Contents

1 Problem 4

Find the average of function $f(\sigma) = |a_1 - a_2| + |a_3 - a_4| + |a_5 - a_6| + |a_7 - a_8|$, where σ is a permutation of $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$.

1.1 Solution

Method is to count the number of different distances noted by $|a_i - a_j|$. If you write 1 to 8 literally as 1, 2, 3, 4, 5, 6, 7, 8, you will note that the distance is from 1 to 7. First let's count the number of distance 1.

distance 1 This is easy: pairs like $(1, 2)$, $(2, 3)$, ... , $(7, 8)$ are distance 1. There are 7 of them.

distance 2 $(1, 3)$, ... , $(6, 8)$. There are 6 of them.

distance 3 $(1, 4)$, ... , $(5, 8)$. There are 5 of them.

distance 4 $(1, 5)$, ... , $(4, 8)$. There are 4 of them.

distance 5 $(1, 6)$, ... , $3, 8)$. There are 3 of them.

distance 6 $(1, 7)$, $(2, 8)$. There are 2 of them.

distance 7 $(1, 7)$. There are 1 of them.

Each pair will appear exactly this many times: $P_6^6 \times 4 \times 2$. Explanation is: once you select a pair, e.g., $(1, 2)$, you can also flip them as $(2, 1)$. So that is for multiplier 2. And then you can place them in each of the 4 pairs location. And lastly, P_6^6 means once you fixed the selection of the pair, the remaining 6 digits will permutate this many times. So total sum of all permutations of $f(\sigma)$ is

$P_6^6 * 4 * 2 * (1 * 7 + 2 * 6 + 3 * 5 + 4 * 4 + 5 * 3 + 6 * 2 + 7 * 1)$. Average is given by this number divided by $P_8^8 = 40320$, which is 12.

See also: problem4.cpp for brutal force calculation. This is to verify the above solution is correct. The output of the program is below

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brutal force method: total = 483840, number of permutations = 40320, average = 12
analytical method: average = 12
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2   Problem 5

Find all real x such that

$$log_{2x}(48\sqrt[3]{3}) = log_{3x}(162\sqrt[3]{2}) \tag{1}$$

2.1   Solution

$$\frac{\ln(48\sqrt[3]{3})}{\ln(2) + \ln(x)} = \frac{\ln(162\sqrt[3]{2})}{\ln(3) + \ln(x)} \tag{2}$$

$$\ln(x) = \frac{\ln(3) \cdot \ln(48\sqrt[3]{3}) - \ln(2) \cdot \ln(162\sqrt[3]{2})}{\ln(162\sqrt[3]{2}) - \ln(48\sqrt[3]{3})} \tag{3}$$

$$\begin{aligned}
x &= \exp\left[\frac{\ln(3) \cdot \ln(48\sqrt[3]{3}) - \ln(2) \cdot \ln(162\sqrt[3]{2})}{\ln(162\sqrt[3]{2}) - \ln(48\sqrt[3]{3})}\right] \tag{4} \\
&= \exp\left[\frac{\ln(3) \cdot \ln(3) - \ln(2) \cdot \ln(2)}{2(\ln(3) - \ln(2))}\right] \tag{5} \\
&= \exp\left[\frac{\ln(6)}{2}\right] \tag{6} \\
x &= \sqrt{6} \tag{7}
\end{aligned}$$

3   Problem 6

Dikshant writes down $2k + 1$ positive integers in a list where k is a positive integer. The integers are not necessarily all distinct, but there are at least three distinct integers in the list. The average of the all $2k + 1$ integers is itself an integer and appears at least once in the list. The average of the smallest $k + 1$ integers in the list and the average of the largest $k + 1$ integers in the list by less than $1/2025$. Determine the smallest possible value of k.

Clarification: if m_1 is the average of the smallest $k + 1$ integers and m_2 is the average of the largest $k + 1$ integers, then the condition is $|m_1 - m_2| < 1/2025$.

3.1   Solution

Let all integers be ranked from the smallest to the largest in the list as: a_1, a_2, ..., a_{2k+1}. So the number a_{k+1} is the median of the array.

$$m_2 - m_1 = \frac{\sum_{i=1}^{k+1}(a_{i+k} - a_i)}{k + 1} < \frac{1}{2025}. \tag{8}$$

In order for the condition to be true, the sum from i to $k + 1$ has to be very small compared to the dinominator $k + 1$. However, each term in the sum, $a_{i+k} - a_i$ is an positive integer or zero, and there are $k + 1$ terms of them. So many of them has to be 0 with the exception of a few very small positive values. This means, the number of distinct value in the list has to be small. In fact, the fewer number of distinct values, the more likely the condition could be satisfied, and the more likely k could be small.

Now assume there are only 3 distinct values, $a < b < c$. According to the problem statement, the average of all values is also an integer and it's in the list. Therefore, the mean must be the

value b. In order for the mean to be integer, the number of value a must be the same as the number of value c. Then let's assume there are l values of a and c, and the number of value b becomes $2k + 1 - 2l$. So coming back to the Eq. (8),

$$m_2 - m_1 = \frac{l(c-a)}{k+1} < \frac{1}{2025}. \tag{9}$$

The smallest the values for l and $c - a$ are 1 and 2, respectively. Plug these in we get

$$\boxed{k_{min} = 4050} \tag{10}$$

4 <2024-07-21 Sun>

4.1 Problem 5

In how many ways can 9 be written as the sum of one or more positive odd integers? (Order of the integers does not matter.)
$(A)8 \quad (B)6 \quad (C)5 \quad (D)9 \quad (E)7$

4.1.1 Solution

9 can only be written as sum of odd number of positive odd integers, so we have several options.

- $9 = 9$

- $9 = 3 + 3 + 3$

- $9 = 1 + 3 + 5$

- $9 = 1 + 1 + 1 + 3 + 3$

- $9 = 1 + 1 + 1 + 1 + 5$

- $9 = 1 + 1 + 1 + 1 + 1 + 1 + 3$

- $9 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$

So answer is (E)

4.2 Problem 6

The sum of the digits of the positive integer n is 123. The sum of the digits of $2n$ is 66. The digits of n include two 3s, six 7s, p 5s, q 6s and no other digits. What is $p^2 + q^2$?
$(A)106 \quad (B)109 \quad (C)160 \quad (D)58 \quad (E)72$

4.2.1 Solution

Let $d_i(n)$ be the i-th digit of n.

$$n = \sum_{i=1}^{2+6+p+q} d_i(n) \cdot 10^{i-1} \tag{11}$$

$$= \sum_p 5 \cdot 10^{i-1} + \sum_q 6 \cdot 10^{i-1} + \sum_2 3 \cdot 10^{i-1} + \sum_6 7 \cdot 10^{i-1} \tag{12}$$

3

$$2n \quad = \quad \sum_p 1 \cdot 10^i + \sum_q 1 \cdot 10^i + \sum_q 2 \cdot 10^{i-1} + \sum_2 6 \cdot 10^{i-1} + \sum_6 1 \cdot 10^i + \sum_6 4 \cdot 10^{i-1} \quad (13)$$

Adding all digits up:

$$\sum_i d_i(n) = 5p + 6q + 2 \cdot 3 + 6 \cdot 7 = 123 \tag{14}$$

$$\sum_i d_i(2n) = p + q + 2q + 2 \cdot 6 + 6 \cdot 1 + 6 \cdot 4 = 66 \tag{15}$$

We get $p = 9$, $q = 5$, so

$$\boxed{p^2 + q^2 = 106} \tag{16}$$

Answer is (A)