

POSSIBLE ASSIGNMENTS OF THE $X(3872)$, $Z_c(3900)$ AND $Z_b(10610)$ AS AXIAL-VECTOR MOLECULAR STATES

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Abstract

In this article, we construct both the color singlet-singlet type and octet-octet type currents to interpolate the $X(3872)$, $Z_c(3900)$, $Z_b(10610)$, and calculate the vacuum condensates up to dimension-10 in the operator product expansion. Then we study the axial-vector hidden charmed and hidden bottom molecular states with the QCD sum rules, explore the energy scale dependence of the QCD sum rules for the heavy molecular states in details, and use the formula $\mu = \sqrt{M_{X/Y/Z}^2 - (2M_Q)^2}$ with the effective masses M_Q to determine the energy scales. The numerical results support assigning the $X(3872)$, $Z_c(3900)$, $Z_b(10610)$ as the color singlet-singlet type molecular states with $J^{PC} = 1^{++}$, 1^{+-} , 1^{+-} , respectively, more theoretical and experimental works are still needed to distinguish the molecule and tetraquark assignments; while there are no candidates for the color octet-octet type molecular states.

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1 Introduction

In 2003, the Belle collaboration reported the first observation of the charmonium-like state $X(3872)$ in the $\pi^+\pi^-J/\psi$ mass spectrum in the exclusive processes $B^\pm \rightarrow K^\pm\pi^+\pi^-J/\psi$ [1]. The evidences for the decay modes $X(3872) \rightarrow \gamma J/\psi, \gamma\psi'$ imply the positive charge conjugation $C = +$ [2], while angular correlations between the final state particles in the $\pi^+\pi^-J/\psi$ support the $J^{PC} = 1^{++}$ assignment, and strongly disfavor (or exclude) the 0^{++} , 0^{-+} , 1^{-+} , 2^{-+} assignments [3]. The $X(3872)$ has been extensively studied since its first observation, for more articles on this subject, one can consult the reviews [4].

In 2011, the Belle collaboration reported the first observation of the $Z_b(10610)$ and $Z_b(10650)$ in the $\pi^\pm\Upsilon(1, 2, 3S)$ and $\pi^\pm h_b(1, 2P)$ mass spectra in the exclusive processes $\Upsilon(5S) \rightarrow \Upsilon(1, 2, 3S)\pi^+\pi^-$, $h_b(1, 2P)\pi^+\pi^-$ [5]. The quantum numbers (isospin, G-parity, spin and parity) $I^G(J^P) = 1^+(1^+)$ are favored [5]. Later, the Belle collaboration updated the parameters $M_{Z_b(10610)} = (10607.2 \pm 2.0)$ MeV, $M_{Z_b(10650)} = (10652.2 \pm 1.5)$ MeV, $\Gamma_{Z_b(10610)} = (18.4 \pm 2.4)$ MeV and $\Gamma_{Z_b(10650)} = (11.5 \pm 2.2)$ MeV [6]. In 2013, the Belle collaboration reported the first observation of the decay processes $\Upsilon(5S) \rightarrow \Upsilon(1, 2, 3S)\pi^0\pi^0$, and obtained the neutral particle $Z_b^0(10610)$ in a Dalitz analysis of the decays to $\Upsilon(2, 3S)\pi^0$ [7]. There have been several assignments of the $Z_b(10610)$ and $Z_b(10650)$, such as the molecular states [8], tetraquark states [9], threshold cusps [10], rescattering effects [11], etc.

In 2013, the BESIII collaboration reported the first observation of the structure $Z_c(3900)$ in the $\pi^\pm J/\psi$ mass spectrum in the process $e^+e^- \rightarrow \pi^+\pi^-J/\psi$ [12]. The mass and decay width are $(3899.0 \pm 3.6 \pm 4.9)$ MeV and $(46 \pm 10 \pm 20)$ MeV, respectively [12]. Then the $Z_c(3900)$ was confirmed by the Belle and CLEO collaborations [13, 14]. There have been several assignments, such as the molecular state [15], tetraquark state [16], hadro-charmonium [17], rescattering effect [18], etc.

In this article, we will focus on the scenario of molecular states. In Ref.[19], S. H. Lee et al take the $X(3872)$ as the $D^{*0}\bar{D}^0 - D^0\bar{D}^{*0}$ molecular state with $J^{PC} = 1^{++}$, study its mass with the QCD sum rules by calculating the vacuum condensates up to dimension-6 in the operator product

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expansion, and obtain the value $M_{X(3872)} = (3.88 \pm 0.06)$ GeV. In Ref.[20], J. R. Zhang and M. Q. Huang study the masses of the $Q\bar{q}Qq$ type molecular states with QCD sum rules in a systematic way by calculating the vacuum condensates up to dimension-6. In Ref.[21], R. D. Matheus et al take the $X(3872)$ as a mixture between charmonium and exotic molecular state with $J^{PC} = 1^{++}$, study the mass $M_{X(3872)}$ and decay width $\Gamma_{X(3872) \rightarrow J/\psi\pi^+\pi^-}$ with the QCD sum rules, and conclude that the $X(3872)$ is approximately 97% a charmonium state $\bar{c}c$ and 3% a molecular state $D^*\bar{D}$. In Ref.[22], J. R. Zhang et al take the $Z_b(10610)$ as a bottomonium-like molecular state $B^*\bar{B}$, study its mass with the QCD sum rules by calculating the vacuum condensates up to dimension-6, and obtain the value $M_{Z_b} = (10.54 \pm 0.22)$ GeV. In Ref.[23], W. Chen et al take the $X(3872)$ as the $J^{PC} = 1^{++}$ mixed state of the charmonium hybrid and $D^*\bar{D}$ molecular state, study its mass with the QCD sum rules, and observe that the mixing is robust. In Ref.[24], J. R. Zhang takes the $Z_c(3900)$ as the $D^*\bar{D}$ molecular state without distinguishing its charge conjugation, study the mass with the QCD sum rules by calculating the vacuum condensates up to dimension-9, and obtain the value $M_{Z_c} = (3.86 \pm 0.27)$ GeV.

In all those works [19, 20, 21, 22, 23, 24], the \overline{MS} masses are taken, however, the energy scales at which the QCD spectral densities are calculated are either not shown explicitly or not specified, and the energy scale dependence of the QCD sum rules is not studied. In the QCD sum rules for the hidden charmed (or bottom) tetraquark states and molecular states, the integrals

$$\int_{4m_Q^2}^{s_0} ds \rho_{QCD}(s) \exp\left(-\frac{s}{T^2}\right), \quad (1)$$

are sensitive to the heavy quark masses m_Q , where the $\rho_{QCD}(s)$ denotes the QCD spectral densities and the T^2 denotes the Borel parameters. Variations of the heavy quark masses lead to changes of integral ranges $4m_Q^2 - s_0$ of the variable ds besides the QCD spectral densities, therefore changes of the Borel windows and predicted masses and pole residues. Furthermore, in Refs.[19, 20, 21, 22, 23, 24], the higher dimensional vacuum condensates are neglected in one way or another. The higher dimensional vacuum condensates play an important role in determining the Borel windows, although they play a less important role in the Borel windows.

In Refs.[25, 26, 27, 28], we focus on the scenario of tetraquark states, distinguish the charge conjugations of the interpolating currents, calculate the vacuum condensates up to dimension-10 in the operator product expansion, study the diquark-antidiquark type scalar, vector, axial-vector, tensor hidden charmed tetraquark states and axial-vector hidden bottom tetraquark states systematically with the QCD sum rules, make reasonable assignments of the $X(3872)$, $Z_c(3900)$, $Z_c(3885)$, $Z_c(4020)$, $Z_c(4025)$, $Z(4050)$, $Z(4250)$, $Y(4360)$, $Y(4630)$, $Y(4660)$, $Z_b(10610)$ and $Z_b(10650)$. Furthermore, we explore the energy scale dependence of the QCD sum rules for the hidden charmed and hidden bottom tetraquark states in details for the first time, and suggest a formula,

$$\mu = \sqrt{M_{X/Y/Z}^2 - (2\mathbb{M}_Q)^2}, \quad (2)$$

with the effective masses $\mathbb{M}_c = 1.80$ GeV and $\mathbb{M}_b = 5.13$ GeV to determine the energy scales of the QCD spectral densities, which works well.

In this article, we take the $X(3872)$, $Z_c(3900)$, $Z_b(10610)$ as the axial-vector hadronic molecular states, distinguish the charge conjugations, construct both the color singlet-singlet type currents and color octet-octet type currents to interpolate them. We calculate the contributions of the vacuum condensates up to dimension-10, study the masses and pole residues, and explore the energy scale dependence in details so as to see whether or not the formula $\mu = \sqrt{M_{X/Y/Z}^2 - (2\mathbb{M}_Q)^2}$ survives in the case of the molecular states, and make tentative assignments of the $X(3872)$, $Z_c(3900)$, $Z_b(10610)$ in the scenario of molecular states.

The article is arranged as follows: we derive the QCD sum rules for the masses and pole residues of the axial-vector molecular states in section 2; in section 3, we present the numerical results and discussions; section 4 is reserved for our conclusion.

