# A Study on Further Properties of Wireless Channel Capacity

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***Abstract*—Future wireless communication calls for exploration of more efficient use of wireless channel capacity to meet the increasing demand on higher data rate and less latency. However, while ergodic capacity and instantaneous capacity are fundamental properties of a wireless channel, they are in many cases not sufficient for use in assessing if data transmission over the channel meets the quality of service (QoS) requirements. To address this limitation, we focus on the concept of "cumulative capacity", which is the cumulated capacity over a time period, and study its properties. Specifically, the *moment generating function*, the *stochastic service curve*, and the *Mellin transform* properties of *cumulative capacity* are investigated and related results are derived. It is appealing that with these properties, QoS assessment of data transmission over the channel can be further performed based on stochastic network calculus, a newly developed theory for (stochastic) QoS analysis. To demonstrate the derived properties, a Rayleigh fading channel is considered and numerical results are provided.**

## I. Introduction

In future wireless communication, there will be a continuing wireless data explosion and an increasing demand on higher data rate and less latency. It has been predicted [1] that the amount of IP data handled by wireless networks will exceed $500$ exabytes by 2020, the aggregate data rate and edge rate will increase respectively by $1000\times$ and $100\times$ from 4G to 5G, and the round-trip latency needs to be less than 1ms in 5G. Evidently, it becomes more and more crucial to explore the ultimate capacity that the wireless channel can provide and to guarantee pluralistic quality of service (QoS) for seamless user experience.

To date, wireless channel capacity has mostly been analyzed for its average rate in the asymptotic regime, e.g., ergodic capacity, or in one time instant/short time slot, e.g. instantaneous capacity. However, these properties of wireless channel capacity are in many cases not sufficient for use in assessing if data transmission over the channel meets its QoS requirements [2]. This calls for studying other properties of wireless channel capacity, which can be easily used for QoS analysis.

The focus of this paper is on a wireless channel capacity concept useful for finite time regimes [2], which is the time-period "cumulative capacity" function or the cumulated capacity over the time period. In particular, we investigate its *moment generating function* [3], *stochastic service curve* [4], and *Mellin transform* [5] properties. Based on these properties, QoS assessment of data transmission over the channel can be further performed by exploiting the stochastic network calculus theory (see e.g. [4]). Specifically, based on the *moment generating function*, the *stochastic service curve*, and the *Mellin transform* of cumulative capacity, various QoS performance analysis results can be readily found respectively from [3], [4] and [5].

In the investigation, both the independent and the more general possibly dependent cases are considered. For the independent case, central limit theorem is applied to obtain simplified result expression, while for the dependent case, copula bounds are made use of. To demonstrate the analysis, the cumulative capacity properties of a Rayleigh fading channel are exemplified with numerical results.

The remainder of the paper is structured as follows. In Sec. II, the wireless channel model and cumulative capacity are introduced. Also in this section, the probability distribution function property of cumulative capacity is investigated. In Sec. III, further properties of cumulative capacity are derived, which include its moment generating function, stochastic strict service curve, and Merlin transform properties. These results are exemplified for a Rayleigh fading channel. Numerical results are presented in Sec. IV. Finally, conclusion is made in Sec. V.

## II. Cumulative Capacity and its Distribution Function Property

### A. Cumulative Capacity

Consider a wireless channel. We assume discrete time $t = 1, 2, \ldots$, and that the instantaneous capacity [6] or mutual information [7] $C(t)$ of the channel at time $t$ can be expressed as a function of the instantaneous SNR $\gamma_t$ at this time [8]:

$$C(t) = \log_2(g(\gamma_t)) \text{ (bits/s/Hz)}. \quad (1)$$

For single input single output (SISO) channels, if CSI is only known at the receiver, the instantaneous capacity or the mutual information of the channel, assuming flat fading, can be expressed as

$$C(t) = \log_2(1 + \gamma|h(t)|^2), \quad (2)$$

where $h(t)$ is a stochastic process describing the fading behavior, $|h(t)|$ denotes the envelope of $h(t)$, $\gamma = P/N_0W$ denotes the average received SNR per complex degree of freedom, $P$ is the average transmission power per complex symbol, $N_0/2$ is the power spectral density of AWGN, and $W$ is the channel bandwidth.

We define the *cumulative capacity* through period $(s,t]$ as

$$S(s,t) \equiv \sum_{i=s+1}^{t} C(i), \tag{3}$$

where $C(i)$ is the instantaneous capacity at time $i$.

### B. Distribution Function of Cumulative Capacity

For random variable $X$, we denote by $F_X$, $f_X$, $E[X]$ and $\sigma^2[X]$ its cumulative distribution function (CDF), probability distribution function (PDF), mean and variance respectively.

