg 1.0$. Arguments are presented why the gluon-exchange interaction may be suppressed for $X(3872)$ and in this case $\Gamma_1 = 42.7$ keV, $\Gamma_2 = 70.5$ keV, and $R_{\psi\gamma} = 1.65$ are predicted for the minimal value $c_X(\min) = 0.185$, while for the maximal value $c_X = 0.215$ we obtained $\Gamma_1 = 30.8$ keV, $\Gamma_2 = 73.2$ keV, and $R_{\psi\gamma} = 2.38$, which agrees with the LHCb data.

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I. INTRODUCTION

In 2003 the Belle collaboration discovered the $X(3872)$ as a narrow peak in the $J/\psi\pi\pi$ invariant mass distribution in the decays $B \rightarrow J/\psi\pi\pi K$ [1]. Now its characteristics, like the mass, the strict restriction on the width, $\Gamma \lesssim 1.2$ MeV, and the charge parity $C = +$, are well established [2–5]. In recent CDF and LHCb experiments the quantum numbers of $X(3872)$ were determined to be $J^{PC} = 1^{++}$ [6, 7]. Still, discussions about the nature of $X(3872)$ continue and to understand its exotic properties a special role is played by the radiative decays, $X(3872) \rightarrow J/\psi\gamma$ and $X(3872) \rightarrow \psi(3686)\gamma$, which are sensitive to the behavior of the $X(3872)$ wave function (w.f.) at medium and large distances.

The first evidence for the decay $X(3872) \rightarrow J/\psi\gamma$ was obtained by the Belle collaboration [8] and confirmed by the BaBar collaboration [9]. Later BaBar has also observed the radiative decay $X(3872) \rightarrow \psi(3686)\gamma$ and determined the branching ratio fraction $R_{\psi\gamma} = \frac{\mathcal{B}(X(3872) \rightarrow \psi(3686)\gamma)}{\mathcal{B}(X(3872) \rightarrow J/\psi\gamma)} = 3.4 \pm 1.4$ [10]. However, Belle has not found evidence for the radiative decay $X(3872) \rightarrow \psi(3686)\gamma$ and put an upper limit for the ratio [11],

$$R_{\psi\gamma} < 2.1. \quad (1)$$

Recently the LHCb group [12] has observed the decay $X(3872) \rightarrow \psi(3686)\gamma$ with a good statistics and determined its value to be

$$R_{\psi\gamma} = 2.46 \pm 0.64 \pm 0.29, \quad (2)$$

while in Ref. [13] the weighted average over three groups of measurements was determined to be $\bar{R}_{\psi\gamma} = 2.31 \pm 0.57$.

Unfortunately, theoretical predictions for the partial widths Γ_1 and Γ_2 of the radiative decays $X(3872) \rightarrow J/\psi\gamma$ and $X(3872) \rightarrow \psi(3686)\gamma$, respectively, vary widely in different models [14–22] (see also the recent reviews [23, 24]). If $X(3872)$ is considered as a pure 2^3P_1 charmonium state, then a large value $R_{\psi\gamma} \simeq 5$ is obtained as shown in

*Electronic address: badalian@itep.ru

†Electronic address: simonov@itep.ru

‡Electronic address: b.l.g.bakker@vu.nl

Refs. [16–19], exceeding the LHCb result. On the contrary, in a molecular picture the transition $X(3872) \rightarrow \psi(3686)\gamma$ is suppressed and the ratio $R_{\psi\gamma}$ should be much smaller [14, 15]. Also in many-channel models, both in the 3P_0 model [17, 18] and the Cornell model [19], large values $R_{\psi\gamma} \geq 5.0$ were predicted.

It is surprising that the predicted widths Γ_1 and Γ_2 strongly differ even within the single-channel approximation (SCA), i.e., when $X(3872)$ is supposed to be the $2\,{}^3P_1$ charmonium state [16–19]. It occurs because different parameters in the potentials and different kinematics are used. Therefore it remains unclear to what degree the theoretical predictions refer to specific features of the radiative decays or to the choice of fitting parameters and kinematics. Meanwhile, during the last decade within the field correlator method [25–27] and in lattice QCD [28, 29] the static potential was shown to be universal in the region $r \lesssim 1.5$ fm, where there is no creation of light $q\bar{q}$ pairs. Here we shall use this information and perform a parameter-free analysis of the charmonium states with the use of the relativistic string Hamiltonian (RSH) [25].

It was already underlined by Li and Chao [16] and in Ref. [24] that the use of relativistic kinematics is very important for the radiative decays and in nonrelativistic approximation the overlap integrals, determining the partial widths, may significantly differ from those in relativistic calculations.

In our previous paper [21] the rates and the ratio $R_{\psi\gamma}$ were calculated considering $X(3872)$ (with $J^{PC} = 1^{++}$) in the coupled-channels (CC) approach, where the coupling to the $D\bar{D}^*$ channels is determined by the relativistic string-breaking mechanism [20–22]. This mechanism was successfully applied to $X(3872)$, explaining it as the $2\,{}^3P_1$ charmonium state shifted down due to CC effects and appearing as a sharp peak just at the $D^0\bar{D}^{*0}$ threshold [20]. Owing to such CC effects the w.f. of $X(3872)$ acquires an admixture $c_X(E_R)$ from the $1\,{}^3P_1$ component, equal to ~ 0.20 . Although this mixing parameter is not very large, it nevertheless strongly affects the value of the partial width Γ_1 , in which the overlap integral is small and very sensitive to different corrections.

However, in Ref. [21] in the overlap integral $K_1 = \langle \varphi(2P)|g(r)|\varphi(1S) \rangle$ the multipole moment $g(r)$ was replaced by r , which is a common practice. In this approximation $\tilde{K}_1 = \langle \varphi(2P)|r|\varphi(1S) \rangle = 0.21$ has a small value, close to those given in Refs. [18, 24]. It appears that the magnitude of K_1 , defined using the multipole function $g(r)$, is a factor of two larger than \tilde{K}_1 , increasing the partial width Γ_1 by about four times. Thus the replacement of $g(r)$ by r cannot be applied to this radiative decay, proceeding with a large photon energy.

We also show that the integral K_1 strongly depends on the gluon-exchange (GE) part of the static potential and consider two possibilities: First, when in $X(3872)$ the GE potential is taken the same as for the low-lying states (below threshold). In this case the results are different in the SCA and in the CC approach. The second possibility implies that in higher charmonium states (above threshold), including $X(3872)$, the GE interaction is suppressed. For such a $c\bar{c}$ dynamics the partial width Γ_1 increases, while the partial width $\Gamma_2 = \Gamma(X(3872) \rightarrow \psi(3686)\gamma)$ weakly depends on the GE interaction.

