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1 Problem 4

Find the average of function $f(\sigma) = |a_1 - a_2| + |a_3 - a_4| + |a_5 - a_6| + |a_7 - a_8|$, where σ is a permutation of $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$.

1.1 Solution

Method is to count the number of different distances noted by $|a_i - a_j|$. If you write 1 to 8 literally as 1, 2, 3, 4, 5, 6, 7, 8, you will note that the distance is from 1 to 7. First let's count the number of distance 1.

distance 1 This is easy: pairs like (1, 2), (2, 3), . . . , (7, 8) are distance 1. There are 7 of them.

distance 2 (1, 3), . . . , (6, 8). There are 6 of them.

distance 3 (1, 4), . . . , (5, 8). There are 5 of them.

distance 4 (1, 5), . . . , (4, 8). There are 4 of them.

distance 5 (1, 6), . . . , (3, 8). There are 3 of them.

distance 6 (1, 7), (2, 8). There are 2 of them.

distance 7 (1, 8). There are 1 of them.

Each pair will appear exactly this many times: $P_6^6 \times 4 \times 2$. Explanation is: once you select a pair, e.g., (1, 2), you can also flip them as (2, 1). So that is for multiplier 2. And then you can place them in each of the 4 pairs location. And lastly, P_6^6 means once you fixed the selection of the pair, the remaining 6 digits will permute this many times. So total sum of all permutations of $f(\sigma)$ is

$P_6^6 * 4 * 2 * (1 * 7 + 2 * 6 + 3 * 5 + 4 * 4 + 5 * 3 + 6 * 2 + 7 * 1)$. Average is given by this number divided by $P_8^8 = 40320$, which is 12.

See also: problem4.cpp for brutal force calculation. This is to verify the above solution is correct. The output of the program is below

brutal force method: total = 483840, number of permutations = 40320, average = 12
analytical method: average = 12

2 Problem 5

Find all real x such that

$$\log_{2x}(48\sqrt[3]{3}) = \log_{3x}(162\sqrt[3]{2}) \quad (1)$$

2.1 Solution

$$\frac{\ln(48\sqrt[3]{3})}{\ln(2) + \ln(x)} = \frac{\ln(162\sqrt[3]{2})}{\ln(3) + \ln(x)} \quad (2)$$

$$\ln(x) = \frac{\ln(3) \cdot \ln(48\sqrt[3]{3}) - \ln(2) \cdot \ln(162\sqrt[3]{2})}{\ln(162\sqrt[3]{2}) - \ln(48\sqrt[3]{3})} \quad (3)$$

$$x = \exp \left[\frac{\ln(3) \cdot \ln(48\sqrt[3]{3}) - \ln(2) \cdot \ln(162\sqrt[3]{2})}{\ln(162\sqrt[3]{2}) - \ln(48\sqrt[3]{3})} \right] \quad (4)$$

$$= \exp \left[\frac{\ln(3) \cdot \ln(3) - \ln(2) \cdot \ln(2)}{2(\ln(3) - \ln(2))} \right] \quad (5)$$

$$= \exp \left[\frac{\ln(6)}{2} \right] \quad (6)$$

$$x = \sqrt{6} \quad (7)$$