Adaptive Tracking and Stabilization of Nonholonomic Mobile Robots With Input Saturation

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***Abstract*—This article investigates the control problems of tracking and stabilization simultaneously for nonholonomic mobile robots subjected to input constraints and parameter uncertainties. A saturated time-varying controller is developed by applying a novel error state modification with bounded auxiliary variables. To improve the estimation performance of unknown kinematic and dynamic parameters, two projection-type adaptation laws are presented, while radial basis function approximations are used to estimate uncertain dynamics. Simulations are conducted to verify the effectiveness of the proposed controller.**

***Index Terms*—Adaptive control, constrained control, robotics, tracking and stabilization.**

I. Introduction

In the past two decades, many research works have been presented to solve the problems of tracking and stabilization simultaneously for nonholonomic systems. The most challenging and interesting parts of this topic are due to the facts that the motion of nonholonomic systems has more degrees of freedom than the independent inputs and the target to be tracked can be any type trajectory whether it is persistently excited (PE) or not. The former one, due to Brockett's theorem [1], makes such system cannot be stabilized asymptotically by any continuous time invariant feedback control law. The latter one requires the control objective can be achieved by using a single controller without any PE restriction on the reference signals.

The existing simultaneous tracking and stabilization control schemes can be roughly divided into two types: single parameter perturbation type and multiple parameter perturbation type. In the single parameter perturbation type, there is a single auxiliary time-varying function, which plays the key role in the control laws to achieve convergence of tracking and stabilization errors to zero simultaneously. For instance, Lee *et al.* [2] designed a single kinematic controller with the aid of one time-varying signal to achieve global convergence of tracking and stabilization errors simultaneously for a class of unicycle-modeled mobile robot, where the reference linear velocity of the robots was restricted to be nonnegative. To remove this restriction, Do *et al.* [3] introduced a coordinate transformation with a new time-varying signal and presented a new unified adaptive controller to simultaneously solve the tracking and stabilization problems for mobile robots with unknown parameters, and similar control laws were applied in [4]–[6] for a class of nonholonomic systems. In [7], with the aid of a delicately designed signal, a simplenonswitching kinematic controller was proposed to achieve asymptotic convergence of tracking or stabilization errors for nonholonomic mobile robots. However, the controls in the single parameter perturbation type also exhibited slow error convergence and high dependence on the initial conditions. As a contrast, due to application of multiple auxiliary signals, the control schemes in the multiple parameter perturbation type can achieve fast error convergence and robust stability. For instance, Dixon *et al.* [8] designed a kinematic controller with the aid of a two variables dynamic-oscillator to achieve a bounded solution of the tracking and stabilization problems for wheeled mobile robots, which was also extended to the dynamic level for underactuated ships [9] and a wheeled mobile robot [10]. Inspired by the work [8], the transverse function approach involved more auxiliary variables was developed in [11], which can used to achieve practical stabilization of arbitrary reference trajectories for nonholonomic systems [12]–[14]. In order to unify the existing results on tracking and stabilization of nonholonomic systems, Li [15] introduced a unified controller by using a new error state transformation with the aid of two auxiliary time-varying signals to solve the tracking and stabilization problems simultaneously for underactuated ships, and similar ideas were used in [16] and [17]. However, it should be noted that the asymptotic convergence of error states was excluded by using the controls in the multiple parameter perturbation type.

Motivated by the abovementioned observations, this work aims to pick a unified control scheme for nonholonomic wheeled mobile robots to simultaneously achieve tracing and stabilization convergence to zero, and while, for realistic consideration, assumes that the robot is subjected to input saturation constraints, parameter uncertainties, and external disturbances. By fully exploiting the aforementioned results in single or multiple parameter perturbation types, a novel unified time-varying feedback controller is designed by applying two auxiliary time-varying variables, while two new projection-type adaptation laws are presented to estimate the unknown parameters. The main contributions of this article can be summarized as two aspects. First, with the aid of a combination of a two variables dynamic-oscillator and an auxiliary time-varying signal, the proposed control scheme can achieve fast convergence and asymptotic stability at the same time. Second, two new projection-type adaption frameworks are designed for the unknown physical parameters such that the parameter estimates are always bounded and their estimation errors have more acceptable accuracy than that rely on the standard gradient update laws.

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II. Preliminaries and Problem Statement

A. Preliminaries

1) Notations: Throughout this note, $\Re^n$ represents the n-dimensional Euclidean space, which can be simplified as $\Re$ when $n = 1$. The one-dimensional (1-D) Euclidean space consisting of all nonnegative real numbers is denoted by $\Re^+$. $\mathrm{diag}(\cdot)$ denotes a diagonal matrix. $\|\cdot\|$ denotes the Euclidean norm of a vector or a matrix. $|\cdot|$

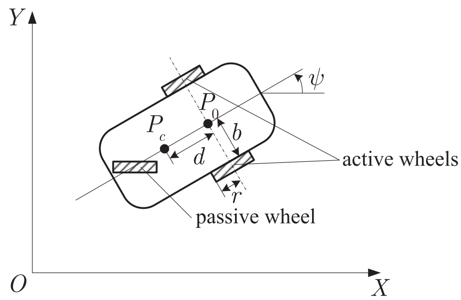

Fig. 1. Nonholonomic mobile robot with two actuated wheels.

denotes the absolute value of a scalar. The superscript T denotes the transpose of a matrix or a vector.

2) Radial basis function (RBF) approximation: According to the universal approximation property [18], RBF network is able to approximate any continuous function $f(X) : \Re^{n_i} \to \Re$ over a compact set $\Omega_X \subset \Re^{n_i}$ with arbitrary accuracy such that

$$\begin{aligned} f(X) &= W^{*T}\xi(X) + \varepsilon^*(X) \\ \xi(X) &= [\xi_1(X), \xi_2(X), \ldots, \xi_{n_h}(X)]^T \\ \xi_i(X) &= \exp\left(-\|X-\mu_i\|^2/\sigma_i^2\right), \quad i = 1, 2, \ldots, n_h \end{aligned} \tag{1}$$

where $W^* \in \Re^{n_h}$ is the ideal constant weight matrix satisfying $\|W^*\| \le W_0$ with W_0 an appropriate positive constant; n_i and n_h denote the numbers of input-layer nodes and hidden-layer nodes, respectively; $\xi_i(X)$ denotes the ith Gaussian basis function; $\mu_i \in \Re^{n_i}$ and $\sigma_i \in \Re$ are the center and standard derivation of the ith hidden node; $\varepsilon^*(X)$ is the ideal approximation error satisfying $\|\varepsilon^*(X)\| \le \varepsilon_0$, ε_0 can be any given positive constant.

B. Problem Statement

In this note, a nonholonomic mobile robot with two actuated wheels is considered (see Fig. 1) whose mathematic model is given by [3] and [19]

$$\dot{q} = S(q)\omega, \quad M\dot{\omega} = F(\dot{q}) + \tau + \tau_d \tag{2}$$

with $q = [x, y, \psi]^T$, $\omega = [\omega_1, \omega_2]^T$, $F(\dot{q}) = [F_1, F_2]^T$ and

$$S(q) = \frac{r}{2}\begin{bmatrix} \cos\psi & \sin\psi & b^{-1} \\ \cos\psi & \sin\psi & -b^{-1} \end{bmatrix}^T, \quad M = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{11} \end{bmatrix}$$

$$m_{11} = \frac{r^2}{4b^2}[(m_c + 2m_w)b^2 + I_0] + I_w$$

$$m_{12} = \frac{r^2}{4b^2}[(m_c + 2m_w)b^2 - I_0]$$

$$I_0 = m_c d^2 + 2m_w b^2 + I_c + 2I_m$$

$$F_1 = -\frac{r^2 m_c d}{2b}\dot{\psi}\omega_2 - d_{11}\omega_1, \quad F_2 = \frac{r^2 m_c d}{2b}\dot{\psi}\omega_1 - d_{22}\omega_2$$

where (x, y) is the position of P_0 in the reference frame OXY, ψ is the heading angle of the robot, ω_1 and ω_2 are the angular velocities of the wheels, $\tau = [\tau_1, \tau_2]^T$ with $\tau_i (i = 1, 2)$ the active control torques, $\tau_d = [\tau_{d1}, \tau_{d2}]^T$ represents external disturbances and unmodeled dynamics; r, b and d are defined in Fig. 1; m_c and m_w are the masses of the body and wheel, respectively; I_c, I_w, and I_m are the inertia of the body about the vertical axis through P_c, the wheel about its axis, the wheel about its diameter, respectively; $d_{ii} > 0 (i = 1, 2)$ are the damping terms.