The cumulative capacity $S(s,t)$ is essentially a random variable that is fully described by its distribution functions. In this paper, we focus on the CDF of $S(s,t)$, which is

$$F_{S(s,t)}(x) \equiv P\{S(s,t) \le x\}, \tag{4}$$

where $x > 0$ by the nature of $S(s,t)$ that is non-negative.

*1) Independent Case:* If $C(i)$ and $C(j)$, $i \neq j$, are independent, $f_{S(s,t)} = f_{C(s+1)} * \ldots * f_{C(t)}$, where $*$ denotes the convolution operation. Hence,

$$F_{S(s,t)}(x) = \int_{-\infty}^{x} f_{S(s,t)}(y)dy. \tag{5}$$

In addition, when the length of the period, $t-s$, is large, according to the central limit theorem, $F_{S(s,t)}(x)$ approaches a normal distribution with mean $E[S(s,t)]$ and variance $\sigma^2[S(s,t)]$(under certain general conditions) [9], i.e.,

$$F_{S(s,t)}(x) \approx G\left(\frac{x - E[S(s,t)]}{\sigma^2[S(s,t)]}\right), \tag{6}$$

where $E[S(s,t)] = \sum_{i=s+1}^{t} E[C(i)]$, $\sigma^2[S(s,t)] = \sum_{i=s+1}^{t} \sigma^2[C(i)]$, and $G(x) \equiv \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}dy$.

*2) Dependent Case:* If it is unknown if $C(i)$ and $C(j)$, $i \neq j$, are independent, the above independent case analysis is no more applicable. In the following, instead of trying to find accurate representation of $F_{S(s,t)}$, we turn to investigate bounds on it. In particular, the following copula inequalities are utilized.

*Lemma 1:* Let $G$ denote the distribution function of $X_1 + \ldots + X_N$, where $X_n$, $n = 1, \ldots, N$, are random variables with distribution functions $F_{X_n}$. Then there holds [10], [11]

$$\check{G}(z) \le G(z) \le \hat{G}(z), \tag{7}$$

where

$$\check{G}(z) = \sup_{x_1+\ldots+x_N=z} \{W(F_{X_1}(x_1), \ldots, F_{X_N}(x_N))\}, \tag{8}$$

$$\hat{G}(z) = \inf_{x_1+\ldots+x_N=z} \left\{\tilde{W}(F_{X_1}(x_1), \ldots, F_{X_N}(x_N))\right\}, \tag{9}$$

with $W(u_1, \ldots, u_N) = \left[\sum_{i=1}^{N} u_i - (N-1)\right]^+$, and $\tilde{W}(u_1, \ldots, u_N) = \sum_{i=1}^{N} u_i - W(u_1, \ldots, u_N) = \left[\sum_{i=1}^{N} u_i\right]_1$.

With Lemma 1, it is easily verified that the CDF of the channel cumulative capacity satisfies the following inequalities:

$$F^l_{S(s,t)}(r) \le F_{S(s,t)}(r) \le F^u_{S(s,t)}(r), \tag{10}$$

where

$$F^u_{S(s,t)}(r) \equiv \inf_{\sum_{i=s+1}^{t} r_i = r} \left[\sum_{i=s+1}^{t} F_{C(i)}(r_i)\right]_1, \tag{11}$$

$$F^l_{S(s,t)}(r) \equiv \sup_{\sum_{i=s+1}^{t} r_i = r} \left[\sum_{i=s+1}^{t} F_{C(i)}(r_i) - (t-s)\right]^+ \tag{12}$$

### C. Discussion

Note that, in the literature, PDF or CDF of the instantaneous capacity is available for various types of channels, e.g. Rayleigh channel [12], Rice channel [13], Nakagami-$m$ channel [14], Suzuki channel [15], and more [8]. Specifically, the CDF of the Rayleigh channel instantaneous capacity is expressed as [12]

$$F_{C(t)}(r) = 1 - e^{-(2^r-1)/\gamma}. \tag{13}$$

With the analysis in this section, the results can be readily extended to the cumulative capacity.

However, up to this point of the paper, it is still not clear yet how such PDF / CDF results for instantaneous capacity and cumulative capacity may be used for QoS analysis, e.g. delay analysis. To bridge the gap, we study in the next section the moment generating function, the stochastic strict service curve, and the Mellin transform properties of the cumulative capacity, since, with these properties, various QoS performance analysis results can be readily found respectively from [3], [4] and [5] of the stochastic network calculus literature.

## III. Further Properties of Cumulative Capacity

In this section, further properties of wireless channel capacity are investigated. Specifically, they are the moment generating function, stochastic strict service curve, and Mellin transform properties of cumulative capacity. To exemplify the analysis, a Rayleigh channel is used.