We use here the CC approach, where the admixture from the continuum to the w.f. of $X(3872)$ is defined by the mixing parameter $c_X(E_R = 3.872 \text{ GeV})$, which is calculated here with a good accuracy: $c_X = 0.200 \pm 0.015$. Although its value is not large, it nevertheless gives an important contribution to Γ_1 .

Also in our analysis we lay the stress on the choice of the c -quark mass, the QCD constant $\Lambda(n_f = 3)$, and the string tension, which cannot be arbitrary parameters lest the physical picture would be distorted.

II. THE $c\bar{c}$ INTERACTION BELOW AND ABOVE THE THRESHOLD

Before studing the $X(3872)$ in the many-channel approach one needs to know its w.f. in the SCA. The rate of the electric dipole ($E1$) transition between an initial state i : $n\,{}^3P_1$ and a final state f : $m\,{}^3S_1$ is given by the expression [23, 24],

$$\Gamma(i \xrightarrow{\text{E1}} f + \gamma) = \frac{4}{3} \alpha e_Q^2 k_\gamma^3 (2J_f + 1) S_{if}^E |\mathcal{E}_{if}|^2, \quad (3)$$

which contains the overlap integral \mathcal{E}_{if} (called also the m.e.), calculated here via the relativistic w.f. The photon energy k_γ is defined as $k_\gamma = \frac{M_i^2 - M_f^2}{2M_i}$, where $M_i(M_f)$ is the mass of the initial (final) state. In Eq. (3) the statistical factor $S_{if}^E = S_{fi}^E$ is given by

$$S_{if}^E = \max(l, l') \left\{ \begin{array}{ccc} J & 1 & J' \\ l' & s & l \end{array} \right\}^2, \quad (4)$$

which for the $E1$ transitions between the $n\,{}^3P_J$ and $m\,{}^3S_1$ ($m\,{}^3D_1$) states, with the same spin $S = 1$, is equal to $\frac{1}{9}$ ($\frac{1}{18}$).

The overlap integral \mathcal{E}_{if} , given by

$$\mathcal{E}_{if} = \int dr r^2 R_{n_i, l_i}(r) g(r) R_{n_f, l_f}(r) = \langle n_i, l_i | g(r) | n_f, l_f \rangle, \quad (5)$$

is determined by the relativistic radial w.f. $R_{n_i, l_i}(r)$ and $R_{n_f, l_f}(r)$. The function $g(r)$ [24] is:

$$g(r) = \frac{3}{k_\gamma} [y j_0(y) - j_1(y)] = \frac{3}{2} \frac{r}{y} \left\{ \sin y \left(1 - \frac{1}{y^2} \right) + \frac{1}{y} \cos y \right\}, \quad (6)$$

where $j_n(y)$ ($n = 1, 2$) are the spherical Bessel functions and the variable $y = k_\gamma r/2$. It can be easily shown that the replacement of $g(r)$ by r ($g(r) < r$), used in many analyses, is a good approximation for $y \leq 1.0$, while in the range, $1.1 \leq y \leq 1.3$, the difference $g(r) - r$ can already reach $\leq 30\%$. The zero of the function $g(r)$ occurs at $y_0 = 2.74$, corresponding to very large r , $r \gtrsim 10 \text{ GeV}^{-1}$, even for the photon energy, $k_\gamma = 0.697 \text{ GeV}$ for the $X(3872) \rightarrow J/\psi \gamma$ radiative decay. In r -space the difference $g(r) - r$ remains small for photon energies $k_\gamma \lesssim 0.40 \text{ GeV}$.

The overlap integrals, involving the $1P$ and $2P$ states, are denoted below as

$$\begin{aligned} I_m &\equiv I_{1P, mS} = \int dr r^2 R_{1P}(r) g(r) R_{mS}(r), \quad (m = 1, 2), \\ K_m &\equiv K_{2P, mS} = \int dr r^2 R_{2P}(r) g(r) R_{mS}(r), \quad (m = 1, 2). \end{aligned} \quad (7)$$

In Eq. (7) the relativistic radial w.f. are calculated with the use of the RSH H_0 , derived in Ref. [25] and applied many times to different effects in quarkonia [30, 31]. For heavy quarkonia H_0 has a very simple form:

$$H_0 = 2\sqrt{\mathbf{p}^2 + m_c^2} + V_B(r), \quad (8)$$

where its kinetic term is similar to that in the relativized quark model (RQM) [32]. However, in H_0 , by derivation, the fundamental value of the c -quark mass is equal to its pole mass, $m_c \simeq 1.42 \text{ GeV}$. (It takes into account corrections perturbative in $\alpha_s(m_c)$ and corresponds to the conventional current mass $\bar{m}_c(\bar{m}_c) \sim 1.23 \text{ GeV}$ [33]). In the RQM the c -quark mass is considered as a fitting parameter, e.g., $m_c = 1.628 \text{ GeV}$ is used in Ref. [32] and $m_c = 1.562 \text{ GeV}$ in Ref. [17].

In the RSH the universal static potential $V_B(r)$ contains the confining and GE terms,

$$V_B(r) = \sigma r - \frac{4\alpha_B(r)}{3r}, \quad (9)$$

where the string tension $\sigma = 0.18 \text{ GeV}^2$ is fixed by the slope of the Regge trajectories for light mesons and therefore cannot be considered as a fitting parameter.

An important step is to make the correct choice of the vector coupling $\alpha_B(r)$ [34]. To this end, the parameters defining the coupling, are here taken in correspondence to the existing data for the strong coupling $\alpha_s(r)$ in the \overline{MS} scheme in perturbative QCD [35]. The asymptotic-freedom behavior of α_B is determined by the “vector” QCD constant Λ_B , which, however, is not a new parameter but expressed through $\Lambda_{\overline{MS}}$. In particular, for $n_f = 3$ the relation $\Lambda_B(n_f = 3) = 1.4753 \Lambda_{\overline{MS}}(n_f = 3)$ is valid. In pQCD

$$\Lambda_{\overline{MS}}(n_f = 3) = 339 \pm 10 \text{ MeV}, \quad (10)$$

was extracted with the use of the matching procedure at the quark-mass thresholds [35]. This value $\Lambda_{\overline{MS}}(n_f = 3) = (339 \pm 10) \text{ MeV}$ gives a rather large value of $\Lambda_B(n_f = 3) = (500 \pm 15) \text{ MeV}$, while in Ref. [34] from the analysis of the bottomonium spectrum, a smaller $\Lambda_B(n_f = 3) = (465 \pm 20) \text{ MeV}$ was shown to be preferable; its value corresponds to $\Lambda_{\overline{MS}}(n_f = 3) = (315 \pm 14) \text{ MeV}$, in agreement with the lower limit in Eq. (10).