In practice, the control inputs of nonholonomic mobile robots are always subjected to saturation constraints. Under this consideration, we assume the following inequalities hold for the actual control torques as

$$|\tau_i| \le \tau_M, \quad i = 1, 2 \tag{3}$$

where $\tau_M > 0$ is a known torque limit of τ_i.

Additionally, it can be easily seen from (2) that there exists the following nonholonomic constraint:

$$\dot{x}\sin\psi - \dot{y}\cos\psi = 0. \tag{4}$$

Let $v = [v_1, v_2]^T = B^{-1}\omega$ be the new angular velocities with $B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Substituting v into (2) and left multiplying B^T both sides of the second equation of (2), we get

$$\begin{aligned} &\dot{x} = \phi_1 v_1 \cos\psi, \quad \dot{y} = \phi_1 v_1 \sin\psi, \quad \dot{\psi} = \phi_2 v_2 \\ &N_1\dot{v}_1 = \tau_{s1} + G_1(t, v_1, v_2), \quad N_2\dot{v}_2 = \tau_{s2} + G_2(t, v_1, v_2) \end{aligned} \tag{5}$$

with

$$\begin{aligned} G_1(t, v_1, v_2) &= -cv_2^2 - D_1 v_1 - D_2 v_2 + \tau_{d1} + \tau_{d2} \\ G_2(t, v_1, v_2) &= cv_1 v_2 - D_2 v_1 - D_1 v_2 + \tau_{d1} - \tau_{d2} \end{aligned} \tag{6}$$

where $\phi_1 = r$, $\phi_2 = b^{-1}r$, $N_1 = 2(m_{11} + m_{12})$, $N_2 = 2(m_{11} - m_{12})$, $\tau_{s1} = \tau_1 + \tau_2$, $\tau_{s2} = \tau_1 - \tau_2$, $c = \frac{r^3 m_c d}{2b^2}$, $D_1 = d_{11} + d_{22}$, and $D_2 = d_{11} - d_{22}$.

Because of measurement inaccuracies and modeling uncertainties, the physical parameters in (5), i.e., ϕ_i, N_i and $G_i(t, v_1, v_2)(i = 1, 2)$, usually are unknown. Thus, to facilitate the control design, the following two assumptions are used throughout this article.

Assumption 1: The unknown constant parameters ϕ_i and N_i can be expressed, respectively, as $\phi_i = \phi_{i0} + \Delta_{\phi i}$ and $N_i = N_{i0} + N_{\delta i}$, where ϕ_{i0} and N_{i0} are known nominal values, $\Delta_{\phi i}$ and $N_{\delta i}$ are unknown constants satisfying $|\Delta_{\phi i}| \le \sigma_\phi$ and $|N_{\delta i}| \le \sigma_N$, where σ_ϕ and σ_N are two known positive constants.

Assumption 2: The angular velocities and disturbances, v_i and $\tau_{di}(i = 1, 2)$, are continuous and bounded.

The reference trajectory $q_d = [x_d, y_d, \psi_d]^T$ to be tracked is generated by

$$\dot{x}_d = \phi_{10}v_{1\,d}\cos\psi_d, \quad \dot{y}_d = \phi_{10}v_{1\,d}\sin\psi_d, \quad \dot{\psi}_d = \phi_{20}v_{2\,d} \tag{7}$$

where $v_{1\,d}$ and $v_{2\,d}$ are the reference angular velocities. Without loose of generality, the reference trajectory (7) can be any type of admissible one satisfying the following assumption.

Assumption 3: The reference angular velocities $(v_{1\,d}, v_{2\,d})$ and their time derivatives $(\dot{v}_{1\,d}, \dot{v}_{2\,d})$ are bounded.

Control objective: Under Assumptions 1–3, design a control law for the bounded inputs τ_1 and τ_2 such that the mobile robot (2) as well as (5) asymptotically track the reference trajectory (7), i.e., $\lim_{t\to\infty}(q(t) - q_d(t)) = [0, 0, 2n\pi]^T$ with n being any integer.

Remark 1: Unlike usually done in the other works, in this article, we aim to realize $\lim_{t\to\infty}(\psi(t) - \psi_d(t)) = 2n\pi$ rather than $\lim_{t\to\infty}(\psi(t) - \psi_d(t)) = 0$. This is because the angles 0 and $2n\pi$ are actually same for any integer n, which makes it reasonable to force the errors $\psi(t) - \psi_d(t)$ to converge to a closer one between 0 and $2n\pi$. ■

III. Control Development

A. Control Design for the Closed-Loop System

1) Kinematic Control Design: In order to overcome the difficulties caused by the nonholonomic constraint (4), we introduce a modification of error states here. Let $q_e = [x_e, y_e, \psi_e]^T$ be the modified tracking errors, which are given by [17]

$$\begin{bmatrix} x_e \\ y_e \end{bmatrix} = R(\psi_a) \begin{bmatrix} x - x_d \\ y - y_d \end{bmatrix} + \begin{bmatrix} \delta_1 \\ 0 \end{bmatrix}, \quad \psi_e = \psi - \psi_d + \delta_2 \tag{8}$$

with

$$R(\psi_a) = \begin{bmatrix} \cos\psi_a & \sin\psi_a \\ -\sin\psi_a & \cos\psi_a \end{bmatrix}$$

where $\psi_a = \psi + \delta_2$, δ_1 and δ_2 are bounded auxiliary variables to be determined later. Note that $R(\psi_a)$ is an orthogonal matrix, i.e., $R^{-1}(\psi_a) = R^T(\psi_a)$.

Differentiating (8) along the solutions of the first three equations of (5) and (7) yields

$$\begin{aligned} \dot{x}_e &= \dot{\psi}_a y_e + \cos\delta_2\left((\phi_{10} + \Delta_{\phi_1})v_1 - \phi_{10}v_{1\,d}\right) + \dot{\delta}_1 + f_c \sin\frac{\psi_e}{2} \\ \dot{y}_e &= -\dot{\psi}_a(x_e - \delta_1) - \sin\delta_2\left((\phi_{10}+\Delta_{\phi_1})v_1 + \phi_{10}v_{1\,d}\right) + f_s\sin\frac{\psi_e}{2} \\ \dot{\psi}_e &= (\phi_{20} + \Delta_{\phi_2})v_2 - \phi_{20}v_{2\,d} + 2\dot{\delta}_2 \end{aligned} \tag{9}$$

where $f_c = -2\phi_{10}v_{1\,d}\sin\psi_e'$ and $f_s = 2\phi_{10}v_{1\,d}\cos\psi_e'$ with $\psi_e' = \frac{\psi_e}{2} - \delta_2$. It can be easily obtained that $|f_c| \le 2\phi_{10}|v_{1\,d}|$ and $|f_s| \le 2\phi_{10}|v_{1\,d}|$.

To stabilize x_e and ψ_e, we choose v_1 and v_2 as the virtual controls which are designed as follows:

$$\begin{aligned} \alpha_{v1} &= \frac{1}{\phi_{10} + \hat{\Delta}_{\phi_1}}\left(\phi_{10}v_{1\,d} - \cos^{-1}\delta_2(k_x\varpi x_e + \dot{\delta}_1)\right) \\ \alpha_{v2} &= \frac{1}{\phi_{20} + \hat{\Delta}_{\phi_2}}\left(\phi_{20}v_{2\,d} - 2\dot{\delta}_2 - k_\psi\sin\frac{\psi_e}{2} - \gamma_\psi^{-1}f_s\varpi y_e\right) \end{aligned} \tag{10}$$

where α_{v1} and α_{v2} are the desired values of v_1 and v_2, k_x, k_ψ, and γ_ψ are positive constants, $\varpi = (1 + x_e^2 + y_e^2)^{-\frac{1}{2}}$, $\hat{\Delta}_{\phi i}(i = 1, 2)$ are the estimates of $\Delta_{\phi i}$, respectively.