2 QCD sum rules for the $J^P = 1^+$ molecular states

In the following, we write down the two-point correlation functions $\Pi_{\mu\nu}(p)$ in the QCD sum rules,

$$\Pi_{\mu\nu}(p) = i \int d^4x e^{ip \cdot x} \langle 0 | T \{ J_\mu(x) J_\nu^\dagger(0) \} | 0 \rangle, \quad (3)$$

$$J_\mu^0(x) = \frac{\bar{u}(x)i\gamma_5 Q(x)\bar{Q}(x)\gamma_\mu d(x) + t\bar{u}(x)\gamma_\mu Q(x)\bar{Q}(x)i\gamma_5 d(x)}{\sqrt{2}}, \quad (4)$$

$$J_\mu^8(x) = \frac{\bar{u}(x)i\gamma_5 \lambda^a Q(x)\bar{Q}(x)\gamma_\mu \lambda^a d(x) + t\bar{u}(x)\gamma_\mu \lambda^a Q(x)\bar{Q}(x)i\gamma_5 \lambda^a d(x)}{\sqrt{2}}, \quad (5)$$

where $t = \pm 1$, $J_\mu(x) = J_\mu^0(x)$, $J_\mu^8(x)$, the λ^a is the Gell-Mann matrix. We construct the color singlet-singlet type (0-0 type) currents $J_\mu^0(x)$ (see Refs.[19, 20, 21, 22, 23, 24]) and color octet-octet type (8-8 type) currents $J_\mu^8(x)$ (see Ref.[29]) to study the hadronic molecular states $X(3872)$ (to be more precise, the charged partner of the $X(3872)$), $Z_c(3900)$, $Z_b(10610)$, etc. We can rearrange the 8-8 type currents $J_\mu^8(x)$ in terms of the following 0-0 type currents,

$$\begin{aligned} J_\mu^8(x) = & \frac{\sqrt{2}}{4} \bar{u}(x)i\gamma_5 d(x)\bar{Q}(x)\gamma_\mu Q(x) + t \frac{\sqrt{2}}{4} \bar{Q}(x)i\gamma_5 Q(x)\bar{u}(x)\gamma_\mu d(x) \\ & + \frac{\sqrt{2}i}{4} \bar{u}(x)\gamma_5 \gamma_\alpha d(x)\bar{Q}(x)\gamma_\mu \gamma^\alpha Q(x) + t \frac{\sqrt{2}i}{4} \bar{Q}(x)\gamma_5 \gamma_\alpha Q(x)\bar{u}(x)\gamma_\mu \gamma^\alpha d(x) \\ & + \frac{\sqrt{2}i}{8} \bar{u}(x)\gamma_5 \sigma_{\alpha\beta} d(x)\bar{Q}(x)\gamma_\mu \sigma^{\alpha\beta} Q(x) + t \frac{\sqrt{2}i}{8} \bar{Q}(x)\gamma_5 \sigma_{\alpha\beta} Q(x)\bar{u}(x)\gamma_\mu \sigma^{\alpha\beta} d(x) \\ & + \frac{\sqrt{2}i}{4} \bar{u}(x)\gamma_\alpha d(x)\bar{Q}(x)\gamma_\mu \gamma^\alpha \gamma_5 Q(x) + t \frac{\sqrt{2}i}{4} \bar{Q}(x)\gamma_\alpha Q(x)\bar{u}(x)\gamma_\mu \gamma^\alpha \gamma_5 d(x) \\ & + \frac{\sqrt{2}i}{4} \bar{u}(x)d(x)\bar{Q}(x)\gamma_\mu \gamma_5 Q(x) + t \frac{\sqrt{2}i}{4} \bar{Q}(x)Q(x)\bar{u}(x)\gamma_\mu \gamma_5 d(x) \\ & - \frac{\sqrt{2}}{3} \bar{u}(x)i\gamma_5 Q(x)\bar{Q}(x)\gamma_\mu d(x) - t \frac{\sqrt{2}}{3} \bar{u}(x)\gamma_\mu Q(x)\bar{Q}(x)i\gamma_5 d(x), \end{aligned} \quad (6)$$

with the identity,

$$\lambda_{ij}^a \lambda_{mn}^a = 2\delta_{in}\delta_{mj} - \frac{2}{3}\delta_{ij}\delta_{mn}, \quad (7)$$

in the color space. The 8-8 type current can be taken as a special superposition of the 0-0 type currents. Under charge conjugation transform \hat{C} , the currents $J_\mu(x)$ have the properties,

$$\hat{C} J_\mu(x) \hat{C}^{-1} = \mp J_\mu(x) \mid_{u \leftrightarrow d} \text{ for } t = \pm 1. \quad (8)$$

The values $t = \mp 1$ correspond to the positive and negative charge conjugations, respectively.

We can insert a complete set of intermediate hadronic states with the same quantum numbers as the current operators $J_\mu(x)$ into the correlation functions $\Pi_{\mu\nu}(p)$ to obtain the hadronic representation [30, 31]. After isolating the ground state contributions from the pole terms, we get the following results,

$$\Pi_{\mu\nu}(p) = \frac{\lambda_{X/Z}^2}{M_{X/Z}^2 - p^2} \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} \right) + \dots = \Pi(p) \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} \right) + \dots, \quad (9)$$

where the pole residues (or couplings) $\lambda_{X/Z}$ are defined by

$$\langle 0 | J_\mu(0) | X/Z(p) \rangle = \lambda_{X/Z} \varepsilon_\mu, \quad (10)$$

the ε_μ are the polarization vectors of the axial-vector mesons $X(3872)$, $Z_c(3900)$, $Z_b(10610)$, etc.

Here we take a short digression to discuss the possible contaminations originate from the higher resonances and continuum states. In the following, we will discuss the hidden-charmed systems for simplicity, the conclusion survives in the hidden-bottom systems. In the nonrelativistic and heavy quark limit, the $C = +$ currents are reduced to the forms,

$$\begin{aligned}\bar{u}\gamma^5 c \bar{c} \gamma^j d - \bar{u}\gamma^j c \bar{c} \gamma^5 d &\propto \xi_u^\dagger \xi_c \xi_c^\dagger \frac{\sigma^j}{2} \xi_d - \xi_u^\dagger \frac{\sigma^j}{2} \xi_c \xi_c^\dagger \xi_d, \\ \bar{u}\gamma^j c \bar{c} \gamma^k d + \bar{u}\gamma^k c \bar{c} \gamma^j d &\propto \xi_u^\dagger \frac{\sigma^j}{2} \xi_c \xi_c^\dagger \frac{\sigma^k}{2} \xi_d + \xi_u^\dagger \frac{\sigma^k}{2} \xi_c \xi_c^\dagger \frac{\sigma^j}{2} \xi_d,\end{aligned}\quad (11)$$

while the $C = -$ currents are reduced to the forms,

$$\begin{aligned}\bar{u}\gamma^5 c \bar{c} \gamma^j d + \bar{u}\gamma^j c \bar{c} \gamma^5 d &\propto \xi_u^\dagger \xi_c \xi_c^\dagger \frac{\sigma^j}{2} \xi_d + \xi_u^\dagger \frac{\sigma^j}{2} \xi_c \xi_c^\dagger \xi_d, \\ \bar{u}\gamma^j c \bar{c} \gamma^k d - \bar{u}\gamma^k c \bar{c} \gamma^j d &\propto \xi_u^\dagger \frac{\sigma^j}{2} \xi_c \xi_c^\dagger \frac{\sigma^k}{2} \xi_d - \xi_u^\dagger \frac{\sigma^k}{2} \xi_c \xi_c^\dagger \frac{\sigma^j}{2} \xi_d,\end{aligned}\quad (12)$$

where the $\xi_{c,u,d}$ are the two-component quark fields and the σ^i are the pauli matrixes. The bilinear fields $\xi_i^\dagger \xi_j$ and $\xi_i^\dagger \frac{\sigma^k}{2} \xi_j$ have the spins 0 and 1, respectively, and couple to (pseudo-) scalar and (axial-) vector meson fields, respectively. The currents J_μ^0 with $C = \pm$ couple potentially to the $\frac{D\bar{D}^* \mp D^*\bar{D}}{\sqrt{2}}$ molecular or scattering states, while the currents $J_{\mu\nu}^0 = \bar{u}\gamma_\mu c \bar{c} \gamma_\nu d \pm \bar{u}\gamma_\nu c \bar{c} \gamma_\mu d$ with $C = \pm$ couple potentially to the $D^*\bar{D}^*$ molecular or scattering states.

On the other hand, the octet currents are reduced into the following forms,

$$\begin{aligned}\bar{u}i\gamma^5 \lambda^a c \bar{c} \lambda^a \gamma^j d &\propto \xi_u^\dagger \lambda^a \xi_c \xi_c^\dagger \lambda^a \frac{\sigma^j}{2} \xi_d = 4\xi_u^\dagger \frac{\sigma^k}{2} \frac{\sigma^j}{2} \xi_d \xi_c^\dagger \frac{\sigma^k}{2} \xi_c + \xi_u^\dagger \frac{\sigma^j}{2} \xi_d \xi_c^\dagger \xi_c - \frac{2}{3} \xi_u^\dagger \xi_c \xi_c^\dagger \frac{\sigma^j}{2} \xi_d \\ &= \xi_u^\dagger \xi_d \xi_c^\dagger \frac{\sigma^j}{2} \xi_c - 2\epsilon^{ijk} \xi_u^\dagger \frac{\sigma^i}{2} \xi_d \xi_c^\dagger \frac{\sigma^k}{2} \xi_c + \xi_u^\dagger \frac{\sigma^j}{2} \xi_d \xi_c^\dagger \xi_c - \frac{2}{3} \xi_u^\dagger \xi_c \xi_c^\dagger \frac{\sigma^j}{2} \xi_d, \\ \bar{u}\lambda^a \gamma^j c \bar{c} \lambda^a \gamma^k d &\propto \xi_u^\dagger \lambda^a \frac{\sigma^j}{2} \xi_c \xi_c^\dagger \lambda^a \frac{\sigma^k}{2} \xi_d \\ &= 4\xi_u^\dagger \frac{\sigma^i}{2} \frac{\sigma^k}{2} \xi_d \xi_c^\dagger \frac{\sigma^i}{2} \frac{\sigma^j}{2} \xi_c + \xi_u^\dagger \frac{\sigma^k}{2} \xi_d \xi_c^\dagger \frac{\sigma^j}{2} \xi_c - \frac{2}{3} \xi_u^\dagger \frac{\sigma^j}{2} \xi_c \xi_c^\dagger \frac{\sigma^k}{2} \xi_d \\ &= \frac{\delta_{jk}}{4} \xi_u^\dagger \xi_d \xi_c^\dagger \xi_c + \frac{1}{2} \epsilon^{jkm} \xi_u^\dagger \frac{\sigma^m}{2} \xi_d \xi_c^\dagger \xi_c + \frac{1}{2} \epsilon^{kjm} \xi_u^\dagger \xi_d \xi_c^\dagger \frac{\sigma^m}{2} \xi_c \\ &\quad + \epsilon^{ikm} \epsilon^{ijn} \xi_u^\dagger \frac{\sigma^m}{2} \xi_d \xi_c^\dagger \frac{\sigma^n}{2} \xi_c + \xi_u^\dagger \frac{\sigma^k}{2} \xi_d \xi_c^\dagger \frac{\sigma^j}{2} \xi_c - \frac{2}{3} \xi_u^\dagger \frac{\sigma^j}{2} \xi_c \xi_c^\dagger \frac{\sigma^k}{2} \xi_d.\end{aligned}\quad (13)$$

The octet current $J_\mu^8 = \bar{u}i\gamma^5 \lambda^a c \bar{c} \lambda^a \gamma_\mu d$ couples potentially to the $J/\psi\pi$, $\psi(3770)\pi$, $\eta_c\rho$, $J/\psi\rho$, $D\bar{D}^*$ molecular or scattering states. The octet current $J_{\mu\nu}^8 = \bar{u}\lambda^a \gamma_\mu c \bar{c} \lambda^a \gamma_\nu d$ couples potentially to the $\eta_c\pi$, $\eta_c\rho$, $J/\psi\pi$, $\psi(3770)\pi$, $J/\psi\rho$, $D^*\bar{D}^*$ molecular or scattering states. In this article, we take the currents $J_\mu^{0,8}$ not the currents $J_{\mu\nu}^{0,8}$, the $D^*\bar{D}^*$ molecular or scattering states have no contaminations.