Also in Ref. [34] the infrared regulator M_B was introduced, since for $n_f = 3$ the characteristic momenta q^2 are rather small and nonperturbative effects become important. As shown in Ref. [36], the regulator M_B is not a new parameter, but expressed via the string tension σ , according to the relation $M_B^2 = 2\pi\sigma$, derived with an accuracy $\sim 10\%$. Then for $\sigma = 0.18 \text{ GeV}^2$ one finds that $M_B = 1.06 \pm 0.10 \text{ GeV}$; here the value $M_B = 1.10 \text{ GeV}$ is used.

Thus the potential $V_B(r)$ contains only fundamental quantities: the current (pole) c -quark mass, the string tension from the Regge trajectories, and $\Lambda_{\overline{MS}}(n_f = 3)$. In our calculations the following set of parameters is used:

$$m_c = 1.425 \text{ GeV}, \quad \sigma = 0.18 \text{ GeV}^2, \quad M_B = 1.10 \text{ GeV}, \quad \Lambda_B(n_f = 3) = 465 \text{ MeV}. \quad (11)$$

For these values of $\Lambda_B(n_f = 3)$ and M_B we find the frozen (asymptotic) value of the coupling $\alpha_{\text{crit}} = 0.6086$.

A remarkable property of the RSH is that in heavy quarkonium the centroid masses $M_{\text{cog}}(nl)$ of the nl multiplet exactly coincide with the eigenvalues of the spinless Salpeter equation (SSE):

$$H_0 \varphi_{nl} = M_0(nl) \varphi_{nl}. \quad (12)$$

In the charmonium spectrum not many levels are below the open charm threshold, where the universal static potential $V_B(r)$ may be tested. However, above the $D\bar{D}$ threshold the $c\bar{c}$ interaction may differ from that for low-lying states, due to open channels or light-quark pair creation. In the string-breaking picture it is manifested as a flattening of the static potential at relatively large distances and this phenomenon is seen in lattice QCD [37] and also observed in the radial excitations of light mesons [38]. For that reason for higher states (above threshold) we consider two possibilities: the first one, when the GE potential is taken the same as in Eq. (9) and the second case, when the GE interaction is supposed to be suppressed for higher states, so that the $c\bar{c}$ dynamics is totally determined by the confining potential. There are several arguments in favor of the latter assumption:

1. Suppression of the GE interaction follows from the analysis of the orbital Regge trajectories in charmonium [39, 40], which are not compatible with a significant GE contribution to the masses of higher states.
2. The fine-structure effects, caused by the GE interaction, are not seen for the $2P$ charmonium multiplet. For example, the measured mass difference $\delta M(2P) = M(\chi_{c2}(2P)) - M(\chi_{c0}(2P)) \leq 10$ MeV, is very small compared to the same mass difference for the $1P$ multiplet: $\delta M(1P) = M(\chi_{c2}(1P)) - M(\chi_{c0}(1P)) = 142$ MeV [33], being smaller by an order of magnitude. It is difficult to explain such a suppression of the fine-structure effects, if a universal GE potential is used both for high- and low-lying states. (The suppression of the fine-structure effects does not contradict possible hyperfine splittings of the higher S -wave charmonium states due to a short-range hyperfine potential [27]).
3. Our calculations of the charmonium spectrum show that the masses of higher states are practically the same for the linear and linear+GE potentials, if the correct choice of the c -quark mass is made (see below Table I).

The confining potential used

$$V_C(r) = \sigma r, \quad (13)$$

has the same string tension $\sigma = 0.18$ GeV². However, for the linear potential the c -quark mass may be different, since for a totally suppressed GE interaction there are no perturbative corrections to the c -quark mass and in this case m_c is equal to the current mass (here $m_c(\bar{m}_c) = (1.27 - 1.29)$ GeV is used).

In Table I we give the centroid masses $M_{\text{cog}}(nl)$ for both potentials, Eqs. (7) and (9).

TABLE I: The centroid masses $M_{\text{cog}}(nl)$ (in MeV) of higher multiplets for the potentials $V_B(r)$ and $V_C(r)$

State	Potential $V_B(r)$	Potential $V_C(r)$	experiment
	$m_c = 1.425$ GeV	$m_c = 1.290$ GeV	
3S	4092	4112	4039 ± 1
4S	4447	4448	4424 ± 4
2P	3949	3949	$3927 \pm 3^a)$
3P	4319	4298	absent
1D	3802	3788	3773 ± 3
2D	4187	4155	4153 ± 3

^{a)} For $M_{\text{cog}}(2P)$ the experimental mass $M(\chi_{c2}(2^3P_2))$ is given.

From Table I one can see that both spectra coincide within $\lesssim 20$ MeV accuracy. Moreover, the masses $M(1D)$, $M(2D)$, $M(2P)$, and $M(4S)$, calculated in SCA, are close to the experimental masses of $\psi(3770)$, $\psi(4160)$, $\chi_{c2}(2P)$, and $\psi(4420)$, respectively, and one can expect that their hadronic shifts are not large, $\lesssim 20$ MeV. It is not so for the $3S$ state, where $M(3S)$ is about (60 ± 10) MeV larger than $M(\psi(4040))$, indicating the importance of the CC effects for $\psi(4040)$.

Thus our analysis of the charmonium spectrum does not allow to draw a definite conclusion what type of the $c\bar{c}$ interaction is preferable above threshold. Therefore other phenomena in charmonium have to be studied. One of them is just the radiative decay $X(3872) \rightarrow J/\psi\gamma$, which, as was already underlined by E. Swanson [15], “is particularly sensitive to model details”.