In order to stabilize y_e, we can substitute (10) into the second equation of (9) and obtain

$$\begin{aligned} \dot{y}_e = &-\dot{\psi}_a x_e + \Big(-k_\psi\sin\frac{\psi_e}{2} + \phi_2 v_{2e} + \tilde{\Delta}_{\phi_2}\alpha_{v2} + \phi_{20}v_{2\,d} \\ &-\dot{\delta}_2 - \gamma_y^{-1}f_s\varpi y_e\Big)\delta_1 + (k_x\varpi x_e + \dot{\delta}_1)\tan\delta_2 \\ &-\sin\delta_2\left(2\phi_{10}v_{1\,d} + \phi_1 v_{1e} + \tilde{\Delta}_{\phi_1}\alpha_{v1}\right) + f_s\sin\frac{\psi_e}{2} \end{aligned} \tag{11}$$

where the relationship $\dot{\psi}_a = \dot{\psi}_e + \dot{\psi}_d - \dot{\delta}_2$ has been used according to the direct differentiation of the last equation of (8), $v_{ie} = v_i - \alpha_{vi}$ and $\tilde{\Delta}_{\phi_i} = \Delta_{\phi_i} - \hat{\Delta}_{\phi_i}(i = 1, 2)$ denote the virtual control errors and the estimation errors, respectively.

Then, we choose the auxiliary variables δ_1 and δ_2 as additional virtual controls and define them as follows:

$$\begin{bmatrix} \dot{\delta}_1 \\ \dot{\delta}_2 \end{bmatrix} = \begin{bmatrix} -k_\delta & -\mathcal{S} \\ \mathcal{S} & -k_\delta \end{bmatrix}\begin{bmatrix} \delta_1 \\ \tan\delta_2 \end{bmatrix} + \begin{bmatrix} \gamma_y \\ 0 \end{bmatrix}\rho(t)\varpi y_e \tag{12}$$

where k_δ and γ_y are two positive constants, $\mathcal{S} = \frac{1}{\Omega}(k_y\varpi y_e + A/\Omega)$ with $k_y > \gamma_y$, $\Omega = \sqrt{\delta_1^2 + \tan^2\delta_2}$ and $A = (\phi_{20}v_{2\,d} - k_\psi\sin\frac{\psi_e}{2} - \gamma_y^{-1}f_s\varpi y_e)\delta_1 + k_x\varpi x_e\delta_2 - 2\phi_{10}v_{1\,d}\sin\delta_2$. $\rho(t)$ is a bounded function given by

$$\begin{aligned} &T_\rho\dot{\rho} + \rho = \rho_0, \quad \rho(0) = 0 \\ &\rho_0 = \begin{cases} 1, & \sqrt{v_{1\,d}^2 + v_{2\,d}^2} \le \sigma_0 \\ 0, & \text{else} \end{cases} \end{aligned} \tag{13}$$

where T_ρ and σ_0 are two appropriate small positive constants. Then, after a simple calculation, one can obtain from (13) that $\rho \in [0, 1)$, and specifically, $\rho \approx 1$ when $\sqrt{v_{1\,d}^2 + v_{2\,d}^2} \le \sigma_0$ and $\rho \approx 0$ when $\sqrt{v_{1\,d}^2 + v_{2\,d}^2} > \sigma_0$.

Substituting (10) and (12) into (9) yields the error kinematics as follows:

$$\begin{aligned} \dot{x}_e &= \dot{\psi}_a y_e - k_x\varpi x_e + \cos\delta_2\left(\phi_1 v_{1e} + \tilde{\Delta}_{\phi_1}\alpha_{v1}\right) + f_c\sin\frac{\psi_e}{2} \\ \dot{y}_e &= -\dot{\psi}_a x_e - (k_y\Omega - \rho\gamma_y\tan\delta_2)\varpi y_e + (\phi_2 v_{2e} \\ &\quad + \tilde{\Delta}_{\phi_2}\alpha_{v2})\delta_1 - \sin\delta_2(\phi_1 v_{1e} + \tilde{\Delta}_{\phi_1}\alpha_{v1}) + f_s\sin\frac{\psi_e}{2} \\ \dot{\psi}_e &= -k_\psi\sin\frac{\psi_e}{2} + \phi_2 v_{2e} + \tilde{\Delta}_{\phi_2}\alpha_{v2} - \gamma_\psi^{-1}f_s\varpi y_e \end{aligned} \tag{14}$$

2) Dynamic Controller Design: In this section, a robust adaptive controller will be designed for the actual controls τ_1 and τ_2 to stabilize the velocity errors $v_{ie}(i = 1, 2)$. To begin, by differentiating v_{ie} along the solutions of (5) and recalling Assumption 2, the error dynamics can be expressed as

$$N_i\dot{v}_{ie} = \tau_{si} + \Upsilon_i(X_{gi}), \quad i = 1, 2 \tag{15}$$

where $X_{gi} = [v_1, v_2, \dot{\alpha}_{vi}, 1]^T$ and $\Upsilon_i(X_{gi}) = G_i(t, v_1, v_2) - N_i\dot{\alpha}_{vi}$. Due to Assumption 2, one can see that there exist positive constants $v_{iM}(i = 1, 2)$ such that $|v_i| \le v_{iM}$. Besides, by differentiating (10) along the solutions of (9), (12), (13), (24) and according to Assumptions 2 and 3 and Proposition 1, one can conclude that $\dot{\alpha}_{vi}(i = 1, 2)$ are both bounded, i.e., there exists positive constants $\dot{\alpha}_{viM}(i = 1, 2)$ such that $|\dot{\alpha}_{vi}| \le \dot{\alpha}_{viM}$. Thus, there exist two compact sets $\Xi_i = \{(v_1, v_2, \dot{\alpha}_{vi}, 1)||v|_1 \le v_{1\,M}, |v|_2 \le v_{2\,M}, |\dot{\alpha}_{vi}| \le \dot{\alpha}_{viM}\}(i = 1, 2)$ satisfying $X_{gi} \in \Xi_i$. Under this consideration, we can use two RBF approximations to estimate $\Upsilon_i(X_{gi})$ as

$$\Upsilon_i(X_{gi}) = W_{gi}^{*T}\xi_{gi} + \varepsilon_{gi}^*, \quad i = 1, 2 \tag{16}$$

where $\xi_{gi} = \xi_g(X_{gi}) \in \Re^{n_h}(i = 1, 2)$ are the RBF vectors with the form depicted in (1), $W_{gi}^* \in \Re^{n_h}$ and ε_{gi}^*, respectively, are the ideal weight matrix and the ideal approximation error satisfying $\|W_{gi}^*\| \le W_{g0}$ and $\|\varepsilon_{gi}^*\| \le \varepsilon_{g0}$ with W_{g0} and ε_{g0} being two known positive constants.

Then, we choose the inputs $\tau_{si}(i = 1, 2)$ as

$$\tau_{si} = \frac{-k_v v_{ie}}{\sqrt{1 + v_{ie}^2}} - (\gamma_v + \varepsilon_{g0})\text{sgn}(v_{ie}) - \hat{W}_{gi}^T\xi_{gi} \tag{17}$$

where k_v and γ_v are positive constants, $\hat{W}_{gi}$ is the estimate of W_{gi}^*.

According to the definitions of $\tau_{si}(i = 1, 2)$, we get

$$\tau_1 = \frac{1}{2}(\tau_{s1} + \tau_{s2}), \quad \tau_2 = \frac{1}{2}(\tau_{s1} - \tau_{s2}). \tag{18}$$

Substituting (16) and (17) into (15), we can yield the error dynamics as

$$N_i\dot{v}_{ie} = \frac{-k_v v_{ie}}{\sqrt{1 + v_{ie}^2}} - (\gamma_v + \varepsilon_{g0})\text{sgn}(v_{ie}) + \tilde{W}_{gi}^T\xi_{gi} + \varepsilon_{gi}^*, i = 1, 2 \tag{19}$$

with $\tilde{W}_{gi} = W_{gi}^* - \hat{W}_{gi}$ being the estimation error.

B. Parameter Estimation Design

In this section, we will perform the bounded adaptation laws for the unknown kinematic and dynamic parameters.