In the scenario of meta-stable Feshbach resonances, the $X(3872)$, $Z_c(3900)$, $Z_c(4025)$, $Z_b(10610)$, $Z_b(10650)$ are taken as the $J/\psi\rho - D\bar{D}^*$, $\psi(3770)\pi - D\bar{D}^*$, $h_c(2P)\pi - D^*\bar{D}^*$, $\chi_{b0}\rho - B\bar{B}^*$, $\chi_{b1}\rho - B^*\bar{B}^*$ hadrocharmonium-molecule mixed states, respectively, where the $\chi_{b0}\rho$ and $\chi_{b1}\rho$ are P-wave systems [32]. The hadrocharmonium system admits bound states giving rise to a discrete spectrum of levels, a resonance occurs if one of such levels falls close to some open-charm threshold, as the coupling between channels leads to an attractive interaction and favors the formation of a meta-stable Feshbach resonance. The couplings of the currents J_μ to the near-threshold hadrocharmonium states $J/\psi\rho$, $\psi(3770)\pi$ and $\chi_{b0}\rho$ contribute to the molecular states $X(3872)$, $Z_c(3900)$ and $Z_b(10610)$, respectively.

Now we study the contributions of the intermediate meson-loops (or the scattering states DD^* , $J/\psi\pi$, $J/\psi\rho$, etc) to the correlation functions $\Pi_{\mu\nu}(p)$,

$$\begin{aligned}\Pi_{\mu\nu}(p) &= -\frac{\widehat{\lambda}_{X/Z}^2}{p^2 - \widehat{M}_{X/Z}^2} \tilde{g}_{\mu\nu}(p) - \frac{\widehat{\lambda}_{X/Z}}{p^2 - \widehat{M}_{X/Z}^2} \tilde{g}_{\mu\alpha}(p) \Sigma_{DD^*}(p) \tilde{g}^{\alpha\beta}(p) \tilde{g}_{\beta\nu}(p) \frac{\widehat{\lambda}_{X/Z}}{p^2 - \widehat{M}_{X/Z}^2} \\ &\quad - \frac{\widehat{\lambda}_{X/Z}}{p^2 - \widehat{M}_{X/Z}^2} \tilde{g}_{\mu\alpha}(p) \Sigma_{J/\psi\pi}(p) \tilde{g}^{\alpha\beta}(p) \tilde{g}_{\beta\nu}(p) \frac{\widehat{\lambda}_{X/Z}}{p^2 - \widehat{M}_{X/Z}^2} \\ &\quad - \frac{\widehat{\lambda}_{X/Z}}{p^2 - \widehat{M}_{X/Z}^2} \tilde{g}_{\mu\alpha}(p) \Sigma_{J/\psi\rho}^{\alpha\beta}(p) \tilde{g}_{\beta\nu}(p) \frac{\widehat{\lambda}_{X/Z}}{p^2 - \widehat{M}_{X/Z}^2} + \dots, \\ &= -\frac{\widehat{\lambda}_{X/Z}^2}{p^2 - \widehat{M}_{X/Z}^2 - \Sigma_{DD^*}(p) - \Sigma_{J/\psi\pi}(p) - \Sigma_{J/\psi\rho}(p) + \dots} \tilde{g}_{\mu\nu}(p) + \dots,\end{aligned}\tag{14}$$

where

$$\Sigma_{DD^*}(p) = i \int \frac{d^4 q}{(2\pi)^4} \frac{G_{X/ZDD^*}^2}{[q^2 - M_D^2][(p-q)^2 - M_{D^*}^2]},\tag{15}$$

$$\Sigma_{J/\psi\pi}(p) = i \int \frac{d^4 q}{(2\pi)^4} \frac{G_{X/ZJ/\psi\pi}^2}{[q^2 - M_{J/\psi}^2][(p-q)^2 - M_\pi^2]},\tag{16}$$

$$\begin{aligned}\Sigma_{J/\psi\rho}^{\alpha\beta}(p) &= i \int \frac{d^4 q}{(2\pi)^4} \frac{G_{X/ZJ/\psi\rho}^2 \epsilon^{\alpha\theta\sigma\tau} \epsilon^{\beta\theta'\sigma'\tau'} p_\tau p_{\tau'} \tilde{g}_{\theta\theta'}(q) \tilde{g}_{\sigma\sigma'}(p-q)}{[q^2 - M_{J/\psi}^2][(p-q)^2 - M_\rho^2]}, \\ &= \Sigma_{J/\psi\rho}(p) \tilde{g}^{\alpha\beta}(p) + \Sigma_{J/\psi\rho}^1(p) \frac{p^\alpha p^\beta}{p^2},\end{aligned}\tag{17}$$

$\tilde{g}_{\mu\nu}(p) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}$, the G_{X/ZDD^*} , $G_{X/ZJ/\psi\pi}$, $G_{X/ZJ/\psi\rho}$ are hadronic coupling constants, the $\widehat{\lambda}_{X/Z}$ and $\widehat{M}_{X/Z}$ are bare quantities to absorb the divergences in the self-energies $\Sigma_{DD^*}(p)$, $\Sigma_{J/\psi\pi}(p)$, $\Sigma_{J/\psi\rho}(p)$, etc.

The renormalized self-energies contribute a finite imaginary part to modify the dispersion relation,

$$\Pi_{\mu\nu}(p) = -\frac{\lambda_{X/Z}^2}{p^2 - M_{X/Z}^2 + i\sqrt{p^2}\Gamma(p^2)} \tilde{g}_{\mu\nu}(p) + \dots,\tag{18}$$

the physical widths $\Gamma_{Z_c(3900)}(M_Z^2) = (46 \pm 10 \pm 20) \text{ MeV}$ and $\Gamma_{X(3872)}(M_X^2) < 1.2 \text{ MeV}$ are small enough, the zero width approximation in the hadronic spectral densities works. The discussion survives in the hidden-bottom systems according to the small physical widths $\Gamma_{Z_b(10610)} = (18.4 \pm 2.4) \text{ MeV}$ and $\Gamma_{Z_b(10650)} = (11.5 \pm 2.2) \text{ MeV}$. The contaminations of the intermediate meson-loops are expected to be small.

In the following, we briefly outline the operator product expansion for the correlation functions $\Pi_{\mu\nu}(p)$ in perturbative QCD. We contract the quark fields in the correlation functions $\Pi_{\mu\nu}(p)$ with

Wick theorem, obtain the results:

$$\begin{aligned}\Pi_{\mu\nu}^0(p) &= -\frac{i}{2}\delta_{jk}\delta_{mn}\delta_{k'j'}\delta_{n'm'} \int d^4x e^{ip\cdot x} \\ &\quad \left\{ \text{Tr} \left[\gamma_5 S_Q^{kk'}(x)\gamma_5 S^{j'j}(-x) \right] \text{Tr} \left[\gamma_\mu S^{nn'}(x)\gamma_\nu S_Q^{m'm}(-x) \right] \right. \\ &\quad + \text{Tr} \left[\gamma_\mu S_Q^{kk'}(x)\gamma_\nu S^{j'j}(-x) \right] \text{Tr} \left[\gamma_5 S^{nn'}(x)\gamma_5 S_Q^{m'm}(-x) \right] \\ &\quad \pm \text{Tr} \left[\gamma_\mu S_Q^{kk'}(x)\gamma_5 S^{j'j}(-x) \right] \text{Tr} \left[\gamma_5 S^{nn'}(x)\gamma_\nu S_Q^{m'm}(-x) \right] \\ &\quad \left. \pm \text{Tr} \left[\gamma_5 S_Q^{kk'}(x)\gamma_\nu S^{j'j}(-x) \right] \text{Tr} \left[\gamma_\mu S^{nn'}(x)\gamma_5 S_Q^{m'm}(-x) \right] \right\}, \quad (19)\end{aligned}$$

$$\Pi_{\mu\nu}^8(p) = \Pi_{\mu\nu}^0(p) |_{\delta_{jk}\delta_{mn}\delta_{k'j'}\delta_{n'm'} \rightarrow \lambda_{jk}^a \lambda_{mn}^a \lambda_{k'j'}^b \lambda_{n'm'}^b} \quad (20)$$

where the \mp correspond the positive and negative charge conjugations, respectively, the $S^{ij}(x)$ and $S_Q^{ij}(x)$ are the full light and heavy quark propagators, respectively,

$$\begin{aligned}S^{ij}(x) &= \frac{i\delta_{ij}\not{x}}{2\pi^2 x^4} - \frac{\delta_{ij}\langle\bar{q}q\rangle}{12} - \frac{\delta_{ij}x^2\langle\bar{q}g_s\sigma G q\rangle}{192} - \frac{ig_s G_{\alpha\beta}^a t_{ij}^a (\not{x}\sigma^{\alpha\beta} + \sigma^{\alpha\beta}\not{x})}{32\pi^2 x^2} - \frac{i\delta_{ij}x^2\not{x}g_s^2\langle\bar{q}q\rangle^2}{7776} \\ &\quad - \frac{\delta_{ij}x^4\langle\bar{q}q\rangle\langle g_s^2 GG \rangle}{27648} - \frac{1}{8}\langle\bar{q}_j\sigma^{\mu\nu}q_i\rangle\sigma_{\mu\nu} - \frac{1}{4}\langle\bar{q}_j\gamma^\mu q_i\rangle\gamma_\mu + \dots, \quad (21)\end{aligned}$$

$$\begin{aligned}S_Q^{ij}(x) &= \frac{i}{(2\pi)^4} \int d^4k e^{-ik\cdot x} \left\{ \frac{\delta_{ij}}{\not{k}-m_Q} - \frac{g_s G_{\alpha\beta}^n t_{ij}^n}{4} \frac{\sigma^{\alpha\beta}(\not{k}+m_Q) + (\not{k}+m_Q)\sigma^{\alpha\beta}}{(k^2-m_Q^2)^2} \right. \\ &\quad \left. + \frac{g_s D_\alpha G_{\beta\lambda}^n t_{ij}^n (f^{\lambda\beta\alpha} + f^{\lambda\alpha\beta})}{3(k^2-m_Q^2)^4} - \frac{g_s^2 (t^a t^b)_{ij} G_{\alpha\beta}^a G_{\mu\nu}^b (f^{\alpha\beta\mu\nu} + f^{\alpha\mu\beta\nu} + f^{\alpha\mu\nu\beta})}{4(k^2-m_Q^2)^5} + \dots \right\}, \\ f^{\lambda\alpha\beta} &= (\not{k}+m_Q)\gamma^\lambda(\not{k}+m_Q)\gamma^\alpha(\not{k}+m_Q)\gamma^\beta(\not{k}+m_Q), \\ f^{\alpha\beta\mu\nu} &= (\not{k}+m_Q)\gamma^\alpha(\not{k}+m_Q)\gamma^\beta(\not{k}+m_Q)\gamma^\mu(\not{k}+m_Q)\gamma^\nu(\not{k}+m_Q), \quad (22)\end{aligned}$$