III. THE RADIATIVE DECAYS OF $X(3872)$ IN SINGLE-CHANNEL APPROXIMATION

Here we calculate the partial widths of the radiative transitions, $X(2^3P_1) \rightarrow J/\psi\gamma$ and $X(2^3P_1) \rightarrow \psi(2S)\gamma$ in SCA, i.e., assuming that $X(3872)$ is the 2^3P_1 state. Then the widths (in units GeV) can be written as

$$\begin{aligned}\Gamma_1^0 &= 1.4418 \cdot 10^{-3} k_{1\gamma}^3 |K_1|^2, \\ \Gamma_2^0 &= 1.4418 \cdot 10^{-3} k_{2\gamma}^3 |K_2|^2,\end{aligned}\quad (14)$$

where the number $1.4418 \cdot 10^{-3}$ is the product of known factors in Eq. (3), $k_{1\gamma} = 0.6974$ GeV, $k_{2\gamma} = 0.1815$ GeV, and for $X(3872)$ as the 2^3P_1 charmonium state, the integrals K_1 , K_2 (in the units GeV^{-1}) are defined in Eq. (7).

From Eq. (7) and for the potential $V_B(r)$ we obtained

$$K_1 = K_{2P,1S} = -0.418 \text{ GeV}^{-1}, \quad (15)$$

which in SCA gives a large partial width:

$$\Gamma_1 = 85.4 \text{ keV}, \quad (16)$$

This integral K_1 , defined with $g(r)$, has a magnitude that is two times larger than $\tilde{K}_1 = \langle R_{2P}|r|R_{1S} \rangle$, where $g(r)$ is approximated by r :

$$\tilde{K}_1 = -0.210 \text{ GeV}^{-1}, \quad \tilde{\Gamma}_1 = 21.6 \text{ keV}. \quad (17)$$

Notice, that even smaller magnitudes of \tilde{K}_1 were obtained in other models [17], [21] (see Table II). (The choice of the negative sign of K_1 will be explained in Sect. V). Thus, in the overlap integral determining the radiative decay $X(2^3P_1) \rightarrow J/\psi\gamma$ with rather large photon energy, $k_\gamma = 0.697$ GeV, the replacement of $g(r)$ by r has a poor accuracy. On the contrary, the differences between the overlap integral K_2 and the width Γ_2 ,

$$K_2 = \langle 2P|g(r)|2S \rangle = 2.886 \text{ GeV}^{-1}, \quad \Gamma_2 = 71.8 \text{ keV}, \quad (18)$$

and the approximate quantities \tilde{K}_2 and $\tilde{\Gamma}_2$

$$\tilde{K}_2 = \langle 2P|r|2S \rangle = 3.017 \text{ GeV}^{-1}, \quad \tilde{\Gamma}_2 = 78.5 \text{ keV} \quad (19)$$

are rather small, $\lesssim 8\%$. Then, because of the small value of $\tilde{\Gamma}_1$, the ratio of the partial widths $\tilde{\Gamma}_2$ to $\tilde{\Gamma}_1$ is large,

$$\tilde{R}_{\psi\gamma} = 3.6. \quad (20)$$

On the contrary, if the integrals K_1 and K_2 are calculated with $g(r)$: $\Gamma_1 = 85.4$ keV, $\Gamma_2 = 71.8$ keV, then

$$R_{\psi\gamma} = 0.84, \quad (21)$$

is four times smaller. This ratio is not compatible with the BaBar [9] and the LHCb results, Eqs. (2), but does not contradict the Belle restriction, $R_{\psi\gamma} \lesssim 2.1$.

In Table II we give the partial widths Γ_1 and Γ_2 , predicted in several relativistic models. One can see that in the SCA, even with relativistic kinematics, the predicted values of Γ_1 vary in a wide range, (11 – 85) keV.

Surprisingly, the m.e. K_1 in Eq. (15) and Γ_1 in Eq. (16) calculated here with a relativistic Hamiltonian, practically coincide with the approximate m.e. $\langle 2P|r|1S \rangle = 0.416 \text{ GeV}^{-1}$ and $\Gamma_1 = 84.6$ keV, calculated in a nonrelativistic model with the logarithmic potential in Ref. [42]; the predicted value found there $R_{\psi\gamma} = 1.1$ is also close to our number in Eq. (21).

As a test of the w.f., defined by the RSH and used here, we also calculated the partial widths of the $E1$ transitions: $\chi_{cJ}(1P) \rightarrow J/\psi\gamma$ ($J = 0, 1, 2$), where all three states lie far below the $D\bar{D}$ threshold and are described by the linear+GE potential. For $\chi_{c1}(1P) \rightarrow J/\psi\gamma$ and $\chi_{c2}(1P) \rightarrow J/\psi\gamma$ the calculated widths appeared to be in precise agreement with the experimental data. For these radiative decays the calculated integral $I_{1P,1S} \equiv I(1^3P_{cJ}, 1S) = 1.849 \text{ GeV}^{-1}$ is taken the same for $J = 0, 1, 2$. For $\psi(3686)$ its w.f. contains an admixture from the $1D$ state [43] and therefore this radiative decay cannot be used as a test.

For all three radiative transitions the photon energies are not large, $k_\gamma \lesssim 0.40$ GeV, and therefore the differences between the m.e. $\langle g(r) \rangle$ and $\langle r \rangle$ are rather small, $\lesssim 7\%$. Nevertheless, with the use of the function $g(r)$ the partial widths of those radiative decays decrease by $\sim 10\%$, improving the agreement with experiment.

From Table III one can see that the calculated values $\Gamma(\chi_{c0}(1P) \rightarrow J/\psi\gamma) = 137$ keV, $\Gamma(\chi_{c1}(1P) \rightarrow J/\psi\gamma) = 290$ keV and $\Gamma(\chi_{c2}(1P) \rightarrow J/\psi\gamma) = 390$ keV are in precise agreement with the experimental widths: $\Gamma_{\text{exp}}(\chi_{c0} \rightarrow J/\psi\gamma) = 133 \pm 10$ keV, $\Gamma_{\text{exp}}(\chi_{c1} \rightarrow J/\psi\gamma) = 296 \pm 13$ keV, and $\Gamma_{\text{exp}}(\chi_{c2} \rightarrow J/\psi\gamma) = 386 \pm 17$ keV [33].

TABLE II: The partial widths of the $X(3872)$ radiative decays (in keV) in relativistic models ^{a)}

Transition	k_γ	[41]	[16]	[17]	[18]	our paper	our paper
Method	in GeV	SCA	CCM	SCA	SCA	SCA	CCM
$2^3P_1 \rightarrow 1^3S_1 + \gamma$	0.697	33	45	11	11	85.4	37 ± 6
$2^3P_1 \rightarrow 2^3S_1 + \gamma$	0.181	146	60	70	63.9	71.8	72 ± 1
$2^3P_1 \rightarrow 1^3D_1 + \gamma$	0.098	7.9		4.0	3.7	5.8	

^{a)} The partial widths are calculated either in single-channel approximation (SCA) or within a coupled-channels method (CCM).