1) Estimation Design of the Kinematic Parameters: In order to derive the estimation of the unknown kinematic parameters, we can rewrite the first three equations of (5) in the following equivalent form:

$$\dot{q} = J(q,v)(\Phi_0 + \Delta_\phi) \tag{20}$$

with $\Phi_0 = [\phi_{10}, \phi_{20}]^T$, $\Delta_\phi = [\Delta_{\phi 1}, \Delta_{\phi 2}]^T$ and

$$J(q,v) = \begin{bmatrix} v_1 \cos\psi & v_1 \sin\psi & 0 \\ 0 & 0 & v_2 \end{bmatrix}^T. \tag{21}$$

Inspired by the work in [20], we define an auxiliary matrix $\mathcal{P}_q \in \Re^{2\times 2}$ and an vector $\mathcal{Q}_q \in \Re^2$ as follows:

$$\begin{aligned} \dot{\mathcal{P}}_q &= -\ell_q \mathcal{P}_q + J_M, \mathcal{P}_q(0) = 0 \\ \dot{\mathcal{Q}}_q &= -\ell_q \mathcal{Q}_q + J^T[\dot{q} - J(\Phi_0 + \hat{\Delta}_\phi)] - \mathcal{P}_q \dot{\hat{\Delta}}_\phi, \mathcal{Q}_q(0) = 0 \end{aligned} \tag{22}$$

where $\ell_q > 0$ is the forgetting factor, $J_M = J^T J = \mathrm{diag}(v_1^2, v_2^2)$, $\hat{\Delta}_\phi = [\hat{\Delta}_{\phi 1}, \hat{\Delta}_{\phi 2}]^T$ denotes the estimation of Δ_ϕ.

Let $\tilde{\Delta}_\phi = [\tilde{\Delta}_{\phi 1}, \tilde{\Delta}_{\phi 2}]^T = \Delta_\phi - \hat{\Delta}_\phi$ be the estimation error. Then, after a simple calculation, one can easily obtain from (22) that

$$\mathcal{P}_q(t) = \int_0^t e^{-\ell_q(t-\varsigma)} J_M(\varsigma) d\varsigma, \quad \mathcal{Q}_q(t) = \mathcal{P}_q \tilde{\Delta}_\phi. \tag{23}$$

In order to guarantee the estimates $\hat{\Delta}_{\phi i} (i = 1, 2)$ to be bounded and due to Assumption 1 and (14), we introduce a discontinuous projection-type adaptation law for them as follows:

$$\dot{\hat{\Delta}}_\phi = \mathrm{Proj}_{\sigma_\phi}\left(\hat{\Delta}_\phi, \gamma_q^{-1}\mathcal{H}_q + \frac{\gamma_q^{-1} k_q \mathcal{P}_q^T \mathcal{Q}_q}{\|\mathcal{Q}_q\| + \epsilon_\delta}\right) \tag{24}$$

where γ_q and k_q are two positive constants to be chosen later, $\mathcal{H}_q = [\alpha_{v1}\varpi(y_e \sin\delta_2 - x_e \cos\delta_2), \alpha_{v2}(\varpi y_e \delta_1 - \gamma_\psi \sin\frac{\psi_e}{2})]^T$, ϵ_δ is a small positive constant, $\mathrm{Proj}_a(b,c)$ is a discontinuous projection operator such that, for two vectors b and c, and a constant $a > 0$

$$\mathrm{Proj}_a(b,c) = \begin{cases} -a^{-1} b\|c\| + c, & \text{if } \|b\| \geq a \text{ \& } b^T c \geq 0 \\ c, & \text{else.} \end{cases} \tag{25}$$

It can be concluded from (25) that, for the differential equation $\dot{b} = \mathrm{Proj}_a(b,c)$, the following properties hold: 1) when $\|b(0)\| \leq a$, then $\|b(t)\| \leq a + \epsilon_0, \forall t \geq 0$, where $\epsilon_0 > 0$ is an appropriate constant; and 2) $(b_0 - b)^T(\mathrm{Proj}_a(b,c) - c) \geq 0, \forall \|b_0\| \leq a$.

2) Estimation Design of the Dynamic Parameters: To obtain the estimation of the unknown dynamic parameters without using the joint accelerations $\dot{v}_{ie}$ and controls $\tau_{si} (i = 1, 2)$, and motivated by the works in [10] and [20], we use a stable and strictly proper linear filter with the form $(\cdot)_f = 1/(\kappa s + 1)$ is introduced on both sides of (15) and (16) as follows:

$$N_i \dot{v}_{ie,f} = \tau_{si,f} + W_{gi}^{*T} \xi_{gi,f} + \varepsilon_{gi,f}, \quad i = 1, 2 \tag{26}$$

with the following filtered parameters:

$$\begin{aligned} \kappa \dot{v}_{ie,f} + v_{ie,f} &= v_{ie}, \quad v_{ie,f}(0) = 0 \\ \kappa \dot{\tau}_{si,f} + \tau_{si,f} &= \tau_{si}, \quad \tau_{si,f}(0) = 0 \\ \kappa \dot{\xi}_{gi,f} + \xi_{gi,f} &= \xi_{gi}, \quad \xi_{gi,f}(0) = 0 \\ \kappa \dot{\varepsilon}_{gi,f} + \varepsilon_{gi,f} &= \varepsilon_{gi}^*, \quad \varepsilon_{gi,f}(0) = 0. \end{aligned} \tag{27}$$

Then, due to the fact that N_i is an unknown matrix, we can rewrite (26) in a concise form as

$$N_{0i} \dot{v}_{ie,f} = \tau_{si,f} + \bar{W}_i^T \bar{\xi}_{if} + \varepsilon_{gi,f}, \quad i = 1, 2 \tag{28}$$

where $\bar{W}_i = [W_{gi}^{*T}, -N_{\delta i}]^T$ and $\bar{\xi}_{if} = [\xi_{gi,f}^T, \dot{v}_{ie,f}]^T$.

Define an auxiliary matrix $\mathcal{P}_{vi} \in \Re^{n_h \times n_h}$ and a vector $\mathcal{Q}_{vi} \in \Re^{n_h}$ as follows:

$$\begin{aligned} \dot{\mathcal{P}}_{vi} &= -\ell_v \mathcal{P}_{vi} + \bar{\xi}_{if}\bar{\xi}_{if}^T, \quad \mathcal{P}_{vi}(0) = 0 \\ \dot{\mathcal{Q}}_{vi} &= -\ell_v \mathcal{Q}_{vi} + \bar{\xi}_{if}\left(N_{0i}\dot{v}_{ie,f} - \tau_{si,f} - \hat{\bar{W}}_i^T \bar{\xi}_{if}\right) \\ &\quad - \mathcal{P}_{vi}\dot{\hat{\bar{W}}}_i, \quad \mathcal{Q}_{vi}(0) = 0 \end{aligned} \tag{29}$$

where $\ell_v > 0$, $\hat{\bar{W}}_i$ denotes the estimation of $\bar{W}_i$ to be chosen later. Let $\tilde{\bar{W}}_i = \bar{W}_i - \hat{\bar{W}}_i$ be the estimation error matrix. It can be easily obtained from (29) that

$$\begin{aligned} \mathcal{P}_{vi}(t) &= \int_0^t e^{-\ell_v(t-\varsigma)} \bar{\xi}_{if}(\varsigma)\bar{\xi}_{if}^T(\varsigma) d\varsigma \\ \mathcal{Q}_{vi}(t) &= \mathcal{P}_{vi}\tilde{\bar{W}}_i + \chi_{vi} \end{aligned} \tag{30}$$

where $\chi_{vi} = \int_0^t e^{-\ell_v(t-\varsigma)}\bar{\xi}_{if}(\varsigma)\varepsilon_{gi,f}(\varsigma)d\varsigma$ is an unknown time-varying vector. Since the terms τ_{si}, ξ_{gi}, and ε_{gi}^* are bounded, then we can obtain from (27) that $\bar{\xi}_{if}$ and $\varepsilon_{gi,f}$ are both bounded, which implies that χ_{vi} is also bounded such that $|\chi_{vi}| \leq \chi_{v0}$ with χ_{v0} being the upper bound of χ_{vi}.

Then, due to Assumption 1 and (19), the updating law of $\hat{\bar{W}}_i$ is designed as follows:

$$\dot{\hat{\bar{W}}}_i = \mathrm{Proj}_{\bar{W}_0}\left(\hat{\bar{W}}_i, -\gamma_W^{-1}\bar{\xi}_i v_{ie} + \frac{\gamma_W^{-1} k_W \mathcal{P}_{vi}^T \mathcal{Q}_{vi} v_{ie}^2}{(\|\mathcal{Q}_{vi}\| + \epsilon_\delta)(1 + v_{ie}^2)}\right) \tag{31}$$

where γ_W and k_W are two positive constants, $\bar{\xi}_i = [\xi_g^T(X_{gi}), 0]^T$, and $\bar{W}_0 = \sqrt{W_{g0}^2 + \sigma_N^2}$.