and $t^n = \frac{\lambda^n}{2}$, $D_\alpha = \partial_\alpha - ig_s G_\alpha^n t^n$ [31], then compute the integrals both in the coordinate and momentum spaces, and obtain the correlation functions $\Pi_{\mu\nu}(p)$ therefore the spectral densities at the level of quark-gluon degrees of freedom. In Eq.(21), we retain the terms $\langle\bar{q}_j\sigma_{\mu\nu}q_i\rangle$ and $\langle\bar{q}_j\gamma_\mu q_i\rangle$ originate from the Fierz re-ordering of the $\langle q_i\bar{q}_j\rangle$ to absorb the gluons emitted from the heavy quark lines to form $\langle\bar{q}_jg_sG_{\alpha\beta}^at_{mn}^a\sigma_{\mu\nu}q_i\rangle$ and $\langle\bar{q}_j\gamma_\mu q_i g_s D_\nu G_{\alpha\beta}^a t_{mn}^a\rangle$ so as to extract the mixed condensate and four-quark condensates $\langle\bar{q}g_s\sigma G q\rangle$ and $g_s^2\langle\bar{q}q\rangle^2$, respectively.

Once analytical results are obtained, we can take the quark-hadron duality and perform Borel transform with respect to the variable $P^2 = -p^2$ to obtain the following QCD sum rules:

$$\lambda_{X/Z}^2 \exp\left(-\frac{M_{X/Z}^2}{T^2}\right) = \int_{4m_Q^2}^{s_0} ds \rho^{0/8}(s) \exp\left(-\frac{s}{T^2}\right), \quad (23)$$

where

$$\rho^{0/8}(s) = \rho_0^{0/8}(s) + \rho_3^{0/8}(s) + \rho_4^{0/8}(s) + \rho_5^{0/8}(s) + \rho_6^{0/8}(s) + \rho_7^{0/8}(s) + \rho_8^{0/8}(s) + \rho_{10}^{0/8}(s), \quad (24)$$

$$\rho_0^0(s) = \frac{1}{4096\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz yz (1-y-z)^3 (s - \overline{m}_Q^2)^2 (35s^2 - 26s\overline{m}_Q^2 + 3\overline{m}_Q^4), \quad (25)$$

$$\rho_0^8(s) = \frac{1}{1152\pi^6} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz yz (1-y-z)^3 (s - \overline{m}_Q^2)^2 (35s^2 - 26s\overline{m}_Q^2 + 3\overline{m}_Q^4), \quad (26)$$

$$\rho_3^0(s) = -\frac{3m_Q \langle \bar{q}q \rangle}{256\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz (y+z)(1-y-z) (s - \overline{m}_Q^2) (7s - 3\overline{m}_Q^2), \quad (27)$$

$$\rho_3^8(s) = -\frac{m_Q \langle \bar{q}q \rangle}{24\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz (y+z)(1-y-z) (s - \overline{m}_Q^2) (7s - 3\overline{m}_Q^2), \quad (28)$$

$$\begin{aligned} \rho_4^0(s) &= -\frac{m_Q^2}{3072\pi^4} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left(\frac{z}{y^2} + \frac{y}{z^2} \right) (1-y-z)^3 \{ 8s - 3\overline{m}_Q^2 + \overline{m}_Q^4 \delta(s - \overline{m}_Q^2) \} \\ &\quad + \frac{1}{1024\pi^4} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz (y+z)(1-y-z)^2 s (5s - 4\overline{m}_Q^2), \end{aligned} \quad (29)$$

$$\begin{aligned} \rho_4^8(s) &= -\frac{m_Q^2}{864\pi^4} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left(\frac{z}{y^2} + \frac{y}{z^2} \right) (1-y-z)^3 \{ 8s - 3\overline{m}_Q^2 + \overline{m}_Q^4 \delta(s - \overline{m}_Q^2) \} \\ &\quad - \frac{1}{2304\pi^4} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz (y+z)(1-y-z)^2 s (5s - 4\overline{m}_Q^2) \\ &\quad + t \frac{m_Q^2}{1152\pi^4} \langle \frac{\alpha_s GG}{\pi} \rangle \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz (s - \overline{m}_Q^2) \left\{ 7 - 2 \left(\frac{1}{y} + \frac{1}{z} \right) (1-y-z) \right. \\ &\quad \left. + \frac{7(1-y-z)^2}{2yz} - \frac{7(1-y-z)}{2} + \left(\frac{1}{y} + \frac{1}{z} \right) \frac{(1-y-z)^2}{2} - \frac{7(1-y-z)^3}{12yz} \right\}, \end{aligned} \quad (30)$$

$$\begin{aligned} \rho_5^0(s) &= \frac{3m_Q \langle \bar{q}g_s \sigma G q \rangle}{512\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz (y+z) (5s - 3\overline{m}_Q^2) \\ &\quad - \frac{3m_Q \langle \bar{q}g_s \sigma G q \rangle}{256\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left(\frac{y}{z} + \frac{z}{y} \right) (1-y-z) (2s - \overline{m}_Q^2), \end{aligned} \quad (31)$$

$$\begin{aligned} \rho_5^8(s) &= \frac{m_Q \langle \bar{q}g_s \sigma G q \rangle}{48\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz (y+z) (5s - 3\overline{m}_Q^2) \\ &\quad + \frac{m_Q \langle \bar{q}g_s \sigma G q \rangle}{192\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left(\frac{y}{z} + \frac{z}{y} \right) (1-y-z) (2s - \overline{m}_Q^2) \\ &\quad + t \frac{m_Q \langle \bar{q}g_s \sigma G q \rangle}{576\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left(\frac{y}{z} + \frac{z}{y} \right) (1-y-z) (5s - 3\overline{m}_Q^2), \end{aligned} \quad (32)$$

$$\begin{aligned} \rho_6^0(s) &= \frac{m_Q^2 \langle \bar{q}q \rangle^2}{16\pi^2} \int_{y_i}^{y_f} dy + \frac{g_s^2 \langle \bar{q}q \rangle^2}{864\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz yz \{ 8s - 3\overline{m}_Q^2 + \overline{m}_Q^4 \delta(s - \overline{m}_Q^2) \} \\ &\quad - \frac{g_s^2 \langle \bar{q}q \rangle^2}{576\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz (1-y-z) \left\{ \left(\frac{z}{y} + \frac{y}{z} \right) (7s - 4\overline{m}_Q^2) \right. \\ &\quad \left. + \frac{1}{3} \left(\frac{z}{y^2} + \frac{y}{z^2} \right) m_Q^2 [7 + 5\overline{m}_Q^2 \delta(s - \overline{m}_Q^2)] - \frac{1}{3} (y+z) (4s - 3\overline{m}_Q^2) \right\}, \end{aligned} \quad (33)$$

$$\begin{aligned}
\rho_6^8(s) = & \frac{2m_Q^2 \langle \bar{q}q \rangle^2}{9\pi^2} \int_{y_i}^{y_f} dy + \frac{g_s^2 \langle \bar{q}q \rangle^2}{243\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz yz \left\{ 8s - 3\overline{m}_Q^2 + \overline{m}_Q^4 \delta(s - \overline{m}_Q^2) \right\} \\
& + \frac{g_s^2 \langle \bar{q}q \rangle^2}{1296\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz (1-y-z) \left\{ \left(\frac{z}{y} + \frac{y}{z} \right) (7s - 4\overline{m}_Q^2) \right. \\
& \left. + \frac{1}{3} \left(\frac{z}{y^2} + \frac{y}{z^2} \right) m_Q^2 [7 + 5\overline{m}_Q^2 \delta(s - \overline{m}_Q^2)] - \frac{1}{3}(y+z)(4s - 3\overline{m}_Q^2) \right\} \\
& - \frac{g_s^2 \langle \bar{q}q \rangle^2}{1944\pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz (1-y-z) \left\{ 3 \left(\frac{z}{y} + \frac{y}{z} \right) (2s - \overline{m}_Q^2) \right. \\
& \left. + \left(\frac{z}{y^2} + \frac{y}{z^2} \right) m_Q^2 [1 + \overline{m}_Q^2 \delta(s - \overline{m}_Q^2)] + 2(y+z)[8s - 3\overline{m}_Q^2 + \overline{m}_Q^4 \delta(s - \overline{m}_Q^2)] \right\} \quad (34)
\end{aligned}$$

$$\begin{aligned}
\rho_7^0(s) = & \frac{m_Q^3 \langle \bar{q}q \rangle}{1536\pi^2} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left(\frac{y}{z^3} + \frac{z}{y^3} + \frac{1}{y^2} + \frac{1}{z^2} \right) (1-y-z) \\
& \left(1 + \frac{2\overline{m}_Q^2}{T^2} \right) \delta(s - \overline{m}_Q^2) \\
& - \frac{3m_Q \langle \bar{q}q \rangle}{256\pi^2} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left(\frac{y}{z^2} + \frac{z}{y^2} \right) (1-y-z) \left\{ 1 + \frac{2\overline{m}_Q^2}{3} \delta(s - \overline{m}_Q^2) \right\} \\
& - \frac{m_Q \langle \bar{q}q \rangle}{128\pi^2} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left\{ 1 + \frac{2\overline{m}_Q^2}{3} \delta(s - \overline{m}_Q^2) \right\} \\
& - \frac{m_Q \langle \bar{q}q \rangle}{512\pi^2} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int_{y_i}^{y_f} dy \left\{ 1 + \frac{2\tilde{m}_Q^2}{3} \delta(s - \tilde{m}_Q^2) \right\}, \quad (35)
\end{aligned}$$