TABLE III: The overlap integrals and the partial widths of the radiative decays $\chi_{cJ}(1P) \rightarrow J/\psi\gamma$.

Transition	k_γ (GeV)	$\langle r \rangle$ (GeV $^{-1}$)	$\langle g(r) \rangle^a)$ (GeV $^{-1}$)	$\Gamma(\text{th})$	$\Gamma(\text{exp})$
$1^3P_{c2} \rightarrow J/\psi + \gamma$	0.429	1.976	1.849	390	386 ± 17
$1^3P_{c1} \rightarrow J/\psi + \gamma$	0.389	1.976	1.849	290	296 ± 13
$1^3P_{c0} \rightarrow J/\psi + \gamma$	0.303	1.976	1.849	137	133 ± 10

^{a)} Calculations refer to the variant with $m_c = 1.425$ GeV, $\Lambda_B(n_f = 3) = 465$ MeV, and $M_B = 1.1$ GeV.

IV. DYNAMICS OF COUPLED CHANNELS AND AN ADMIXTURE OF NEIGHBORING STATES

In the CC approach $X(3872)$ is treated as the original 2^3P_1 ($Q\bar{Q}$) state, shifted down by the CC interaction with the $D\bar{D}^*(\bar{D}\bar{D}^*)$ channel. Moreover, in Ref. [20] explicit parameters of this interaction were found, which put the pole of $X(3872)$ in the multi-channel Green's function exactly at the position of the open channel threshold. It is also important that within our approach the parameters are calculated and found to lie in a narrow range, i.e., they are not fitted. Below we use these fixed parameters to predict the admixture generated by the CC interaction, to the original 2^3P_1 state.

Leaving the details of the CC formalism to the Appendix and the original papers [20–22], we recapitulate briefly the essence of the mixing problem. We define the extra (open)-channel interaction by the function $V_{CC}(\mathbf{q}, \mathbf{q}', E)$ in momentum space, which depends on the total energy E and contains all thresholds of the open channels. Then we are writing the full Green's function $G_{Q\bar{Q}}^{(BB)}$, which describes the evolution of a $Q\bar{Q}$ pair (produced in the e^+e^- annihilation or the B decay process), interacting in the intermediate states with $D\bar{D}^*(\bar{D}\bar{D}^*)$ and finally ending up again as a $Q\bar{Q}$ pair (in a similar e^+e^- annihilation or the B decay process), as

$$G_{Q\bar{Q}}^{(BB)} = \sum_{n,m} (\hat{B}\psi_n) \frac{1}{\hat{E} - E + \hat{w}} (\hat{B}\psi_m)^\dagger, \quad (22)$$

where the matrices are defined as $\hat{E} = E_n \delta_{nm}$, and $\hat{w} = w_{nm}$ with

$$w_{nm}(E) = \int \psi_n^\dagger(\mathbf{q}) V_{cc}(\mathbf{q}, \mathbf{q}', E) \psi_m(\mathbf{q}') \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{d^3\mathbf{q}'}{(2\pi)^2}. \quad (23)$$

Here the symbol \hat{B} in $(\hat{B}\psi_n)$ stands for an operator, defining the production (annihilation) process. In the case when the pure $Q\bar{Q}$ state is already formed, one can put $\hat{B} \equiv 1$, while in the e^+e^- annihilation process the operator \hat{B} is proportional to γ_μ .

Note, that $G_{Q\bar{Q}}^{(BB)}$ contains all CC contributions in the intermediate states, but the experimental conditions require these contributions to be projected onto the $Q\bar{Q}$ states. We now include in $G_{Q\bar{Q}}^{(BB)}$ the possibility to emit any particle (γ, π, \dots) in the intermediate state. As shown in Ref. [21] (see Appendix), this can be done inserting the self-energy part with the photon loop to \hat{E} and \hat{w} : $\hat{E} + \hat{w} \rightarrow \hat{E}(\gamma) + \hat{w}(\gamma)$, so that the probability of this emission is given by the

discontinuity of the self-energy part of $G_{Q\bar{Q}}^{(BB)}$, e.g.

$$\begin{aligned} Y_\gamma(E) &= \frac{1}{2} \Delta G_{Q\bar{Q}}^{(BB)} \\ &= \sum_{n,m,l} (\hat{B}\psi_n) \left(\frac{1}{\hat{E} - E + \hat{w}} \right)_{nm} \frac{1}{2i} \Delta(\hat{E}(\gamma) + \hat{w}(\gamma))_{ml} \left(\frac{1}{\hat{E} - E + \hat{w}} \right)_{lq} (\hat{B}\psi_q)^\dagger \\ &= \sum_{k,n} a_k \psi_k \frac{1}{2i} \Delta(\hat{E} + \hat{w})_{kn} \psi_n^\dagger a_n^\dagger. \end{aligned} \quad (24)$$

In this way one can start in e^+e^- , or the B decay, with the creation of any combination $\{\hat{B}\psi_n\}$, which finally, at the moment of the γ detection, becomes the set of the states $(\psi_k + \gamma)$ with the probability amplitude $a_k(\hat{B}\psi_n)$.

The amplitude a_k is given by

$$a_k = \sum_l \left(\frac{1}{\hat{E} - E + \hat{w}} \right)_{kl} (B\psi_l). \quad (25)$$

Since \hat{E} is diagonal, $\hat{E}_{ik} = E_i \delta_{ik}$, all matrix structure is due to \hat{w} . This matrix was calculated in Ref. [21] for the 1^{++} states, giving the values shown in the Table IV in the Appendix. here one can see, that only two 1^{++} states are strongly connected by w_{nm} , namely 1^3P_1 and 2^3P_1 , while higher states yield very small admixtures.