### A. Moment Generating Function

The moment generating function (MGF) of $S(s,t) = \sum_{i=s+1}^{t} C(i)$, denoted by $M_{S(s,t)}(\theta)$, is [3], [4]

$$M_{S(s,t)}(\theta) \equiv E\left[e^{\theta S(s,t)}\right] = \int_{-\infty}^{\infty} e^{\theta r} dF_{S(s,t)}(r), \tag{14}$$

where $\theta$ is a real variable. We denote $\overline{M}_{S(s,t)}(\theta) = M_{S(s,t)}(-\theta) = E\left[e^{-\theta S(s,t)}\right]$.

With $F_{S(s,t)}(x)$ derived in the previous section, $M_{S(s,t)}(\theta)$ is readily obtained. Specifically, for the independent case, we have

$$\overline{M}^{i}_{S(s,t)}(\theta) = \Pi_{i=s+1}^{t}\overline{M}_{C(i)}(\theta) \tag{15}$$

$$\approx \int_{-\infty}^{\infty} e^{-\theta r} dG\left(\frac{r-E[S(s,t)]}{\sigma^2[S(s,t)]}\right), \tag{16}$$

where for the independent and identically distributed (i.i.d.) case, (15) becomes

$$\overline{M}^{ii}_{S(s,t)}(\theta) = (\overline{M}_{C(i)}(\theta))^{t-s}. \tag{17}$$

For the more general case,

$$\overline{M}^{dl}_{S(s,t)}(\theta) \leq \overline{M}^{d}_{S(s,t)}(\theta) \leq \overline{M}^{du}_{S(s,t)}(\theta) \tag{18}$$

with

$$\overline{M}^{dl}_{S(s,t)}(\theta) = \int_{-\infty}^{\infty} e^{-\theta r} dF^{l}_{S(s,t)}(r), \tag{19}$$

$$\overline{M}^{du}_{S(s,t)}(\theta) = \int_{-\infty}^{\infty} e^{-\theta r} dF^{u}_{S(s,t)}(r). \tag{20}$$

*1) Rayleigh channel as an example:* For the i.i.d. case,

$$\overline{M}^{ii}_{S(s,t)}(\theta) = \left(\int_0^{\infty} e^{-\theta r} d\left\{1-e^{-(2^r-1)/\gamma}\right\}\right)^{t-s} \tag{21}$$

$$= \left(\frac{\ln 2}{\gamma}\int_0^{\infty} e^{-\theta r} 2^r e^{-(2^r-1)/\gamma} dr\right)^{t-s}. \tag{22}$$

For the general (possibly dependent) case, the upper bound and the lower bound of the MGF $\overline{M}^{d}_{S(s,t)}(\theta)$ are expressed as

$$\overline{M}^{du}_{S(s,t)}(\theta)$$

$$= \int_0^{\infty} e^{-\theta r} d\left\{\left[(t-s)\left(1-e^{-\left(2^{\frac{r}{t-s}}-1\right)/\gamma}\right)\right]_1\right\} \tag{23}$$

$$= (t-s)\left(1-\exp\left(\frac{1-2^{\Omega_u/(t-s)}}{\gamma}-\theta\Omega_u\right)\right)$$

$$-\theta(t-s)\int_0^{\Omega_u} \exp\left(\frac{1-2^{r/(t-s)}}{\gamma}-\theta r\right) dr, \tag{24}$$

$$\overline{M}^{dl}_{S(s,t)}(\theta)$$

$$= \int_0^{\infty} e^{-\theta r} d\left\{\left[1-(t-s)e^{-\left(2^{\frac{r}{t-s}}-1\right)/\gamma}\right]^+\right\} \tag{25}$$

$$= (t-s)\exp\left(\frac{1-2^{\Omega_l/(t-s)}}{\gamma}-\theta\Omega_l\right)$$

$$-\theta(t-s)\int_{\Omega_l}^{\infty} \exp\left(\frac{1-2^{r/(t-s)}}{\gamma}-\theta r\right) dr, \tag{26}$$

where $\Omega_u = \tau\log_2(1-\gamma\log(1-1/\tau))$ for $\tau > 1$, $\Omega_l = \tau\log_2(1-\gamma\log(1/\tau))$ for $\tau \geq 1$, and $\tau = t-s$.

*2) Application to effective capacity:* Suppose the service process has stationary increments, i.e., $S(s,s+t) =_{st} S(t)$ for all $s,t \leq 0$, the effective capacity of $S$ is defined as [4], [16]

$$\mathring{r}^{(c)}(\theta) \equiv -\limsup_{t\to\infty}\frac{1}{\theta t}\sup_{s\geq 0}\log E\left[e^{-\theta(S(s,s+t))}\right], \tag{27}$$

where $\theta$ is a real variable.