$$\begin{aligned}
\rho_7^8(s) = & \frac{m_Q^3 \langle \bar{q}q \rangle}{432\pi^2} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left(\frac{y}{z^3} + \frac{z}{y^3} + \frac{1}{y^2} + \frac{1}{z^2} \right) (1-y-z) \\
& \left(1 + \frac{2\overline{m}_Q^2}{T^2} \right) \delta(s - \overline{m}_Q^2) \\
& - \frac{m_Q \langle \bar{q}q \rangle}{24\pi^2} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left(\frac{y}{z^2} + \frac{z}{y^2} \right) (1-y-z) \left\{ 1 + \frac{2\overline{m}_Q^2}{3} \delta(s - \overline{m}_Q^2) \right\} \\
& + \frac{m_Q \langle \bar{q}q \rangle}{288\pi^2} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left\{ 1 + \frac{2\overline{m}_Q^2}{3} \delta(s - \overline{m}_Q^2) \right\} \\
& + t \frac{m_Q \langle \bar{q}q \rangle}{144\pi^2} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left\{ 1 - 7 \left(\frac{1}{y} + \frac{1}{z} \right) \frac{1-y-z}{4} \right\} \left\{ 1 + \frac{2\overline{m}_Q^2}{3} \delta(s - \overline{m}_Q^2) \right\} \\
& - \frac{m_Q \langle \bar{q}q \rangle}{144\pi^2} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle \int_{y_i}^{y_f} dy \left\{ 1 + \frac{2\tilde{m}_Q^2}{3} \delta(s - \tilde{m}_Q^2) \right\}, \quad (36)
\end{aligned}$$

$$\begin{aligned}
\rho_8^0(s) = & - \frac{m_Q^2 \langle \bar{q}q \rangle \langle \bar{q}g_s \sigma Gq \rangle}{32\pi^2} \int_0^1 dy \left(1 + \frac{\tilde{m}_Q^2}{T^2} \right) \delta(s - \tilde{m}_Q^2) \\
& + \frac{m_Q^2 \langle \bar{q}q \rangle \langle \bar{q}g_s \sigma Gq \rangle}{64\pi^2} \int_0^1 dy \left(\frac{1}{y} + \frac{1}{1-y} \right) \delta(s - \tilde{m}_Q^2), \quad (37)
\end{aligned}$$

$$\begin{aligned}
\rho_8^8(s) = & -\frac{m_Q^2 \langle \bar{q}q \rangle \langle \bar{q}g_s \sigma G q \rangle}{9\pi^2} \int_0^1 dy \left(1 + \frac{\tilde{m}_Q^2}{T^2} \right) \delta(s - \tilde{m}_Q^2) \\
& -\frac{m_Q^2 \langle \bar{q}q \rangle \langle \bar{q}g_s \sigma G q \rangle}{144\pi^2} \int_0^1 dy \left(\frac{1}{y} + \frac{1}{1-y} \right) \delta(s - \tilde{m}_Q^2) \\
& -t \frac{\langle \bar{q}q \rangle \langle \bar{q}g_s \sigma G q \rangle}{144\pi^2} \int_{y_i}^{y_f} dy \left\{ 1 + \frac{2\tilde{m}_Q^2}{3} \delta(s - \tilde{m}_Q^2) \right\}, \tag{38}
\end{aligned}$$

$$\begin{aligned}
\rho_{10}^0(s) = & \frac{m_Q^2 \langle \bar{q}g_s \sigma G q \rangle^2}{256\pi^2 T^6} \int_0^1 dy \tilde{m}_Q^4 \delta(s - \tilde{m}_Q^2) \\
& -\frac{m_Q^4 \langle \bar{q}q \rangle^2 \langle \frac{\alpha_s GG}{\pi} \rangle}{288T^4} \int_0^1 dy \left\{ \frac{1}{y^3} + \frac{1}{(1-y)^3} \right\} \delta(s - \tilde{m}_Q^2) \\
& +\frac{m_Q^2 \langle \bar{q}q \rangle^2 \langle \frac{\alpha_s GG}{\pi} \rangle}{96T^2} \int_0^1 dy \left\{ \frac{1}{y^2} + \frac{1}{(1-y)^2} \right\} \delta(s - \tilde{m}_Q^2) \\
& -\frac{m_Q^2 \langle \bar{q}g_s \sigma G q \rangle^2}{256\pi^2 T^4} \int_0^1 dy \left(\frac{1}{y} + \frac{1}{1-y} \right) \tilde{m}_Q^2 \delta(s - \tilde{m}_Q^2) \\
& +t \frac{\langle \bar{q}g_s \sigma G q \rangle^2}{20736\pi^2} \int_0^1 dy \left(1 + \frac{2\tilde{m}_Q^2}{T^2} \right) \delta(s - \tilde{m}_Q^2) \\
& +\frac{m_Q^2 \langle \bar{q}q \rangle^2 \langle \frac{\alpha_s GG}{\pi} \rangle}{288T^6} \int_0^1 dy \tilde{m}_Q^4 \delta(s - \tilde{m}_Q^2), \tag{39}
\end{aligned}$$

$$\begin{aligned}
\rho_{10}^8(s) = & \frac{m_Q^2 \langle \bar{q}g_s \sigma G q \rangle^2}{72\pi^2 T^6} \int_0^1 dy \tilde{m}_Q^4 \delta(s - \tilde{m}_Q^2) \\
& -\frac{m_Q^4 \langle \bar{q}q \rangle^2 \langle \frac{\alpha_s GG}{\pi} \rangle}{81T^4} \int_0^1 dy \left\{ \frac{1}{y^3} + \frac{1}{(1-y)^3} \right\} \delta(s - \tilde{m}_Q^2) \\
& +\frac{m_Q^2 \langle \bar{q}q \rangle^2 \langle \frac{\alpha_s GG}{\pi} \rangle}{27T^2} \int_0^1 dy \left\{ \frac{1}{y^2} + \frac{1}{(1-y)^2} \right\} \delta(s - \tilde{m}_Q^2) \\
& +7t \frac{\langle \bar{q}q \rangle^2 \langle \frac{\alpha_s GG}{\pi} \rangle}{1296} \int_0^1 dy \left(1 + \frac{2\tilde{m}_Q^2}{T^2} \right) \delta(s - \tilde{m}_Q^2) \\
& +\frac{m_Q^2 \langle \bar{q}g_s \sigma G q \rangle^2}{576\pi^2 T^4} \int_0^1 dy \left(\frac{1}{y} + \frac{1}{1-y} \right) \tilde{m}_Q^2 \delta(s - \tilde{m}_Q^2) \\
& +t \frac{\langle \bar{q}g_s \sigma G q \rangle^2}{864\pi^2} \int_0^1 dy \left(1 + \frac{3\tilde{m}_Q^2}{2T^2} + \frac{\tilde{m}_Q^4}{T^4} \right) \delta(s - \tilde{m}_Q^2) \\
& +t \frac{\langle \bar{q}g_s \sigma G q \rangle^2}{5832\pi^2} \int_0^1 dy \left(1 + \frac{2\tilde{m}_Q^2}{T^2} \right) \delta(s - \tilde{m}_Q^2) \\
& +\frac{m_Q^2 \langle \bar{q}q \rangle^2 \langle \frac{\alpha_s GG}{\pi} \rangle}{81T^6} \int_0^1 dy \tilde{m}_Q^4 \delta(s - \tilde{m}_Q^2), \tag{40}
\end{aligned}$$

the subscripts 0, 3, 4, 5, 6, 7, 8, 10 denote the dimensions of the vacuum condensates, the superscripts 0, 8 denote the 0-0 type and 8-8 type interpolating currents respectively; $y_f = \frac{1+\sqrt{1-4m_Q^2/s}}{2}$,

$$y_i = \frac{1-\sqrt{1-4m_Q^2/s}}{2}, z_i = \frac{ym_Q^2}{ys-m_Q^2}, \bar{m}_Q^2 = \frac{(y+z)m_Q^2}{yz}, \tilde{m}_Q^2 = \frac{m_Q^2}{y(1-y)}, \int_{y_i}^{y_f} dy \rightarrow \int_0^1 dy, \int_{z_i}^{1-y} dz \rightarrow \int_0^{1-y} dz$$

when the δ functions $\delta(s - \bar{m}_Q^2)$ and $\delta(s - \tilde{m}_Q^2)$ appear.

In this article, we carry out the operator product expansion to the vacuum condensates up to dimension-10, and assume vacuum saturation for the higher dimensional vacuum condensates. The condensates $\langle \frac{\alpha_s}{\pi} GG \rangle$, $\langle \bar{q}q \rangle \langle \frac{\alpha_s}{\pi} GG \rangle$, $\langle \bar{q}q \rangle^2 \langle \frac{\alpha_s}{\pi} GG \rangle$, $\langle \bar{q}g_s \sigma Gq \rangle^2$ and $g_s^2 \langle \bar{q}q \rangle^2$ are the vacuum expectations of the operators of the order $\mathcal{O}(\alpha_s)$. The four-quark condensate $g_s^2 \langle \bar{q}q \rangle^2$ comes from the terms $\langle \bar{q}\gamma_\mu t^a q g_s D_\eta G_{\lambda\tau}^a \rangle$, $\langle \bar{q}_j D_\mu^\dagger D_\nu^\dagger D_\alpha^\dagger q_i \rangle$ and $\langle \bar{q}_j D_\mu D_\nu D_\alpha q_i \rangle$, rather than comes from the perturbative corrections of $\langle \bar{q}q \rangle^2$. The condensates $\langle g_s^3 GGG \rangle$, $\langle \frac{\alpha_s GG}{\pi} \rangle^2$, $\langle \frac{\alpha_s GG}{\pi} \rangle \langle \bar{q}g_s \sigma Gq \rangle$ have the dimensions 6, 8, 9 respectively, but they are the vacuum expectations of the operators of the order $\mathcal{O}(\alpha_s^{3/2})$, $\mathcal{O}(\alpha_s^2)$, $\mathcal{O}(\alpha_s^{3/2})$ respectively, and discarded. We take the truncations $n \leq 10$ and $k \leq 1$ in a consistent way, the operators of the orders $\mathcal{O}(\alpha_s^k)$ with $k > 1$ are discarded. Furthermore, the values of the condensates $\langle g_s^3 GGG \rangle$, $\langle \frac{\alpha_s GG}{\pi} \rangle^2$, $\langle \frac{\alpha_s GG}{\pi} \rangle \langle \bar{q}g_s \sigma Gq \rangle$ are very small, and they can be neglected safely.

