Contents

1 Problem 4

Find the average of function $f(\sigma) = |a_1 - a_2| + |a_3 - a_4| + |a_5 - a_6| + |a_7 - a_8|$, where σ is a permutation of $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$.

1.1 Solution

Method is to count the number of different distances noted by $|a_i - a_j|$. If you write 1 to 8 literally as 1, 2, 3, 4, 5, 6, 7, 8, you will note that the distance is from 1 to 7. First let's count the number of distance 1.

distance 1 This is easy: pairs like $(1, 2)$, $(2, 3)$, ... , $(7, 8)$ are distance 1. There are 7 of them.

distance 2 $(1, 3)$, ... , $(6, 8)$. There are 6 of them.

distance 3 $(1, 4)$, ... , $(5, 8)$. There are 5 of them.

distance 4 $(1, 5)$, ... , $(4, 8)$. There are 4 of them.

distance 5 $(1, 6)$, ... , $3, 8)$. There are 3 of them.

distance 6 $(1, 7)$, $(2, 8)$. There are 2 of them.

distance 7 $(1, 7)$. There are 1 of them.

Each pair will appear exactly this many times: $P_6^6 \times 4 \times 2$. Explanation is: once you select a pair, e.g., $(1, 2)$, you can also flip them as $(2, 1)$. So that is for multiplier 2. And then you can place them in each of the 4 pairs location. And lastly, P_6^6 means once you fixed the selection of the pair, the remaining 6 digits will permutate this many times. So total sum of all permutations of $f(\sigma)$ is

$P_6^6 * 4 * 2 * (1 * 7 + 2 * 6 + 3 * 5 + 4 * 4 + 5 * 3 + 6 * 2 + 7 * 1)$. Average is given by this number divided by $P_8^8 = 40320$, which is 12.

See also: problem4.cpp for brutal force calculation. This is to verify the above solution is correct. The output of the program is below

```
brutal force method: total = 483840, number of permutations = 40320, average = 12
analytical method: average = 12
```

2 Problem 5

Find all real x such that

$$log_{2x}(48\sqrt[3]{3}) = log_{3x}(162\sqrt[3]{2}) \tag{1}$$

2.1 Solution

$$\frac{\ln(48\sqrt[3]{3})}{\ln(2)+\ln(x)} = \frac{\ln(162\sqrt[3]{2})}{\ln(3)+\ln(x)} \tag{2}$$

$$\ln(x) = \frac{\ln(3)\cdot\ln(48\sqrt[3]{3}) - \ln(2)\cdot\ln(162\sqrt[3]{2})}{\ln(162\sqrt[3]{2}) - \ln(48\sqrt[3]{3})} \tag{3}$$

$$\begin{aligned}
x &= \exp\left[\frac{\ln(3)\cdot\ln(48\sqrt[3]{3}) - \ln(2)\cdot\ln(162\sqrt[3]{2})}{\ln(162\sqrt[3]{2}) - \ln(48\sqrt[3]{3})}\right] && (4)\\
&= \exp\left[\frac{\ln(3)\cdot\ln(3) - \ln(2)\cdot\ln(2)}{2(\ln(3)-\ln(2))}\right] && (5)\\
&= \exp\left[\frac{\ln(6)}{2}\right] && (6)\\
x &= \sqrt{6} && (7)
\end{aligned}$$

3 *<2024-07-21 Sun>*

3.1 Problem 5

In how many ways can 9 be written as the sum of one or more positive odd integers? (Order of the integers does not matter.)

$(A)8 \quad (B)6 \quad (C)5 \quad (D)9 \quad (E)7$

3.1.1 Solution

9 can only be written as sum of odd number of positive odd integers, so we have several options.

- $9 = 9$
- $9 = 3+3+3$
- $9 = 1+3+5$
- $9 = 1+1+1+3+3$
- $9 = 1+1+1+1+5$
- $9 = 1+1+1+1+1+1+3$
- $9 = 1+1+1+1+1+1+1+1+1$

So answer is (E)

3.2 Problem 6

The sum of the digits of the positive integer n is 123. The sum of the digits of $2n$ is 66. The digits of n include two 3s, six 7s, p 5s, q 6s and no other digits. What is $p^2 + q^2$?

$(A)106 \quad (B)109 \quad (C)160 \quad (D)58 \quad (E)72$

3.2.1 Solution

Let $d_i(n)$ be the i-th digit of n.

$$\begin{aligned} n &= \sum_{i=1}^{2+6+p+q} d_i(n) \cdot 10^{i-1} && (8) \\ &= \sum_p 5 \cdot 10^{i-1} + \sum_q 6 \cdot 10^{i-1} + \sum_2 3 \cdot 10^{i-1} + \sum_6 7 \cdot 10^{i-1} && (9) \end{aligned}$$

$$2n = \sum_p 1 \cdot 10^i + \sum_q 1 \cdot 10^i + \sum_q 2 \cdot 10^{i-1} + \sum_2 6 \cdot 10^{i-1} + \sum_6 1 \cdot 10^i + \sum_6 4 \cdot 10^{i-1} \quad (10)$$

Adding all digits up:

$$\sum_i d_i(n) = 5p + 6q + 2 \cdot 3 + 6 \cdot 7 = 123 \quad (11)$$

$$\sum_i d_i(2n) = p + q + 2q + 2 \cdot 6 + 6 \cdot 1 + 6 \cdot 4 = 66 \quad (12)$$

We get $p = 9$, $q = 5$, so

$$\boxed{p^2 + q^2 = 106} \quad (13)$$

Answer is (A)