Solutions to Section 5.1 Homework Problems
Problems 1–33 (odd) and 32.
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1. To determine if 2 is an eigenvalue of the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix},$$

we must determine if the equation $A\mathbf{x} = 2\mathbf{x}$ has non–trivial solutions. To do this, we write $\mathbf{x}$ as $I_2\mathbf{x}$, then write the equation $A\mathbf{x} = 2I_2\mathbf{x}$ as $(A - 2I_2)\mathbf{x} = \mathbf{0}$. Using the usual row reduction approach, we obtain

$$A - 2I_2 = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \backsim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

This shows that the equation $A\mathbf{x} = 2\mathbf{x}$ does have non–trivial solutions and hence that 2 is an eigenvalue of A. We can also see that eigenvector of A corresponding to the eigenvalue $\lambda = 2$ is $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

To check that this is correct, we note that

$$A\mathbf{v} = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

and

$$\lambda\mathbf{v} = 2\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

which shows that $A\mathbf{v} = \lambda\mathbf{v}$.

3. Letting

$$A = \begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}$$

and

$$\mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix},$$

we note that

$$A\mathbf{v} = \begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 29 \end{bmatrix}.$$

Since $A\mathbf{v}$ is not a scalar multiple of $\mathbf{v}$, we conclude that $\mathbf{v}$ is not an eigenvector of A.

5. Letting

$$A = \begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$$

and

$$\mathbf{v} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix},$$

we note that

$$A\mathbf{v} = \begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This shows that **v** is an eigenvector of A with corresponding eigenvalue $\lambda = 0$ (because $A\mathbf{v} = 0\mathbf{v}$).

7. Letting

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix},$$

we want to determine if the equation $(A - 4I_3)\mathbf{x} = \mathbf{0}$ has non–trivial solutions.

$$A - 4I_3 = \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This shows that $(A - 4I_3)\mathbf{x} = \mathbf{0}$ does have non–trivial solutions and that a particular non–trivial solution is

$$\mathbf{v} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

This vector, **v**, is an eigenvector of A corresponding to the eigenvalue $\lambda = 4$ as can be checked:

$$A\mathbf{v} = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$$

and

$$\lambda\mathbf{v} = 4\mathbf{v} = 4\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix}$$

(shows that $A\mathbf{v} = 4\mathbf{v}$).

9. To find the eigenvectors corresponding to the eigenvalue 5, we must solve the equation $(A - 5I)\mathbf{x} = \mathbf{0}$.

$$A - 5I = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix} \backsim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

shows that all solutions of the equation $(A - 5I)\mathbf{x} = \mathbf{0}$ have the form

$$\mathbf{x} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Thus, the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a basis for the eigenspace of A corresponding to the eigenvalue 5.

To find the eigenvectors corresponding to the eigenvalue 1, we must solve the equation $(A - I)\mathbf{x} = \mathbf{0}$.

$$A - I = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

shows that all solutions of the equation $(A - I)\mathbf{x} = \mathbf{0}$ have the form

$$\mathbf{x} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus, the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basis for the eigenspace of A corresponding to the eigenvalue 1.

11. To find the eigenvectors corresponding to the eigenvalue 10, we must solve the equation $(A - 10I)\mathbf{x} = \mathbf{0}$.

$$A - 10I = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} -6 & -2 \\ -3 & -1 \end{bmatrix} \backsim \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{bmatrix}$$

shows that all solutions of the equation $(A - 10I)\mathbf{x} = \mathbf{0}$ have the form

$$\mathbf{x} = \begin{bmatrix} -\frac{1}{3}x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}.$$

Thus, the vector $\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$ is a basis for the eigenspace of A corresponding to the eigenvalue 10.

13. To find the eigenvectors corresponding to the eigenvalue 1, we must solve the equation $(A - I)\mathbf{x} = \mathbf{0}$.

$$A - I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

shows that all solutions of the equation $(A - I)\mathbf{x} = \mathbf{0}$ have the form

$$\mathbf{x} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Thus, the vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a basis for the eigenspace of A corresponding to the eigenvalue 1.

To find the eigenvectors corresponding to the eigenvalue 2, we must solve the equation $(A - 2I)\mathbf{x} = \mathbf{0}$.

$$A - 2I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

shows that all solutions of the equation $(A - 2I)\mathbf{x} = \mathbf{0}$ have the form

$$\mathbf{x} = \begin{bmatrix} -\frac{1}{2}x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}.$$

Thus, the vector $\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$ is a basis for the eigenspace of A corresponding to the eigenvalue 2.

To find the eigenvectors corresponding to the eigenvalue 3, we must solve the equation $(A-3I)\mathbf{x} = \mathbf{0}$.

$$A-3I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

shows that all solutions of the equation $(A-3I)\mathbf{x} = \mathbf{0}$ have the form

$$\mathbf{x} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, the vector $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for the eigenspace of A corresponding to the eigenvalue 3.

15. To find the eigenvectors corresponding to the eigenvalue 3, we must solve the equation $(A-3I)\mathbf{x} = \mathbf{0}$.

$$A-3I = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix} \backsim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

shows that all solutions of the equation $(A-3I)\mathbf{x} = \mathbf{0}$ have the form

$$\mathbf{x} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, the pair of vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

forms a basis for the eigenspace of A corresponding to the eigenvalue 3.

17. Since this matrix is triangular, its eigenvalues are the entries on its main diagonal. Hence its eigenvalues are 0, 2, and -1.