We differentiate Eq.(23) with respect to $\frac{1}{T^2}$, eliminate the pole residues $\lambda_{X/Z}$, and obtain the QCD sum rules for the masses,

$$M_{X/Z}^2 = \frac{\int_{4m_Q^2}^{s_0} ds \frac{d}{d(-1/T^2)} \rho^{0/8}(s) e^{-\frac{s}{T^2}}}{\int_{4m_Q^2}^{s_0} ds \rho^{0/8}(s) e^{-\frac{s}{T^2}}}. \quad (41)$$

3 Numerical results and discussions

The input parameters are taken to be the standard values $\langle \bar{q}q \rangle = -(0.24 \pm 0.01 \text{ GeV})^3$, $\langle \bar{q}g_s \sigma Gq \rangle = m_0^2 \langle \bar{q}q \rangle$, $m_0^2 = (0.8 \pm 0.1) \text{ GeV}^2$, $\langle \frac{\alpha_s GG}{\pi} \rangle = (0.33 \text{ GeV})^4$ at the energy scale $\mu = 1 \text{ GeV}$ [30, 31, 33]. The quark condensate and mixed quark condensate evolve with the renormalization group equation, $\langle \bar{q}q \rangle(\mu) = \langle \bar{q}q \rangle(Q) \left[\frac{\alpha_s(Q)}{\alpha_s(\mu)} \right]^{\frac{4}{9}}$ and $\langle \bar{q}g_s \sigma Gq \rangle(\mu) = \langle \bar{q}g_s \sigma Gq \rangle(Q) \left[\frac{\alpha_s(Q)}{\alpha_s(\mu)} \right]^{\frac{2}{27}}$.

In the article, we take the \overline{MS} masses $m_c(m_c) = (1.275 \pm 0.025) \text{ GeV}$ and $m_b(m_b) = (4.18 \pm 0.03) \text{ GeV}$ from the Particle Data Group [34], and take into account the energy-scale dependence of the \overline{MS} masses from the renormalization group equation,

$$\begin{aligned} m_c(\mu) &= m_c(m_c) \left[\frac{\alpha_s(\mu)}{\alpha_s(m_c)} \right]^{\frac{12}{25}}, \\ m_b(\mu) &= m_b(m_b) \left[\frac{\alpha_s(\mu)}{\alpha_s(m_b)} \right]^{\frac{12}{23}}, \\ \alpha_s(\mu) &= \frac{1}{b_0 t} \left[1 - \frac{b_1 \log t}{b_0^2 t} + \frac{b_1^2 (\log^2 t - \log t - 1) + b_0 b_2}{b_0^4 t^2} \right], \end{aligned} \quad (42)$$

where $t = \log \frac{\mu^2}{\Lambda^2}$, $b_0 = \frac{33-2n_f}{12\pi}$, $b_1 = \frac{153-19n_f}{24\pi^2}$, $b_2 = \frac{2857-\frac{5033}{9}n_f+\frac{325}{27}n_f^2}{128\pi^3}$, $\Lambda = 213 \text{ MeV}$, 296 MeV and 339 MeV for the flavors $n_f = 5, 4$ and 3 , respectively [34].

We tentatively take the threshold parameters of the axial-vector molecular states $X(3872)$ (or $Z_c(3900)$) and $Z_b(10610)$ as $s_0 = (18.5 - 20.5) \text{ GeV}^2$ and $(122 - 126) \text{ GeV}^2$ respectively to avoid the contaminations of the higher resonances and continuum states, and search for the optimal values to satisfy the two criteria (pole dominance and convergence of the operator product expansion) of the QCD sum rules. In this article, we assume that the energy gap between the ground states and the first radial excited states is about $(0.4 - 0.6) \text{ GeV}$, just like that of the conventional mesons.

The correlation functions $\Pi(p)$ can be written as

$$\begin{aligned}\Pi(p) &= \sum_n C_n(p^2, \mu) \langle \mathcal{O}_n(\mu) \rangle = \int_{4m_Q^2(\mu)}^{\infty} ds \frac{\rho_{QCD}(s, \mu)}{s - p^2} \\ &= \int_{4m_Q^2(\mu)}^{s_0} ds \frac{\rho_{QCD}(s, \mu)}{s - p^2} + \int_{s_0}^{\infty} ds \frac{\rho_{QCD}(s, \mu)}{s - p^2},\end{aligned}\quad (43)$$

at the QCD side, where the $C_n(p^2, \mu)$ are the Wilson coefficients and the $\langle \mathcal{O}_n(\mu) \rangle$ are the vacuum condensates of dimension- n . The short-distance contributions at $p^2 > \mu^2$ are included in the coefficients $C_n(p^2, \mu)$, the long-distance contributions at $p^2 < \mu^2$ are absorbed into the vacuum condensates $\langle \mathcal{O}_n(\mu) \rangle$. If $\mu \gg \Lambda_{QCD}$, the Wilson coefficients $C_n(p^2, \mu)$ depend only on short-distance dynamics, while the long-distance effects are taken into account by the vacuum condensates $\langle \mathcal{O}_n(\mu) \rangle$.

The correlation functions $\Pi(p)$ are scale independent,

$$\frac{d}{d\mu} \Pi(p) = 0, \quad (44)$$

which does not mean

$$\frac{d}{d\mu} \int_{4m_Q^2(\mu)}^{s_0} ds \frac{\rho_{QCD}(s, \mu)}{s - p^2} \rightarrow 0, \quad (45)$$

in the present case due to the following two reasons:

- Perturbative corrections are neglected, the higher dimensional vacuum condensates are factorized into lower dimensional ones therefore the energy scale dependence of the higher dimensional vacuum condensates is modified;
- Truncations s_0 set in, the correlation between the threshold $4m_Q^2(\mu)$ and continuum threshold s_0 is unknown, the quark-hadron duality is an assumption.

We perform the Borel transform with respect to the variable $P^2 = -p^2$ at the QCD side and obtain the result,

$$\int_{4m_Q^2(\mu)}^{s_0} ds \frac{\rho_{QCD}(s, \mu)}{s - p^2} \rightarrow \int_{4m_Q^2(\mu)}^{s_0} ds \frac{\rho_{QCD}(s, \mu)}{T^2} \exp\left(-\frac{s}{T^2}\right). \quad (46)$$

The QCD sum rules are characterized by two energy scales μ^2 and T^2 . The Borel parameters T^2 have to be small enough such that the contributions from the higher resonances and continuum states are damped sufficiently. On the other hand, the Borel parameters T^2 must be large enough such that the higher-dimensional vacuum condensates are suppressed sufficiently.

The heavy tetraquark system $Q\bar{Q}q'\bar{q}$ could be described by a double-well potential with two light quarks $q'\bar{q}$ lying in the two wells respectively. In the heavy quark limit, the c (and b) quark can be taken as a static well potential, which binds the light quark q' to form a diquark in the color antitriplet channel or binds the light antiquark \bar{q} to form a meson in the color singlet channel (or a meson-like state in the color octet channel). Then the heavy tetraquark states are characterized by the effective heavy quark masses M_Q (or constituent quark masses) and the virtuality $V = \sqrt{M_{X/Y/Z}^2 - (2M_Q)^2}$ (or bound energy not as robust). The effective masses M_Q have uncertainties, the optimal values in the diquark-antidiquark systems are not necessary the ideal values in the meson-meson systems.

Now the QCD sum rules have three typical energy scales μ^2, T^2, V^2 . It is natural to take the energy scale,

$$\mu^2 = V^2 = \mathcal{O}(T^2). \quad (47)$$

J^{PC}	$T^2(\text{GeV}^2)$	$s_0(\text{GeV}^2)$	pole	$M_{X/Z}(\text{GeV})$	$\lambda_{X/Z}(\text{GeV}^5)$
$1^{++} (\bar{u}c\bar{c}d)_{0-0}$	$2.2 - 2.8$	19.5 ± 1	(49 – 80)%	$3.89^{+0.09}_{-0.09}$	$1.72^{+0.29}_{-0.25} \times 10^{-2}$
$1^{+-} (\bar{u}c\bar{c}d)_{0-0}$	$2.2 - 2.8$	19.5 ± 1	(49 – 80)%	$3.89^{+0.09}_{-0.09}$	$1.72^{+0.29}_{-0.25} \times 10^{-2}$
$1^{++} (\bar{u}b\bar{b}d)_{0-0}$	$7.2 - 8.0$	124 ± 2	(47 – 65)%	$10.61^{+0.10}_{-0.09}$	$1.13^{+0.17}_{-0.14} \times 10^{-1}$
$1^{+-} (\bar{u}b\bar{b}d)_{0-0}$	$7.2 - 8.0$	124 ± 2	(47 – 65)%	$10.61^{+0.10}_{-0.09}$	$1.13^{+0.17}_{-0.14} \times 10^{-1}$
$1^{++} (\bar{u}c\bar{c}d)_{8-8}$	$2.6 - 3.3$	22 ± 1	(51 – 80)%	$4.08^{+0.10}_{-0.10}$	$5.70^{+0.98}_{-0.81} \times 10^{-2}$
$1^{+-} (\bar{u}c\bar{c}d)_{8-8}$	$2.6 - 3.3$	22 ± 1	(50 – 79)%	$4.10^{+0.09}_{-0.10}$	$5.75^{+0.97}_{-0.80} \times 10^{-2}$
$1^{++} (\bar{u}b\bar{b}d)_{8-8}$	$7.4 - 8.2$	126 ± 2	(50 – 67)%	$10.66^{+0.11}_{-0.08}$	$2.63^{+0.37}_{-0.32} \times 10^{-1}$
$1^{+-} (\bar{u}b\bar{b}d)_{8-8}$	$7.4 - 8.2$	126 ± 2	(50 – 67)%	$10.66^{+0.11}_{-0.09}$	$2.63^{+0.37}_{-0.31} \times 10^{-1}$

Table 1: The Borel parameters, continuum threshold parameters, pole contributions, masses and pole residues of the 0-0 type and 8-8 type molecular states.

In Figs.1-3, we will plot only the lines for the 0-0 type molecular states for simplicity. In Fig.1, the masses are plotted with variations of the Borel parameters T^2 and energy scales μ with the threshold parameters $s_0 = 19.5 \text{ GeV}^2$ and 124 GeV^2 for the 0-0 type hidden charmed and hidden bottom molecular states, respectively. From the figure, we can see that the masses decrease monotonously with increase of the energy scales, just like that of the tetraquark states [25, 26, 27, 28]. If the energy scale formula $\mu = \sqrt{M_{X/Y/Z}^2 - (2M_Q)^2}$ with the effective masses $M_c = 1.80 \text{ GeV}$ and $M_b = 5.13 \text{ GeV}$ is also an acceptable choice in the case of the hadronic molecular states, the energy scales $\mu = 1.5 \text{ GeV}$ and 2.7 GeV for the hidden charmed and hidden bottom molecular states respectively should reproduce the experimental values of the masses of the $X(3872)$, $Z_c(3900)$ and $Z_b(10610)$.