Defining the amplitudes a_1 and a_2 for the 1^3P_1 and 2^3P_1 states, respectively, one has

$$\frac{a_1}{a_2} = \frac{(E_2 + w_{22} - E) \frac{B\psi_1}{B\psi_2} - w_{21}}{E_1 + w_{11} - E - w_{12} \frac{B\psi_1}{B\psi_2}}. \quad (26)$$

With the use of the RSH we have $E_2 = 3949 \pm 10$ MeV, $E_1 = 3523 \pm 10$ MeV and then from Table III

$$E_2 + w_{22} = (3850 \pm 10) \text{ MeV}, \quad E_1 + w_{11} = (3203 \pm 10) \text{ MeV}. \quad (27)$$

From Eq. (26) one can see that the solution for a_1/a_2 is sensitive to the original value of E_2 , but depends weakly on $B\psi_1/B\psi_2$ and we take for this ratio the same value as for the e^+e^- process, namely $|\psi_1(0)|^2/|\psi_2(0)|^2 \approx 0.85$. Inserting these numbers into Eq. (26) and taking $E_2 = 3949$ MeV from Table I (the contribution from the D^*D^* channel is neglected) one obtains the value of c_X at the resonance energy, $E_R = 3872$ MeV ($B\psi_1/B\psi_2 = 0$):

$$c_X(E_R) = \frac{a_1}{a_2}(E = 3872 \text{ MeV}) = 0.183. \quad (28)$$

However, this number can be considered as a lower limit, since we have not accounted for the contribution of the closed $D^*\bar{D}^*$ channel to the original levels E_1 and E_2 . The estimate of these $D^*\bar{D}^*$ contributions to the $2^{++}(2^3P_2)$ and $0^{++}(2^3P_0)$ states was made in Ref. [20] for the same CC parameters as for $X(3872)$. It yields a shift around -30 MeV, in good agreement with the $Z(3930)$ as the 2^3P_2 state. In this case in Eq. (28) $E_2 = 3920$ MeV and a larger value of c_X is obtained,

$$c_X = \frac{a_1}{a_2}(E = 3872 \text{ MeV}) \approx 0.215 \quad (29)$$

As a result, we chose to accept the value of c_X from the range, defined by Eqs. (28) and (29)

$$c_X = 0.200 \pm 0.015. \quad (30)$$

V. THE COUPLED-CHANNELS CORRECTIONS TO THE $X(3872)$ WAVE FUNCTION

In the previous section the effects from open $D\bar{D}^*$ channels were calculated, resulting in the contribution from the 1^3P_1 state to the normalized w.f. of $X(3872)$, which can be represented as

$$\varphi(X(3872)) = \sqrt{1 - c_X^2} \varphi(2^3P_1) + c_X \varphi(1P^3P_1). \quad (31)$$

As seen from Eq. (30) the mixing parameter c_X at the energy $E_R = 3.782$ GeV lies in a narrow range; this admixture was not fitted, but calculated within the string breaking mechanism.

Moreover, in the many-channels mechanism the sign of the continuum component cannot be arbitrary anymore and here we choose positive c_X : it takes place if the radial w.f. $R_{2P}(r)$ is chosen to be negative at small distances (for $r \lesssim 0.5$ fm), while the radial w.f. of the ground state $R_{1P}(r)$ is positive in the whole region. Just for this choice a negative value for the overlap integral in Eqs. (15, 17) was obtained.

Due to the CC admixture in the $X(3872)$ w.f., the m.e. in the radiative decays of $X(3872)$ can be presented as a superposition,

$$\begin{aligned}\langle X(3872)|g(r)|J/\psi\rangle &= \sqrt{1 - c_X^2} K_1 + c_X I_1, \\ \langle X(3872)|g(r)|\psi(2S)\rangle &= \sqrt{1 - c_X^2} K_2 + c_X I_2,\end{aligned}\quad (32)$$

where I_n and K_n ($n = 1, 2$) are defined in Eq. (7) and the radial w.f. of $\psi(3686)$ is taken as $R(2S)(r)$, i.e., without mixing with the $1D$ state.

For $X(3872)$ two different $c\bar{c}$ potentials are used below. In first case the w.f. of all states involved in the radiative decays of $X(3872)$, are defined by the RSH with the linear+GE potential, $V_B(r)$, given in Eq. (9). Then in the m.e. Eq. (32) the two terms have different signs and therefore the magnitude of this m.e. is smaller as compared to the number given in Eq. (15). Due to this cancellation this m.e. appears to be very small, even for the minimal value of $c_X(\min) = 0.185$ from Eq. (28):

$$|\langle X(3872, 2^3P_1)|g(r)|J/\psi\rangle| \leq 0.08 \text{ GeV}^{-1}, \quad (33)$$

giving a small partial width $\Gamma_1 = 3.0$ keV.

On the contrary, for the transition $X(3872) \rightarrow \psi(2S)\gamma$ its m.e. increases to 3.30 GeV^{-1} and $\Gamma_2 = 94$ keV is large. Thus, within the CC approach, taking the linear+GE potential for $X(3872)$, one obtains a small partial width $\Gamma_1 \lesssim 3$ keV and very large value for the ratio $R_{\psi\gamma} \gg 1.0$.

The situation is different, if we assume that the GE interaction is totally suppressed for higher charmonium states, i.e., the $c\bar{c}$ dynamics is determined by the confining potential. (Note that such a suppression does not exclude the use of the hyperfine interaction, having a very small range [28], for the higher S -wave charmonium states, like $\psi(4040)$, concentrated near the origin). As a result, the m.e. of the radiative transitions, $X(3872) \rightarrow J/\psi\gamma$ and $X(3872) \rightarrow \psi(3686)\gamma$, are defined by the “mixed” m.e., in which the $2P$ w.f. is calculated with the confining potential, while the w.f. of J/ψ and $\psi(3686)$ are calculated with the linear+GE potential.

These m.e. are denoted as,

$$\begin{aligned}K_1(\text{mix}) &= \int dr r^2 R_{2P}(\text{lin}) |g(r)| R_{1S}(\text{lin + GE}), \\ K_2(\text{mix}) &= \int dr r^2 R_{2P}(\text{lin}) |g(r)| R_{2S}(\text{lin + GE}).\end{aligned}\quad (34)$$

It appears that the magnitude of the m.e. $K_1(\text{mix})$ is larger compared to the number given in Eq. (15) and for the un-regularized w.f. (the solutions of the SSE in Eq. (12)) the m.e. are

$$K_1(\text{mix}) = -0.693 \text{ GeV}^{-1}, \quad K_2(\text{mix}) = 2.053 \text{ GeV}^{-1}. \quad (35)$$

Instead, below we use the m.e. $K_i(\text{mix})$ ($i = 1, 2$), calculated for the regularized w.f. with the use of the Einbein approximation:

$$K_1(\text{mix}) = -0.664 \text{ GeV}^{-1}, \quad K_2(\text{mix}) = 2.475 \text{ GeV}^{-1}. \quad (36)$$