The effective capacity calculated through the instantaneous capacity follows directly from (16) and (18), i,e.,

$$\mathring{r}^{(c)}_{i}(\theta) = -\limsup_{t\to\infty}\frac{1}{\theta t}\sup_{s\geq 0}\log\overline{M}^{i}_{S(s,s+t)}(\theta), \tag{28}$$

$$\mathring{r}^{(c)}_{d}(\theta) = -\limsup_{t\to\infty}\frac{1}{\theta t}\sup_{s\geq 0}\log\overline{M}^{d}_{S(s,s+t)}(\theta), \tag{29}$$

for the independent and dependent cases respectively.

For the Rayleigh fading channel, $\frac{1}{\theta\tau}\log\overline{M}^{ii}_{S(\tau)}(\theta)$ is constant with $\tau$, $\frac{1}{\theta\tau}\log\overline{M}^{du}_{S(\tau)}(\theta)$ increases with $\tau$, while $\frac{1}{\theta\tau}\log\overline{M}^{dl}_{S(\tau)}(\theta)$ decreases with $\tau$. The effective capacity for the i.i.d. case and dependent case are respectively,

$$\mathring{r}^{(c)}_{ii}(\theta) = -\frac{1}{\theta}\log\left(\frac{\ln 2}{\gamma}\int_0^{\infty} e^{-\theta r}2^r e^{-(2^r-1)/\gamma} dr\right), \tag{30}$$

$$\mathring{r}^{(c)}_{dl}(\theta) = -\lim_{\tau\to\infty}\frac{1}{\theta\tau}\log\Big(\tau(1-\exp((1-2^{\Omega_u/\tau})/\gamma$$

$$-\theta\Omega_u)) - \theta\tau\int_0^{\Omega_u}\exp\left((1-2^{r/\tau})/\gamma-\theta r\right) dr\Big), \tag{31}$$

$$\mathring{r}^{(c)}_{du}(\theta) = -\frac{1}{\theta}\log\Big(\tau\exp\left((1-2^{\Omega_l/\tau})/\gamma-\theta\Omega_l\right)$$

$$-\theta\tau\int_{\Omega_l}^{\infty}\exp\left((1-2^{r/\tau})\gamma-\theta r\right) dr\Big). \tag{32}$$

### *B. Stochastic Strict Service Curve*

A system is said to be a stochastic strict server providing stochastic strict service curve (SSSC) $\beta$ with bounding function $g$, if during any period $(s,t]$ the amount of service $S(s,t)$ provided by the system satisfies [4],

$$P\{S(s,t) < \beta(t-s)-x\} \leq g(x), \tag{33}$$

for all $x \geq 0$.

*1) Independent Case:* If $C(i)$ and $C(j)$, $i \neq j$, are independent, applying (6), the expression of SSSC is obtained

$$P\{S(s,t) \leq \beta(t-s)-x\}$$

$$\approx G\left(\frac{(\beta(t-s)-x)-E[S(s,t)]}{\sigma^2[S(s,t)]}\right) = g^{i}_{s,t}(x). \tag{34}$$

If $S(s,t)$ also has stationary increments, i.e., $C(i) =_{st} C(j)$, the SSSC can be expressed as [17]

$$\beta^{i}(t-s) = \frac{1}{-\theta}\log\overline{M}^{i}_{S(s,t)}(\theta) \tag{35}$$

$$= \frac{t-s}{-\theta}\log\overline{M}_{C}(\theta), \tag{36}$$

with bounding function $g(x) = e^{-\theta x}$.

*2) Dependent Case:* If $S(s,t)$ has stationary increments, the SSSC can be expressed as [17]

$$\beta^d(t-s) \quad = \quad \frac{1}{-\theta}\log \overline{M}^d_{S(s,t)}(\theta), \tag{37}$$

with bounding function $g(x) = e^{-\theta x}$.

More generally, if $C(i)$ and $C(j)$, $i \neq j$, are dependent, replacing $r = \beta(t-s) - x$ in (11) and (12), the SSSC of the capacity is obtained

$$\begin{aligned} g^{dl}_{s,t}(x) &\equiv F^l_{S(s,t)}(\beta(t-s)-x) \\ &\leq P\left\{S(s,t) \leq \beta(t-s)-x\right\} \leq \\ &\quad F^u_{S(s,t)}(\beta(t-s)-x) \equiv g^{du}_{s,t}(x). \end{aligned} \tag{38}$$

*3) Rayleigh channel as an example:* For the i.i.d. case, applying (36),

$$\beta^{ii}(t-s) = \frac{t-s}{-\theta}\log \overline{M}_C(\theta) \tag{39}$$

$$= \frac{t-s}{-\theta}\log\left(\frac{\ln 2}{\gamma}\int_0^\infty e^{-\theta r}2^r e^{-(2^r-1)/\gamma}dr\right), \tag{40}$$

with bounding function $g(x) = e^{-\theta x}$.