In calculations, we observe that the effective masses $M_c = 1.80 \text{ GeV}$ and $M_b = 5.13 \text{ GeV}$ are acceptable values (if the uncertainties of the QCD sum rules are taken into account) but not the optimal values to reproduce the experimental values of the masses of the $X(3872)$, $Z_c(3900)$, $Z_c(4020)$, $Z_c(4025)$, $Y(4140)$, $Z_b(10610)$ and $Z_b(10650)$ consistently in the scenario of molecular states [35]. The energy scales $\mu = 1.3 \text{ GeV}$ and 2.6 GeV are the optimal energy scales to reproduce the experimental data $M_{X(3872)} = 3.87 \text{ GeV}$, $M_{Z_c(3900)} = 3.90 \text{ GeV}$, $M_{Z_b(10610)} = 10.61 \text{ GeV}$ (also the experimental values of the masses of the $Z_c(4020)$, $Z_c(4025)$, $Y(4140)$ and $Z_b(10650)$ [35]) approximately. The modified values $M_c = 1.84 \text{ GeV}$ and $M_b = 5.14 \text{ GeV}$ work for the hadronic molecular states, and can be used to update the QCD sum rules for the heavy molecular states [36].

In Fig.2, the contributions of the pole terms are plotted with variations of the threshold parameters s_0 and Borel parameters T^2 at the energy scales $\mu = 1.3 \text{ GeV}$ and 2.6 GeV for the 0-0 type hidden charmed and hidden bottom molecular states, respectively. In Fig.3, the contributions of different terms in the operator product expansion are plotted with variations of the Borel parameters T^2 with the parameters $s_0 = 19.5 \text{ GeV}^2$, $\mu = 1.3 \text{ GeV}$ and $s_0 = 124 \text{ GeV}^2$, $\mu = 2.6 \text{ GeV}$ for the 0-0 type hidden charmed and hidden bottom molecular states, respectively. From the figures, we can choose the optimal Borel parameters and threshold parameters to satisfy the two criteria of the QCD sum rules. The Borel parameters, continuum threshold parameters and the pole contributions are shown explicitly in Table 1.

We take into account all uncertainties of the input parameters, and obtain the values of the masses and pole residues of the molecular states, which are shown in Table 1 and Figs.4-5.

The masses of the 0-0 type molecular states $\bar{u}c\bar{c}d(1^{++})$, $\bar{u}c\bar{c}d(1^{+-})$ and $\bar{u}b\bar{b}d(1^{+-})$ are consistent

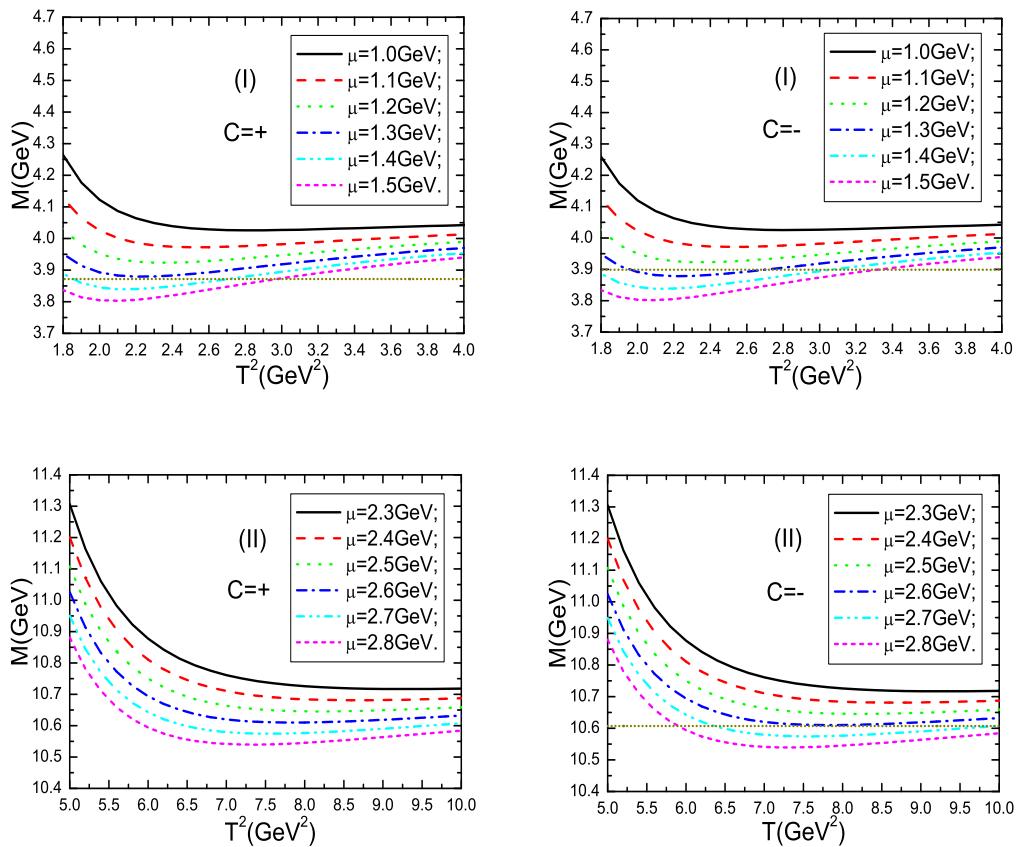


Figure 1: The masses with variations of the Borel parameters T^2 and energy scales μ , where the (I) and (II) denote the 0-0 type hidden charmed and hidden bottom molecular states respectively; the horizontal lines denote the experimental values; the $C = \pm$ denote the charge conjugations.

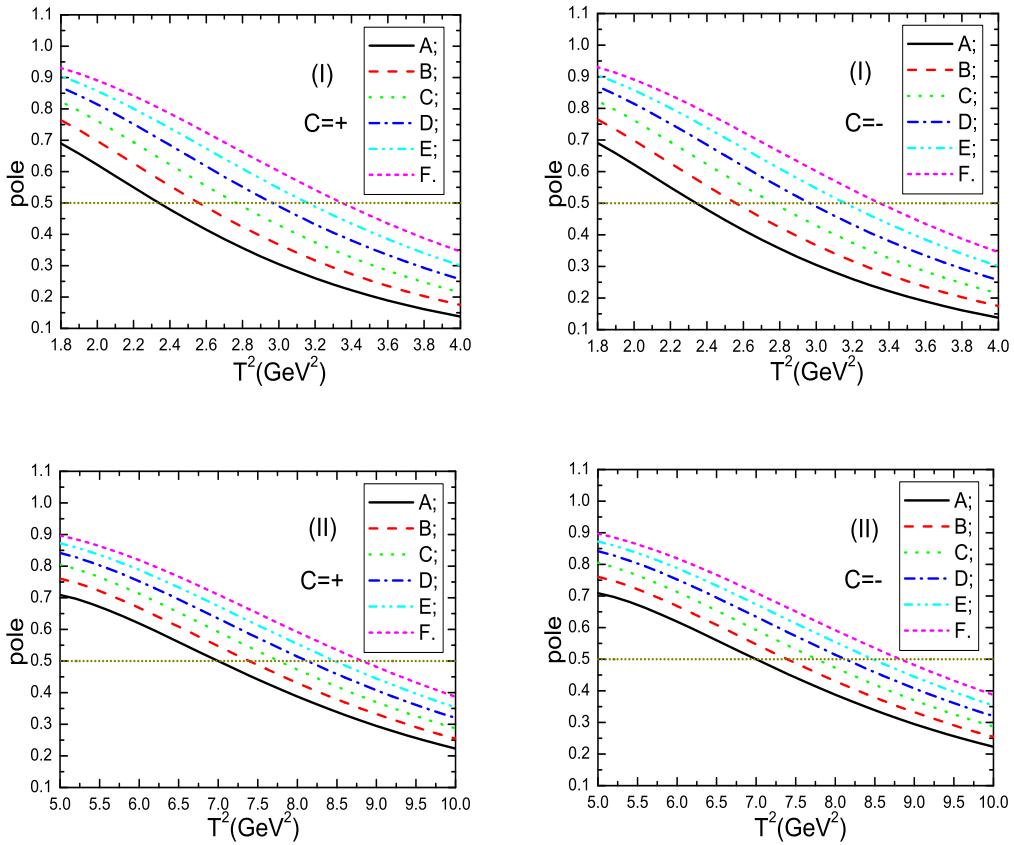


Figure 2: The pole contributions with variations of the Borel parameters T^2 and threshold parameters s_0 , where the (I) and (II) denote the 0-0 type hidden charmed and hidden bottom molecular states respectively; the A, B, C, D, E, F denote the threshold parameters $s_0 = 16.5, 17.5, 18.5, 19.5, 20.5, 21.5 \text{ GeV}^2$ respectively for the hidden charmed molecular states, $s_0 = 118, 120, 122, 124, 126, 128 \text{ GeV}^2$ respectively for the hidden bottom molecular states; the $C = \pm$ denote the charge conjugations.

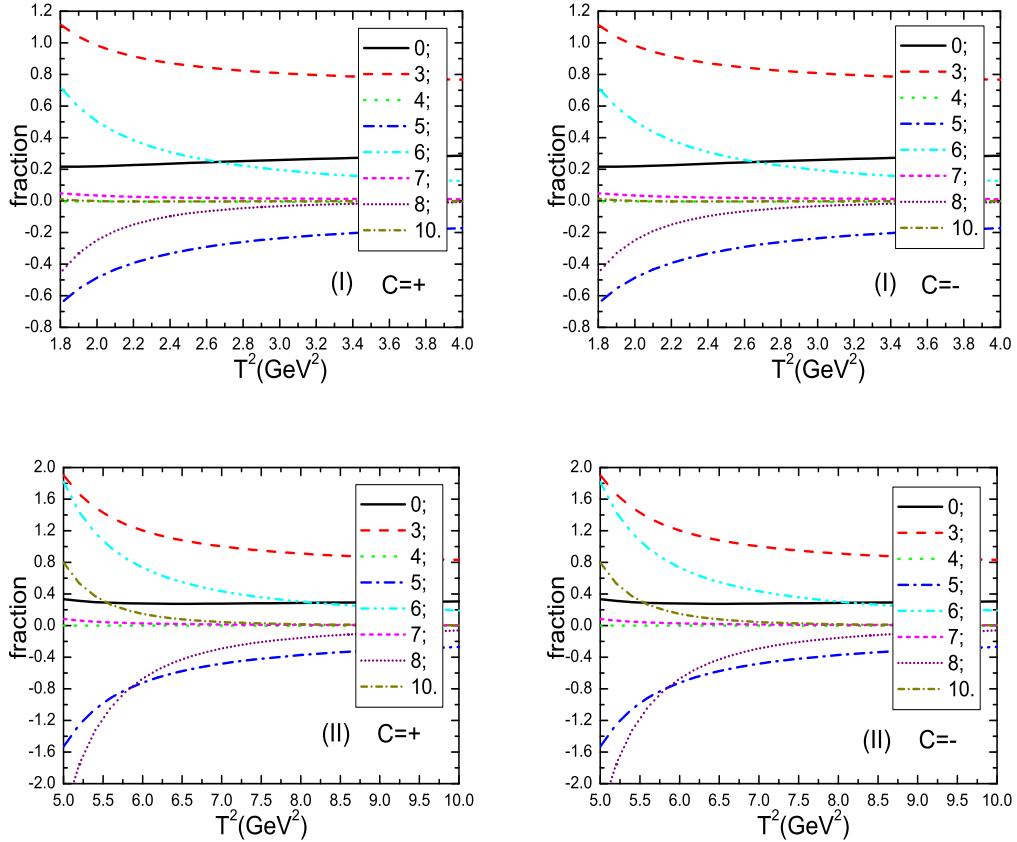


Figure 3: The contributions of different terms in the operator product expansion with variations of the Borel parameters T^2 , where the (I) and (II) denote the 0-0 type hidden charmed and hidden bottom molecular states respectively; the 0, 3, 4, 5, 6, 7, 8, 10 denote the dimensions of the vacuum condensates; the $C = \pm$ denote the charge conjugations.