19. Since the columns of A are clearly linearly dependent, the equation $A\mathbf{x} = \mathbf{0}$ has non–trivial solutions. This means that there is a non–zero vector **v** such that $A\mathbf{v} = \mathbf{0}$. This means that $A\mathbf{v} = 0\mathbf{v}$ and hence that 0 is eigenvalue of A.

21.

- **a.** False. If $A\mathbf{v} = \lambda\mathbf{x}$ for some non–zero vector **x**, then λ is an eigenvalue of A.
- **b.** True.
- **c.** True.
- **d.** True.
- **e.** False. To find the eigenvalue of A, we solve the *characteristic equation* of A. That is coming up in Section 5.2.

22.

- **a.** False. If $\mathbf{x} \neq \mathbf{0}$ and $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ, then **x** is an eigenvector of A.
- **b.** False in general. However, it is true that if λ_1 and λ_2 are eigenvalues of A with $\lambda_1 \neq \lambda_2$ and $\mathbf{v}_1$ is an eigenvector corresponding to λ_1 and $\mathbf{v}_2$ is an eigenvector corresponding to λ_2, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.
- **c.** Not applicable (refers to material in Section 4.9 which we are not covering).
- **d.** False. This is true though if A is a triangular matrix.
- **e.** True. If λ is an eigenvalue of A, then the eigenspace of A corresponding to the eigenvalue λ is the null space of the matrix $A - \lambda I$.

23. Eigenvectors corresponding to distinct eigenvalues are linearly independent. If a 2x2 matrix had, say, three distinct eigenvalues, then this matrix would have three linearly independent eigenvectors. However, since each eigenvector is a vector in $\Re^2$, this is impossible. Any set of three vectors in $\Re^2$ must be linearly dependent. The same reasoning applies in explaining why an nxn matrix can have at most n distinct eigenvalues.

25. We are given that λ is an eigenvalue of an invertible matrix A. Since A is invertible, we know that $\lambda \neq 0$. We also know that there is a non–zero vector **x** such that $A\mathbf{x} = \lambda\mathbf{x}$. From this, we conclude that $A^{-1}(A\mathbf{x}) = A^{-1}(\lambda\mathbf{x})$ and hence that $\mathbf{x} = \lambda(A^{-1}\mathbf{x})$. Multiplying both sides of the latter equation by λ^{-1} and writing the resulting equation in the reverse order, we obtain $A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$. This shows that λ^{-1} is an eigenvalue of A^{-1}.

27. Since A and A^T have the same entries on their main diagonals, we observe that $A^T - \lambda I = (A - \lambda I)^T$.

Now suppose that λ is an eigenvalue of A but not an eigenvalue of A^T. Then the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has non–trivial solutions but the equation $(A^T - \lambda I)\mathbf{x} = \mathbf{0}$ has only the trivial solution. This means that the matrix $A - \lambda I$ is not invertible matrix but the matrix $A^T - \lambda I$ is invertible. However, since $A^T - \lambda I = (A - \lambda I)^T$, then $A^T - \lambda I$ must not be invertible because it is the transpose of a matrix that is not invertible.

Since our original assumption (that λ is an eigenvalue of A but not an eigenvalue of A^T) has led us to a contradiction, we must admit that this assumption is not possible. We conclude that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T.

29. If A is an nxn matrix whose rows all sum to s, then the rows of $A - sI$ all sum to 0. This means that the zero vector is the sum of the columns of $A - sI$ and hence that the zero vector is a non–trivial linear combination of the columns of $A - sI$. Hence, the columns of $A - sI$ must form a linearly dependent set which implies that the matrix $A - sI$ is not invertible and hence that s is an eigenvalue of A.

31. If $T : \Re^2 \to \Re^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation that reflects points across some line through the origin, then any vector $\mathbf{x}$ on that line must satisfy $T(\mathbf{x}) = A\mathbf{x} = \mathbf{x}$. This means that $\lambda = 1$ is an eigenvalue of A whose corresponding eigenspace is the line through which T reflects all points in $\Re^2$.

32. If $T : \Re^3 \to \Re^3$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation that rotates all points in $\Re^3$ about some line through the origin, then any vector $\mathbf{x}$ on that line must satisfy $T(\mathbf{x}) = A\mathbf{x} = \mathbf{x}$. This means that $\lambda = 1$ is an eigenvalue of A whose corresponding eigenspace is the line about which T rotates all points in $\Re^3$.

33. We are given that $A\mathbf{u} = \lambda\mathbf{u}$ and that $A\mathbf{v} = \mu\mathbf{v}$, that c_1 and c_2 are scalars, and that $\mathbf{x}_k = c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v}$ for all $k = 0, 1, 2, \ldots$.

a. By definition, $\mathbf{x}_{k+1} = c_1\lambda^{k+1}\mathbf{u} + c_2\mu^{k+1}\mathbf{v}$.

b.

$$\begin{aligned} A\mathbf{x}_k &= A(c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v}) \\ &= c_1\lambda^k(A\mathbf{u}) + c_2\mu^k(A\mathbf{v}) \\ &= c_1\lambda^k(\lambda\mathbf{u}) + c_2\mu^k(\mu\mathbf{v}) \\ &= c_1\lambda^{k+1}\mathbf{u} + c_2\mu^{k+1}\mathbf{v} \end{aligned}$$

We thus see that $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for all $k = 0, 1, 2, \ldots$.