C. Stability Analysis

Before giving the stability analysis of the resultant error system, a technical lemma and a proposition are first introduced.

The following technical lemma is an extended version of Barbalat's lemma introduced in [21].

Lemma 1(see[21]): Let $\zeta(t) : \Re^+ \to \Re$ be any differentiable function. If $\zeta(t)$ converges to some limit value as $t \to \infty$, and its derivative satisfies

$$\frac{d\zeta}{dt}(t) = \zeta_0(t) + \varrho(t), \quad t \geq 0 \tag{32}$$

where $\zeta_0(t)$ is a uniformly continues function and $\lim_{t\to\infty}\varrho(t) = 0$, then both of $\frac{d\zeta}{dt}(t)$ and $\zeta_0(t)$ tend to zero as $t \to \infty$. ■

Then, an result about the boundedness of the auxiliary variables δ_1 and δ_2 is given by means of the following proposition.

Proposition 1: The auxiliary variables δ_1 and δ_2 in (12) are uniformly bounded. Moreover, if $y_e(t)$ tends to zero as $t \to \infty$, then δ_1 and δ_2 are asymptotically stable. ■

Proof: Define $V_\delta = \frac{1}{2}(\delta_1^2 - \ln(\cos^2\delta_2))$ as the Lyapunov function candidate. Differentiating V_δ along the solutions of (12) yields

$$\dot{V}_\delta = -k_\delta \Omega^2 + \rho\gamma_y \delta_1 \varpi y_e. \tag{33}$$

Since $\rho \in [0, 1)$ and the inequalities $|\varpi y_e| \leq 1$ and $|\delta_1| \leq \Omega$ hold, we have

$$\dot{V}_\delta \leq -(k_\delta\Omega - \gamma_y)\Omega. \tag{34}$$

It can be seen from (34) that $\dot{V}_\delta$ is always negative outside the compact set $\{\Omega \leq k_\delta^{-1}\gamma_y\}$, which implies that V_δ is uniformly bounded, and so do δ_1 and δ_2.

On the other hand, due to the inequalities $\rho \in [0,1)$, $\varpi \leq 1$ and $|\delta_1| \leq \Omega$, we can also obtain from (33) that

$$\dot{V}_\delta \leq -k_\delta \Omega^2 + \gamma_y \Omega |y_e|. \tag{35}$$

Then, according to the result of [22, Th. 3], if $y_e(t)$ tends to zero as $t \to \infty$, we can conclude that Ω is asymptotically stable, and so do δ_1 and δ_2. ■

Now, the main result can be stated by means of the following theorem.

Theorem 1: Consider the nonholonomic mobile robot (2) and suppose Assumptions 1–3 hold. Choose the control parameters k_v and γ_v such that

$$k_v + \gamma_v + \sqrt{n_h(W_{g0}^2 + \sigma_N^2)} + \varepsilon_{g0} \leq \tau_M. \tag{36}$$

Then, the saturated control inputs given by (18) with the virtual control laws (10), (12) and the adaptation laws (24) and (31) can guarantee that (i) the error states $(x - x_d, y - y_d, \psi - \psi_d)$ converge to the equilibrium points $(0, 0, 2n\pi)$ with n being integers, and (ii) the control inputs do not violate the saturation constraints (3). ■

Proof: Consider the following Lyapunov function candidate:

$$V = V_1 + V_2 \tag{37}$$

with

$$V_1 = \varpi^{-1} - 1 + 2\gamma_\psi \left(1 - \cos\frac{\psi_e}{2}\right) + \frac{\gamma_q}{2}\tilde{\Delta}_\phi^T\tilde{\Delta}_\phi \tag{38}$$

$$V_2 = \frac{p_v}{2}\left(N_{10}v_{1e}^2 + N_{20}v_{2e}^2 + \gamma_W(\tilde{\bar{W}}_1^T\tilde{\bar{W}}_1 + \tilde{\bar{W}}_2^T\tilde{\bar{W}}_2)\right) \tag{39}$$

where p_v and p_δ are two appropriate positive constants.

Differentiating (38) along the solutions of (14), (24) and due to the fact that $\dot{\tilde{\Delta}}_\phi = -\dot{\hat{\Delta}}_\phi$, we can yield

$$\begin{aligned}\dot{V}_1 = {} & \varpi x_e\left(-k_x\varpi x_e + \cos\delta_2(\phi_1 v_{1e} + \tilde{\Delta}_{\phi_1}\alpha_{v1}) + f_c\sin\frac{\psi_e}{2}\right)\\ & + \varpi y_e\left(-(k_y\Omega - \rho\gamma_y\tan\delta_2)\varpi y_e + (\phi_2 v_{2e}\right.\\ & \left.+\tilde{\Delta}_{\phi_2}\alpha_{v2})\delta_1 - \sin\delta_2(\phi_1 v_{1e} + \tilde{\Delta}_{\phi_1}\alpha_{v1})\right)\\ & + \gamma_\psi\sin\frac{\psi_e}{2}(-k_\psi\sin\frac{\psi_e}{2} + \phi_2 v_{2e} + \tilde{\Delta}_{\phi_2}\alpha_{v2})\\ & - \gamma_q\tilde{\Delta}_\phi^T\mathrm{Proj}_{\sigma_\phi}\left(\hat{\Delta}_\phi, \gamma_q^{-1}\mathcal{H}_q + \frac{\gamma_q^{-1}k_q\mathcal{P}_q^T\mathcal{Q}_q}{\|\mathcal{Q}_q\| + \epsilon_\delta}\right).\end{aligned} \tag{40}$$

Following the second property of the operator Proj and the relationship $\mathcal{Q}_q(t) = \mathcal{P}_q\tilde{\Delta}_\phi$, we can rewrite (40) as

$$\begin{aligned}\dot{V}_1 \leq {} & -k_x\varpi^2x_e^2 + \varpi x_e\left(\phi_1 v_{1e}\cos\delta_2 + f_c\sin\frac{\psi_e}{2}\right) - (k_y\Omega\\ & - \rho\gamma_y\tan\delta_2)\varpi^2y_e^2 + \varpi y_e(\phi_2 v_{2e}\delta_1\\ & - \phi_1 v_{1e}\sin\delta_2) - \gamma_\psi k_\psi\sin^2\frac{\psi_e}{2} + \gamma_\psi\phi_2 v_{2e}\sin\frac{\psi_e}{2}\\ & - k_q\frac{\|\mathcal{P}_q\tilde{\Delta}_\phi\|^2}{\|\mathcal{P}_q\tilde{\Delta}_\phi\| + \epsilon_\delta}.\end{aligned} \tag{41}$$

Additionally, by recalling the definition of Ω, we know that $|\tan\delta_2| \leq \Omega$. Besides, as mentioned above, we also know that $\rho \in (0,1)$, $k_y > \gamma_y$ and $|f_c| \leq 2\phi_{10}|v_{1\,d}|$. Then, by using the Young's inequality $|f_c\varpi x_e\sin\frac{\psi_e}{2}| \leq \mu_x\varpi^2x_e^2 + \mu_x^{-1}\phi_{10}^2v_{1\,d}^2\sin^2\frac{\psi_e}{2}$ with $\mu_x > 0$ being an appropriate constant, (41) can be rewritten as

$$\begin{aligned}\dot{V}_1 \leq {} & -k_x'\varpi^2x_e^2 - k_y'\Omega\varpi^2y_e^2 - k_\psi'\sin^2\frac{\psi_e}{2} - k_q\frac{\|\mathcal{P}_q\tilde{\Delta}_\phi\|^2}{\|\mathcal{P}_q\tilde{\Delta}_\phi\| + \epsilon_\delta}\\ & + 2\phi_1|v_{1e}| + \phi_2(\gamma_\psi + |\delta_1|)|v_{2e}|\end{aligned} \tag{42}$$

where $k_x' = k_x - \mu_x$, $k_y' = k_y - \gamma_y$, and $k_\psi' = \gamma_\psi k_\psi - \mu_x^{-1}\phi_{10}^2v_{1\,d}^2$ can be always positive by choosing appropriate μ_x, γ_ψ, and k_ψ.