If $S(s,t)$ has stationary increments, the lower bound and upper bound of the SSSC can be expressed as

$$\begin{aligned} \beta^{dl}(\tau) &= \frac{1}{-\theta}\log \overline{M}^{du}_{S(\tau)}(\theta) \\ &= \frac{1}{-\theta}\log\Big(\tau(1-\exp((1-2^{\Omega_u/\tau})/\gamma - \theta\Omega_u)) \\ &\quad - \theta\tau\int_0^{\Omega_u}\exp\Big((1-2^{r/\tau})/\gamma - \theta r\Big)\,dr\Big), \end{aligned} \tag{41}$$

$$\begin{aligned} \beta^{du}(\tau) &= \frac{1}{-\theta}\log \overline{M}^{dl}_{S(\tau)}(\theta) \\ &= \frac{1}{-\theta}\log\Big(\tau\exp\Big((1-2^{\Omega_l/\tau})/\gamma - \theta\Omega_l\Big) \\ &\quad - \theta\tau\int_{\Omega_l}^{\infty}\exp\Big((1-2^{r/\tau})/\gamma - \theta r\Big)\,dr\Big), \end{aligned} \tag{42}$$

where $\tau = t-s$, with bounding function $g(x) = e^{-\theta x}$.

For the dependent case, the lower bound and upper bound of the SSSC are expressed as

$$\begin{aligned} &P\left\{S(s,t) \leq \beta(t-s)-x\right\} \\ &\leq \inf_{\sum_{i=s+1}^{t} r_i = \beta(t-s)-x}\left\{\left[\sum_{i=s+1}^{t}\left(1-e^{-(2^{r_i}-1)/\gamma}\right)\right]_1\right\} \\ &\leq \left[(t-s)\left(1-\exp\left(-\left(2^{\frac{\beta(t-s)-x}{t-s}}-1\right)\Big/\gamma\right)\right)\right]_1 \equiv \dot{g}^{du}_{s,t}(x), \end{aligned}$$

$$\begin{aligned} &P\left\{S(s,t) \leq \beta(t-s)-x\right\} \\ &\geq \sup_{\sum_{i=s+1}^{t} r_i = \beta(t-s)-x}\left[\sum_{i=s+1}^{t}\left(1-e^{-(2^{r_i}-1)/\gamma}\right)-(t-s-1)\right]^+ \\ &\geq \left[1-(t-s)\exp\left(-\left(2^{\frac{\beta(t-s)-x}{t-s}}-1\right)\Big/\gamma\right)\right]^+ \equiv \dot{g}^{dl}_{s,t}(x). \end{aligned}$$

Let $\dot{g}^{du}_{s,t}(x) = \varepsilon$, $\tau = t-s$, and $\beta^d_l(\tau) = \beta(\tau) - x$, we obtain

$$\beta^d_l(\tau) = \tau\log_2\left(1-\gamma\log\left(1-\frac{\varepsilon}{\tau}\right)\right). \tag{43}$$

Let $\dot{g}^{dl}_{s,t}(x) = \varepsilon$, $\tau = t-s$, and $\beta^d_u(\tau) = \beta(\tau) - x$, we obtain

$$\beta^d_u(\tau) = \tau\log_2\left(1-\gamma\log\left(\frac{1-\varepsilon}{\tau}\right)\right). \tag{44}$$

### C. Mellin Transform

If the channel capacity is expressed with the natural logarithm, i.e.,

$$C(t) = \log(g(\gamma_t)) \text{ nats/s/Hz}, \tag{45}$$

in order to circumvent the calculation of logarithm, the cumulative capacity may be transformed to a new domain, e.g., the MGF domain, with the exponential,

$$\mathcal{S}(s,t) = e^{S(s,t)} = \prod_{i=s+1}^{t} g(\gamma_i). \tag{46}$$

Then the MGF property analysis applies.

Analog to the MGF for the cumulative capacity, Mellin transform (MT) has also been proposed [5], i.e.,

$$\mathcal{M}_{\mathcal{S}(s,t)}(\vartheta) \equiv E[\mathcal{S}^{\vartheta-1}(s,t)] \tag{47}$$

$$= \int_{-\infty}^{\infty} r^{\vartheta-1}dF_{\mathcal{S}(s,t)}(r) = \int_{-\infty}^{\infty} r^{\vartheta-1}dF_{S(s,t)}(r^*), \tag{48}$$

where $r^* = \log_2 e \log r$, for any complex variable $\vartheta$ for which the right hand side of (47) exits.