For the calculated $K_1(\text{mix})$ the partial width Γ_1 strongly depends on the value of the mixing parameter c_X , even if it is taken from the narrow range given in Eq. (30). Using the m.e. I_m (which refers to the $1P$ state) from Table III and $K_1(\text{mix})$ from Eq. (36), one has

$$\Gamma_1 = \Gamma(X(3872) \rightarrow J/\psi\gamma) = (37 \pm 6) \text{ keV for } c_X = 0.200 \pm 0.015. \quad (37)$$

Owing to the CC effects the value of Γ_1 appears to be ~ 2.2 times smaller than in SCA. While changing the admissible value of c_X from the minimal to the maximum values, i.e., by 15%, the partial width Γ_1 increases from 31 keV to 43 keV, i.e., 1.4 times. It means that a precise knowledge of Γ_1 from experiment would put a strong restriction on the mixing parameter c_X at the resonance energy $E_R = 3.872$ GeV.

For the radiative decay $X(3872) \rightarrow \psi(3686)\gamma$ we have used the radial w.f. $R_{2S}(r)$ for $\psi(3686)$. Then taking the m.e. $I_{2S,1P}$ from Table III and $K_2(\text{mix})$ from Eq. (36), one obtains

$$\Gamma_2 = (72 \pm 1.0) \text{ keV for } c_X = 0.200 \pm 0.015. \quad (38)$$

Its value weakly depends on the mixing parameter c_X . However, this width may be smaller, if the $2S - 1D$ mixing is taken into account in the w.f. of $\psi(3686)$ [43].

Thus in the CC approach with the minimal value of the mixing parameter, $c_X = 0.185$, we have

$$\Gamma_1 = 42.7 \text{ keV}, \quad \Gamma_2 = 70.5 \text{ keV}, \quad R_{\psi\gamma} = 1.65, \quad (39)$$

while for the maximal value, $c_X = 0.215$,

$$\Gamma_1 = 30.8 \text{ keV}, \quad \Gamma_2 = 73.2 \text{ keV}, \quad R_{\psi\gamma} = 2.38, \quad (40)$$

i.e., this partial width Γ_1 is by 40% smaller than for $c_X(\text{min}) = 0.185$. The ratio of the widths also changes in a rather wide range:

$$R_{\psi\gamma}(\text{th}) = 2.02 \pm 0.36, \quad (41)$$

being 2.4 times larger than that in the SCA Eq. (20). This value is in a good agreement with the LHCb data given in Eq. (2). Also its lower limit (for $c_X = 0.185$) does not contradict the Belle result, $R_{\psi\gamma} < 2.1$.

Thus, in our analysis the agreement with the LHCb result was obtained under two conditions: when the GE interaction is suppressed and the CC admixture to the w.f. of $X(3872)$ is not small, $\sim 20\%$. For the partial width Γ_1 we predict the value 37 ± 6 keV.

VI. CONCLUSIONS

The $c\bar{c}$ dynamics in $X(3872)$ with $J^{PC} = 1^{++}$ was studied within the CC approach, where a coupling to the $D\bar{D}^*$ channels is determined by the string-breaking mechanism and the mixing parameter $c_X(E = E_R) = 0.200 \pm 0.015$ was calculated. Two different variants of the $c\bar{c}$ interaction were considered for the higher charmonium states.

1. The linear+GE potential used is the same for the lower and the higher charmonium states. In this case the partial width $\Gamma_1 = \Gamma(X(3872) \rightarrow J/\psi\gamma) \sim 3.0$ keV is small and gives rise to a very large value of the ratio $R_{\psi\gamma} \gg 1.0$ in the CC approach. On the contrary, in SCA the ratio $R_{\psi\gamma} = 0.84$ is not large.
2. The GE interaction is supposed to be suppressed for higher states. For this dynamics the charmonium spectrum above the $D\bar{D}$ threshold can be described with a good accuracy, if perturbative corrections in the c -quark mass are neglected.

If the suppressed GE potential is used, then the partial width Γ_1 increases, being very sensitive to the value of the mixing parameter c_X . For $c_X = 0.215$ we find $\Gamma_1 = 30.8$ keV, $\Gamma_2 = 73.2$ keV, and the calculated ratio of the partial widths $R_{\psi\gamma}(\text{max}) = 2.38$ is in good agreement with the LHCb result. The predicted minimal value, $R_{\psi\gamma}(\text{min}) = 1.65$, does not contradict the Belle measurements with $R_{\psi\gamma}(\text{Belle}) < 2.1$.

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Appendix A: The coupled-channel mechanism

Here we essentially use the formalism, suggested in Ref. [44] and developed further in Refs. [20, 21], and below we partly reiterate the material from Ref. [21] for the convenience of the reader.

We use here the string-decay Lagrangian of the 3P_0 type for the decay $c\bar{c} \rightarrow (cq)(\bar{c}\bar{q})$ [20]:

$$\mathcal{L}_{\text{sd}} = \int d^4x \bar{\psi}_q M_\omega \psi_q, \quad (\text{A1})$$

where the light quark bispinors are treated in the limit of large m_c , which allows us to go over to the reduced (2×2) form of the decay matrix elements. Moreover, to simplify the calculations the actual w.f. of the $c\bar{c}$ states, calculated in Ref. [21] with the use of the RSH, are fitted here by five (or three) oscillator w.f. (HO), while the D meson w.f. is described by a single HO term, which provides a few percent accuracy with the parameter $\beta_2 \simeq 0.48$. In this case the factor M_ω in (A1) is calculated to be $M_\omega = 4\sqrt{2}\sigma/\sqrt{\pi}\beta_2 \simeq 0.8$ GeV (see below), which produces the correct total width of $\psi(3770)$. The transition m.e. for the decays $(c\bar{c})_n \rightarrow (D_{n_2} \bar{D}_{n_3})$ are denoted here as $n \rightarrow n_2, n_3$, and in the 2×2 formalism this m.e. reduces to

$$J_{nn_2n_3}(\mathbf{p}) = \frac{\gamma}{\sqrt{N_c}} \int \bar{y}_{123} \frac{d^3\mathbf{q}}{(2\pi)^3} \Psi_n^+(\mathbf{p} + \mathbf{q}) \psi_{n_2}(\mathbf{q}) \psi_{n_3}(\mathbf{q}). \quad (\text{A2})$$