Takeing the derivative of V_2 along the solutions of (19) and (31), and recalling the second property of the operator Proj and the facts $\dot{\tilde{\bar{W}}}_i = -\dot{\hat{\bar{W}}}_i (i = 1, 2)$, we can obtain

$$\begin{aligned}\dot{V}_2 \leq p_v\sum_{i=1,2}\Bigg(& v_{ie}\left(\frac{-k_v v_{ie}}{\sqrt{1 + v_{ie}^2}} - (\gamma_v + \varepsilon_{g0})\mathrm{sgn}(v_{ie}) + \varepsilon_{gi}^*\right)\\ & - \frac{k_W\tilde{\bar{W}}_i^T\mathcal{P}_{vi}^T\mathcal{Q}_{vi}v_{ie}^2}{(\|\mathcal{Q}_{vi}\| + \epsilon_\delta)(1 + v_{ie}^2)}\Bigg).\end{aligned} \tag{43}$$

Then, due to the relationship $\mathcal{Q}_{vi}(t) = \mathcal{P}_{vi}\tilde{\bar{W}}_i + \chi_{vi}$ and by using the inequalities $|\varepsilon_{gi}^*| \leq \varepsilon_{g0}$, $|\chi_{vi}| \leq \chi_{v0}$, $\frac{\|\mathcal{Q}_{vi}\|}{\|\mathcal{Q}_{vi}\| + \epsilon_\delta} \leq 1$ and $\frac{v_{ie}^2}{(1+v_{ie}^2)} \leq \frac{v_{ie}^2}{\sqrt{1+v_{ie}^2}}$, (43) can be rewritten as

$$\begin{aligned}\dot{V}_2 \leq {} & p_v\sum_{i=1,2}\Bigg(\frac{-k_v v_{ie}^2}{\sqrt{1 + v_{ie}^2}} - \gamma_v|v_{ie}| - \frac{k_W\|\mathcal{Q}_{vi}\|^2v_{ie}^2}{(\|\mathcal{Q}_{vi}\| + \epsilon_\delta)(1 + v_{ie}^2)}\\ & + \frac{k_W\|\chi_{vi}\|\|\mathcal{Q}_{vi}\|v_{ie}^2}{(\|\mathcal{Q}_{vi}\| + \epsilon_\delta)(1 + v_{ie}^2)}\Bigg)\\ \leq {} & p_v\sum_{i=1,2}\left(\frac{-k_v' v_{ie}^2}{\sqrt{1 + v_{ie}^2}} - \gamma_v|v_{ie}| - \frac{k_W\|\mathcal{Q}_{vi}\|^2v_{ie}^2}{(\|\mathcal{Q}_{vi}\| + \epsilon_\delta)(1 + v_{ie}^2)}\right)\end{aligned} \tag{44}$$

where $k_v' = k_v - k_W\chi_{v0}$ is a positive constant due to appropriate choices of k_v and k_W.

Combining (42) and (44) and according to the definition of V in (37), we can obtain the time derivation of V as

$$\begin{aligned}\dot{V} = \dot{V}_1 + \dot{V}_2 \leq {} & -k_x'\varpi^2x_e^2 - k_y'\Omega\varpi^2y_e^2 - k_\psi'\sin^2\frac{\psi_e}{2} - k_q\frac{\|\mathcal{P}_q\tilde{\Delta}_\phi\|^2}{\|\mathcal{P}_q\tilde{\Delta}_\phi\| + \epsilon_\delta}\\ & + 2\phi_1|v_{1e}| + \phi_2(\gamma_\psi + |\delta_1|)|v_{2e}| + p_v\sum_{i=1,2}\Bigg(\frac{-k_v' v_{ie}^2}{\sqrt{1 + v_{ie}^2}}\\ & -\gamma_v|v_{ie}| - \frac{k_W\|\mathcal{Q}_{vi}\|^2v_{ie}^2}{(\|\mathcal{Q}_{vi}\| + \epsilon_\delta)(1 + v_{ie}^2)}\Bigg).\end{aligned} \tag{45}$$

By choosing an appropriate value of p_v such that $p_v\gamma_v = \max(2(\phi_{10} + \sigma_\phi), (\phi_{20} + \sigma_\phi)(\gamma_\psi + |\delta_1|))$, one can obtain $2\phi_1|v_{1e}| + \phi_2(\gamma_\psi + |\delta_1|)|v_{2e}| - p_v\gamma_v(|v_{1e}| + |v_{2e}|) \leq 0$. Then, we get

$$\begin{aligned}\dot{V} = \dot{V}_1 + \dot{V}_2 \leq {} & -k_x'\varpi^2x_e^2 - k_y'\Omega\varpi^2y_e^2 - k_\psi'\sin^2\frac{\psi_e}{2} - k_q\frac{\|\mathcal{P}_q\tilde{\Delta}_\phi\|^2}{\|\mathcal{P}_q\tilde{\Delta}_\phi\| + \epsilon_\delta}\\ & + p_v\sum_{i=1,2}\left(\frac{-k_v' v_{ie}^2}{\sqrt{1 + v_{ie}^2}} - \frac{k_W\|\mathcal{Q}_{vi}\|^2v_{ie}^2}{(\|\mathcal{Q}_{vi}\| + \epsilon_\delta)(1 + v_{ie}^2)}\right)\\ \leq {} & -\eta\|X\|^2 \leq 0\end{aligned} \tag{46}$$

with

$$\eta = \min\left(k_x'\varpi^2, k_y'\varpi^2, k_\psi{}', \frac{k_q}{\|\mathcal{P}_q\tilde{\Delta}_\phi\| + \epsilon_\delta}, \frac{p_v k_v'}{\sqrt{1+v_{1e}^2}}, \frac{p_v k_v'}{\sqrt{1 + v_{2e}^2}}\right.$$

$$\left.\frac{p_v k_W}{(\|\mathcal{Q}_{v1}\|+\epsilon_\delta)(1+v_{1e}^2)}, \frac{p_v k_W}{(\|\mathcal{Q}_{v2}\|+\epsilon_\delta)(1+v_{2e}^2)}\right)$$

$$X=[x_e, \sqrt{\Omega}y_e, \sin\frac{\psi_e}{2}, \|\mathcal{P}_q\tilde{\Delta}_\phi\|, v_{1e}, v_{2e}, \|\mathcal{Q}_{v1}\|v_{1e}, \|\mathcal{Q}_{v2}\|v_{2e}]^T.$$

From (46), it is obvious that $\dot{V}$ is always nonincreasing, which implies that $V(t)$ is bounded. Furthermore, by applying Barbalat's lemma in [23] to (46), we have $\lim_{t\to\infty}\|X(t)\|\to 0$, i.e., the error states x_e, Ωy_e, $\sin\frac{\psi_e}{2}$, $\mathcal{P}_q\tilde{\Delta}_\phi$, v_{1e}, v_{2e}, $\mathcal{Q}_{v1}v_{1e}$, and $\mathcal{Q}_{v2}v_{2e}$ tend to zero as time goes to infinity. In addition, because $\lim_{t\to\infty}\sin\frac{\psi_e(t)}{2}=0$, we have $\lim_{t\to\infty}\psi_e(t)=2n\pi$ with n being an integer.

Since $\mathcal{P}_q\tilde{\Delta}_\phi$ tends to zero as $t\to\infty$, we know from (23) that $\lim_{t\to\infty}\tilde{\Delta}_{\phi i}(t)\int_0^t e^{-\ell_q(t-\varsigma)}v_i^2(\varsigma)d\varsigma=0 (i=1,2)$. After a simple analysis, we can yield that $\lim_{t\to\infty}\tilde{\Delta}_{\phi i}(t)v_i(t)=0 (i=1,2)$, which, due to $\lim_{t\to\infty}v_{ie}(t)=0$ and $v_i=v_{ie}+\alpha_{vi}(i=1,2)$, also implies that $\lim_{t\to\infty}\tilde{\Delta}_{\phi i}(t)\alpha_{vi}(t)=0(i=1,2)$.

To prove the convergence of δ_1 and δ_2 to zero, according to $\lim_{t\to\infty}\Omega(t)y_e(t)=0$, we know that at least one of y_e and Ω tends to zero as $t\to\infty$. If $\lim_{t\to\infty}y_e(t)=0$, then due to Proposition 1, we get $\lim_{t\to\infty}\Omega(t)=0$. Thus, the convergence of Ω as well as δ_1 and δ_2 to zero can be concluded.