According to (6) and (10), the MT of the cumulative capacity for the independent and dependent case can be derived,

$$\mathcal{M}^i_{\mathcal{S}(s,t)}(\vartheta) \approx \int_{-\infty}^{\infty} r^{\vartheta-1}dG\left(\frac{r^* - E[S(s,t)]}{\sigma^2[S(s,t)]}\right), \tag{49}$$

$$\mathcal{M}^{dl}_{\mathcal{S}(s,t)}(\vartheta) = \int_{-\infty}^{\infty} r^{\vartheta-1}dF^l_{S(s,t)}(r^*) \leq \mathcal{M}^d_{\mathcal{S}(s,t)}(\vartheta) \tag{50}$$

$$\leq \int_{-\infty}^{\infty} r^{\vartheta-1}dF^u_{S(s,t)}(r^*) = \mathcal{M}^{du}_{\mathcal{S}(s,t)}(\vartheta), \tag{51}$$

where (50) and (51) hold when $\vartheta - 1 < 0$.

*1) Rayleigh channel as an example:* For the i.i.d. case [5],

$$\mathcal{M}^{ii}_{\mathcal{S}(s,t)}(\vartheta) = \left(e^{1/\gamma}\gamma^{\vartheta-1}\int_{\gamma^{-1}}^{\infty} r^{\vartheta-1}e^{-r}dr\right)^{t-s}. \tag{52}$$

For the dependent case,

$$\mathcal{M}^{du}_{\mathcal{S}(s,t)}(\vartheta)$$

$$= \int_0^\infty r^{\vartheta-1}d\left\{\left[(t-s)\left(1-e^{-\left(2^{\frac{r^*}{t-s}}-1\right)\Big/\gamma}\right)\right]_1\right\} \tag{53}$$

$$= \frac{\log 2\log_2 e}{\gamma} \times \int_1^{e^{\Omega_u/\log_2 e}} r^{\vartheta-2} \times 2^{\frac{r^*}{t-s}} \times e^{\frac{1-2^{\frac{r^*}{t-s}}}{\gamma}}dr, \tag{54}$$

$$
\begin{aligned}
&\mathcal{M}^{dl}_{\mathcal{S}(s,t)}(\vartheta) \\
&= \int_0^\infty r^{\vartheta-1} d\left\{\left[1-(t-s)e^{-\left(2^{\frac{r^*}{t-s}}-1\right)/\gamma}\right]^+\right\} && (55)\\
&\geq \int_0^\infty r\times e^{(\vartheta-2)r} d\left\{\left[1-(t-s)e^{-\left(2^{\frac{r^*}{t-s}}-1\right)/\gamma}\right]^+\right\} && (56)\\
&= \frac{\log 2\log_2 e}{\gamma}\times \int_{e^{\Omega_l/\log_2 e}}^{\infty} e^{(\vartheta-2)r}\times 2^{\frac{r^*}{t-s}}\times e^{\frac{1-2^{\frac{r^*}{t-s}}}{\gamma}} dr, && (57)
\end{aligned}
$$

where $\vartheta - 1 < 0$, $\Omega_u = \tau\log_2(1-\gamma\log(1-1/\tau))$ for $\tau > 1$, $\Omega_l = \tau\log_2(1-\gamma\log(1/\tau))$ for $\tau \geq 1$, and $\tau = t-s$.

### D. Remark

In this subsection, service guarantees for periodic arrival at the i.i.d. Rayleigh fading channel are illustrated taking advantage of results in the previous subsections. Unit bandwidth is assumed.

For periodic arrival $A(t)$, the source produces $\sigma$ units of workload at times $\{U\eta + n\eta, n = 0, 1, \ldots\}$, where $\eta$ is the period time length and $U$ is the initial start time that is uniformly distributed on the interval $[0, 1]$. For all $t \geq 0$ and $\theta \geq 0$, the MGF of $A(t)$ is known [18]

$$M_{A(t)}(\theta) = e^{\theta\sigma\lfloor\frac{t}{\eta}\rfloor}\left(1+\left(\frac{t}{\eta}-\left\lfloor\frac{t}{\eta}\right\rfloor\right)\left(e^{\theta\sigma}-1\right)\right). \quad (58)$$

Specifically, when $\frac{t}{\eta} = \left\lfloor\frac{t}{\eta}\right\rfloor$, $M_{A(t)}(\theta) = e^{\theta\sigma\frac{t}{\eta}}$. In addition, $A(t)$ is bounded by a deterministic arrival curve [17]

$$\alpha(t) = \frac{\sigma}{\eta}t + \sigma. \quad (59)$$

Moreover, the MT of $\mathcal{A}(t) \equiv e^{A(t)}$ is $\mathcal{M}_{\mathcal{A}(t)}(\vartheta) = M_{A(t)}(\vartheta - 1)$.