The factor \bar{y}_{123} contains a trace of the spin-angular variables (for details see Refs. [20, 21]), and $\gamma = \frac{2M_\omega}{\langle m_q + U_S - V_D + \varepsilon_0 \rangle}$, where the average over the Dirac denominator contains the light quark mass m_q , the scalar $U_S = \sigma r$ and the vector $V_D(r) = -\frac{4\alpha}{3r}$ potentials, and ε_0 , which is the Dirac eigenvalue in the static potential, created by the heavy quark:

$$\vec{\alpha} \cdot \vec{p} + \beta (m_q + U_S(r)) \psi(r) = (\epsilon_0 - V_D(r)) \psi(r). \quad (\text{A3})$$

Knowing the string tension $\sigma = 0.18$ GeV 2 and the averaged momentum distribution β_2 in the w.f. of a heavy-light meson, the value $M_\omega = \frac{4\sqrt{2}\sigma}{\sqrt{\pi}\beta_2}$ was calculated in Ref. [20]. For the $1S$, $2S$, and $3S$ charmonium states M_ω is different and equal to 0.65, 0.80, 1.10, respectively. Using the averaged values of the scalar and vector potentials, one obtains $\gamma \approx 1.4$.

The intermediate decay channels, like DD^* , induce an additional interaction ‘‘potential’’ $V_{CC}(\mathbf{q}, \mathbf{q}', E)$ (the quotation marks imply nonlocality and energy dependence of this potential):

$$V_{CC}(\mathbf{q}, \mathbf{q}', E) = \sum_{n_2n_3} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{X_{n_2n_3}(\mathbf{q}, \mathbf{p}) X_{n_2n_3}^\dagger(\mathbf{q}', \mathbf{p})}{E - E_{n_2n_3}(\mathbf{p})}, \quad (\text{A4})$$

$$X_{n_2n_3}(\mathbf{q}, \mathbf{p}) = \frac{\gamma}{\sqrt{N_c}} \bar{y}_{123}(\mathbf{q}, \mathbf{p}) \psi_{n_2}(\mathbf{q} - \mathbf{p}) \psi_{n_3}(\mathbf{q} - \mathbf{p}). \quad (\text{A5})$$

The new Hamiltonian with account of the CC interaction is

$$H = H_0 + V_{CC} \quad (\text{A6})$$

where H_0 is given in Eqs. (8,9). At this point it is important to stress that V_{CC} contains all thresholds of open channels, and to treat the spectrum of the Hamiltonian H rigorously, one should exploit the Weinberg eigenvalue formalism, developed for this purpose in Ref. [20], second reference. However, for the states below all thresholds (or neglecting the open channel amplitudes in first approximation) one can use an expansion in the complete set of the eigenvalues of H_0 . One can show, as in Ref. [20], that the exact Weinberg formalism reduces to this expansion for the real eigenvalues at or below thresholds. Using that, one obtains for the Green function an expansion,

$$G_{Q\bar{Q}}(1, 2; E) = \sum_{n,m} \psi_n(1) (\hat{E} - E + \hat{w})_{nm}^{-1} \psi_m(2), \quad (\text{A7})$$

where the matrix element of \hat{w} is denoted by w_{nm} and given by

$$\begin{aligned} w_{nm}(E) &= \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{d^3\mathbf{q}'}{(2\pi)^3} \psi_n(\mathbf{q}) V_{CC}(\mathbf{q}, \mathbf{q}', E) \psi_m(\mathbf{q}') \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{n_2n_3} \frac{J_{nn_2n_3}(\mathbf{p}) J_{mn_2n_3}^\dagger(\mathbf{p})}{E - E_{n_2n_3}(\mathbf{p})} \end{aligned} \quad (\text{A8})$$

and the energy eigenvalues are obtained from the determinant condition

$$\det(E - \hat{E} - \hat{w}) = 0, \quad (\text{A9})$$

where $(\hat{E})_{nm} = E_n \delta_{nm}$

Defining as in Eq. (22) the effective $Q\bar{Q}$ w.f.,

$$\psi_{Q\bar{Q}}^{(\mathcal{B})} = \sum_{k,l} \psi_k \left(\frac{1}{\hat{E} - E + \hat{w}} \right)_{kl} (\hat{B} \psi_l) \equiv \sum_k a_k \psi_k, \quad (\text{A10})$$

one has to calculate the resulting amplitudes a_k in the expansion of $X(3872)$ in terms of the $Q\bar{Q}$ states. The values of w_{nm} are given in Table IV, and one can see that only two states, $|1P\rangle$ and $|2P\rangle$, are important.

Keeping only two $Q\bar{Q}$ eigenfunctions, one has

$$a_1 = \frac{E_2 - E + w_{22}}{\det} (\hat{B}\psi_1) - \frac{w_{21}}{\det} (\hat{B}\psi_2) \quad (A11)$$

$$a_2 = \frac{E_1 - E + w_{11}}{\det} (\hat{B}\psi_2) - \frac{w_{12}}{\det} (\hat{B}\psi_1), \quad (A12)$$

where $\det \equiv \det(\hat{E} - E + \hat{w})$. Then one obtains the ratio

$$\frac{a_1}{a_2} = \frac{-w_{21} + (E_2 + w_{22} - E) \frac{(\hat{B}\psi_1)}{(\hat{B}\psi_2)}}{E_1 + w_{11} - E - w_{12} \frac{(\hat{B}\psi_1)}{(\hat{B}\psi_2)}}. \quad (A13)$$

We calculate this ratio for the energy $E_R = 3872$ MeV, taking the values of w_{ik} from the Table IV. Note, that we have corrected the sign of w_{12} in accordance with the relative signs of the wave functions $|1P\rangle$ and $|2P\rangle$. The resulting values of $c_X \equiv \frac{a_1}{a_2}(3872 \text{ MeV})$ are given in Eqs. (28-30).

From Table IV one can see that the 3^3P_1 state gives a negligible admixture. The values of w_{mn} for $E = E_R$ are computed according to Eq. (A8) with the w.f., calculated in Ref. [20], and are given in Table IV.

TABLE IV: The m.e. w_{nm} (in GeV) between n^3P_1 and m^3P_1 states for two approximations of the exact w.f.

$n \ m$	1 1	1 2	2 2	3 2
w_{nm} for 5 HO	-0.320	0.122	- 0.099	-0.0003
w_{nm} for 3 HO	-0.319	0.121	- 0.098	-0.0011

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