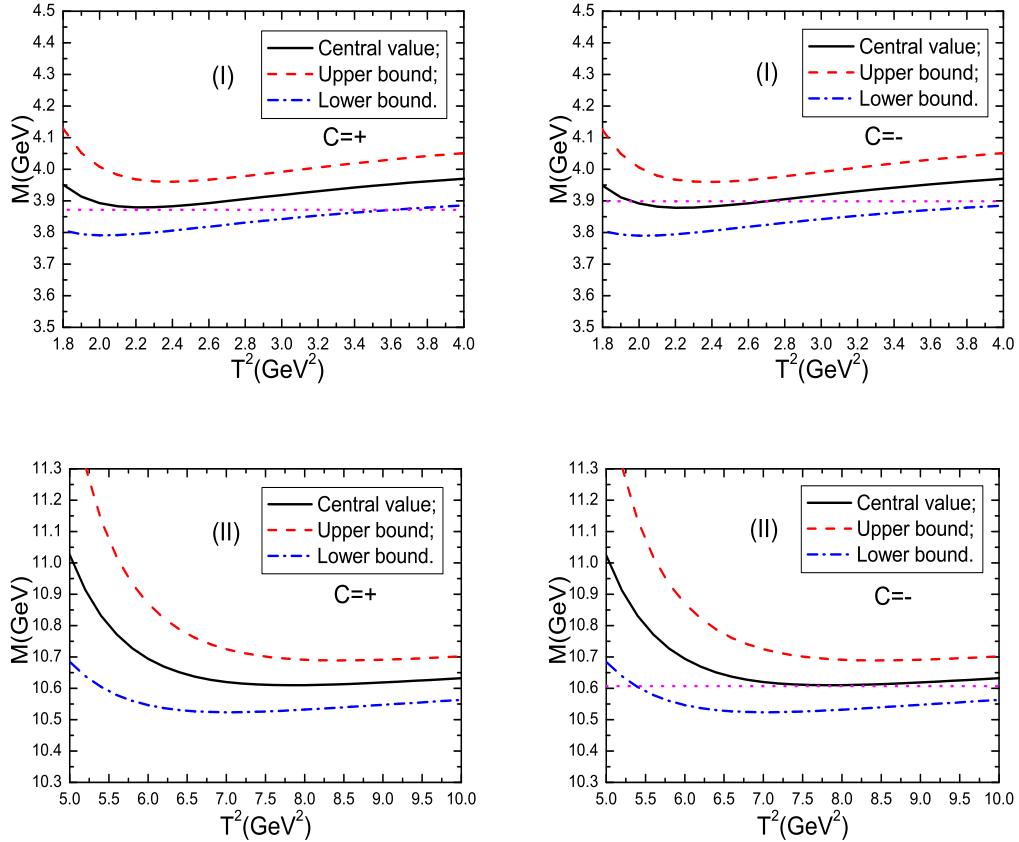


Figure 4: The masses of the 0-0 type molecular states with variations of the Borel parameters T^2 , where the horizontal lines denote the experimental values; the (I) and (II) denote the hidden charmed and hidden bottom molecular states, respectively; the $C = \pm$ denote the charge conjugations.

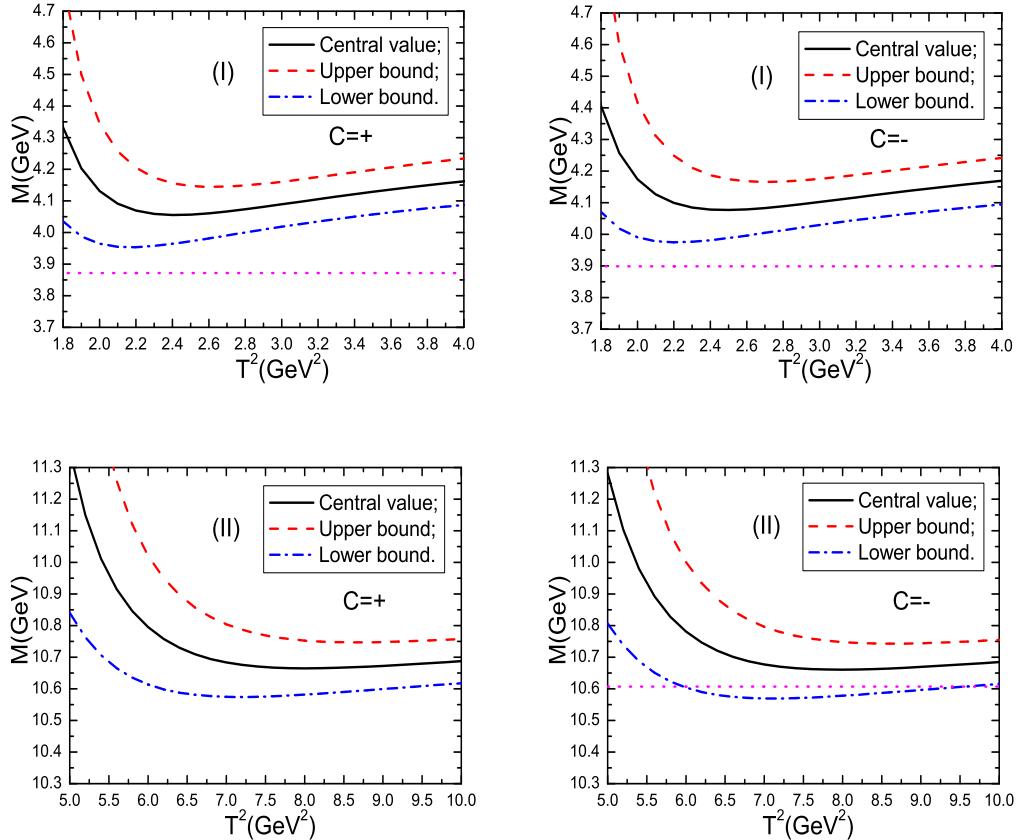


Figure 5: The masses of the 8-8 type molecular states with variations of the Borel parameters T^2 , where the horizontal lines denote the experimental values; the (I) and (II) denote the hidden charmed and hidden bottom molecular states, respectively; the $C = \pm$ denote the charge conjugations.

with that of the $X(3872)$, $Z_c(3900)$ and $Z_b(10610)$ respectively within uncertainties,

$$M_{\bar{u}c\bar{c}d,1^{++},(0-0)} = (3.89_{-0.09}^{+0.09}) \text{ GeV} \approx M_{X(3872)} = (3871.68 \pm 0.17) \text{ MeV (exp)[34]}, \quad (48)$$

$$M_{\bar{u}c\bar{c}d,1^{+-},(0-0)} = (3.89_{-0.09}^{+0.09}) \text{ GeV} \approx M_{Z_c(3900)} = (3899.0 \pm 3.6 \pm 4.9) \text{ MeV (exp)[12]}, \quad (49)$$

$$M_{\bar{u}b\bar{b}d,1^{+-},(0-0)} = (10.61_{-0.09}^{+0.10}) \text{ GeV} \approx M_{Z_b(10610)} = (10607.2 \pm 2.0) \text{ MeV (exp)[6]}. \quad (50)$$

The present predictions favor assigning the $X(3872)$, $Z_c(3900)$, $Z_b(10610)$ as the S -wave $D^*\bar{D}$, $D^*\bar{D}$ and $B^*\bar{B}$ molecular states, respectively, while our previous works favor assigning the $X(3872)$, $Z_c(3900)$, $Z_b(10610)$ as the diquark-antidiquark type tetraquark states [25, 28].

Although the mass is a fundamental parameter in describing a hadron, a hadron cannot be identified unambiguously by the mass alone, more theoretical and experimental works on the productions and decays are still needed to identify the $X(3872)$, $Z_c(3900)$, $Z_b(10610)$. At the present time, it is still a open problem. From Table 1, we can see that the charge conjugation partners have almost degenerate masses, and the 8-8 type molecular states have larger masses than that of the 0-0 type molecular states. The present predictions can be confronted with the experimental data in the future at the BESIII, LHCb and Belle-II.

4 Conclusion

In this article, we take the $X(3872)$, $Z_c(3900)$, $Z_b(10610)$ as the molecular states, construct both the color singlet-singlet type and color octet-octet type currents to interpolate them, and calculate the vacuum condensates up to dimension-10 in the operator product expansion. Then we study the axial-vector hidden charmed and hidden bottom molecular states with the QCD sum rules, explore the energy scale dependence in details for the first time, and use the energy scale formula $\mu = \sqrt{M_{X/Y/Z}^2 - (2\mathbb{M}_Q)^2}$ suggested in our previous works with the modified effective masses $\mathbb{M}_c = 1.84 \text{ GeV}$ and $\mathbb{M}_b = 5.14 \text{ GeV}$ to determine the energy scales of the QCD spectral densities. The energy scale formula works well for both the hidden charmed (or bottom) molecular states and tetraquark states. In the QCD sum rules for the hidden charmed (or bottom) tetraquark states and molecular states, the hadronic masses and pole residues are sensitive to the heavy quark masses m_Q , the energy scale formula has outstanding advantage in determining the m_Q . The numerical results support assigning the $X(3872)$, $Z_c(3900)$ and $Z_b(10610)$ as the 0-0 type molecular states with $J^{PC} = 1^{++}$, 1^{+-} , 1^{+-} , respectively; while there are no candidates for the 8-8 type molecular states. The present predictions can be confronted with the experimental data in the future at the BESIII, LHCb and Belle-II. More theoretical and experimental works on the productions and decays are still needed to distinguish the molecule and tetraquark assignments, as a hadron cannot be identified unambiguously by the mass alone. The pole residues can be taken as basic input parameters to study relevant processes of the $X(3872)$, $Z_c(3900)$ and $Z_b(10610)$ with the three-point QCD sum rules.

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