According to [3] and [5], the backlog $b$ and delay $d$ with violation probability $\epsilon$ can be derived respectively based on MGF and MT with the identical results

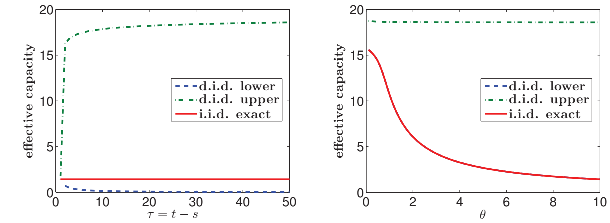
$$
\begin{aligned}
b &= \inf_{\theta>0}\left[\frac{1}{\theta}\left(\log\frac{1}{1-X}-\log\epsilon\right)\right], && (60)\\
d &= \inf_{\theta>0}\left[\log_Z\left(\epsilon(1-X)\right)\right], && (61)
\end{aligned}
$$

under the condition that $X < 1$, where $X = e^{\frac{\theta\sigma}{\eta}}Z$ and $Z = \frac{\ln 2}{\gamma}\int_0^\infty e^{-\theta r}2^r e^{-(2^r-1)/\gamma}dr$.

According to [4], the backlog $b$ and delay $d$ with violation probability $\epsilon$ can be derived based on (stochastic) arrival curve and SSSC respectively

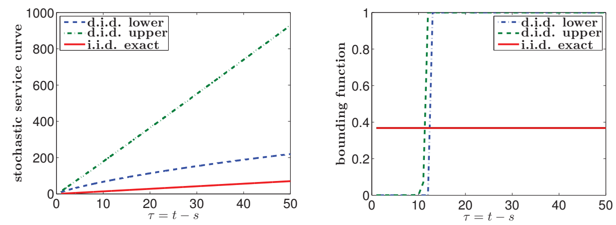
$$
\begin{aligned}
b &= \sigma - \frac{1}{\theta}\log\epsilon, && (62)\\
d &= \frac{-\theta\left(\sigma-\frac{1}{\theta}\log\epsilon\right)}{\log\left(\frac{\ln 2}{\gamma}\int_0^\infty e^{-\theta r}2^r e^{-(2^r-1)/\gamma}dr\right)}, && (63)
\end{aligned}
$$

under the same condition for (60) and (61), and $\theta$ is free parameter for optimization.




(a) Effective capacity subject to $\tau$ (b) Effective capacity subject to $\theta$

Fig. 1. $\gamma = 50$ dB, $\theta = 10$ for (a), and $\gamma = 50$ dB, $\tau = 1$ for $\mathring{r}^{(c)}_{dl}(\theta)$ and $\mathring{r}^{(c)}_{ii}(\theta)$, $\tau = 50$ for $\mathring{r}^{(c)}_{du}(\theta)$ for (b).




(a) SSSC bounds (b) Bounding functions

Fig. 2. $\gamma = 50$ dB, $\varepsilon = 0.01$, and $\theta = 10$.

## IV. Numerical Results

We devote this section to numerical results of MGF, SSSC, and MT of the cumulative capacity of the Rayleigh fading channel. Specifically, the i.i.d. case, stationary increments (s.i.) case, and dependent and identically distributed (d.i.d.) case are included and compared.

### A. Moment Generating Function

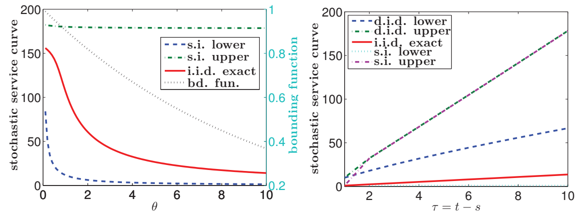
It is illustrated in Fig. 1 (a) that $\frac{1}{\theta\tau}\log\overline{M}^{ii}_{S(\tau)}(\theta)$ is constant with $\tau$, $\frac{1}{\theta\tau}\log\overline{M}^{du}_{S(\tau)}(\theta)$ increases with $\tau$, while $\frac{1}{\theta\tau}\log\overline{M}^{dl}_{S(\tau)}(\theta)$ decreases with $\tau$. It is shown in Fig. 1 (b) that ECs decrease with $\theta$. Particularly, the lower bound is very close to the exact result and they almost coincide, and the upper bound deviates from the exact result by a large extent.

