1. $x(n)=\{1,2,3,4\}$.

4点DFT: $DFT[x(n)]=X(k)=\sum_{n=0}^{3}x(n)W_N^{nk}$

$X(0)=\sum_{n=0}^{3}(n+1)e^{-j\frac{2\pi}{4}0\cdot n}=10$. $X(1)=-2+2j$. $X(2)=-2$ $X(3)=-2-2j$

8点: $DFT[x(n)]=X(k)=\sum_{n=0}^{7}x(n)W_N^{nk}$. $X(0)=10$. $X(1)=(1-\sqrt{2})-3(1+\sqrt{2})j$

$X(2)=-2+2j$ $X(3)=(1+\sqrt{2})+3(1-\sqrt{2})j$ $X(4)=-2$. $X(5)=(1+\sqrt{2})+3(\sqrt{2}-1)j$.

$X(6)=-2-2j$ $X(7)=(1-\sqrt{2})+3(\sqrt{2}+1)j$.

2. $\because f(t)$是周期信号 $\therefore f(t)=\sum_{n=0}^{\infty}F_n e^{jn\Omega_0 t}=\sum_{k=0}^{\infty}F_k e^{j\frac{2\pi k}{T}t}$ 以符合采样定理.

$\therefore$存在频率上限且谐波不超过N. 设$f(t)$在一个周期内采样序列为$x(n)$ ($n\in\mathbb{Z}$且$0\le n<N$).

即$x(n)=f(\frac{n}{N}T)$. 此时 $X(k)=\sum_{n=0}^{N-1}x(n)e^{-j\frac{2\pi n}{N}k}=\sum_{n=0}^{N-1}f(\frac{n}{N}T)e^{-j\frac{2\pi n}{N}k}$.

将求和式中Σ展开: $X(k)=\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}F_k e^{j\frac{2\pi k}{T}\frac{n}{N}T}e^{-j\frac{2\pi n}{N}k_0}=\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}F_k e^{j\frac{2\pi n}{N}(k-k_0)}$.

① $k\ne k_0$时: $\sum_{n=0}^{N-1}F_k e^{j\frac{2\pi n}{N}(k-k_0)}\xlongequal{t=j\frac{2\pi(k-k_0)}{N}}\frac{1-(e^t)^N}{1-e^t}=0$.

② $k=k_0$时: $\sum_{n=0}^{N-1}F_k e^{j\frac{2\pi n}{N}(k-k_0)}=\sum_{n=0}^{N-1}F_k=NF_k$.

$\therefore X(k)=N\cdot F_k$. $k\in\{0,1,\cdots,N-1\}$.

3. 设列向量 $\tilde{X}=\tilde{X}(k)=\sum_{n=0}^{N-1}\tilde{x}(n)e^{-jn\omega_k}$, 则 $\tilde{X}=B\tilde{x}$. $B\in\mathbb{C}^{N\times N}$ 且 $B(k,n)=e^{-jn\omega_k}$.

设列向量 $X=X(k)=\sum_{n=0}^{L-1}x(n)e^{-jn\omega_k}$ 则 $X=Ax$, $A\in\mathbb{C}^{N\times L}$, $A(k,n)=e^{-jn\omega_k}$

设 $L=qN+r$ 则 $A=[\underbrace{B\ B\ \cdots}_{q个B}\ \underbrace{\ }_{B前r列}]$ $\therefore Ax=[B\ B\cdots]x=B(\underbrace{[I\ I\cdots}_{q个B}\ \underbrace{I}_{B前r列}])x=B\tilde{x}$

$\therefore X=\tilde{X}$, 即 $\sum_{n=0}^{N-1}\tilde{x}(n)e^{-jn\omega_k}=\sum_{n=0}^{L-1}x(n)e^{-jn\omega_k}$, $\forall k\in\{0,1,\cdots,N-1\}$.

4. $W_N^k=e^{-j\frac{2\pi k}{N}}$

$X(k)=\sum_{n=0}^{N-1}x(n)W_N^{nk}=\sum_{n=0}$

$G(k)=\sum_{n=0}^{\lfloor\frac{N}{2}\rfloor}g(n)W_{\frac{N}{2}}^{nk}=\sum_{n=0}^{\lfloor\frac{N}{2}\rfloor}x(2n)W_N^{2nk}$.

$H(k)=\sum_{n=0}^{\lfloor\frac{N}{2}\rfloor}h(n)W_{\frac{N}{2}}^{nk}=\sum_{n=0}^{\lfloor\frac{N}{2}\rfloor}x(2n+1)W_N^{2nk}$.

$G(k)+W_N^kH(k)=\sum_{n=0}^{\lfloor N/2\rfloor}\left(x(2n)W_N^{2nk}+x(2n+1)(W_N^{2nk}\cdot W_N^{k})\right)$

$=\sum_{n=0}^{\lfloor N/2\rfloor}\left(x(2n)W_N^{2nk}+x(2n+1)W_N^{(2n+1)k}\right)$

$=\sum_{n=0}^{N-1}x(n)W_N^{nk}=X(k)$. 得证.