To prove that $\lim_{t\to\infty}y_e(t)=0$, we consider two possible cases in the following discussion.

Case 1($\sqrt{v_{1\,d}^2+v_{2\,d}^2}\le\sigma_0$)*:* In this case, according to (13), we have $\rho_0=1$, then $\lim_{t\to\infty}\rho(t)=1$ following the first equation of (13). On the other hand, since δ_1 and δ_2 are uniformly bounded and converge to zero as $t\to\infty$, then by applying Lemma 1 to (12), we can yield $\lim_{t\to\infty}\rho(t)y_e(t)=0$, and thus, the convergence of $y_e(t)$ to zero can be concluded straightforwardly.

Case 2 ($\sqrt{v_{1\,d}^2+v_{2\,d}^2}>\sigma_0$)*:* In this case, we can obtain from (13) that $\rho_0=0$ and $\lim_{t\to\infty}\rho(t)=0$. Then, by applying Lemma 1 to the first and third equations of (14) and recalling the asymptotic convergence of x_e, $\sin\frac{\psi_e}{2}$, v_{1e}, v_{2e}, $\tilde{\Delta}_{\phi1}\alpha_{v1}$, and $\tilde{\Delta}_{\phi2}\alpha_{v2}$, we have $\lim_{t\to\infty}\dot{\psi}_a(t)y_e(t)=0$ and $\lim_{t\to\infty}f_s(t)y_e(t)=0$. Moreover, due to the definitions of $\dot{\psi}_a$ and f_s, one can yield that $\lim_{t\to\infty}\dot{\psi}_a(t)=\lim_{t\to\infty}\phi_{20}v_{2\,d}(t)$ and $\lim_{t\to\infty}f_s(t)y_e(t)=\pm\lim_{t\to\infty}2\phi_{10}v_{1\,d}(t)$. Thus, by summarizing the results of abovementioned limits, we obtain $\lim_{t\to\infty}v_{1\,d}(t)y_e(t)=0$ and $\lim_{t\to\infty}v_{2\,d}(t)y_e(t)=0$, and after a simple calculation, we can yield $\lim_{t\to\infty}\sqrt{v_{1\,d}^2(t)+v_{2\,d}^2(t)}y_e(t)=0$. Then, due to the fact that $\sqrt{v_{1\,d}^2+v_{2\,d}^2}>\sigma_0$, we can conclude that $\lim_{t\to\infty}y_e(t)=0$.

So far, we have proved that $\lim_{t\to\infty}(x_e(t),y_e(t),\psi_e(t))=(0,0,2n\pi)$ and $\lim_{t\to\infty}(\delta_1(t),\delta_2(t))=(0,0)$. Then, according to (8), we can obtain that

$$\begin{bmatrix}x-x_d\\ y-y_d\end{bmatrix}=R^T(\psi_a)\begin{bmatrix}x_e-\delta_1\\ y_e\end{bmatrix},\quad \psi-\psi_d=\psi_e-\delta_2 \tag{47}$$

which implies that

$$\lim_{t\to\infty}\|\begin{bmatrix}x(t)-x_d(t)\\ y(t)-y_d(t)\end{bmatrix}\|^2=\lim_{t\to\infty}\|\begin{bmatrix}x_e(t)-\delta_1(t)\\ y_e(t)\end{bmatrix}\|^2=0$$
$$\lim_{t\to\infty}|\psi(t)-\psi_d(t)|=\lim_{t\to\infty}|\psi_e(t)-\delta_2(t)|=2n\pi \tag{48}$$

and as a result, we can conclude that $\lim_{t\to\infty}(q(t)-q_d(t))=[0,0,2n\pi]^T$.

To investigate the bounds of the actual inputs τ_i, by combining (17) and (18) and using inequalities $\frac{|v_{ie}|}{\sqrt{1+v_{ie}^2}}\le 1$, $|\text{sgn}(\cdot)|\le 1$, $\|\hat{W}_{gi}\|\le\sqrt{W_{g0}^2+\sigma_N^2}$, and $\|\xi_g(\cdot)\|\le\sqrt{n_h}$, we have $|\tau_i|\le k_v+\gamma_v+\sqrt{n_h(W_{g0}^2+\sigma_N^2)}+\varepsilon_{g0}$, $i=1,2$. Then, if (36) holds, we get $|\tau_i|\le\tau_M(i=1,2)$, which implies that the controls would not violate the saturation constraints (3). ■

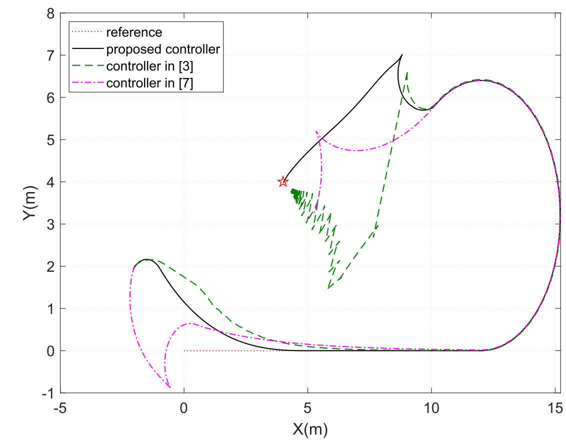

Fig. 2. Reference and actual trajectories.

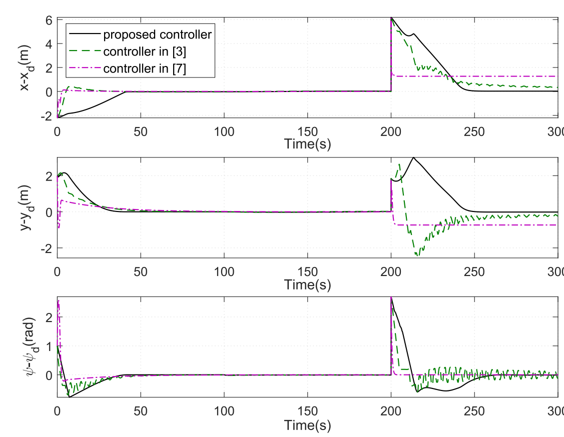

Fig. 3. Time responses of the error states.

IV. Illustrative Example

Consider a nonholonomic mobile robot with the physical parameters [19]: $r=0.15$, $b=0.75$, $d=0.3$, $m_c=30$, $m_w=1$, $I_c=15.625$, $I_w=0.005$, $d_{11}=d_{22}=10$, and $\tau_M=50$. The disturbances are set as $\tau_{d1}=0.01\sin(0.5t+\pi/3)$ and $\tau_{d2}=0.005\sin(1.5t-\pi/6)$. The nominal values of the parameters are $\phi_{10}=0.12$, $\phi_{20}=0.25$, $N_{10}=N_{20}=0.75$. The reference trajectory is selected a combination of a time-varying trajectory for the first 200 s and a standstill point for the rest, where the time-varying trajectory is selected as $(v_{1\,d},v_{2\,d})=(1,0)$ for the first 100 s, i.e., a straight line, then $(v_{1\,d},v_{2\,d})=(1,0.15)$ for the following 100 s, i.e., a circle, and the standstill point is chosen as $(x_d,y_d,\psi_d)=(4,4,\pi/3)$. The initial conditions are chosen as $(x(0),y(0),\psi(0))=(-2,2,1)$ and $(x_d(0),y_d(0),\psi_d(0))=(0,0,0)$. The control parameters are set as follows: $k_x=5$, $k_y=k_\psi=\gamma_y=2$, $k_v=25$, $T_\rho=0.1$, $\varepsilon_{g0}=\sigma_0=\epsilon_\delta=0.1$, $k_\delta=0.5$, $\gamma_\psi=0.01$, $\ell_q=\ell_v=1$, $k_q=250$, $\gamma_q=100$, $\gamma_v=0.1$, $k_W=0.1$, $\gamma_W=20$, $\sigma_\phi=0.05$, $\sigma_N=0.2$. The RBF networks used in this article contain $8(=2^3)$ hidden nodes for input vectors X_{g1} and X_{g2}, i.e., $n_h=8$, while the centers are evenly spaced in $[-1.5,1.5]\times[-1.5,1.5]\times[-2,2]\times 1$, the standard derivation

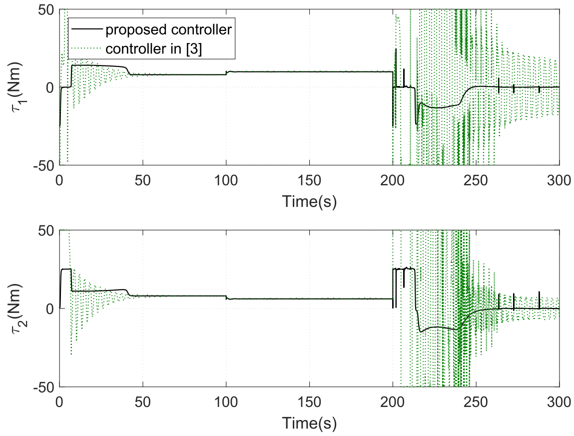

Fig. 4. Time responses of control inputs.