### B. Stochastic Strict Service Curve

The bounds of SSSCs and bounding functions for the d.i.d. case and the i.i.d. case are shown in Fig. 2. It is illustrated in Fig. 2 (a) that SSSCs of the d.i.d. case are upper bounds of that of the i.i.d. case, and in Fig. 2 (b) that the bounding functions of the d.i.d. case increase with $\tau$, while the bounding function of the i.i.d. case is constant with $\tau$.

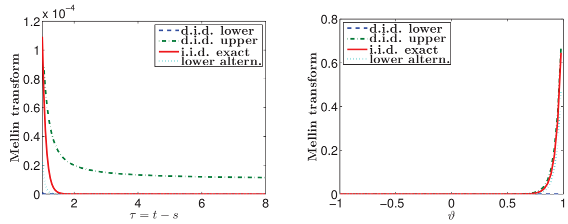
The bounds of SSSCs and bounding functions for the s.i. case and i.i.d. case subject to $\theta$ are shown in Fig. 3 (a). The SSSCs and bounding function decrease with $\theta$, which means that, if there is no additional knowledge or constraint, $\theta$ should be set big for tight SSSCs and bounding functions.

The SSSCs of the i.i.d. case, s.i. case, and d.i.d. case are compared in Fig. 3 (b). The SSSCs for the i.i.d. case and s.i. case are based on MGF, while the SSSC for the d.i.d. case are based on the manipulation of the CDF of the cumulative



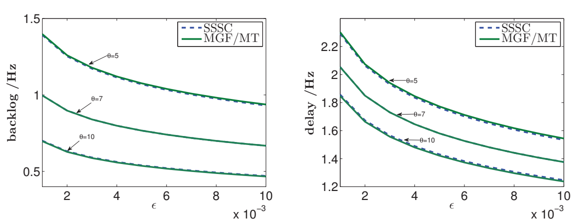

(a) SSSCs vs. bounding functions (b) SSSCs comparision

Fig. 3. $\gamma = 50$ dB, $\tau = 10$, $x = 0.1$ for (a), and $\gamma = 50$ dB, $\varepsilon = 0.01$, $\theta = 10$ for (b).




(a) MT subject to $\tau$ (b) MT subject to $\vartheta$

Fig. 4. $\gamma = 50$ dB, $\vartheta = 0$ for (a), $\gamma = 50$ dB, $\tau = 2$ for (b), and $\Omega^* = 9e^{\Omega_l/\log_2 e}$.




(a) Backlog subject to $\epsilon$ (b) Delay subject to $\epsilon$

Fig. 5. $\gamma = 50$ dB, $\sigma = 0.01$, $\eta = 0.1$, and $\theta = 10$.

capacity. The SSSCs of the d.i.d. case are upper bounds of that of the i.i.d. case, while the SSSCs of the s.i. case are the lower bound and upper bound of that of the i.i.d. case. Particularly, the upper bound of the s.i. case is close to the upper bound of the d.i.d. case.

### C. Mellin Transform

It is illustrated in Fig. 4 that the lower bound of the MT of the cumulative capacity decreases faster with $\tau$ and increases slower with $\vartheta$ than the upper bound of the d.i.d. case and the i.i.d. case, which is because that $e^{(\vartheta-2)r}$ decreases faster than $r^{\vartheta-2}$, i.e., a tighter lower bound requires a tighter lower bound of $r^{\vartheta-1}$ than $r \times e^{\vartheta-2}$. An alternative for the lower bound is to use the integral of (55) on $[e^{\Omega_l/\log_2 e}, \Omega^*]$, $\Omega^* < \infty$, because it is not integrable on $[e^{\Omega_l/\log_2 e}, \infty)$.

### D. Service Guarantees

The backlog and delay of the periodic arrival at the i.i.d. Rayleigh channel are shown in Fig. 5. The assumption of unit bandwidth can be interpreted as the equivalent service guarantees for the arrival based on the unit bandwidth. It is illustrated that the backlog and delay decrease with $\theta$, the results from the two approaches equal when $\theta \approx 7$, the results based on the MGF or MT are tighter when $\theta > 7$, and looser when $\theta < 7$.

## V. Conclusion

In this paper, we investigated a new statistical quantity of wireless channel capacity, i.e., the cumulative capacity that is the cumulated capacity through a period. Based on the CDF of the cumulative capacity, we derived the MGF, SSSC, and MT of the cumulative capacity. These further properties of wireless channel capacity are essential building blocks for QoS analysis with stochastic network calculus. Application to the Rayleigh fading channel and service guarantees for periodic arrival are illustrated as concretization of the generic analysis.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