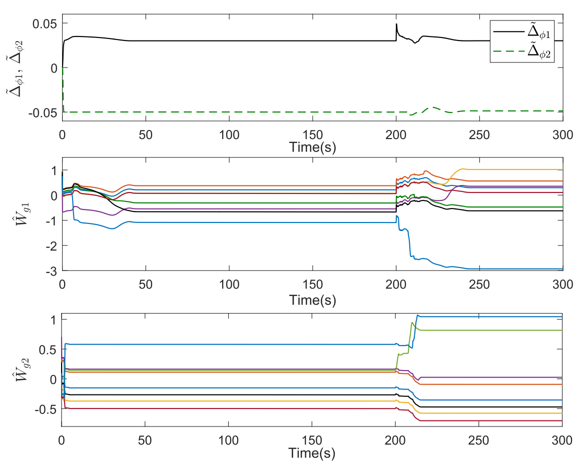

Fig. 5. Time responses of the estimated parameters.

$\sigma_i = 1$, $W_{g0} = 5$. Results are plotted in Figs. 2–5. Besides, we also illustrate the controllers presented in [3], [7] as a comparison.

It can be seen from Figs. 2 and 3 that all of controllers have similar performance in the case of tracking time-varying trajectory. But in the case of tracking a standstill point, the proposed controller yields much better performance than the others. Since the controller presented in [7] is only on the kinematic level, one can see from Fig. 4 that the time responses of actual inputs of the proposed controller are more acceptable than that of the one in [3], and the proposed controller does not violate the input saturation constraints. Fig. 5 shows the parameter estimation performance, where one can see that all estimates are guaranteed to be bounded, and particularly, the estimated kinematic parameters, i.e., $\hat{\Delta}_{\phi1}$ and $\hat{\Delta}_{\phi2}$ converge to their true values.

V. Conclusion

A single time-varying controller has been presented to simultaneously solve the tracking and stabilization problems for nonholonomic mobile robots with input constraints. The key feature of the proposed controller is a novel error state transformation with the usage of two bounded auxiliary variables. Two projection-type adaption laws have been presented to estimate the unknown kinematic and dynamic parameters, where the latter one has applied RBF approximations. Our future works will focus on the application and extension of the proposed control scheme to a class of nonholonomic robots.

References

[1] R. W. Brockett, “Asymptotic stability and feedback stabilization,” in *Differential Geometric Control Theory*, R. W. Brockett, R. S. Millman, and H. J. Sussmann, Eds. Boston, MA, USA: Birkhäuser, 1983, pp. 181–191.

[2] T.-C. Lee, K.-T. Song, C.-H. Lee, and C.-C. Teng, “Tracking control of unicycle-modeled mobile robots using a saturation feedback controller,” *IEEE Trans. Control Syst. Technol.*, vol. 9, no. 1, pp. 305–318, Mar. 2001.

[3] K. D. Do, Z.-P. Jiang, and J. Pan, “Simultaneous tracking and stabilization of mobile robots: An adaptive approach,” *IEEE Trans. Autom. Control*, vol. 49, no. 7, pp. 1147–1152, Jul. 2004.

[4] J. Huang, C. Wen, W. Wang, and Z.-P. Jiang, “Adaptive stabilization and tracking control of a nonholonomic mobile robot with input saturation and disturbance,” *Syst. Control Lett.*, vol. 62, no. 3, pp. 234–241, 2013.

[5] B. Li, Y. Fang, G. Hu, and X. Zhang, “Model-free unified tracking and regulation visual servoing of wheeled mobile robots,” *IEEE Trans. Control Syst. Technol.*, vol. 24, no. 4, pp. 1328–1339, Jul. 2016.

[6] Z. Wang, G. Li, X. Chen, H. Zhang, and Q. Chen, “Simultaneous stabilization and tracking of nonholonomic WMRs with input constraints: Controller design and experimental validation,” *IEEE Trans. Ind. Electron.*, vol. 66, no. 7, pp. 5343–5352, Jul. 2019.

[7] Y. Wang, Z. Miao, H. Zhong, and Q. Pan, “Simultaneous stabilization and tracking of nonholonomic mobile robots: A Lyapunov-based approach,” *IEEE Trans. Control Syst. Technol.*, vol. 23, no. 4, pp. 1440–1450, Jul. 2015.

[8] W. E. Dixon, D. M. Dawson, E. Zergeroglu, and F. Zhang, “Robust tracking and regulation control for mobile robots,” *Int. J. Robust Nonlinear Control*, vol. 10, pp. 199–216, 2000.

[9] A. Behal, D. M. Dawson, W. E. Dixon, and Y. Fang, “Tracking and regulation control of an underactuated surface vessel with nonintegrable dynamics,” *IEEE Trans. Autom. Control*, vol. 47, no. 3, pp. 495–500, Mar. 2002.

[10] W. E. Dixon, M. S. de Queiroz, D. M. Dawson, and T. J. Flynn, “Adaptive tracking and regulation of a wheeled mobiled robot with controller/update law modularity,” *IEEE Trans. Control Syst. Technol.*, vol. 12, no. 1, pp. 138–147, Jan. 2004.

[11] P. Morin and C. Samson, “Practical stabilization of driftless systems on Lie groups: The transverse function approach,” *IEEE Trans. Autom. Control*, vol. 48, no. 9, pp. 1496–1508, Sep. 2003.

[12] P. Morin and C. Samson, “Control of nonholonomic mobile robots based on the transverse function approach,” *IEEE Trans. Robot.*, vol. 25, no. 5, pp. 1058–1073, Oct. 2009.

[13] K. D. Do, “Practical control of underactuated ships,” *Ocean Eng.*, vol. 37, pp. 1111–1119, 2010.

[14] T. Hamel and C. Samson, “Transverse function control of a motorboat,” *Automatica*, vol. 65, pp. 132–139, 2016.

[15] J.-W. Li, “Robust tracking control and stabilization of underactuated ships,” *Asian J. Control*, vol. 20, no. 6, pp. 2143–2153, 2018.

[16] H. Huang, J. Zhou, Q. Di, J. Zhou, and J. Li, “Robust neural network-based tracking control and stabilization of a wheeled mobile robot with input saturation,” *Int. J. Robust Nonlinear Control*, vol. 29, pp. 375–392, 2019.

[17] J.-W. Li, “Robust adaptive control of underactuated ships with input saturation,” *Int. J. Control*, vol. 94, no. 7, pp. 1784–1793, 2021.

[18] J. Park and I. W. Sandberg, “Universal approximation using radial-basis-function networks,” *Neural Comput.*, vol. 3, no. 2, pp. 246–257, 1991.

[19] T. Fukao, H. Nakagawa, and N. Adachi, “Adaptive tracking control of a nonholomic mobile robot,” *IEEE Trans. Robot. Autom.*, vol. 16, no. 5, pp. 609–615, Oct. 2000.

[20] J. Na, M. N. Mahyuddin, G. Hermann, X. Ren, and P. Barber, “Robust adaptive finite-time parameter estimation and control for robotic systems,” *Int. J. Robust Nonlinear Control*, vol. 25, pp. 3045–3071, 2015.

[21] C. Samson, “Control of chained systems application to path following and time-varying point-stabilization of mobile robots,” *IEEE Trans. Autom. Control*, vol. 40, no. 1, pp. 64–77, Jan. 1995.

[22] E. Panteley and A. Loría, “Growth rate conditions for stability of cascaded time-varying systems and their applications,” *Automatica*, vol. 37, pp. 453–460, 2001.

[23] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ, USA: Prentice-Hall, 2002.